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Labour Motivation: An Axiomatic Vector Model

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Título: Motivación en el trabajo: Un modelo de vectores axiomáticos.
Resumen: En el presente artículo se da una lista de axiomas necesarios para la construcción de una teoría matemática de la motivación humana. Se propone un modelo matemático de la motivación en el trabajo. La motivación se representa como un vector resultante de la motivación parcial generada por grupos específicos de necesidades. El modelo de Vroom se incluye en el modelo propuesto como ejemplo de motivación. Se establece una correlación entre los gastos de motivación, el nivel de motivación y el nivel de productividad.
Palabras clave: Motivación; motivación en el trabajo; manejo de la motivación; vector; necesidades; productividad

Abstract: The present article gives a list of axioms needed for building up a mathematical theory of human motivation. A mathematical model of labour motivation is proposed. Motivation is represented as a resulting vector of partial motivation generated by specific groups of needs. Vroom's model is included in the proposed model as instant motivation. A correlation between the expenses on motivation, the level of motivation and the level of productivity is established.
Key words: Motivation; labour motivation; motivation management; vector; needs; productivity.

Effective control of an economical or managerial phenomenon requires a mathematical model able to provide a reliable qualitative description of this phenomenon and to ensure quantitative forecasts of its characteristics. Our understanding of the nature of human motivation did gradually improve over the years (we have now many more or less consistent theories giving adequate descriptions of motivation processes), but a good mathematical model of labour motivation is not available. Actually the only example of mathematical approach towards this problem is Victor Vroom's model, but it is rather qualitative than mathematical, as it limits itself to only one formula:

$$M=RBW,$$

where
M – motivation;
R – employee's expectations that his/her efforts will lead to necessary results;
B – expectations that the results will lead to the expected remuneration;
W – expected value of the remuneration.

I would say that even this formula is not mathematical in the proper meaning of this word as it is extremely difficult to correctly measure all variables it includes. It is rather a quasi-mathematical way to explain to readers that motivation does depend on employee's expectations.

It is obvious that absence of mathematical tools in this field of human resources management is absolutely unacceptable nowadays as a mathematical model of motivation is necessary for motivation management. I will try to fill in this gap in the present paper.

First of all, it is essential to remember that a mathematical theory should be built on a basis of a set of axioms. A thorough analysis of motivation theories shows that all of them contain – either explicitly or implicitly – a set of basic rules. It is logical to consider these rules as motivation axioms and use them as a basis for a mathematical theory of labour motivation (I omit the detailed overview of existing motivation theories in order to spare time; a good description can be found in Mescon, Albert, and Khedouri (1992)), as well as in Wikipedia, article "Motivation". These axioms are as follows (we also will use the basic notion of needs):

1. Human behaviour is determined by human motivation – actually this is the general basis of the theory of human behaviour and therefore deserves to be mentioned as the first axiom.
2. Human motivation is determined by human needs. This hypothesis appears in virtually all motivation theories – both content and process ones. For example, it is completely clear that the variable "Expected value of the remuneration" in Vroom's theory simply means "Capacity to satisfy one's needs".
3. Human needs can be divided into separate more or less homogenous groups, each one of these groups contributing to the general motivation (generating a partial motivation). I will refer to these groups of needs as to motivation factors. Homogeneity means in this context that these needs have at least one criterion in common.

It is easy to deduct from the axioms 1-3 the following lemma 1.1: in general, the number of groups of needs influencing a person's behaviour in the moment t is superior to 0. Mathematically it can be expressed as follows: if N is the number of separate groups of needs, N>0, and if n is the number of groups of needs determining an individual's behaviour in the moment t, then

$$0 < n \leq N \ .$$

This lemma does not appear explicitly in any one of existing theories of motivation. It is easy to demonstrate that this lemma is correct: according to the axiom 1, all acts (and inaction) of a human being is determined by his/her motivation. Motivation, according to the axiom 2, depends on his/her needs. As people is constantly doing something, it means that their behaviour depends on at least one group of needs which is the lemma 1.1.

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4. Every group of needs can be described by two characteristics – intensity and importance. Intensity is the quantity of goods required to satisfy this need. Importance is the priority of a certain group of needs over other groups. These characteristics can be expressed in quantitative form, therefore, it is possible to build up a quantitative theory of motivation, so the importance of this axioms cannot be underestimated. The existence of intensity is obvious, the existence of importances can be easily deducted from any hierarchy of needs (like Maslow's one).
5. The general motivation directly depends on groups of needs (motivation factors) – that is, if the intensity and/or the importance of a given group of needs is growing, the overall motivation will grow too.
6. Groups of needs are independent from each other, in other words, there is generally no correlation between changes of importance/intensity of different groups of needs. It means that the overall motivation of a person can be described by an additive function.
7. Needs from each group can be satisfied by specific goods, typical for this group only. It means that goods (material, non-material or abstract) that satisfy needs belonging to a certain group of needs cannot satisfy needs from other groups (need in security cannot be satisfied by a hamburger, creative needs cannot be satisfied by purchase of trousers). I would like to highlight that this is true in general – in some specific cases goods can satisfy different needs. An obvious example is money that can be used for purchase of virtually all goods for all types of needs. A "Rolls-Royce" is not simply a car (that satisfies transport needs) – it can also satisfy one's needs in prestige and luxury. But in general different needs are satisfied by different goods. We can conclude thereof that different needs make people act differently in order to achieve different goals (we can see a clear connection between the axiom 6 and the axiom 7).
8. Contributions of all groups of needs (motivation factors) into overall motivation are described by the same mathematical functions (I could have said instead that the psychological mechanism of correlation between a need and the motivation is the same for all types of needs). We can draw this conclusion from the fact that no motivation theories – either content or process – suppose a specificity of any group of needs, motivation mechanisms are described the same way for all groups of needs.
9. The overall motivation is equal to zero in only one case – if all contribution of all groups of needs are equal to zero. In other words, if a partial motivation (generated by a group of needs) is equal to zero (that is, if its importance and/or necessity is/are equal to zero), it does not mean that the overall motivation will be equal to zero too. Conclusion – the overall motivation cannot be calculated by multiplication of partial motivations.
10. There are no universal constants describing motivation, that is, we cannot calculate motivation theoretically, not using empirical data.

All these principles should be respected in a theory of motivation. If we do not take into account ant one of them, the theory will be over-simplistic or simply incorrect.

- A practical model of motivation should:
- Give a correct description of group of needs (motivation factors) determining human behaviour;
- Explain (adequately enough) the mechanism of correlation between motivation factors and human behaviour (that is, explain the formation of motivation);
- Give a quantitative evaluation of human motivation;
- Provide a correlation between an employee's motivation and an employee's productivity.

From the mathematical point of view it is necessary to build up a model of motivation that should be as general as possible, not limited by specific characteristics of employees interviewed and that could be easily adapted to a concrete situation. Thus, let's suppose that human needs are divided into N groups – but not specifying what criteria we use to make this division (it is obvious that according to our goals and depending on the concrete situation needs can be divided in groups in different ways).

The contradiction between the practical requirement to provide descriptions of groups of needs and of mechanism of their influence on human motivation and the mathematical decision not to specify number of groups of needs and criteria used for subdivision (this decision is due to the necessity to make our theory as general as possible) – this contradiction in reality does not exist. It can be easily solved thanks to the following conclusions: a company may and should divide needs into groups according to its goals (I will propose below my own scheme of possible division), and, according to the axiom 8 above, the mechanism of influence of needs on human behaviour is the same, therefore, we can describe the mechanism for any group of needs using mathematical tools and then we will simply apply this model to other groups of needs).

If we use a physical analogy, we can describe the motivation as a force that incites people to act in a certain way. Force in physics is a vector and therefore has not only a quantitative measure (length) but also a direction. We supposed above that a man acts under influence of N groups of needs. I will mark the absolute value (the intensity of motivation factors) of these needs on axes of a Cartesian coordinate system $x_1x_{2...}x_N$. Each motivation factor will generate a partial motivation – that is a radius vector of a point corresponding to the intensity of a motivation factor. This passage from scalar representation to vector one reflects the fact that different motivation factors make people act in different ways (needs of a certain group cannot be satisfied by goods satisfying needs from other groups – axiom 7; vectors of partial motivation are oriented in different directions). The fact that values of different groups of needs are marked on different axes means that groups of needs are independent from each other (axiom 6). It's worth mentioning that radius vectors of partial motivation lie on positive parts of Cartesian axes, as from the economical point of view it is extremely difficult to imagine a person having needs with negative intensity (however, the model does allow introducing such needs). The overall motivation is calculated as a resulting vector of these partial motivations – or, in other words, the overall motivation is itself a vector described by a motivation vector. Let us introduce the following symbols:

m_i – value of the i-th motivation factor (a scalar value);

$\mathbf{M_i}$ – value of the motivation generated by the i-th motivation factor. Its value is equal to m_i, but it is a vector;

M –overall value of motivation (resulting motivation, is also a vector). Its absolute value is equal (in an N-dimension coordinate system):

$$M = \sqrt{\sum_{i=1}^{N} m_i^2} \quad (1)$$

The vector formula of the resulting motivation (the formula of the motivation vector) is:

$$\mathbf{M} = \sum_{\mathbf{i}=1}^{\mathbf{N}} \mathbf{M}_{\mathbf{i}} \quad (2)$$

The motivation vector in the chosen coordinate system is written as M (m_1, m_2, …, m_N).

But this approach just takes into account the absolute value of motivation factors (their intensity) – but not their importance. In order to introduce their importance into the model I will take correction coefficients varying from 0 to 1. The final formula for calculation of absolute value of motivation (after introduction of correction coefficients) is:

$$M = \sqrt{\sum_{i=1}^{N} \alpha_i m_i^2}, \quad (3)$$

α_t – coefficients reflecting the importance of a corresponding group of needs for an employee.

From the economical point of view the formula 3 is much more precise in describing human motivation (as it takes into account the axiom 4), but it does not correspond to the mathematical logic of presenting motivation as a vector in a Cartesian system of coordinates. However, this seeming inconsistency can be easily solved, as intensity of motivation factors can be marked on axes of a rectilinear – but not Cartesian – system of coordinates, that is (m_1, m_2, …, m_N) are rectilinear, but not Cartesian coordinates. The passage from rectilinear to Cartesian coordinates is made according to the following formula:

$$x_i = m_i \cos \varphi_i, \quad (4)$$

x_i – the i-th Cartesian coordinate;

m_i – the i-th rectilinear coordinate;

φ_i – angle between the corresponding axes of the Cartesian and the rectilinear systems of coordinates.

It is absolutely obvious therefore that

$$\alpha_i = \cos^2 \varphi_i, \quad (5)$$

The formula 4 takes into account only one projection of a rectilinear axis – however, in order to be mathematically correct, the formula should include projection on all N axes. But this fact, while being odd from the mathematical point of view, does not contradict the logic of the model, as, supposing that different groups of needs are independent, we cannot connect one partial vector with different axes. This phenomenon helps to understand the limits of this model and to propose ways for its development.

Therefore, all multiplicators from the formula 3 have a clear mathematical, economical and psychological meaning.

This model is developed on a basis of presentation of motivation as a vector in an N-dimension space. This N-dimension space can be defined as motivation space in which the motivation vector lies. The number N corresponds to the number of groups needs are divided to. This number is defined by researchers' and managers' goals and it is possible to study space with different N while studying motivation. It is necessary to mention that the number of groups of needs and criteria used for division have no influence on the model structure.

As vectors of partial motivations lies on axes of a Cartesian system of coordinates, the absolute value of the overall motivation will be inferior to the sum of absolute values of vectors of partial motivation (according to the formula 3). It reflects the fact that different needs lead people to different actions (as reflected by divergence of vectors of partial motivation), while the overall motivation is a compromise between different partial motivations.

Overall motivation of a group of people (group motivation) can be calculated as the vector sum of motivations of all members of the group. A company is interested in having maximum individual and group motivation of its employees. The absolute value will be maximal if all individual motivation vectors are collinear. It is possible to check if this condition is respected by calculating the angle between these vectors. As their coordinates are known, the angle can be found through their scalar product.

I will below show this calculation for 2 employees, but this method can be easily generalized for n employees.

$\mathbf{M^1}$, $\mathbf{M^2}$ – motivation vectors of the first and the second employee (their absolute values are M1, M2 respectively). Their coordinates in the N-dimension Cartesian motivation space are:

$$\mathbf{M}^2(X_1^2, X_2^2, ..., X_N^2)\ \mathbf{M}^1(X_1^1, X_2^1, ..., X_N^1) \quad (6)$$

It is important that

$$X_i^2 \equiv \alpha_i^2 m_i^2,\ X_i^1 \equiv \alpha_i^1 m_i^1, \quad (7)$$

β – angle between motivation vectors:

$$\beta = \arccos \frac{\sum_{i=1}^{N} X_1^1 X_2^2}{M^1 M^2} \qquad (8)$$

The angle β is a measure of divergence of motivation vectors of different employees, that is, a measure of qualitative difference of their needs (a measure of quantitative difference is the difference of their absolute values).

In order to effectively manage a group of employees it is very important to have a good idea about the group motivation. Each Cartesian coordinate of the group motivation vector $\mathbf{M_R}$ is calculated according to the following formula:

$$x_i^{M_R} = \sum_{j=1}^{K} \sum_{i=1}^{N} x_i^j = \sum_{j=1}^{K} \sum_{i=1}^{N} m_i^j \cos \varphi_i^j, \qquad (9)$$

K – number of employees in the group.

The formula of average motivation:

$$\overline{\mathbf{M}} = \frac{1}{K} \mathbf{M}_R \qquad (10)$$

The passing from individual and group motivation to average motivation is very important: it gives the manager a possibility to evaluate influence of different changes of values of motivation factors on motivation of an average employee without diving into analysis of all individual vectors. But this average approach gets closer to the real situation when motivation profiles of different employees get closer to each other (in other words, when divergence between their motivation vectors gets smaller). Motivation profile stands here for a list of intensities and importances of motivation factors of an employee.

It is possible to set up for every position within every department of company an ideal motivation vector, that is, the motivation vector of an ideal employee who completely share the company's goals, mission and expectations (importances of motivation factors for the company and for this employee are equal) and who is perfectly happy with values of motivation factors offered by the company (the intensities are equal too). The Cartesian coordinates of such vector are:

$$M\ (\alpha_1^{id} m_1^{id}, \alpha_2^{id} m_2^{id}, ..., \alpha_N^{id} m_N^{id}), \quad (11)$$

α_i^{id} – ideal (perfectly corresponding to the company's expectations) importance of the i-th motivation factor;

m_i^{id} – ideal (perfectly corresponding to the company's budget limitations) intensity of the i-th motivation factor.

The correspondence of motivation vectors of newly hired employees to the ideal vectors defined for their positions will help to ensure the maximal convergence between the company's and the employee's interests and to avoid motivation losses during the passing from individual motivation to group motivation.

Of course, the full convergence of company's expectations and employee's interests is a utopia, therefore the company should define the maximal acceptable angle β_{max} between the motivation vector of an employee and the ideal motivation vector defined for his position. After that the company can defined the corporate solid angle Ω_{sol} within which all employees' motivation vectors should be located and which is equal to $2\beta_{max}$. The ideal motivation vector goes through the centre of the N-dimension circle formed by this solid angle on the surface of a N-dimension sphere with the radius equal to the value of the ideal motivation vector. A possible indication for definition of the angle could be the fact that the employees' productivity reaches 100% when the convergence between their expectations of the company and the company expectations of employees reaches 60%. (Solomanidina and Solomanidin, 2005).

As indicated above, the precise set of groups of needs should be defined according to the study's tasks – different goals require different distribution of needs. However, in general it might be logical to divide needs into 3 major groups: material consumptive (related to satisfaction of material – physiological and social – needs), non-material consumptive (related with needs in respect, in good relations with colleagues, in good social climate at work) and creative (relative with need to realize one's potential).

I noted in the very beginning of this article that the only example of application of mathematical tools to motivation analysis was Vroom's formula. It would be logical to try to include it in the present model of motivation.

The proposed model of motivation is a long-term model – it is supposed that employees' motivation profiles remain unchanged during a relatively long period of time. It describes general aspirations of an employee but provides no information about his/her reaction to a concrete task. However, in practice it might be important to forecast employee's behaviour not only in long and middle term, but also in short term, even better – in a given moment of time. This goal could be reached thanks to the notion of instant motivation that is understood as the probability that the employee will perform necessary actions in order to fulfill the task he/she is in charge of. The formula is:

$$Mp=LSCP \qquad (12)$$

M_P – instant motivation (motivation in probabilistic form);

L – measure of how the task is hard to perform;

S – self-reliance of the employee, his/her belief in his/her forces and potential;

C – belief of the employee in justice and honesty of the person responsible for remuneration (this index is closely related to the degree of satisfaction of non-material consumptive needs of this employee within this company);

P – probability of the fact that the remuneration will satisfy this employee's needs (depends on how the remuneration offered corresponds to importances and intensities of this employee's motivation factors).

It is obvious that the formula 12 is absolutely equivalent to Vroom's formula indicated in the beginning of the article – the only difference is that the index R in Vroom's formula (expectations that the efforts will lead to the necessary results) is decomposed into L and S.

However, the most important question for the human resources management is not the absolute value of motivation of an employee, but the correlation between the motivation budget of an employee, his/her motivation and productivity.

First of all, we have to define the correlation between the motivation budget and an employee's motivation. It is logical to consider the salary as a stimulus, and we know that the correlation between stimuli and feeling is described by Weber-Fechner's law (Javorskij, and D'etlaf , 1979). If we extrapolate this approach, we will see that:

$$m_i = z \ln MB, \qquad (13)$$

m_i – intensity of the i-th motivation factor proposed by the company (as it is perceived by the employee);
z – a coefficient;
MB – motivation budget dedicated to the i-th motivation factor.

The correlation between overall motivation and the intensity of work (as the productivity depends on a set of specific factors) may be formalized this way:

$$I = AM_{ov}, \qquad (14)$$

I – intensity of work of an employee;
A – a coefficient;
M_{OV} – overall motivation.

It is very important to highlight that the formulae 13-14 need a special empirical study and are proposed here as a hypothesis only.

In full analogy with extinction process, the function z(t) will be exponential asymptotic (due to the fact that the employee gets accustomed o the constant values of intensities of motivation factors offered by the company and to the natural change of his/her priorities over the time, that is, due to the growth of divergence of his/her and company's interests):

$$z(t) = z_0 e^{-Yt} \qquad (15)$$

Z_0* - value of the multiplicator Z* in the moment t=0;
Y – constant of weakening of perception of motivation budget (its mathematical meaning – it is the inverse value to the period of time during which the multiplicator z decreases by e times).

The limits of this rule are defined by psychological and physical potential of the employee and minimal level of his/her needs.

Traditional approach to motivation management consists in modification of intensities of motivation factors offered by the employer (the employer decreases or increases the sets of goods transmitted to employees in exchange for their work and satisfying their needs). This approach can me defined as extensive. The approach concentrating on motivation management through modification of importances of motivation factors of individual employees can be considered intensive.

Decreasing or increasing the set of goods means for a company expenses or economy of a resource (money). Intensities of motivation factors can therefore be presented as functions of money.

Main problems of motivation management:

1. Replace one motivation factor by another ensuring that the motivation remains unchanged and calculate the economical effect of this replacement (additional expenses or economy of a resource R):

$$M = const \;, \alpha_i \delta m_i = -\alpha_k \delta m_k \text{, define } \delta R \qquad (15)$$

2. There is an amount of the resource R, that should be used to modify motivation factors so that the motivation increase be maximal. The inverse task: we have to reduce the amount of the resource R used for motivation so that the motivation decrease be minimal.

3. It is necessary to increase the motivation M by δM so that the expenses of the resource R be minimal (the inverse task – to reduce the motivation by a defined value so that the economy of the resource R be maximal)

$$\delta M = \sqrt{\sum_{i=1}^{N} \alpha_i^2 m_i^2 (R_{i0} + \delta R_i)} - \sqrt{\sum_{i=1}^{N} \alpha_i^2 m_i^2 (R_{i0})} = const; \delta R \rightarrow \min \qquad (16)$$

In order to give a quantitative evaluation of different groups of motivation factors and to ensure there comparison it would be logical to use special scales in which motivation factors are measured by experts' evaluation and weighing. These scales are not universal.

Importances of motivation factors can be measured through psychological tests and interviews in order to define employees' priorities. The recommended scale for importance is from 0 to 1.

Intensities of motivation factors can be measured on a scale from 0 to 100. In case of material consumptive factors (that are directly measured as the employee's wages) 0 is equal to zero wage, 100 – maximal wage existing within this company.

Calculation of intensities of non-material consumptive and creative groups requires clear definition of the structure of these groups. As soon as the precise list of needs composing each of these 2 groups is defined, each need gets a weight and then the degree of satisfaction of every need is measured from 0 to 100 (via interviews). After that the average weighed satisfaction of the corresponding group is calculated, and this average satisfaction represents the intensity of the corresponding motivation factor offered by the company. The respective formula is as follows:

$$m = \frac{\sum_{j=1}^{n} \psi_j \mu_j}{\sum_{j=1}^{n} \psi_j}, \qquad (17)$$

where

m – intensity of the motivation factor;
n – number of sub-needs;
ψ – weight of the corresponding sub-need;
μ –intensity of the corresponding sub-need.

This mathematical model of motivation is consistent and can provided HR professionals with a reliable tool for calculation and forecasts of motivation level (both individual and group). This model can also be easily expanded from the mathematical point of view, as basic axioms can be reviewed – this development will be offered in another paper.

References

Mescon, M.H., Albert, M., and Khedouri, F. (1992). Osnovy menedžm'enta. *Djelo*, Moskva.

Solomanidina, T.O., and Solomanidin, V.G. (2005). Motivacija trudovoj d'ejatel'nosti p'ersonala. *Žurnal upravl'enija p'ersonalom*. Moskva.

Javorskij, B.M., and D'etlaf, A.A. (1979). Spravočnik po fizik'e dl'a inžen'erov I stud'entov vuzov. *Nauka*. Moskva

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