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On discrete Brunn-Minkowski type inequalities
(Desigualdades de tipo Brunn-Minkowski discretas)

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Contents

Resumen

Relacionando el volumen, $\mathrm{vol}(\cdot)$ (es decir, la medida de Lebesgue), y la suma de Minkowski (o suma vectorial) $+$ de conjuntos compactos, nos encontramos con la famosa *desigualdad de Brunn-Minkowski*. Ésta puede enunciarse diciendo que si $K, L \subset \mathbb{R}^n$ son compactos y no vacíos, entonces

$$\mathrm{vol}(K+L)^{1/n} \geq \mathrm{vol}(K)^{1/n} + \mathrm{vol}(L)^{1/n}, \tag{$*$}$$

dándose la igualdad, cuando $\mathrm{vol}(K)\mathrm{vol}(L) > 0$, si, y sólo si, K y L son conjuntos convexos compactos homotéticos. Existen diversas versiones geométricas equivalentes de este resultado, siendo especialmente interesante la llamada *forma de Blaschke* de la desigualdad de Brunn-Minkowski: si $K, L \subset \mathbb{R}^n$ son conjuntos convexos y compactos, y B_K, B_L son las bolas (cerradas y centradas en el origen) tales que $\mathrm{vol}(B_K) = \mathrm{vol}(K)$ y $\mathrm{vol}(B_L) = \mathrm{vol}(L)$, entonces

$$\mathrm{vol}(K+L) \geq \mathrm{vol}(B_K + B_L). \tag{$\dagger$}$$

La desigualdad de Brunn-Minkowski es uno de los resultados más potentes en Geometría Convexa y más allá, y tiene extensiones en otros campos. El equivalente analítico para funciones de $(*)$ nos lleva de forma natural a la llamada *desigualdad de Borell-Brascamp-Lieb*: dado $p \geq -1/n$, $p \neq 0$, si $f, g, h : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ son funciones medibles no-negativas (con integrales no nulas), tales que $h(x+y) \geq \big(f(x)^p + g(y)^p\big)^{1/p}$ para cualesquiera $x, y \in \mathbb{R}^n$ con $f(x)g(y) > 0$, entonces

$$\int_{\mathbb{R}^n} h(x)\,\mathrm{d}x \geq \left[\left(\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x\right)^q + \left(\int_{\mathbb{R}^n} g(x)\,\mathrm{d}x\right)^q\right]^{1/q}, \tag{$\ddagger$}$$

donde $q = p/(np+1)$; aquí, y en lo que sigue, los casos $p = \pm\infty$ deben entenderse como los casos límite usuales.

Hoy en día existe un creciente interés por obtener equivalentes discretos de desigualdades geométricas clásicas, es decir, versiones discretas en las cuales los conjuntos compactos y el volumen se reemplazan por conjuntos finitos de puntos y por el cardinal, respectivamente. Desde luego, la desigualdad de Brunn-Minkowski es uno de los primeros resultados a considerar en el contexto discreto.

Es muy sencillo comprobar que no existe una desigualdad de Brunn-Minkowski para el cardinal en su forma clásica $(*)$. También merece la pena observar que no es fácil obtener dicha desigualdad

a partir de la desigualdad de Brunn-Minkowski, como apuntaron Gardner y Gronchi en [8, páginas 3996-3997]. Por lo tanto, una versión discreta de $(*)$ debería, o bien tener una estructura diferente, o bien involucrar modificaciones de los conjuntos. En la primera línea de trabajo, Gardner y Gronchi [8] obtuvieron un análogo discreto de la forma de Blaschke (†) de la desigualdad de Brunn-Minkowski: demostraron que si A, B son subconjuntos finitos del retículo entero $\mathbb{Z}^n$, con $\dim B = n$, entonces

$$|A + B| \geq \left|\mathrm{D}^{\mathrm{B}}_{|\mathrm{A}|} + \mathrm{D}^{\mathrm{B}}_{|\mathrm{B}|}\right|. \qquad (\S)$$

Aquí $\mathrm{D}^{\mathrm{B}}_{|\mathrm{A}|}$ es un *B-segmento inicial*: *grosso modo*, como una intersección de un símplice con $\mathbb{Z}^n$.

Esta tesis doctoral está dedicada a determinar nuevas versiones discretas de las desigualdades de Brunn-Minkowski y de Borell-Brascamp-Lieb, manteniendo la estructura de las correspondientes desigualdades clásicas, y por tanto, modificando uno de los conjuntos involucrados. También probaremos que estas nuevas desigualdades implican las (pertinentes) versiones clásicas.

El trabajo comienza con una primera sección en la que establecemos la notación e introducimos los conceptos y resultados que serán necesarios más adelante. También recordamos varias desigualdades para sumas que jugarán un papel relevante en las demostraciones de nuestros resultados. A continuación presentamos la desigualdad de Brunn-Minkowski en el contexto clásico (continuo), a saber, $(*)$, junto con varias versiones (equivalentes) de la misma. Nos trasladamos entonces al marco discreto: recopilamos algunas desigualdades relevantes que, aun cambiando la estructura de la desigualdad de Brunn-Minkowski, pueden verse como versiones discretas de la misma. Hacemos especial énfasis en la desigualdad de Gardner y Gronchi (§), y concluimos el capítulo proporcionando un nuevo método para calcular, algorítmicamente, la cota que aparece en el resultado de Gardner y Gronchi.

El segundo capítulo está dedicado al estudio de nuevas desigualdades discretas de tipo Brunn-Minkowski en las que se preserva la estructura del resultado clásico, teniéndose, por tanto, que modificar alguno de los conjuntos involucrados. Con este fin introducimos dos construcciones diferentes: a partir de un conjunto finito $A \subset \mathbb{Z}^n$ podemos, o bien añadir puntos ($\bar{A}$), o eliminar puntos ($\mathrm{r}(A)$) de un modo preciso. Esto nos va a permitir obtener dos versiones discretas (equivalentes) de la desigualdad clásica de Brunn-Minkowski:

Teorema A. *Sean $A, B \subset \mathbb{Z}^n$ finitos, $A, B \neq \emptyset$. Entonces*

$$\left|\bar{A} + B\right|^{1/n} \geq |A|^{1/n} + |B|^{1/n}$$

y

$$|A + B|^{1/n} \geq \left|\mathrm{r}(A)\right|^{1/n} + |B|^{1/n}.$$

Ambas desigualdades son óptimas.

A continuación demostramos que a partir de estas versiones equivalentes se puede obtener la desigualdad de Brunn-Minkowski clásica. Concluimos el capítulo probando que el número de puntos adicionales en $\bar{A}$ no puede ser demasiado grande, dependiendo sólo de (la estructura de) A y de la dimensión. De forma más precisa, proporcionamos cotas superiores e inferiores tanto para la razón $|\bar{A}|/|A|$ como para la diferencia $|\bar{A}|-|A|$. Resultados similares se obtendrán para la segunda construcción $\mathrm{r}(A)$.

En el último capítulo investigamos versiones discretas de la desigualdad de Borell-Brascamp-Lieb (‡). Comenzamos el capítulo con una primera sección dedicada a recordar esta importante desigualdad, tanto en términos de la p-suma (véase (‡)) como de su versión para p-medias. Seguidamente, y con el fin de obtener tales desigualdades de Borell-Brascamp-Lieb discretas, generalizamos una de las construcciones previamente mencionadas: eliminando puntos de nuestro conjunto finito inicial $A \subset \mathbb{Z}^n$, $A \neq \emptyset$, definimos un nuevo conjunto reducido $\mathrm{r}_f(A)$ conforme a una *función peso* particular $f : \mathbb{Z}^n \longrightarrow \mathbb{R}_{\geq 0}$. Utilizando esta transformación demostramos el resultado principal de la sección:

Teorema B. *Sean $A, B \subset \mathbb{Z}^n$ conjuntos finitos no vacíos. Sea $-1/n \leq p \leq \infty$, $p \neq 0$, y sean $f, g, h : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ funciones no-negativas tales que*

$$h(x+y) \geq \bigl(f(x)^p + g(y)^p\bigr)^{1/p}$$

para cualesquiera $x \in A$, $y \in B$ con $f(x)g(y) > 0$. Entonces

$$\sum_{z\in A+B} h(z) \geq \left[\left(\sum_{x\in \mathrm{r}_f(A)} f(x)\right)^q + \left(\sum_{y\in B} g(y)\right)^q\right]^{1/q},$$

donde $q = p/(np+1)$ (siempre y cuando las sumas del lado derecho de la desigualdad no sean cero).

También probamos que la desigualdad clásica de Borell-Brascamp-Lieb (‡) puede deducirse a partir de la versión discreta (Teorema B) bajo la hipótesis débil (pero necesaria) de que las funciones f, g sean integrables Riemann.

En la última sección consideramos otra importante medida discreta asociada a un conjunto $K \subset \mathbb{R}^n$ y al retículo entero, el llamado *enumerador de puntos del retículo* $\mathrm{G}_n(K) = |K \cap \mathbb{Z}^n|$, y trabajamos con combinaciones convexas de conjuntos, $(1-\lambda)K + \lambda L$, $\lambda \in (0,1)$, en lugar de con la suma $K + L$. La pregunta natural que se plantea es si se puede conseguir una desigualdad de tipo Brunn-Minkowski para $\mathrm{G}_n(K)$. De nuevo, no va a ser posible obtener dicha desigualdad en su forma clásica ($*$), y por tanto hay que intentar determinar el mejor conjunto M, con $(1-\lambda)K + \lambda L \subset M$, para el cual se verifique que

$$\mathrm{G}_n\bigl(M\bigr)^{1/n} \geq (1-\lambda)\mathrm{G}_n(K)^{1/n} + \lambda \mathrm{G}_n(L)^{1/n}$$

para todo $\lambda \in (0,1)$. A este respecto demostramos el siguiente resultado:

Teorema C. *Sean* $K, L \subset \mathbb{R}^n$ *conjuntos acotados no vacíos tales que* $\mathrm{G}_n(K)\mathrm{G}_n(L) > 0$*, y sea* $\lambda \in (0,1)$*. Entonces*

$$\mathrm{G}_n\big((1-\lambda)K + \lambda L + (-1,1)^n\big)^{1/n} \geq (1-\lambda)\mathrm{G}_n(K)^{1/n} + \lambda \mathrm{G}_n(L)^{1/n}.$$

La desigualdad es óptima.

Este teorema se obtendrá como consecuencia de un resultado funcional (más general), el cual puede verse como otra versión discreta de la desigualdad de Borell-Brascamp-Lieb, ahora para p-medias; y de nuevo, esta versión discreta implicará la desigualdad clásica de Borell-Brascamp-Lieb para p-medias bajo hipótesis concretas sobre las funciones involucradas.

Los resultados originales que se recogen en esta tesis doctoral pueden encontrarse en las referencias [12, 13, 14, 15].

Preface

Relating the volume, $\mathrm{vol}(\cdot)$ (i.e., the Lebesgue measure), with the Minkowski (vectorial) addition $+$ of compact sets, one is led to the famous *Brunn-Minkowski inequality*. One form of it states that if $K, L \subset \mathbb{R}^n$ are compact and non-empty, then

$$\mathrm{vol}(K+L)^{1/n} \geq \mathrm{vol}(K)^{1/n} + \mathrm{vol}(L)^{1/n}, \tag{¶}$$

with equality, when $\mathrm{vol}(K)\mathrm{vol}(L) > 0$, if and only if K and L are homothetic compact convex sets. There are several equivalent geometric versions of this result, and specially interesting is the so-called *Blaschke form* of the Brunn-Minkowski inequality: if $K, L \subset \mathbb{R}^n$ are compact convex sets, and B_K, B_L are the (closed) balls (centered at the origin) such that $\mathrm{vol}(B_K) = \mathrm{vol}(K)$ and $\mathrm{vol}(B_L) = \mathrm{vol}(L)$, then

$$\mathrm{vol}(K+L) \geq \mathrm{vol}(B_K + B_L). \tag{$\|$}$$

The Brunn-Minkowski inequality is one of the most powerful results in Convex Geometry and beyond, and with various extensions to other fields. Regarding an analytical counterpart for functions of (¶), one is naturally led to the so-called *Borell-Brascamp-Lieb inequality*: for $p \geq -1/n$, $p \neq 0$, if $f, g, h : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ are non-negative measurable functions (with non-zero integrals), such that $h(x+y) \geq \big(f(x)^p + g(y)^p\big)^{1/p}$ for all $x, y \in \mathbb{R}^n$ such that $f(x)g(y) > 0$, then

$$\int_{\mathbb{R}^n} h(x)\,\mathrm{d}x \geq \left[\left(\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x\right)^q + \left(\int_{\mathbb{R}^n} g(x)\,\mathrm{d}x\right)^q\right]^{1/q}, \tag{$**$}$$

where $q = p/(np+1)$; here, and in the following, the cases $p = \pm\infty$ have to be understood as the natural limit cases.

There is nowadays a growing interest in getting discrete counterparts of classical geometric inequalities, i.e., discrete analogs in which compact sets and volume are replaced by finite sets of points and cardinality, respectively. Of course, the Brunn-Minkowski inequality is one of the first results to be considered in this discrete setting.

It can be easily seen that one cannot expect to obtain a Brunn-Minkowski inequality for the cardinality in the classical form (¶). It is also worth remarking that getting such an inequality from the classical Brunn-Minkowski inequality is not an easy task, as pointed out by Gardner and Gronchi in [8, pages 3996-3997]. Therefore, a discrete version of (¶) should either have a different structure or involve modifications of the sets. In the first line, Gardner and Gronchi [8] obtained a discrete analog of the Blaschke form (‖) of the Brunn-Minkowski inequality: they proved that if A, B are finite subsets of the integer lattice $\mathbb{Z}^n$, with dimension $\dim B = n$, then

$$|A+B| \geq \left|\mathrm{D}^{\mathrm{B}}_{|\mathrm{A}|} + \mathrm{D}^{\mathrm{B}}_{|\mathrm{B}|}\right|. \qquad (\dagger\dagger)$$

Here $\mathrm{D}^{\mathrm{B}}_{|\mathrm{A}|}$ is a *B-initial segment*: roughly speaking, like an intersection of a simplex with $\mathbb{Z}^n$.

This dissertation is devoted to provide new discrete Brunn-Minkowski and Borell-Brascamp-Lieb type inequalities, preserving the structure of their classical form, and so, modifying one of the sets involved. We also prove that these new inequalities imply the (corresponding) classical ones.

The work starts with an introductory first section where we establish the notation and introduce the concepts and results that will be needed later on. We also recall some inequalities involving sums that will play a relevant role in the proofs of our results. Next, we present the Brunn-Minkowski inequality in the classical (continuous) setting, namely, (¶), together with several (equivalent) versions of it. Then, we move to the discrete setting: we collect some relevant inequalities that, even changing the structure of the Brunn-Minkowski inequality, can be seen as discrete versions of it. We make special emphasis on the inequality of Gardner and Gronchi (††), and we end the chapter providing with a new method for algorithmically computing the bound in the Gardner&Gronchi result.

The second chapter is devoted to study new discrete Brunn-Minkowski type inequalities preserving the structure of the classical result, but so involving some modifications on one of the sets. To this aim, we introduce two different constructions: starting from a finite set $A \subset \mathbb{Z}^n$, we can either adding points ($\bar{A}$), or removing points ($\mathrm{r}(A)$) in a precise way. This will allow us to get two (equivalent) discrete versions of the classical Brunn-Minkowski inequality:

Theorem A. *Let $A, B \subset \mathbb{Z}^n$ be finite, $A, B \neq \emptyset$. Then*

$$\left|\bar{A}+B\right|^{1/n} \geq |A|^{1/n} + |B|^{1/n}$$

and

$$|A+B|^{1/n} \geq \left|\mathrm{r}(A)\right|^{1/n} + |B|^{1/n}.$$

Both inequalities are sharp.

Next, we prove that these equivalent versions can be used to infer the Brunn-Minkowski inequality in its classical form. We conclude the chapter by showing that the number of additional points in $\bar{A}$ cannot be too large, and depends only on (the structure of) A and on the dimension. More precisely, we provide upper and lower bounds for both the ratio $|\bar{A}|/|A|$ and the difference $|\bar{A}| - |A|$. Similar results will be obtained for the second construction $\mathrm{r}(A)$.

In the last chapter, we investigate discrete versions of the Borell-Brascamp-Lieb inequality ($**$). We start the chapter with a first section devoted to recalling this important inequality, both in terms of the p-sum (cf. $(**)$) and its version for p-means. Next, in order to get such a discrete Borell-Brascamp-Lieb inequality, we generalize one of the constructions previously mentioned: by removing points from our original finite set $A \subset \mathbb{Z}^n$, $A \neq \emptyset$, we define a new reduced set $\mathrm{r}_f(A)$ according to a particular *weight function* $f : \mathbb{Z}^n \longrightarrow \mathbb{R}_{\geq 0}$. Using this transformation we prove our main result in this section:

Theorem B. *Let $A, B \subset \mathbb{Z}^n$ be non-empty finite sets. Let $-1/n \leq p \leq \infty$, $p \neq 0$, and let $f, g, h : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ be non-negative functions such that*

$$h(x+y) \geq \big(f(x)^p + g(y)^p\big)^{1/p}$$

for all $x \in A$, $y \in B$ with $f(x)g(y) > 0$. Then

$$\sum_{z \in A+B} h(z) \geq \left[\left(\sum_{x \in \mathrm{r}_f(A)} f(x)\right)^q + \left(\sum_{y \in B} g(y)\right)^q\right]^{1/q},$$

with $q = p/(np+1)$ (provided that both sums in the right-hand side are non-zero).

We also show that the classical Borell-Brascamp-Lieb inequality $(**)$ can be obtained from the discrete version (Theorem B) under the mild (but necessary) assumption that the functions f, g are Riemann integrable.

In the last section we consider another important discrete measure associated both to a set $K \subset \mathbb{R}^n$ and to the integer lattice, the so-called *lattice point enumerator* $\mathrm{G}_n(K) = |K \cap \mathbb{Z}^n|$, and we deal with convex combinations of sets, $(1-\lambda)K + \lambda L$, $\lambda \in (0,1)$, instead of $K + L$. The natural question arises whether one can get a Brunn-Minkowski type inequality for the lattice point enumerator. Again it will be not possible to obtain such an inequality in the classical form (¶), and so, we look for the best set M, with $(1-\lambda)K + \lambda L \subset M$, satisfying

$$\mathrm{G}_n\big(M\big)^{1/n} \geq (1-\lambda)\mathrm{G}_n(K)^{1/n} + \lambda \mathrm{G}_n(L)^{1/n}$$

for all $\lambda \in (0,1)$. In this regard we prove the following result:

Theorem C. *Let $K, L \subset \mathbb{R}^n$ be non-empty bounded sets such that $\mathrm{G}_n(K)\mathrm{G}_n(L) > 0$, and let $\lambda \in (0,1)$. Then*

$$\mathrm{G}_n\big((1-\lambda)K + \lambda L + (-1,1)^n\big)^{1/n} \geq (1-\lambda)\mathrm{G}_n(K)^{1/n} + \lambda \mathrm{G}_n(L)^{1/n}.$$

The inequality is sharp.

This theorem will be obtained as a consequence of a functional (and more general) result, which can be seen as another discrete version of the Borell-Brascamp-Lieb inequality, but now for p-means; and again, this discrete version will imply the classical Borell-Brascamp-Lieb inequality for p-means under particular assumptions on the functions involved.

The original results which are contained in this dissertation can be found in the papers [12, 13, 14, 15].

Chapter 1

On the discrete Brunn-Minkowski inequality of Gardner and Gronchi

The Brunn-Minkowski inequality is one of the most powerful results in Convex Geometry, and has unexpected applications in many other fields. Roughly speaking, it states that the n-th root of the volume is a concave function. Nowadays there is an increasing interest for getting discrete analogs of classical results (in continuous settings), being an important example of this a beautiful discrete version of the Brunn-Minkowski inequality due to Gardner and Gronchi.

In this first chapter we will give a brief overview of both, the classical Brunn-Minkowski inequality, and the Gardner&Gronchi result, having as a final aim to provide with a new method for algorithmically computing the bounds appearing in the Gardner&Gronchi inequality. We will start the chapter with a first section containing the notation, definitions and basic results that will be needed throughout the rest of the dissertation.

The original work that we collect here can be found in [13].

1.1 Notation, definitions and basic results

As usual, we write $\mathbb{R}^n$ to represent the n-dimensional Euclidean space, and we denote by e_i the i-th canonical unit vector. For $x, y \in \mathbb{R}^n$, $[x, y]$ will represent the closed segment with endpoints x and y, namely,

$$[x, y] = \big\{(1 - \lambda)x + \lambda y : \, 0 \leq \lambda \leq 1\big\},$$

and analogously, we set

$$(x, y) = \{(1-\lambda)x + \lambda y : 0 < \lambda < 1\}$$

for the open segment with endpoints x, y.

We write $\pi_{i_1,\dots,i_k}$, $1 \leq i_1, \dots, i_k \leq n$, to denote the orthogonal projection onto the k-dimensional coordinate plane $\mathbb{R}e_{i_1} + \cdots + \mathbb{R}e_{i_k}$. For the sake of brevity we just write H_i to represent the i-th coordinate hyperplane, namely, $H_i = \mathbb{R}e_1 + \cdots + \mathbb{R}e_{i-1} + \mathbb{R}e_{i+1} + \cdots + \mathbb{R}e_n$, and $\pi_{(i)} = \pi_{1,\dots,i-1,i+1,\dots,n}$ for the corresponding orthogonal projection onto H_i.

Given a subset $M \subset \mathbb{R}^n$, we denote by $\dim M$, $\operatorname{int} M$ and $\operatorname{cl} M$ the dimension of M (i.e., the dimension of the smallest affine subspace of $\mathbb{R}^n$ containing M), the interior of M and its closure, respectively. Moreover, we will use χ_M to represent the characteristic function of M, i.e.,

$$\chi_M(x) = \begin{cases} 1 & \text{if } x \in M, \\ 0 & \text{otherwise.} \end{cases}$$

For $x \in \mathbb{R}^n$, $d(x, M)$ will stand for the Euclidean distance between $x \in \mathbb{R}^n$ and M. Furthermore, if $\dim M = k$, $k \in \{1, \dots, n\}$, and it is contained in some affine subspace that we identify with $\mathbb{R}^k$ then, for each $z \in \mathbb{R}^r$, $r \in \{1, \dots, k-1\}$, we will write $M(z)$ to denote the section of M at z (in the corresponding subspace) orthogonal to the coordinate plane $\mathbb{R}e_{k-r+1} + \cdots + \mathbb{R}e_k$, this is,

$$M(z) = \{x \in \mathbb{R}^{k-r} : (x, z) \in M\}. \tag{1.1}$$

Finally, as usual in the literature, by $\lfloor t \rfloor$ we will denote the floor function of a real number $t \in \mathbb{R}$, i.e., the greatest integer less than or equal to t; similarly, $\lceil t \rceil$ will represent the ceiling function of t, namely, the smallest integer greater than or equal to t. In addition, we will write $\mathbb{R}_{\geq 0}$ (respectively, $\mathbb{Q}_{\geq 0}$) to denote the set of non-negative real (respectively, rational) numbers.

Next we state several definitions and inequalities which are well known and can be found in any book on Convexity; among many others we refer to [10, 26] as source books.

The Brunn-Minkowski inequality arises from combining two key notions for (compact) sets: volume and Minkowski addition. We recall briefly their definitions.

Definition 1.1. *Let $M \subset \mathbb{R}^n$ be a compact set. The* volume *of M, denoted as $\operatorname{vol}(M)$, is the (n-dimensional) Lebesgue measure of M.*

When integrating, as usual, $\mathrm{d}x$ will stand for $\mathrm{dvol}(x)$.

Definition 1.2. *Let $A, B \subset \mathbb{R}^n$. The* Minkowski (vectorial) addition *of A and B is defined as*

$$A + B = \{a + b : a \in A, b \in B\}$$

(see Figure 1.1).

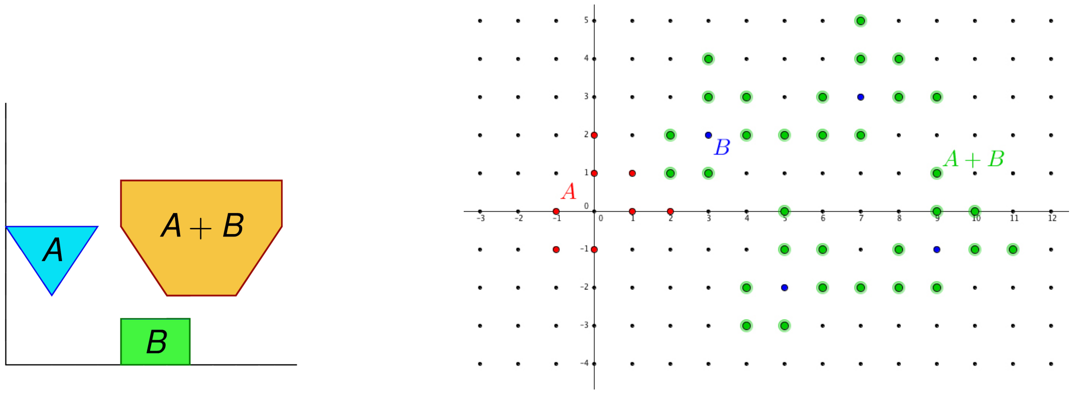

Figure 1.1: Two examples of Minkowski addition.

Moreover, for $\lambda \in \mathbb{R}$ we write $\lambda A = \{\lambda a : a \in A\}$ and, we will shorten by $x + A := \{x\} + A$.

Definition 1.3. *A (non-empty) set $K \subset \mathbb{R}^n$ is said to be* convex *if, given two points $x, y \in K$, the line segment $[x, y]$ is contained in K, i.e., if any convex combination $(1-\lambda)x + \lambda y \in K$, $0 \le \lambda \le 1$.*

For an arbitrary $M \subset \mathbb{R}^n$, the *convex hull* of M, denoted by $\operatorname{conv} M$, is the smallest convex set containing M.

Next we move to the discrete setting. The discrete counterpart of the volume for finite sets is the *cardinality*, which will be denoted by $|A|$ for a finite subset $A \subset \mathbb{R}^n$.

As usual in the literature we represent by $\mathbb{Z}^n$ the (n-dimensional) *integer lattice*, i.e., the lattice of all points with integer coordinates in $\mathbb{R}^n$, and we write

$$\mathbb{Z}^n_{\ge 0} := \big\{x = (x_1, \dots, x_n) \in \mathbb{Z}^n : x_i \ge 0,\ i = 1, \dots, n\big\}.$$

In general, a *(geometric) lattice* Λ in $\mathbb{R}^n$ is the family of all integer linear combinations of n linearly independent vectors $\{v_1, \dots, v_n\} \subset \mathbb{R}^n$, which is called a basis of Λ.

Another important discrete measure associated both to a set $M \subset \mathbb{R}^n$ and to the integer lattice is the so-called *lattice point enumerator*, namely,

$$\mathrm{G}_n(M) = |M \cap \mathbb{Z}^n|$$

(it may be also defined with respect to an arbitrary lattice).

Special sets that will appear throughout this dissertation are the lattice sets: a finite set $A \subset \mathbb{Z}^n$ is a (convex) *lattice set* if $A = (\operatorname{conv} A) \cap \mathbb{Z}^n$. In particular, we will denote by C^n_r, for $r \in \mathbb{N}$, the *lattice cube*

$$\mathrm{C}^n_r = r[0,1]^n \cap \mathbb{Z}^n,$$

with $r + 1$ integer points in its edges.

1.1.1 Some inequalities involving sums

We conclude this first introductory section by recalling some important inequalities for sums that will play a relevant role in the proofs of our results. We start by stating the well-known *arithmetic-geometric mean inequality* (see e.g. [10, Corollary 1.2]):

Theorem 1.1.1 (The arithmetic-geometric mean inequality). *Let* $a_1,\dots,a_n \geq 0$ *and let* $\alpha_1,\dots,\alpha_n \geq 0$ *be such that* $\sum_{i=1}^n \alpha_i = 1$. *Then*

$$\prod_{i=1}^n a_i^{\alpha_i} \leq \sum_{i=1}^n \alpha_1 a_i.$$

The classical and the reverse Hölder inequalities (see [5, Theorem 1 in p. 178]) will be also needed later.

Theorem 1.1.2 (Hölder's inequality). *Let* $a_1,\dots,a_n,b_1,\dots,b_n \geq 0$ *and let* $1 \leq p \leq \infty$. *Then*

$$\sum_{i=1}^n a_i b_i \leq \left(\sum_{i=1}^n a_i^p\right)^{1/p} \left(\sum_{i=1}^n b_i^q\right)^{1/q},$$

where $1 \leq q \leq \infty$ *is such that* $1/p + 1/q = 1$.

In fact, from the above result the so-called reverse Hölder inequality can be deduced:

Corollary 1.1.1 (Reverse Hölder's inequality). *Let* $a_1,\dots,a_n,b_1,\dots,b_n \geq 0$ *and let* $0 < p \leq 1$. *Then*

$$\sum_{i=1}^n a_i b_i \geq \left(\sum_{i=1}^n a_i^p\right)^{1/p} \left(\sum_{i=1}^n b_i^q\right)^{1/q},$$

where $-\infty \leq q < 0$ *is such that* $1/p + 1/q = 1$.

In the above results, for $p = \pm\infty$ we set

$$\left(\sum_{i=1}^n a_i^p\right)^{1/p} = \begin{cases} \max\{a_i : i = 1,\dots,n\} & \text{if } p = \infty, \\ \min\{a_i : i = 1,\dots,n\} & \text{if } p = -\infty. \end{cases}$$

1.2 The Brunn-Minkowski inequality. First discrete versions

Relating the notions of volume and Minkowski addition (of compact sets), one is led to the famous Brunn-Minkowski inequality.

Theorem 1.2.1. *Let* $K, L \subset \mathbb{R}^n$ *be compact sets. Then*

$$\mathrm{vol}(K+L)^{1/n} \geq \mathrm{vol}(K)^{1/n} + \mathrm{vol}(L)^{1/n}, \tag{1.2}$$

with equality if $\mathrm{vol}(K)\mathrm{vol}(L) > 0$ *if and only if* K *and* L *are homothetic compact convex sets.*

The Brunn-Minkowski inequality is one of the most powerful results in Convex Geometry and beyond: for instance, its equivalent analytic version (the Prékopa-Leindler inequality, see e.g. [10, Theorem 8.14]) and the fact that the convexity/compactness assumption can be weakened to Lebesgue measurability (see [17]), have allowed it to move to much wider fields. It implies very important inequalities such as the isoperimetric and the Urysohn inequalities (see e.g. [26, page 382]), and it has been the starting point for new developments like the L_p-Brunn-Minkowski theory (see e.g. [18, 19]), or a reverse Brunn-Minkowski inequality (see e.g. [21]), among many others. It would not be possible to collect here all references regarding versions, applications and/or generalizations of the Brunn-Minkowski inequality. For extensive and beautiful surveys on them we refer to [1, 7].

There are several equivalent geometric versions of the Brunn-Minkowski inequality. Among others, we find:

$$\begin{aligned} \mathrm{vol}\big((1-\lambda)K+\lambda L\big)^{1/n} &\geq (1-\lambda)\mathrm{vol}(K)^{1/n}+\lambda\mathrm{vol}(L)^{1/n} && \textit{(additive form)},\\ \mathrm{vol}\big((1-\lambda)K+\lambda L\big) &\geq \mathrm{vol}(K)^{1-\lambda}\mathrm{vol}(L)^{\lambda} && \textit{(multiplicative form)},\\ \mathrm{vol}\big((1-\lambda)K+\lambda L\big) &\geq \min\big\{\mathrm{vol}(K),\mathrm{vol}(L)\big\} && \textit{(minimal form)}, \end{aligned} \tag{1.3}$$

for $0<\lambda<1$. Specially interesting for what follows is the equivalent version called the *Blaschke form* of the Brunn-Minkowski inequality, which reads as follows (see e.g. [7]):

Theorem 1.2.2 (Blaschke form of the Brunn-Minkowski inequality). *Let $K, L \subset \mathbb{R}^n$ be compact convex sets. If B_K, B_L are the (closed) balls (centered at the origin) with $\mathrm{vol}(B_K) = \mathrm{vol}(K)$ and $\mathrm{vol}(B_L) = \mathrm{vol}(L)$, then*

$$\mathrm{vol}(K+L) \geq \mathrm{vol}(B_K+B_L). \tag{1.4}$$

Next we move to the discrete setting, i.e., we consider finite sets of integer points which are not necessarily full-dimensional unless indicated otherwise. At this point we would like to observe that, in order to bound the cardinality of a sum $A+B$ of finite sets (not necessarily of integer points), throughout this chapter, we can work without loss of generality with subsets A, B of $\mathbb{Z}^n$, due to the following lemma which was proved in [8, Corollary 3.3]:

Lemma 1.2.1. *Let $A, B \subset \mathbb{R}^n$ be finite sets. Then there exist $A', B' \subset \mathbb{Z}^n$ satisfying*

i) $|A'| = |A|$, $|B'| = |B|$ *and* $|A'+B'| = |A+B|$, *and*

ii) $\dim A' = \dim A$ *and* $\dim(A'+B') = \dim(A+B)$.

It can be easily seen that one cannot expect to obtain a Brunn-Minkowski inequality for the cardinality in the classical form. Indeed, simply taking $A = \{0\}$ to be the origin and any finite set $B \subset \mathbb{Z}^n$, then

$$|A+B|^{1/n} = |B|^{1/n} < 1 + |B|^{1/n} = |A|^{1/n} + |B|^{1/n}.$$

Another easy example is obtained by considering two lattice cubes $\mathrm{C}^n_{r_1}$ and $\mathrm{C}^n_{r_2}$: indeed, since $\mathrm{C}^n_{r_1} + \mathrm{C}^n_{r_2} = \mathrm{C}^n_{r_1+r_2}$, then

$$\left|\mathrm{C}^n_{r_1} + \mathrm{C}^n_{r_2}\right|^{1/n} = \left|\mathrm{C}^n_{r_1+r_2}\right|^{1/n} = r_1 + r_2 + 1 < r_1 + r_2 + 2 = \left|\mathrm{C}^n_{r_1}\right|^{1/n} + \left|\mathrm{C}^n_{r_2}\right|^{1/n}.$$

Therefore, a discrete Brunn-Minkowski type inequality should either have a different structure or involve modifications of the sets. A first example is the following simple but relevant inequality (see e.g. [27, Chapter 2]) which, in particular, provides us with a 1-dimensional discrete Brunn-Minkowski inequality.

Lemma 1.2.2. *Let $A, B \subset \mathbb{Z}^n$ be finite sets. Then*

$$|A + B| \geq |A| + |B| - 1. \tag{1.5}$$

Indeed, since the cardinality is translation invariant, we can place the maximum point of A and the minimum point of B in the lexicographical order at the origin of coordinates (see Figure 1.2). Then $A + B \supset A \cup B$, and hence

$$|A + B| \geq |A \cup B| = |A| + |B| - 1.$$

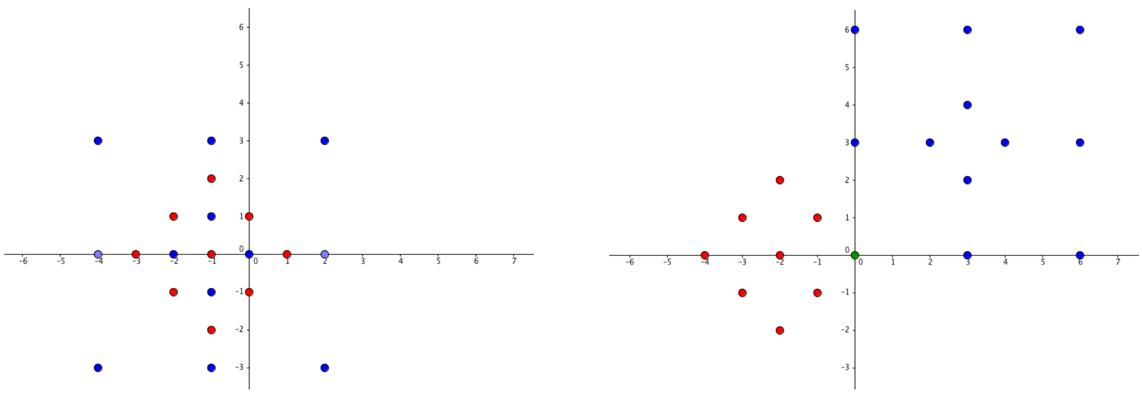

Figure 1.2: Translating two sets to place the maximum/minimum at the origin.

In [24], Ruzsa improved the above result as follows:

Proposition 1.2.1. *Let $A, B \subset \mathbb{Z}^n$ be finite sets with $|B| \leq |A|$ and $\dim(A + B) = n$. Then*

$$|A + B| \geq |A| + \sum_{i=1}^{|B|-1} \min\{n, |A| - i\} \tag{1.6}$$

and

$$|A + B| \geq |A| + n|B| - \frac{n(n+1)}{2}. \tag{1.7}$$

In fact, one can obtain (1.7) from (1.6) after a simple computation. He also proved that, *under the above assumptions*, (1.6) is tight and there is no improvement of (1.7) which is linear in $|A|$, i.e., no inequality of the form

$$|A+B| \geq c|A| + f_1\big(|B|\big) + f_2(n)$$

may hold with $c > 1$.

1.3 The discrete version of Gardner and Gronchi

In [8], Gardner and Gronchi obtained a beautiful and powerful discrete Brunn-Minkowski inequality: they proved a discrete analog of its Blaschke form (Theorem 1.2.2). To state this result we need several previous definitions.

Definition 1.4. *Let $B \subset \mathbb{Z}^n$ be finite with $|B| \geq n+1$. The B-*weight *function $w_B : \mathbb{Z}^n \longrightarrow \mathbb{R}$ is defined as*

$$w_B(x) = \frac{x_1}{|B|-n} + \sum_{i=2}^{n} x_i.$$

This function allows us to define a special order in $\mathbb{Z}^n_{\geq 0}$ depending on $|B|$ (see Figure 1.3):

Definition 1.5. *Let $B \subset \mathbb{Z}^n$ be finite with $|B| \geq n+1$. Given $x, y \in \mathbb{Z}^n_{\geq 0}$, $x \neq y$, we say that $x <_B y$ if*

- *$w_B(x) < w_B(y)$ or*
- *$w_B(x) = w_B(y)$ and there exists $j \in \{1, \ldots, n\}$ such that $x_j > y_j$ and $x_i = y_i$ for all $i < j$.*

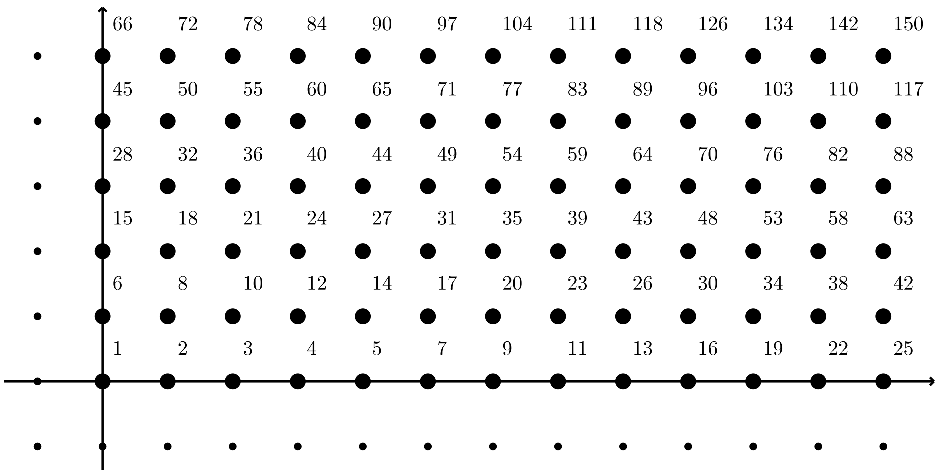

Figure 1.3: The B-order in $\mathbb{Z}^2_{\geq 0}$ for $|B| = 6$.

We note that the minimum of $\mathbb{Z}^n_{\geq 0}$ in any B-order is always the origin. Moreover, one can check that the first $|B|$ points in $\mathbb{Z}^n_{\geq 0}$ for any B-order are

$$0 <_B \mathrm{e}_1 <_B 2\mathrm{e}_1 <_B 3\mathrm{e}_1 \cdots <_B (|B| - n)\mathrm{e}_1 <_B \mathrm{e}_2 <_B \mathrm{e}_3 <_B \cdots <_B \mathrm{e}_n. \tag{1.8}$$

We are now ready to introduce the key sets in the inequality of Gardner and Gronchi: the B-initial segments. They will be the analog to the balls in (1.4).

Definition 1.6. *Let $B \subset \mathbb{Z}^n$ be finite with $|B| \geq n+1$. For $m \in \mathbb{N}$, we define the B-*initial segment *(of order m), denoted by $\mathrm{D}^{\mathrm{B}}_{\mathrm{m}}$, as the set of the first m points of $\mathbb{Z}^n_{\geq 0}$ in the B-order.*

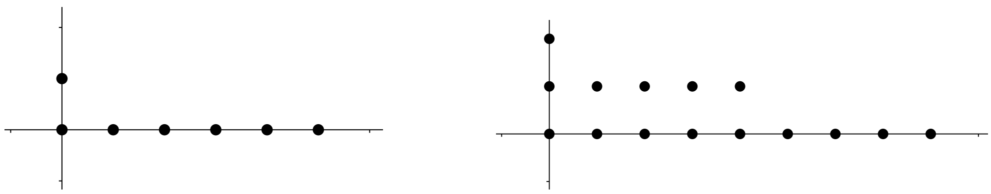

Figure 1.4: B-initial segments for the B-order in Figure 1.3 for $|A| = 7$ (left) and $|A| = 15$ (right).

We notice that all definitions above depend only on the cardinality of B, and not on B itself. Another important observation is that initial segments behave well with the Minkowski addition: Gardner and Gronchi proved that if $F = \mathrm{D}^{\mathrm{B}}_{\mathrm{m}}$ is the m-initial segment in the B-order, then $F + \mathrm{D}^{\mathrm{B}}_{|\mathrm{B}|}$ is also an initial segment in the same B-order (see [8, Lemma 5.12]).

Now we can finally state the announced discrete Brunn-Minkowski type inequality of Gardner and Gronchi.

Theorem 1.3.1 (Gardner&Gronchi's inequality). *Let $A, B \subset \mathbb{Z}^n$ be finite sets with $\dim B = n$. Then*

$$|A + B| \geq \left|\mathrm{D}^{\mathrm{B}}_{|\mathrm{A}|} + \mathrm{D}^{\mathrm{B}}_{|\mathrm{B}|}\right|. \tag{1.9}$$

As consequences of (1.9) they get two additional discrete Brunn-Minkowski type inequalities:

$$|A + B|^{1/n} \geq |A|^{1/n} + \frac{1}{(n!)^{1/n}}\big(|B| - n\big)^{1/n} \tag{1.10}$$

and, if $|B| \leq |A|$, then

$$|A + B| \geq |A| + (n-1)|B| + \big(|A| - n\big)^{(n-1)/n}\big(|B| - n\big)^{1/n} - \frac{n(n-1)}{2}.$$

These inequalities improve previous results obtained by Ruzsa in [24, 25].

1.4 Computing the cardinality $\left|\mathrm{D}^{\mathrm{B}}_{|\mathrm{A}|} + \mathrm{D}^{\mathrm{B}}_{|\mathrm{B}|}\right|$

In this section we collect the new results we have obtained regarding the computation of the bound $\left|\mathrm{D}^{\mathrm{B}}_{|\mathrm{A}|} + \mathrm{D}^{\mathrm{B}}_{|\mathrm{B}|}\right|$. They can be found in [13].

If we want to estimate the cardinality of the Minkowski addition of two finite sets $A, B \subset \mathbb{Z}^n$ from below, one might have the impression that inequality (1.9) cannot help us, because we are replacing the problem of estimating $|A + B|$ by the one of computing the cardinality of another Minkowski addition, namely, $\left|\mathrm{D}^{\mathrm{B}}_{|\mathrm{A}|} + \mathrm{D}^{\mathrm{B}}_{|\mathrm{B}|}\right|$.

However, $\mathrm{D}^{\mathrm{B}}_{|\mathrm{A}|}$ and $\mathrm{D}^{\mathrm{B}}_{|\mathrm{B}|}$ are very special sets: they are B-initial segments, and therefore, their sum $\mathrm{D}^{\mathrm{B}}_{|\mathrm{A}|} + \mathrm{D}^{\mathrm{B}}_{|\mathrm{B}|}$ is also a B-initial segment. So, in order to know its cardinality, it is enough to find the point $p \in \mathrm{D}^{\mathrm{B}}_{|\mathrm{A}|} + \mathrm{D}^{\mathrm{B}}_{|\mathrm{B}|}$ of maximum position in the B-order, because

$$x \in \mathrm{D}^{\mathrm{B}}_{|\mathrm{A}|} + \mathrm{D}^{\mathrm{B}}_{|\mathrm{B}|} \text{ if and only if } x <_B p \text{ or } x = p;$$

or equivalently, $|\mathrm{D}^{\mathrm{B}}_{|\mathrm{A}|} + \mathrm{D}^{\mathrm{B}}_{|\mathrm{B}|}|$ is the position of the "last" point $p \in \mathrm{D}^{\mathrm{B}}_{|\mathrm{A}|} + \mathrm{D}^{\mathrm{B}}_{|\mathrm{B}|}$ in the B-order.

We also note that $p = a + b \in \mathrm{D}^{\mathrm{B}}_{|\mathrm{A}|} + \mathrm{D}^{\mathrm{B}}_{|\mathrm{B}|}$ is the maximum position point in the B-order if and only if $a \in \mathrm{D}^{\mathrm{B}}_{|\mathrm{A}|}$ and $b \in \mathrm{D}^{\mathrm{B}}_{|\mathrm{B}|}$ are the maximum position points (in the B-order) of A and B, respectively. Indeed, this is an immediate consequence of the fact that

$$p + x <_B q + x \quad \text{if} \quad p <_B q,$$

which is derived from the linearity of the B-weight function w_B together with the relations

$$p_j + x_j > q_j + x_j \quad \text{and} \quad p_i + x_i = q_i + x_i$$

for all $i < j$, provided that $p_j > q_j$ and $p_i = q_i$ for all $i < j$.

Moreover, we already know that the maximum position point in $\mathrm{D}^{\mathrm{B}}_{|\mathrm{B}|}$ is always e_n (cf. (1.8)).

Thus, the problem of computing the cardinality $|\mathrm{D}^{\mathrm{B}}_{|\mathrm{A}|} + \mathrm{D}^{\mathrm{B}}_{|\mathrm{B}|}|$ is reduced to have a method which allows us to know the position in the B-order of any point of $\mathbb{Z}^n_{\geq 0}$, and viceversa. Once we have such a method, we may compute $|\mathrm{D}^{\mathrm{B}}_{|\mathrm{A}|} + \mathrm{D}^{\mathrm{B}}_{|\mathrm{B}|}|$ as follows:

Algorithm 1: Obtaining the bound in (1.9)

Step 1 Find the point $a \in \mathbb{Z}^n_{\geq 0}$ with position $|A|$ in the B-order.
Step 2 Compute the position s in the B-order of $p = a + \mathrm{e}_n$.
Step 3 Then $|A + B| \geq s$.

Example 1.1. *In the example of Figure 1.3, if* $|A| = 54$ *then* $a = (6, 3)$*, and so* $p = a + \mathrm{e}_2 = (6, 4)$*. Therefore,*

$$|A + B| \geq \left|\mathrm{D}^{\mathrm{B}}_{|\mathrm{A}|} + \mathrm{D}^{\mathrm{B}}_{|\mathrm{B}|}\right| = 77.$$

If we want to know the position of a point in a certain B-order, we just need to "B-order" the points of $\mathbb{Z}^n_{\geq 0}$, taking also into account that the B-order depends only on the cardinality of B but not on its structure.

To this aim, we will directly code the points in $\mathbb{Z}^n_{\geq 0}$ according to the precise definition of the B-order. Following this idea we propose Algorithm 2 as an efficient (computationally speaking) way to compute the next point (in the B-order) to a given one in $\mathbb{Z}^n_{\geq 0}$.

Algorithm 2: Finding the next point to a given one (in the B-order)

Step 1 Find i such that $\big((x_i >= 1 \text{ if } i > 1) \text{ or } (x_i >= |B| - n \text{ if } i = 1)\big)$ and $x_j = 0$ for all $i < j < n$.

Step 2 We deduce the next point depending on i:

case *There is no i satisfying the conditions of Step 1* **do**
  Since $x = (m, 0 \dots, 0, k)$ with $m < |B| - n$, then the next point is
  $y = (m + k * (|B| - n) + 1, 0 \dots, 0, 0)$

case $i = 1$ *satisfies the conditions of Step 1* **do**
  Since $x = (m, 0, 0 \dots, 0, k)$ with $m \geq |B| - n$, then the next point is
  $y = (m - (|B| - n), k + 1, 0, \dots, 0, 0)$

case $i > 1$ *satisfies the conditions of Step 1* **do**
  Since $x = (\dots, m, 0, 0, \dots, 0, k)$, then the next point is $y = (\dots, m - 1, k + 1, 0, \dots, 0, 0)$

So, together with the ideas of Algorithm 1, we propose the next algorithm as a way to compute the bound in (1.9).

Algorithm 3: Obtaining the bound in (1.9)

Step 1 Starting from the origin, apply Algorithm 2, $|A| - 1$ times, to get the point $a \in \mathbb{Z}^n_{\geq 0}$ with position $|A|$ in the B-order.

Step 2 Storage t: the amount of times we have to apply Algorithm 2 until we find $p = a + \mathrm{e}_n$.

Step 3 Then $|A + B| \geq |A| + t$.

We conclude this section by pointing out another way (computationally speaking) of counting the points of $\mathbb{Z}^n_{\geq 0}$ in increasing B-order. This will further allow us to directly know the precise position of a given point (with no need of counting from the origin till the given point).

We start noting that one can group the points of $\mathbb{Z}^n_{\geq 0}$ according to their B-weight. Following this idea we may use the sets P_m, also a key point in the proof of Gardner and Gronchi, which are defined as

$$P_m = \left\{ x \in \mathbb{Z}^n_{\geq 0} : w_B(x) = \frac{m}{|B| - n} \right\}, \quad m \in \mathbb{N} \cup \{0\}.$$

Since the B-order organizes the points according to their B-weight, if we know the cardinality of each set P_m, $m \in \mathbb{N} \cup \{0\}$, then, for any $s \in \mathbb{N} \cup \{0\}$, we will also know the B-weight of the point

$x \in \mathbb{Z}^n_{\geq 0}$ occupying the s-th position; in fact,

$$x \in P_m \quad \text{if and only if} \quad \sum_{i=0}^{m-1} |P_i| < s \leq \sum_{i=0}^{m} |P_i|. \tag{1.11}$$

In this regard, we prove the following result.

Proposition 1.4.1 ([13]). *Let $m \in \mathbb{N} \cup \{0\}$ and let $B \subset \mathbb{Z}^n_{\geq 0}$ be finite. Then*

$$|P_m| = \begin{pmatrix} n + \left\lfloor \frac{m}{|B|-n} \right\rfloor - 1 \\ \left\lfloor \frac{m}{|B|-n} \right\rfloor \end{pmatrix}. \tag{1.12}$$

Proof. First we show that if m is such that

$$r := \frac{m}{|B|-n} \in \mathbb{N} \cup \{0\},$$

then (1.12) holds. We observe that, for any $x \in P_m$,

$$\frac{x_1}{|B|-n} = r - \sum_{i=2}^{n} x_i \in \mathbb{N} \cup \{0\},$$

and hence we may consider the function $c_m : P_m \longrightarrow \{0,1\}^{n+r-1}$ given by

$$c_m(x) = \left(0, \overset{\left(\frac{x_1}{|B|-n}\right)}{\ldots}, 0, 1, 0, \overset{(x_2)}{\ldots}, 0, 1, \ldots, 1, 0, \overset{(x_n)}{\ldots}, 0\right).$$

We note that c_m is a bijective mapping between P_m and the subset of all $(n+r-1)$-tuples in $\{0,1\}^{n+r-1}$ with r entries equal zero, and therefore $|P_m| = \binom{n+r-1}{r}$.

To conclude the proof it is enough to show that if m is such that $m/\big(|B|-n\big) \in \mathbb{N} \cup \{0\}$ then $P_{m+k} = P_m + k\mathrm{e}_1$ for all $0 < k < |B|-n$, which clearly implies that $|P_{m+k}| = |P_m|$. To this end, if $y \in P_{m+k}$ then

$$w_B(y) = \frac{y_1}{|B|-n} + \sum_{i=2}^{n} y_i = \frac{m+k}{|B|-n},$$

and so

$$\frac{y_1 - k}{|B|-n} \in \mathbb{Z}.$$

Moreover, since $0 < k < |B|-n$ and

$$0 \leq \frac{y_1}{|B|-n} = \frac{y_1-k}{|B|-n} + \frac{k}{|B|-n},$$

we infer that $y_1 - k \geq 0$ and thus

$$y' = y - k\mathrm{e}_1 = (y_1 - k, y_2, \ldots, y_n) \in \mathbb{Z}^n_{\geq 0}.$$

Now, since $w_B(y') = m/\big(|B|-n\big)$, we have that $y' \in P_m$ and thus $P_{m+k} - k\mathrm{e}_1 \subset P_m$, i.e., $P_{m+k} \subset P_m + k\mathrm{e}_1$. The opposite inclusion is trivial. □

Assuming, without loss of generality, that $m/\big(|B|-n\big) \in \mathbb{N}\cup\{0\}$ (for which we have the relation $P_{m+k} = P_m + k\mathrm{e}_1$ for all $0 < k < |B| - n$), we note that the "coding function" c_m appearing in the proof can be also used to B-order the points within each P_m. Indeed, given $x, y \in P_m$, then $x <_B y$ if and only if $c_m(x) < c_m(y)$ in the lexicographical order. Moreover, since the points in P_{m+1} are "B-greater" than those of P_m, and since we know the value of $|P_m|$ (see Proposition 1.4.1), the function c_m allows us to determine, as a consequence, the position of any point of $\mathbb{Z}^n_{\geq 0}$ in the B-order.

Furthermore, we can exploit these techniques to obtain the next point (in the B-order) to a given one in $\mathbb{Z}^n_{\geq 0}$. Roughly speaking, we should compute the weight function of the point, check its image under the corresponding "coding function" c_m, obtain the next point in $c_m(P_m)$ with respect to the lexicographical order and get the inverse image of the latter under c_m (we notice that if the corresponding image under c_m is the maximum of $c_m(P_m)$, we must give the first point of P_{m+1}, which is $(m+1)\mathrm{e}_1$).

1.4.1 Appendix: the algorithms

Algorithm: Code of Algorithm 2

Input: the dimension n; the cardinality of B $cardB$; the coordinates of an integer point with non-negative entries $x_1, \dots, x_n$

Output: the coordinates of the next point $y_1, \dots, y_n$

Step 1 Find i such that $\big((x_i >= 1$ if $i > 1)$ or $(x_i >= cardB - n$ if $i = 1)\big)$ and $x_j = 0$ for all $i < j < n$.

```
i ← 0   /* To consider the case in which such an i does not exist */
y_n ← x_n   /* Along this process we will set y_i = x_i */
for j = n − 1 to 2 do
    y_j ← x_j
    if x_j >= 1 and i = 0 then
        i ← j
/* For i = 1 we need to check that x_1 >= |B| − n */
y_1 ← x_1
if x_1 >= cardB − n and i = 0 then
    i ← 1
```

Step 2 We deduce the next point depending on i

```
case i = 0 do
    /* Then x = (m, 0 ..., 0, k) with m < |B| − n and so y = (m + k ∗ (|B| − n) + 1, 0 ..., 0, 0) */
    y_1 ← x_1 + x_n ∗ (cardB − n) + 1
    y_n ← 0
case i >= 1 do
    /* If i = 1 then x = (m, 0, ..., 0, k) with m >= cardB − n and so
       y = (m − (cardB − n), k + 1, ..., 0) else x = (..., m, 0, 0, ..., 0, k) and thus
       y = (..., m − 1, k + 1, 0, ..., 0, 0) */
    if i = 1 then
        y_1 ← x_1 − (cardB − n)
    else
        y_i ← x_i − 1
    y_{i+1} ← x_n + 1
    if i + 1 < n then
        y_n ← 0
return The coordinates y_1, ..., y_n of the next point to x in the B-order.
```

Algorithm: Code of Algorithm 3

Input: the dimension n; the cardinality of A $cardA$, the cardinality of B $cardB$

Output: the bound in (1.9)

Step 1 Starting from the origin, apply Algorithm 2 $cardA - 1$ times to know the point $a \in \mathbb{Z}^n_{\geq 0}$ with position $cardA$ in the B-order.

```
x ← (0, ..., 0)
for i = 1 to cardA − 1 do
    x ← Algorithm 2 (n, cardB, x)
```

Step 2 Storage t: the amount of times we have to apply Algorithm 2 until we find $p = a + \mathrm{e}_n$.

```
p ← x + e_n
t ← 0
while x ≠ p do
    x ← Algorithm 2 (n, cardB, x)
    t ← t + 1
```

Step 3 **return** $cardA + t$

Chapter 2

New discrete versions of the Brunn-Minkowski inequality

As mentioned in Section 1.2, in many cases the classical Brunn-Minkowski inequality for the cardinality is not satisfied. Also lattice cubes or elongated simplices do not verify it. There are, however, special sets or families of sets for which the inequality keeps its usual form. Next we show some examples.

Example 2.1. *Let* $A = \big\{(x,y) \in \mathbb{Z}^2 : x, y \neq 0, |x| + 2|y| \leq 7\big\}$ *and* $B = \mathrm{C}_1^2 \subset \mathbb{Z}^2$*. Then we have*

$$|A+B|^{1/2} = \sqrt{72} > 8 = \sqrt{36} + \sqrt{4} = |A|^{1/2} + |B|^{1/2}.$$

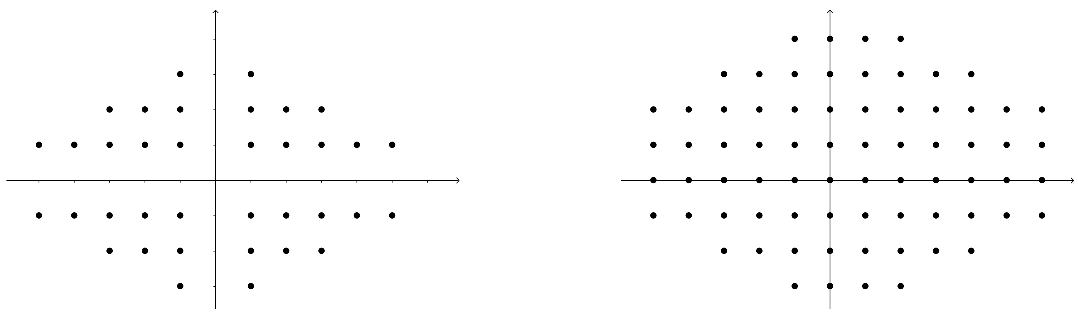

Figure 2.1: The discrete sets A (left) and $A + B$ (right).

Example 2.2. *For finite* $A, B \subset \mathbb{Z}^n$*, the relation* $|A+B| \leq |A|\,|B|$ *trivially holds (see e.g. [27, Chapter 2]), and it is easy to check that equality holds if and only if any point of* $A + B$ *has a* *unique expression as a sum of a point of A and a point of B. Under this assumption, i.e., if $|A+B| = |A|\,|B|$, and furthermore, if $|A|, |B| \geq 2^n$ (they are large enough), then A, B satisfy a classical Brunn-Minkowski type inequality:*

$$|A+B|^{1/n} = |A|^{1/n}|B|^{1/n} \geq \max\left\{2|A|^{1/n}, 2|B|^{1/n}\right\} \geq |A|^{1/n} + |B|^{1/n}.$$

We are interested in obtaining new Brunn-Minkowski type inequalities for the cardinality valid for all (finite and non-empty) sets. This will be the main aim of this chapter. The original work that we collect here can be found in [12].

2.1 Different ways to transform a discrete set

As we saw in Sections 1.2 and 1.3, in order to obtain a discrete Brunn-Minkowski type inequality, one option is to change the structure of the inequality. An alternative way to get a classical Brunn-Minkowski type inequality will be to modify one of the sets involved in the problem, either by adding or removing some points. Then the question arises as to how many points one should add/remove to ensure the reliability of the inequality.

2.1.1 Transforming one set by adding extra points

In order to guess how many points one should add, we consider two lattice cubes $\mathrm{C}^n_{r_1}$ and $\mathrm{C}^n_{r_2}$: since $\mathrm{C}^n_{r_1} + \mathrm{C}^n_{r_2} = \mathrm{C}^n_{r_1+r_2}$, then

$$\left|\mathrm{C}^n_{r_1} + \mathrm{C}^n_{r_2}\right|^{1/n} = r_1 + r_2 + 1 < r_1 + r_2 + 2 = \left|\mathrm{C}^n_{r_1}\right|^{1/n} + \left|\mathrm{C}^n_{r_2}\right|^{1/n}.$$

So, in order to reverse the above inequality we must add to $\mathrm{C}^n_{r_1}$, say, a suitable amount of points, such that the new set $\bar{\mathrm{C}}^n_{r_1}$ satisfies

$$\left|\bar{\mathrm{C}}^n_{r_1} + \mathrm{C}^n_{r_2}\right|^{1/n} \geq r_1 + r_2 + 2. \tag{2.1}$$

We do it by means of a recursive procedure, as follows. Let $\Lambda \subset \mathbb{Z}^k$ (finite), $k \in \{1, \dots, n\}$, and let $m_0 \in \pi_k(\Lambda) \subset \mathbb{Z}$ be such that

$$\left|\Lambda(m_0)\right| = \max_m \left|\Lambda(m)\right| \tag{2.2}$$

(cf. (1.1)). Certainly the integer m_0 providing the maximum section is not necessarily unique. In that case, one can choose arbitrarily any possibility. In order to establish a criterion for the construction we set

$$m_0 = \max\left\{m' \in \pi_k(\Lambda) : \left|\Lambda(m')\right| = \max_m \left|\Lambda(m)\right|\right\}.$$

Finally, we define the function $\sigma_k : \{\Lambda \subset \mathbb{Z}^k : \Lambda \text{ finite}\} \longrightarrow \{\Lambda \subset \mathbb{Z}^k : \Lambda \text{ finite}\}$ given by

$$\sigma_k(\Lambda) = \begin{cases} \Lambda \cup \{\max \Lambda + 1\} & \text{if } k = 1, \\ \Lambda \cup \Big(\Lambda(m_0) \times \big\{\max\{\pi_k(\Lambda)\} + 1\big\}\Big) & \text{if } k > 1; \end{cases}$$

i.e., σ_k acts on Λ by just adding a copy of the maximum section $\Lambda(m_0)$ to the set in the position $\max\{\pi_k(\Lambda)\}+1$. As before this choice is irrelevant, and the maximum section $\Lambda(m_0)$ can be placed at any $m \notin \pi_k(\Lambda)$.

We are now ready to recursively define $\bar{A}$ for a finite set $A \subset \mathbb{Z}^n$. In a first step, we construct a new set A_1^+ by means of its sections: $A_1^+ = \sigma_n(A)$ (see Figure 2.2).

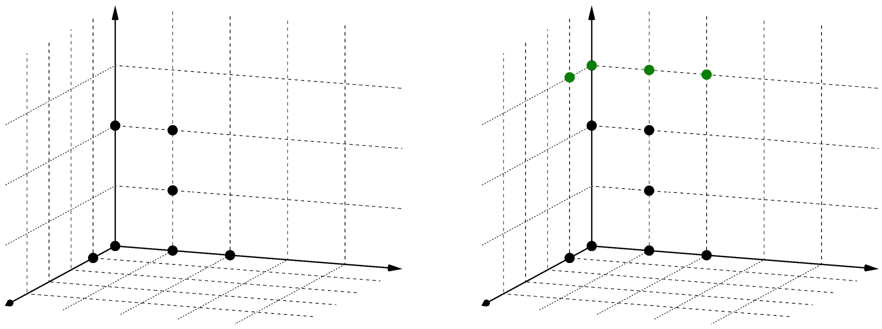

Figure 2.2: A discrete set A (left) and the set A_1^+ (right).

In the second one we take (see Figure 2.3)

$$A_2^+ = \bigcup_{m \in \pi_n(A_1^+)} \Big(\sigma_{n-1}\big(A_1^+(m)\big) \times \{m\}\Big).$$

In the k-th step, $k \geq 2$, we have

$$A_k^+ = \bigcup_{m \in \pi_{n-k+2,\ldots,n}(A_{k-1}^+)} \Big(\sigma_{n-k+1}\big(A_{k-1}^+(m)\big) \times \{m\}\Big).$$

Then we define $\bar{A} = A_n^+$.

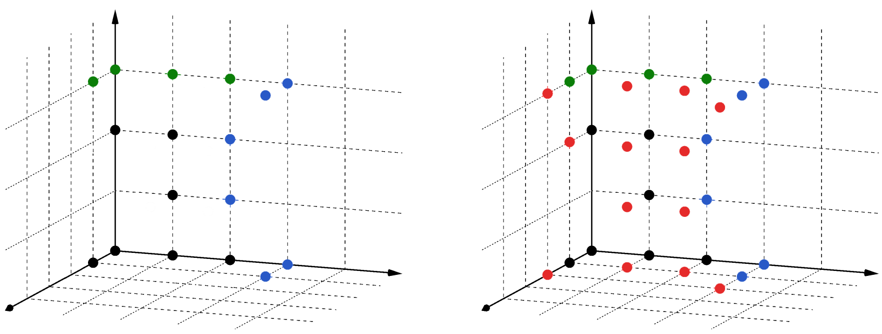

Figure 2.3: The sets A_2^+ (left) and $\bar{A} = A_3^+$ (right) for the discrete set A in Figure 2.3.

In the case of a lattice cube we have $\bar{\mathrm{C}}^n_{r_1} = \mathrm{C}^n_{r_1+1}$. Therefore $\bar{\mathrm{C}}^n_{r_1} + \mathrm{C}^n_{r_2} = \mathrm{C}^n_{r_1+r_2+1}$, and thus (2.1) holds with equality.

We note the recursive nature of the construction of $\bar{A}$, in which the action of adding the maximum section to the given set is repeatedly used onto every successive section of the original set A. Therefore, the following two properties are evident: for all $k = 1, \dots, n$,

$$\begin{aligned} &\text{i) } \pi_{n-k+1}\bigl(\bar{A}\bigr) = \pi_{n-k+1}(A_k^+) \quad \text{and} \\ &\text{ii) } \bar{A}(m) = \overline{A_k^+(m)},\ m \in \mathbb{Z}^k. \end{aligned} \tag{2.3}$$

In Section 2.4 we will show that the number of additional points in $|A|$ is somehow controlled. Moreover, upper and lower bounds for the ratio $\bigl|\bar{A}\bigr|/|A|$ and the difference $\bigl|\bar{A}\bigr| - |A|$ can be provided. In the first case only the dimension will play a role, whereas for the difference it will depend on the structure and the cardinality of A.

Remark 2.1. *The set $\bar{A}$ can be different (both its structure and cardinality) when either the role of the coordinate axes is interchanged in its construction, or if we use a different criterion for the choice of m_0, or even if we add as a "doubled" maximum section an arbitrary point set with the same cardinality. In any case, the number of additional points is controlled (see Proposition 2.4.3). Moreover, as we will see, the above choices for the construction of $\bar{A}$ will be not relevant for the proofs of the results.*

2.1.2 Transforming one set by removing points

Similarly, instead of adding points to the original (finite) set $A \subset \mathbb{Z}^n$, $A \neq \emptyset$, we may *reduce* it to define a new set $\mathrm{r}(A)$ in such a way that

$$\mathrm{r}\bigl(\bar{A}\bigr) = A. \tag{2.4}$$

To this aim, we define the function

$$\varrho_k : \{\Lambda \subset \mathbb{Z}^k : \Lambda \text{ finite}\} \longrightarrow \{\Lambda \subset \mathbb{Z}^k : \Lambda \text{ finite}\}$$

given by

$$\varrho_k(\Lambda) = \begin{cases} \Lambda \setminus \{m_0\} & \text{if } k = 1, \\ \Lambda \setminus \Bigl(\Lambda(m_0) \times \bigl\{m_0\bigr\}\Bigr) & \text{if } k > 1; \end{cases} \tag{2.5}$$

i.e., ϱ_k acts on Λ just removing the maximum section $\Lambda(m_0)$ from the set. To complete the picture we set $\varrho_k(\emptyset) = \emptyset$. In this way, ϱ_k is the left inverse function of σ_k.

Now, for $1 \leq k < n$, we write

$$A_k^- = \bigcup_{m \in \pi_{k+1,\dots,n}(A_{k-1}^-)} \Bigl(\varrho_k\bigl(A_{k-1}^-(m)\bigr) \times \{m\}\Bigr),$$

with $A_0^- = A$ (see Figure 2.4). Then we define

$$\mathrm{r}(A) = \varrho_n\bigl(A_{n-1}^-\bigr).$$

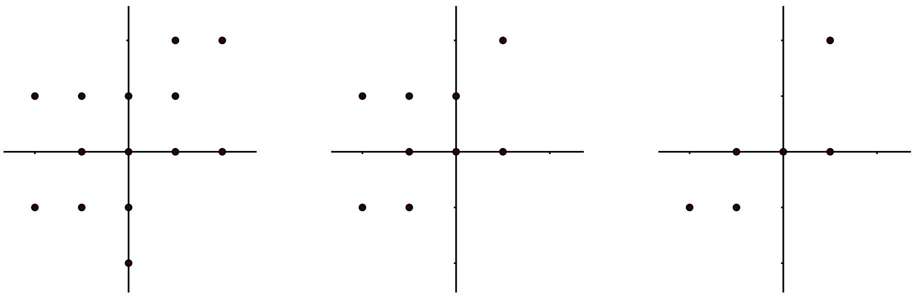

Figure 2.4: Transforming a discrete set A (left) into $\mathrm{r}(A)$ (right).

Remark 2.2. *We note on the one hand that, from the definition of* $\mathrm{r}(A)$*,* (2.4) *holds because* $\varrho_k\big(\sigma_k(\Lambda)\big)=\Lambda$ *for all* $k=1,\dots,n$*. On the other hand, since there are different ways to construct* $\overline{\mathrm{r}(A)}$*, (cf. Remark 2.1), it is possible to add every successive maximum section in such a way that*

$$\overline{\mathrm{r}(A)}\subset A. \tag{2.6}$$

Remark 2.3. *We observe that* $\mathrm{r}(A)$ *might be the empty set. Actually,* $\mathrm{r}(A)\neq\emptyset$ *necessarily implies that both* $|A|\geq 2^n$ *and* $\big|\pi_i(A)\big|\geq 2$ *for all* $1\leq i\leq n$*. Indeed,* A_{n-1}^- *must contain at least two points to assure that* $\mathrm{r}(A)\neq\emptyset$*; this yields that at least four points belong to* A_{n-2}^- *and, recursively, that* $|A|\geq 2^n$*. In the same way, if* $\big|\pi_i(A)\big|\leq 1$ *for some* $1\leq i\leq n$*, then we would have that* A_i^- *is empty and thus* $\mathrm{r}(A)=\emptyset$*.*

Now we are in a position to establish and prove the main results of this chapter.

2.2 New discrete Brunn-Minkowski type inequalities

Using the constructions provided in Section 2.1, we state the following theorems.

Theorem 2.2.1 ([12]). *Let* $A,B\subset\mathbb{Z}^n$ *be finite,* $A,B\neq\emptyset$*. Then*

$$\big|\bar{A}+B\big|^{1/n}\geq|A|^{1/n}+|B|^{1/n}. \tag{2.7}$$

Equality holds when A *and* B *are lattice cubes.*

As we mentioned in Remark 2.1, the set $\bar{A}$ can be constructed in different ways. Thus, in order to bound from above $|A|^{1/n}+|B|^{1/n}$ in Theorem 2.2.1, one can choose in the definition of $\bar{A}$ the options (for m_0 and the axis order) making $\big|\bar{A}+B\big|$ minimum, which surely will depend on the original sets A and B.

Regarding the second construction, the corresponding discrete Brunn-Minkowski inequality reads as follows:

Theorem 2.2.2 ([12]). *Let $A, B \subset \mathbb{Z}^n$ be finite, $A, B \neq \emptyset$. Then*

$$|A+B|^{1/n} \geq \left|\mathrm{r}(A)\right|^{1/n} + |B|^{1/n}. \tag{2.8}$$

Equality holds when A and B are lattice cubes.

In fact we will prove that the discrete inequalities (2.7) and (2.8) are equivalent (see Proposition 2.2.1).

Before getting fully into the proofs of our theorems, we will make a few previous considerations and remarks.

2.2.1 On the different Brunn-Minkowski type inequalities

Before starting the proofs of our main theorems, we observe that inequalities (2.7) and (1.9) (or even (1.10)) are not comparable. For instance, if $A = B = \{0, \mathrm{e}_1, \mathrm{e}_2\}$, then $D^B_{|A|} = A$ and $D^B_{|B|} = B$, and obviously equality holds in (1.9), but we have a strict inequality in (2.7). Therefore (1.9) provides a stronger bound than (2.7). However, if $A = B = \mathrm{C}^2_2$ then $A + B = \mathrm{C}^2_4$ and, moreover, $D^B_{|A|} = D^B_{|B|}$ is the lattice simplex $\mathrm{conv}\{0, 7\mathrm{e}_1, \mathrm{e}_2\} \cap \mathbb{Z}^2$ (see Figure 2.5). Hence

$$|A+B| = 25 > \left|D^B_{|A|} + D^B_{|B|}\right| = 24,$$

whereas we have equality in (2.7). In this case, the bound provided by (2.7) is stronger than (1.9) (or (1.10)).

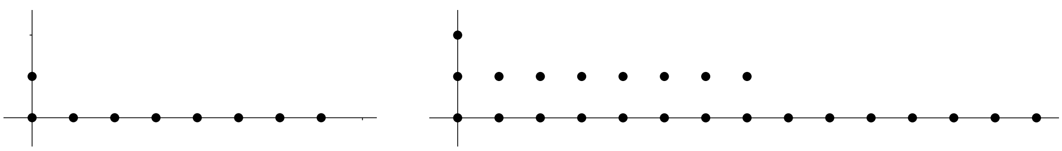

Figure 2.5: $D^B_{|A|}$ for $A = B = \mathrm{C}^2_2$ (left) and $D^B_{|A|} + D^B_{|B|}$ (right).

On the other hand, in [20], Matolcsi and Ruzsa consider the sum set

$$A + kB := A + B + \overset{(k)}{\ldots} + B,$$

$k \in \mathbb{N}$, and provide a lower bound for its cardinality when $\dim B = n$ and $A \subset \mathrm{conv}\, B$:

$$|A + kB| \geq \left(|A| - \frac{kn}{k+1}\right)\binom{n+k}{k}.$$

In [3], Böröczky, Santos and Serra characterize the sets A and B for which equality holds. As a direct consequence of Theorem 2.2.2 another bound for the cardinality $|A + kB|$ can be obtained, without additional conditions on the sets A and B:

Corollary 2.2.1 ([12]). *Let $A, B \subset \mathbb{Z}^n$ be finite, $A, B \neq \emptyset$. Then*

$$|A + kB|^{1/n} \geq |A|^{1/n} + k\big|\mathrm{r}(B)\big|^{1/n}.$$

Equality holds when A and B are lattice cubes.

It can also be easily seen, by just considering the corresponding extremal sets, that the bound of Matolcsi and Ruzsa in [20] for $|A+kB|$ and the one provided by Corollary 2.2.1 are not comparable.

In order to conclude this subsection, we would like to do a final remark. We note that although the cardinality of $\bar{A}$ is obviously enlarged, in many cases the difference between $\big|\bar{A} + B\big|$ and $|A+B|$ may be not too big; indeed one can have $\big|\bar{A} + B\big| = |A+B|$. An illustration of this is Example 2.1, which can be generalized to the following families of sets:

Example 2.3. *A compact set $K \subset \mathbb{R}^n$ is called* unconditional *if, for any $(x_1, \dots, x_n) \in K$, then $(\varepsilon_1 x_1, \dots, \varepsilon_n x_n) \in K$ for all $\varepsilon_i \in [-1, 1]$, $i = 1, \dots, n$. We consider the following sets. Given an unconditional compact set $K \subset \mathbb{R}^n$, let*

$$A = \left(K \setminus \bigcup_{i=1}^{n} H_i\right) \cap \mathbb{Z}^n.$$

Furthermore, let $B \subset \mathbb{Z}^n$ (finite) satisfy the following condition: if $(x_1, \dots, x_n) \in B$, there exist $\varepsilon_i \in \{-1, 1\}$, $i = 1, \dots, n$, such that $(x_1, \dots, x_{i-1}, x_i + \varepsilon_i, x_{i+1}, \dots, x_n) \in B$ for all $i = 1, \dots, n$ (see Figure 2.6).

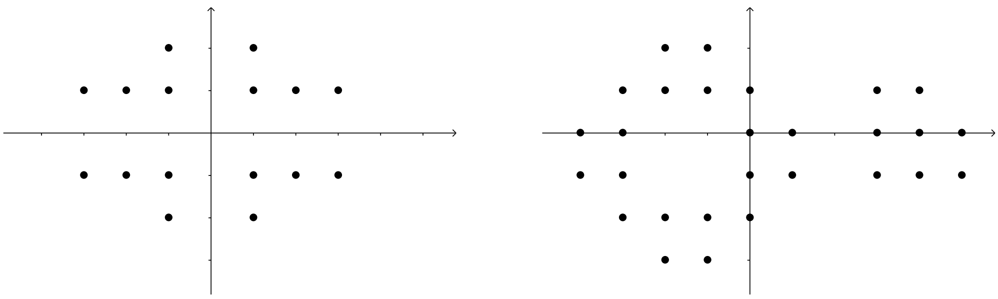

Figure 2.6: An example of a set A (left) and B (right) in the above constructions.

As mentioned in Remark 2.1, there are different ways of constructing $\bar{A}$, and in the case of $A = (K \setminus \bigcup_{i=1}^{n} H_i) \cap \mathbb{Z}^n$, we place the successive maximum sections on the coordinates hyperplanes. In this way we even have

$$\bar{A} + B = A + B.$$

Indeed, given $x = (x_1, \dots, x_n) \in \bar{A} \setminus A$ and $b = (b_1, \dots, b_n) \in B$ let $I \subset \{1, \dots, n\}$ be such that $x_i = 0$ if $i \in I$ and $x_i \neq 0$ otherwise. On one hand, there exist $\varepsilon_i \in \{-1, 0, 1\}$, $i = 1, \dots, n$, such that $(b_1 + \varepsilon_1, \dots, b_n + \varepsilon_n) \in B$, and with $\varepsilon_i = 0$ if and only if $i \notin I$. On the other hand, denoting by

$\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$, we have $x - \varepsilon \in A$ because K is unconditional. Then $x + b = (x - \varepsilon) + (b + \varepsilon) \in A + B$, which shows that $\bar{A} + B \subset A + B$. The reverse inclusion is obvious.

Therefore, although we are adding points in the construction of $\bar{A}$, the cardinality of $\bar{A} + B$ does not increase (with respect to that of $A + B$). Hence, Theorem 2.2.1 yields

$$|A + B|^{1/n} \geq |A|^{1/n} + |B|^{1/n}.$$

We also note that for the constructed set A, the above inequality does not hold for arbitrary B (for instance, it is enough to consider $B = \{0\}$).

2.2.2 The proofs of the main theorems

Before the proof of Theorem 2.2.1 we have to state two auxiliary results. The first one may be regarded as a discrete counterpart of the *layer cake formula*.

Lemma 2.2.1 ([12]). *Let $\Omega \subset \mathbb{Z}$ be finite and let $f : \Omega \longrightarrow \mathbb{Z}_{\geq 0}$. Then*

$$\sum_{m \in \Omega} f(m) = \sum_{t=1}^{\max_\Omega f} \Big|\big\{m \in \Omega : f(m) \geq t\big\}\Big|.$$

Proof. Let $N = \max_{m \in \Omega} f(m)$, and we consider variables x_i, $i = 1, \dots, N$. Then we have the relation

$$\sum_{m \in \Omega} \big(x_1 + x_2 + \dots + x_{f(m)}\big) = \sum_{t=1}^{N} x_t \Big|\big\{m \in \Omega : f(m) \geq t\big\}\Big|,$$

because the variable x_t appears in the left-hand side expression if and only if $f(m) \geq t$. Then, setting $x_1 = \dots = x_N = 1$, we get the result. □

Lemma 2.2.2 ([12]). *Let $\Omega \subset \mathbb{Z}$ be finite and let $f : \Omega \longrightarrow \mathbb{Z}_{\geq 0}$. Then, for any $r, N \in \mathbb{N}$, we have*

$$r \sum_{t=1}^{N} \Big|\big\{m \in \Omega : f(m) \geq t\big\}\Big| = \sum_{t = \frac{1}{rN}, \frac{2}{rN}, \dots, 1} \left|\left\{m \in \Omega : \frac{f(m)}{N} \geq t\right\}\right|.$$

Proof. First we rewrite

$$\begin{aligned}
\sum_{t = \frac{1}{rN}, \frac{2}{rN}, \dots, 1} \left|\left\{m \in \Omega : \frac{f(m)}{N} \geq t\right\}\right| = & \sum_{t = \frac{1}{rN}, \dots, \frac{1}{N}} \left|\left\{m \in \Omega : \frac{f(m)}{N} \geq t\right\}\right| \\
& + \sum_{t = \frac{r+1}{rN}, \dots, \frac{2}{N}} \left|\left\{m \in \Omega : \frac{f(m)}{N} \geq t\right\}\right| \\
& + \dots + \sum_{t = \frac{(N-1)r+1}{rN}, \dots, 1} \left|\left\{m \in \Omega : \frac{f(m)}{N} \geq t\right\}\right|.
\end{aligned}$$

We note that, for each of the above sums, i.e., for all $i = 0, \dots, N-1$,

$$\begin{aligned}\sum_{t=\frac{ir+1}{rN},\dots,\frac{i+1}{N}} \left|\left\{m \in \Omega : \frac{f(m)}{N} \geq t\right\}\right| &= \sum_{t=\frac{ir+1}{r},\dots,i+1} \left|\left\{m \in \Omega : f(m) \geq t\right\}\right| \\ &= \sum_{t=i+\frac{1}{r},\dots,i+1} \left|\left\{m \in \Omega : f(m) \geq i+1\right\}\right| \\ &= r\left|\left\{m \in \Omega : f(m) \geq i+1\right\}\right|,\end{aligned}$$

and thus we can conclude that

$$\sum_{t=\frac{1}{rN},\frac{2}{rN},\dots,1} \left|\left\{m \in \Omega : \frac{f(m)}{N} \geq t\right\}\right| = r\sum_{i=0}^{N-1}\left|\left\{m \in \Omega : f(m) \geq i+1\right\}\right|.$$

This proves the result. □

Now we are in a position to prove our first discrete version of the Brunn-Minkowski inequality.

Proof of Theorem 2.2.1. We will show (2.7) by (finite) induction on the dimension n. The case $n = 1$ is a direct consequence of (1.5):

$$\left|\bar{A} + B\right| \geq \left|\bar{A}\right| + |B| - 1 = |A| + |B|.$$

So, we will suppose that the inequality is true for $n-1$. We first observe that for all $m_1, m_2 \in \mathbb{Z}$, it is clear that

$$\left(\bar{A} + B\right)(m_1 + m_2) \supset \bar{A}(m_1) + B(m_2).$$

Then, taking $m_1 \in \pi_n\big(\bar{A}\big) = \pi_n(A_1^+)$ (cf. (2.3) i)) and $m_2 \in \pi_n(B)$, and applying induction hypothesis (i.e., (2.7) in $\mathbb{Z}^{n-1}$), we get (see also (2.3) ii))

$$\begin{aligned}\left|\left(\bar{A} + B\right)(m_1 + m_2)\right| &\geq \left|\bar{A}(m_1) + B(m_2)\right| = \left|\overline{A_1^+(m_1)} + B(m_2)\right| \\ &\geq \left(\left|A_1^+(m_1)\right|^{1/(n-1)} + \left|B(m_2)\right|^{1/(n-1)}\right)^{n-1}.\end{aligned} \tag{2.9}$$

For the sake of brevity we denote by

$$c_A = \max_{m\in\mathbb{Z}}\left|A(m)\right| > 0, \qquad c_B = \max_{m\in\mathbb{Z}}\left|B(m)\right| > 0,$$

and let

$$c = \left(c_A^{1/(n-1)} + c_B^{1/(n-1)}\right)^{n-1} \quad \text{and} \quad \theta = \frac{c_B^{1/(n-1)}}{c_A^{1/(n-1)} + c_B^{1/(n-1)}} \in (0,1).$$

We observe that also $c_A = \max_{m\in\mathbb{Z}}\left|A_1^+(m)\right|$. Furthermore, let

$$p, q \in \mathbb{N} \quad \text{satisfy } \frac{p}{q} \leq c. \tag{2.10}$$

Finally, for $M = A, A_1^+, B$ or $\bar{A}+B$, we denote by $f_M : \mathbb{Z} \longrightarrow \mathbb{Q}_{\geq 0}$ the functions given by

$$f_A(m) = \frac{\bigl|A(m)\bigr|}{c_A}, \quad f_{A_1^+}(m) = \frac{\bigl|A_1^+(m)\bigr|}{c_A}, \quad f_B(m) = \frac{\bigl|B(m)\bigr|}{c_B}, \quad \text{and}$$
$$f_{\bar{A}+B}(m) = \frac{q}{p}\Bigl|\bigl(\bar{A}+B\bigr)(m)\Bigr|.$$

Using (2.9) we get

$$\begin{aligned}
\Bigl|\bigl(\bar{A}+B\bigr)\,(m_1+m_2)\Bigr| &\geq \Bigl(\bigl|A_1^+(m_1)\bigr|^{1/(n-1)} + \bigl|B(m_2)\bigr|^{1/(n-1)}\Bigr)^{n-1} \\
&= c\left(\frac{c_A^{1/(n-1)}}{c^{1/(n-1)}} f_{A_1^+}(m_1)^{1/(n-1)} + \frac{c_B^{1/(n-1)}}{c^{1/(n-1)}} f_B(m_2)^{1/(n-1)}\right)^{n-1} \\
&= c\Bigl((1-\theta) f_{A_1^+}(m_1)^{1/(n-1)} + \theta f_B(m_2)^{1/(n-1)}\Bigr)^{n-1} \\
&\geq c\,\min\Bigl\{f_{A_1^+}(m_1), f_B(m_2)\Bigr\} \geq \frac{p}{q}\min\Bigl\{f_{A_1^+}(m_1), f_B(m_2)\Bigr\}.
\end{aligned}$$

Thus, we have obtained the functional inequality

$$f_{\bar{A}+B}(m_1+m_2) \geq \min\Bigl\{f_{A_1^+}(m_1), f_B(m_2)\Bigr\}. \tag{2.11}$$

Now we observe, on one hand, that the super-level sets

$$\bigl\{m \in \mathbb{Z} : f_A(m) \geq t\bigr\}, \quad \Bigl\{m \in \mathbb{Z} : f_{A_1^+}(m) \geq t\Bigr\}, \quad \bigl\{m \in \mathbb{Z} : f_B(m) \geq t\bigr\}$$

are non-empty for all $t \in [0,1]$ and, moreover, the definition of A_1^+ yields

$$\Bigl|\Bigl\{m \in \mathbb{Z} : f_{A_1^+}(m) \geq t\Bigr\}\Bigr| = \Bigl|\bigl\{m \in \mathbb{Z} : f_A(m) \geq t\bigr\}\Bigr| + 1.$$

On the other hand, (2.11) implies that

$$\bigl\{m \in \mathbb{Z} : f_{\bar{A}+B}(m) \geq t\bigr\} \supset \Bigl\{m \in \mathbb{Z} : f_{A_1^+}(m) \geq t\Bigr\} + \bigl\{m \in \mathbb{Z} : f_B(m) \geq t\bigr\},$$

and then, using (1.5) for $n = 1$ and the above identity, we get

$$\begin{aligned}
\Bigl|\bigl\{m \in \mathbb{Z} : f_{\bar{A}+B}(m) \geq t\bigr\}\Bigr| &\geq \Bigl|\Bigl\{m \in \mathbb{Z} : f_{A_1^+}(m) \geq t\Bigr\}\Bigr| + \Bigl|\bigl\{m \in \mathbb{Z} : f_B(m) \geq t\bigr\}\Bigr| - 1 \\
&= \Bigl|\bigl\{m \in \mathbb{Z} : f_A(m) \geq t\bigr\}\Bigr| + \Bigl|\bigl\{m \in \mathbb{Z} : f_B(m) \geq t\bigr\}\Bigr|.
\end{aligned} \tag{2.12}$$

We also observe that the cardinality of $\bigl|\bar{A}+B\bigr|$ can be expressed as

$$\bigl|\bar{A}+B\bigr| = \sum_{m\in\mathbb{Z}} \Bigl|\bigl(\bar{A}+B\bigr)\,(m)\Bigr| = \sum_{m\in\mathbb{Z}} \frac{p}{q}\, f_{\bar{A}+B}(m),$$

where we write the sum over $\mathbb{Z}$ for the sake of brevity. Analogously,

$$|A| = \sum_{m\in\mathbb{Z}} c_A f_A(m) \quad \text{and} \quad |B| = \sum_{m\in\mathbb{Z}} c_B f_B(m).$$

Lemma 2.2.1 applied to the (integer) function $f(m) = p\, f_{\bar{A}+B}(m)$ leads to

$$\left|\bar{A}+B\right| = \frac{1}{q}\sum_{m\in\mathbb{Z}} p\, f_{\bar{A}+B}(m) = \frac{1}{q}\sum_{t=1}^{p\,\max_{\mathbb{Z}} f_{\bar{A}+B}} \left|\left\{m\in\mathbb{Z} : p\, f_{\bar{A}+B}(m)\geq t\right\}\right|,$$

and since $\max_{m\in\mathbb{Z}} f_{\bar{A}+B}(m)\geq 1$ by (2.11), we get

$$\left|\bar{A}+B\right| \geq \frac{1}{q}\sum_{t=1}^{p}\left|\left\{m\in\mathbb{Z} : p\, f_{\bar{A}+B}(m)\geq t\right\}\right|.$$

Let $c' = p\, c_A c_B$. Applying Lemma 2.2.2 to the above sum for $N = p$ and $r = c_A c_B$, and then using (2.12), we obtain

$$\begin{aligned} \left|\bar{A}+B\right| &\geq \frac{1}{q}\frac{1}{c_A c_B}\sum_{t=\frac{1}{c'},\frac{2}{c'},\dots,1}\left|\left\{m\in\mathbb{Z} : f_{\bar{A}+B}(m)\geq t\right\}\right| \\ &\geq \frac{1}{q}\frac{1}{c_A c_B}\sum_{t=\frac{1}{c'},\dots,1}\left[\left|\left\{m\in\mathbb{Z} : f_A(m)\geq t\right\}\right| + \left|\left\{m\in\mathbb{Z} : f_B(m)\geq t\right\}\right|\right]. \end{aligned} \tag{2.13}$$

Now, Lemma 2.2.2 for $N = c_A$, $r = p\, c_B$ and Lemma 2.2.1 yield

$$\begin{aligned} \sum_{t=\frac{1}{c'},\dots,1}\left|\left\{m\in\mathbb{Z} : f_A(m)\geq t\right\}\right| &= p\, c_B\sum_{t=1}^{c_A}\left|\left\{m\in\mathbb{Z} : c_A f_A(m)\geq t\right\}\right| \\ &= p\, c_B\sum_{m\in\mathbb{Z}} c_A f_A(m) = p\, c_B|A|, \end{aligned} \tag{2.14}$$

and analogously (now $N = c_B$ and $r = p\, c_A$ in Lemma 2.2.2),

$$\sum_{t=\frac{1}{c'},\dots,1}\left|\left\{m\in\mathbb{Z} : f_B(m)\geq t\right\}\right| = p\, c_A|B|. \tag{2.15}$$

Then, (2.13), (2.14) and (2.15) together, give

$$\left|\bar{A}+B\right| \geq \frac{1}{q}\frac{1}{c_A c_B}\Big(p\, c_B|A| + p\, c_A|B|\Big) = \frac{p}{q}\left(\frac{|A|}{c_A}+\frac{|B|}{c_B}\right). \tag{2.16}$$

Since (2.16) holds for any rational number $p/q\leq c$ (cf. (2.10)), by a limit procedure we also get inequality (2.16) for the real positive number c. And then, applying the (reverse) Hölder inequality (Corollary 1.1.1) with parameters $1/n$ and $-1/(n-1)$, we conclude that

$$\left|\bar{A}+B\right| \geq c\left(\frac{|A|}{c_A}+\frac{|B|}{c_B}\right) \geq \left(|A|^{1/n}+|B|^{1/n}\right)^n.$$

Finally we prove that the inequality is sharp. Indeed, let A, B be the lattice cubes $A = \mathrm{C}^n_{m_1}$ and $B = \mathrm{C}^n_{m_2}$. Then $\bar{A} = \mathrm{C}^n_{m_1+1}$, and hence $\bar{A}+B = \mathrm{C}^n_{m_1+m_2+1}$. Therefore,

$$\left|\bar{A}+B\right|^{1/n} = m_1+m_2+2 = |A|^{1/n}+|B|^{1/n}. \qquad \square$$

Remark 2.4. *We observe that our approach involves not only finite sets of* $\mathbb{Z}^n$*, but can be extended to general (finite) sets of* $\mathbb{R}^n$ *by suitably defining* $\bar{A}$*. Therefore, Theorem 2.2.1 can be stated for any pair of finite (non-empty) sets of* $\mathbb{R}^n$*.*

The following corollary is a direct consequence of Theorem 2.2.1.

Corollary 2.2.2 ([12]). *Let* A, B *be finite subsets of* $\mathbb{Z}^n$*,* $A, B \neq \emptyset$*. Then*

$$|A+B| \geq \left(|A|^{1/n} + |B|^{1/n}\right)^n - \left|\left(\bar{A}+B\right) \setminus (A+B)\right|.$$

We conclude this section by proving the second version of our discrete Brunn-Minkowski inequality.

Proof of Theorem 2.2.2. If $\mathrm{r}(A) = \emptyset$ then the inequality $|A+B|^{1/n} \geq |B|^{1/n}$ trivially holds. So we assume that $\mathrm{r}(A) \neq \emptyset$. In this case, Theorem 2.2.1 applied to the sets $\mathrm{r}(A)$ and B, together with (2.6) yields

$$|A+B|^{1/n} \geq \left|\overline{\mathrm{r}(A)} + B\right|^{1/n} \geq \left|\mathrm{r}(A)\right|^{1/n} + |B|^{1/n}.$$

The equality case is a consequence of the equality case in Theorem 2.2.1. □

Moreover, it is easy to see that (2.7) and (2.8) are indeed equivalent:

Proposition 2.2.1 ([12]). *Let* $A, B \subset \mathbb{Z}^n$ *be finite,* $A, B \neq \emptyset$*. Then* (2.7) *and* (2.8) *are equivalent.*

Proof. In the proof of Theorem 2.2.2 we have already proved that (2.7) implies (2.8). In order to prove the converse we just have to note that the operator $\mathrm{r}(\cdot)$ has been defined in such a way that $\mathrm{r}\big(\bar{A}\big) = A$ for any $A \neq \emptyset$ (cf. (2.4)). Therefore, applying (2.8) to $\bar{A}$ and B we get

$$\left|\bar{A}+B\right|^{1/n} \geq \left|\mathrm{r}\big(\bar{A}\big)\right|^{1/n} + |B|^{1/n} = |A|^{1/n} + |B|^{1/n}.$$ □

2.3 From the discrete version to the continuous one

We note that it is not possible to directly obtain any of the above discrete Brunn-Minkowski inequalities from the classical one (1.2) by using the method of replacing the points by suitable compact sets. As pointed out by Gardner and Gronchi in [8, pages 3996–3997],

> "*it is worth remarking that the obvious idea of replacing the points in the two finite sets by small congruent balls and applying the classical Brunn-Minkowski inequality to the resulting compact sets is doomed to failure. The fact that the sum of two congruent balls is a ball of twice the radius introduces an extra factor of* $1/2$ *that renders the resulting bound weaker than even the trivial bound (11) below*".

We clarify that (11) in [8] coincides with (1.5) of the present dissertation.

This section is devoted to showing that, however, the classical Brunn-Minkowski inequality for compact sets (1.2) can be obtained as a consequence of Theorem 2.2.2.

For each $k \in \mathbb{N}$, we consider the family of all (closed) cubes of edge-length 2^{-k}, with vertices in the lattice $2^{-k}\mathbb{Z}^n$. This family tessellates the whole space, i.e., covers $\mathbb{R}^n$ and its elements have disjoint interiors.

Definition 2.1. *Let $K \subset \mathbb{R}^n$. The k-*discretization *of K, $k \in \mathbb{N}$, is defined as*

$$K_k = \left\{ x \in 2^{-k}\mathbb{Z}^n : \left(x + \left[0, 2^{-k}\right]^n \right) \cap K \neq \emptyset \right\}.$$

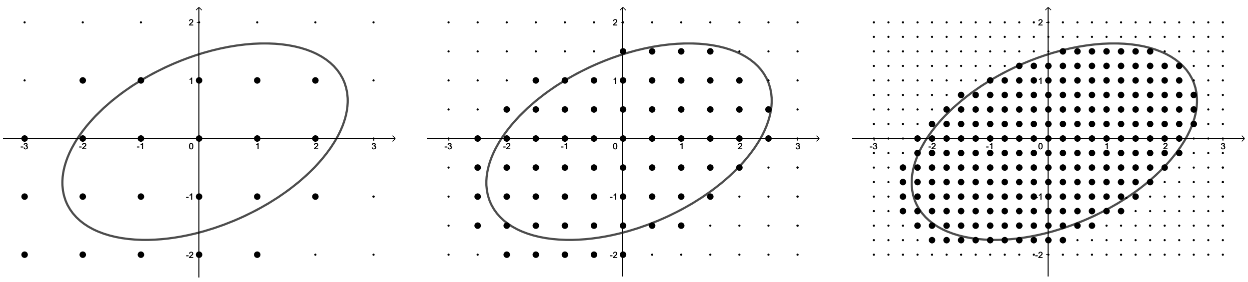

Figure 2.7: Examples of a k-discretization of an ellipse for $k = 0, 1, 2$.

Standard arguments allow to show the following lemma, which will be a key point in order to get the main result of this section. We include its proof here for completeness.

Lemma 2.3.1 ([12]). *Let $K \subset \mathbb{R}^n$ be a non-empty compact set. Then*

$$\mathrm{vol}(K) = \lim_{k\to\infty} \frac{|K_k|}{2^{kn}}.$$

Proof. Given a compact set $K \subset \mathbb{R}^n$, a standard straightforward computation shows that

$$K = \bigcap_{k=1}^{\infty} \left(K_k + \left[0, 2^{-k}\right]^n \right). \tag{2.17}$$

Indeed, let $x \in \cap_{k=1}^{\infty} \left(K_k + [0, 2^{-k}]^n\right)$. Then, $x \in \left(K_k + [0, 2^{-k}]^n\right)$ for all $k \in \mathbb{N}$ and thus, since the diameter of each cube of $K_k + [0, 2^{-k}]^n$ (namely, the largest distance between any two points therein) equals $2^{-k}\sqrt{n}$, we have that $d(x, K) \leq 2^{-k}\sqrt{n}$ for all $k \in \mathbb{N}$. This implies that $d(x, K) = 0$ and then $x \in K$ (since K is closed). The reverse inclusion is trivial.

On the other hand, the fact that $\left(K_k + [0, 2^{-k}]^n\right)_{k\in\mathbb{N}}$ is a decreasing sequence satisfying that $\mathrm{vol}\left(K_1 + [0, 2^{-1}]^n\right) < \infty$ yields (see e.g. [6, Proposition 1.2.5 (b)])

$$\mathrm{vol}\left(\bigcap_{k=1}^{\infty} \left(K_k + \left[0, 2^{-k}\right]^n \right) \right) = \lim_{k\to\infty} \mathrm{vol}\left(K_k + \left[0, 2^{-k}\right]^n \right),$$

and therefore

$$\mathrm{vol}(K) = \mathrm{vol}\left(\bigcap_{k=1}^{\infty}\left(K_k + \left[0, 2^{-k}\right]^n\right)\right) = \lim_{k\to\infty} \mathrm{vol}\left(K_k + \left[0, 2^{-k}\right]^n\right) = \lim_{k\to\infty} \frac{|K_k|}{2^{kn}}. \qquad \square$$

Theorem 2.3.1 ([12]). *The discrete Brunn-Minkowski inequality* (2.8) *implies the classical Brunn-Minkowski inequality* (1.2).

Proof. For each $k \in \mathbb{N}$, let K_k, L_k be the k-discretizations of K, L, respectively. Since K and L are compact, both K_k, L_k are finite sets, and we can use (2.8) to deduce that, for any $k \in \mathbb{N}$, we have

$$|K_k + L_k|^{1/n} \geq \left|\mathrm{r}(K_k)\right|^{1/n} + |L_k|^{1/n}.$$

Therefore

$$\lim_{k\to\infty}\left(\frac{|K_k + L_k|}{2^{kn}}\right)^{1/n} \geq \lim_{k\to\infty}\left(\frac{\left|\mathrm{r}(K_k)\right|}{2^{kn}}\right)^{1/n} + \lim_{k\to\infty}\left(\frac{|L_k|}{2^{kn}}\right)^{1/n}. \tag{2.18}$$

Now, for $k \in \mathbb{N}$, we define the set

$$F_k = \left(K_k + \left[0, 2^{-k}\right]^n\right) + \left(L_k + \left[0, 2^{-k}\right]^n\right).$$

It is clear that $F_1 \supset F_2 \supset \dots$ and, moreover,

$$K + L = \bigcap_{k=1}^{\infty} F_k.$$

Hence

$$\mathrm{vol}(K + L) = \mathrm{vol}\left(\bigcap_{k=1}^{\infty} F_k\right) = \lim_{k\to\infty} \mathrm{vol}(F_k)$$

(see e.g. [6, Proposition 1.2.5 (b)]) and then, from

$$F_k = K_k + L_k + \left[0, 2^{-k+1}\right]^n \supset K_k + L_k + \left[0, 2^{-k}\right]^n$$

we obtain

$$\mathrm{vol}(K + L)^{1/n} = \lim_{k\to\infty} \mathrm{vol}(F_k)^{1/n} \geq \lim_{k\to\infty}\left(\frac{|K_k + L_k|}{2^{kn}}\right)^{1/n}.$$

Now, using (2.18) and Lemma 2.3.1 we immediately get

$$\mathrm{vol}(K + L)^{1/n} \geq \lim_{k\to\infty}\left(\frac{\left|\mathrm{r}(K_k)\right|}{2^{kn}}\right)^{1/n} + \lim_{k\to\infty}\left(\frac{\left|L_k\right|}{2^{kn}}\right)^{1/n} = \lim_{k\to\infty}\left(\frac{\left|\mathrm{r}(K_k)\right|}{2^{kn}}\right)^{1/n} + \mathrm{vol}(L)^{1/n}.$$

Thus, in order to finish the proof, it suffices to show that

$$\lim_{k\to\infty} \frac{\left|\mathrm{r}(K_k)\right|}{2^{kn}} = \mathrm{vol}(K). \tag{2.19}$$

For the sake of brevity we denote by $K_{k,i} = (K_k)^-_{i-1} \setminus (K_k)^-_i$, $i = 1, \ldots, n$, i.e., the set of all points removed from K_k in the i-th step of the construction of $\mathrm{r}(K_k)$. Then it is clear that

$$\mathrm{vol}\left(K_{k,i} + \left[0, 2^{-k}\right]^n\right) = \mathrm{vol}\left(\pi_{(i)}(K_{k,i}) + \left[0, 2^{-k}\right]^n\right)$$

and hence

$$\begin{aligned}\frac{|K_{k,i}|}{2^{kn}} &= \mathrm{vol}\left(K_{k,i} + \left[0, 2^{-k}\right]^n\right) = \mathrm{vol}\left(\pi_{(i)}(K_{k,i}) + \left[0, 2^{-k}\right]^n\right) \\ &\leq \mathrm{vol}\left(\pi_{(i)}(K_k) + \left[0, 2^{-k}\right]^n\right) = \frac{\left|\pi_{(i)}(K_k)\right|}{2^{kn}}.\end{aligned}$$

Moreover, $\left(\pi_{(i)}(K)\right)_k = \pi_{(i)}(K_k)$, and then Lemma 2.3.1 yields

$$0 = \mathrm{vol}\left(\pi_{(i)}(K)\right) = \lim_{k\to\infty} \frac{\left|\left(\pi_{(i)}(K)\right)_k\right|}{2^{kn}} = \lim_{k\to\infty} \frac{\left|\pi_{(i)}(K_k)\right|}{2^{kn}} \geq \lim_{k\to\infty} \frac{|K_{k,i}|}{2^{kn}},$$

which implies that

$$\lim_{k\to\infty} \frac{|K_{k,i}|}{2^{kn}} = 0.$$

With Lemma 2.3.1 again, this shows that

$$\begin{aligned}\lim_{k\to\infty} \frac{\left|\mathrm{r}(K_k)\right|}{2^{kn}} &= \lim_{k\to\infty} \frac{|K_k| - \left|K_k \setminus \mathrm{r}(K_k)\right|}{2^{kn}} = \lim_{k\to\infty} \frac{|K_k| - \sum_{i=1}^n |K_{k,i}|}{2^{kn}} \\ &= \lim_{k\to\infty} \frac{|K_k|}{2^{kn}} - \sum_{i=1}^n \lim_{k\to\infty} \frac{|K_{k,i}|}{2^{kn}} = \mathrm{vol}(K).\end{aligned}$$

This proves (2.19) and concludes the proof. □

2.4 Bounding the cardinality of the sets $\bar{A}$ and $\mathrm{r}(A)$

Let $A \subset \mathbb{Z}^n$ be finite and non-empty. In this section we will show that the number of additional points in $\bar{A}$ cannot be too large, and depends only on (the structure of) A and on the dimension. More precisely, we will provide upper and lower bounds for the ratio $\left|\bar{A}\right|/|A|$ and the difference $\left|\bar{A}\right| - |A|$. Similarly, we will give bounds for both $|\mathrm{r}(A)|/|A|$ and $|A| - |\mathrm{r}(A)|$. We start with the first case, where only the dimension plays a role.

Proposition 2.4.1 ([12]). *Let $n \geq 1$ and let $A \subset \mathbb{Z}^n$ be finite and non-empty. Then*

$$1 < \frac{|\bar{A}|}{|A|} \leq 2^n. \tag{2.20}$$

Both inequalities are sharp.

Proof. First we prove the upper bound. It is clear that $|A_1^+| \leq 2|A|$, and equality holds if and only if $\bigl|\pi_n(A)\bigr| = 1$. Moreover, $|A_{i+1}^+| \leq 2|A_i^+|$ for all $i = 1, \dots, n-1$, with equality if and only if $\bigl|\pi_{n-i}(A_i^+(m))\bigr| = 1$ for every $m \in \pi_{n-i+1,\dots,n}(A_i^+)$. Then we get $\bigl|\bar{A}\bigr|/|A| \leq 2^n$, and equality holds if and only if A is a singleton.

Finally we observe that the ratio $\bigl|\bar{A}\bigr|/|A| > 1$ trivially, and that it may be arbitrarily small, as it is shown by considering $A = \mathrm{C}_r^n$, $r \in \mathbb{N}$. In this case,

$$\frac{\bigl|\bar{A}\bigr|}{|A|} = \frac{|\mathrm{C}_{r+1}^n|}{|\mathrm{C}_r|} = \left(1 + \frac{1}{r+1}\right)^n,$$

which tends to 1 when $r \to \infty$. This shows that the lower bound in (2.20) is tight. □

Next we deal with the second case, that of getting bounds for $\bigl|\bar{A}\bigr| - |A|$, which will also depend on the structure and the cardinality of the original set A. We are going to present two different upper bounds for the difference $|\bar{A}| - |A|$ because, depending on the distribution of the points of A, each one can be sharper than the other. Roughly speaking, if A is somehow "close" to a lattice orthogonal box, then Proposition 2.4.3 will give a more precise bound for $\bigl|\bar{A}\bigr| - |A|$; on the contrary, if the points of A lie in different hyperplanes orthogonal to the coordinate axis, then Proposition 2.4.5 will provide a better approximation. Nevertheless, a suitable "mixture" of both results will lead to a third sharper upper bound (see Theorem 2.4.1).

We start noticing that if we intend to control the number of points that we add to A, first we must determine how many new points we have in the first step A_1^+. Clearly,

$$\bigl|\pi_i(A_1^+)\bigr| = \begin{cases} \bigl|\pi_i(A)\bigr| & \text{for } i = 1, \dots, n-1, \\ \bigl|\pi_n(A)\bigr| + 1 & \text{for } i = n. \end{cases} \tag{2.21}$$

Proposition 2.4.2 ([12]). *Let $n \geq 2$ and let $A \subset \mathbb{Z}^n$ be finite and non-empty. Then*

$$|A_1^+| - |A| \leq \prod_{i=1}^{n-1} \bigl|\pi_i(A)\bigr|. \tag{2.22}$$

Proof. Since $A \subset A_1^+$, then $|A_1^+| - |A| = |A_1^+ \setminus A| = \max_{m\in\mathbb{Z}} \bigl|A(m)\bigr|$. So, it suffices to prove that for all $m \in \mathbb{Z}$,

$$\bigl|A(m)\bigr| \leq \prod_{i=1}^{n-1} \bigl|\pi_i(A)\bigr|, \tag{2.23}$$

which follows from the (discrete) Loomis-Whitney inequality: it can be seen by replacing each point in $A(m)$ by a small cube with edges parallel to the coordinate lines $\mathbb{R}e_i$, $i = 1, \dots, n$, that

$$\bigl|A(m)\bigr| \leq \prod_{i=1}^{n-1} \Bigl|\pi_i\bigl(A(m)\bigr)\Bigr|$$

(see e.g. [9, Section 5] and the references within). This shows (2.23). □

In order to establish the first announced upper bound for the cardinality of $\bar{A}$, we also need the following general identity for natural numbers. For the sake of brevity we set the meaningless products to be 1. We use this convention here and throughout the rest of the dissertation.

Lemma 2.4.1 ([12]). *Let $a_1, \ldots, a_n \in \mathbb{N}$. Then*

$$\sum_{k=1}^{n}\left(\prod_{i=1}^{n-k} a_i \prod_{i=n-k+2}^{n}(a_i+1)\right) = \prod_{i=1}^{n}(a_i+1) - \prod_{i=1}^{n} a_i.$$

Proof. A recursive procedure shows that

$$\begin{aligned}\prod_{i=1}^{n} a_i + \sum_{k=1}^{n}\left(\prod_{i=1}^{n-k} a_i \prod_{i=n-k+2}^{n}(a_i+1)\right) &= \prod_{i=1}^{n} a_i + \prod_{i=1}^{n-1} a_i + \sum_{k=2}^{n}\left(\prod_{i=1}^{n-k} a_i \prod_{i=n-k+2}^{n}(a_i+1)\right)\\ &= \prod_{i=1}^{n-1} a_i(a_n+1) + \prod_{i=1}^{n-2} a_i(a_n+1) + \sum_{k=3}^{n}\left(\prod_{i=1}^{n-k} a_i \prod_{i=n-k+2}^{n}(a_i+1)\right)\\ &= \prod_{i=1}^{n-2} a_i \prod_{i=n-1}^{n}(a_i+1) + \sum_{k=3}^{n}\left(\prod_{i=1}^{n-k} a_i \prod_{i=n-k+2}^{n}(a_i+1)\right) = \cdots = \prod_{i=1}^{n}(a_i+1).\end{aligned}$$

□

We are now ready to provide the first upper bound, as well as a lower bound, for the difference $|\bar{A}| - |A|$.

Proposition 2.4.3 ([12]). *Let $n \geq 1$ and let $A \subset \mathbb{Z}^n$ be finite and non-empty. Then*

$$2^n - 1 \leq |\bar{A}| - |A| \leq \prod_{i=1}^{n}\Big(|\pi_i(A)| + 1\Big) - \prod_{i=1}^{n}|\pi_i(A)|. \tag{2.24}$$

Both inequalities are sharp.

Proof. For the lower bound, since A is non-empty, we clearly have $|A_i^+| - |A_{i-1}^+| \geq 2^{i-1}$ for all $i = 1, \ldots, n$, and so $|\bar{A}| - |A| \geq 2^n - 1$. Equality holds if and only if A is singleton.

For the upper bound, if $n = 1$ then $|\bar{A}| = |A| + 1$, and (2.24) trivially holds. Therefore we assume that $n \geq 2$. We observe that, in order to construct $\bar{A}$, we first add the new points corresponding to A_1^+, then the new points of $\sigma_{n-1}\big(A_1^+(m)\big)$ for each $m \in \pi_n(A_1^+)$, and so on. Therefore:

1st step. By (2.22) we add, at most, $\prod_{i=1}^{n-1}|\pi_i(A)|$ points.

2nd step. Using again (2.22) in the corresponding section, we can assure that we add $|\pi_n(A_1^+)| = |\pi_n(A)| + 1$ times (cf. (2.21)), at most, $\prod_{i=1}^{n-2}|\pi_i(A)|$ points.

k-th step. In short, for $k = 1, \ldots, n$ we are adding, at most,

$$\prod_{i=1}^{n-k}|\pi_i(A)| \prod_{i=n-k+2}^{n}\Big(|\pi_i(A)| + 1\Big) \tag{2.25}$$

new points.

Altogether, and using Lemma 2.4.1, we conclude that

$$|\bar{A}| - |A| \leq \sum_{k=1}^{n}\left(\prod_{i=1}^{n-k}|\pi_i(A)| \prod_{i=n-k+2}^{n}\Big(|\pi_i(A)|+1\Big)\right) = \prod_{i=1}^{n}\Big(|\pi_i(A)|+1\Big) - \prod_{i=1}^{n}|\pi_i(A)|.$$

In order to show that the upper bound in (2.24) may be attained it is enough to consider a lattice orthogonal box A (see Figure 2.8). □

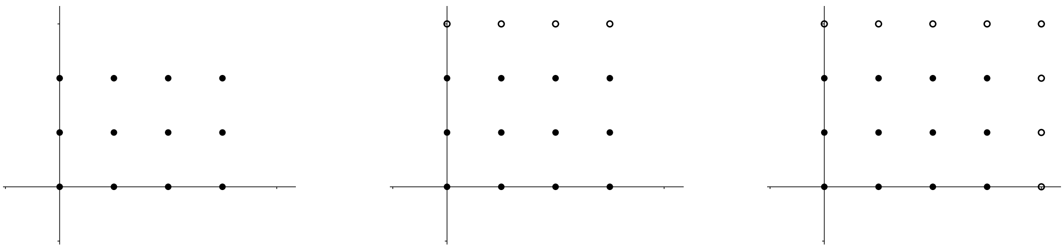

Figure 2.8: The upper bound in (2.24) is sharp: a lattice box A (left), A_1^+ (middle) and $\bar{A}$ (right).

Similar inequalities to those collected in Propositions 2.4.1 and 2.4.3 for the set $\bar{A}$ can be derived for $\mathrm{r}(A)$. This is the content of the following result.

Proposition 2.4.4. *Let $n \geq 1$ and let $A \subset \mathbb{Z}^n$ be finite with $\mathrm{r}(A) \neq \emptyset$. Then*

$$0 < \frac{|\mathrm{r}(A)|}{|A|} < 1 \tag{2.26}$$

and

$$2^n - 1 \leq |A| - |\mathrm{r}(A)| \leq \prod_{i=1}^{n}|\pi_i(A)| - \prod_{i=1}^{n}\big(|\pi_i(A)|-1\big). \tag{2.27}$$

All the inequalities are sharp.

Before showing the result, we would like to point out that, in principle, one cannot directly derive (2.27) from (2.24). Indeed, for an arbitrary set $A \subset \mathbb{Z}^n$ there does not exist in general a set $B \subset \mathbb{Z}^n$ such that $\bar{B} = A$ (we recall that $\mathrm{r}\big(\bar{A}\big) = A$ but, in general, $\overline{\mathrm{r}(A)}$ does not coincide with A). Moreover, even if there exists such a set B for a given one A, from (2.24) we would have

$$|A| - \big|\mathrm{r}(A)\big| = \big|\bar{B}\big| - |B| \leq \prod_{i=1}^{n}\Big(\big|\pi_i(B)\big|+1\Big) - \prod_{i=1}^{n}\big|\pi_i(B)\big|;$$

but the right-hand side of the above inequality does not yield in general the upper bound in (2.27) because $\big|\pi_i(A)\big| = \big|\pi_i(\bar{B})\big|$ may be greater than $\big|\pi_i(B)\big| + 1$ for $1 \leq i \leq n-1$.

We also note that, in the above statement, the assumption $\mathrm{r}(A) \neq \emptyset$ is necessary for the lower bound in (2.27), and also allows us to write a strict inequality in the left-hand side of (2.26).

Proof of Proposition 2.4.4. Both bounds in (2.26) are trivial. In order to show that the lower bound is tight, we consider $A = \{0\} \cup \mathrm{C}_r^n \setminus \mathrm{C}_{r-1}^n$, $r \in \mathbb{N}$. In this case, $\mathrm{r}(A) = \{0\}$ and thus

$$\frac{|\mathrm{r}(A)|}{|A|} = \frac{1}{(r+1)^n - r^n + 1},$$

which tends to 0 when $r \to \infty$. For the upper bound we take $A = \mathrm{C}_r^n$, $r \in \mathbb{N}$, and then

$$\frac{|\mathrm{r}(A)|}{|A|} = \frac{|\mathrm{C}_{r-1}^n|}{|\mathrm{C}_r^n|} = \left(\frac{r}{r+1}\right)^n,$$

which goes to 1 when $r \to \infty$.

Next, to show the lower bound in (2.27), we notice that A_{n-1}^- must contain at least two points to assure that $\mathrm{r}(A) \neq \emptyset$. Then, we are removing at least one point in the n-th step (of the recursive construction of $\mathrm{r}(A)$), which yields that at least four points belong to A_{n-2}^-, and hence we cut down at least two points in the $(n-1)$-th step. Recursively, we get that at least 2^{n-k} points are erased in the k-th step and thus we remove at least $\sum_{k=1}^n 2^{n-k} = 2^n - 1$ points from A to get $\mathrm{r}(A)$. Equality is attained only if $\mathrm{r}(A)$ is a singleton (and $|A| = 2^n$), which holds, for instance, if $A = \mathrm{C}_1^n$.

Finally, we prove the upper bound in (2.27). We observe that, for $1 \leq k \leq n-1$,

$$\left|\pi_{k+1,\dots,n}(A_{k-1}^-)\right| \leq \prod_{i=k+1}^n \left|\pi_i(A)\right|$$

(cf. (2.23)), which can be seen by applying the (discrete) Loomis-Whitney inequality. Therefore:

1st step. We remove 1 point for each $m \in \pi_{2,\dots,n}(A)$, i.e., at most $\prod_{i=2}^n \left|\pi_i(A)\right|$ points.

2nd step. We remove at most $\left|\pi_1(A_1^-)\right| \leq \left|\pi_1(A)\right| - 1$ points for each $m \in \pi_{3,\dots,n}(A_1^-)$, i.e., at most $\left(\left|\pi_1(A)\right| - 1\right) \prod_{i=3}^n \left|\pi_i(A)\right|$ points.

k-th step. In short, for $k = 1, \dots, n$ we are removing, at most,

$$\prod_{i=1}^{k-1} \left(\left|\pi_i(A)\right| - 1\right) \prod_{i=k+1}^n \left|\pi_i(A)\right|$$

points.

Altogether, and using Lemma 2.4.1, we conclude that

$$|A| - \left|\mathrm{r}(A)\right| \leq \sum_{k=1}^n \left(\prod_{i=1}^{k-1} \left(\left|\pi_i(A)\right| - 1\right) \prod_{i=k+1}^n \left|\pi_i(A)\right|\right) = \prod_{i=1}^n \left|\pi_i(A)\right| - \prod_{i=1}^n \left(\left|\pi_i(A)\right| - 1\right).$$

In order to show that the upper bound in (2.27) may be attained it is enough to consider a lattice orthogonal box A. □

We conclude this section by showing other possible upper bounds for the difference $|\bar{A}| - |A|$. To this aim, first we prove the following result.

Proposition 2.4.5. *Let $n \geq 1$ and let $A \subset \mathbb{Z}^n$ be finite and non-empty. Then*

$$|\bar{A}| - |A| \leq \sum_{i=1}^{n} 2^{n-i} \left(|A| - \left|\pi_{n-i+1}(A)\right| + \prod_{j=n-i+2}^{n} \left|\pi_j(A)\right| \right). \tag{2.28}$$

The inequality is sharp.

Proof. If $n = 1$ then $|\bar{A}| = |A| + 1$, and since $|A| = \left|\pi_1(A)\right|$, (2.28) trivially holds. Therefore, we assume that $n \geq 2$.

First we observe that $|A_1^+ \setminus A| = \max_{m\in\pi_n(A)}\left|A(m)\right|$, and clearly,

$$\max_{m\in\pi_n(A)}\left|A(m)\right| \leq |A| - \left|\pi_n(A)\right| + 1.$$

Hence, in the first step we have, at most,

$$z_1(A) := |A| - \left|\pi_n(A)\right| + 1$$

new points. For the successive steps we will use the same kind of argument.

Thus, in the second step, we are adding $\left|\sigma_{n-1}\big(A_1^+(m)\big) \setminus A_1^+(m)\right|$ new points for each integer $m \in \pi_n(A_1^+) = \pi_n(A) \cup \big\{\max \pi_n(A)+1\big\}$, i.e., "in each section orthogonal to $\mathbb{R}e_n$". So we distinguish two cases.

On one hand, if $m \in \pi_n(A)$ then $A_1^+(m) = A(m)$, and we bound the above cardinality by using the previous technique to obtain

$$\left|\sigma_{n-1}\big(A(m)\big) \setminus A(m)\right| = \max_{s\in\pi_{n-1}(A(m))}\left|A(m)(s)\right| \leq \left|A(m)\right| - \left|\pi_{n-1}\big(A(m)\big)\right| + 1. \tag{2.29}$$

Then, since

$$\pi_{n-1}(A) = \bigcup_{m\in\pi_n(A)} \pi_{n-1}\big(A(m)\big),$$

we have

$$\begin{aligned}\sum_{m\in\pi_n(A)} \left|\sigma_{n-1}\big(A(m)\big) \setminus A(m)\right| &\leq \sum_{m\in\pi_n(A)} \Big(\left|A(m)\right| - \left|\pi_{n-1}\big(A(m)\big)\right| + 1\Big) \\ &\leq |A| - \left|\pi_{n-1}(A)\right| + \left|\pi_n(A)\right| =: z_2'(A).\end{aligned}$$

On the other hand, if $m = \max \pi_n(A) + 1$ then $A_1^+(m)$ is the set of points added in the first step, and so

$$\left|\sigma_{n-1}\big(A_1^+(m)\big) \setminus A_1^+(m)\right| \leq z_1(A).$$

Altogether, the total number of points added in the second step can be bounded by

$$z_2(A) := z_1(A) + z_2'(A).$$

Using this argument recursively we observe that in the k-th step we are adding "$(n-k)$-dimensional sections", some of them deriving from the original set A, and some other arising from previously added sections (of higher dimensions). As before, we bound them separately:

(i) In the first case, the cardinality of each new section can be bounded as (cf. (2.29))

$$\max_{s\in\pi_{n-k+1}(A(m_1)\dots(m_{k-1}))}\bigl|A(m_1)\dots(m_{k-1})(s)\bigr| \\ \leq \bigl|A(m_1)\dots(m_{k-1})\bigr| - \Bigl|\pi_{n-k+1}\bigl(A(m_1)\dots(m_{k-1})\bigr)\Bigr| + 1,$$

where $m_1\in\pi_n(A)$, $m_2\in\pi_{n-1}\bigl(A(m_1)\bigr)$ and so on, till

$$m_{k-1}\in\pi_{n-k+2}\bigl(A(m_1)\dots(m_{k-2})\bigr).$$

Thus, summing over these sets (we just write the index in each summand for the sake of brevity) we get the following upper bound for the number of new points arising from A:

$$\sum_{m_1}\sum_{m_2}\cdots\sum_{m_{k-1}}\max_{s\in\pi_{n-k+1}(A(m_1)\dots(m_{k-1}))}\bigl|A(m_1)\dots(m_{k-1})(s)\bigr| \\ \leq |A| - \bigl|\pi_{n-k+1}(A)\bigr| + \prod_{j=n-k+2}^{n}\bigl|\pi_j(A)\bigr| =: z'_k(A).$$

(ii) In the second case, we just may take, as an upper bound,

$$\sum_{j=1}^{k-1} z_j(A).$$

To sum up, (i) and (ii) together give, for the k-th step, $k=1,\dots,n$, a total of at most

$$z_k(A) := z'_k(A) + \sum_{j=1}^{k-1} z_j(A) \tag{2.30}$$

new points. Therefore we can conclude that

$$\bigl|\bar{A}\bigr| \leq |A| + \sum_{k=1}^{n} z_k(A).$$

Finally, from (2.30), we observe that

$$\sum_{k=1}^{n} z_k(A) = \sum_{k=1}^{n}\left(z'_k(A) + \sum_{i=1}^{k-1} z_i(A)\right) = \sum_{k=1}^{n}\left(z'_k(A) + \sum_{i=1}^{k-1} 2^{k-1-i} z'_i(A)\right) = \sum_{i=1}^{n} 2^{n-i} z'_i(A) \\ = \sum_{i=1}^{n} 2^{n-i}\left(|A| - \bigl|\pi_{n-i+1}(A)\bigr| + \prod_{j=n-i+2}^{n}\bigl|\pi_j(A)\bigr|\right).$$

This finishes the proof of (2.28). In order to show that the bound in (2.28) may be attained, we can consider a lattice segment (see Figure 2.9). □

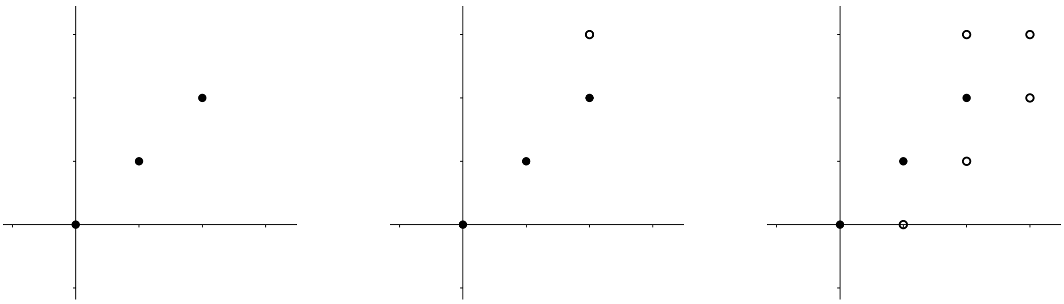

Figure 2.9: Equality holds in (2.28): a lattice segment A (left), A_1^+ (middle) and $\bar{A}$ (right).

As we mentioned before, the upper bounds in (2.24) and (2.28) are not comparable, in the sense that depending on the structure of the set A, each of them can be sharper than the other. For instance, we already know that the lattice box in Figure 2.8 gives equality in (2.24), namely, $|\bar{A}| - |A| = 8$, whereas the bound obtained using (2.28) is 31. On the other hand, for the lattice segment in Figure 2.9 the inequality (2.28) holds with equality, $|\bar{A}| - |A| = 5$; however, if we compute in this case the bound given in (2.24) we get 7.

Although computing (2.24) or (2.28) is not in general an easy task, a better upper bound for $|\bar{A}| - |A|$ is obtained using, for each step in the construction of $\bar{A}$, the minimum of the previous two bounds.

Theorem 2.4.1. *Let $A \subset \mathbb{Z}^n$, $n \geq 1$, be finite and non-empty, and let*

$$\bar{z}_i(A) = \min\left\{\prod_{j=1}^{n-i}|\pi_j(A)| \prod_{j=n-i+2}^{n}\Bigl(|\pi_j(A)| + 1\Bigr), |A| - |\pi_{n-i+1}(A)| + \prod_{j=n-i+2}^{n}|\pi_j(A)| + \sum_{j=1}^{i-1}\bar{z}_j(A)\right\}$$

for $i = 1, \dots, n$. Then

$$|\bar{A}| - |A| \leq \sum_{i=1}^{n}\bar{z}_i(A). \tag{2.31}$$

Proof. The case $n = 1$ again reduces to $|\bar{A}| = |A| + 1$. For $n \geq 2$ the proof is a consequence of (2.25) and (2.30): in the construction of $\bar{A}$, in the k-th step we have two different bounds for the number of new points, namely, (2.25) and (2.30), and hence we can refine both just by taking the minimum between them; but even more, (2.30) can be improved by considering

$$|A| - |\pi_{n-k+1}(A)| + \prod_{j=n-k+2}^{n}|\pi_j(A)| + \sum_{j=1}^{k-1}\bar{z}_j(A).$$

This proves the result. □

The following table shows the number of points which are added in each step in the construction of $\bar{A}$, for $A = \{e_1, 2e_1, 3e_1, e_2, 2e_2\}$ (see Figure 2.10), as well as the bounds provided by (2.24), (2.28) and (2.31). In this example equality holds in (2.31), whereas we have strict inequalities in (2.24) and (2.28).

Step	Added points	Bounds in (2.24)	Bounds in (2.28)	Bounds in (2.31)
1	3	4	3	3
2	4	4	7	4
Total	12	13	15	12

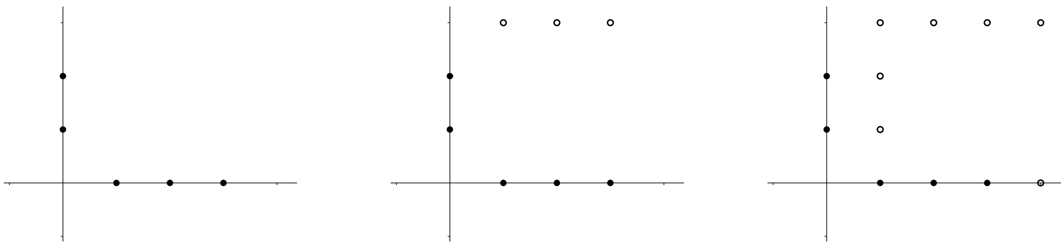

Figure 2.10: Comparing the bounds in (2.24), (2.28) and (2.31).

Chapter 3

Discrete Borell-Brascamp-Lieb type inequalities

Regarding an analytical counterpart for functions of the Brunn-Minkowski inequality, one is naturally led to the so-called *Borell-Brascamp-Lieb inequality*, originally proved in [2] and [4]. This inequality can be seen, not only as an analytical extension of the Brunn-Minkowski inequality, but also as a way to obtain different Brunn-Minkowski type inequalities depending on a parameter p. This powerful result will be the starting point in this chapter, being our main aim to present new discrete Borell-Brascamp-Lieb type inequalities.

The original work that we collect here can be found in [14, 15].

3.1 The classical Borell-Brascamp-Lieb inequality

In order to introduce the Borell-Brascamp-Lieb inequality, we first recall the definition of p-sum of two non-negative numbers, where $p \neq 0$ is a parameter varying in $\mathbb{R} \cup \{\pm\infty\}$ (for a general reference for p-sums of non-negative numbers, we refer the reader to the classic text of Hardy, Littlewood, and Pólya [11] and to the handbook [5]).

Definition 3.1. *Let $a, b \geq 0$ and let $p \in \mathbb{R} \cup \{\pm\infty\}$, $p \neq 0$. The* p-sum $\mathcal{S}_p(a, b)$ *of a, b is defined as*

$$\mathcal{S}_p(a,b) = \begin{cases} (a^p + b^p)^{1/p} & \text{if } p \neq \pm\infty, \\ \max\{a,b\} & \text{if } p = \infty, \\ \min\{a,b\} & \text{if } p = -\infty, \end{cases}$$

for $ab > 0$, and $\mathcal{S}_p(a, b) = 0$ when $ab = 0$ for all $p \in \mathbb{R} \cup \{\pm\infty\}$, $p \neq 0$.

Note that $\mathcal{S}_p(a,b)=0$, if $ab=0$, is redundant for all $p<0$, however it is relevant for $p>0$. The reason to modify in this way the definition of p-sum given in [11] is due to the classical statement of the Borell-Brascamp-Lieb inequality, which is collected below. In fact, without such a modification, the thesis of the latter result would not have mathematical interest.

Similarly we can define the *p-mean* of two non-negative numbers, which will be also needed it in following sections.

Definition 3.2. *Let $a,b\geq 0$, $\lambda\in[0,1]$ and $p\in\mathbb{R}\cup\{\pm\infty\}$. We define the* p-mean $\mathcal{M}_p^\lambda(a,b)$ *of a,b with respect to λ as*

$$\mathcal{M}_p^\lambda(a,b)=\begin{cases}\big((1-\lambda)a^p+\lambda b^p\big)^{1/p}, & \text{if } p\neq 0,\pm\infty,\\ a^{1-\lambda}b^\lambda & \text{if } p=0,\\ \max\{a,b\} & \text{if } p=\infty,\\ \min\{a,b\} & \text{if } p=-\infty,\end{cases}$$

for $ab>0$, with $\mathcal{M}_p^\lambda(a,b)=0$ when $ab=0$ and $p\in\mathbb{R}\cup\{\pm\infty\}$.

The following theorem (see also [7] for a detailed presentation), as previously stated, can be regarded as the functional counterpart of the Brunn-Minkowski inequality. In fact, a straightforward proof of (1.2) can be obtained by applying (3.1) to the characteristic functions $f=\chi_K$, $g=\chi_L$ and $h=\chi_{K+L}$ of compact sets K,L of positive measure, with $p=\infty$.

Theorem 3.1.1 (The Borell-Brascamp-Lieb inequality (p-sums version)). *Let $p\geq -1/n$, $p\neq 0$, and let $f,g,h:\mathbb{R}^n\longrightarrow\mathbb{R}_{\geq 0}$ be non-negative measurable functions such that*

$$h(x+y)\geq\mathcal{S}_p\big(f(x),g(y)\big)$$

for all $x,y\in\mathbb{R}^n$. Then

$$\int_{\mathbb{R}^n}h(x)\,\mathrm{d}x\geq\mathcal{S}_{\frac{p}{np+1}}\left(\int_{\mathbb{R}^n}f(x)\,\mathrm{d}x,\int_{\mathbb{R}^n}g(x)\,\mathrm{d}x\right). \tag{3.1}$$

The Borell-Brascamp-Lieb inequality has an equivalent version in terms of the p-means:

Theorem 3.1.2 (The Borell-Brascamp-Lieb inequality (p-means version)). *Let $p\geq -1/n$ and $\lambda\in[0,1]$, and let $f,g,h:\mathbb{R}^n\longrightarrow\mathbb{R}_{\geq 0}$ be non-negative measurable functions such that*

$$h\big((1-\lambda)x+\lambda y\big)\geq\mathcal{M}_p^\lambda\big(f(x),g(y)\big)$$

for all $x,y\in\mathbb{R}^n$. Then

$$\int_{\mathbb{R}^n}h(x)\,\mathrm{d}x\geq\mathcal{M}^\lambda_{\frac{p}{np+1}}\left(\int_{\mathbb{R}^n}f(x)\,\mathrm{d}x,\int_{\mathbb{R}^n}g(x)\,\mathrm{d}x\right). \tag{3.2}$$

Using the p-means version of the Borell-Brascamp-Lieb inequality applied to the characteristic functions $f = \chi_{_K}$, $g = \chi_{_L}$ and $h = \chi_{_{(1-\lambda)K+\lambda L}}$ of compact sets K, L of positive measure, the most general Brunn-Minkowski inequality

$$\mathrm{vol}\big((1-\lambda)K + \lambda L\big) \geq \big((1-\lambda)\mathrm{vol}(K)^q + \lambda \mathrm{vol}(L)^q\big)^{1/q}$$

can be obtained for $q = p/(np+1) \leq 1/n$. In the particular cases $p = \infty$, $p = 0$ and $p = -1/n$ we obtain the additive form, multiplicative form and minimal form of the Brunn-Minkowski inequality, respectively, that we gathered in (1.3).

The main aim of this chapter will be to present new discrete versions of (3.1) and (3.2). We note however that, since it is not possible to get a Brunn-Minkowski inequality in its classical form for the cardinality (see the example in page 5), one cannot expect to obtain a discrete analogue of the Borell-Brascamp-Lieb inequality just by replacing integrals by sums. Therefore, discrete counterparts for the Borell-Brascamp-Lieb inequality should again have a different structure or involve modifications of the sets.

Moreover, in contrast to the continuous setting, where one may directly obtain (3.2) from (3.1) (and viceversa) because of the homogeneity of the volume, one cannot expect to derive in a similar way a discrete version of (3.2) from that of (3.1). The issue relies on the lack of homogeneity of our ways of measuring in the discrete setting: the cardinality and the lattice point enumerator.

3.2 A first discrete Borell-Brascamp-Lieb type inequality

Our first approach in order to get a discrete Borell-Brascamp-Lieb type inequality will be to discretize Theorem 3.1.1 in the spirit of Theorem 2.2.2. To this end, we generalize one of the constructions shown in Section 2.1: by removing points from our original finite set $A \subset \mathbb{Z}^n$, $A \neq \emptyset$, we define a new reduced set $\mathrm{r}_f(A)$ according to a particular function f.

Now, if $\Lambda \subset \mathbb{Z}^k$ is finite, $k \in \{1, \ldots, n\}$, and $f : \Lambda \longrightarrow \mathbb{R}_{\geq 0}$ is a non-negative function (which will be often referred to as a *weight function*), let $m_0 = m_0(\Lambda, f) \in \pi_k(\Lambda)$ be such that

$$\sum_{x \in A(m_0)} f(x, m_0) = \max_m \sum_{x \in \Lambda(m)} f(x, m)$$

(cf. (2.2)). Certainly, the integer m_0 providing the "maximum section" with respect to the weight function f does not necessarily have to be unique. In that case, we define

$$m_0 = \max\left\{ m' \in \pi_k(\Lambda) : \sum_{x \in \Lambda(m')} f(x, m') = \max_m \sum_{x \in \Lambda(m)} f(x, m) \right\}. \tag{3.3}$$

Then, the function ϱ_k defined in (2.5) acts on Λ by removing the "maximum section" $\Lambda(m_0)$, but now with respect to the weight function f, from the set. Then, for our set $A \subset \mathbb{Z}^n$, we define

$$\mathrm{r}_f(A) = \varrho_n\big(A^-_{n-1}\big).$$

In other words, $\mathrm{r}_f(A)$ is given by

$$\mathrm{r}_f(A) = \bigcup_{m\in\pi_n(A)\setminus\{m_0(A_{n-1}^-,f)\}} \Big(\mathrm{r}_f\big(A(m)\big) \times \{m\}\Big). \tag{3.4}$$

We observe that the reduced set $\mathrm{r}(A)$ constructed in Section 2.1 is just $\mathrm{r}_\varphi(A)$ when $\varphi : \mathbb{Z}^n \longrightarrow \mathbb{R}_{\geq 0}$ is the constant weight function given by $\varphi(x) = 1$ for all $x \in \mathbb{Z}^n$.

Using this construction we can state our main result in this section:

Theorem 3.2.1 ([14]). *Let $A, B \subset \mathbb{Z}^n$ be non-empty finite sets. Let $-1/n \leq p \leq \infty$, $p \neq 0$, and let $f, g, h : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ be non-negative functions such that*

$$h(x + y) \geq \mathcal{S}_p\big(f(x), g(y)\big)$$

for all $x \in A$, $y \in B$. Then

$$\sum_{z\in A+B} h(z) \geq \mathcal{S}_{\frac{p}{np+1}}\left(\sum_{x\in \mathrm{r}_f(A)} f(x), \sum_{y\in B} g(y)\right). \tag{3.5}$$

As in the continuous setting, inequality (3.5) can be seen as a functional extension of the discrete Brunn-Minkowski inequality (2.8), just by considering the characteristic functions $f = \chi_A$, $g = \chi_B$ and $h = \chi_{A+B}$, and taking $p = \infty$. We also note that, as in the case of Theorem 2.2.2, the above result holds for finite subsets $A, B \subset \mathbb{R}^n$, via a suitable construction of the set $\mathrm{r}_f(A)$. We state and prove Theorem 3.2.1 in the case of $\mathbb{Z}^n$ for the sake of simplicity.

Before starting with the proof of Theorem 3.2.1, we observe that as a straightforward consequence of it we get the following Brunn-Minkowski type inequality for discrete measures associated to *p-additive functions*, in the spirit of (2.8). Indeed, it is enough to apply Theorem 3.2.1 to the functions $f = \chi_A\phi$, $g = \chi_B\phi$ and $h = \chi_{A+B}\phi$.

Corollary 3.2.1 ([14]). *Let $-1/n \leq p \leq \infty$, $p \neq 0$, and let $\phi : \mathbb{Z}^n \longrightarrow \mathbb{R}_{\geq 0}$ be a non-negative function such that*

$$\phi(x + y) \geq \mathcal{S}_p\big(\phi(x), \phi(y)\big)$$

for any $x, y \in \mathbb{Z}^n$. Let μ be the discrete measure on $\mathbb{Z}^n$ with mass function ϕ, i.e., such that

$$\mu(M) = \sum_{x\in M} \phi(x)$$

for any finite set $M \subset \mathbb{Z}^n$, and let $A, B \subset \mathbb{Z}^n$ be finite. Then

$$\mu(A + B) \geq \mathcal{S}_{\frac{p}{np+1}}\Big(\mu\big(\mathrm{r}_\phi(A)\big), \mu(B)\Big).$$

3.2.1 The proof of the discrete Borell-Brascamp-Lieb inequality

Before the proof of Theorem 3.2.1 we need to state some auxiliary results. The following lemma can be regarded as a discrete analog of the well-known *Cavalieri Principle*.

Lemma 3.2.1 ([14]). *Let $\Omega \subset \mathbb{Z}^n$ be finite, let $f : \Omega \longrightarrow \mathbb{R}_{\geq 0}$ and set $f(\Omega) \subset \{k_0, k_1, \dots, k_r\}$ where $0 = k_0 < k_1 < \cdots < k_r$. Then*

$$\sum_{x\in\Omega} f(x) = \sum_{i=1}^{r} (k_i - k_{i-1}) \Big| \big\{x \in \Omega : f(x) \geq k_i\big\} \Big| = \int_0^{\infty} \Big| \big\{x \in \Omega : f(x) \geq t\big\} \Big| \, \mathrm{d}t.$$

Proof. The second equality is immediate and, hence, we will show the first one. To this aim, let $x \in \Omega$ and consider $k_s = f(x)$ for some $s \in \{1, \dots, r\}$ (we may assume, without loss of generality, that $f(x) > 0$). Then, with

$$\delta_i(x) = \begin{cases} 1 & \text{if } f(x) \geq k_i, \\ 0 & \text{otherwise,} \end{cases}$$

we have

$$f(x) = \sum_{i=1}^{s} (k_i - k_{i-1}) = \sum_{i=1}^{r} (k_i - k_{i-1}) \delta_i(x),$$

and thus we can conclude that

$$\begin{aligned} \sum_{x\in\Omega} f(x) &= \sum_{x\in\Omega} \sum_{i=1}^{r} (k_i - k_{i-1}) \delta_i(x) = \sum_{i=1}^{r} (k_i - k_{i-1}) \sum_{x\in\Omega} \delta_i(x) \\ &= \sum_{i=1}^{r} (k_i - k_{i-1}) \Big| \big\{x \in \Omega : f(x) \geq k_i\big\} \Big|. \end{aligned}$$

□

Remark 3.1. *We note that, under the conditions of the above result, on one hand we may ensure that for any $k' \in (k_{i-1}, k_i)$, since $k' \notin f(\Omega)$,*

$$(k_i - k_{i-1}) \Big| \big\{x \in \Omega : f(x) \geq k_i\big\} \Big| = (k_i - k') \Big| \big\{x \in \Omega : f(x) \geq k_i\big\} \Big| + (k' - k_{i-1}) \Big| \big\{x \in \Omega : f(x) \geq k'\big\} \Big|.$$

On the other hand, for every $k' > k_m = \max_{x\in\Omega} f(x)$ we clearly have $\Big| \big\{x \in \Omega : f(x) \geq k'\big\} \Big| = 0$. Therefore, the set $\{k_0, k_1, \dots, k_r\}$ is not relevant.

The following result essentially yields the case $n = 1$ of Theorem 3.2.1 and will be used to derive (3.5). Note, however, that it holds not only for 1-dimensional sets but also for n-dimensional sets, in contrast to the case $n = 1$ of the classical Borell-Brascamp-Lieb inequality.

Lemma 3.2.2 ([14]). *Let $\Omega_1, \Omega_2 \subset \mathbb{Z}^n$ be two finite sets. Let $-1 \leq p \leq \infty$, $p \neq 0$, and let $f, g, h : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ be non-negative functions such that*

$$h(x + y) \geq \mathcal{S}_p\big(f(x), g(y)\big)$$

for all $x \in \Omega_1$, $y \in \Omega_2$. *Then*

$$\sum_{z \in \Omega_1 + \Omega_2} h(z) \geq \mathcal{S}_{\frac{p}{p+1}} \left(\sum_{x \in \Omega_1 \setminus \{x_0\}} f(x), \sum_{y \in \Omega_2} g(y) \right),$$

where $x_0 \in \Omega_1$ *is such that* $f(x_0) = \max_{x \in \Omega_1} f(x)$.

Proof. Clearly, we may assume that both $\sum_{x \in \Omega_1 \setminus \{x_0\}} f(x), \sum_{y \in \Omega_2} g(y) > 0$. We consider the non-negative functions $F, G, H : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ given by

$$F(x) = \frac{f(x)}{a}, \qquad G(y) = \frac{g(y)}{b}, \qquad H(z) = \frac{h(z)}{c_p},$$

where

$$a = \max_{x \in \Omega_1} f(x) > 0, \qquad b = \max_{y \in \Omega_2} g(y) > 0 \qquad \text{and} \qquad c_p = \mathcal{S}_p(a, b) > 0.$$

Then

$$\max_{x \in \Omega_1} F(x) = \max_{y \in \Omega_2} G(y) = 1.$$

First, we show that, for any $x \in \Omega_1$, $y \in \Omega_2$, we have

$$H(x + y) \geq \min\bigl\{F(x), G(y)\bigr\}. \tag{3.6}$$

To this aim, it is enough to consider $x \in \Omega_1$ and $y \in \Omega_2$ with $f(x)g(y) > 0$. For $p \neq \infty$, and writing $\theta = b^p / c_p^p \in (0, 1)$, we get

$$\begin{aligned} h(x + y) &\geq \bigl(f(x)^p + g(y)^p\bigr)^{1/p} = c_p \left(\frac{F(x)^p a^p + G(y)^p b^p}{c_p^p} \right)^{1/p} \\ &= c_p\bigl((1 - \theta)F(x)^p + \theta G(y)^p\bigr)^{1/p} \geq c_p \min\{F(x), G(y)\}. \end{aligned}$$

For $p = \infty$, $h(x + y) \geq \max\bigl\{f(x), g(y)\bigr\} \geq c_\infty \min\bigl\{F(x), G(y)\bigr\}$ clearly holds. Therefore, we have shown (3.6).

The definition of F and G now implies that the level sets

$$\bigl\{x \in \Omega_1 : F(x) \geq t\bigr\}, \qquad \bigl\{y \in \Omega_2 : G(y) \geq t\bigr\}$$

are non-empty for any $t \in [0, 1]$. Moreover, from (3.6) we deduce that

$$\bigl\{z \in \Omega_1 + \Omega_2 : H(z) \geq t\bigr\} \supset \bigl\{x \in \Omega_1 : F(x) \geq t\bigr\} + \bigl\{y \in \Omega_2 : G(y) \geq t\bigr\}$$

and thus, by (1.5) together with the fact that $f(x_0) = \max_{x \in \Omega_1} f(x)$, we have

$$\begin{aligned} \Bigl|\bigl\{z \in \Omega_1 + \Omega_2 : H(z) \geq t\bigr\}\Bigr| &\geq \Bigl|\bigl\{x \in \Omega_1 : F(x) \geq t\bigr\}\Bigr| + \Bigl|\bigl\{y \in \Omega_2 : G(y) \geq t\bigr\}\Bigr| - 1 \\ &= \Bigl|\bigl\{x \in \Omega_1 \setminus \{x_0\} : F(x) \geq t\bigr\}\Bigr| + \Bigl|\bigl\{y \in \Omega_2 : G(y) \geq t\bigr\}\Bigr| \end{aligned}$$

for all $t \in [0, 1]$.

Finally, set

$$\{k_0, k_1, \dots, k_r\} \supset F\big(\Omega_1\big) \cup G\big(\Omega_2\big) \cup H\big(\Omega_1 + \Omega_2\big),$$

with $0 = k_0 < k_1 < \cdots < k_r$ where, for some $s \in \{1, \dots, r\}$,

$$k_s = \max_{y \in \Omega_2} G(y) = 1 \geq \max_{x \in \Omega_1 \setminus \{x_0\}} F(x).$$

Then, by the above inequality, and using Lemma 3.2.1, we get

$$\begin{aligned}
\sum_{z \in \Omega_1 + \Omega_2} h(z) &= \sum_{z \in \Omega_1 + \Omega_2} c_p\, H(z) = c_p \sum_{i=1}^{r} (k_i - k_{i-1}) \Big| \big\{ z \in \Omega_1 + \Omega_2 : H(z) \geq k_i \big\} \Big| \\
&\geq c_p \sum_{i=1}^{s} (k_i - k_{i-1}) \Big| \big\{ z \in \Omega_1 + \Omega_2 : H(z) \geq k_i \big\} \Big| \\
&\geq c_p \sum_{i=1}^{s} (k_i - k_{i-1}) \Big(\Big| \big\{ x \in \Omega_1 \setminus \{x_0\} : F(x) \geq k_i \big\} \Big| + \Big| \big\{ y \in \Omega_2 : G(y) \geq k_i \big\} \Big| \Big) \\
&= c_p \left(\sum_{x \in \Omega_1 \setminus \{x_0\}} F(x) + \sum_{y \in \Omega_2} G(y) \right) = c_p \left(\frac{1}{a} \sum_{x \in \Omega_1 \setminus \{x_0\}} f(x) + \frac{1}{b} \sum_{y \in \Omega_2} g(y) \right) \\
&\geq \mathcal{S}_{\frac{p}{p+1}} \left(\sum_{x \in \Omega_1 \setminus \{x_0\}} f(x), \sum_{y \in \Omega_2} g(y) \right).
\end{aligned}$$

The last inequality follows from the reverse Hölder inequality (see Corollary 1.1.1) with parameter $p/(p+1) \leq 1$, just by taking $a_1 = \sum_{x \in \Omega_1 \setminus \{x_0\}} f(x)$, $a_2 = \sum_{y \in \Omega_2} g(y)$, $b_1 = 1/a$ and $b_2 = 1/b$. $\square$

Now we are in a position to prove Theorem 3.2.1. The main idea of the proof we present here is exploiting the above result (for $n = 1$) via an inductive procedure, and it goes back to the classical proof of the Borell-Brascamp-Lieb inequality (see e.g. [2, 4, 16, 22]). We develop it here in detail for the sake of completeness.

Proof of Theorem 3.2.1. We may assume, without loss of generality, that both $\sum_{x \in \mathrm{r}_f(A)} f(x) > 0$ and $\sum_{y \in B} g(y) > 0$.

If $n = 1$, the result follows immediately from Lemma 3.2.2 with $\Omega_1 = A$, $\Omega_2 = B$ and noticing that $\mathrm{r}_f(A) = A \setminus \{m_0\}$; recall that $f(m_0) = \max_{m \in A} f(m)$, cf. (3.3).

Now we suppose that $n > 1$ and that the theorem holds for dimension $n - 1$. Let $m_A \in \pi_n(A)$ and $m_B \in \pi_n(B)$. Let $\Omega_1 = A(m_A) \subset \mathbb{Z}^{n-1}$ and $\Omega_2 = B(m_B) \subset \mathbb{Z}^{n-1}$, and consider the functions $f_1, g_1, h_1 : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}_{\geq 0}$ given by

$$f_1(x) = f(x, m_A), \quad g_1(x) = g(x, m_B), \quad h_1(x) = h(x, m_A + m_B),$$

for any $x \in \mathbb{R}^{n-1}$. Since for all $x \in \Omega_1$, $y \in \Omega_2$ we have

$$h_1(x + y) = h(x + y, m_A + m_B) \geq \mathcal{S}_p\left(f(x, m_A), g(y, m_B)\right) = \mathcal{S}_p\left(f_1(x), g_1(y)\right),$$

we may assert that

$$\sum_{z\in A(m_A)+B(m_B)} h_1(z) \geq \mathcal{S}_{\frac{p}{(n-1)p+1}}\left(\sum_{x\in \mathrm{r}_f(A(m_A))} f_1(x), \sum_{y\in B(m_B)} g_1(y)\right).$$

This, together with the fact that

$$(A+B)(m_A+m_B) \supset A(m_A)+B(m_B),$$

yields, in terms of f, g and h,

$$\sum_{z\in(A+B)(m_A+m_B)} h(z,m_A+m_B) \geq \mathcal{S}_{\frac{p}{(n-1)p+1}}\left(\sum_{x\in \mathrm{r}_f(A(m_A))} f(x,m_A), \sum_{y\in B(m_B)} g(y,m_B)\right). \quad (3.7)$$

Now, let $f_2, g_2, h_2 : \mathbb{Z} \longrightarrow \mathbb{R}_{\geq 0}$ be the functions defined by

$$f_2(m) = \sum_{x\in \mathrm{r}_f(A(m))} f(x,m), \qquad g_2(m) = \sum_{y\in B(m)} g(y,m) \quad \text{and}$$
$$h_2(m) = \sum_{z\in(A+B)(m)} h(z,m).$$

Let $m_0 = m_0(A^-_{n-1}, f) \in \pi_n(A)$ be the value for which

$$\mathrm{r}_f(A) = \bigcup_{m\in\pi_n(A)\setminus\{m_0\}} \Big(\mathrm{r}_f\big(A(m)\big) \times \{m\}\Big)$$

holds (see (3.4)). Then we clearly have $f_2(m_0) = \max_{m\in\pi_n(A)} f_2(m)$. Hence, (3.7) yields, in terms of f_2, g_2 and h_2,

$$h_2(m_A+m_B) \geq \mathcal{S}_{\frac{p}{(n-1)p+1}}\big(f_2(m_A), g_2(m_B)\big)$$

for any $m_A \in \pi_n(A)$, $m_B \in \pi_n(B)$, and thus we may use Lemma 3.2.2 with $\Omega_1 = \pi_n(A)$, $\Omega_2 = \pi_n(B)$ and the functions f_2, g_2 and h_2 to obtain

$$\sum_{m\in\pi_n(A)+\pi_n(B)} h_2(m) \geq \mathcal{S}_{\frac{p}{np+1}}\left(\sum_{m_A\in\pi_n(A)\setminus\{m_0\}} f_2(m_A), \sum_{m_B\in\pi_n(B)} g_2(m_B)\right).$$

This, together with the relations

$$\sum_{m\in\pi_n(A)+\pi_n(B)} h_2(m) = \sum_{z\in A+B} h(z), \qquad \sum_{m_A\in\pi_n(A)\setminus\{m_0\}} f_2(m_A) = \sum_{x\in \mathrm{r}_f(A)} f(x),$$
$$\sum_{m_B\in\pi_n(B)} g_2(m_B) = \sum_{y\in B} g(y),$$

finishes the proof. □

3.2.2 From the discrete version to the continuous one

In order to conclude this section, we also show that the classical Borell-Brascamp-Lieb inequality (3.1) can be obtained from the discrete version (3.5) under the mild (but necessary) assumption that the functions f, g are Riemann integrable.

Theorem 3.2.2 ([14]). *The discrete Borell-Brascamp-Lieb inequality* (3.5) *implies the classical Borell-Brascamp-Lieb inequality* (3.1)*, provided that the functions f, g are Riemann integrable.*

Proof. We assume that $f, g, h : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ are non-negative measurable functions, with f, g Riemann integrable, such that

$$h(x+y) \geq \mathcal{S}_p\left(f(x), g(y)\right)$$

for all $x, y \in \mathbb{R}^n$. Let $m \in \mathbb{N}$ and let $K = (-m, m)^n$. For each $k \in \mathbb{N}$, let K_k be the k-discretization of K (see Definition 2.1). Since K is bounded, we know that K_k is a finite set.

We define the functions $f_k, g_k : K_k \longrightarrow \mathbb{R}_{\geq 0}$ given by

$$f_k(x) = \inf_{z \in x + [0, 2^{-k}]^n} f(z), \qquad g_k(x) = \inf_{z \in x + [0, 2^{-k}]^n} g(z),$$

and let $h_k : K_k + K_k \longrightarrow \mathbb{R}_{\geq 0}$ be the function defined by

$$h_k(x) = \inf_{z \in x + [0, 2^{-k}]^n} h(z).$$

We note that, for any $x, y \in K_k$, we have

$$\begin{aligned} h_k(x+y) &= \inf_{z \in x+y+[0,2^{-k}]^n} h(z) \geq \inf_{z \in x+[0,2^{-k}]^n + y + [0,2^{-k}]^n} h(z) = \inf_{\substack{z_1 \in x+[0,2^{-k}]^n \\ z_2 \in y+[0,2^{-k}]^n}} h(z_1+z_2) \\ &\geq \inf_{\substack{z_1 \in x+[0,2^{-k}]^n \\ z_2 \in y+[0,2^{-k}]^n}} \mathcal{S}_p\left(f(z_1), g(z_2)\right) \geq \mathcal{S}_p\left(\inf_{z_1 \in x+[0,2^{-k}]^n} f(z_1), \inf_{z_2 \in y+[0,2^{-k}]^n} g(z_2)\right) \\ &= \mathcal{S}_p\left(f_k(x), g_k(y)\right), \end{aligned}$$

and thus, since K_k is a finite set, we can use Theorem 3.2.1 to deduce that, for any $k \in \mathbb{N}$, we have

$$2^{-kn} \sum_{z \in K_k + K_k} h_k(z) \geq \mathcal{S}_{\frac{p}{np+1}}\left(2^{-kn} \sum_{x \in \mathrm{r}_f(K_k)} f_k(x), 2^{-kn} \sum_{y \in K_k} g_k(y)\right). \tag{3.8}$$

First, we clearly have

$$\int_{\mathbb{R}^n} h(x)\,\mathrm{d}x \geq \sum_{z \in K_k + K_k} 2^{-kn} h_k(z). \tag{3.9}$$

Now, using (3.9), (3.8) and taking into account that g is a Riemann integrable function, we immediately get

$$\begin{aligned}\int_{\mathbb{R}^n} h(x)\,\mathrm{d}x &\geq \lim_{k\to\infty} \mathcal{S}_{\frac{p}{np+1}}\left(2^{-kn}\sum_{x\in \mathrm{r}_f(K_k)} f_k(x), 2^{-kn}\sum_{y\in K_k} g_k(y)\right)\\ &\geq \mathcal{S}_{\frac{p}{np+1}}\left(\lim_{k\to\infty} 2^{-kn}\sum_{x\in \mathrm{r}_f(K_k)} f_k(x), \lim_{k\to\infty} 2^{-kn}\sum_{y\in K_k} g_k(y)\right) \qquad (3.10)\\ &= \mathcal{S}_{\frac{p}{np+1}}\left(\lim_{k\to\infty} 2^{-kn}\sum_{x\in \mathrm{r}_f(K_k)} f_k(x), \int_K g(x)\,\mathrm{d}x\right),\end{aligned}$$

because $2^{-kn}\sum_{y\in K_k} g_k(y)$ is a *lower sum* of g for the partition $\left\{x+\left[0,2^{-k}\right]^n : x\in K_k\right\}$ of $\mathrm{cl}\,K$.

In the following, we show that

$$\lim_{k\to\infty} 2^{-kn}\sum_{x\in \mathrm{r}_f(K_k)} f_k(x) = \int_K f(x)\,\mathrm{d}x. \qquad (3.11)$$

Since the function f is Riemann integrable and non-negative, it is bounded, and then there exists a constant $c\in\mathbb{R}_{\geq 0}$ such that $f(x)\leq c$ for all $x\in K$, which implies that $f_k(x)\leq c$ for any $x\in K_k$. For the sake of brevity let $K_{k,i}=(K_k)^-_{i-1}\setminus (K_k)^-_i$, $i=1,\dots,n$, i.e., the set of all points removed from K_k in the i-th step of the construction of $\mathrm{r}_f(K_k)$. Then it is clear that

$$\left|K_{k,i}\right| = \left|\pi_{(i)}(K_{k,i})\right| \leq \left|\pi_{(i)}(K_k)\right|,$$

and moreover, $\left(\pi_{(i)}(K)\right)_k = \pi_{(i)}(K_k)$. So we have

$$0=\int_{\pi_{(i)}(K)} c\,\mathrm{d}x = \lim_{k\to\infty} 2^{-kn}\sum_{x\in\left(\pi_{(i)}(K)\right)_k} c \geq \lim_{k\to\infty} 2^{-kn}\sum_{x\in K_{k,i}} c,$$

which implies that

$$\lim_{k\to\infty} 2^{-kn}\sum_{x\in K_{k,i}} f_k(x) = 0.$$

This shows that

$$\begin{aligned}\lim_{k\to\infty} 2^{-kn}\sum_{x\in \mathrm{r}_f(K_k)} f_k(x) &= \lim_{k\to\infty}\left(2^{-kn}\sum_{x\in K_k} f_k(x) - 2^{-kn}\sum_{x\in K_k\setminus \mathrm{r}_f(K_k)} f_k(x)\right)\\ &= \lim_{k\to\infty}\left(2^{-kn}\sum_{x\in K_k} f_k(x) - 2^{-kn}\sum_{i=1}^n\sum_{x\in K_{k,i}} f_k(x)\right)\\ &= \lim_{k\to\infty} 2^{-kn}\sum_{x\in K_k} f_k(x) - \sum_{i=1}^n \lim_{k\to\infty} 2^{-kn}\sum_{x\in K_{k,i}} f_k(x)\\ &= \int_K f(x)\,\mathrm{d}x.\end{aligned}$$

This proves (3.11) and then, by (3.10),

$$\int_{\mathbb{R}^n} h(x)\,\mathrm{d}x \geq \mathcal{S}_{\frac{p}{np+1}}\left(\int_K f(x)\,\mathrm{d}x, \int_K g(x)\,\mathrm{d}x\right).$$

Since this is true for $K=(-m,m)^n$, for every $m \in \mathbb{N}$, the proof is now concluded because

$$\lim_{m\to\infty}\int_{(-m,m)^n} \phi(x)\,\mathrm{d}x = \int_{\mathbb{R}^n}\phi(x)\,\mathrm{d}x$$

for every non-negative measurable function $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$. □

Remark 3.2. *Note that the assumption that the functions f and g are Riemann integrable in Theorem 3.2.2 seems to be necessary. Indeed, in order to derive the classical Borell-Brascamp-Lieb inequality* (3.1) *from the discrete version* (3.5)*, one needs to consider some functions to which one may apply* (3.5)*, and then the corresponding finite sums should approximate in some sense the integrals of f and g. The point is that, whereas these sums may be seen as Riemann sums over uniform partitions, there seems to be no natural way to involve integrals of arbitrary (measurable) simple functions.*

3.3 A different approach to a discrete Borell-Brascamp-Lieb type inequality

In this section we deal with convex combinations of two sets, $(1-\lambda)K+\lambda L$, $\lambda \in (0,1)$, instead of just the addition $K+L$. In this regard, we already know that the classical Brunn-Minkowski inequality (1.2) has an equivalent version in this form, namely,

$$\mathrm{vol}\big((1-\lambda)K+\lambda L\big)^{1/n} \geq (1-\lambda)\mathrm{vol}(K)^{1/n} + \lambda \mathrm{vol}(L)^{1/n}, \qquad \lambda \in (0,1), \tag{3.12}$$

for $K, L \subset \mathbb{R}^n$ compact sets (see (1.3)).

Aiming to get a discrete version of (3.12) it is worth noting that, in contrast to (1.2), this equivalent version (3.12) admits a straightforward discrete counterpart, which is due, in a sense, to the lack of homogeneity of the cardinality function: indeed, for any pair of non-empty finite sets $A, B \subset \mathbb{R}^n$, using (1.5) (and since $|A|, |B| \geq 1$), we get

$$\begin{aligned}\big|(1-\lambda)A+\lambda B\big| &\geq \big|(1-\lambda)A\big| + |\lambda B| - 1 = |A|+|B|-1\\ &= (1-\lambda)|A| + \lambda|B| + \lambda|A| + (1-\lambda)|B| - 1\\ &\geq (1-\lambda)|A| + \lambda|B| \geq \Big((1-\lambda)|A|^{1/n} + \lambda|B|^{1/n}\Big)^n,\end{aligned}$$

where the last inequality follows from the convexity of the function $t \mapsto t^n$ for $t \geq 0$. Nevertheless this inequality is meaningless from a geometric point of view: the point is that while the quantities

$|A|, |B|$ on the right-hand side are reduced by the factors $(1-\lambda)$ and λ, the sets $(1-\lambda)A$ and λB on the left-hand side have the same cardinality as A and B, respectively.

A possible solution would be to involve a more natural way to "count points" according to dilatations, namely, using the lattice point enumerator G_n (for compact subsets of $\mathbb{R}^n$) instead of the cardinality $|\cdot|$ (for finite subsets of $\mathbb{R}^n$).

Again, one cannot expect to obtain a Brunn-Minkowski inequality for G_n in the classical form (3.12) (which, as we have mentioned before, would be a similar situation to what happens for $|\cdot|$ regarding a discrete version of (1.2)). Indeed, even for $n = 1$, just by taking $\lambda = 1/2$, $K = [0, m-\varepsilon]$ and $L = [0, m+\varepsilon/2]$, with $m \in \mathbb{N}$ and $0 < \varepsilon < 1$, one gets

$$\mathrm{G}_1\left(\frac{K+L}{2}\right) = \mathrm{G}_1\left(\left[0, m-\frac{\varepsilon}{4}\right]\right) = m < m + \frac{1}{2} = \frac{\mathrm{G}_1(K)+\mathrm{G}_1(L)}{2}. \tag{3.13}$$

Thus, as in (2.7), an alternative to get such an inequality for the lattice point enumerator would be to consider a certain extension of $(1-\lambda)K + \lambda L$. So, we pose the following question:

Question 3.3.1. *Given compact sets $K, L \subset \mathbb{R}^n$ containing some integer point, which is the best set M with $(1-\lambda)K + \lambda L \subset M$ for which*

$$\mathrm{G}_n\big(M\big)^{1/n} \geq (1-\lambda)\mathrm{G}_n(K)^{1/n} + \lambda \mathrm{G}_n(L)^{1/n}$$

holds for all $\lambda \in (0,1)$?

Here we give a positive answer to Question 3.3.1, providing a discrete counterpart to (3.12):

Theorem 3.3.1 ([15]). *Let $K, L \subset \mathbb{R}^n$ be non-empty bounded sets such that $\mathrm{G}_n(K)\mathrm{G}_n(L) > 0$, and let $\lambda \in (0,1)$. Then*

$$\mathrm{G}_n\big((1-\lambda)K + \lambda L + (-1,1)^n\big)^{1/n} \geq (1-\lambda)\mathrm{G}_n(K)^{1/n} + \lambda \mathrm{G}_n(L)^{1/n}. \tag{3.14}$$

The inequality is sharp.

Theorem 3.3.1 will be obtained as a consequence of a functional (and more general) result, which can be seen as a discrete version of the Borell-Brascamp-Lieb inequality for p-means (Theorem 3.1.2). To state this main result and henceforth, we will need the following notation: for a non-negative function $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$, we denote by $\phi^* : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ the function given by

$$\phi^*(z) = \sup_{u \in (-1,1)^n} \phi(z+u) \quad \text{for all } z \in \mathbb{R}^n.$$

Theorem 3.3.2 ([15]). *Let $\lambda \in (0,1)$ and let $-1/n \leq p \leq \infty$. Let $K, L \subset \mathbb{R}^n$ be non-empty bounded sets, and let $f, g, h : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ be non-negative functions such that*

$$h\big((1-\lambda)x + \lambda y\big) \geq \mathcal{M}_p^{\lambda}\left(f(x), g(y)\right)$$

for all $x \in K$ *and* $y \in L$*. Then*

$$\sum_{z\in M\cap\mathbb{Z}^n} h^*(z) \geq \mathcal{M}^{\lambda}_{\frac{p}{np+1}}\left(\sum_{x\in K\cap\mathbb{Z}^n} f(x), \sum_{y\in L\cap\mathbb{Z}^n} g(y)\right), \tag{3.15}$$

where $M = (1-\lambda)K + \lambda L + (-1,1)^n$.

Given a basis $\mathcal{B} = \{v_1 \dots, v_n\}$ of an n-dimensional lattice $\Lambda \subset \mathbb{R}^n$ we will write, for short,

$$D_{\mathcal{B}} = \sum_{i=1}^{n} (-v_i, v_i).$$

We note that $D_{\mathcal{C}} = (-1,1)^n$ for the canonical basis $\mathcal{C} = \{\mathrm{e}_1 \dots, \mathrm{e}_n\}$. Let $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be the linear (bijective) map given by $\varphi(x) = \sum_{i=1}^n x_i v_i$ for all $x = (x_1, \dots, x_n) \in \mathbb{R}^n$.

Under the same assumptions (and notation) of the above result, we may consider the auxiliary functions $\bar{f}, \bar{g}, \bar{h} : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ defined by

$$\bar{f}(x) = f\bigl(\varphi(x)\bigr), \qquad \bar{g}(x) = g\bigl(\varphi(x)\bigr) \quad \text{and} \quad \bar{h}(x) = h\bigl(\varphi(x)\bigr)$$

for all $x \in \mathbb{R}^n$. Then, as an immediate consequence of Theorem 3.3.2 (applied to the functions $\bar{f}, \bar{g}$ and $\bar{h}$), we get that

$$\sum_{z\in M\cap\Lambda} h^{*_{\mathcal{B}}}(z) \geq \mathcal{M}^{\lambda}_{\frac{p}{np+1}}\left(\sum_{x\in K\cap\Lambda} f(x), \sum_{y\in L\cap\Lambda} g(y)\right),$$

where $M = (1-\lambda)K + \lambda L + D_{\mathcal{B}}$ and $h^{*_{\mathcal{B}}}(z) = \sup_{u\in D_{\mathcal{B}}} h(z+u)$ for all $z \in \mathbb{R}^n$. So, Theorem 3.3.2 holds also in the setting of a general n-dimensional lattice Λ.

3.3.1 Some auxiliary results

We start with a lemma that will allow us to get the one-dimensional version of the above-mentioned Brunn-Minkowski inequality for the lattice point enumerator (cf. Question 3.3.1 and Theorem 3.3.1).

Lemma 3.3.1 ([15]). *For* $\lambda \in (0,1)$*, let* $K, L, M \subset \mathbb{R}$ *be non-empty sets with* $(1-\lambda)K + \lambda L \subset M$*. If* $M = \bigcup_{i=1}^r [a_i, b_i]$ *is a finite union of (pairwise disjoint) compact intervals then*

$$\mathrm{G}_1(M) + \Delta(M) \geq (1-\lambda)\mathrm{G}_1(K) + \lambda \mathrm{G}_1(L), \tag{3.16}$$

where $\Delta(M)$ *denotes the number of non-integer extreme points of* M*, namely*

$$\Delta(M) = \bigl|\{a_i \notin \mathbb{Z} : 1 \leq i \leq r\}\bigr| + \bigl|\{b_i \notin \mathbb{Z} : 1 \leq i \leq r\}\bigr|.$$

Proof. We show the result by induction on the number r of intervals of M. For the case $r=1$, i.e., when $M=[a_1,b_1]$ is a (non-empty) compact interval, we have on the one hand that

$$\mathrm{G}_1(M)=\lfloor b_1\rfloor-\lceil a_1\rceil+1.$$

Moreover, denoting by $i_{_K}=\inf K$, $s_{_K}=\sup K$, $i_{_L}=\inf L$ and $s_{_L}=\sup L$, we clearly get

$$\begin{aligned}\mathrm{G}_1(K)&\leq \mathrm{G}_1\big([i_{_K},s_{_K}]\big)=\lfloor s_{_K}\rfloor-\lceil i_{_K}\rceil+1 \quad \text{and}\\ \mathrm{G}_1(L)&\leq \mathrm{G}_1\big([i_{_L},s_{_L}]\big)=\lfloor s_{_L}\rfloor-\lceil i_{_L}\rceil+1.\end{aligned}$$

On the other hand, the inclusion $(1-\lambda)K+\lambda L\subset M$ implies that

$$\lfloor b_1\rfloor\geq b_1-\chi_{_{\mathbb{R}\setminus\mathbb{Z}}}(b_1)\geq(1-\lambda)\lfloor s_{_K}\rfloor+\lambda\lfloor s_{_L}\rfloor-\chi_{_{\mathbb{R}\setminus\mathbb{Z}}}(b_1)$$

and

$$-\lceil a_1\rceil\geq -a_1-\chi_{_{\mathbb{R}\setminus\mathbb{Z}}}(a_1)\geq-(1-\lambda)\lceil i_{_K}\rceil-\lambda\lceil i_{_L}\rceil-\chi_{_{\mathbb{R}\setminus\mathbb{Z}}}(a_1),$$

and thus

$$\lfloor b_1\rfloor-\lceil a_1\rceil+1\geq(1-\lambda)\big(\lfloor s_{_K}\rfloor-\lceil i_{_K}\rceil+1\big)+\lambda\big(\lfloor s_{_L}\rfloor-\lceil i_{_L}\rceil+1\big)-\Delta(M).$$

This, together with the above upper bounds for the lattice point enumerator G_1 of K and L, yields $\mathrm{G}_1(M)\geq(1-\lambda)\mathrm{G}_1(K)+\lambda\mathrm{G}_1(L)-\Delta(M)$, which shows the case $r=1$.

So, we suppose that the inequality is true for r and assume that $M=\bigcup_{i=1}^{r+1}[a_i,b_i]$, where $b_i<a_{i+1}$ for all $1\leq i\leq r$.

Denoting by $M_1=[a_1,b_1]$ and $M_2=\bigcup_{i=2}^{r+1}[a_i,b_i]$, we may assume, without loss of generality, that $M_1\cap\big((1-\lambda)K+\lambda L\big)\neq\emptyset$; otherwise, the result follows directly from applying the induction hypothesis to the sets K, L and M_2. Hence, we may define $m=\sup\Big(M_1\cap\big((1-\lambda)K+\lambda L\big)\Big)$ and then, since K and L are bounded (because $(1-\lambda)K+\lambda L\subset M$), there exist $k\in\operatorname{cl}K$ and $l\in\operatorname{cl}L$ such that $(1-\lambda)k+\lambda l=m$. Thus, considering the sets

$$K_1=\{x\in K: x\leq k\},\quad K_2=K\setminus K_1,\quad L_1=\{x\in L: x\leq l\}\quad\text{and}\quad L_2=L\setminus L_1,$$

we clearly have that $(1-\lambda)K_1+\lambda L_1\subset M_1$ and $(1-\lambda)K_2+\lambda L_2\subset M_2$. Therefore, applying the induction hypothesis, and taking into account that M_1 are M_2 are disjoint, we get

$$\begin{aligned}\mathrm{G}_1(M)+\Delta(M)&=\mathrm{G}_1(M_1)+\mathrm{G}_1(M_2)+\Delta(M_1)+\Delta(M_2)\\ &\geq(1-\lambda)\big(\mathrm{G}_1(K_1)+\mathrm{G}_1(K_2)\big)+\lambda\big(\mathrm{G}_1(L_1)+\mathrm{G}_1(L_2)\big)\\ &=(1-\lambda)\mathrm{G}_1(K)+\lambda\mathrm{G}_1(L),\end{aligned}$$

as desired. □

Remark 3.3. *One might think that if* $(1-\lambda)K+\lambda L$ *is as a finite union of (pairwise disjoint) compact intervals, the set* $M=(1-\lambda)K+\lambda L$ *would yield a tighter inequality in* (3.16). *However, this is not true in general: if we consider* $K=[-2m,-1]\cup[1,2m]$ *with* $m\in\mathbb{N}$, $L=\{0\}$ *and* $\lambda=1/2$, *then for* $M=(1-\lambda)K+\lambda L=[-m,-1/2]\cup[1/2,m]$ *we get* $\mathrm{G}_1(M)+\Delta(M)=2m+2$, *whereas for* $M'=[-m,m]$ *we have* $\mathrm{G}_1(M')+\Delta(M')=2m+1$.

We notice that, as shown in (3.13), the quantity $\Delta(M)$ cannot be (in general) omitted. However, we can rewrite (3.16) to provide an answer to Question 3.3.1 for $n=1$ (also in the case in which $\mathrm{G}_1(K)\mathrm{G}_1(L)=0$):

Lemma 3.3.2 ([15]). *Let* $\lambda\in(0,1)$ *and let* $K,L\subset\mathbb{R}$ *be non-empty bounded sets. Then*

$$\mathrm{G}_1\big((1-\lambda)K+\lambda L+(-1,1)\big)\geq(1-\lambda)\mathrm{G}_1(K)+\lambda\mathrm{G}_1(L). \tag{3.17}$$

The inequality is sharp.

Proof. Let $M=\bigcup_{x\in(1-\lambda)K+\lambda L}\big[\lfloor x\rfloor,\lceil x\rceil\big]$. Clearly, since K and L are bounded, M is a finite union of compact intervals containing $(1-\lambda)K+\lambda L$. From Lemma 3.3.1 we then obtain

$$\mathrm{G}_1(M)+\Delta(M)\geq(1-\lambda)\mathrm{G}_1(K)+\lambda\mathrm{G}_1(L)$$

which, together with the facts that $\Delta(M)=0$ and $M\cap\mathbb{Z}=\big((1-\lambda)K+\lambda L+(-1,1)\big)\cap\mathbb{Z}$, yields the desired inequality (3.17).

Finally, in order to show that the equality may be attained, we consider $K=L=[0,m]$ with $m\in\mathbb{N}$, for which

$$\mathrm{G}_1\big((1-\lambda)K+\lambda L+(-1,1)\big)=m+1=(1-\lambda)\mathrm{G}_1(K)+\lambda\mathrm{G}_1(L)$$

for all $\lambda\in(0,1)$. □

Remark 3.4. *Since both sides on* (3.17) *remain invariant under translations by integers of the sets* K *and* L, *we may replace* $(-1,1)$ *(in* (3.17)*) by any other interval* $(m,m+2)$, *with* $m\in\mathbb{Z}$.

We note however that the solution to Question 3.3.1 provided by Lemma 3.3.2 for $n=1$, *via* $M=(1-\lambda)K+\lambda L+(-1,1)$, *cannot be in general improved by means of any other interval strictly contained in* $(-1,1)$. *Indeed, by considering* $(-1,a]$, *for* $0<a<1$, *and taking* $K=[0,1]$, $L=[0,2]$ *and* $\lambda\in(0,1)$ *such that* $\lambda+a<1$, *we get that*

$$\mathrm{G}_1\big((1-\lambda)K+\lambda L+(-1,a]\big)=\mathrm{G}_1\big((-1,1+\lambda+a]\big)=2<2+\lambda=(1-\lambda)\mathrm{G}_1(K)+\lambda\mathrm{G}_1(L).$$

The case $[a,1)$, *with* $-1<a<0$, *is completely analogous and thus, no interval smaller than* $(-1,1)$ *(with respect to set inclusion) can be taken into account. Furthermore, since the approach we will carry out throughout this section relies on induction on the dimension* n, *the solution (to Question 3.3.1) given by the sum of the set* $(-1,1)$, *for* $n=1$, *will turn into adding* $(-1,1)^n$, *for arbitrary* n *(see Theorem 3.3.1).*

The following corollary follows directly from Lemma 3.2.1.

Corollary 3.3.1 ([15]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded set, let $f : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ be a non-negative function and set $f(\Omega \cap \mathbb{Z}^n) \subset \{k_0, k_1, \dots, k_r\}$ with $0 = k_0 < k_1 < \cdots < k_r$. Then*

$$\sum_{x \in \Omega \cap \mathbb{Z}^n} f(x) = \sum_{i=1}^{r} (k_i - k_{i-1}) \mathrm{G}_n\big(\{x \in \Omega : f(x) \geq k_i\}\big).$$

Next result yields the case $n = 1$ of Theorem 3.3.2 and will be used to derive it.

Lemma 3.3.3 ([15]). *Let $\lambda \in (0,1)$ and let $-1 \leq p \leq \infty$. Let $\Omega_1, \Omega_2 \subset \mathbb{R}$ be non-empty bounded sets and let $f, g, h : \mathbb{R} \longrightarrow \mathbb{R}_{\geq 0}$ be non-negative functions such that*

$$h\big((1-\lambda)x + \lambda y\big) \geq \mathcal{M}_p^{\lambda}\left(f(x), g(y)\right)$$

for all $x \in \Omega_1$, $y \in \Omega_2$. Then

$$\sum_{z \in \Omega \cap \mathbb{Z}} h^*(z) \geq \mathcal{M}^{\lambda}_{\frac{p}{p+1}} \left(\sum_{x \in \Omega_1 \cap \mathbb{Z}} f(x), \sum_{y \in \Omega_2 \cap \mathbb{Z}} g(y) \right),$$

where $\Omega = (1-\lambda)\Omega_1 + \lambda\Omega_2 + (-1, 1)$.

Proof. Clearly, we may assume that $\sum_{x \in \Omega_1 \cap \mathbb{Z}} f(x), \sum_{y \in \Omega_2 \cap \mathbb{Z}} g(y) > 0$. We consider the non-negative functions $F, G, H, H^* : \mathbb{R} \longrightarrow \mathbb{R}_{\geq 0}$ given by

$$F(x) = \frac{f(x)}{a}, \qquad G(y) = \frac{g(y)}{b}, \qquad H(z) = \frac{h(z)}{c_p}, \qquad H^*(z) = \frac{h^*(z)}{c_p},$$

where

$$a = \max_{x \in \Omega_1 \cap \mathbb{Z}} f(x) > 0, \qquad b = \max_{y \in \Omega_2 \cap \mathbb{Z}} g(y) > 0 \qquad \text{and} \quad c_p = \mathcal{M}_p^{\lambda}(a, b) > 0.$$

Then

$$\max_{x \in \Omega_1 \cap \mathbb{Z}} F(x) = \max_{y \in \Omega_2 \cap \mathbb{Z}} G(y) = 1.$$

First, we show that, for any $x \in \Omega_1$, $y \in \Omega_2$, we have that

$$H\big((1-\lambda)x + \lambda y\big) \geq \min\big\{F(x), G(y)\big\}. \tag{3.18}$$

To this aim, it is enough to consider $x \in \Omega_1$, $y \in \Omega_2$ with $f(x)g(y) > 0$. If $p \neq 0$ and $p \neq \infty$, writing $\theta = \lambda b^p / c_p^p \in (0,1)$, we get

$$\begin{aligned} h\big((1-\lambda)x + \lambda y\big) &\geq \big((1-\lambda)f(x)^p + \lambda g(y)^p\big)^{1/p} = c_p \left(\frac{(1-\lambda)a^p F(x)^p + \lambda b^p G(y)^p}{c_p^p} \right)^{1/p} \\ &= c_p \big((1-\theta)F(x)^p + \theta G(y)^p\big)^{1/p} \geq c_p \min\big\{F(x), G(y)\big\}. \end{aligned}$$

For $p = 0$ we have

$$h\big((1-\lambda)x+\lambda y\big) \geq f(x)^{1-\lambda}g(y)^{\lambda} = a^{1-\lambda}b^{\lambda}F(x)^{1-\lambda}G(y)^{\lambda} \geq c_0 \min\big\{F(x), G(y)\big\},$$

whereas for $p = \infty$,

$$h\big((1-\lambda)x+\lambda y\big) \geq \max\big\{f(x), g(y)\big\} \geq c_{\infty} \min\big\{F(x), G(y)\big\}$$

clearly holds. Therefore, we have shown (3.18).

The definition of F and G now implies that, for any $t \in [0,1]$, the level sets

$$\big\{x \in \Omega_1 : F(x) \geq t\big\}, \qquad \big\{y \in \Omega_2 : G(y) \geq t\big\}$$

are non-empty. Moreover, writing $\Omega_\lambda = (1-\lambda)\Omega_1 + \lambda\Omega_2$, from (3.18) we deduce that

$$\big\{z \in \Omega_\lambda : H(z) \geq t\big\} \supset (1-\lambda)\big\{x \in \Omega_1 : F(x) \geq t\big\} + \lambda\big\{y \in \Omega_2 : G(y) \geq t\big\}$$

and thus, by Lemma 3.3.2, we have

$$\mathrm{G}_1\Big(\big\{z\in \Omega_\lambda : H(z) \geq t\big\}+(-1,1)\Big) \geq (1-\lambda)\mathrm{G}_1\Big(\big\{x\in \Omega_1 : F(x) \geq t\big\}\Big)+\lambda\mathrm{G}_1\Big(\big\{y\in \Omega_2 : G(y) \geq t\big\}\Big) \quad (3.19)$$

for all $t \in [0,1]$.

We note that, since $H^*(z+u) \geq H\big((z+u)-u\big) = H(z)$ for all $u \in (-1,1)$, we also have

$$\big\{z \in \Omega : H^*(z) \geq t\big\} \supset \big\{z \in \Omega_\lambda : H(z) \geq t\big\} + (-1,1). \quad (3.20)$$

Finally, set $\{k_0, k_1, \dots, k_r\} \supset F\big(\Omega_1 \cap \mathbb{Z}\big) \cup G\big(\Omega_2 \cap \mathbb{Z}\big) \cup H^*\big(\Omega \cap \mathbb{Z}\big)$, with $0 = k_0 < k_1 < \cdots < k_r$ where, for some $s \in \{1, \dots, r\}$,

$$k_s = \max_{x\in\Omega_1\cap\mathbb{Z}} F(x) = \max_{y\in\Omega_2\cap\mathbb{Z}} G(y) = 1.$$

Then, by (3.19), (3.20) and using Corollary 3.3.1, we get

$$\begin{aligned}
\sum_{z\in\Omega\cap\mathbb{Z}} h^*(z) &= \sum_{z\in\Omega\cap\mathbb{Z}} c_p H^*(z) = c_p \sum_{i=1}^{r} (k_i - k_{i-1})\mathrm{G}_1\Big(\big\{z \in \Omega : H^*(z) \geq k_i\big\}\Big) \\
&\geq c_p \sum_{i=1}^{s} (k_i - k_{i-1})\mathrm{G}_1\Big(\big\{z \in \Omega : H^*(z) \geq k_i\big\}\Big) \\
&\geq c_p \sum_{i=1}^{s} (k_i - k_{i-1})\Big[(1-\lambda)\mathrm{G}_1\Big(\big\{x \in \Omega_1 : F(x) \geq k_i\big\}\Big) + \lambda\mathrm{G}_1\Big(\big\{y \in \Omega_2 : G(y) \geq k_i\big\}\Big)\Big] \\
&= c_p \left((1-\lambda)\sum_{x\in\Omega_1\cap\mathbb{Z}} F(x) + \lambda \sum_{y\in\Omega_2\cap\mathbb{Z}} G(y)\right) = c_p \left(\frac{1-\lambda}{a}\sum_{x\in\Omega_1\cap\mathbb{Z}} f(x) + \frac{\lambda}{b}\sum_{y\in\Omega_2\cap\mathbb{Z}} g(y)\right) \\
&\geq \mathcal{M}^{\lambda}_{\frac{p}{p+1}}\left(\sum_{x\in\Omega_1\cap\mathbb{Z}} f(x), \sum_{y\in\Omega_2\cap\mathbb{Z}} g(y)\right).
\end{aligned}$$

If $p \neq 0$, the last inequality follows from the reverse Hölder inequality (Corollary 1.1.1) with parameter $p/(p+1) \leq 1$, just by taking

$$a_1 = (1-\lambda)^{(p+1)/p} \sum_{x\in\Omega_1\cap\mathbb{Z}} f(x), \qquad a_2 = \lambda^{(p+1)/p} \sum_{y\in\Omega_2\cap\mathbb{Z}} g(y),$$
$$b_1 = \big((1-\lambda)^{1/p}a\big)^{-1} \quad \text{and} \quad b_2 = (\lambda^{1/p}b)^{-1}.$$

The case $p = 0$ follows from the arithmetic-geometric mean inequality (Theorem 1.1.1). □

3.3.2 The proofs of Theorems 3.3.1 and 3.3.2

We start with the proof of the more general Theorem 3.3.2.

Proof of Theorem 3.3.2. We do it by induction on the dimension. If $n = 1$, the result follows immediately from Lemma 3.3.3. Now we suppose that $n > 1$ and we assume that the theorem holds for dimension $n-1$. Let $t_K \in \pi_n(K)$, $t_L \in \pi_n(L)$ and we set, for the sake of brevity, $t_\lambda = (1-\lambda)t_K + \lambda t_L$. Moreover, we denote by

$$M_\lambda = (1-\lambda)K + \lambda L \quad \text{and} \quad M_{n-1} = (1-\lambda)K(t_K) + \lambda L(t_L) + (-1,1)^{n-1}.$$

We consider the non-negative functions $f_1, g_1, h_1 : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}_{\geq 0}$ given by

$$f_1(x) = f(x, t_K), \qquad g_1(x) = g(x, t_L) \quad \text{and} \quad h_1(x) = h(x, t_\lambda)$$

for all $x \in \mathbb{R}^{n-1}$. Since

$$\begin{aligned} h_1\big((1-\lambda)x + \lambda y\big) &= h\big((1-\lambda)x + \lambda y, (1-\lambda)t_K + \lambda t_L\big) \\ &\geq \mathcal{M}_p^\lambda\left(f(x,t_K), g(y,t_L)\right) = \mathcal{M}_p^\lambda\left(f_1(x), g_1(y)\right) \end{aligned}$$

for any $x \in K(t_K)$ and $y \in L(t_L)$, we may assert that

$$\sum_{z\in M_{n-1}\cap\mathbb{Z}^{n-1}} h_1^*(z) \geq \mathcal{M}^\lambda_{\frac{p}{(n-1)p+1}}\left(\sum_{x\in K(t_K)\cap\mathbb{Z}^{n-1}} f_1(x), \sum_{y\in L(t_L)\cap\mathbb{Z}^{n-1}} g_1(y)\right).$$

Now, since

$$M_\lambda(t_\lambda) = \big((1-\lambda)K + \lambda L\big)\big((1-\lambda)t_K + \lambda t_L\big) \supset (1-\lambda)K(t_K) + \lambda L(t_L),$$

we have that

$$\Big(M_\lambda + \big((-1,1)^{n-1} \times \{0\}\big)\Big)(t_\lambda) \supset M_{n-1},$$

and hence, the above inequality yields, in terms of f, g and h,

$$\sum_{z\in\left[M_\lambda+(-1,1)^{n-1}\times\{0\}\right]\cap\mathbb{Z}^{n-1}} h^{**}(z, t_\lambda) \geq \mathcal{M}^\lambda_{\frac{p}{(n-1)p+1}}\left(\sum_{x\in K(t_K)\cap\mathbb{Z}^{n-1}} f(x,t_K), \sum_{y\in L(t_L)\cap\mathbb{Z}^{n-1}} g(y,t_L)\right), \quad (3.21)$$

where $h^{**}:\mathbb{R}^n\longrightarrow\mathbb{R}_{\geq 0}$ is the function given by

$$h^{**}(z)=\sup_{v\in M_\lambda+\bigl((-1,1)^{n-1}\times\{0\}\bigr)} h(z+v),$$

for which we have $h^{**}(x,t_\lambda)=h_1^*(x)$ for all $x\in\mathbb{R}^{n-1}$.

Now, let $f_2,g_2,h_2:\mathbb{R}\longrightarrow\mathbb{R}_{\geq 0}$ be the functions defined by

$$f_2(t)=\sum_{x\in K(t)\cap\mathbb{Z}^{n-1}} f(x,t),\qquad g_2(t)=\sum_{y\in L(t)\cap\mathbb{Z}^{n-1}} g(y,t)\quad\text{and}$$
$$h_2(t)=\sum_{z\in\bigl(M_\lambda+(-1,1)^{n-1}\times\{0\}\bigr)(t)\cap\mathbb{Z}^{n-1}} h^{**}(z,t).$$

Then, (3.21) yields, in terms of f_2, g_2 and h_2,

$$h_2\bigl((1-\lambda)t_K+\lambda t_L\bigr)\geq\mathcal{M}^\lambda_{\frac{p}{(n-1)p+1}}\bigl(f_2(t_K),g_2(t_L)\bigr)$$

for any $t_K\in\pi_n(K)$ and $t_L\in\pi_n(L)$, and thus we may use Lemma 3.3.3 with the sets $\pi_n(K)$ and $\pi_n(L)$ and the functions f_2, g_2 and h_2 to obtain

$$\sum_{t\in\Omega\cap\mathbb{Z}} h_2^*(t)\geq\mathcal{M}^\lambda_{\frac{p}{np+1}}\left(\sum_{t_K\in\pi_n(K)\cap\mathbb{Z}} f_2(t_K),\sum_{t_L\in\pi_n(L)\cap\mathbb{Z}} g(t_L)\right),$$

where $\Omega=(1-\lambda)\pi_n(K)+\lambda\pi_n(L)+(-1,1)$.

In the following we prove that $\sum_{t\in\Omega\cap\mathbb{Z}}h_2^*(t)=\sum_{z\in M\cap\mathbb{Z}^n}h^*(z)$, and hence the above inequality, together with the relations

$$\sum_{t_K\in\pi_n(K)\cap\mathbb{Z}} f_2(t_K)=\sum_{x\in K\cap\mathbb{Z}^n} f(x)\qquad\text{and}\qquad\sum_{t_L\in\pi_n(L)\cap\mathbb{Z}} g_2(t_L)=\sum_{y\in L\cap\mathbb{Z}^n} g(y),$$

shows the result.

Indeed, from the fact that $(u,-w)\in(-1,1)^n$ for any $(u,0)\in(-1,1)^{n-1}\times\{0\}$ and $w\in(-1,1)$, we have that $\Bigl(M_\lambda+\bigl((-1,1)^{n-1}\times\{0\}\bigr)\Bigr)(t+w)\subset M(t)$ for all $w\in(-1,1)$ and $t\in\mathbb{R}$: indeed, if t is so that $t+w\in\pi_n(M_\lambda)$, then the inclusion holds; otherwise it is trivial. Thus, we get

$$\begin{aligned}\sum_{t\in\Omega\cap\mathbb{Z}} h_2^*(t)&=\sum_{t\in\Omega\cap\mathbb{Z}}\sup_{w\in(-1,1)} h_2(t+w)\\&=\sum_{t\in\Omega\cap\mathbb{Z}}\sup_{w\in(-1,1)}\sum_{x\in\bigl(M_\lambda+(-1,1)^{n-1}\times\{0\}\bigr)(t+w)\cap\mathbb{Z}^{n-1}} h^{**}(x,t+w)\\&\leq\sum_{t\in\Omega\cap\mathbb{Z}}\sum_{x\in M(t)\cap\mathbb{Z}^{n-1}}\sup_{w\in(-1,1)} h^{**}(x,t+w)\\&=\sum_{t\in\Omega\cap\mathbb{Z}}\sum_{x\in M(t)\cap\mathbb{Z}^{n-1}}\sup_{w\in(-1,1)}\sup_{v\in(-1,1)^{n-1}} h(x+v,t+w)\\&=\sum_{z\in M\cap\mathbb{Z}^n}\sup_{u\in(-1,1)^n} h(z+u)=\sum_{z\in M\cap\mathbb{Z}^n} h^*(z),\end{aligned}$$

as claimed. This finishes the proof. $\square$

We conclude this section by proving Theorem 3.3.1, which answers Question 3.3.1.

Proof. The inequality is an immediate consequence of Theorem 3.3.2, just by taking $p = \infty$, $f = \chi_K$, $g = \chi_L$ and $h = \chi_{(1-\lambda)K+\lambda L}$, for which we clearly have that $h^* = \chi_{(1-\lambda)K+\lambda L+(-1,1)^n}$.

Now, in order to show that the equality may be attained, we consider $K = L = [0, m]^n$ with $m \in \mathbb{N}$. Then

$$\mathrm{G}_n\big((1-\lambda)K + \lambda L + (-1,1)^n\big)^{1/n} = m + 1 = (1-\lambda)\mathrm{G}_n(K)^{1/n} + \lambda \mathrm{G}_n(L)^{1/n}$$

for all $\lambda \in (0, 1)$. □

3.3.3 From the discrete version to the continuous one

Again, in order to conclude the section, we show that the classical (p-means version of the) Borell-Brascamp-Lieb inequality (3.2) can be obtained from the discrete version (3.15) under some mild assumptions for the functions there involved.

Theorem 3.3.3 ([15]). *The discrete Borell-Brascamp-Lieb inequality* (3.15) *implies the classical Borell-Brascamp-Lieb inequality* (3.2)*, provided that the functions f, g are Riemann integrable and h is upper semicontinuous.*

In the following, for a function $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$, we write $\phi^{*_k} : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ to denote the function given by $\phi^{*_k}(z) = \sup_{u \in (-2^{-k}, 2^{-k})^n} \phi(z + u)$ for all $z \in \mathbb{R}^n$.

Proof. We assume that $f, g, h : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ are non-negative measurable functions, with f, g Riemann integrable and h upper semicontinuous, such that

$$h\big((1-\lambda)x + \lambda y\big) \geq \mathcal{M}_p^{\lambda}\left(f(x), g(y)\right)$$

for all $x, y \in \mathbb{R}^n$. Let $m \in \mathbb{N}$ and let $K = [-m, m]^n$. We will first show that

$$\int_K h(x)\mathrm{d}x \geq \mathcal{M}^{\lambda}_{\frac{p}{np+1}}\left(\int_K f(x)\mathrm{d}x, \int_K g(x)\mathrm{d}x\right), \tag{3.22}$$

for which we may assume (multiplying by χ_K if necessary) that f, g and h vanish outside K.

For each $k \in \mathbb{N}$, we define the functions $f_k, g_k, h_k : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ given by

$$f_k(x) = \inf_{z \in x+[0,2^{-k}]^n} f(z), \qquad g_k(x) = \inf_{z \in x+[0,2^{-k}]^n} g(z) \quad \text{and} \quad h_k(x) = \inf_{z \in x+[0,2^{-k}]^n} h(z).$$

We note that for any $x, y \in \operatorname{int} K$ we have

$$\begin{aligned}
h_k\big((1-\lambda)x + \lambda y\big) &= \inf_{z \in (1-\lambda)x+\lambda y+[0,2^{-k}]^n} h(z) = \inf_{z \in (1-\lambda)\left(x+[0,2^{-k}]^n\right)+\lambda\left(y+[0,2^{-k}]^n\right)} h(z) \\
&= \inf_{\substack{z_1 \in x+[0,2^{-k}]^n \\ z_2 \in y+[0,2^{-k}]^n}} h\big((1-\lambda)z_1 + \lambda z_2\big) \geq \inf_{\substack{z_1 \in x+[0,2^{-k}]^n \\ z_2 \in y+[0,2^{-k}]^n}} \mathcal{M}_p^{\lambda}\left(f(z_1), g(z_2)\right) \\
&\geq \mathcal{M}_p^{\lambda}\left(\inf_{z_1 \in x+[0,2^{-k}]^n} f(z_1), \inf_{z_2 \in y+[0,2^{-k}]^n} g(z_2)\right) = \mathcal{M}_p^{\lambda}\left(f_k(x), g_k(y)\right),
\end{aligned}$$

and thus, we can use Theorem 3.3.2 for $2^{-k}\mathbb{Z}^n$ to deduce that, for any $k \in \mathbb{N}$, we have

$$2^{-kn} \sum_{z\in K\cap 2^{-k}\mathbb{Z}^n} (h_k)^{*_k}(z) \geq \mathcal{M}^{\lambda}_{\frac{p}{np+1}} \left(2^{-kn} \sum_{x\in(\operatorname{int} K)\cap 2^{-k}\mathbb{Z}^n} f_k(x), 2^{-kn} \sum_{y\in(\operatorname{int} K)\cap 2^{-k}\mathbb{Z}^n} g_k(y) \right), \quad (3.23)$$

where, on the left-hand side, we have used that

$$\left(\operatorname{int} K + \left(-2^{-k}, 2^{-k}\right)^n\right) \cap 2^{-k}\mathbb{Z}^n = K \cap 2^{-k}\mathbb{Z}^n$$

because $K = [-m, m]^n$.

The level sets $\{x \in K : h(x) \geq t\}$ are closed, because h is upper semicontinuous and K is closed (see [23, Theorem 1.6]), and then a standard straightforward computation shows that

$$\{x \in K : h(x) \geq t\} = \bigcap_{k=1}^{\infty} \left(\{x \in K : h(x) \geq t\} + \left(-2^{-k}, 2^{-k}\right)^n \right).$$

Moreover, since h vanishes outside K, we have

$$\{x \in K : h(x) > t\} + \left(-2^{-k}, 2^{-k}\right)^n \supset \left\{x \in K + \left[0, 2^{-k}\right]^n : h^{*_k}(x) > t\right\}$$

for all $t > 0$. Thus, by using Fubini's theorem and the monotone convergence theorem, we get

$$\begin{aligned}
\int_K h(x)\mathrm{d}x &= \int_0^\infty \operatorname{vol}\Big(\{x \in K : h(x) \geq t\}\Big)\,\mathrm{d}t \\
&= \int_0^\infty \operatorname{vol}\left(\bigcap_{k=1}^{\infty} \left(\{x \in K : h(x) \geq t\} + \left(-2^{-k}, 2^{-k}\right)^n\right)\right)\mathrm{d}t \\
&= \int_0^\infty \lim_{k\to\infty} \operatorname{vol}\left(\{x \in K : h(x) \geq t\} + \left(-2^{-k}, 2^{-k}\right)^n\right)\mathrm{d}t \\
&= \lim_{k\to\infty} \int_0^\infty \operatorname{vol}\left(\{x \in K : h(x) \geq t\} + \left(-2^{-k}, 2^{-k}\right)^n\right)\mathrm{d}t \\
&\geq \lim_{k\to\infty} \int_0^\infty \operatorname{vol}\Big(\{x \in K + [0, 2^{-k}]^n : h^{*_k}(x) > t\}\Big)\,\mathrm{d}t \\
&= \lim_{k\to\infty} \int_{K+[0,2^{-k}]^n} h^{*_k}(x)\,\mathrm{d}x.
\end{aligned} \quad (3.24)$$

Now we show that, given $z \in \mathbb{R}^n$, $h^{*_k}(x) \geq (h_k)^{*_k}(z)$ for all $x \in z + [0, 2^{-k}]^n$. Indeed, we have

$$\begin{aligned}
h^{*_k}(x) &= \sup_{u\in(-2^{-k},2^{-k})^n} h(x+u) \geq \sup_{u\in(-2^{-k},2^{-k})^n} \inf_{v\in[0,2^{-k}]^n} h(z+v+u) \\
&\geq \sup_{u\in(-2^{-k},2^{-k})^n} h_k(z+u) = (h_k)^{*_k}(z).
\end{aligned}$$

This, together with (3.24) and the fact that $K + \left[0, 2^{-k}\right]^n = K \cap 2^{-k}\mathbb{Z}^n + \left[0, 2^{-k}\right]^n$, implies that

$$\int_K h(x)\,\mathrm{d}x \geq \lim_{k\to\infty} \int_{K+[0,2^{-k}]^n} h^{*_k}(x)\,\mathrm{d}x \geq \lim_{k\to\infty} 2^{-kn} \sum_{z\in K\cap 2^{-k}\mathbb{Z}^n} (h_k)^{*_k}(z).$$

Furthermore, since f is Riemann integrable and $2^{-nk}\sum_{x\in(\operatorname{int}K)\cap 2^{-k}\mathbb{Z}^n} f_k(x)$ is a *lower sum* of $f\cdot\chi_{(-m,m]^n}$ for the partition $\left\{x+[0,2^{-k}]^n\subset K : x\in 2^{-k}\mathbb{Z}^n\right\}$ of K, it is clear that

$$\lim_{k\to\infty} 2^{-kn}\sum_{x\in(\operatorname{int}K)\cap 2^{-k}\mathbb{Z}^n} f_k(x)=\int_K f(x)\,\mathrm{d}x.$$

Here we observe that it was crucial to work with $\operatorname{int}K$ in order to get a lower sum of $f\cdot\chi_{(-m,m]^n}$ for the above partition. We also point out the necessity of considering the characteristic function $\chi_{(-m,m]^n}$ instead of $\chi_{[-m,m]^n}$, which has no influence when computing the above integral: in this way, the function $f\cdot\chi_{(-m,m]^n}$ vanishes on the points of the corresponding facets of the cube.

The same holds for the function g and then, taking limits in both sides of (3.23), we get (3.22). Since (3.22) is true for $K=[-m,m]^n$ and every $m\in\mathbb{N}$, the proof is now concluded because

$$\int_{\mathbb{R}^n}\phi(x)\,\mathrm{d}x=\lim_{m\to\infty}\int_{[-m,m]^n}\phi(x)\,\mathrm{d}x$$

for every non-negative measurable function $\phi:\mathbb{R}^n\to\mathbb{R}_{\geq 0}$. □

It is well-known that a function is Riemann integrable if and only if it is continuous almost everywhere. Since the boundary of a convex set has null measure (and from the characterization of the upper semicontinuity in terms of the level sets) we get the following result, as a straightforward consequence of the previous one.

Corollary 3.3.2. *The discrete Brunn-Minkowski inequality* (3.14) *implies the classical Brunn-Minkowski inequality* (3.12) *for compact convex sets K and L with non-empty interior.*

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