anales de psicología
2008, vol. 24, nº 2 (diciembre), 262-270

ISSN edición impresa: 0212-9728. ISSN edición web (www.um.es/analesps): 1695-2294

TEMA MONOGRÁFICO: Psicología de las matemáticas

Rational Numbers and Intensive Quantities: Challenges and Insights to Pupils' Implicit Knowledge

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Abstract: This paper analyzes the difficulties that pupils have in learning to use rational numbers to refer to quantities as well as those involved in understanding equivalence and order in rational numbers. It is shown that pupils' performance in all these tasks is not a matter of an all-or-nothing knowledge. Rational numbers are used in different situations, which can help or hinder pupils' insights into the logic of rational numbers. The concluding section suggests that there is less awareness of the logic of rational numbers because culture often is organised to facilitate our functioning even if we don't understand rational numbers well. We argue that it is important to develop this understanding in school so that young people are not at a loss when this knowledge is required.
Key words: Rational numbers; fractions; proportional judgement; conceptual and procedural knowledge; implicit and explicit knowledge.

Título: Números racionales y cantidades intensivas: Retos y comprensión para el conocimiento implícito de los alumnos.
Resumen: Este trabajo analiza las dificultades que tienen los alumnos para aprender a usar los números racionales al referirse a cantidades, así como las implicadas en la comprensión de la equivalencia y el orden en los números racionales. Se demuestra que el rendimiento de los alumnos en todas estas tareas no es una cuestión de un conocimiento todo o nada. Los números racionales se utilizan en diferentes situaciones, que pueden ayudar u obstaculizar la comprensión de los alumnos sobre la lógica de los números racionales. En la última sección se sugiere que hay menos conciencia de la lógica de los números racionales debido a que la cultura a menudo se organiza para facilitar nuestro funcionamiento, aun cuando no entendamos bien los números racionales. Creemos que es importante desarrollar esta comprensión en la escuela para que los jóvenes no sean incapaces cuando se les requiera este conocimiento.
Palabras clave: Números racionales; fracciones; razonamiento proporcional; conocimiento conceptual y procedimental; conocimiento implícito y explícito.

A major challenge facing policy makers is to define what a mathematics curriculum for all pupils should contain, so that anyone leaving basic education could expect to master concepts and skills required for everyday life and work. We suggest that, in the domain of numbers, research on pupils' learning and on everyday concepts now makes it possible to define such a core curriculum with clarity. Two conceptual domains, whole and rational numbers, and two types of quantities, extensive and intensive quantities, emerge from research in mathematics education as the basic ideas that such a core curriculum should cover. In this paper, we focus on rational numbers and intensive quantities, which receive less attention in the current aims for mathematics education in England (as described by the attainment levels), than whole numbers and extensive quantities do.

In the first section of the paper we discuss the differences between answering the question "how much?" when the answer is a natural number or a rational number. It is argued that children find the procedure used to attach a natural number to a quantity easy whereas the procedure required to attach a fraction to a quantity is very difficult. We also distinguish between different situations in which fractions are used and show that these situations have an impact on whether children think that a natural number is a good answer. In the second section we consider what children gain by learning to use natural and rational numbers to represent quantities and how these representations are related to equivalence and order relations between the quantities. The third section examines briefly the evidence for the importance of using situations as a starting point in the teaching of rational numbers. The fifth section addresses the question: are fractions worth the effort? Finally, the last section presents an overview of the main points made in this paper.

Using numbers to refer to quantities

Most children have little difficulty in forming some idea of the meaning of natural numbers because natural numbers are what you get from counting (Hartnett & Gelman, 1998). When first starting primary school, at age about four and a half or five years, most children can answer the question "how many sweets do you have?" if the number of sweets is within their counting range. They know a procedure – counting – that leads to a number which says how many sweets they have. This procedure is applied easily by children to anything that can be counted – objects (sweets, marbles, stones, coins, dogs etc.) as well as events (steps, jumps, turns in a game etc.). What is being counted does not have an effect on how the number label will be used. This knowledge, of course, is not sufficient to say that children really understand natural numbers but it shows how accessible the first ideas about natural numbers are to children.

The ease of using natural number signs to refer to quantities contrasts strongly with the difficulty of finding the

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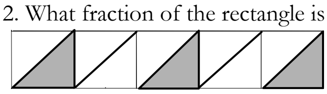
right number sign for quantities that must be represented by rational numbers. Even after they have been taught to use fractions to represent quantities, many pupils in Years 4 and 5 of primary school continue to have difficulties in using fraction labels. We carried out a survey with pupils in Year 4 (N = 74; mean age about 8y6m) and Year 5 (N = 49; mean age about 9y6m) from six primary schools (which covered a varied intake in terms of socio-economic levels) in Oxford. Our survey contained 12 items with the question "what fraction?" The mean number of correct responses for pupils in Year 4 was 2.64 (SD=3.58) and for Year 5 pupils was 6.1 (SD=4.96).

In order to convey a better picture of the pupils' difficulties, Table 1 presents a sample of items and the percentage of correct responses for each item by year group. All items were presented to the children with the support of pictures. For example, the first item, which describes a situation where three girls are going to share a pie, was illustrated with a drawing of a pie and three girls; the pupils were asked to indicate what fraction of the pie each one would receive, if the girls shared it fairly. The survey included items investigating other aspects of fractions' knowledge but only the right use of fraction labels will be discussed at this point.

Table 1: Percentage of correct responses by Year Group to items asking pupils to identify the fraction names.

Items	% correct
1. This pie has to be shared equally between 3 girls. What fraction of the pie does each girl get?	Y4: 27 Y5: 53
2. What fraction of the rectangle is shaded? 	Y4: 32 Y5: 51
3. A picture showed 4 chocolate muffins and 2 cream muffins. What fraction of the muffins is made with chocolate?	Y4: 12 Y5: 49
4. A picture showed that a mixture of paint was made with three bottles of white paint and three same size bottles of blue paint. What fraction of the mixture was made with blue paint?	Y4: 20 Y5: 50

It is remarkable that, even after being taught fractions in school, only about half of the pupils in Year 5 respond correctly to the easiest item, which is about the unitary fraction 1/3. Although it is true that the fractions had to be written, whereas identifying a natural number to describe a quantity by counting objects is an oral activity, writing the numbers used for the numerator and the denominator of the fractions could not be an obstacle for pupils in Year 5: not one of these was larger than 10. Why should it be so difficult to learn to use rational numbers to describe quantities when it is so easy to use natural numbers?

Rational numbers represent a quantity by setting two quantities in relation to each other. Thus in contrast to natural numbers, two values – the numerator and the denominator – are used. Counting is the procedure used to come up with the values that will be the numerator and the denominator, but knowing what to count and which is the numerator and the denominator requires more than counting. So in order to come up with the right fraction, pupils need to understand how situations are represented by fractions.

In *item 1* (Table 1), the situation described is "one pie will be shared fairly by three girls". The fraction represents the number of items being shared – 1 (pie) the numerator – divided by the number of recipients – 3 (girls), the denominator. The amount of pie that each girl eats, when they share it fairly, is 1 (pie) divided by 3 (girls): 1/3. This numerical representation could be read as "one divided by three" or "one third". In this situation, the relation between the numerator and the denominator can easily be thought of as division; for this reason, mathematics education researchers refer to this type of situation as *quotient situations* (for different classifications of fractions situations, see, for example, Kieren, 1988; Behr, Harel, Post, & Lesh, 1993). However, it is also possible to ignore the idea of division, and to think only that each girl will eat one part out of three. It is quite likely that the way pupils are taught about fractions influences whether they interpret the fraction 1/3 in this situation both as "one divided by three" and as "one part out of three" or exclusively as "one part out of three".

Item 2 presents a situation often used in English primary schools in the teaching of fractions. The fraction of the rectangle that has been shaded is labelled by the number of shaded parts (3) as the numerator and the total number of parts (10), shaded and unshaded, as the denominator: 3/10. This type of item exemplifies *part-whole situations,* because of the relation expressed between the numerator (a part) and the denominator (the whole).

It could be argued that distinguishing between part-whole and quotient situations is really like splitting hairs: perhaps pupils can't really see any difference between the two. However, we hypothesise that pupils realise more easily that a whole number is not an adequate answer to item 1 than to item 2. They realise that there are fewer pies than girls, so the answer should not be a whole number. In contrast, in part-whole situations, pupils could simply count the number of shaded parts and ignore the unpainted parts.

An analysis of how children used numbers in these two situations to answer the question "what fraction?" supports this hypothesis. The percentages of correct responses to the two items are very similar – 35% and 31% correct responses, respectively, to items 1 and 2, combining the data for both year groups. The most frequent error to both items was a natural number. However, 21% of pupils used a natural number as a response to item 1 whereas 34% used a natural number as an answer to item 2. The difference between these two proportions was significant according to a binomial test. So, we conclude that pupils do realise more easily that a natural number is not an appropriate answer to item 1, even though they are not much better at giving the right answer to item 1 than to item 2.

The natural number responses to item 1 were varied: 12% of the pupils answered "3" (the number of children or the number of pieces into which the pie would have to be divided); 6% answered "1" (presumably referring to the number of pieces that each child would receive); and 3% answered either "2" or "4" (we are not sure how to interpret these answers but they could result from subtracting or adding the number of pies and number of girls, something that children do when they have no other idea of what to do with the numbers). Only 8% of the pupils left the item blank. The remaining 72% percent gave answers that were a fraction or something like a fraction. Some children (13%) wrote their answers in words, using expressions that indicated that they realised that fractions were called for but they did not know how to label the fraction. Examples of these answers are "3 big pieces", "3 halves", "3 quarters" and "quarters". Other children (22%) used fractional representation and offered a variety of wrong fractions as the answer: 9% of these were fractions using the numbers 1 and 3 (1/1, 3/1, and 3/3) and the remaining 13% used other fractions (1/2, 1/4, 3/4 etc.) which included numbers not immediately identifiable in the situation. Thus, only about one third of the pupils knew what the right fraction was, even though the majority realised that a natural number was not a correct answer. This difficulty that pupils aged 8 and 9 have in answering the question "what fraction?" is in stark contrast with young children's ability to answer the question "how many?"

In item 2, all the pupils (34%) who answered with a natural number simply counted the number of shaded parts – they answered "3". The next most frequent mistake was 1/3, where the number of shaded parts is used as the denominator and 1 as the numerator; this error was displayed by 11% of the pupils. This suggests that the pupils knew that they should be using a fraction and that the number "3" should be part of it, but they did not know what function the "3" should have when they were setting up the fractional number. Errors involving different combinations of the numbers 1, 3, 7 and 10, which appear in the situation, represented 5% of the total number of responses. Only 4% of the children left the item blank. This leaves 14% of the children so bewildered that they provided answers that included numbers that are not immediately perceived in the situation – such as 1/2, 1/4, 1/5, 6/8, 3/5 and 2/3.

In part-whole situations, such as the one illustrated in item 2, the whole is a continuous quantity. These situations are often distinguished from others, where the whole is a discrete quantity, such as those illustrated in *item 3*. Although there are similarities between these two situations, the reason for differentiating them becomes clear when we consider that there are two possible correct answers for item 3: 4/6 or 2/3. To arrive at 4/6, pupils can use the same reasoning as that used in part-whole situations: 4 (chocolate muffins) out of 6 (the total number of muffins, chocolate plus cream muffins). In order to obtain the answer 2/3, pupils must operate on the numbers that are immediately perceptible, 4 and 6, and realise that there is a simpler way of expressing the relation between them: 2 out of every 3 are chocolate muffins. Situations where it is necessary to operate on the numbers in order to identify the fraction that describes the quantity are referred to as *operator situations*. In English primary schools, operator situations are used often in teaching.

An analysis of the different responses to this item shows that 27% of the answers, combining the data across year groups, were correct: 20% of the children answered 4/6 and 7% answered 2/3. But the most frequent response – 45% of all responses - was the use of a natural number instead of a fraction. The number "4" was the most common answer (42%); the other natural numbers were "6" (total number of muffins) and "3". The next most frequent error was to use the number of chocolate muffins as the denominator and 1 as the numerator: 11% of all the responses. The percentage of children using a fraction with other combinations of the numbers 1, 2, 4 and 6 was 6% and only 2% left the item blank. About 13% of the children seem to have no idea how to represent the fraction of muffins that is chocolate flavoured, giving answers such as 1/3, 3/4, 1/8 and 6/8.

Item 4 is, in many ways, similar to item 3: both refer to a fraction of discrete quantities, so labelling them involves basically the same process. We only distinguish between these two types of situations because they refer to different types of quantities, a difference that matters on other occasions in which we pose problems about quantities represented by fractions. In item 3, when the set of muffins changes, the quantity represented by the fraction also changes: 2/3 of 6 muffins is different from 2/3 of 12 muffins. This type of quantity is termed *extensive quantity*. In item 4, the quantity does not change when the whole changes: if I make a shade of blue paint using half blue and half white paint, the shade of blue should be the same irrespective of whether I use 6 or 12 bottles of paint to make the mixture. This type of quantity is termed *intensive quantity*. Intensive quantities may seem unfamiliar to people when we present them in this context but they are very important in everyday life and in science education. Often the same situation involves both an intensive and an extensive quantity, as in the case of paint mixtures. A mixture of 3 bottles of white paint with 3 bottles of blue paint gives a *different amount of paint* than a mixture of 6 bottles of white and 6 of blue paint – the extensive quantity differs – but gives *the same shade*. The intensive quantity is the same.

The distinction between intensive and extensive quantities is also important for how we operate on these quantities. If a child ate 1/3 of a pie in the morning and 1/3 of a pie in the afternoon, the child ate, altogether, 2/3 of the pie. The fractions of the extensive quantity are added. If you make a mixture of paint which is 1/3 blue paint in the morning and add to it another mixture which is also 1/3 blue paint in the afternoon, the mixture is still made with 1/3 blue paint. The fractions of intensive quantities are not added when we put them together. Past research has shown that pupils often treat intensive quantities as if they were extensive quantities, and thus arrive at wrong conclusions about what happens when they are added. In the survey that we have been describing, we asked the pupils what would happen if we mixed two tins of paint which had the same shade of blue; they provided the response by pointing to a shade of blue which varied from very light blue to very dark blue. Only one pupil left the item blank; 37% correctly showed the same shade of blue; 4% indicated a lighter shade; the majority – 58% – chose a darker shade as resulting from the mixture of two tins of paint of the same shade. This suggests that they were treating the colour of the mixture in the same way that they would treat the amount of paint: when the two tins are mixed, they thought that there would be more of it and it would be darker. In fact, although there is more paint, the colour remains the same.

These differences between intensive and extensive quantities did not seem to affect radically the answers to the question "what fraction?" The percentage of correct responses to the two items is very similar – 27% and 37% correct, respectively, for item 3 and item 4. Both items show two types of correct responses, one which uses the numbers in the situations directly in the fractional representation and the other which uses the simpler fraction: 17% of the pupils wrote 1/2, 3% wrote "half", 16% wrote 3/6; one pupil wrote 3/6=1/2, displaying an awareness of the possibility of both answers. Although half is considered an easy fraction, the rate of correct responses in this item is unimpressive, particularly when we consider that pupils in both year groups had been taught about half in school.

For both items, the most frequent response was a natural number. In the case of item 4, 39% of the pupils used a natural number: 34% answered "3" (the number of bottles with blue paint) and 5% used other natural numbers (6, 5, 4, and 1). The error of using the number of blue bottles as the denominator and 1 as the numerator was also observed in this item, but with a slightly lower frequency (6%) than that observed for the previous items. In short, the difference between extensive and intensive quantities does not seem to affect significantly how well children can answer the question "what fraction?", even though this is a significant distinction when it comes to comparing and adding fractions.

In conclusion, this survey showed that, even after pupils have been taught procedures to answer the question "what fraction?", they still find it difficult to implement these procedures and identify the correct fraction to represent a quantity. This contrasts with the ease with which young children can answer the question "how many" and is a clear symptom of the problems that pupils have in forming even the simplest ideas about rational numbers. The rate of correct responses in identifying a fraction did not vary significantly across situations but we could not make systematic comparisons in this survey because the fractions that had to be named were not controlled for. However, it was possible to document that some situations signal more clearly to pupils than others that a natural number is not an appropriate answer: the error of producing a natural number as a response was least frequent in quotient situations, even though the children were no better at producing the right answer to these items, and most frequent in operator and intensive quantity situations, where approximately 40% of the pupils answered the question "what fraction?" with a natural number.

The importance of number labels for understanding numbers

The ability to answer the question "how many" is a weak criterion for assessing children's understanding of natural number. Although some have argued that this ability indicates that the child understands cardinality (e.g., Gelman & Gallistel, 1978), most researchers would support the need for further analyses of pupils' understanding of number, natural or rational.

In the domain of natural numbers, Piaget's (1952) pioneering work defines two criteria as essential for crediting children with the understanding of number: understanding equivalence and understanding order. He argued that 'number is at the same time both class and asymmetrical relation; it does not derive from one or the other of the logical operations, but from their union' (1952, p. ix).

Following Piaget's analysis of natural numbers, we suggest that it is necessary to investigate how children come to understand the logic of classes and the system of asymmetrical relations in the context of rational numbers.

In the case of natural numbers, learning to count and answer the question "how many" is a procedure that can help children both with the logic of classes and the understanding of order. If a child counts two sets of objects and both have 8 items, this could help the child understand that the two sets are equivalent – that is, belong to the class of sets with the same cardinal. Also, learning to count gives children help in understanding the ordering of these classes: the order in which we say the natural numbers is an ascending order.

The role of numerical signs in the domain of rational numbers is clearly different, because the signs cannot offer the same type of support to children. Fractions that refer to the same quantity can be designated by different words – one half, two quarters – and different written numerical signs – 1/2, 2/4. Moreover, one can refer to different quantities by using the same words or written numerical signs: half (or 1/2) of 8 and half (or 1/2) of 12 are not the same quantity. As indicated earlier on, rational numbers express relations, and so it is not sufficient to consider the signs in rational numbers in order to know what quantity they represent: it is necessary to interpret what type of relation is represented. If pupils focus their attention entirely on the signs, they could both miss the fact that two different fractions can be equivalent and also the fact that the same fraction might refer to different quantities.

Ordering natural numbers can be accomplished by perception or by counting. When two sets are visibly different in number – say, one has 3 and the other has 10 elements – perception suffices. When a perceptual judgment is not sufficient, the order of the counting words maps onto the order of the size of sets. In contrast, there are three cases that have to be considered in ordering fractions. If the denominator is the same in the two fractions, the larger the numerator, the larger is the fraction. If the numerator is the same, the larger the denominator, the smaller is the fraction. If both the numerator and the denominator differ, it is necessary to consider the fractions by means of proportional relations. The order of counting words could help in the first case, but of course success in items that require pupils to compare two fractions with the same denominator would not be sufficient to credit them with understanding the ordering of fractions. Previous research demonstrates that pupils have difficulty in ordering fractions when the numerator is the same and the denominator differs: they would have to think of an inverse relation between the denominator and the quantity represented by the fraction but they often do not, and judge the fraction with the larger denominator as "the larger number". Much research has documented this misconception both in the U.K. (see, for example, Hart, 1984; Kerslake, 1986) and the U.S. (Behr *et al.*, 1984).

These surveys in the U.K. were carried out about 20 years ago, and mostly with pupils aged 11 to 16 years. It is possible that changes in the curriculum and teaching methods in the last two decades have had an impact on pupils' understanding of fractions in primary school. So we included in our survey questions where pupils were asked to compare relatively simple fractions. The fractions were presented without any reference to a situational context; the pupils were asked first to tick a box to show whether the fractions indicated the same amount or different amounts. Those who ticked "different" were then asked to circle the bigger fraction. The fractions and the percentage of responses by year group are presented in Table 2.

Table 2: Percentage of correct responses by Year Group to items asking pupils to compare two fractions and percentage making the most frequent error.

Item	Year 4		Year 5	
	% correct	**Most frequent error**	**% correct**	**Most frequent error**
2/4 and 3/4	84	8% - same	82	14% - 2/4 bigger
1/3 and 1/2	31	47% - 1/3 bigger	59	35% - 1/3 bigger
1/3 and 2/4	76	10% - 1/3 bigger	82	10% - 1/3 bigger
2/4 and 3/6	15	66% - 3/6 bigger	45	41% - 3/6 bigger
1/2 and 3/6	19	63% - 3/6 bigger	45	41% - 3/6 bigger

We chose these fractions for comparison to allow us to assess the basis for the pupils' judgments. In the *first two items*, either the numerator or the denominator is the same for the fractions being compared. Pupils who base their judgment simply on the value of the numbers that differ would get the first item correct and the second one wrong, by making the mistake of saying that 1/3 is bigger than 1/2. Almost half of the pupils in Year 4 and about 1/3 of the pupils in Year 5 made this mistake. This shows that the percentage of correct responses to item 1 tells us little about pupils' understanding of fractions: they can get this answer correct for the wrong reason.

The percentage of correct answers to *item 3* also gives an optimistic view of pupils' understanding of fractions: both the numerator and the denominator vary, but it is likely that pupils get this item right not because they use proportional reasoning to compare the fractions but because they are still judging the value from the general impression caused by the numbers: many more pupils say that 2/4 is bigger than 1/3 than that 1/2 is bigger than 1/3 (item 2), although 1/2 and 2/4 are equivalent fractions. So the comparison between 1/3 and 2/4 gives a large proportion of "false positives" in the assessment of pupils' understanding of the order of fractions. We intentionally chose the equivalent fractions 1/2 and 2/4 for these items because this equivalence is emphasised in the teaching of fractions to Year 4 pupils. However, this did not prevent the same pupils from saying that 2/4 is more than 1/3 but 1/3 is more than 1/2.

In *items 4 and 5*, the fractions are equivalent; this should be a simple a comparison of equivalent fractions because both fractions are equivalent to 1/2. Past research has shown that 1/2 and its equivalent fractions are easier to understand than other families of equivalent fractions, regardless of whether this involves the comparison of numbers (Parrat-Dayan, 1985), visual quantities divided in half (Spinillo & Bryant, 1991), or in the context of more difficult concepts such as probability (Piaget & Inhelder, 1975). Yet, less than half of the pupils in Year 5 and less than 1/5 of the pupils in Year 4 succeeded in identifying these as equivalent fractions. A large proportion of responses still seemed to be based on the general impression given by the numbers, leading to the answer that 3/6 is bigger than 2/4 and also bigger than 1/2.

We confirmed this difficulty in Year 4 and 5 pupils' later in a larger survey, which involved 318 pupils from 8 schools in Oxford and London: only 28% of the pupils were able to indicate that 3/4 is greater than 3/5. Again in this survey, the pupils seemed to find it difficult to understand the inverse relation between the denominator and the size of the number, when the numerator is the same.

In short, the role that linguistic and mathematical signs play in supporting pupils' understanding of equivalence and ordering differs between natural numbers and fractions. Much more conceptual understanding is required for the understanding of equivalence and order in the domain of fractions, where procedures based on the number labels do not suffice. The survey suggests that pupils in Years 4 and 5 have considerable difficulty with very basic of ideas about number when they have to use them in the context of rational numbers. Their difficulty is manifested when they need to use a fraction to represent a quantity and also when they have to identify equivalences between fractions or order fractions by magnitude.

Should we conclude then that their difficulty is so basic that rational numbers should not be taught in primary school? This question is briefly addressed in the section that follows.

The role of situational contexts in understanding number concepts

Research with natural numbers shows that children succeed in solving addition and subtraction problems in simple situational contexts before they succeed in solving problems without reference to a context. Hughes (1981; 1986) was the first researcher to demonstrate this without ambiguity. The same children who could say how many bricks would be in a box if he put in 3 bricks and then 2 more were unable to say how much is 3 plus 2.

Our surveys suggest that the same applies to rational number: the same children perform better in tasks of fraction comparison in certain situations than they do when asked to compare fractions outside a situational context.

Using the same methodology of presenting the items with the support of pictures, we posed the following question: "One group of 5 children will share 2 cartons of orange juice; they will drink it all and share it fairly. There is a second group, with 4 children, who will also share 2 cartons of orange juice; they will also drink it all and share fairly. Will each child in one group drink as much as each child in the other group? If not, circle the group that will drink more." The cartons of orange juice in the pictures were identical to each other. The fractions that would represent these situations would be 2/5 and 2/4 – so, if the pupils followed the general impression given by the numbers, they would answer that the children in the group with 5 children would drink more. However, the rate of correct responses to this question was quite high, in comparison with the previous items: 92% and 88%, respectively, for years 4 and 5. Thus, in this division situation, the majority of the pupils realised that, the more children sharing, the less each one would receive. They do have an insight into the inverse relation between the number of recipients and the size of the share each one will get – the same insight which they need to compare fractions with the same numerator and different denominators.

This finding is not confined to our survey. Previous research on primary school pupils' understanding of the inverse relation between the divisor and the quotient has shown that this is a relation understood by the majority of 7-year-olds. Pupils show this knowledge both when the dividend is a discrete quantity (Correa, 1995; Correa, Nunes, & Bryant, 1998) and when it is a continuous quantity (Kornilaki & Nunes, 2005).

We also analysed the pupils' ability to compare fractions when the fractions were equivalent. One survey involved 130 pupils in Years 4 and 5, from 5 different schools in Oxfordshire. Our comparison focused on part-whole and quotient situations and both items were presented with the support of drawings. In the *part-whole item*, the pupils were told that Peter and Alan had identical chocolate bars. The bars were too large to be eaten at once so Peter cut his into 4 identical parts and ate 2. Alan cut his into 8 identical parts and ate 4. The pupils were asked to choose one of the following alternatives: Peter ate more chocolate than Alan; Alan ate more than Peter; they ate the same amount. In the *quotient item*, the pupils were told that two groups of children had identical pies to share. The first group had 4 children and they had 1 pie to share among themselves, which they shared fairly. The second group had 2 pies to share among 8 children, and they shared them fairly. The pupils were asked to choose one of the following alternatives: Each child in the first group eats more than each child in the second group; each child in the second group eats more than each child in the first group; the children in both groups eat the same amount of pie. The rate of correct responses to the part-whole item were 46% for the part-whole item and 77% for the quotient item. These percentages differ significantly according to a binomial test.

In our larger survey with 318 Year 4 and 5 pupils, we asked pupils to judge whether two fractions, 1/3 and 2/6, were equivalent or not. The items were presented simply as numbers, without context, and also in the context of part-whole and quotient situations. Pupils were most successful in quotient situations (68% correct), followed by part-whole situations (41% correct) followed by numerical problems without context (39% correct).

In summary, all the comparisons we have carried out so far indicate that the situation in which fractions problems are presented to pupils has a significant impact on the rate of correct responses. Our items have not covered all the possible contrasts, but the indication is that pupils perform better when comparing fractions in problems posed in quotient situations, followed by part-whole and then by problems with fractional numbers without context. Thus, pupils can use their understanding of the logic of a situation to support their problem solving efforts in the domain of rational numbers before they can work with the numbers outside a context.

Why pupils should know about rational numbers at the end of primary school

After discussing the difficulties that pupils have with rational numbers, it seems timely to ask: are rational numbers worth the trouble? We have come across teachers who strongly held the view that they are not. What would pupils be missing, if they leave primary school without a good understanding of rational numbers?

Many of the situations we encounter in everyday life and in work settings are best understood if they are represented by rational numbers. When we go to the supermarket, for example, we should be able to know the difference between the price of an item – i.e. how much we have to pay – and the cost of an item – i.e. its price relative to amount. Of course we pay more for 2 kilos of fish than for one kilo; the question is whether the relative cost of the two packages is the same. The cost per kilo of a package with 2 kilos might be lower than the cost per kilo of a package with 1 kilo, even if the price of the 2 kg package is greater. Government regulations help the shopper in this respect, if the shopper is willing to read the labels on shelves: supermarkets are now required to display the price per unit. So, cultural arrangements are such that the rational numbers could be avoided in judging best buys in the supermarket.

There are many situations where consumers are offered "cultural protection" against the need to understand rational numbers because the denominator is fixed. Fuel consumption – another use of rational numbers in everyday life – is rated in miles per gallon when a car is described, so all we need to do is consider the numerator – how many miles (per gallon). Speed is described in miles per hour, simplifying the problem for those who might not understand rational numbers. Pay rates are described per hour, another simplification of rational numbers. This cultural protection may allow us to avoid understanding, or to remain only partially aware of the value represented by the denominator in these rational numbers.

But it does not work always. Discounts on products are described in percentages – but here the simplification is not helpful, because 10% off on a more expensive product may still leave the product with a higher price than 8% off on a product that was less expensive to begin with.

Our culture has plenty of devices to simplify situations where we use rational numbers, so we are less aware of how often we encounter them in everyday life. However, once we have to calculate with rational numbers, our lack of understanding is revealed. Sometimes we can add percentages; sometime we can't. For example, we can find the percentage of pupils in a school who don't use motorised transport to go to school by adding those who walk with those who cycle, but we can't find the percentage of pupils who pass an item by adding the percentage of pupils who succeeded in Year 4 with the percentage of pupils who succeeded in Year 5. Why?

Sometimes we add the numerators and the denominators of rational numbers, sometimes we don't. If a team won 5/8 games in the first half of the championship and 6/8 in the second half, we can add the numerators and denominators, and find the team's rate of success – 11/16. But we can't add the numerator and the denominator of the fractions of pizza that we ate from the first and the second pizza that the waiter brought to find out what is the fraction of pizza that we ate altogether. Why?

In short, we can get by in many situations in life without knowing about rational numbers as long as we don't have to use them to calculate. When we use them to calculate, we have to know more than simple rules: we have to understand.

We think that the arguments for developing a real understanding of rational numbers are very compelling, once we begin to think of how pervasive they are in everyday life. However, the arguments are even more compelling in the case of scientific concepts. Many concepts in science involve the relational thinking that is represented by rational numbers, not by natural numbers. When making comparisons between cities, for example, we consider indicators such as recreation area per person, per capita income (income by number of people), number of GPs by population. We need rational numbers to make these comparisons because it would not be a "fair" comparison to consider the natural number that represents the total recreation area of the city, the total income received by everyone in the city, and the number of practising GPs (medical doctors) in the city. A city with a larger population is likely to have a greater recreational area, total income and more GPs than a small city, but may have a lower quality of life when these values are considered in relation to the population. What is of interest is the relative amount of recreation space, the income and number of GPs per person. So we make all the comparisons relative to the number of people in the city.

The need for a "fair" comparison between cities is easily understood – perhaps more easily understood in social science in general and in everyday life than in less familiar scientific contexts. But relative concepts are everywhere in science too. Concepts such as speed (distance travelled by unit of time), density (mass per volume), temperature (energy by mass), solution (amount of solute by amount of solvent), pressure (perpendicular force by area), and probability (number of favourable cases by total number of cases) – to name just a few – are all described by rational numbers because they involve the same sort of relational concepts. Many studies have shown that pupils often treat these concepts as if they were represented by natural numbers. In studies about temperature, for example, it has been shown that pupils add the temperature of two volumes of water when these are put together – and indicate that if water from two containers, each at the temperature 20º, is poured into a third container, the water in the third container will be at 40º (Stavy, Strauss, Orpaz, & Carmi, 1982; Driver, Guesne, & Tiberghien, 1985). Other studies show that pupils might focus on one of the variables and fail to attend to the other when the concept requires establishing a relation between the two: for example, young children attempt to explain buoyancy by focusing on size, and believe that larger objects sink while smaller ones float (or vice-versa, for other children), whereas slightly older children focus on mass (which they refer to as "weight"), and believe that heavy objects sink and light ones float. In both cases, they are dealing with a quantity that would be represented by a natural number rather than the relation between the two, which represents the intensive quantity, density (see Bryant et al, this volume).

Rational numbers are extremely important in science because so many quantities studied in science education are intensive, rather than extensive – and intensive quantities can only be represented by rational numbers. Pupils develop intuitive notions of intensive quantities in everyday life – see Howe *et al.* (this volume) – but the numerical representation of these concepts often eludes them, and so knowledge of them remains implicit. It is unlikely that such implicit knowledge, which cannot be represented mathematically, can provide a sufficient basis for pupils to extend their understanding of everyday intensive quantities to the less familiar scientific concepts. It can be hypothesised – but there is certainly no evidence for this so far – that mathematics teaching could offer a stronger basis for science learning if pupils became more aware of the relative nature of rational numbers and their connection to the quantification of intensive quantities.

To conclude, we suggest that the investment in helping pupils understand rational numbers is presently too low. Pupils start to learn to count in order to attach a natural number to a quantity from the time they start school. Five years later, when they are nine or ten years old, the focus of the curriculum is still on natural numbers and the operations with natural numbers. The teaching of decimal fractions is presented as an extension of the number system, to include units smaller than one. Pupils do not need to think of rational numbers in order to extend the conception of the number system to smaller and smaller units, which can be counted just like natural numbers. The teaching of decimals can be carried out exclusively in the context of extensive quantities – and it often is. For example, pupils can learn that the value after the decimal point refers to pence in the context of money and to decimetres and centimetres in the context of length. This is useful knowledge but does not challenge pupils' natural number concepts – it only extends them to the representation of smaller units, which, by convention, appear after the decimal point. This teaching of decimals can be successful but it does not provide a basis for the discussion of rational number concepts in the broader way that we have argued is necessary. It could reinforce the identification of rational numbers with the idea of "parts smaller than the whole", which is a step towards understanding rational numbers but does not reflect the relative nature of these numbers (Vosniadou ref here).

A solid understanding of rational numbers should allow pupils to connect them with at least the four situations which we described in the first section of this paper – quotient, part-whole, operator and intensive quantities. Our ongoing investigations, which cannot be described in detail here, suggest that teaching in one situation does not easily generalise to all the other situations. Pupils who were taught about fractions in the context of quotient situations made more progress than those who received the traditional classroom instruction on part-whole concepts, but only in solving problems in quotient and part-whole situations: they were not able to transfer their learning to operator and intensive quantity situations.

We believe that the evidence presently available suggests that pupils benefit from starting to think about rational numbers in the context of quotient situations. These should soon be connected with part-whole situations – and pupils often make these connections spontaneously when they discuss their solutions do different problems. However, our investigations show that they will need support to extend their understanding to operator and intensive quantities situations, which are more difficult and more easily misconceived as passive of representation by natural numbers. Progression in rational numbers should extend pupils' understanding to include operator and intensive quantities situations, which will prepare them for using this new conception of number in a variety of contexts. This progression would involve pupils in the study of rational numbers for at least the same five years that are presently dedicated to the study of natural numbers.

In summary

This analysis of rational numbers has focused on several aspects of the differences between natural and rational numbers. The first was related to the ability to use numbers to represent quantities. Children find it very easy to learn to count and use numbers to answer the question "how many?" but, even after formal instruction in school, they continue to have difficulties in using rational numbers to represent quantities and answer the question "what fraction?"

Once pupils learn to represent natural numbers, they can use this representation to help them understand the equivalence and order of quantities. Quantities represented by the same natural number are equivalent and those represented by different numbers are not. The role of signs – written notation and oral numbers – is different in the domain of rational numbers: two different fractions can represent the same amount (e.g. 1/2 and 2/4) and the same number can represent different quantities (e.g. 1/2 of 8 and 1/2 of 12). Thus teaching should not focus on signs as the basis for understanding fractions: understanding the logic of the situations represented is essential.

It was documented more that 20 years ago that children can solve addition and subtraction problems with natural numbers in the context of some situations before they can solve the same problems when these are presented simply as numbers, without reference to situations. We documented the same phenomenon about rational numbers: pupils were more successful in comparing fractions in the context of some situations than when the fractions were presented as numbers, without reference to a situation.

Four types of situations in which fractions are used were distinguished here: quotient, part-whole, operator, and intensive quantities. The basis for distinguishing between these situations was an analysis of what the numerator and the denominator of the fractions refer to and the importance of the whole. In quotient situations, the numerator refers to a quantity to be divided and the denominator refers to the divisor: for example, one pie to be divided by three children is represented as 1/3. In part-whole situations, the denominator refers to the total number of parts into which a whole was divided and the numerator refers to the number of parts taken: for example, if a figure was divided into 3 parts and 1 is painted, the fraction painted is 1/3. In operator situations, the numerator and the denominator indicate groups of discrete quantities: for example, if Ali has 24 marbles and lost 1/3, we divide 24 by 3 and multiply by 1 to find out how many marbles he lost. Intensive quantity situations use the relation between two extensive quantities, represented by the numerator and the denominator, to represent a third quantity, which is the intensive quantity; here, the whole does not matter. For example, a juice made with 1/3 orange concentrate and 2/3 water tastes the same irrespective of whether the total amount of juice is 6 or 24 cups.

Pupils find it easiest to recognise that quotient situations cannot be represented by a natural number, even if they do not know its exact representation. They might use words like "parts", "pieces", "halves" and "quarters" to show that they realise that a natural number is not a good answer in these situations. They use natural numbers more often in response to the question "what fraction?" in the context of the other three situations. They also find it easier to compare fractions in quotient situations than in the other three situations, irrespective of whether the fractions to be compared are equivalent or different. This suggests that pupils' insights into these situations could be used to support their learning of rational numbers.

Finally, it is emphasised that pupils should be given the opportunity to progress in their understanding of rational numbers by learning how to use them in all of these four situations. If they were to be exposed to only some of these, their concept would be restricted, because it is unlikely that they would generalise across situations spontaneously. Progress in understanding rational number concepts requires exploration of increasingly more difficult situations in which these numbers are used.

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(Article received: 8-4-2008; accepted: 15-5-2008)