UNIVERSIDAD DE MURCIA

ESCUELA INTERNACIONAL DE DOCTORADO

Integration, Geometry and Topology in
Banach Spaces

Integración, Geometría y Topología en
Espacios de Banach

D. Gonzalo Martínez Cervantes
2017

Doctoral Thesis

Integration, Geometry and Topology in Banach Spaces

Integración, Geometría y Topología en Espacios de Banach

Gonzalo Martínez Cervantes

supervised by

Antonio Avilés López
and
José Rodríguez Ruiz

Es habitual dedicar una página al comienzo de una tesis doctoral a las personas que de una forma u otra forman parte de ella. Me siento obligado a comenzar esta lista de agradecimientos con mis grandes mentores Antonio Avilés y José Rodríguez. El lector habrá leído en la contraportada de esta tesis que la misma ha sido supervisada por ellos. Realmente el verbo "supervisar" no hace justicia al trabajo que estas dos personas han hecho conmigo. A ellos les agradezco de corazón que me hayan guiado en cada paso y que hayan asumido un rol muchísimo más amplio que el de simples supervisores o tutores.

Doy las gracias también a todas las personas que convirtieron mi estancia en Polonia en una experiencia muy positiva y, en especial, a Piotr Koszmider por su hospitalidad y dedicación. También doy las gracias a Grzegorz Plebanek, sin el cual no existiría gran parte del contenido de esta tesis.

De igual manera agradezco a mi gente de la sala Lázaro Cánovas y, en general, a todas las personas de la facultad, la ayuda que siempre me han aportado cuando lo he necesitado.
Aprovecho también para agradecer el apoyo y los momentos de evasión tan gratos a mis amigos y familiares. Finalmente doy las gracias a Patricia por todo lo compartido y por haber estado siempre ahí.

Resumen

Esta memoria ha sido elaborada durante el período de disfrute de una beca FPU de la Universidad de Murcia. Una ayuda complementaria de dicho programa ha permitido al autor realizar una estancia en el *Institute of Mathematics of the Polish Academy of Sciences* en Varsovia (abril-junio de 2016).

Esta investigación también ha sido financiada parcialmente por los proyectos de investigación 19275/PI/14 de la Fundación Séneca - Agencia de Ciencia y Tecnología de la Región de Murcia y por el Ministerio de Economía y Competitividad y FEDER (proyecto MTM2014-54182-P).

La memoria recoge a lo largo de tres capítulos la investigación realizada por el autor durante su doctorado.

El primer capítulo está dedicado a la integral de Riemann para funciones tomando valores en un espacio de Banach. El estudio de la relación entre la integrabilidad Riemann y la continuidad en espacios de Banach comenzó en 1927 cuando L.M. Graves mostró en [Gra27] la existencia de una función integrable Riemann con valores en un espacio de Banach que no es continua en casi todo punto (c.t.p.). Surge así el siguiente problema:

> *Dado un espacio de Banach X, determinar condiciones suficientes y necesarias para la integrabilidad Riemann de una función $f\colon [0,1] \to X$.*

Un espacio de Banach X en el que cualquier función integrable Riemann $f\colon [0,1] \to X$ es continua en c.t.p. se dice que tiene la propiedad de Lebesgue (LP). Ningún espacio de Banach clásico de dimensión infinita, excepto ℓ_1, tiene la LP. La sección 1.2 recoge gran parte del conocimiento existente sobre la LP y su relación con otras propiedades. En particular, algunos resultados de K.M. Naralenkov, A. Pełczyńsky, G.C. da Rocha Filho, J. Bourgain and H. Rosenthal relacionan la LP con espacios de Banach asintóticamente ℓ_1 y la propiedad de Schur. En esta sección también estudiamos operadores que transforman funciones integrables Riemann en funciones continuas en c.t.p.

Por otro lado, A. Alexiewicz y W. Orlicz construyeron en 1951 una función integrable Riemann que no era continua en c.t.p. [AO51]. Un espacio de Banach X

se dice que tiene la propiedad débil de Lebesgue (WLP) si toda función integrable Riemann $f\colon [0,1] \to X$ es débil continua en c.t.p. Esta propiedad fue definida por primera vez en [Wan96]. Todo espacio de Banach con dual separable tiene la WLP y el ejemplo de [AO51] muestra que $\mathcal{C}([0,1])$ no tiene la WLP. Otros espacios con la WLP, como $L^1([0,1])$, aparecen en [CRSP10] y [WW01].

En la sección 1.3 presentamos nuevos resultados sobre la WLP. En particular, probamos que el espacio de James JT no tiene la WLP (Teorema 1.3.7) y estudiamos cuándo $\ell_p(\Gamma)$ y $c_0(\Gamma)$ tienen la WLP en el caso no separable (Teorema 1.3.13). Además, probamos que la WLP es estable bajo ℓ_1-sumas (Teorema 1.3.18) y como consecuencia obtenemos que el espacio $\mathcal{C}(K)^*$ tiene la WLP siempre que K sea un espacio compacto de la clase MS, es decir, siempre que toda medida de probabilidad regular y de Borel sobre K sea separable (Corolario 1.3.21).

Alexiewicz y Orlicz también aportaron en [AO51] un ejemplo de una función débil continua la cual no es integrable Riemann. V. Kadets probó en [Kad94] que un espacio de Banach X tiene la propiedad de Schur si y sólo si toda función continua $f\colon [0,1] \to X$ es integrable Riemann. C. Wang y Z. Yang extendieron este resultado en [WY00] a cualquier topología localmente convexa más débil que la topología de la norma. En la sección 1.4 damos una versión cuantitativa de estos resultados que, en particular, responde postivamente una pregunta formulada por M.A. Sofi en [Sof12].

Parte del material de este capítulo se encuentra publicado en [MC16b].

En el segundo capítulo estudiamos algunas clases de espacios compactos asociadas a clases de espacios de Banach. En particular estudiamos la clase de los compactos débil Radon-Nikodým (WRN) y su relación con las clases de compactos Radon-Nikodým, compactos de Corson y compactos de Eberlein. Un espacio compacto es WRN si es homeomorfo a un subconjunto compacto en la topología débil* del dual de un espacio de Banach sin copias de ℓ_1. Es inmediato que todo compacto Radon-Nikodým es WRN. La clase de compactos WRN fue estudiada por E. Glasner y M. Megrelishvili en [GM12] y [GM14]. Sin embargo, esta clase de compactos también fue previamente considerada por C. Stegall en [Ste90] bajo el nombre de compactos *dual non* ℓ_1. Además, esta clase coincide con la clase de espacios compactos asociada a la clase de espacios de Banach débilmente precompactamente generados considerada por R. Haydon en [Hay81] (véase la sección 2.1 y el corolario 2.1.6).

La clase de espacios compactos WRN está íntimamente relacionada con la clase de compactos Radon-Nikodým; ambas clases tienen una caracterización en términos de fragmentabilidad (cf. sección 2.2). Una pregunta habitual a la hora de estudiar una clase de espacios compactos es su estabilidad mediante imágenes continuas. Son resultados clásicos que las clases de espacios compactos de Eberlein y las clases de espacios compactos de Corson son cerradas bajo imágenes continuas. E. Glasner and M. Megrelishvili preguntaron en [GM14] si la imagen continua de un compacto WRN es WRN. La pregunta análoga para compactos Radon-Nikodým fue planteada en 1987 por I. Namioka en [Nam87]. La pregunta de Namioka provocó la aparición en la literatura de varias superclases de la clase de imágenes continuas de compactos Radon-Nikodým, e.g. la clase de compactos numerablemente inferiormente fragmentables, la clase de compactos Quasi Radon-Nikodým y la clase de los compactos fuertemente fragmentables. Sin embargo, I. Namioka [Nam02] y A. Avilés [Avi05] probarón que estas tres clases son realmente la misma, aunque sigue siendo un problema abierto si la clase de compactos Quasi Radon-Nikodým coincide con la clase de imágenes continuas de compactos Radon-Nikodým.

De manera similar, en la sección 2.3 definimos la clase de los compactos Quasi WRN, que también es una superclase de la clase de imágenes continuas de compactos WRN. Al igual que sucede con la clase de los compactos Radon-Nikodým y los compactos Quasi Radon-Nikodým, probamos que esta clase es estable bajo imágenes continuas (Teorema 2.3.6) y que todo compacto cero-dimensional Quasi WRN es WRN (Teorema 2.3.8). Además, la sección 2.4 contiene algunos resultados y ejemplos de compactos WRN y compactos Quasi WRN que relacionan estas clases con las clases de compactos de Eberlein, Corson y Radon-Nikodým. En particular, en esta sección se incluye un ejemplo de un espacio compacto Corson y WRN que no es Eberlein y se prueba que todo espacio de Filippov es WRN.

La pregunta de Namioka fue finalmente respondida negativamente por A. Avilés y P. Koszmider en [AK13]; dieron una construcción de una imagen continua de un compacto Radon-Nikodým que no era Radon-Nikodým. Usando esta construcción aportamos una respuesta negativa a la pregunta de Glasner y Megrelishvili en la sección 2.5. Concretamente, una modificación de esta construcción da un ejemplo de una imagen continua de un compacto Radon-Nikodým que no es WRN.

Un espacio compacto WRN no es necesariamente sucesionalmente compacto. R. Haydon preguntó en [Hay81] si cualquier compacto infinito WRN contiene una sucesión convergente no trivial. Esta pregunta motiva el estudio de los compactos cero-dimensionales WRN desde el punto de vista de las álgebras de Boole (Sección 2.6). En particular, decimos que un álgebra de Boole es WRN si el compacto asociado mediante la dualidad de Stone es WRN. La proposición 2.6.3 caracteriza esta clase de álgebras y nos ayuda a entender mejor su comportamiento. Por ejemplo, una consecuencia inmediata es que un compacto WRN no puede contener copias de $\beta\mathbb{N}$. Esta y otras propiedades de las álgebras de Boole WRN son compartidas con la clase de las álgebras de Boole mínimamente generadas. Además, las álgebras de Boole mínimamente generadas han resultado ser una herramienta muy útil durante las últimas décadas para construir, bajo ciertos axiomas adicionales, ejemplos de espacios compactos infinitos sin sucesiones convergentes no triviales.

No obstante, a pesar de las propiedades similares que comparten la clase de álgebras de Boole WRN y la clase de álgebras de Boole mínimamente generadas, concluímos que ambas clases son incomparables (véase la sección 2.7).

En la sección 2.8 introducimos dos subclases de la clase de álgebras de Boole WRN; la clase de las álgebras uniformemente WRN y la clase de las álgebras fuertemente WRN. Estas clases de álgebras de Boole son el resultado de endurecer la caracterización de las álgebras de Boole WRN obtenida en la proposición 2.6.3. A lo largo de esta sección obtenemos algunos resultados que relacionan estás álgebras con la clase de álgebras mínimamente generadas y probamos que ambas clases son incomparables.

Finalmente, en la sección 2.9 estudiamos la existencia de ciertas medidas en estas subclases y en la clase de compactos WRN motivados por un resultado de J. Rodríguez que establece que toda medida de probabilidad regular y de Borel en un compacto WRN es separable (Proposición 2.9.9).

Los resultados principales de este capítulo pueden encontrarse en [MC15] y en el trabajo [AMCP16] realizado conjuntamente por A. Avilés, G. Plebanek y el autor.

En el tercer capítulo estudiamos distintas propiedades secuenciales de espacios topológicos. La más restrictiva de ellas es la propiedad de Fréchet-Urysohn (FU). Un espacio topológico es FU si todo punto en la clausura de un subespacio es el límite de una sucesión en el subespacio. Dos propiedades más débiles son la secuencialidad y la estrechez numerable. Un espacio topológico es secuencial si todo subespacio sucesionalmente cerrado (es decir, cerrado a través de límites de sucesiones convergentes) es cerrado. Por otro lado, un espacio topológico tiene estrechez numerable si todo punto en la clausura de un subespacio está también en la clausura de un subconjunto numerable del subespacio. Puede probarse fácilmente que todo espacio topológico secuencial tiene estrechez numerable (Lema 3.1.1). Además, si estas dos propiedades son iguales sobre la clase de los espacios topológicos compactos es indecidible en ZFC. Si nos restringimos a la clase de espacios topológicos compactos, entonces secuencialidad también implica compacidad secuencial. A lo largo de este capítulo se estudia cuándo la bola dual de un espacio de Banach con la topología débil* tiene alguna de estas propiedades. También consideramos versiones convexas de todas estas propiedades. En la página 68 puede verse un diagrama con las relaciones entre estas y otras propiedades.

En la sección 3.2 estudiamos aquellos espacios de Banach cuya bola dual es débil*-sucesionalmente compacta. J. Diestel dedicó el Capítulo XIII de su libro [Die84] a este tema, donde escribió lo siguiente:

> *Hasta la fecha no hay ninguna caracterización de aquellos espacios de Banach X que no tienen bola dual débil*-sucesionalmente com-*

pacta. Además, parece que ninguna de las clases de espacios de Banach estudiadas en la actualidad ofrece ninguna esperanza de ser un candidato viable para la caracterización de los espacios con bola dual débil-sucesionalmente compacta.*

Por supuesto, cualquier espacio de Banach separable tiene bola dual débil*-metrizable y por tanto débil*-sucesionalmente compacta. Es también inmediato que todo espacio de Banach reflexivo tiene bola dual débil*-sucesionalmente compacta. Más aún, una consecuencia del Teorema ℓ_1 de Rosenthal es que todo espacio de Banach cuyo dual no contiene a ℓ_1 tiene bola dual débil*-sucesionalmente compacta. Además, todo espacio de Banach WLD y todo espacio de Asplund tienen bola dual débil*-sucesionalmente compacta.

Por otro lado, si un espacio de Banach contiene a ℓ_∞ o a $\ell_1(\mathfrak{c})$ entonces su bola dual con la topología débil* contiene una copia homeomorfa de $\beta\mathbb{N}$ y por tanto no es débil*-sucesionalmente compacta. Además, J. Hagler y E. Odell e independientemente R. Haydon dieron ejemplos de espacios de Banach sin copias de ℓ_1 que no tienen bola dual débil*-sucesionalmente compacta. Parece por tanto que las palabras de Diestel siguen siendo válidas en el presente; estamos todavía lejos de encontrar una caracterización de los espacios de Banach con bola dual débil*-sucesionalmente compacta. En la sección 3.2 tratamos este tema atendiendo a las clases de compactos estudiadas en el segundo capítulo. Finalmente nos centramos en la pregunta de Haydon sobre la existencia de compactos infinitos WRN sin sucesiones convergentes no triviales, obteniendo una respuesta negativa en el caso de los compactos asociados a la clase más restrictiva de álgebras de Boole uniformemente WRN (corolario 3.2.5) También consideramos en esta sección espacios de Banach con bola dual débil*-compacta bloque convexa, que no es más que la versión convexa de la compacidad secuencial. En particular, el Teorema 3.2.11 muestra la relación entre esta propiedad y otras propiedades mencionadas anteriormente.

La sección 3.3 se centra en los espacios de Banach con bola dual débil*-secuencial. Puede verse fácilmente que todo espacio de Banach WLD tiene bola dual débil*-secuencial (incluso débil*-FU). En esta sección damos condiciones suficientes para que un espacio de Banach tenga bola dual débil*-secuencial.

Estos resultados son aplicados en la sección 3.4 para contestar una pregunta de A. Plichko sobre la existencia de espacios de Banach con bola dual débil*-secuencial sin dual débil*-angelical. Concretamente probamos que estos resultados se pueden aplicar para obtener que el espacio de Johnson-Lindenstrauss JL_2 tiene bola dual débil*-secuencial con orden secuencial 2 (Teorema 3.4.2).

Otra consecuencia de los resultados obtenidos es que $\mathcal{C}(K)$ tiene bola dual débil*-secuencial para todo espacio compacto disperso K de altura numerable (véase el teorema 3.4.5). Sin embargo, es un problema abierto si existe en ZFC un espacio compacto secuencial de orden estrictamente mayor que 2. Usando una construcción de A.I. Baškirov y otra de A. Dow concluímos la consistencia de la existencia de espacios de Banach con bola dual débil*-secuencial de orden mayor que 2 y de órdenes numerables arbitrariamente grandes (véase el corolario 3.4.6).

Parte del material de este capítulo puede encontrarse en [MC16a] y [AMCP16].

En el segundo capítulo se han usado algunos resultados combinatorios que, por su interés, hemos decidido incluir en este trabajo en los apéndices finales A y B.

Concretamente en el Apéndice A se prueban algunas propiedades del árbol de Todorcevic que, junto con el Lema 2.4.7, ayudan a construir un ejemplo de espacio compacto WRN y Corson que no es Eberlein. En el Apéndice B se demuestra el Lema de Sauer-Shelah, que resulta imprescindible para la demostración del Teorema 2.6.6.

Abstract

This work gathers in three chapters the research done by the author during his PhD Thesis. During this period the author was supported by a FPU grant from the University of Murcia. A fellowship from the same program allowed the author to carry out a stay in the *Institute of Mathematics of the Polish Academy of Sciences* in Warsaw from April to June 2016.

This research was also partially supported by the research project 19275/PI/14 funded by Fundación Séneca - Agencia de Ciencia y Tecnología de la Región de Murcia within the framework of PCTIRM 2011-2014 and by Ministerio de Economía y Competitividad and FEDER (project MTM2014-54182-P).

The first chapter examines the Riemann integral of vector-valued functions. The study of the relation between Riemann integrability and continuity in Banach spaces started in 1927, when Graves showed in [Gra27] the existence of a vector-valued Riemann integrable function not continuous almost everywhere (a.e. for short). Thus, the following problem arises:

> *Given a Banach space X, determine necessary and sufficient conditions for the Riemann integrability of a function $f\colon [0,1] \to X$.*

A Banach space X for which every Riemann integrable function $f\colon [0,1] \to X$ is continuous a.e. is said to have the *Lebesgue property* (LP for short). All classical infinite-dimensional Banach spaces except ℓ_1 do not have the LP. Section 1.2 surveys the knowledge about the LP and its relation with other properties. In particular, we include some results from K.M. Naralenkov, A. Pełczyńsky, G.C. da Rocha Filho, J. Bourgain and H. Rosenthal which relate the LP to spreading models, asymptotic ℓ_1 Banach spaces and the Schur property. In this section we also study operators which transform Riemann integrable functions in a.e. continuous functions.

On the other hand, A. Alexiewicz and W. Orlicz constructed in 1951 a Riemann integrable function which is not weakly continuous a.e. [AO51]. A Banach space X is said to have the weak Lebesgue property (WLP for short) if every Riemann integrable function $f\colon [0,1] \to X$ is weakly continuous a.e. This property was

introduced in [Wan96]. Every Banach space with separable dual has the WLP and the example of [AO51] shows that $\mathcal{C}([0,1])$ does not have the WLP. Other spaces with the WLP, such as $L^1([0,1])$, can be found in [CRSP10] and [WW01]. In Section 1.3 we present new results on the WLP. In particular, we prove that the James tree space JT does not have the WLP (Theorem 1.3.7) and we study when $\ell_p(\Gamma)$ and $c_0(\Gamma)$ have the WLP in the nonseparable case (Theorem 1.3.13). Moreover, we prove that the WLP is stable under ℓ_1-sums (Theorem 1.3.18) and we apply this result to obtain that $\mathcal{C}(K)^*$ has the WLP whenever K is a compact space in the class MS, i.e. whenever K is a compact space such that every regular Borel probability measure on K is separable (Corollary 1.3.21).

Alexiewicz and Orlicz also provided in [AO51] an example of a weakly continuous non-Riemann integrable function. V. Kadets proved in [Kad94] that a Banach space X has the Schur property if and only if every weakly continuous function $f\colon [0,1] \to X$ is Riemann integrable. C. Wang and Z. Yang extended this result in [WY00] to arbitrary locally convex topologies weaker than the norm topology. In Section 1.4 we give an operator theoretic form of these results that, in particular, provides a positive answer to a question posed by M.A. Sofi in [Sof12].

The material of this chapter is published in [MC16b].

In the second chapter we study some classes of compact spaces associated to classes of Banach spaces. In particular we study the class of weakly Radon-Nikodým (WRN) compact spaces and its relation with the classes of Radon-Nikodým compacta, Corson compacta and Eberlein compacta. A compact space is WRN if it is homeomorphic to a weak*-compact subset of the dual of a Banach space containing no copy of ℓ_1. Obviously every Radon-Nikodým compact space is WRN. The class of WRN compact spaces was studied by E. Glasner and M. Megrelishvili in [GM12] and [GM14]. Nevertheless, this class of compact spaces was also considered by C. Stegall in [Ste90] under the name of *dual non* ℓ_1 compact spaces. Moreover, this class coincides with the class of compact spaces associated to weakly precompactly generated Banach spaces considered by R. Haydon in [Hay81] (cf. Section 2.1 and Corollary 2.1.6).

The class of WRN compact spaces is closely related to the class of Radon-Nikodým compact spaces; both classes have a characterization in terms of fragmentability (cf. Section 2.2). E. Glasner and M. Megrelishvili asked in [GM14] whether the continuous image of a WRN compact space is WRN. The analogous question for Radon-Nikodým compact spaces was posed in 1987 by I. Namioka [Nam87]. Namioka's question caused the apparition in the literature of several superclasses of the class of continuous images of Radon-Nikodým compact spaces, e.g. the class of countably lower fragmentable compact spaces, the class of quasi

Radon-Nikodým compact spaces and the class of strongly fragmentable compact spaces. Nevertheless, I. Namioka [Nam02] and A. Avilés [Avi05] proved that these classes are all the same. However, it is still an open problem whether every Quasi Radon-Nikodým compact space is a continuous image of a Radon-Nikodým compact space.

In a similar way, in Section 2.3 we define the class of Quasi WRN compact spaces. We prove that this class is stable under continuous images (Theorem 2.3.6) and that every zero-dimensional Quasi WRN compact space is WRN (Theorem 2.3.8). Moreover, Section 2.4 contains some results and examples of WRN and Quasi WRN compact spaces that relates these classes to the classes of Eberlein, Corson and Radon-Nikodým compacta. In particular, this section includes an example of a Corson WRN compact space which is not Eberlein and it is also proved that Filippov spaces are WRN.

Namioka's question was answered negatively by A. Avilés and P. Koszmider in [AK13]; they constructed a continuous image of a Radon-Nikodým compact space which is not Radon-Nikodým. Using this construction we are able to provide a negative answer to Glasner and Megrelishvili's question in Section 2.5. Namely, this construction provides an example of a continuous image of a Radon-Nikodým compact space which is not WRN.

R. Haydon asked in [Hay81] whether every infinite WRN compact space contains a nontrivial convergent sequence. This question motivates us to study WRN compact spaces from the Boolean algebra setting (Section 2.6). Namely, we say that a Boolean algebra is WRN if the compact space associated through the Stone duality is WRN. Proposition 2.6.3 provides a useful characterization of this class of Boolean algebras. In particular, it follows that a WRN compact space cannot contain a copy of $\beta\mathbb{N}$. The class of minimally generated Boolean algebras share some properties (like not containing $\beta\mathbb{N}$) with the class of WRN Boolean algebras. Furthermore, minimally generated Boolean algebras provide an important tool for constructing infinite compact spaces containing no nontrivial convergent sequences. Nevertheless, despite the similarities between these classes of Boolean algebras, we conclude in Section 2.7 that both classes are uncomparable.

In Section 2.8 we introduce two subclasses of the class of WRN Boolean algebras; the class of uniformly WRN algebras and the class of strongly WRN algebras. These classes appears in a natural way by strengthening the characterization of WRN Boolean algebras obtained in Proposition 2.6.3. Throughout this section we obtain some results relating these classes to the class of minimally generated Boolean algebras and we prove that both subclasses are uncomparable.

Finally, in Section 2.9 we study measures on these subclasses and on the class of WRN compact spaces motivated by a result of J. Rodríguez which states that every regular Borel probability measure on a WRN compact space is separable (Proposition 2.9.9).

The main results of this chapter correspond to [MC15] and to the paper [AMCP16], which is a joint work with A. Avilés and G. Plebanek.

In the third chapter we study sequential properties. A topological space is said to be Fréchet-Urysohn (FU) if the closure of every subspace coincides with the set of limits of sequences in the subspace. A generalization of FU property are sequentiality and countable tightness. A topological space is sequential if every nonclosed subspace contains a sequence converging to a point which is not in the subspace. On the other hand, a topological space has countable tightness if the closure of every subspace coincides with the union of closures of countable subsets of the subspace. It can be easily checked that every sequential topological space has countable tightness (Lemma 3.1.1). Furthermore, whether every compact space with countable tightness is sequential is known as the Moore-Mrowka Problem and it is undecidable in ZFC. Notice that every sequential compact space is sequentially compact. Throughout this chapter we study Banach spaces whose dual ball with the weak*-topology has some of these properties. We also consider convex versions of the previous properties. A diagram with the relations among these properties can be seen in page 68.

In Section 3.2 we study Banach spaces with weak*-sequentially compact dual ball. J. Diestel devoted Chapter XIII of [Die84] to this topic, where he wrote the following:

> *To date, there is no characterization of those Banach spaces X having weak* sequentially compact dual balls. Furthermore, it appears that none of the classes of Banach spaces presently under study offers any hope of a viable candidate for the characterization of spaces with sequentially compact dual balls.*

Of course, every separable Banach space has weak*-metrizable dual ball and therefore weak*-sequentially compact dual ball. It is also immediate that every reflexive Banach space has weak*-sequentially compact dual ball. It follows from Rosenthal's ℓ_1 Theorem that if a dual Banach space X^* does not contain ℓ_1 then X has weak*-sequentially compact dual ball. Moreover, every WLD and every Asplund space have weak*-sequentially compact dual ball.

On the other hand, if a Banach space X contains ℓ_∞ or $\ell_1(\mathfrak{c})$ then the dual ball of X contains a homeomorphic copy of $\beta\mathbb{N}$ and therefore it is not weak*-sequentially compact. Furthermore, J. Hagler and E. Odell and, independently, R. Haydon provided examples of Banach spaces without a copy of ℓ_1 whose dual ball is not weak*-sequentially compact. It seems that Diestel's words remain valid at present; we are still far from finding a characterization of those Banach spaces with weak*-sequentially compact dual ball. In Section 3.2 we treat this topic looking at those classes of compact spaces studied in the second chapter and finally we focus on Haydon's question concerning the existence of infinite WRN compact spaces without nontrivial convergent sequences. In particular we obtain a negative answer for the class of compact spaces associated to the class of uniformly WRN algebras (Corollary 3.2.5). We also consider Banach spaces with weak*-convex block compact dual ball, which can be seen as the convex version of sequential compactness. In particular, Theorem 3.2.11 shows the relation among this property and the properties mentioned above.

In Section 3.3 we focus on Banach spaces with weak*-sequential dual ball. It can be easily seen that every WLD Banach space has weak*-sequential dual ball (even weak*-FU). We also provide sufficient conditions for a Banach space to have weak*-sequential dual ball.

These results are applied in Section 3.4 to answer a question of A. Plichko about the existence of Banach spaces with weak*-sequential dual ball which do not have weak*-angelic dual. Namely, we show that these results can be applied to prove that the Johnson-Lindenstrauss space JL_2 has weak*-sequential dual ball with sequential order 2 (Theorem 3.4.2). Moreover, we obtain that $\mathcal{C}(K)$ has weak*-sequential dual ball whenever K is a scattered compact space with countable height (Theorem 3.4.5). Nevertheless, it is an open problem whether there exist in ZFC compact spaces of sequential order greater than 2. Using constructions of A.I. Baškirov and A. Dow we conclude the consistency of the existence of Banach spaces with weak*-sequential dual ball with sequential order greater than 2 (Corollary 3.4.6).

The material of this chapter is based on [MC16a] and [AMCP16].

Some interesting combinatorial results used along the second chapter have been included in Appendices A and B of this work. Namely, in Appendix A we prove some properties of the Todorcevic tree that, together with Lemma 2.4.7, provide an example of a WRN and Corson compact space which is not Eberlein. In Appendix B it is proved the Sauer-Shelah Lemma, which is used in the proof of Theorem 2.6.6.

Terminology

For any set A in a topological space, we denote by $\overline{A}$ the closure of A and by $Int(A)$ the interior of A. The density character $\mathrm{dens}(T)$ of a topological space T is the minimal cardinality of a dense subset.

$\mathfrak{c}$ denotes the cardinality of the continuum. By ω we denote the least infinite ordinal and by ω_1 the least uncountable ordinal.

All Banach spaces are assumed to be real. In what follows, X^* denotes the dual of a Banach space X. The weak and weak* topologies of a Banach space will be denoted by w and w^* respectively. For any set A in a Banach space, $\mathrm{span}(A)$ denotes the linear subspace generated by the elements of A and by $\overline{\mathrm{span}}(A)$ we denote $\overline{\mathrm{span}(A)}$.

Let $(x_i)_{i=1}^\infty$ be a sequence in a Banach space X. $(x_i)_{i=1}^\infty$ is said to be normalized if $\|x_i\| = 1$ for every $i \in \mathbb{N}$. $(x_i)_{i=1}^\infty$ is said to be a basis in X if for every $x \in X$ there is a unique sequence of scalars $(a_i)_{i=1}^\infty$ such that the series $\sum a_i x_i$ converges to x in norm. If $(x_i)_{i=1}^\infty$ is a basis in X, then there are functionals $(x_i^*)_{i=1}^\infty$ in X^* such that $x_i^*(x_j) = 0$ if $i \neq j$ and $x_i^*(x_i) = 1$ for every i. In this case, $(x_i^*)_{i=1}^\infty$ are called the biorthogonal functionals of $(x_i)_{i=1}^\infty$. A basis $(x_i)_{i=1}^\infty$ is said to be shrinking if the biorthogonal functionals $(x_i^*)_{i=1}^\infty$ are a basis in X^*. If $(x_i)_{i=1}^\infty$ is a basis in $\overline{\mathrm{span}}\{x_i : i \in \mathbb{N}\}$, then we say that $(x_i)_{i=1}^\infty$ is a basic sequence.

The series $\sum x_i$ is said to be unconditionally convergent if for every sequence $\varepsilon_i \in \{-1, 1\}$, the series $\sum \varepsilon_i x_i$ converges. It is said to be absolutely convergent if $\sum \|x_i\|$ converges. A nonzero vector x of the form $\sum_{i=m}^n a_i x_i$ is said to be a block with respect to the sequence $(x_i)_{i=1}^\infty$ and the set of integers i for which $a_i \neq 0$ is denoted by $\mathrm{supp}\, x$. Moreover, $\{y_i\}_{i=1}^n$ are called successive blocks with respect to $(x_i)_{i=1}^\infty$ if $\max \mathrm{supp}\, y_i < \min \mathrm{supp}\, y_{i+1}$ for every $i = 1, 2, \ldots, n-1$.

If $\{X_i : i \in \Gamma\}$ is a family of Banach spaces, then the ℓ_p-sum of this family is denoted by $(\bigoplus_{i\in\Gamma} X_i)_{\ell_p}$, i.e.

$$\left(\bigoplus_{i\in\Gamma} X_i\right)_{\ell_p} = \{(x_i)_{i\in\Gamma} : x_i \in X_i, (\|x_i\|)_{i\in\Gamma} \in \ell_p(\Gamma)\}.$$

The c_0-sum of a family of Banach spaces is analogously defined.

Throughout this work, by a compact space we mean a compact Hausdorff topological space. For any compact space K, we denote by $\mathcal{C}(K)$ the Banach space consisting of real continuous functions on K equipped with the supremum norm.

A property P of Banach spaces is said to be a three-space property if whenever $Y \subseteq X$ are Banach spaces such that Y and X/Y have property P then X has property P.

Contents

Chapter 1

Riemann integrability of vector-valued functions

Section 1.1
Notation and preliminaries

In this chapter we study the Riemann integral of functions taking values in a Banach space. We start with some notation. We denote the Lebesgue measure in $\mathbb{R}$ by μ. A partition of the interval $[a,b] \subseteq \mathbb{R}$ is a finite collection of nonoverlapping closed subintervals covering $[a,b]$. A tagged partition of the interval $[a,b]$ is a partition $\{[t_{i-1},t_i] : 1 \leq i \leq N\}$ of $[a,b]$ together with a set of points $\{s_i : 1 \leq i \leq N\}$ that satisfy $s_i \in (t_{i-1},t_i)$ for each i.

Let $\mathcal{P} = \{(s_i,[t_{i-1},t_i]) : 1 \leq i \leq N\}$ be a tagged partition of $[a,b]$ and X a Banach space. For every function $f\colon [a,b] \to X$ we denote by $f(\mathcal{P})$ the Riemann sum

$$f(\mathcal{P}) := \sum_{i=1}^{N} (t_i - t_{i-1}) f(s_i).$$

The norm of the tagged partition $\mathcal{P}$ is

$$\|\mathcal{P}\| := \max\{t_i - t_{i-1} : 1 \leq i \leq N\}.$$

We say that a function $f\colon [a,b] \to X$ is Riemann integrable, with integral $x \in X$, if for every $\varepsilon > 0$ there is $\delta > 0$ such that $\|f(\mathcal{P}) - x\| < \varepsilon$ for all tagged partitions $\mathcal{P}$ of $[a,b]$ with norm less than δ. In this case x is unique and it is denoted by $\int_a^b f(t)dt$ or $\int_a^b f$.

Some of the classical results of the Riemann integral in the real case remain valid in the general case. We summarize in the next theorem some basic properties which will be used along this chapter:

Theorem 1.1.1 ([Gor91]). *Let X be a Banach space and $f\colon [a,b] \to X$ a Riemann integrable function.*

1. *The function f is Riemann integrable on every subinterval of $[a,b]$.*
2. *If $\|f(t)\| \leq M$ for every $t \in [a,b]$ then $\|\int_a^b f(t)dt\| \leq M(b-a)$.*
3. *If $T\colon X \to Y$ is a bounded linear operator, then Tf is Riemann integrable on $[a,b]$ and $\int_a^b Tf = T(\int_a^b f)$. In particular, for each $x^* \in X^*$, x^*f is continuous almost everywhere.*

From now on we work with the unit interval $[0,1]$. The following criterion will be our main tool for proving the Riemann integrability of many functions:

Theorem 1.1.2 ([Gor91]). *Let X be a Banach space and $f\colon [0,1] \to X$ a function. The following statements are equivalent:*

1. *The function f is Riemann integrable.*
2. *For each $\varepsilon > 0$ there exists a partition $\mathcal{P}_\varepsilon$ of $[0,1]$ with $\|f(\mathcal{P}_1) - f(\mathcal{P}_2)\| < \varepsilon$ for all tagged partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ of $[0,1]$ that have the same intervals as $\mathcal{P}_\varepsilon$.*
3. *There is $x \in X$ such that for every $\varepsilon > 0$ there exists a partition $\mathcal{P}_\varepsilon$ of $[0,1]$ such that $\|f(\mathcal{P}) - x\| < \varepsilon$ whenever $\mathcal{P}$ is a tagged partition of $[0,1]$ with the same intervals as $\mathcal{P}_\varepsilon$.*

For functions taking values in the real line, the Riemann integral can also be defined using upper and lower sums. For functions taking values in a Banach space we can provide a similar definition using the oscillation of a function:

Definition 1.1.3. *Let X be a Banach space and $f\colon [0,1] \to X$ a function.*

- *For any subinterval I of $[0,1]$, we denote the oscillation of f on I by*

$$Osc(f,I) := \sup\{\|f(t) - f(t')\| : t,t' \in I\}.$$

- *For each $t \in (0,1)$, we denote by*

$$Osc(f,t) := \lim_{\varepsilon \to 0^+} Osc(f,[t-\varepsilon,t+\varepsilon])$$

the oscillation of f at t. Analogously, $Osc(f,0) := \lim_{\varepsilon\to 0^+} Osc(f,[0,\varepsilon])$ and $Osc(f,1) := \lim_{\varepsilon\to 0^+} Osc(f,[1-\varepsilon,1])$.

- *f is Darboux integrable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\sum_{i=1}^{n} \mu(I_i) Osc(f, I_i) < \varepsilon$$

whenever $\mathcal{P} = \{I_1, I_2, \ldots, I_n\}$ is a partition of $[0,1]$ with $\|\mathcal{P}\| < \delta$.

Notice that if $f \colon [0,1] \to X$ is a function taking values in a Banach space X, then the set of points of discontinuity of f is

$$Disc(f) = \{t \in [0,1] : Osc(f,t) > 0\}.$$

It is easy to see that every Darboux integrable function is Riemann integrable. Moreover, a standard argument shows the following:

Theorem 1.1.4. *A function $f \colon [0,1] \to X$ taking values in a Banach space X is Darboux integrable if and only if it is bounded and continuous almost everywhere.*

Section 1.2
The Lebesgue property

The well-known Lebesgue's criterion for Riemann integrability states that a real function is Riemann integrable if and only if it is bounded and continuous almost everywhere (a.e. for short). However, L.M. Graves showed in [Gra27] the existence of a Riemann integrable function $f \colon [0,1] \to \ell_\infty([0,1])$ which is everywhere discontinuous. A Banach space X for which every Riemann integrable function $f \colon [0,1] \to X$ is continuous a.e. is said to have the *Lebesgue property* (LP for short)[1]. The spaces c_0, ℓ_p with $p > 1$ and $L_p([0,1])$ with $p \geq 1$ does not have the LP. Nevertheless, A.S. Nemirovski, M.Ju. Ochan and R. Redjouani [NOR73] and G.C. da Rocha Filho [dRF79] proved independently that ℓ_1 has the LP. Moreover, da Rocha Filho proved that the Tsirelson space also has the LP. The behavior of the Lebesgue property has been well studied in [dRF79] and [Piz89]. We summarize here some relevant results concerning this property:

Theorem 1.2.1. *1. The LP is separably determined, i.e. a Banach space has the LP if and only if every separable subspace has the LP. Moreover, a Banach space has the LP if and only if every subspace with a basis has the LP ([Piz89, Theorem 1.14]).*

[1] This property is also known in the literature as the *Darboux property*.

2. *The LP is stable under* ℓ_1*-sums ([dRF79, Theorem 3.b.14]).*

3. *The LP is a three-space property ([Piz89, Proposition 1.19]).*

The following properties are related with the LP:

Definition 1.2.2.

- *A Banach space* X *has the* Schur property *if every weakly convergent sequence in* X *is norm convergent.*

- *A Banach space is* asymptotic ℓ_1 with respect to a normalized basis $(e_i)_{i=1}^{\infty}$ *if there exists a constant* $C \geq 1$ *such that for every* $n \in \mathbb{N}$ *there is a function* $F_n : \mathbb{N} \cup \{0\} \to \mathbb{N}$ *with*

$$C^{-1} \sum_{i=1}^{n} |a_i| \leq \left\| \sum_{i=1}^{n} a_i x_i \right\|$$

for all normalized successive blocks $\{x_i\}_{i=1}^{n}$ *with respect to* $(e_i)_{i=1}^{\infty}$ *that satisfy* $F_n(0) \leq \operatorname{supp} x_1$ *and* $F_n(\max \operatorname{supp} x_i) < \min \operatorname{supp} x_{i+1}$ *for all* $i = 1, 2, ..., n-1$ *and for all* $\{a_i\}_{i=1}^{n} \subseteq \mathbb{R}$. *In this case the basis* $(e_i)_{i=1}^{\infty}$ *is said to be an* asymptotic ℓ_1 basis. *Moreover, if we can take* $F_n(k) = k$ *for every* $n, k \in \mathbb{N}$, *then the basis* $(e_i)_{i=1}^{\infty}$ *is said to be* stabilized asymptotic ℓ_1.

- *Let* $(e_i)_{i=1}^{\infty}$ *be a normalized basic sequence in a Banach space. A basic sequence* $(x_i)_{i=1}^{\infty}$ *is said to be a* spreading model *of* $(e_i)_{i=1}^{\infty}$ *if there is a decreasing sequence of positive numbers* ε_n *converging to zero such that*

$$\left| \left\| \sum_{i=1}^{n} a_i e_{k_i} \right\| - \left\| \sum_{i=1}^{n} a_i x_i \right\| \right| < \varepsilon_n$$

for all $n \leq k_1 \leq ... \leq k_n$ *and all* $\{a_i\}_{i=1}^{n} \subseteq [-1, 1]$.

Theorem 1.2.3. *[Nar07] Let* X *be a Banach space.*

1. *If* X *is asymptotic* ℓ_1 *with respect to its normalized basis* $\{e_i\}$, *then* X *has the LP.*

2. *If* X *has the LP, then each spreading model of* X *is equivalent to the standard unit vector basis of* ℓ_1 *(A. Pełczyński and G.C. da Rocha Filho).*

Moreover, if a Banach space X has the Schur property then each spreading model of X is equivalent to the standard unit vector basis of ℓ_1. Thus, we have the following implications among these properties:

X is asymptotic ℓ_1 $\Rightarrow$ X has the LP $\Rightarrow$ Every spreading model of X is equivalent to the ℓ_1 basis

$\Uparrow$

X has the Schur property

A. Pełczyńsky and G.C. da Rocha Filho proved that if X is a subspace of $L^1([0,1])$ such that every spreading model of X is equivalent to the standard unit vector basis of ℓ_1, then X has both the LP and the Schur property. Moreover, J. Bourgain and H. Rosenthal [BR80] constructed a subspace of $L^1([0,1])$ with the Schur property (so with the LP too) which fails the Radon-Nikodým property. Since every stabilized asymptotic ℓ_1 basis is boundedly complete [Dew02, Proposition 3.3.1] and every Banach space with a boundedly complete basis has the Radon-Nikodým property [DU77, page 64, Theorem 6], the Banach space constructed by J. Bourgain and H. Rosenthal is a Banach space with the LP but with no *stabilized* asymptotic ℓ_1 basis.

On the other hand, some examples of Banach spaces with the Schur property failing the LP were constructed by E. Odell, R. Haydon and K.M. Naralenkov (see [Nar07] and [Hay84]). We provide here another example of a Schur space which fails the LP. Moreover, our example is the dual of a Banach space with an unconditional shrinking basis. We represent the dyadic tree by

$$T = \{(n,k) : n = 0,1,2,\dots \text{ and } k = 1,2,\dots,2^n\}.$$

A node $(n,k) \in T$ has two immediate successors $(n+1, 2k-1)$ and $(n+1, 2k)$. Moreover, for every node $p = (n,k) \in T$, we write $|p| = n$. The order in the dyadic tree is given by the relation $p \leq q$ if and only if there is a finite sequence $\{p_1, \dots, p_m\}$ in T such that $p_1 = p$, $p_m = q$ and p_{j+1} is an immediate successor of p_j for every $j = 1,2,\dots,m-1$. For any $p \in T$ and $n = 0,1,2,...$, we denote by $p|_n$ the unique element $q \in T$ such that $|q| = n$ and $q \leq p$. We define the Banach space T_1 as the completion of $c_{00}(T)$ with the norm

$$\|x\| = \sup_{n\geq 1}\left\{\sum_{|p|=n} \sup\{|x(q)| : q \geq p\}\right\}.$$

This space is defined in [Lin04, Section 5.5] and it is a generalization of the space constructed by Talagrand in [Tal83]. For every $(n,k) \in T$, we denote by $e_{(n,k)}$ the characteristic function of $(n,k) \in T$.

Theorem 1.2.4. *[Lin04, Theorem 5.5.3] T_1 has a shrinking unconditional basis $\{e_{(n,k)} : (n,k) \in T\}$ and the dual space T_1^* has the Schur property.*

Theorem 1.2.5. *T_1^* does not have the LP.*

Proof. Set $\{e^*_{(n,k)} : (n,k) \in T\}$ the biorthogonal functionals of $\{e_{(n,k)} : (n,k) \in T\}$. Define $f : [0,1] \to T_1^*$ as follows:

$$f(t) = \begin{cases} e^*_{(n-1,k)} & \text{if } t = \frac{2k-1}{2^n} \text{ with } n \in \mathbb{N} \text{ and } k = 1,2,\dots,2^{n-1}; \\ 0 & \text{in any other case.} \end{cases}$$

The function f is not continuous at any point, since the dyadic points $\{\frac{2k-1}{2^n} : (k,n) \in T\}$ are dense in $[0,1]$. Therefore, it is enough to show that f is Riemann integrable. Notice that if $p_1, p_2, \dots, p_m$ are elements in T such that $p_i|_n \neq p_j|_n$ whenever $i \neq j$, then

$$\left\| \sum_i a_i e^*_{p_i} \right\| \leq \sup_{x \in B_{T_1}} \sum_i |a_i x(p_i)| \leq \max\{|a_i|\} \sup_{x \in B_{T_1}} \left\{ \sum_i |x(p_i)| \right\} \leq$$

$$\leq \max\{|a_i|\} \sup_{x \in B_{T_1}} \left\{ \sum_{|p|=n} \{|x(q)| : q \geq p\} \right\} \leq \max |a_i|,$$

where the last inequality follows from the definition of the norm in T_1. Therefore,

$$\left\| \sum a_i e^*_{p_i} \right\| \leq \max |a_i| \tag{1.1}$$

Take $N \in \mathbb{N}$ and $\{I_1, I_2, \dots, I_{2^N-1}\}$ a family of disjoint intervals of $[0,1]$ such that

$$\sum_{1 \leq n \leq 2^N-1} \mu(I_n) \leq \frac{1}{2^N} \text{ and } \frac{n}{2^N} \in \operatorname{Int}(I_n) \text{ for every } 1 \leq n \leq 2^N - 1. \tag{1.2}$$

Let $J_1, J_2, \dots, J_{2^N}$ be the closed disjoint intervals of $[0,1]$ determined by

$$[0,1] \setminus \bigcup_{1 \leq n \leq 2^N-1} \operatorname{Int}(I_n).$$

It follows from (1.1) that for any points $t_i \in J_i$, $i = 1,2,\dots,2^N$,

$$\left\| \sum \mu(J_i) f(t_i) \right\| \leq \max \mu(J_i) \leq \frac{1}{2^N},$$

where the last inequality follows from (1.2).

Therefore, every tagged partition $\mathcal{P}_N$ with intervals $J_1, I_1, J_2, \dots, I_{2^N-1}, J_{2^N}$ and points $t_1, t_1', t_2, \dots, t'_{2^N-1}, t_{2^N}$ satisfies

$$\|f(\mathcal{P}_N)\| \leq \left\| \sum \mu(J_i) f(t_i) \right\| + \left\| \sum \mu(I_i) f(t_i') \right\| \leq \frac{1}{2^N} + \frac{1}{2^N} = \frac{1}{2^{N-1}}.$$

Since the previous inequality holds for any $N \in \mathbb{N}$, it follows from Theorem 1.1.2 that f is Riemann integrable with integral zero. □

Another related problem is to study which operators transform Riemann integrable functions in a.e. continuous functions. A. Pełzyński and G.C. da Rocha Filho studied this class of operators and called them *Darboux operators*.

Definition 1.2.6. *Let X, Y be Banach spaces and $T\colon X \to Y$ a bounded linear operator*

1. *T is Darboux if for every Riemann integrable function $f\colon [0,1] \to X$, the function Tf is continuous a.e.*

2. *T is compact if $T(B_X)$ is relatively compact.*

3. *T is absolutely summing if for every sequence $(x_n)_{n=1}^{\infty}$ in X such that $\sum x_n$ converges unconditionally, the series $\sum Tx_n$ converges absolutely.*

In particular, it is mentioned without proof in [PdRF80] that every compact operator and every absolutely summing operator is Darboux. We include here a proof of the last fact:

Theorem 1.2.7. *Let X, Y be Banach spaces and $T\colon X \to Y$ an absolutely summing operator. Then T is Darboux.*

Proof. Let $f\colon [0,1] \to X$ be a Riemann integrable function. We prove that Tf is continuous a.e. Notice that T is absolutely summing if and only if there exists $C > 0$ such that for every $n \in \mathbb{N}$ and every $x_1, x_2, ..., x_n \in X$,

$$\sum_{i \le n} \|Tx_i\| \le C \max_{S \subseteq \{1,...,n\}} \left\| \sum_{i \in S} x_i \right\|.$$

Let $E = \{t \in [0,1] : \mathrm{Osc}(Tf, t) > 0\}$ and $E_n = \{t \in [0,1] : \mathrm{Osc}(Tf, t) \ge \frac{1}{n}\}$ for every $n \in \mathbb{N}$. Then, $E = \bigcup_{n \in \mathbb{N}} E_n$. Since each E_n is closed, the set E is measurable and we must show that $\mu(E) = 0$. If $\mu(E) \neq 0$, then there exists $N \in \mathbb{N}$ such that $\mu(E_N) > 0$. Let $\mathcal{P} = \{I_1, I_2, ..., I_p\}$ be any partition of $[0,1]$ with $\|\mathcal{P}\| < \frac{1}{N}$ and let $J = \{j \le p : \ \mu(E_N \cap \mathrm{Int}(I_j)) > 0\}$. For each $j \in J$, we can take $t_j, t_j' \in E_N \cap \mathrm{Int}(I_j)$ such that $\|Tf(t_j) - Tf(t_j')\| > \frac{1}{2N}$. The latter implies

$$\sum_{j \le p} \mu(I_j) \|Tf(t_j) - Tf(t_j')\| > \frac{\mu(E_N)}{2N}.$$

On the other hand, since T is an absolutely summing operator, there exists $C > 0$ such that

$$\sum_{j \le p} \mu(I_j) \|Tf(t_j) - Tf(t_j')\| \le C \max_{S \subseteq \{1,...,p\}} \|\sum_{j \in S} \mu(I_j)(f(t_j) - f(t_j'))\| =$$

$$= C\|f(\mathcal{P}_1) - f(\mathcal{P}_2)\|$$

for certain tagged partitions $\mathcal{P}_1, \mathcal{P}_2$ with $\|\mathcal{P}_1\| = \|\mathcal{P}_2\| = \|\mathcal{P}\|$. Due to the Riemann integrability of f, we can choose $\mathcal{P}$ such that

$$\sum_{j\leq p} \mu(I_j)\|Tf(t_j) - Tf(t'_j)\| < \frac{\mu(E_N)}{2N},$$

in contradiction with $\sum\limits_{j\leq p} \mu(I_j)\|Tf(t_j) - Tf(t'_j)\| > \frac{\mu(E_N)}{2N}$.

□

Section 1.3

The weak Lebesgue property

Definition 1.3.1. *A Banach space X is said to have the weak Lebesgue property (WLP for short) if every Riemann integrable function $f\colon [0,1] \to X$ is weakly continuous a.e.*

The WLP property was introduced in [Wan96]. In 1951 A. Alexiewicz and W. Orlicz constructed a Riemann integrable function $f\colon [0,1] \to \mathcal{C}([0,1])$ which is not weakly continuous a.e. [AO51]. In particular, they showed that $\mathcal{C}([0,1])$ does not have the WLP. Nevertheless, every Banach space with separable dual has the WLP. Every Banach space with the LP property has the WLP property and, moreover, if a Banach space has the Schur property then it has the LP if and only if it has the WLP. Other spaces with the WLP, such as $L^1([0,1])$, can be found in [CRSP10] and [WW01].

The following lemma provides a useful tool in the study of weak continuity:

Lemma 1.3.2. *Let X be a Banach space, $D = \{x_i^*\}_{i\in\Gamma}$ be a dense subset of X^* (or, more generally, whose linear span is dense in X^*), $f\colon [0,1] \to X$ be a bounded function, $E \subseteq [0,1]$ be the set of points of weak discontinuity of f and E_i be the set of points of discontinuity of $x_i^* f\colon [0,1] \to \mathbb{R}$ for each $i \in \Gamma$. Then, $E = \bigcup_{i\in\Gamma} E_i$.*

Proof. Notice that $E_i \subseteq E$ for every $i \in \Gamma$. Since the set of points of discontinuity of any linear combination of $\{x_i^* f\}_{i\in\Gamma}$ is contained in $\bigcup_{i\in\Gamma} E_i$, we may suppose that D is dense in X^*. We prove that f is weakly continuous at every point of $(\bigcup_{i\in\Gamma} E_i)^c$. Let $x^* \in X^*$ and let M be an upper bound for $\{\|f(t)\| : t \in [0,1]\}$. Fix $\varepsilon > 0$ and $t \in (\bigcup_{i\in\Gamma} E_i)^c$. Then, there exists $x_i^* \in D$ such that $\|x_i^* - x^*\| < \frac{\varepsilon}{3M}$.

Since $t \notin E_i$, there exists a neighborhood U of t such that $|x_i^* f(t) - x_i^* f(t')| < \frac{\varepsilon}{3}$ for every $t' \in U$. Thus,

$$|x^* f(t) - x^* f(t')| \leq |x^* f(t) - x_i^* f(t)| + |x_i^* f(t) - x_i^* f(t')| + |x_i^* f(t') - x^* f(t')| < \varepsilon$$

for every $t' \in U$. Therefore, $x^* f$ is continuous at each $t \in (\bigcup_{i\in\Gamma} E_i)^c$ for every $x^* \in X^*$. $\square$

We need to introduce some cardinal invariants. By $\mathrm{cov}(\mathcal{M})$ we denote the smallest cardinal κ such that there exist κ nowhere dense sets in $[0,1]$ whose union is the interval $[0,1]$. We are interested in the following characterization of this cardinal:

Theorem 1.3.3 ([BS92]). *The cardinal* $\mathrm{cov}(\mathcal{M})$ *coincides with the smallest cardinal* κ *such that there exist* κ *closed sets in* $[0,1]$ *with Lebesgue measure zero whose union does not have Lebesgue measure zero.*

A set $A \subseteq \mathbb{R}$ is said to be *strongly null* if for every sequence of positive reals $(\varepsilon_n)_{n=1}^\infty$ there exists a sequence of intervals $(I_n)_{n=1}^\infty$ such that $\mu(I_n) < \varepsilon_n$ for every $n \in \mathbb{N}$ and $A \subseteq \bigcup_{n\in\mathbb{N}} I_n$. The following result provides a useful characterization of strongly null sets:

Theorem 1.3.4 ([Paw96]). *A set* $A \subseteq \mathbb{R}$ *is strongly null if and only if for every closed set* F *with Lebesgue measure zero, the set* $A+F = \{a+z : a \in A \text{ and } z \in F\}$ *has Lebesgue measure zero.*

We will denote by $\mathrm{non}(\mathcal{SN})$ the smallest cardinal of a nonstrongly null set. We have

$$\aleph_1 \leq \mathrm{cov}(\mathcal{M}) \leq \mathrm{non}(\mathcal{SN}) \leq \mathfrak{c}.$$

Furthermore, under Martin's axiom, and therefore under the Continuum Hypothesis too, $\mathrm{non}(\mathcal{SN}) = \mathrm{cov}(\mathcal{M}) = \mathfrak{c}$. Furthermore, if $\mathfrak{b} = \mathfrak{c}$ then $\mathrm{non}(\mathcal{SN}) = \mathrm{cov}(\mathcal{M})$, where $\mathfrak{b}$ denotes the bounding number[2]. However, there exist models of ZFC satisfying $\mathrm{cov}(\mathcal{M}) < \mathrm{non}(\mathcal{SN})$. For further references and results on this cardinals we refer the reader to [BJ95].

It is known that every Banach space with separable dual has the WLP [WW01]. Next theorem gives a generalization in terms of $\mathrm{cov}(\mathcal{M})$.

Theorem 1.3.5. *Every Banach space* X *such that* $\mathrm{dens}(X^*) < \mathrm{cov}(\mathcal{M})$ *has the WLP.*

[2] $\mathfrak{b}$ is the smallest cardinality of an unbounded set in $(\mathbb{N}^\mathbb{N}, \leq^*)$, where $f \leq^* g$ if and only if $f(n) \leq g(n)$ for all but finitely many $n \in \mathbb{N}$.

Proof. Let $D = \{x_i^*\}_{i\in\Gamma}$ be a dense subset in X^* with $|\Gamma| < \mathrm{cov}(\mathcal{M})$ and take $f\colon [0,1] \to X$ a Riemann integrable function. We prove that f is weakly continuous a.e. Let E_i be the set of points of discontinuity of $x_i^* f$ for every $i \in \Gamma$. Since every function $x_i^* f$ is Riemann integrable, each E_i is a countable union of closed sets with measure zero. Since $|\Gamma| < \mathrm{cov}(\mathcal{M})$, it follows from Theorem 1.3.3 that $E := \bigcup_{i\in\Gamma} E_i$ has measure zero. By Lemma 1.3.2, E is the set of points of weak discontinuity of f, so f is weakly continuous a.e. □

Corollary 1.3.6. *Every Banach space with separable dual has the WLP.*

The space ℓ_1 has the WLP because it has the LP. Using Theorem 1.2.3, it is easy to find a separable Banach space with nonseparable dual such that it does not contain an isomorphic copy of ℓ_1 but it has the LP (for example, the space defined by Odell in [Ode85]). On the other hand, the James tree space JT (see [AK06, Section 13.4]) is a separable Banach space with nonseparable dual such that it does not contain an isomorphic copy of ℓ_1 and it does not have the WLP:

Theorem 1.3.7. *The James tree space does not have the WLP.*

Proof. Let us recall the definition of the James tree space. We consider again the dyadic tree

$$T = \{(n,k) : n = 0,1,2,\dots \text{ and } k = 1,2,\dots,2^n\}.$$

A segment of T is a finite sequence $\{p_1,\dots,p_m\}$ such that p_{j+1} is an immediate successor of p_j for every $j = 1,2,\dots,m-1$. The James tree space JT is the completion of $c_{00}(T)$ with the norm

$$\|x\| = \sup \sqrt{\sum_{j=1}^{l} \left(\sum_{(n,k)\in S_j} x(n,k) \right)^2} < \infty,$$

where the supremum is taken over all $l \in \mathbb{N}$ and all sets of pairwise disjoint segments $S_1, S_2, \dots, S_l$. Let $\{e_{(n,k)}\}_{(n,k)\in T}$ be the canonical basis of JT, i.e. $e_{(n,k)}$ is the characteristic function of $(n,k) \in T$. Define $f\colon [0,1] \to JT$ as follows:

$$f(t) = \begin{cases} e_{(n-1,k)} & \text{if } t = \frac{2k-1}{2^n} \text{ with } n \in \mathbb{N} \text{ and } k = 1,2,\dots,2^{n-1} \\ 0 & \text{in any other case.} \end{cases}$$

We claim that f is Riemann integrable. Fix $N \in \mathbb{N}$ and let $\{I_1, I_2, \dots, I_{2^N-1}\}$ be a family of closed disjoint intervals of $[0,1]$ with

$$\sum_{1\le n\le 2^N-1} \mu(I_n) \le \frac{1}{2^N} \text{ and } \frac{n}{2^N} \in Int(I_n) \text{ for each } 1 \le n \le 2^N - 1.$$

Let $J_1, J_2, \dots, J_{2^N}$ be the closed disjoint intervals of $[0,1]$ determined by

$$[0,1] \setminus \bigcup_{1\le n\le 2^N-1} \mathrm{Int}(I_n).$$

Then, $\mu(J_n) \le \frac{1}{2^N}$ and $\| \sum_{n=1}^{2^N} a_n f(t_n)\| \le \sqrt{\sum_{n=1}^{2^N} a_n^2}$ for every $a_n \in \mathbb{R}$ and every $t_n \in J_n$ due to the definition of the norm in JT. Thus, any tagged partition $\mathcal{P}_N$ with intervals $J_1, I_1, J_2, \dots, I_{2^N-1}, J_{2^N}$ and points $t_1, t_1', t_2, \dots, t_{2^N-1}', t_{2^N}$ satisfies

$$\|f(\mathcal{P}_N)\| \le \left\| \sum_{n=1}^{2^N} \mu(J_n) f(t_{2n-1}) \right\| + \sum_{n=1}^{2^N-1} \mu(I_n) \le$$

$$\le \sqrt{\sum_{n=1}^{2^N} \mu(J_n)^2} + \frac{1}{2^N} \le \sqrt{\sum_{n=1}^{2^N} \frac{1}{2^{2N}}} + \frac{1}{2^N} \le \frac{2}{\sqrt{2^N}}.$$

Hence, $\|f(\mathcal{P}_N)\| \xrightarrow{N\to\infty} 0$ and f is Riemann integrable with integral zero.

We show that f is not weakly continuous at any irrational point $t \in [0,1]$. Fix a irrational point $t \in [0,1]$. There exists a sequence of dyadic points $\left(\frac{2k_j-1}{2^{n_j}}\right)_{j=1}^{\infty}$ converging to t with $(n_j-1, k_j)_{j=1}^{\infty}$ a sequence in T such that $(n_{j+1}-1, k_{j+1})$ is an immediate successor of (n_j-1, k_j) for every $j \in \mathbb{N}$. Then, $\sum_{j=1}^{\infty} e^*_{(n_j-1,k_j)}$ is a functional in JT^*, so the sequence $f(\frac{2k_j-1}{2^{n_j}}) = e_{(n_j-1,k_j)}$ is not weakly null and f is not weakly continuous at t. □

Corollary 1.3.8 ([AO51])**.** *$\mathcal{C}([0,1])$ does not have the WLP.*

Proof. Since every subspace of a Banach space with the WLP has the WLP and every separable Banach space is isometrically isomorphic to a subspace of $\mathcal{C}([0,1])$, it follows from the previous theorem and the separability of JT that $\mathcal{C}([0,1])$ does not have the WLP. □

Corollary 1.3.9. *Let K be a compact Hausdorff space.*

1. *If K is metrizable, then $\mathcal{C}(K)$ has the WLP if and only if K is countable.*

2. *If $\mathcal{C}(K)$ has the WLP then K is scattered. The converse is not true since $c_0(\mathfrak{c})$ does not have the WLP (Theorem 1.3.13) and it is isomorphic to a $\mathcal{C}(K)$ space with K scattered.*

Proof. If K is a countable compact metric space, then $\mathcal{C}(K)^*$ is separable [FHH+11, Theorem 14.24], so $\mathcal{C}(K)$ has the WLP (Theorem 1.3.5). If K is an uncountable compact metric space, then Milutin's Theorem states that $\mathcal{C}(K)$ is isomorphic to $\mathcal{C}([0,1])$ [AK06, Theorem 4.4.8], so $\mathcal{C}(K)$ does not have the WLP (Corollary 1.3.8). Finally, if K is not scattered, then $\mathcal{C}(K)$ has a subspace isomorphic to $\mathcal{C}([0,1])$ (see the proof of [FHH+11, Theorem 14.26]), so $\mathcal{C}(K)$ does not have the WLP. □

Remark 1.3.10. *Let $\{X_i\}_{i\in\Gamma}$ be a family of Banach spaces and $X := (\bigoplus_{i\in\Gamma} X_i)_{\ell_p}$ with $1 < p < \infty$ or $X := (\bigoplus_{i\in\Gamma} X_i)_{c_0}$. If $f\colon [0,1] \to X$ is a bounded function with $f(t) = (f_i(t))_{i\in\Gamma}$ and $f_i(t) \in X_i$, then, due to Lemma 1.3.2, its set of points of weak discontinuity is $E = \bigcup_{i\in\Gamma} E_i$, where E_i is the set of points of weak discontinuity of the corresponding f_i. Thus, the countable ℓ_p-sum or c_0-sum of Banach spaces with the WLP has the WLP. We cannot extend this result to uncountable ℓ_p-sums or c_0-sums even when $X_i = \mathbb{R}$ for every $i \in \Gamma$ (Theorem 1.3.13).*

Now, we study the WLP for the spaces of the form $c_0(\kappa)$ and $\ell_p(\kappa)$ with κ a cardinal.

Theorem 1.3.11. *For any cardinal κ and any $1 < p < \infty$, if $c_0(\kappa)$ or $\ell_p(\kappa)$ has the WLP, then X has the WLP for every Banach space X with* $\operatorname{dens}(X^*) \leq \kappa$.

Proof. Suppose X is a Banach space without the WLP and such that $\operatorname{dens}(X^*) = \kappa$. Therefore, there exists a Riemann integrable function $f\colon [0,1] \to X$ which is not weakly continuous a.e. Set $\{x^*_\alpha\}_{\alpha<\kappa}$ a dense set in the dual ball B_{X^*} and fix $f_\alpha = x^*_\alpha f$ for every $\alpha < \kappa$. Let E^n_α be the set of points where f_α has oscillation strictly bigger than $\frac{1}{n}$ for every $n \in \mathbb{N}$. Note that each E^n_α has Lebesgue measure zero. Since f is not weakly continuous a.e., Lemma 1.3.2 asserts that $\bigcup_{\alpha<\kappa} \left(\bigcup_{n\in\mathbb{N}} E^n_\alpha\right)$ does not have Lebesgue measure zero, so there exists $n \in \mathbb{N}$ such that $\bigcup_{\alpha<\kappa} E^n_\alpha$ does not have Lebesgue measure zero.

Set $F_0 := E^n_0$ and $F_\alpha := E^n_\alpha \setminus \left(\bigcup_{\beta<\alpha} E^n_\beta\right)$ for every $\alpha \in \kappa \setminus \{0\}$. The sets F_α are pairwise disjoint. Let $\chi_{F_\alpha}\colon [0,1] \to \{0,1\}$ be the characteristic function of F_α for every $\alpha < \kappa$ and $g\colon [0,1] \to c_0(\kappa)$ the function defined by the formula $g(t) = \sum_{\alpha<\kappa} \chi_{F_\alpha}(t) e_\alpha$ for every $t \in [0,1]$, where $\{e_\alpha\}_{\alpha<\kappa}$ is the canonical basis of $c_0(\kappa)$.

Notice that g is not weakly continuous a.e. since each χ_{F_α} is not continuous at any point of F_α (because $\mu(F_\alpha) = 0$) and $\bigcup_{\alpha<\kappa} F_\alpha = \bigcup_{\alpha<\kappa} E^n_\alpha$ is not Lebesgue null. We claim that g is Riemann integrable. Let $\varepsilon > 0$. Since f is Riemann integrable, there exists a partition $\mathcal{P}_\varepsilon$ of $[0,1]$ such that $\|f(\mathcal{P}_1) - f(\mathcal{P}_2)\| < \frac{\varepsilon}{n}$ for all tagged partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ of $[0,1]$ that have the same intervals as $\mathcal{P}_\varepsilon$. For every $\alpha < \kappa$ and any tagged partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ of $[0,1]$ that have the same intervals as $\mathcal{P}_\varepsilon$,

$$|\chi_{F_\alpha}(\mathcal{P}_1) - \chi_{F_\alpha}(\mathcal{P}_2)| \leq \sum_{i=1}^{N} \mu(I_i) \leq n|f_\alpha(\mathcal{P}'_1) - f_\alpha(\mathcal{P}'_2)| \leq n\|f(\mathcal{P}'_1) - f(\mathcal{P}'_2)\| < \varepsilon$$

for suitable tagged partitions $\mathcal{P}'_1$ and $\mathcal{P}'_2$ of $[0,1]$ with the same intervals as $\mathcal{P}_\varepsilon$, where $I_1, I_2, \ldots, I_N$ are the intervals of $\mathcal{P}_\varepsilon$ whose interior has non-empty intersection with E^n_α. Therefore, g is Riemann integrable and $c_0(\kappa)$ does not have the WLP.

Fix $1 < p < \infty$. We have to prove that $\ell_p(\kappa)$ does not have the WLP. Let $h\colon [0,1] \to \ell_p(\kappa)$ be the function given by the formula $h(t) = \sum_{\alpha<\kappa} \chi_{F_\alpha}(t) \tilde{e}_\alpha$, where $\{\tilde{e}_\alpha\}_{\alpha<\kappa}$ is the canonical basis of $\ell_p(\kappa)$. Since the sets F_α are pairwise disjoint, the function h is well-defined. Moreover, h is not weakly continuous a.e. because $I \circ h = g$, where $I\colon \ell_p(\kappa) \to c_0(\kappa)$ is the canonical inclusion. Set $F = \bigcup_{\alpha<\kappa} F_\alpha$ and $\phi\colon F \to \kappa$ such that $\phi(t) = \alpha$ if $t \in F_\alpha$. We claim that h is Riemann integrable with integral zero. Let $\varepsilon > 0$ and $\mathcal{P}_\varepsilon = \{I_1, I_2, \dots, I_M\}$ be a partition of $[0,1]$ such that $\|g(\mathcal{P}')\| < \varepsilon$ for any tagged partition $\mathcal{P}'$ of $[0,1]$ with the same intervals as $\mathcal{P}_\varepsilon$. Notice that

$$\mu\left(\bigcup_{\mathrm{Int}(I_i)\cap F_\alpha\neq\emptyset} I_i\right) < \varepsilon \text{ for every } \alpha < \kappa. \tag{1.3}$$

Thus, for any tagged partition $\mathcal{P} = \{(s_i, I_i)\}_{i=1}^M$ the following inequalities hold:

$$\|h(\mathcal{P})\| = \left\|\sum_{s_i\in F} \mu(I_i)\tilde{e}_{\phi(s_i)}\right\| = \left\|\sum_{\alpha<\kappa} \mu\Bigg(\bigcup_{\phi(s_i)=\alpha} I_i\Bigg)\tilde{e}_\alpha\right\| =$$

$$= \left(\sum_{\alpha<\kappa} \mu\Bigg(\bigcup_{\phi(s_i)=\alpha} I_i\Bigg)^p\right)^{\frac{1}{p}} = \left(\sum_{\alpha<\kappa} \mu\Bigg(\bigcup_{\phi(s_i)=\alpha} I_i\Bigg)^{p-1} \mu\Bigg(\bigcup_{\phi(s_i)=\alpha} I_i\Bigg)\right)^{\frac{1}{p}} \leq$$

$$\overset{(1.3)}{\leq} \varepsilon^{\frac{p-1}{p}} \left(\sum_{\alpha<\kappa} \mu\Bigg(\bigcup_{\phi(s_i)=\alpha} I_i\Bigg)\right)^{\frac{1}{p}} \leq \varepsilon^{\frac{p-1}{p}}$$

Therefore, h is Riemann integrable with Riemann integral zero. □

Since $\mathrm{dens}(c_0(\kappa)^*) = \mathrm{dens}(\ell_p(\kappa)^*) = \kappa$ for any cardinal κ and any $1 < p < \infty$, the following result is an immediate consequence of the previous theorem:

Corollary 1.3.12. *For any cardinal κ and any $1 < p < \infty$, $c_0(\kappa)$ has the WLP if and only if $\ell_p(\kappa)$ has the WLP.*

The LP is separably determined [Piz89]. Nevertheless, it follows from the following theorem that the WLP is not separably determined, since every separable infinite-dimensional subspace of $\ell_2(\kappa)$ is isomorphic to ℓ_2 (which has separable dual).

Theorem 1.3.13. *Let κ be a cardinal and $X = c_0(\kappa)$ or $X = \ell_p(\kappa)$ with $1 < p < \infty$.*

1. *If $\kappa < \mathrm{cov}(\mathcal{M})$ then X has the WLP.*
2. *If $\kappa \geq \mathrm{non}(\mathcal{SN})$ then X does not have the WLP.*

Proof. It is enough to prove the result when $X = c_0(\kappa)$ due to Corollary 1.3.12. Since $\mathrm{dens}(c_0(\kappa)^*) = \kappa$, it follows from Theorem 1.3.5 that $c_0(\kappa)$ has the WLP if $\kappa < \mathrm{cov}(\mathcal{M})$.

Suppose $\mathrm{non}(\mathcal{SN}) \leq \kappa \leq \mathfrak{c}$. Due to Theorem 1.3.4, there exist a closed Lebesgue null set F and a set $E = \{x_\alpha\}_{\alpha<\kappa}$ in $\mathbb{R}$ such that $E + F$ does not have Lebesgue measure zero. Without loss of generality, we may assume that $E, F \subseteq [0, \frac{1}{2}]$ and consequently $(E + F) \subseteq [0, 1]$. Set $F_0 := x_0 + F$ and $F_\alpha := (x_\alpha + F) \setminus \left(\bigcup_{\beta<\alpha} F_\beta\right)$ for every $0 < \alpha < \kappa$. Let $\chi_{F_\alpha} : [0, 1] \to \{0, 1\}$ be the characteristic function of F_α for every $\alpha < \kappa$ and $f : [0, 1] \to c_0(\kappa)$ the function defined by the formula $f(t) = \sum_{\alpha<\kappa} \chi_{F_\alpha}(t)e_\alpha$ for every $t \in [0, 1]$, where $\{e_\alpha\}_{\alpha<\kappa}$ is the canonical basis of $c_0(\kappa)$.

Since the sets F_α are pairwise disjoint, the function f is well-defined. Each χ_{F_α} is not continuous at F_α, since F_α cannot contain an interval of $[0, 1]$. Therefore, f is not weakly continuous a.e. because $\bigcup_{\alpha<\kappa} F_\alpha = E + F$ does not have Lebesgue measure zero.

We claim that f is Riemann integrable. For every $\alpha < \kappa$ and every tagged partition $\mathcal{P} = \{(s_i, I_i)\}_{i=1}^N$ we have

$$\chi_{F_\alpha}(\mathcal{P}) = \sum_{i=1}^{N} \mu(I_i)\chi_{F_\alpha}(s_i) \leq \sum_{i=1}^{N} \mu(I_i - x_\alpha)\chi_F(s_i - x_\alpha) = \chi_F(\mathcal{P}')$$

for a suitable tagged partition $\mathcal{P}'$ with $\|\mathcal{P}\| = \|\mathcal{P}'\|$. Since F is a closed Lebesgue null set, the characteristic function χ_F is Riemann integrable due to Lebesgue's Theorem. Then, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\chi_F(\mathcal{P}) < \varepsilon$ for every tagged partition $\mathcal{P}$ with $\|\mathcal{P}\| < \delta$. Therefore, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\chi_{F_\alpha}(\mathcal{P}) < \varepsilon$ for all tagged partitions $\mathcal{P}$ with $\|\mathcal{P}\| < \delta$ and for every $\alpha < \kappa$. Thus, f is Riemann integrable since $\|f(\mathcal{P})\| = \sup_{\alpha<\kappa} \chi_{F_\alpha}(\mathcal{P}) < \varepsilon$ for every tagged partition $\mathcal{P}$ of $[0, 1]$ with $\|\mathcal{P}\| < \delta$. □

The facts that the countable ℓ_1-sum of spaces with the WLP has the WLP (Theorem 1.3.16) and that $L^1(\lambda)$ has the WLP if $\mathrm{dens}(L^1(\lambda)) < \mathrm{cov}(\mathcal{M})$ (Theorem 1.3.17) will be a consequence of the following lemma.

Lemma 1.3.14. *Let* $(\Omega, \Sigma, \lambda)$ *be a probability space and* $\mathfrak{P} = \{P_A : X \to X : A \in \Sigma\}$ *a family of bounded linear operators on a Banach space* X *such that*

(1) $P_A + P_{\Omega\setminus A} = P_\Omega = id_X$ *for every* $A \in \Sigma$.

(2) $\|P_A(x)\| \leq \|x\|$ *for every* $x \in X$ *and every* $A \in \Sigma$.

(3) $\|P_A(x)\| + \|P_B(x')\| \leq \max\{\|x + x'\|, \|x - x'\|\}$ *for every* $x, x' \in X$ *whenever* $A \cap B = \emptyset$.

(4) $\lim_{\lambda(A)\to 0}\|P_A(x)\| = 0$ *for every* $x \in X$.

Let $f : [0,1] \to X$ *be a Riemann integrable function. Then there is a measurable set* $E \subseteq [0,1]$ *with* $\mu(E) = 1$ *such that, for every sequence* $(t_n)_{n=1}^{\infty}$ *in* $[0,1]$ *converging to some* $t \in E$*, the set* $\{f(t_n) : n \in \mathbb{N}\}$ *is* $\mathfrak{P}$*-uniformly integrable, in the sense that*

$$\lim_{\lambda(A)\to 0}\sup_{n\in\mathbb{N}}\big\|P_A(f(t_n))\big\| = 0.$$

Proof. The proof is similar to that of [CRSP10, Lemma 2.3] and [WW01, Lemma 3]. Fix $\beta > 0$ and denote by E_β the set of points $t \in [0,1]$ such that for every $\delta > 0$ there exist $t' \in [0,1]$ with $|t' - t| < \delta$ and a set $A \in \Sigma$ with $\lambda(A) < \delta$ such that

$$\|P_A(f(t) - f(t'))\| > \beta.$$

Let μ^* be the Lebesgue outer measure in $[0,1]$. We show that $\mu^*(E_\beta) = 0$ with a proof by contradiction. Suppose $\mu^*(E_\beta) > 0$. Since f is Riemann integrable, we can choose a partition $\mathcal{P} = \{J_1, \ldots, J_m\}$ of $[0,1]$ such that

$$\left\|\sum_{j=1}^{m}\mu(J_j)(f(\xi_j) - f(\xi'_j))\right\| < \beta\mu^*(E_\beta) \tag{1.4}$$

for all choices $\xi_j, \xi'_j \in J_j, 1 \le j \le m$. Let $S = \{j \in \{1, \ldots, m\} : I_j \cap E_\beta \neq \emptyset\}$, where $I_j = \mathrm{Int}(J_j)$ for each $j = 1, \ldots, m$. Thus,

$$\sum_{j\in S}\mu^*(I_j \cap E_\beta) = \mu^*(E_\beta). \tag{1.5}$$

It is not restrictive to suppose $S = \{1, \ldots, n\}$ for some $1 \le n \le m$.

Because of the definition of E_β and I_1, there exist points $t_1 \in I_1 \cap E_\beta$ and $t'_1 \in I_1$ such that $\|f(t_1) - f(t'_1)\| \ge \|P_A(f(t_1) - f(t'_1))\| > \beta$ for some $A \in \Sigma$, hence $\|\mu(I_1)(f(t_1) - f(t'_1))\| > \beta\mu(I_1)$.

Fix $1 \le k < n$ and assume that we have already chosen points $t_j, t'_j \in I_j$ for all $1 \le j \le k$ with the property that

$$\left\|\sum_{j=1}^{k}\mu(I_j)(f(t_j) - f(t'_j))\right\| > \beta\left(\sum_{j=1}^{k}\mu(I_j)\right).$$

Define $x := \sum_{j=1}^{k}\mu(I_j)(f(t_j) - f(t'_j)) \in X$ and

$$\alpha := \|x\| - \beta\left(\sum_{j=1}^{k}\mu(I_j)\right) > 0.$$

Due to (4), we can choose $\delta > 0$ such that $\|P_A(x)\| < \alpha$ whenever $A \in \Sigma$ satisfies $\lambda(A) < \delta$. Take $t_{k+1}, t'_{k+1} \in I_{k+1}$ and a set $A \in \Sigma$ with $\lambda(A) < \delta$ such that $\|P_A(f(t_{k+1}) - f(t'_{k+1}))\| > \beta$, so $y := \mu(I_{k+1})(f(t_{k+1}) - f(t'_{k+1}))$ satisfies

$$\|P_A(y)\| > \beta\mu(I_{k+1}).$$

By the choice of A, (1) and (3), we also have (interchanging the role of t_{k+1} and t'_{k+1} if necessary)

$$\left\|\sum_{j=1}^{k+1} \mu(I_j)(f(t_j) - f(t'_j))\right\| \geq \|P_A(y)\| + \|P_{A^c}(x)\| \geq \|P_A(y)\| + \|x\| - \|P_A(x)\| >$$

$$> \beta\mu(I_{k+1}) + \alpha + \beta\sum_{j=1}^{k} \mu(I_j) - \|P_A(x)\| > \beta\sum_{j=1}^{k+1} \mu(I_j).$$

Thus, there exist $t_j, t'_j \in I_j$ for all $1 \leq j \leq n$ such that

$$\left\|\sum_{j=1}^{n} \mu(I_j)(f(t_j) - f(t'_j))\right\| > \beta\left(\sum_{j=1}^{n} \mu(I_j)\right) \overset{(1.5)}{\geq} \beta\mu^*(E_\beta),$$

which contradicts the inequality (1.4). So we can conclude that $\mu^*(E_\beta) = 0$.

Therefore, $E := [0,1] \setminus \bigcup_{n\in\mathbb{N}} E_{\frac{1}{n}}$ is measurable with $\mu(E) = 1$. Fix $t \in E$ and $m \in \mathbb{N}$. Since $t \notin E_{\frac{1}{m}}$, there exists $\delta_m > 0$ such that for every $t' \in [0,1]$ with $|t' - t| < \delta_m$ and every set $A \in \Sigma$ with $\lambda(A) < \delta_m$,

$$\|P_A(f(t) - f(t'))\| \leq \frac{1}{m}.$$

Thus, for every $m \in \mathbb{N}$, every sequence $(t_n)_{n=1}^{\infty}$ converging to t and every $A \in \Sigma$ with $\lambda(A) < \delta_m$,

$$\|P_A(f(t_n))\| \leq \|P_A(f(t))\| + \frac{1}{m} \text{ for } n \text{ big enough depending only on } m.$$

Now the conclusion follows from (4). □

Let $\{X_i\}_{i\in\Gamma}$ be a family of Banach spaces. We denote by $\pi_j\colon (\bigoplus_{i\in\Gamma} X_i)_{\ell_1} \to X_j$ the canonical projection onto X_j for each $j \in \Gamma$.

We will need the following property of ℓ_1-sums and the space $L_1(\lambda)$ for Theorems 1.3.16 and 1.3.17:

Lemma 1.3.15. *Let $(\Omega, \Sigma, \lambda)$ be a probability space and $\{X_i\}_{i\in\Gamma}$ a family of Banach spaces. Then:*

1. $\max\{\|x+y\|,\|x-y\|\} \geq \sum_{i\in A}\|\pi_i(x)\| + \sum_{i\in B}\|\pi_i(y)\|$ *for every vectors* $x,y\in(\bigoplus_{i\in\Gamma}X_i)_{\ell_1}$ *and any disjoint sets* $A,B\subseteq\Gamma$.
2. $\max\{\|f+g\|,\|f-g\|\} \geq \int_A|f|d\lambda + \int_B|g|d\lambda$ *for any* $f,g\in L_1(\lambda)$ *and any disjoint sets* $A,B\in\Sigma$.

Proof. The second part is essentially Lemma 2 of [WW01]. The proof of the first part is analogous and we include it for the sake of completeness. Let $x,y\in(\bigoplus_{i\in\Gamma}X_i)_{\ell_1}$ and $A,B\subseteq\Gamma$ be disjoint sets. The conclusion follows from the following elementary consequence of the triangle inequality $\frac{1}{2}(\|u+v\|+\|u-v\|)\geq\max\{\|u\|,\|v\|\}$ and the following inequalities:

$$\begin{aligned}\max\{\|x+y\|,\|x-y\|\} &\geq \frac{1}{2}(\|x+y\|+\|x-y\|)\\ &= \sum_{i\in\Gamma}\frac{1}{2}(\|\pi_i x+\pi_i y\|+\|\pi_i x-\pi_i y\|)\\ &\geq \sum_{i\in\Gamma}\max\{\|\pi_i x\|,\|\pi_i y\|\}\\ &\geq \sum_{i\in A}\max\{\|\pi_i x\|,\|\pi_i y\|\}+\sum_{i\in B}\max\{\|\pi_i x\|,\|\pi_i y\|\}\\ &\geq \sum_{i\in A}\|\pi_i x\|+\sum_{i\in B}\|\pi_i y\|.\end{aligned}$$

□

Theorem 1.3.16. *Let* $\{X_i\}_{i\in\mathbb{N}}$ *be Banach spaces with the WLP. Then the space* $X:=(\bigoplus_{i\in\mathbb{N}}X_i)_{\ell_1}$ *has the WLP.*

Proof. We are going to apply Lemma 1.3.14. Take $\Omega:=\mathbb{N}$, $\Sigma:=\mathcal{P}(\mathbb{N})$ the power set of $\mathbb{N}$, $\lambda(A):=\sum_{n\in A}2^{-n}$ and $\mathfrak{P}=\{P_A: A\in\Sigma\}$ with

$$\pi_i(P_A(x))=\begin{cases}\pi_i(x) & \text{if } i\in A\\ 0 & \text{if } i\notin A\end{cases}$$

for every $A\in\Sigma$ and every $x\in X$. Property (3) of Lemma 1.3.14 is Lemma 1.3.15(1) and property (4) holds because if $\lambda(A)<\frac{1}{2^n}$, then $A\subseteq\{n,n+1,\dots\}$, so

$$\|P_A(x)\|=\sum_{i\in A}\|\pi_i(x)\|\leq\sum_{i\geq n}\|\pi_i(x)\|$$

for every $x\in X$. Therefore, we can apply Lemma 1.3.14, so there exists a measurable set $E\subseteq[0,1]$ with $\mu(E)=1$ such that for every sequence $(t_n)_{n=1}^{\infty}$ in $[0,1]$ converging to some $t\in E$ the set $\{f(t_n):n\in\mathbb{N}\}$ is $\mathfrak{P}$-uniformly integrable. We can assume that, for each $i \in \mathbb{N}$, the map $t \mapsto \pi_i(f(t))$ is weakly continuous at each point of E because each X_i has the WLP.

It is a well-known fact that a sequence $(x_n)_{n=1}^{\infty}$ in X converges weakly to $x \in X$ if and only if it satisfies the following two conditions:

(i) $\pi_i(x_n) \to \pi_i(x)$ weakly in X_i for every $i \in \mathbb{N}$;

(ii) for every $\varepsilon > 0$ there is a finite set $J \subseteq \mathbb{N}$ such that $\sup_{n\in\mathbb{N}} \|P_{\mathbb{N}\setminus J}(x_n)\| \le \varepsilon$.

Since $\mathfrak{P}$-uniform integrability is equivalent to (ii), it follows that f is weakly continuous at each point of E. □

A similar idea to that of Theorem 1.3.16 lets us prove the following theorem, which improves [WW01, Theorem 5] and [CRSP10, Proposition 2.10].

Theorem 1.3.17. *Let $(\Omega, \Sigma, \lambda)$ be a probability space.*

1. *If* $\mathrm{dens}(L^1(\lambda)) < \mathrm{cov}(\mathcal{M})$ *then* $L^1(\lambda)$ *has the WLP.*
2. *If* $\mathrm{dens}(L^1(\lambda)) \ge \mathrm{non}(\mathcal{SN})$ *then* $L^1(\lambda)$ *does not have the WLP.*

Proof. Fix a Riemann integrable function $f\colon [0,1] \to L^1(\lambda)$. Take $P_A(x) := x\chi_A$ for every $A \in \Sigma$ and every $x \in L^1(\lambda)$. The family of operators $\{P_A\colon A \in \Sigma\}$ fulfills the requirements of Lemma 1.3.14 (bear in mind Lemma 1.3.15). Then $\mathfrak{P}$-uniform integrability is the usual uniform integrability and therefore a set is bounded and $\mathfrak{P}$-uniformly integrable if and only if it is relatively weakly compact due to Dunford's Theorem (see [AK06, Theorem 5.2.9]). Lemma 1.3.14 ensures that there exist a measurable set $E \subseteq [0,1]$ with $\mu(E) = 1$ such that for every sequence $(t_n)_{n=1}^{\infty}$ in $[0,1]$ converging to some $t \in E$, the set $\{f(t_n)\colon n \in \mathbb{N}\}$ is relatively weakly compact.

Let $\mathcal{C} \subseteq \Sigma$ be a dense family of λ-measurable sets, i.e. such that

$$\inf_{C\in\mathcal{C}} \lambda(A \bigtriangleup C) = 0 \text{ for every } A \in \Sigma.$$

Let $(h_n)_{n=1}^{\infty}$ be a relatively weakly compact sequence in $L^1(\lambda)$ and $h \in L^1(\lambda)$. Since $\mathcal{C}$ is a dense family of λ-measurable sets, if $\int_C h_n\, d\mu \to \int_C h\, d\mu$ for every $C \in \mathcal{C}$, then $h = w\text{-}\lim h_n$.

Suppose $\mathrm{dens}(L^1(\lambda)) < \mathrm{cov}(\mathcal{M})$. Then $\mathcal{C}$ can be taken such that $|\mathcal{C}| < \mathrm{cov}(\mathcal{M})$. Therefore, we can assume that, for each $C \in \mathcal{C}$, the Riemann integrable map $t \mapsto \int_C f(t)\, d\lambda$ is continuous at each point of E. Then, for every sequence $(t_n)_{n=1}^{\infty}$ in $[0,1]$ converging to a point $t \in E$, we have $f(t) = w\text{-}\lim f(t_n)$.

Now suppose $\nu = \mathrm{dens}(L^1(\lambda)) \ge \mathrm{non}(\mathcal{SN})$. Due to Maharam's Theorem (see [Lac12, p. 127, Theorem 9]), $L^1(\lambda)$ contains an isometric copy of $L^1(\mu_\nu)$, where μ_ν is the usual product probability measure on $\{0,1\}^\nu$. Since $L^1(\mu_\nu)$ contains an isomorphic copy of $\ell_2(\nu)$ (see [Lac12, p. 128, Theorem 12]) and $\ell_2(\nu)$ does not have the WLP (Theorem 1.3.13), we conclude that $L^1(\lambda)$ does not have the WLP. □

Theorem 1.3.16 can be extended to arbitrary ℓ_1-sums:

Theorem 1.3.18. *The arbitrary ℓ_1-sum of a family of Banach spaces with the WLP has the WLP.*

Proof. The proof uses some ideas of [MdRF84]. Let $f\colon [0,1] \to X := (\bigoplus_{i\in\Gamma} X_i)_{\ell_1}$ be a Riemann integrable function, where $\{X_i\}_{i\in\Gamma}$ is a family of Banach spaces with the WLP. For each $J \subseteq \Gamma$, we denote by $P_J\colon X \to X$ the function defined by $\pi_i(P_J(x)) = \pi_i(x)$ if $i \in J$ and $\pi_i(P_J(x)) = 0$ in any other case. Let $(r_n)_{n=1}^{\infty}$ be an enumeration of the rational numbers in $[0,1]$ and fix a countable set $L \subseteq \Gamma$ such that $P_L(f(r_n)) = f(r_n)$ for every $n \in \mathbb{N}$. Then, $f = (f - P_L f) + P_L f$. Since $P_L f$ is Riemann integrable and takes values in the space

$$X|_L := \{x \in X : \pi_i(x) = 0 \text{ for each } i \notin L\},$$

which is isomorphic to a countable ℓ_1-sum of spaces with the WLP, by Theorem 1.3.16 $P_L f$ is weakly continuous almost everywhere.

Therefore, we can assume that $\int_0^1 f(t)dt = 0$ and that f is null over a dense set. Let

$$A_n^J := \{t \in [0,1] \colon \|P_{J^c}(f(t))\| \geq \frac{1}{n}\}$$

for each $n \in \mathbb{N}$ and each subset $J \subseteq \Gamma$. If $J_1 \subseteq J_2 \subseteq \Gamma$, then $A_n^{J_2} \subseteq A_n^{J_1}$.

Claim: *For every $n \in \mathbb{N}$ there exists a countable set $J \subseteq \Gamma$ with $\mu\left(\overline{A_n^J}\right) = 0$.*

Suppose this is not the case. Then, there exist $n \in \mathbb{N}$ and $\delta > 0$ with $\mu\left(\overline{A_n^J}\right) > \delta$ for every countable subset $J \subseteq \Gamma$ (if for every $m \in \mathbb{N}$ we can take a countable set $J_m \subseteq \Gamma$ with $\mu\left(\overline{A_n^{J_m}}\right) < \frac{1}{m}$, then $J = \bigcup_{m\in\mathbb{N}} J_m$ satisfies $\mu\left(\overline{A_n^J}\right) = 0$). Let $\mathcal{P} = \{I_1, I_2, \ldots, I_N\}$ be a partition of $[0,1]$ such that

$$\left\|\sum_{j=1}^{N} \mu(I_j)(f(\xi_j) - f(\xi_j'))\right\| < \frac{\delta}{n} \text{ for all choices } \xi_j, \xi_j' \in I_j, 1 \leq j \leq N. \tag{1.6}$$

Notice that since we can choose any arbitrary subcollection of ξ_j, ξ_j' from the dense subset where $f = 0$, the same inequality as (1.6) remains valid for any arbitrary partial sum $\sum_{j\in M} \mu(I_j)(f(\xi_j) - f(\xi_j'))$ with $M \subseteq \{1, ..., N\}$. Let $J \subseteq \Gamma$ be a countable subset. Since $\sum_{j=1}^{N} \mu\left(I_j \cap \overline{A_n^J}\right) = \mu\left(\overline{A_n^J}\right) > \delta$ and f is null over a dense set, we can suppose that there exist $\xi_1 \in \operatorname{Int}(I_1) \cap A_n^J$ and $\xi_1' \in I_1$ such that $\|\mu(I_1)(f(\xi_1) - f(\xi_1'))\| \geq \frac{1}{n}\mu(I_1)$. Let $J_1 = \operatorname{supp} f(\xi_1) \cup \operatorname{supp} f(\xi_1')$. By (1.6) we have $\mu(I_1) < \delta < \sum_{j=1}^N \mu\left(I_j \cap \overline{A_n^{J_1}}\right)$ and so it is not restrictive to suppose $\mathrm{Int}(I_2) \cap \overline{A_n^{J_1}} \neq \emptyset$. Thus, due to Lemma 1.3.15, we can choose $\xi_2, \xi_2' \in I_2$ such that

$$\|\mu(I_1)(f(\xi_1) - f(\xi_1')) + \mu(I_2)(f(\xi_2) - f(\xi_2'))\| \geq \frac{1}{n}(\mu(I_1) + \mu(I_2)).$$

Fix $1 \leq k < N$ and assume that we have already chosen points $\xi_j, \xi_j' \in I_j$ for all $1 \leq j \leq k$ with the property that

$$\left\|\sum_{j=1}^k \mu(I_j)(f(\xi_j) - f(\xi_j'))\right\| \geq \frac{1}{n}\left(\sum_{j=1}^k \mu(I_j)\right).$$

Set $J_k := \bigcup_{j=1}^k \operatorname{supp} f(\xi_j) \cup \operatorname{supp} f(\xi_j')$, which is countable. By (1.6) we have $\sum_{j=1}^k \mu(I_j) < \delta < \sum_{j=1}^N \mu\left(I_j \cap \overline{A_n^{J_k}}\right)$, hence it is not restrictive to suppose that $\mathrm{Int}(I_{k+1}) \cap \overline{A_n^{J_k}} \neq \emptyset$ and therefore that there exist points $\xi_{k+1}, \xi_{k+1}' \in I_{k+1}$ such that

$$\left\|\sum_{j=1}^{k+1} \mu(I_j)(f(\xi_j) - f(\xi_j'))\right\| \geq \frac{1}{n}\left(\sum_{j=1}^{k+1} \mu(I_j)\right).$$

Since $\sum_{j=1}^N \mu(I_j) = 1 > \delta$, it follows that there exist $\xi_j, \xi_j' \in I_j$ for every $1 \leq j \leq N$ such that

$$\left\|\sum_{j=1}^N \mu(I_j)(f(\xi_j) - f(\xi_j'))\right\| \geq \frac{\delta}{n}.$$

But this is a contradiction with (1.6). Therefore, the **Claim** is proved.

Thus, for every $n \in \mathbb{N}$ there exists a countable set J_n such that $\mu\left(\overline{A_n^{J_n}}\right) = 0$. Fix $J := \bigcup_{n\in\mathbb{N}} J_n$. Theorem 1.3.16 guarantees the existence of a set $F \subseteq [0,1]$ of measure one such that $P_J(f)$ is weakly continuous at every point of F. Let $E = F \setminus \bigcup_{n\in\mathbb{N}} \overline{A_n^J}$. Then, $\mu(E) = 1$, $f = P_J(f) + P_{J^c}(f)$, $P_J(f)$ is weakly continuous at each point of E and $P_{J^c}(f)$ is norm continuous at each point of E (if $t_n \to t \in E$ then $P_{J^c}(f)(t) = 0$ and for every $m \in \mathbb{N}$, $t_n \notin A_m^J$ for n big enough so $\|P_{J^c}(f)(t_n)\| < \frac{1}{m}$). □

Corollary 1.3.19 ([Piz89, NOR73]). *$\ell_1(\kappa)$ has the LP for any cardinal κ.*

Proof. Since $\ell_1(\kappa)$ has the Schur property, $\ell_1(\kappa)$ has the LP if and only it has the WLP. Therefore, the conclusion follows from Theorem 1.3.18. □

As an application of 1.3.18 we also obtain the following result:

Corollary 1.3.20. *Let K be a compact Hausdorff space. Then, $\mathcal{C}(K)^*$ has the WLP if* $\operatorname{dens}(L^1(\lambda)) < \operatorname{cov}(\mathcal{M})$ *for every regular Borel probability measure λ on K.*

Proof. For every compact Hausdorff space K, the Banach space $\mathcal{C}(K)^*$ is isometric to an ℓ_1-sum of $L^1(\lambda)$ spaces, where each λ is a regular Borel probability measure on K (see the proof of [AK06, Proposition 4.3.8]). Thus, $\mathcal{C}(K)^*$ has the WLP if each space $L^1(\lambda)$ has the WLP, due to Theorem 1.3.18. Hence, the result follows from Theorem 1.3.17. □

Corollary 1.3.21. *If K is a compact Hausdorff space in the class MS (i.e. $L^1(\lambda)$ is separable for every regular Borel probability measure on K), then $\mathcal{C}(K)^*$ has the WLP.*

Some classes of compact spaces in the class MS are metric compacta, Eberlein compacta, Radon-Nikodým compacta, Rosenthal compacta and weakly Radon-Nikodým compacta (see next chapter for definitions). For more details on this class, we refer the reader to [DK95], [MP12] and [PS15].

The LP is a three-space property, i.e. if X is a Banach space and Y is a subspace of X such that Y and X/Y have the LP, then X has the LP [Piz89, Proposition 1.19]. This result follows from Michael's Selection Theorem. However, as far as we are concerned, it is not known whether the WLP is a three-space property. We have a positive result in the following case:

Theorem 1.3.22. *Let X be a Banach space and Y a subspace of X. If Y is reflexive,* $\operatorname{dens}(Y) < \operatorname{cov}(\mathcal{M})$ *and X/Y has the WLP, then X has the WLP.*

Proof. Let $Q\colon X \to X/Y$ be the quotient operator and $\phi\colon X/Y \to X$ be a norm-norm continuous map such that $Q\phi = 1_{X/Y}$ given by Michael's Selection Theorem (see [FHH$^+$11, Section 7.6]). Let $f\colon [0,1] \to X$ be a Riemann integrable function. Then, since Qf is Riemann integrable and X/Y has the WLP, there exists a set $E \subseteq [0,1]$ with $\mu(E) = 1$ such that Qf is weakly continuous at every point of E. Set

$$C = \{x \in X : \exists\, (t_n)_{n=1}^{\infty} \text{ converging to some } t \in E \text{ with } x = w\text{-}\lim f(t_n)\}. \quad (1.7)$$

First we are going to see that $\operatorname{dens}(C) < \operatorname{cov}(\mathcal{M})$. Let $x \in C$ and $(t_n)_{n=1}^{\infty}$ as in (1.7). Then $Qx = w\text{-}\lim Qf(t_n) = Qf(t)$. Therefore, $x = \phi(Qx) + (x - \phi(Qx))$ with $\phi(Qx) \in \phi(Qf(E))$ and $x - \phi(Qx) \in Y$. Notice that $\phi(Qf(E))$ is separable because of the w-separability of $Qf(E)$ and Mazur's Lemma. Thus, $C \subseteq \phi(Qf(E)) + Y$ satisfies $\operatorname{dens}(C) < \operatorname{cov}(\mathcal{M})$.

Let $\{x_\alpha^*\}_{\alpha\in\Gamma} \subseteq X^*$ be a set separating points of C with $|\Gamma| < \operatorname{cov}(\mathcal{M})$. Set $E_0 \subseteq E$ with $\mu(E_0) = 1$ such that $x_\alpha^* f$ is continuous at every point of E_0 for every

$\alpha \in \Gamma$. Notice that this can be done because the set of discontinuity points of each $x_\alpha^* f$ is an F_σ Lebesgue null set and $|\Gamma| < \operatorname{cov}(\mathcal{M})$. We claim that f is weakly continuous at each point of E_0. Let $t \in E_0$ and $(t_n)_{n=1}^\infty$ be a sequence converging to t. Since $Qf(t) = w\text{-}\lim Qf(t_n)$, the set $\{Qf(t_n) \colon n \in \mathbb{N}\}$ is relatively weakly compact in X/Y. From the reflexivity of Y, it follows that Q is a Tauberian operator (i.e. ${Q^{**}}^{-1}(X/Y) \subseteq X$), so $\{f(t_n) \colon n \in \mathbb{N}\}$ is relatively weakly compact in X (see [GMA10, Theorem 2.1.5 and Corollary 2.2.5]). Therefore, it is enough to prove the uniqueness of the limit of the subsequences of $(f(t_n))_{n=1}^\infty$. Let $x = w\text{-}\lim_k f(t_{n_k})$. Then, $x, f(t) \in C$ and $x_\alpha^*(x) = \lim_k x_\alpha^*(f(t_{n_k})) = x_\alpha^*(f(t))$ for every $\alpha \in \Gamma$, so $x = f(t)$. □

Section 1.4

Weak continuity does not imply integrability

It is not true that every weakly continuous function is Riemann integrable [AO51]. In fact, V. Kadets proved the following theorem:

Theorem 1.4.1 ([Kad94]). *If X is a Banach space without the Schur property, then there is a weakly continuous function $f \colon [0,1] \to X$ which is not Riemann integrable.*

The proof of the previous theorem together with Josefson-Nissenzweig Theorem (see [Die84, Chapter XII]) gives the following corollary:

Corollary 1.4.2. *Given an infinite-dimensional Banach space X, there always exists a weak* continuous function $f \colon [0,1] \to X^*$ which is not Riemann integrable.*

In [WY00], Wang and Yang extend the previous result to a general locally convex topology weaker than the norm topology. In this section, we generalize these results in Theorem 1.4.4.

Following the terminology used in [EGS14], we say that a subset M of a Banach space is *spaceable* if $M \cup \{0\}$ contains a closed infinite-dimensional subspace.

We start with the definitions of τ-Dunford-Pettis operators and the τ-Schur property, which coincide with the classical definitions of Dunford-Pettis or completely continuous operator and the Schur property when τ is the weak topology.

Definition 1.4.3. *Let X and Y be Banach spaces and τ a locally convex topology on X weaker than the norm topology. An operator $T \colon X \to Y$ is said to be τ-Dunford-Pettis (τ-DP for short) if it carries bounded τ-null sequences to norm null sequences. A Banach space X is said to have the τ-Schur property if the identity operator $I \colon X \to X$ is τ-DP.*

Theorem 1.4.4. *Let X and Y be Banach spaces and τ be a locally convex topology on X weaker than the norm topology. If $T\colon X \to Y$ is an operator which is not τ-DP, then the family of all bounded τ-continuous functions $f\colon [0,1] \to X$ such that Tf is not Riemann integrable is spaceable in $\ell_\infty([0,1], X)$, the space of all bounded functions from $[0,1]$ to X with the supremum norm.*

Proof. The proof uses ideas from [Kad94]. Since T is not τ-DP, we can take a bounded sequence $(x_n)_{n=1}^\infty$ that is τ-convergent to zero such that $\|Tx_n\| = 1$ for all $n \in \mathbb{N}$.

Let $K \subseteq [0,1]$ be a copy of the Cantor set constructed by removing from $[0,1]$ an open interval I_1^1 in the middle of $[0,1]$ and removing open intervals $I_1^n, I_2^n, \dots I_{2^n}^n$ from the middles of the remaining intervals in each step. Suppose that the removed intervals are so small that $\mu(K) > \frac{2}{3}$. Let $\mathcal{C}_a([0,1])$ be the closed subspace of $\mathcal{C}([0,1])$ consisting of all continuous functions $g\colon [0,1] \to \mathbb{R}$ antisymmetric with respect to the axis $x = \frac{1}{2}$ and with $g(0) = g(1) = 0$. For every $g \in \mathcal{C}_a([0,1])$ and every open interval $I = (a,b)$ in $[0,1]$, we define the functions $g_I\colon [0,1] \to \mathbb{R}$ and $f_g\colon [0,1] \to X$ as follows

$$g_I(t) = \begin{cases} 0 & \text{if } t \notin (a,b), \\ g(\frac{t-a}{b-a}) & \text{if } t \in [a,b]. \end{cases}$$

$$f_g(t) = \begin{cases} 0 & \text{if } t \in K, \\ g_{I_k^n}(t)x_n & \text{if } t \in I_k^n. \end{cases}$$

The function $\phi\colon \mathcal{C}_a([0,1]) \to \ell_\infty([0,1], X)$ given by the formula $\phi(g) := f_g$ for every $g \in \mathcal{C}_a([0,1])$ is a linear map and satisfies $\|\phi(g)\| = (\sup_n \|x_n\|)\|g\|$ for every $g \in \mathcal{C}_a([0,1])$. Therefore, ϕ is a multiple of an isometry. Thus, $V := \phi(\mathcal{C}_a([0,1])$ is an infinite-dimensional closed subspace of $\ell_\infty([0,1], X)$.

We are going to check that each function $f_g \neq 0$ is τ-continuous but Tf_g is not Riemann integrable. Since g is continuous, $g(0) = g(1) = 0$ and $x_n \xrightarrow{\tau} 0$, f_g is τ-continuous. Suppose Tf_g is Riemann integrable. Then,

$$y^*\left(\int_0^1 Tf_g(t)dt\right) = \int_0^1 y^*Tf_g(t)dt = \sum_{k,n} y^*(Tx_n)\int_{I_k^n} g_{I_k^n}(t)dt = 0$$

for each $y^* \in Y^*$. The only possible value for the Riemann integral of Tf_g is 0 due to the above equality. Choose a partition $\mathcal{P} = \{J_1, J_2, \dots, J_N\}$ of $[0,1]$. Let $A = \{j : 1 \leq j \leq N,\ \operatorname{Int} J_j \cap K \neq \emptyset\}$. We can take $m \in \mathbb{N}$ such that if $j \in A$ then J_j contains some interval I_k^m. Hence, if $j \in A$, there is $t_j \in J_j$ such that $f_g(t_j) = \|g\|x_m$. If $j \notin A$, then we pick any $t_j \in \operatorname{Int} J_j$. From the inequality $\sum_{j\in A} \mu(J_j) \geq \mu(K) > \frac{2}{3}$, we deduce

$$\left\|\sum_{j=1}^N \mu(J_j)Tf_g(t_j)\right\| = \left\|\sum_{j\in A} \mu(J_j)Tf_g(t_j) + \sum_{j\notin A} \mu(J_j)Tf_g(t_j)\right\| \geq$$

$$\geq \left\| \sum_{j \in A} \|g\| \mu(J_j) T x_m \right\| - \left\| \sum_{j \notin A} \mu(J_j) T f_g(t_j) \right\| > \frac{2}{3}\|g\| - \frac{1}{3} \sup_{t \in [0,1]} \|T f_g(t)\| = \frac{1}{3}\|g\|.$$

Then, Tf_g is Riemann integrable if and only if $g = 0$ if and only if $f_g = 0$. □

The next corollary gives an affirmative answer to a question posed by Sofi in [Sof12].

Corollary 1.4.5. *Given an infinite-dimensional Banach space X, the set of all weak* continuous functions $f\colon [0,1] \to X^*$ which are not Riemann integrable is spaceable in $\ell_\infty([0,1], X^*)$.*

Proof. X^* is not w^*-Schur for any infinite-dimensional Banach space X due to the Josefson-Nissenzweig Theorem. Thus, the conclusion follows from Theorem 1.4.4. □

Given a Banach space X, a function $f\colon [0,1] \to X$ is said to be scalarly Riemann integrable if every composition $x^* f$ with $x^* \in X^*$ is Riemann integrable.

We can also characterize Dunford-Pettis operators thanks to Theorem 1.4.4. The equivalence (1) ⇔ (3) in the following corollary was mentioned without proof in [PdRF80].

Corollary 1.4.6. *Let X and Y be Banach spaces and $T\colon X \to Y$ be an operator. The following statements are equivalent:*

1. *T is Dunford-Pettis.*
2. *Tf is Riemann integrable for every w-continuous function $f\colon [0,1] \to X$.*
3. *Tf is Riemann integrable for every scalarly Riemann integrable function $f\colon [0,1] \to X$.*

Proof. (2) ⇒ (1) is a consequence of Theorem 1.4.4. Since every w-continuous function $f\colon [0,1] \to X$ is scalarly Riemann integrable, (3) implies (2). Therefore, it remains to prove (1) ⇒ (3). Suppose T is Dunford-Pettis and fix $(\mathcal{P}_n)_{n=1}^\infty$ a sequence of tagged partitions of $[0,1]$ with $\|\mathcal{P}_n\| \xrightarrow{n} 0$. Let $f\colon [0,1] \to X$ be a scalarly Riemann integrable function. Then, $x^* f(\mathcal{P}_n) \xrightarrow{n} \int_0^1 x^* f(t)dt$ for every $x^* \in X^*$. Thus, $f(\mathcal{P}_n)$ is a w-Cauchy sequence in X, so $Tf(\mathcal{P}_n)$ is norm convergent to some $y \in Y$. The limit y does not depend on the sequence of tagged partitions, since if $(\mathcal{P}'_n)_{n=1}^\infty$ is any other sequence of tagged partitions with $\|\mathcal{P}'_n\| \xrightarrow{n} 0$, then $f(\mathcal{P}_n) - f(\mathcal{P}'_n)$ is weakly null and this in turn implies that $\|Tf(\mathcal{P}_n) - Tf(\mathcal{P}'_n)\| \xrightarrow{n} 0$. Thus, Tf is Riemann integrable. □

Chapter 2

Weakly Radon-Nikodým compact spaces

Section 2.1
Introduction

In [Nam87], I. Namioka defined a compact space K to be **Radon-Nikodým** (RN for short) if and only if it is homeomorphic to a weak*-compact subset of a dual Banach space with the Radon-Nikodým property. C. Stegall proved that the dual X^* of a Banach space X has the Radon-Nikodým property if and only if every separable subspace of X has separable dual. In this case X is said to be an Asplund space.

Since every Asplund Banach space does not contain an isomorphic copy of ℓ_1, the class of weakly Radon-Nikodým compact spaces generalizes the class of RN compact spaces. In [GM12], E. Glasner and M. Megrelishvili define a compact space to be **weakly Radon-Nikodým** (WRN for short) if and only if it is homeomorphic to a weak*-compact subset of the dual of a Banach space not containing an isomorphic copy of ℓ_1. The name of this class is motivated by a characterization of dual Banach spaces with the weak Radon-Nikodým property[1] as those Banach spaces whose predual does not contain an isomorphic copy of ℓ_1.

In this chapter we focus on the class of WRN compacta. Nevertheless, we also study some other classes of compact spaces associated to classes of Banach spaces.

Definition 2.1.1. *Let X be a Banach space.*

[1] The weak Radon-Nikodým property is analogous to the Radon-Nikodým property but with respect to Pettis integral instead of Bochner integral.

- *A sequence* $(x_n)_{n=1}^\infty$ *in* X *is said to be weakly Cauchy if* $x^*(x_n)$ *is convergent for every* $x^* \in X^*$.

- *A set* $W \subseteq X$ *is said to be weakly precompact if every sequence in* W *has a weakly Cauchy subsequence. Notice that, by the Eberlein-Šmulian Theorem, every weakly precompact set is weakly compact.*

- X *is said to be weakly compactly generated (WCG for short) if there exists a weakly compact set* $W \subseteq X$ *such that* $\overline{\mathrm{span}}(W) = X$.

- X *is said to be weakly precompactly generated (WPG for short) if there exists a weakly precompact set* $W \subseteq X$ *such that* $\overline{\mathrm{span}}(W) = X$.

- X *is said to be Asplund if every separable subspace of* X *has separable dual.*

- X *is said to be Asplund generated if there exist an Asplund space* Y *and a bounded linear operator* $T\colon Y \to X$ *with dense range.*

- X *is said to be weakly Lindelöf determined (WLD) if* (X^*, w^*) *can be topologically embedded in a* Σ*-product*

$$\Sigma(\Gamma) = \{x \in \mathbb{R}^\Gamma : \mathrm{supp}(x) \textit{ is countable}\}$$

  with the topology of pointwise convergence.

These classes of Banach spaces are related with the following classes of compact spaces (see [Kos15b]):

Definition 2.1.2. *Let* K *be a compact space.*

- K *is said to be in the class* MS *if every regular Borel measure on* K *is separable.*

- K *is said to be Eberlein if it is homeomorphic to a weakly compact subset of a Banach space.*

- K *is said to be Corson if it can be embedded in a* Σ*-product* $\Sigma(\Gamma)$ *for some set* Γ.

Theorem 2.1.3. *Let* X *be a Banach space and* K *a compact space.*

- K *is Eberlein if and only if* $\mathcal{C}(K)$ *is WCG.*

- *If* X *is WCG then* (B_{X^*}, w^*) *is Eberlein. Moreover,* X *is a subspace of a WCG space if and only if* (B_{X^*}, w^*) *is Eberlein.*

- K *is Corson and it is in the class* MS *if and only if* $\mathcal{C}(K)$ *is WLD.*

- *X is WLD if and only if (B_{X^*}, w^*) is Corson.*
- *K is RN if and only if $\mathcal{C}(K)$ is Asplund generated.*
- *If X is Asplund generated then (B_{X^*}, w^*) is RN.*

The class of WPG Banach spaces was introduced by R. Haydon in [Hay81]. In this paper he considered the class of compact spaces K such that $\mathcal{C}(K)$ is WPG. Namely, he said the following:

> *The present author would be interested to know whether WPG spaces have any of the good properties of these other classes, and whether there is a nice characterization of those compact spaces T for which $\mathcal{C}(T)$ is WPG. One obvious question is whether every such space T contains a nontrivial convergent sequence.*

We finish this section with results analogous to Theorem 2.1.3 for WPG Banach spaces and WRN compact spaces. In particular, we prove that the class of compact spaces considered by R. Haydon is exactly the class of WRN compacta.

Theorem 2.1.4. *Let K be a WRN compact space. Then $\mathcal{C}(K)$ is WPG.*

Proof. Without loss of generality, we suppose that K is a subspace of (B_{X^*}, w^*) for some Banach space X not containing ℓ_1. Then, Rosenthal's ℓ_1-theorem 2.2.9 asserts that B_X is a weakly precompact set.

Let $T\colon X \to \mathcal{C}(K)$ be the bounded linear operator determined by the formula $(Tx)(x^*) = x^*(x)$ for every $x^* \in K$ and every $x \in X$. Notice that every bounded linear operator transforms weakly Cauchy sequences into weakly Cauchy sequences and therefore weakly precompact sets into weakly precompact sets. Thus, $T(B_X) \subseteq \mathcal{C}(K)$ is a weakly precompact set in $\mathcal{C}(K)$ separating the points of K. Set $W = T(B_X) \cup \{1\}$, which is also a weakly precompact set in $\mathcal{C}(K)$, and take

$$W_n = \frac{W^n}{2^n} = \left\{ \frac{f_1 f_2 \dots f_n}{2^n} \in \mathcal{C}(K) : f_1, \dots f_n \in W \right\}.$$

Each W_n is also weakly precompact. Fix $L = \bigcup_{n\in\mathbb{N}} W_n$. Since every sequence in L has a subsequence convergent to zero or a subsequence in some W_n, the set L is also weakly precompact. Notice that $\mathrm{span}(L)$ is a subspace of $\mathcal{C}(K)$ closed under multiplication which contains constant functions and separates points of K. By the Stone-Weierstrass Theorem, we have $\overline{\mathrm{span}} L = \mathcal{C}(K)$ and therefore $\mathcal{C}(K)$ is WPG. □

Theorem 2.1.5. *If X is a WPG Banach space then (B_{X^*}, w^*) is WRN.*

Proof. An analogous version of the Davis-Figiel-Johnson-Pełczińsky Factorization Method [DFJP74] states that a Banach space X is WPG if and only if there exists a Banach space Y not containing ℓ_1 and a bounded linear operator $T\colon Y \to X$ with dense range (see for example [AGL+16, Section 4]). Then $T^*\colon X^* \to Y^*$ restricted to B_{X^*} is a weak*-embedding from B_{X^*} into the dual of a Banach space not containing ℓ_1, so B_{X^*} is WRN. □

Corollary 2.1.6. *A compact space K is WRN if and only if $\mathcal{C}(K)$ is WPG.*

Proof. Since every subspace of a WRN compact space is WRN and K is homeomorphic to a subset of $B_{\mathcal{C}(K)^*}$, the result is a consequence of Theorems 2.1.4 and 2.1.5. □

It is not true in general that if X has WRN dual ball then X is WPG. A Banach space is said to be weakly sequentially complete if every weakly Cauchy sequence in it is weakly convergent. Therefore, by the Eberlein-Šmulian Theorem, a weakly sequentially complete Banach space is WPG if and only if it is WCG. For every measure μ, the space $L^1(\mu)$ is weakly sequentially complete. H. Rosenthal constructed a probability measure μ and a nonWCG subspace X of $L^1(\mu)$ [Ros74b]. Therefore, this space is an example of a nonWPG Banach space with WRN dual ball (B_{X^*} is Eberlein since it is a continuous image of the dual ball of $L^1(\mu)$ and $L^1(\mu)$ is WCG).

Section 2.2
Topological characterizations of WRN compact spaces

Some of the classes introduced in the previous section have nice topological characterizations. For example, D. Amir and J. Lindenstrauss proved that a compact space K is Eberlein if and only if it can be embedded in $c_0(\Gamma)$ for some set Γ, where $c_0(\Gamma) \subseteq \mathbb{R}^\Gamma$ is equipped with the topology of pointwise convergence.

Moreover, I. Namioka provided the following characterization of RN compact spaces.

Definition 2.2.1. *Let K be a compact space and $d\colon K \times K \to \mathbb{R}$ a metric on K, not necessarily related with the topology on K.*

- *The metric d is said to fragment K if for every $\varepsilon > 0$ and every nonempty closed set $F \subseteq K$ there is an open set $U \subseteq K$ such that $U \cap F \neq \emptyset$ and*

$$\mathrm{diam}_d(U \cap F) = \sup\{d(x,y) : x, y \in U \cap F\} < \varepsilon.$$

- *For any sets $U, V \subseteq K$ we denote*

$$d(U,V) := \inf\{d(x,y) : x \in U, y \in V\}.$$

- *The metric d is said to be lower semicontinuous (l.s.c.) if for every distinct points $x, y \in K$ and every $0 < \delta < d(x,y)$ there are open sets U, V with $x \in U$ and $y \in V$ such that $d(U,V) > \delta$.*

Theorem 2.2.2. *[Nam87] A compact space K is RN if and only if there is a l.s.c. metric on K which fragments K.*

We are going to see a characterization of WRN compact spaces in terms of fragmented families of functions.

Definition 2.2.3 ([GM12])**.** *Let K be a compact space. A family of functions $\mathcal{F} \subseteq \mathcal{C}(K)$ is said to be* fragmented *if for every nonempty subset A of K and every $\varepsilon > 0$ there exists an open subset U in K such that $U \cap A$ is nonempty and $f(U \cap A)$ has diameter smaller than ε for every $f \in \mathcal{F}$. $\mathcal{F}$ is said to be* eventually fragmented *if every sequence in F has a subsequence which is a fragmented family on K.*

The equivalence among conditions *2*, *3* and *4* in the following theorem is classical; see for example [Tal84, Theorem 14.1.7].

Theorem 2.2.4. *([GM12, Fact 4.3 and Proposition 4.6]) Let K be a compact space and $\mathcal{F} \subseteq \mathcal{C}(K)$ a uniformly bounded family of continuous functions. Then the following conditions are equivalent:*

1. *$\mathcal{F}$ is an eventually fragmented family.*
2. *$\mathcal{F}$ is weakly precompact.*
3. *Every sequence in $\mathcal{F}$ has a pointwise convergent subsequence.*
4. *$\mathcal{F}$ does not contain ℓ_1-sequences.*

Theorem 2.2.5 ([GM12, Theorem 6.5])**.** *Let K be a compact space. Then K is WRN if and only if there exists an eventually fragmented uniformly bounded family of continuous functions $\mathcal{F} \subseteq \mathcal{C}(K)$ which separates the points of K.*

Another useful characterization of WRN compact spaces is given in terms of independent sequences:

Definition 2.2.6. *A sequence $(A_n^0, A_n^1)_{n\in\mathbb{N}}$ of disjoint pairs of subsets of a set S is said to be* independent *if $\bigcap_{k=1}^n A_k^{\epsilon(k)} \neq \emptyset$ for every $n \in \mathbb{N}$ and every $\epsilon\colon \{1,2,...,n\} \to \{0,1\}$.*

A sequence of functions $(f_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^S$ is said to be independent *if there exist real numbers $p < q$ such that the sequence $(A_n^0, A_n^1)_{n\in\mathbb{N}}$ is independent, where $A_n^0 = \{s \in S : f_n(s) < p\}$ and $A_n^1 = \{s \in S : f_n(s) > q\}$ for every $n \in \mathbb{N}$.*

Remark 2.2.7. *A sequence $(A_n^0, A_n^1)_{n\in\mathbb{N}}$ of disjoint pairs of subsets of a set S is independent if and only if for every $n \in \mathbb{N}$ and every $\epsilon\colon \{1,2,...,n\} \to \{0,1\}$ the set $\bigcap_{k=1}^n A_k^{\epsilon(k)}$ is infinite, since if $\left|\bigcap_{k=1}^n A_k^{\epsilon(k)}\right| = m$ is finite then we can extend ϵ to $\epsilon'\colon \{1,2,...,n+m\} \to \{0,1\}$ satisfying*

$$\left|\bigcap_{k=1}^{n+j} A_k^{\epsilon'(k)}\right| \leq m - j$$

for every $0 \leq j \leq m$.

Theorem 2.2.8 ([Ros74b])**.** *Let S be a set and $f_n\colon S \to \mathbb{R}$ a uniformly bounded sequence of functions. Then f_n has a pointwise convergent subsequence or an independent subsequence.*

Since every uniformly bounded independent sequence of functions is equivalent in the supremum norm to the usual ℓ_1-basis, an easy application of the previous result yields the well-known Rosenthal's ℓ_1-Theorem:

Theorem 2.2.9 (Rosenthal's ℓ_1-Theorem, [Ros74b])**.** *Let X be a Banach space and $(x_n)_{n=1}^\infty$ a bounded sequence in X. Then $(x_n)_{n=1}^\infty$ contains a weakly Cauchy subsequence or a subsequence equivalent to the ℓ_1-basis.*

Thus, we obtain the following characterization of WRN compact spaces:

Theorem 2.2.10. *A compact space K is WRN if and only if there exist a set Γ and a homeomorphic embedding $e\colon K \to [0,1]^\Gamma$ such that for every $p < q$, the family of disjoint pairs of subsets $(A_\alpha^0, A_\alpha^1)_{\alpha\in\Gamma}$ does not contain independent sequences, where $A_\alpha^0 = \{x \in K : e(x)_\alpha < p\}$ and $A_\alpha^1 = \{x \in K : e(x)_\alpha > q\}$ for every $\alpha \in \Gamma$.*

Proof. If there exists such an embedding, then the family of functions $\mathcal{F} = \{f_\alpha\}_{\alpha\in\Gamma}$ is a weakly precompact family of continuous functions separating the points of K, where $f_\alpha(x) = e(x)_\alpha$ for every $x \in K$ and every $\alpha \in \Gamma$. Therefore, K is WRN due to Theorems 2.2.4 and 2.2.5.

On the other hand, if K is WRN then there exists a uniformly bounded family $\mathcal{F} \subseteq \mathcal{C}(K)$ without independent sequences and separating the points of K. If $\|f\| \leq M$ for every $f \in \mathcal{F}$, then set

$$\Gamma = \frac{\mathcal{F} + M}{2M} = \left\{\frac{f + M}{2M} : f \in \mathcal{F}\right\} \subseteq [0,1]^K$$

and define $e\colon K \to [0,1]^\Gamma$ such that $e(x)_\alpha = \alpha(x)$ for every $\alpha \in \Gamma$. Since Γ separates the points of K and it does not contain an independent sequence of functions, e is an embedding which satisfies the conditions of the theorem. $\square$

We can extend the previous characterization by generalizing the definition of independent sequence of functions:

Definition 2.2.11. *Let K, L be compact spaces. A sequence of functions $f_n\colon K \to L$ is said to be* L-independent *if there exist closed disjoint sets C, C' in L such that the sequence $(f_n^{-1}(C), f_n^{-1}(C'))_{n\in\mathbb{N}}$ is independent.*

Lemma 2.2.12. *Let K, L be compact spaces and $f_n\colon K \to L$ an L-independent sequence of continuous functions. Then f_n does not have pointwise convergent subsequences.*

Proof. Take C, C' closed sets as in the Definition 2.2.11. Take any subsequence f_{n_k}. Since $\left(f_{n_k}^{-1}(C), f_{n_k}^{-1}(C')\right)_{n\in\mathbb{N}}$ is an independent sequence consisting of compact subsets of K, we can take $t \in \bigcap_{k\in\mathbb{N}} \left(f_{n_{2k}}^{-1}(C) \cap f_{n_{2k+1}}^{-1}(C')\right)$. Thus, $f_{n_k}(t)$ cannot be a convergent sequence since C and C' are disjoint closed sets. $\square$

Lemma 2.2.13. *Let S be a set, M a metric compact space and $f_n\colon S \to M$ a sequence of functions. Then f_n has a pointwise convergent subsequence or an M-independent subsequence.*

Proof. Take $q\colon M \to [0,1]^{\mathbb{N}}$ an embedding from M into the Hilbert cube and denote by q_n the nth-coordinate function of q. Suppose f_n does not have an M-independent subsequence. Then, $q_1 \circ f_n$ does not have an independent subsequence. By Theorem 2.2.8 there exists a convergent subsequence of $q_1 \circ f_n$. A standard diagonal argument provides a subsequence f_{n_k} such that $(q_m \circ f_{n_k})_{k=1}^{\infty}$ converges for every $m \in \mathbb{N}$. Thus, f_{n_k} is a convergent subsequence of f_n. $\square$

Theorem 2.2.14. *Let K, M be compact spaces and $\mathcal{F}$ a family of continuous functions from K to M which separates the points of K. If M is metrizable and $\mathcal{F}$ does not have M-independent sequences, then K is WRN.*

Proof. As in the previous proof, take $q\colon M \to [0,1]^{\mathbb{N}}$ an embedding from M into the Hilbert cube, with q_n the coordinate functions of q. Set $\mathcal{F}_n = \{\frac{q_n \circ f}{n} : f \in \mathcal{F}\}$ and $\mathcal{F}' = \bigcup_{n\in\mathbb{N}} \mathcal{F}_n$. Then, each $\mathcal{F}_n$ does not contain independent sequences of functions and it is immediate that $\mathcal{F}'$ is a family of continuous functions which separates the points of K and with no independent subsequences. Therefore, K is WRN. $\square$

Section 2.3
Quasi WRN compact spaces

In [Arv02] and [FHM98] two superclasses of continuous images of RN compacta are defined. In [Avi05] it is proved that both superclasses are equal. Compact spaces of these superclasses are called QRN.

Definition 2.3.1. *Let $\varepsilon > 0$ and K a compact space.*

- *A pseudometric $d\colon K \times K \to \mathbb{R}$ is a symmetric nonnegative function which satisfies the triangle inequality and such that $d(x,x) = 0$ for every $x \in K$, but we may have $d(x,y) = 0$ for distinct x, y.*
- *A pseudometric $d\colon K \times K \to \mathbb{R}$ on K is said to ε-fragment K if for each nonempty subset L of K there is a nonempty relatively open subset U of L of d-diameter smaller than ε.*
- *A compact space K is QRN if and only if there are uniformly bounded sets $\{A_{n,p} : n, p \in \mathbb{N}\}$ in $\mathcal{C}(K)$ such that $\mathcal{C}(K) = \bigcup_{n\in\mathbb{N}} A_{n,p}$ for every $p \in \mathbb{N}$, and K is $\frac{1}{p}$-fragmented by the pseudometric $d_{A_{n,p}}$, where $d_{A_{n,p}}(x,y) = \sup_{f\in A_{n,p}} |f(x) - f(y)|$ for every $x, y \in K$ and every $n, p \in \mathbb{N}$.*

We will use this definition of QRN compacta in Theorem 2.3.3, although some other equivalent definitions of QRN compact spaces might be sometimes more convenient (see for example the comment that follows Definition 2.5.1).

In [Arv02] it is proved that zero-dimensional QRN compact spaces are RN and that continuous images of QRN compact spaces are QRN. In essence, our definition of QWRN compact spaces is analogous to the definition of QRN compacta given in [FHM98]. In this section, we prove similar results for QWRN compact spaces.

Definition 2.3.2. *A compact space K is* ***quasi WRN*** *(QWRN for short) if there exists a homeomorphic embedding $e\colon K \to [0,1]^\Gamma$ such that for every $\varepsilon > 0$ there exists a decomposition $\Gamma = \bigcup_{n\in\mathbb{N}} \Gamma_n^\varepsilon$ such that for every $p < q$ with $q - p > \varepsilon$, the family of pairs $(A^0_\alpha, A^1_\alpha)_{\alpha\in\Gamma_n^\varepsilon}$ does not contain independent sequences for every $n \in \mathbb{N}$, where $A^0_\alpha = \{x \in K : e(x)_\alpha < p\}$ and $A^1_\alpha = \{x \in K : e(x)_\alpha > q\}$ for every $\alpha \in \Gamma$.*

It follows from Theorem 2.2.10 that every WRN compact space is QWRN. Moreover, it follows from this characterization of QRN compact spaces that every QRN compact space is QWRN:

Theorem 2.3.3. *Every QRN compact space is QWRN.*

Proof. Let K be a QRN compact space. It follows from Definition 2.3.1 that there exist uniformly bounded sets $\{A_{n,p} : n, p \in \mathbb{N}\}$ in $\mathcal{C}(K)$ such that $\mathcal{C}(K) = \bigcup_{n\in\mathbb{N}} A_{n,p}$ for every $p \in \mathbb{N}$ and such that K is $\frac{1}{p}$-fragmented by the pseudometric $d_{A_{n,p}}$, where $d_{A_{n,p}}(x, y) = \sup_{f\in A_{n,p}} |f(x) - f(y)|$ for every $x, y \in K$ and every $n, p \in \mathbb{N}$.

Take $\Gamma = \{f \in \mathcal{C}(K) : f(K) \subseteq [0, 1]\}$. Since the functions of Γ separate the points of K, the function $e\colon K \to [0, 1]^\Gamma$ given by $e(x)_\alpha = \alpha(x)$ determines a homeomorphic embedding. Fix $\varepsilon > 0$ and take $p \in \mathbb{N}$ such that $\frac{1}{p} < \varepsilon$. Then $\Gamma = \bigcup_n (A_{n,p} \cap \Gamma) = \bigcup_n \Gamma_n^\varepsilon$, where $\Gamma_n^\varepsilon = A_{n,p} \cap \Gamma$ for every $n \in \mathbb{N}$. Fix $q_1 < q_2$ with $q_2 - q_1 > \varepsilon$ and take $A_\alpha^0 = \{x \in K : e(x)_\alpha \leq q_1\}$ and $A_\alpha^1 = \{x \in K : e(x)_\alpha \geq q_2\}$ for every $\alpha \in \Gamma$. We are going to prove that $(A_\alpha^0, A_\alpha^1)_{\alpha\in\Gamma_n^\varepsilon}$ does not contain independent sequences by contradiction.

Suppose there exists an independent sequence $\left(A_{\alpha_m}^0, A_{\alpha_m}^1\right)_{m\in\mathbb{N}}$ with $\alpha_m \in \Gamma_n^\varepsilon$ for every $m \in \mathbb{N}$. By Zorn's Lemma there is a closed subspace L of K such that $\left(A_{\alpha_m}^0 \cap L, A_{\alpha_m}^1 \cap L\right)_{m\in\mathbb{N}}$ is an independent sequence and if $L' \subsetneq L$ is a proper closed subspace of L, then the sequence $\left(A_{\alpha_m}^0 \cap L', A_{\alpha_m}^1 \cap L'\right)_{m\in\mathbb{N}}$ is not independent. Now, since K is $\frac{1}{p}$-fragmented by the pseudometric $d_{A_{n,p}}$, there exists a nonempty relative open subset U of L with diameter smaller than $\frac{1}{p}$ with respect to $d_{A_{n,p}}$. Take $L' = L \cap U^c$, which is a proper closed subset of L. By hypothesis, there exist finite disjoint sets S_1, S_2 of $\mathbb{N}$ such that $\left(\bigcap_{k\in S_1} A_{\alpha_k}^0\right) \cap \left(\bigcap_{k\in S_2} A_{\alpha_k}^1\right) \cap L' = \emptyset$. Take any element $m \in \mathbb{N} \setminus (S_1 \cup S_2)$. Notice that, since U has diameter smaller than $\frac{1}{p} < \varepsilon$, we have that $U \cap A_{\alpha_m}^0$ or $U \cap A_{\alpha_m}^1$ is empty. Without loss of generality, suppose $U \cap A_{\alpha_m}^0 = \emptyset$. Then,

$$\left(\bigcap_{k\in S_1\cup\{m\}} A_{\alpha_k}^0\right) \cap \left(\bigcap_{k\in S_2} A_{\alpha_k}^1\right) \cap L = \emptyset$$

and therefore $\left(A_{\alpha_m}^0 \cap L, A_{\alpha_m}^1 \cap L\right)_{m\in\mathbb{N}}$ is not an independent sequence, in contradiction with our assumption. $\square$

A useful characterization of QWRN compact spaces is given by the following lemma:

Lemma 2.3.4. *A compact space K is QWRN if and only if there exists a homeomorphic embedding $e\colon K \to [0, 1]^\Gamma$ satisfying that for every $p < q$ there exists a countable decomposition $\Gamma = \bigcup_{n\in\mathbb{N}} \Gamma_n^{p,q}$ such that the family of pairs $(A_\alpha^0, A_\alpha^1)_{\alpha\in\Gamma_n^{p,q}}$ does not contain independent sequences for every $n \in \mathbb{N}$, where $A_\alpha^0 = \{x \in K : e(x)_\alpha < p\}$ and $A_\alpha^1 = \{x \in K : e(x)_\alpha > q\}$ for every $\alpha \in \Gamma$.*

Proof. If K is QWRN, then we can take $\Gamma_n^{p,q} = \Gamma_n^{\frac{q-p}{2}}$ for every $n \in \mathbb{N}$.

Now we prove the other implication. Fix $\varepsilon > 0$. There exist $0 \leq p_1 < p_2 < ... < p_m \leq 1$ such that for every $p < q$ with $q - p > \varepsilon$, there exist $p < p_j < p_{j+1} < q$ for some $j < m$. Thus, we can obtain a countable decomposition of

$$\Gamma = \bigcup_{(n_1,...,n_{m-1})\in\mathbb{N}^{m-1}} \bigcap_{j=1}^{m-1} \Gamma_{n_j}^{p_j,p_{j+1}},$$

with each $\bigcap_{j=1}^{m-1} \Gamma_{n_j}^{p_j,p_{j+1}}$ satisfying that for every $p < q$ with $q - p > \varepsilon$, the family of pairs $(A_\alpha^0, A_\alpha^1)_{\alpha\in\bigcap_{j=1}^{m-1}\Gamma_{n_j}^{p_j,p_{j+1}}}$ does not contain independent sequences, where $A_\alpha^0 = \{x \in K : e(x)_\alpha < p\}$ and $A_\alpha^1 = \{x \in K : e(x)_\alpha > q\}$ for every $\alpha \in \Gamma$. □

The following lemma is a modification of Lemma 3 in [Ros74a] due to S. Todorcevic.

Lemma 2.3.5 ([GM14, Lemma 9.5])**.** *Let $(A_n^0, A_n^1)_{n\in\mathbb{N}}$ be an independent sequence of disjoint pairs of subsets of a set S. Suppose there exist $N \in \mathbb{N}$ and N sequences of disjoint pairs $(A_{n,j}^0, A_{n,j}^1)_{n\in\mathbb{N}}$ with $j = 1, 2, ..., N$ such that*

$$A_n^0 \times A_n^1 \subseteq \bigcup_{j=1}^{N} A_{n,j}^0 \times A_{n,j}^1 \text{ for every } n \in \mathbb{N}.$$

Then, there is $j_0 \in \{1, 2, ..., N\}$ and a subsequence $(n_k)_{k\in\mathbb{N}}$ of $\mathbb{N}$ such that $\left(A_{n_k,j_0}^0, A_{n_k,j_0}^1\right)_{k\in\mathbb{N}}$ is an independent sequence.

Theorem 2.3.6. *The continuous image of a QWRN compact space is QWRN.*

Proof. Let $f : L \to K$ be a continuous surjective function with $K \subseteq [0, 1]^\Gamma$, $L \subseteq [0, 1]^\Lambda$ and Λ satisfying the conditions of Definition 2.3.2. We are going to prove that Γ satisfies the conditions of Lemma 2.3.4.

Fix $p < p' < q' < q$ and $A_\alpha^0 = \{x \in K : x_\alpha < p\}$, $A_\alpha^1 = \{x \in K : x_\alpha > q\}$ for every $\alpha \in \Gamma$. A basis for the topology of L is given by the open sets

$$U_{(\beta,r,s)} = \{y \in L : r_i < y_{\beta_i} < s_i \text{ for each } i = 1, ..., n\}$$

with $\beta = (\beta_1, ..., \beta_n) \in \Lambda^n$, $r = (r_1, ..., r_n), s = (s_1, ..., s_n) \in [-1, 2]^n$ and $n \in \mathbb{N}$. Therefore,

$$f^{-1}(A_\alpha^0) \subseteq f^{-1}(\{x \in K : x_\alpha \leq p\}) \subseteq f^{-1}(\{x \in K : x_\alpha < p'\}) = \bigcup_{(\beta,r,s)\in S'_\alpha} U_{(\beta,r,s)}$$

for some set S'_α. Due to the compactness of $f^{-1}(\{x \in K : x_\alpha \leq p\})$, there exists a finite set $S_\alpha \subseteq S'_\alpha$ such that $f^{-1}(A_\alpha^0) \subseteq \bigcup_{(\beta,r,s)\in S_\alpha} U_{(\beta,r,s)}$. Similarly, there exists a finite set S_α^1 such that

$$f^{-1}(A_\alpha^1) \subseteq \bigcup_{(\beta,r,s)\in S_\alpha^1} U_{(\beta,r,s)} \subseteq f^{-1}(\{x \in K : x_\alpha > q'\}.$$

Without loss of generality, we can take for every $\alpha \in \Gamma$ a natural number n_α such that $|S_\alpha| = |S^1_\alpha| = n_\alpha$. Set

$$\{U_{(\beta,r,s)} : (\beta, r, s) \in S_\alpha\} =: \{U_i^{\alpha,0} : i = 1, 2, ..., n_\alpha\}$$

and

$$\{U_{(\beta,r,s)} : (\beta, r, s) \in S^1_\alpha\} =: \{U_i^{\alpha,1} : i = 1, 2, ..., n_\alpha\}.$$

For every $U_{(\beta,r,s)}$ and $m \in \mathbb{N}$, we define

$$m(U_{(\beta,r,s)}) := \left\{ y \in L : y_{\beta_i} < r_i - \frac{1}{m} \text{ or } y_{\beta_i} > s_i + \frac{1}{m} \text{ for some } i \right\}.$$

Notice that $\overline{U_i^{\alpha,0}} \cap \overline{U_j^{\alpha,1}} = \emptyset$ for every $i, j = 1, 2, ..., n_\alpha$. Therefore, for each $\alpha \in \Gamma$ we can fix $m_\alpha \in \mathbb{N}$ such that

$$U_j^{\alpha,1} \subseteq m_\alpha(U_i^{\alpha,0}) \text{ for every } i, j = 1, 2, ..., n_\alpha.$$

For every $\varepsilon > 0$, we have a decomposition $\Lambda = \bigcup_n \Lambda_n^\varepsilon$ with each Λ_n^ε satisfying the conditions of Definition 2.3.2. For every $n, m, N \in \mathbb{N}$, define $\Gamma_{n,m,N} \subseteq \Gamma$ the set of all points $\alpha \in \Gamma$ with $n_\alpha = n$, $m_\alpha = m$, $|\beta| \le N$ and $\beta_i \in \bigcup_{k=1}^N \Lambda_k^{\frac{1}{2m}}$ for every $(\beta, r, s) \in S_\alpha$.

We claim that $(A^0_\alpha, A^1_\alpha)_{\alpha\in\Gamma_{n,m,N}}$ has no independent sequences. If not, the family

$$\left(f^{-1}\left(A^0_\alpha\right), f^{-1}\left(A^1_\alpha\right)\right)_{\alpha\in\Gamma_{n,m,N}}$$

contains an independent sequence too. Since

$$f^{-1}(A^0_\alpha) \times f^{-1}(A^1_\alpha) \subseteq \bigcup_{i,j=1,...,n} U_i^{\alpha,0} \times U_j^{\alpha,1},$$

there exist $i, j \in \{1, 2, ..., n\}$ such that the family $\left(U_i^{\alpha,0}, U_j^{\alpha,1}\right)_{\alpha\in\Gamma_{n,m,N}}$ contains an independent sequence, due to Lemma 2.3.5. Therefore, $\left(U_i^{\alpha,0}, m\left(U_i^{\alpha,0}\right)\right)_{\alpha\in\Gamma_{n,m,N}}$ contains an independent sequence. By definition,

$$m(U_i^{\alpha,0}) = \bigcup_{t=1,...,k} \left\{ y \in L : y_{\beta_t^\alpha} < r_t^\alpha - \frac{1}{m} \right\} \cup \left\{ y \in L : y_{\beta_t^\alpha} > s_t^\alpha + \frac{1}{m} \right\},$$

where $U_i^{\alpha,0} = U_{(\beta,r,s)}$ and $\beta = (\beta_1^\alpha, ..., \beta_k^\alpha)$, $r = (r_1^\alpha, ..., r_k^\alpha)$ and $s = (s_1^\alpha, ..., s_k^\alpha)$. Without loss of generality, we suppose that

$$\left(U_i^{\alpha,0}, \left\{ y \in L : y_{\beta_t^\alpha} < r_t^\alpha - \frac{1}{m} \right\} \right)_{\alpha\in\Gamma_{n,m,N}}$$

contains an independent sequence. Since $U_i^{\alpha,0} \subseteq \{y \in L : y_{\beta_t^\alpha} > r_t^\alpha\}$, the family of pairs $\left(\{y \in L : y_{\beta_t^\alpha} > r_t^\alpha\}, \{y \in L : y_{\beta_t^\alpha} < r_t^\alpha - \frac{1}{m}\}\right)_{\alpha \in \Gamma_{n,m,N}}$ contains an independent sequence. Therefore, there exists an independent sequence of the form

$$\left(\{y \in L : y_{\beta_k} > r_k\}, \left\{y \in L : y_{\beta_k} < r_k - \frac{1}{m}\right\}\right)_{k \in \mathbb{N}}$$

with $\beta_k \in \Lambda_1^{\frac{1}{2m}} \cup \Lambda_2^{\frac{1}{2m}} \cup ... \cup \Lambda_N^{\frac{1}{2m}}$ for every $k \in \mathbb{N}$. Taking a subsequence if necessary, we can suppose that there exist $t \in \{1, 2, ..., N\}$ such that $\beta_k \in \Lambda_t^{\frac{1}{2m}}$ for every k and that r_k converges to $r \in [-1, 2]$. Therefore,

$$\left(\left\{y \in L : y_\beta > r - \frac{1}{8m}\right\}, \left\{y \in L : y_\beta < r - \frac{3}{4m}\right\}\right)_{\beta \in \Lambda_t^{\frac{1}{2m}}}$$

contains an independent sequence. This is a contradiction with the definition of $\Lambda_t^{\frac{1}{2m}}$ because $r - \frac{1}{8m} - (r - \frac{3}{4m}) = \frac{5}{8m} > \frac{1}{2m}$.

Thus, $\Gamma = \bigcup_{n,m,N \in \mathbb{N}} \Gamma_{n,m,N}$ and $(A_\alpha^0, A_\alpha^1)_{\alpha \in \Gamma_{n,m,N}}$ does not contain independent sequences for any $n, m, N \in \mathbb{N}$. □

Since we have not made any assumption on the family Γ, the previous argument applied to $L = K$ proves the following criterion:

Lemma 2.3.7. *A compact space K is QWRN if and only if* ***for every*** *homeomorphic embedding $e \colon K \to [0,1]^\Gamma$ and for every $\varepsilon > 0$ there exists a countable decomposition $\Gamma = \bigcup_{n \in \mathbb{N}} \Gamma_n^\varepsilon$ such that for every $p < q$ with $q - p > \varepsilon$, the family of pairs $(A_\alpha^0, A_\alpha^1)_{\alpha \in \Gamma_n^\varepsilon}$ does not contain independent sequences for every $n \in \mathbb{N}$, where $A_\alpha^0 = \{x \in K : e(x)_\alpha < p\}$ and $A_\alpha^1 = \{x \in K : e(x)_\alpha > q\}$ for every $\alpha \in \Gamma$.*

As in the case of QRN and RN compact spaces, every zero-dimensional QWRN compact space is WRN.

Theorem 2.3.8. *Let K be a zero-dimensional QWRN compact space. Then K is WRN.*

Proof. Since K is zero-dimensional, we can suppose that $K \subseteq \{0,1\}^\Gamma$ for some set Γ. Due to Lemma 2.3.7, there exists a decomposition $\Gamma = \bigcup_{n \in \mathbb{N}} \Gamma_n$ such that for every $p < q$ with $q - p > \frac{1}{2}$, the family of pairs $A_\alpha^0 = \{x \in K : x_\alpha < p\}$, $A_\alpha^1 = \{x \in K : x_\alpha > q\}$ with $\alpha \in \Gamma_n$ does not contain independent sequences. Since $K \subseteq \{0,1\}^\Gamma$, each Γ_n does not contain an independent sequence of functions. Let $\mathcal{F} = \{f_\alpha\}_{\alpha \in \Gamma} \subseteq \mathcal{C}(K)$, where $f_\alpha(x) = \frac{x_\alpha}{n}$ for every $n \in \mathbb{N}$, $\alpha \in \Gamma_n$ and $x \in K$. The family $\mathcal{F}$ separates the points of K and it does not contain an independent sequence of functions, so K is WRN. □

As a corollary of Theorems 2.3.6 and 2.3.8, we obtain the following result:

Corollary 2.3.9. *Zero-dimensional continuous images of WRN compact spaces are also WRN.*

Section 2.4

Examples and relations with other classes of compact spaces

In [GM14], E. Glasner and M. Megrelishvili studied the class of WRN compact spaces and proved the following result, from which it follows that the split interval is WRN (but it is not RN):

Theorem 2.4.1 ([GM14, Theorem 8.7]). *Every linearly ordered compact space is WRN.*

Proof. Let K be a WRN compact space. Set $\mathcal{F}$ the set of increasing functions from K to $[0,1]$. $\mathcal{F}$ separates the points of K [Nac65]. Moreover, for any two functions $f, g \in \mathcal{F}$ and any $p < q$, we have

$$\{x \in K : f(x) < p\} \cap \{x \in K : g(x) > q\} = \emptyset$$

or

$$\{x \in K : f(x) > q\} \cap \{x \in K : g(x) < p\} = \emptyset.$$

Namely, if $x' \in \{x \in K : f(x) < p\} \cap \{x \in K : g(x) > q\}$, then for every $y > x'$ we have $g(y) > q$ and for every $y < x'$ we have $f(y) < p$, so $\{x \in K : f(x) > q\} \cap \{x \in K : g(x) < p\} = \emptyset$. Thus, $\mathcal{F}$ does not contain independent sequences and K is WRN. □

The split interval is the space $[0,1] \times \{0,1\} \setminus \{(0,0),(1,1)\}$ endowed with the lexicographic order.

Corollary 2.4.2 ([GM14, Corollary 8.8]). *The split interval is WRN but not RN.*

Proof. The split interval is WRN because of Theorem 2.4.1. I. Namioka proved that it is not RN because it is not fragmented by a metric [Nam87]. □

On the other hand, it follows from a result of Talagrand [Tal81] that the Stone-Čech compactification of the natural numbers $\beta\mathbb{N}$ is not WRN (another proof of S. Todorcevic is included in [GM14, Appendix]). This fact is also a consequence of Corollary 2.6.5.

In order to study the relation between WRN compact spaces and the classes of compact spaces introduced in the first section, we need some classical results. V. Farmaki gave the following characterization of Corson compacta which are Eberlein:

Theorem 2.4.3 ([Far87]). *A compact space $K \subseteq \Sigma(\Gamma)$ is Eberlein if and only if for every $\varepsilon > 0$ there exist a countable decomposition $\Gamma = \bigcup_{n\in\mathbb{N}} \Gamma_n^\varepsilon$ such that for every $x \in K$ and every $n \in \mathbb{N}$, the set $\{\alpha \in \Gamma_n^\varepsilon : |x_\alpha| > \varepsilon\}$ is finite.*

Talagrand's compact is defined as the subspace of $\{0,1\}^{\mathbb{N}^{\mathbb{N}}}$ which consists of all characteristic functions 1_A with $A \subseteq \mathbb{N}^{\mathbb{N}}$ such that there exists $n \in \mathbb{N}$ with $x(k) = y(k)$ for every $k = 1, 2, ..., n$ and $x(n+1) \neq y(n+1)$ for every $x, y \in A$ with $x \neq y$. Talagrand's compact is an example of a Corson compact space which is not Eberlein [Tal79].

However, every Corson and RN compact space is Eberlein:

Theorem 2.4.4 ([Ste94, OSV91])**.** *A compact K is Eberlein if and only if it is Corson and RN.*

It is proved in [Arv02] that the previous theorem can be extended to QRN compact spaces, i.e. K is Eberlein if and only if it is Corson and QRN. We define a compact space $K \subseteq \mathbb{R}^\Gamma$ to be *solid* if for every finite subset $A \subseteq \Gamma$ and every $x \in K$, the element $x1_A$ belongs to K, where

$$x1_A(\gamma) = \begin{cases} x(\gamma) & \text{if } \gamma \in A; \\ 0 & \text{in any other case.} \end{cases}$$

Next theorem improves last theorem of [Ste90]. If the compact space is solid, then Theorem 2.4.4 can also be extended to QWRN compact spaces:

Theorem 2.4.5. *Let $K \subseteq \Sigma(\Gamma)$ be a solid Corson compact space. Then, K is WRN if and only if it is QWRN if and only if it is Eberlein.*

Proof. Since every Eberlein compact space is WRN and every WRN compact space is QWRN, we have to show that if K is QWRN then it is Eberlein. Suppose K is QWRN and fix $\varepsilon > 0$. There exists a decomposition $\Gamma = \bigcup_{n\in\mathbb{N}} \Gamma_n$ such that the family $\left(A^0_\gamma, A^1_\gamma\right)_{\gamma\in\Gamma_n}$ does not contain independent sequences for every $n \in \mathbb{N}$, where $A^0_\gamma = \{x \in K : x_\gamma > \varepsilon\}$ and $A^1_\gamma = \{x \in K : x_\gamma < \frac{\varepsilon}{2}\}$. Let $x \in K$. We are going to see that $\{\gamma \in \Gamma_N : |x_\gamma| > \varepsilon\}$ is finite for every $N \in \mathbb{N}$. Suppose $(\gamma_n)_{n\in\mathbb{N}} \subseteq \Gamma_N$ is a sequence of coordinates with $|x_{\gamma_n}| > \varepsilon$. Since K is solid, for every $\delta\colon \{1, 2, ..., n\} \to \{0, 1\}$, the element $x1_{\{\gamma_k:\delta(k)=0\}}$ is in K and, therefore $(A^0_{\gamma_n}, A^1_{\gamma_n})$ is independent, since

$$x1_{\{\gamma_k:\delta(k)=0\}} \in \bigcap_{k=1}^{n} A^{\delta(k)}_{\gamma_k}.$$

Thus, $\{\alpha \in \Gamma_N : |x_\alpha| > \varepsilon\}$ is finite for every $N \in \mathbb{N}$ and K is Eberlein due to Farmaki's Theorem 2.4.3. □

Corollary 2.4.6. *Talagrand's compact is not QWRN.*

Proof. Since Talagrand's compact is solid, Corson and not Eberlein, the conclusion follows from Theorem 2.4.5. □

Nevertheless, Theorem 2.4.4 cannot be extended to WRN. The existence of Corson WRN compact spaces which are not Eberlein follows from the results in [Arg92]. Namely, Argyros proved that the family of all segments of Todorcevic tree and Reznichenko's family satisfy the conditions of the following lemma:

Lemma 2.4.7. *Let Γ be a set and $\mathcal{S}$ a family of subsets of Γ which satisfy the following conditions:*

(1) The space $K = \{1_s : s \in \mathcal{S}\} \subseteq \{0,1\}^\Gamma$ is compact.

(2) Each $s \in \mathcal{S}$ is countable.

(3) For every $s \in \mathcal{S}$, the set $L_s = \{s \cap t : t \in \mathcal{S}\}$ is countable.

(4) For every countable decomposition $\Gamma = \bigcup_n \Gamma_n$ there exist a set $s \in \mathcal{S}$ and $n_0 \in \mathbb{N}$ such that $s \cap \Gamma_{n_0}$ is infinite.

Then the compact space K is Corson and WRN but it is not Eberlein.

Proof. Condition (2) implies that K is Corson and it follows from Farmaki's Theorem 2.4.3 that condition (4) implies that K is not Eberlein. We prove that condition (3) implies that K is WRN. We consider the elements of Γ as functions $\gamma\colon K \to \{0,1\}$, with $\gamma(1_s) = 1_s(\gamma)$. Since Γ separates the points of K, it is enough to prove that it does not contain independent sequences. Suppose γ_n is an independent sequence in Γ. Thus, for every infinite set $N \subseteq \mathbb{N}$ there exists an element $s_N \in \mathcal{S}$ such that $\gamma_n(1_{s_N}) = 1$ if and only if $n \in N$. Then

$$L_{s_\mathbb{N}} = \{s_\mathbb{N} \cap t : t \in \mathcal{S}\} \supseteq \{s_\mathbb{N} \cap s_N : N \subseteq \mathbb{N}\},$$

which gives a contradiction with property (3) since the set on the right-hand side is uncountable. □

For the sake of completeness we include in Appendix A the construction of Todorcevic tree and we prove that the family of all segments in it satisfies the hypothesis of Lemma 2.4.7.

We finish the section by studying split compact spaces. In Section 2.5 we use split compact spaces to construct a continuous image of an RN compact space which is not WRN. Furthermore, the class of split compact spaces was used by P. Koszmider in [Kos16] and [Kos15a] for constructing counterexamples to some classical problems. In particular, he provided consistent examples of nonmetrizable compact spaces K such that all zero-dimensional subspaces of all continuous images of K are metrizable. A compact space with this property cannot be Eberlein neither Rosenthal, but it is consistent that there are Corson compact spaces with this property (see [Kos16][Proposition 4.2]). Nevertheless, Koszmider also proved that under Martin's axiom and the negation of the Continuum Hypothesis, compact spaces with this property cannot be Corson. Notice that any Souslin line is a WRN compact space (it is a linearly ordered compact space) with this property (see the proof of [Kos16][Proposition 4.2 (5)]). Moreover, we prove that there are WRN compact spaces with this property even under Martin's axiom and the negation of the Continuum Hypothesis (Corollary 2.4.11).

Definition 2.4.8. *Let M be a metric compact space, L a compact space, κ an ordinal, $\{r_\xi : \xi < \kappa\}$ a family of distinct points of L and $f_\xi \colon L \setminus \{r_\xi\} \to M$ a continuous function for every $\xi < \kappa$. The split L induced by $\{f_\xi : \xi < \kappa\}$ is the subspace K of $L^{\{*\}} \times M^\kappa$ consisting of points of the form*

$$\{x_{\xi,t} : \xi < \kappa,\ t \in M\} \cup \{x_r : r \in L \setminus \{r_\xi : \xi < \kappa\}\},$$

where

- $x_{\xi,t}(*) = r_\xi$, $x_{\xi,t}(\xi) = t$ *and* $x_{\xi,t}(\eta) = f_\eta(r_\xi)$ *if* $\eta \neq \xi$.
- $x_r(*) = r$ *and* $x_r(\xi) = f_\xi(r)$ *for all* $r \in L \setminus \{r_\xi : \xi < \kappa\}$ *and every* $\xi < \kappa$.

Notice that if K is the split L induced by $\{f_\xi : \xi < \kappa\}$, where L and $\{f_\xi : \xi < \kappa\}$ are as in Definition 2.4.8, then the complement of K in $L^{\{*\}} \times M^\kappa$ consists of points $x \in L^{\{*\}} \times M^\kappa$ such that $x(*) = r_\xi$ but $x(\eta) \neq f_\eta(r_\xi)$ for some $\xi,\ \eta < \kappa$ with $\eta \neq \kappa$ or $x(*) \notin \{r_\xi : \xi < \kappa\}$ but $x(\eta) \neq f_\eta(x(*))$ for some $\eta < \kappa$. It follows from the continuity of the functions f_ξ that K^c is open and therefore K is compact.

The classical split interval is an example of a split compact space of this form. Moreover, the example of Theorem 2.7.10 is also a split Cantor space. We provide in Theorem 2.4.9 a sufficient condition for a split compact space to be WRN.

We say that the sequence of functions $f_{\xi_n} \colon L \setminus \{r_{\xi_n}\} \to M$ is M-independent if there exists an extension (probably not continuous) $g_{\xi_n} \colon L \to M$ of f_{ξ_n} for each $n \in \mathbb{N}$ such that the sequence g_{ξ_n} is M-independent. Notice that f_{ξ_n} is M-independent if and only if every extension provides an M-independent sequence, i.e. if g_n and h_n are different extensions of f_{ξ_n} then the sequence g_n is M-independent if and only if h_n is M-independent. Namely, if g_n is not M-independent then there are closed disjoint sets $C,\ C'$ of M and disjoint finite subsets $S_1,\ S_2$ of $\mathbb{N}$ such that

$$\left(\bigcap_{k \in S_1} g_k^{-1}(C)\right) \cap \left(\bigcap_{k' \in S_2} g_{k'}^{-1}(C')\right) = \emptyset.$$

But then

$$\left(\bigcap_{k \in S_1} h_k^{-1}(C)\right) \cap \left(\bigcap_{k' \in S_2} h_{k'}^{-1}(C')\right) \subseteq \{r_{\xi_n} : n \in S_1 \cup S_2\}$$

is a finite set and therefore h_n is not M-independent because of Remark 2.2.7.

Theorem 2.4.9. *Let K be the split L induced by $\{f_\xi : \xi < \kappa\}$, where L and $\{f_\xi : \xi < \kappa\}$ are as in Definition 2.4.8. If $\{f_\xi : \xi < \kappa\}$ does not contain M-independent sequences and L is WRN, then K is WRN.*

Proof. Denote by $\pi_* \colon K \to L$ the projection onto the first coordinate (i.e. $\pi_*(x) = x(*)$ for every $x \in K \subseteq L^{\{*\}} \times M^\kappa$) and by $\pi_\xi \colon K \to M$ the projection onto the ξ-coordinate of M, i.e. $\pi_\xi(x) = x(\xi)$ for every $x \in K$, $\xi < \kappa$. We claim that $\{\pi_\xi : \xi < \kappa\}$ does not contain M-independent sequences. Take a sequence π_{ξ_n}. Since $\{f_\xi : \xi < \kappa\}$ does not contain M-independent sequences, by Lemma 2.2.13 we may suppose that f_{ξ_n} is pointwise convergent, in the sense that $(f_{\xi_n}(x))_{n\in\mathbb{N},\xi_n\neq x}$ converges for every $x \in L$. Notice that for every $x \in K$, $\pi_{\xi_n}(x) = f_{\xi_n}(x(*))$ for all except at most one $n \in \mathbb{N}$. Thus, the sequence π_{ξ_n} is pointwise convergent and therefore it does not contain M-independent subsequences due to Lemma 2.2.12. Hence $\{\pi_\xi : \xi < \kappa\}$ does not contain M-independent sequences. Since L is WRN, there exists a family $\mathcal{F}$ of continuous functions from L to $[0, 1]$ separating points and with no independent sequences. Since the family of functions

$$\{\pi_\xi : \xi < \kappa\} \cup \{f \circ \pi_* : f \in \mathcal{F}\}$$

separates the points of K, a similar argument to the one used in Theorem 2.2.14 proves that K is WRN. □

Example 2.4.10. *Set $L = [0, 1]^2$, $M = \mathbb{S}$, where $\mathbb{S}$ is the unit sphere in $\mathbb{R}^2$ with the Euclidean metric, $\{r_\xi : \xi < \kappa\} \subseteq L$ and $f_\xi \colon L \setminus \{r_\xi\} \to M$ defined as $f_\xi(x) = \frac{x-r_\xi}{d(x,r_\xi)}$ for every $\xi < \kappa$, where d is the Euclidean distance in $[0, 1]^2$. Let K be the split L induced by $\{f_\xi : \xi < \kappa\}$. K is said to be a* Filippov space. *We claim that K is WRN. By Theorem 2.4.9, it is enough to check that every sequence f_{ξ_n} does not contain an M-independent subsequence. However, since r_{ξ_n} is a sequence in $[0, 1]^2$, we may suppose without loss of generality that r_{ξ_n} converges to some $r \in [0, 1]^2$. But then, notice that $f_{\xi_n}(x) = \frac{x-r_{\xi_n}}{d(x,r_{\xi_n})}$ converges to $\frac{x-r}{d(x,r)}$ for every $x \neq r$. Passing to a subsequence if necessary, we may suppose that the sequence $f_{\xi_n}(r)$ is also convergent. Thus, f_{ξ_n} does not contain M-independent subsequences and we conclude that K is WRN.*

Corollary 2.4.11. *Under Martin's axiom and the negation of the Continuum Hypothesis there is a WRN nonmetrizable compact space K such that all zero-dimensional subspaces of all continuous images of K are metrizable.*

Proof. It is a consequence of Example 2.4.10 and [Kos16, Theorem 4.5], where it is proved that there is a Filippov space with this property. □

Section 2.5

A continuous image of an RN compact space which is not WRN

The class of Corson compact spaces and the class of Eberlein compact spaces are closed under continuous images (see [BRW77] and [Gul77]).

In [Nam87] I. Namioka asked whether the class of RN compact spaces is closed under continuous images. This question was solved negatively by A. Avilés and P. Koszmider in [AK13]. E. Glasner and M. Megrelishvili posed the same question for the class of WRN compact spaces in [GM14]. In this section we answer negatively the question of E. Glasner and M. Megrelishvili by proving that a modification of the construction given in [AK13] provides an example of a continuous image of an RN compact space which is not WRN.

Definition 2.5.1. *Let K be a compact space. A metric $d\colon K \times K \to \mathbb{R}$ is Reznichenko if for every distinct points $x, y \in K$ there are open sets U, V in K with $x \in U$, $y \in V$ and $d(U,V) = \inf\{d(u,v) : u \in U,\ v \in V\} > 0$.*

By a result of Namioka [Nam02], a compact space K is QRN if and only if there exists a Reznichenko metric which fragments K.

During this section we denote by $\Delta = 2^{\mathbb{N}} = \{0,1\}^{\mathbb{N}}$ the Cantor set with the topology induced by the metric $\rho\colon \Delta\times\Delta \to \mathbb{R}$ given by $\rho(x,y) = 2^{-\min\{k\colon\, x_k \neq y_k\}}$ if $x \neq y$ and by $q\colon \Delta \to [0,1]$ the continuous surjective function given by the formula $q(t_1, t_2, \ldots) = \sum_{k\in\mathbb{N}} \frac{t_k}{2^k}$. We are going to construct an RN compact space $\mathbb{L}_0$, a nonWRN compact space $\mathbb{L}_1$ and a surjective continuous function $\pi\colon \mathbb{L}_0 \to \mathbb{L}_1$ in a way similar to [AK13]. In particular, we construct a zero-dimensional RN compact space L, a set $B \subseteq L$ and continuous functions $g_x\colon L\setminus\{x\} \to \Delta$, $f_x = q\circ g_x$ for every $x \in B$. Then, $\mathbb{L}_0$ is the split L induced by $\{g_x : x \in B\}$ and $\mathbb{L}_1$ is the split L induced by $\{f_x : x \in B\}$. Since L and Δ are zero-dimensional, $\mathbb{L}_0$ is also zero-dimensional. Arvanitakis [Arv02] proved that continuous images of QRN compact spaces are QRN and that a zero-dimensional compact space is QRN if and only if it is RN. The fact that $\mathbb{L}_0$ is RN will be a consequence of the previous results of Arvanitakis and the following lemma:

Lemma 2.5.2. *Let K be the split L induced by $\{f_\xi : \xi < \kappa\}$, where L and $\{f_\xi : \xi < \kappa\}$ are as in Definition 2.4.8. Take d' a metric on M. Suppose there is a Reznichenko metric $d\colon L\times L \to \mathbb{R}$ which fragments L (i.e. L is QRN). Moreover, suppose that for every r_ξ there exists an open set U_ξ in L such that:*

(1) $U_\xi \cap \{r_{\xi'} : \xi' < \kappa\} = \{r_\xi\}$;

(2) if u_n, v_n are sequences in $U_\xi\setminus\{r_\xi\}$ with $d(u_n, v_n)$ converging to zero, then there exist subsequences u_{n_k}, v_{n_k} of u_n, v_n such that $d'(f_\xi(u_{n_k}), f_\xi(v_{n_k}))$ also converges to zero;

(3) $d(r_\xi, u) \geq 1$ *for every* $u \in L$ *with* $u \neq r_\xi$.

Then K *is QRN.*

Proof. Without loss of generality, $d' \leq 1$. Consider the map $\delta\colon K \times K \to \mathbb{R}$ defined as:

- $\delta(x, x) = 0$ for every $x \in K$;
- $\delta(x, y) = d(x(*), y(*))$ if $x(*) \neq y(*)$;
- $\delta(x, y) = d'(x(\xi), y(\xi))$ if $x(*) = y(*) = r_\xi$.

Obviously δ is nonnegative, symmetric and it vanishes on (x, y) if and only if $x = y$. Since $d' \leq 1$, condition (3) ensures that δ is a metric. We prove first that δ is Reznichenko. Take two points $x, y \in K$ with $0 < \delta(x, y)$. If $x(*) \neq y(*)$, then there exist open sets U_1, U_2 in L such that $x(*) \in U_1$, $y(*) \in U_2$, $d(U_1, U_2) > 0$ and we can take $V_1 = (U_1 \times M^\kappa) \cap K$, $V_2 = (U_2 \times M^\kappa) \cap K$ which are open sets in K with $x \in V_1$, $y \in V_2$ and $\delta(V_1, V_2) > 0$. If $x(*) = y(*) = r_\xi$, then there are open sets U_1, U_2 in M such that $x(\xi) \in U_1$, $y(\xi) \in U_2$ and $d'(U_1, U_2) > 0$. Set $V_1 = \pi_\xi^{-1}(U_1)$, $V_2 = \pi_\xi^{-1}(U_2)$, where $\pi_\xi\colon K \to M$ is the projection in the ξ-coordinate of M. Finally take $U_1' = (U_\xi \times M^\kappa) \cap V_1$, $U_2' = (U_\xi \times M^\kappa) \cap V_2$, which are open sets in K with $x \in U_1'$ and $y \in U_2'$. We claim that $\delta(U_1', U_2') > 0$. If not, there are sequences x_n in U_1' and y_n in U_2' with $\delta(x_n, y_n)$ converging to zero. Since $d'(U_1, U_2) > 0$, we have that $x_n(*), y_n(*) \in U_\xi \setminus \{r_\xi\}$ for all except at most finitely many $n \in \mathbb{N}$. Therefore, we may suppose that $\delta(x_n, y_n) = d(x_n(*), y_n(*))$ for every $n \in \mathbb{N}$. Thus, condition (2) provides subsequences x_{n_k}, y_{n_k} with $d'(x_{n_k}(\xi), y_{n_k}(\xi))$ converging to zero. But this contradicts the fact that $d'(U_1, U_2) > 0$, since $x_{n_k}(\xi) \in U_1$ and $y_{n_k}(\xi) \in U_2$ for every $k \in \mathbb{N}$. Thus, δ is Reznichenko.

We prove now that δ fragments K. Take $F \subseteq K$ and $1 > \varepsilon > 0$. Set $F(*) := \{x(*) \in L : x \in F\}$. Since d fragments L, there exists an open set U in L with $U \cap F(*) \neq \emptyset$ and $\operatorname{diam}_d(U \cap F(*)) < \varepsilon$. Since $\varepsilon < 1$, condition (3) implies that $U \cap F(*) = \{r_\xi\}$ for some $\xi < \kappa$ or $U \cap F(*) \cap \{r_\xi : \xi < \kappa\} = \emptyset$. If $U \cap F(*) = \{r_\xi\}$, then we can take some open set V in M with $V \cap \pi_\xi(F \cap (U \times M^\kappa)) \neq \emptyset$ and $\operatorname{diam}_{d'}(V) < \varepsilon$. Then, the open set $U' = (U \times M^\kappa) \cap K \cap \pi_\xi^{-1}(V)$ satisfy $U' \cap F \neq \emptyset$ and $\operatorname{diam}_\delta(U' \cap F) < \varepsilon$. On the other hand, if $U \cap F(*) \cap \{r_\xi : \xi < \kappa\} = \emptyset$, then the open set $U' = (U \times M^\kappa) \cap K$ satisfy $U' \cap F \neq \emptyset$ and $\operatorname{diam}_\delta(U' \cap F) = \operatorname{diam}_d(U \cap F(*)) < \varepsilon$. Thus, we conclude that δ is a Reznichenko metric which fragments K and therefore K is QRN. □

Now we are going to construct the RN compact space L. Let $T = 2^{<\omega}$ denotes the set of all finite sequences of 0's and 1's and, for every $t \in T$, $|t|$ denotes the length of t. For every $t = (t_1, t_2, ..., t_n) \in T$ and $z = (z_1, z_2, ...) \in T \cup \Delta$, we write

$$t^\frown z := (t_1, t_2, ..., t_n, z_1, z_2, ...) \in T \cup \Delta.$$

We consider $T \cup \Delta$ equipped with the lexicographical order. By G we denote all finite sets $g = \{s_1, ..., s_n\}$ such that $s_1, ..., s_n \in T$ satisfy $|s_1| = ... = |s_n|$. Given $t \in T$ and $g \in G$ we define $\Gamma^t_g \colon \Delta \to \Delta$ as:

- $\Gamma^t_g(s^\frown\lambda) = t^\frown\lambda$ if $s \in g$ and $\lambda \in \Delta$;
- $\Gamma^t_g(z) = t^\frown(0, 0, ...)$ in the rest of points.

Notice that each Γ^t_g is continuous.

The main difference between our construction and the one in [AK13] is the choice of the functions Γ^t_g with $g \in G$ and $t \in T$. This choice lets us prove that $\mathbb{L}_1$ is not WRN.

Let $K = \bigcup_{t\in T} A_t \cup B \cup C$ be a scattered compact space such that:

(1) all points of $A = \bigcup_{t\in T} A_t$ are isolated in K and the sets $\{A_t : t \in T\}$ are pairwise disjoint;

(2) for every $x \in B$ there exists an infinite set $B_x \subseteq A$ such that $\overline{B_x} = B_x \cup \{x\}$ and moreover, $\overline{B_x}$ is open in K;

(3) there exists a function $\psi \colon B \to G^T$ such that for any family of subsets of A of the form $\{X^t_g : g \in G, t \in T\}$ with $A_t = \bigcup_{g\in G} X^t_g$ for every $t \in T$, there exists $x \in B$ such that $B_x \cap X^t_{\psi(x)[t]}$ is infinite for all $t \in T$.

A. Avilés and P. Koszmider called a compact space of the previous form a *basic space* and they provided some examples of such compact spaces. Notice that, since K is scattered, it is RN.

Consider $L = (A \times \Delta) \cup B \cup C$. We equip L with the following topology. A basic neighborhood of a point (a, t) in L is of the form $\{a\} \times U$ where U is a neighborhood of t in Δ. A basic neighborhood of a point $x \in B \cup C$ is of the form $((U \cap A) \times \Delta) \cup (U \setminus A)$, where U is a neighborhood of x in K. Notice that L is a zero-dimensional compact space. Consider the metric $\delta' \colon L \times L \to [0, 1]$ given by the formula

(i) $\delta'(x, y) = 0$ if $x = y$;

(ii) $\delta'((a, t), (a, t')) = \rho(t, t')$ for every $a \in A$, $t \in \Delta$;

(iii) $\delta'(x, y) = 1$ in any other case.

It is immediate that δ' is a l.s.c. metric which fragments L and therefore L is RN.

Consider $\mathbb{L}_0$, $\mathbb{L}_1$ and $\pi \colon \mathbb{L}_0 \to \mathbb{L}_1$ defined in the same way as in [AK13]:

- For every $x \in B$, $g_x \colon L \setminus \{x\} \to \Delta$ is the continuous function given by the formula $g_x(a, z) = \Gamma^t_{\psi(x)[t]}(z)$ for every $a \in A_t \cap B_x$, $z \in \Delta$ and $g_x(y) = 0$ in any other case.
- For every $x \in B$, $f_x \colon L \setminus \{x\} \to [0, 1]$ is the continuous function $f_x = q \circ g_x$.
- $\mathbb{L}_0 = \{[u, v] \in L \times \Delta^B : g_x(u) = v_x \text{ for all } x \in B \setminus \{u\}\}$, i.e. $\mathbb{L}_0$ is the split L induced by $\{g_x : x \in B\}$.
- $\mathbb{L}_1 = \{[u, v] \in L \times [0, 1]^B : f_x(u) = v_x \text{ for all } x \in B \setminus \{u\}\}$, i.e. $\mathbb{L}_1$ is the split L induced by $\{f_x : x \in B\}$.
- $\pi \colon \mathbb{L}_0 \to \mathbb{L}_1$ is the continuous function given by the formula $\pi[u, v] = [u, q\,(v_x)_{x \in B}]$.

It is clear that π is continuous and surjective. Although L is an RN compact space, Theorem 2.4.9 cannot be applied in this case because $\{g_x : x \in B\}$ and $\{f_x : x \in B\}$ contain independent sequences.

Lemma 2.5.3. *$\mathbb{L}_0$ is RN.*

Proof. Since $\mathbb{L}_0$ is zero-dimensional, it is enough to prove that δ' satisfies the conditions of Lemma 2.5.2. For $x \in B$, take the open set in L, $U_x = (B_x \times \Delta) \cup \{x\}$. Then $U_x \cap B = \{x\}$. Moreover, for every $x \in B$ we have $\delta'(x, y) = 1$ for every $x \neq y$. Thus, conditions (1) and (3) of Lemma 2.5.2 are satisfied and we only have to prove that if $u_n = (a_n, z_n)$, $v_n = (a'_n, z'_n)$ are sequences in $U_x \setminus \{x\} = B_x \times \Delta$ with $\delta'(u_n, v_n)$ converging to zero, then there exists subsequences u_{n_k}, v_{n_k} such that $\rho(g_x(u_{n_k}), g_x(v_{n_k}))$ converges to zero. Notice that since $\delta'(u_n, v_n)$ converges to zero, we may assume that $a_n = a'_n$ for every $n \in \mathbb{N}$. Thus, either there is a subsequence of a_n which is contained in some A_t with $t \in T$, or there is a subsequence a_{n_k} with $a_{n_k} \in A_{t_k}$ and $|t_k|$ converges to infinity. In the first case, we may suppose that $a_n \in A_t$ for every $n \in \mathbb{N}$ and then $g_x((a_n, z_n)) = \Gamma^t_{\psi(x)[t]}(z_n)$ and $g_x((a_n, z'_n)) = \Gamma^t_{\psi(x)[t]}(z'_n)$ for every $n \in \mathbb{N}$. But since $\Gamma^t_{\psi(x)[t]}$ is continuous and $\delta'(u_n, v_n) = \rho(z_n, z'_n)$ converges to zero, we conclude that $\rho(g_x(u_n), g_x(v_n)) = \rho\left(\Gamma^t_{\psi(x)[t]}(z_n), \Gamma^t_{\psi(x)[t]}(z'_n)\right)$ also converges to zero. In the second case we have a subsequence $a_{n_k} \in A_{t_k}$ with $|t_k|$ converging to infinity. But then $\operatorname{diam}_\rho\left(\Gamma^{t_n}_{\psi(x)[t_n]}(\Delta)\right)$ converges to zero and we can also conclude in this case that $\rho(g_x(u_n), g_x(v_n))$ converges to zero, so the proof is complete. $\square$

Since the continuous image of an RN compact space is QRN, $\mathbb{L}_1$ is QRN. Nevertheless, it is not WRN:

Theorem 2.5.4. *$\mathbb{L}_1$ is not WRN.*

Proof. Suppose that $\mathcal{F} \subseteq \mathcal{C}(\mathbb{L}_1)$ is an eventually fragmented uniformly bounded family. We will find two points that are not separated by $\mathcal{F}$. For every $a \in A$ and every $z_1 \leq z_2 \in \Delta$, we denote by $a + z_1$ the point of $\mathbb{L}_1$ of the form $[(a, z_1), v]$, where $v_x = f_x(a, z_1)$ for every $x \in B$. We denote by $a + [z_1, z_2]$ the set of points $a + z$ with $z \in [z_1, z_2]$, where the order in Δ is the lexicographical order. Similarly, for every $x \in B$ and every $\xi_1 \leq \xi_2 \in [0,1]$ we denote by $x \oplus \xi_1 = [x, v] \in \mathbb{L}_1$ the point given by the formula $v_y = f_y(x)$ for every $y \in B \setminus \{x\}$ and $v_x = \xi_1$ and by $x \oplus [\xi_1, \xi_2]$ we denote the set of points $x \oplus \xi$ with $\xi \in [\xi_1, \xi_2]$. For every $a \in A_t$ and $f \in \mathcal{F}$, we can find $s_f(a) \in T$ such that

$$\operatorname{diam}\left(f(a + [s_f(a)^\frown(0,0,...), s_f(a)^\frown(1,1,...)])\right) < \frac{1}{4^{|t|}}.$$

For each $a \in A$ and $f \in \mathcal{F}$ fix $s_f(a)$ with the previous property and $|s_f(a)|$ minimum.

Then, the set $\{s_f(a) : f \in \mathcal{F}\}$ is finite for every $a \in A$. Namely, if this set were not finite, there would exist $a \in A_t$ for some $t \in T$ and a sequence $\{f_n\}_{n\in\mathbb{N}}$ in $\mathcal{F}$ such that $|s_{f_n}(a)| \to \infty$, so $\{f_n : n \in \mathbb{N}\}$ would not have a fragmented subsequence, since for any open subset O of $a + \Delta$ there would exist N such that $f_n(O)$ has diameter bigger than $\frac{1}{4^{|t|}}$ for every $n > N$. Therefore, $\{s_f(a) : f \in \mathcal{F}\}$ is finite for every $a \in A$ due to the eventual fragmentability of $\mathcal{F}$. Thus, for every $a \in A_t$ there exists $g_a \in G$ such that for every $f \in \mathcal{F}$ we can find $s \in g_a$ with

$$\operatorname{diam}\left(f(a + [s^\frown(0,0,...), s^\frown(1,1,...)])\right) < \frac{1}{4^{|t|}}.$$

Let $X_g^t = \{a \in A_t : g_a = g\}$ for every $t \in T$ and every $g \in G$. These sets satisfy $A_t = \bigcup_{g\in G} X_g^t$ for every $t \in T$. Due to property (3) of the basic space K, there exists $x \in B$ such that $B_x \cap X_{\psi(x)[t]}^t$ is infinite for every $t \in T$.

We are going to prove that $\mathcal{F}$ does not separate the points of $\mathbb{L}_1$ by showing that $f(x \oplus 0) = f(x \oplus 1)$ for every $f \in \mathcal{F}$. Fix $f \in \mathcal{F}$ and an infinite subset $\{a_n : n \in \mathbb{N}\} \subseteq B_x \cap X_{\psi(x)[t]}^t$. Since $g_{a_n} = \psi(x)[t] \in G$ for every $n \in \mathbb{N}$ and $\psi(x)[t]$ is finite, there exist a subsequence $\{a_{n_k}\}_{k\in\mathbb{N}}$ and $s \in \psi(x)[t]$ such that

$$\operatorname{diam}\left(f(a_{n_k} + [s^\frown(0,0,...), s^\frown(1,1,...)])\right) < \frac{1}{4^{|t|}} \text{ for every } k \in \mathbb{N}.$$

Notice that

$$f_x(a_{n_k} + s^\frown(i,i,...)) = q(\Gamma^t_{\psi(x)[t]}(s^\frown(i,i,...)) = q(t^\frown(i,i,...)) =: t^i$$

for every $i \in \{0,1\}$. Taking limits we obtain $a_{n_k} + s^\frown(i,i,...) \to x \oplus \xi^i$ for every $i \in \{0,1\}$, where

$$\xi^i = \lim_n f_x(a_{n_k} + s^\frown(i,i,...)) = t^i.$$

For every $\xi_0, \xi_1 \in [t^0, t^1]$, there exist $\lambda_0, \lambda_1 \in \Delta$ such that $q(t^\frown \lambda_i) = \xi_i$ and therefore $a_{n_k} + s^\frown \lambda_i \to x \oplus \xi_i$, so

$$d(f(x \oplus \xi_0), f(x \oplus \xi_1)) = \lim_n d(f(a_{n_k} + s^\frown \lambda_0), f(a_{n_k} + s^\frown \lambda_1)) \leq \frac{1}{4^{|t|}}.$$

Thus,

$$\operatorname{diam}\big(f(x \oplus [t^0, t^1])\big) \leq \frac{1}{4^{|t|}}.$$

Now, since for every $m \in \mathbb{N}$

$$\{[t^0, t^1] : t \in T, |t| = m\} = \{[(k-1)2^{-m}, k2^{-m}] : k = 1, 2, ..., 2^m\},$$

it follows that $\operatorname{diam} f(x \oplus [0,1]) \leq 2^m \frac{1}{4^m} = \frac{1}{2^m}$ for every $m \in \mathbb{N}$. Therefore, $f(x \oplus 0) = f(x \oplus 1)$ and $\mathcal{F}$ does not separate $x \oplus 0$ and $x \oplus 1$. □

Since the continuous image of an RN compact space is QRN, $\mathbb{L}_1$ is an example of a QRN compact space which is not WRN.

Section 2.6
WRN Boolean algebras

In this section we study Boolean algebras associated to zero-dimensional WRN compact spaces. We consider abstract Boolean algebras $\mathfrak{A}, \mathfrak{B}, \ldots$, keeping the usual set-theoretic notation. In particular, a^c denotes the complement of $a \in \mathfrak{A}$, but we shall also write $a^1 = a$ and $a^0 = a^c$ when convenient.

Given an algebra $\mathfrak{A}$ and any $\mathcal{G} \subseteq \mathfrak{A}$, we denote by $\langle \mathcal{G} \rangle$ the algebra generated by $\mathcal{G}$, i.e. the smallest subalgebra of $\mathfrak{A}$ containing $\mathcal{G}$. If $\langle \mathcal{G} \rangle = \mathfrak{A}$ then $\mathcal{G}$ is called a generating family.

For an algebra $\mathfrak{A}$, by $\operatorname{ult}(\mathfrak{A})$ we denote its Stone space (of ultrafilters on $\mathfrak{A}$) and

$$\mathfrak{A} \ni a \to \widehat{a} \in \operatorname{Clop}(\operatorname{ult}(\mathfrak{A}))$$

is the Stone isomorphism between $\mathfrak{A}$ and the algebra of clopen subsets of its Stone space.

Recall that an indexed family of elements $\{a_i : i \in I\}$ in an algebra $\mathfrak{A}$ is *independent* if

$$\bigcap_{i \in I'} a_i \cap \bigcap_{i \in I''} a_i^c \neq 0,$$

for every pair I', I'' of finite disjoint subsets of I. As we have seen, independent sequences play a fundamental role in the study of WRN compact spaces. Moreover, the existence of independent sequences can be deduced from the existence of measures with certain properties. By a measure μ on an algebra $\mathfrak{A}$ we mean a finitely additive nonnegative probability functional $\mathfrak{A} \to [0,1]$. We say that a family $\mathcal{E}$ of elements of a Boolean algebra $\mathfrak{A}$ is ε-separated by a measure μ if $\mu(a \triangle b) \geq \varepsilon$ for all distinct $a, b \in \mathcal{E}$, where by $a \triangle b$ we denote the symmetric difference $a \triangle b = (a \cap b^c) \cup (a^c \cap b)$.

Lemma 2.6.1 ([FP04]). *If $\varepsilon > 0$ and $\mathcal{E}$ is an infinite family ε-separated by some measure μ then $\mathcal{E}$ contains an infinite independent sequence.*

Note that, in turn, if $(a_n)_n$ is an independent sequence in some Boolean algebra $\mathfrak{A}$ then there is a measure μ on $\mathfrak{A}$ such that $\mu(a_n) = 1/2$ for every n and a_n are (stochastically) μ-independent. In particular, if $n \neq k$ then $\mu(a_n \triangle a_k) = 1/2$ so a_n are separated by μ.

Definition 2.6.2. *A Boolean algebra $\mathfrak{A}$ is* weakly Radon-Nikodým *(WRN) if there is a family $\mathcal{G} \subseteq \mathfrak{A}$ generating $\mathfrak{A}$ such that $\mathcal{G}$ can be written as $\mathcal{G} = \bigcup_{n\in\mathbb{N}} \mathcal{G}_n$, where, for every n, $\mathcal{G}_n$ contains no infinite independent sequence.*

The name of this class of Boolean algebras is motivated by the following Theorem:

Proposition 2.6.3 ([AMCP16]). *The following conditions are equivalent for a Boolean algebra $\mathfrak{A}$:*

(i) *$\mathfrak{A}$ is WRN;*

(ii) *there is a decomposition $\mathfrak{A} = \bigcup_{n\in\mathbb{N}} \mathcal{E}_n$ such that, for every n, $\mathcal{E}_n$ contains no infinite independent sequence;*

(iii) *the Stone space* $\mathrm{ult}(\mathfrak{A})$ *of $\mathfrak{A}$ is WRN.*

Although Proposition 2.6.3 can be deduced from Lemma 2.3.7 and Theorem 2.3.8, in this section we are going to prove it by studying the behavior of families containing no infinite independent sequences or independent sequences of size n for some $n \in \mathbb{N}$.

Note that condition (*ii*) provides an equivalent definition of WRN algebras that is sometimes more convenient; for instance it yields the following.

Corollary 2.6.4. *If $\mathfrak{A}$ is a WRN algebra then every subalgebra $\mathfrak{B} \subseteq \mathfrak{A}$ is WRN too.*

Clearly every countable algebra is WRN. More generally, if $\mathfrak{A}$ is an interval algebra, that is $\mathfrak{A} = \langle \mathcal{G} \rangle$ where the family $\mathcal{G}$ is linearly ordered, then $\mathfrak{A}$ is WRN. It follows from Corollary 2.6.4 that every subalgebra of an interval algebra is WRN.

This is also a consequence of a result due to Heindorf [Hei97], stating that an algebra $\mathfrak{A}$ embeds into some interval algebra if and only if $\mathfrak{A} = \langle \mathcal{G} \rangle$, where $\mathcal{G}$ has the property that any two elements of $\mathcal{G}$ are either comparable or disjoint.

Note that a Boolean algebra $\mathfrak{A}$ does not contain an infinite independent sequence if and only if the space $\mathrm{ult}(\mathfrak{A})$ is scattered, i.e. $\mathfrak{A}$ is superatomic. Moreover, we say that a Boolean algebra $\mathfrak{A}$ is RN if $\mathrm{ult}(\mathfrak{A})$ is RN. It follows from [Arv02, Corollary 3.7] that $\mathfrak{A}$ is RN if and only if there is a decomposition $\mathfrak{A} = \bigcup_{n\in\mathbb{N}} \mathfrak{A}_n$ such that, for every n, $\mathfrak{A}_n$ is a *Boolean subalgebra* containing no independent sequence. Therefore, every RN Boolean algebra is WRN.

To give some examples of Boolean algebras that are not WRN note another obvious consequence of Proposition 2.6.3.

Corollary 2.6.5. *If $\mathfrak{A}$ is a WRN algebra then $\mathfrak{A}$ contains no uncountable independent sequence.*

Hence $P(\omega)$, $P(\omega)/fin$, $\mathrm{Clop}(2^{\omega_1})$ are not WRN; moreover, no infinite complete algebra can be WRN. In particular, the Stone-Čech compactification of the natural numbers $\beta\mathbb{N}$ is not WRN, as it was mentioned after Corollary 2.4.2.

We start studying families with no independent families of size n for some $n \in \mathbb{N}$. The proof of the next result uses the Sauer-Shelah Lemma B.1, which is recalled together with a proof in the Appendix B.

Theorem 2.6.6 ([AMCP16]). *Let $\mathcal{E}$ be a family in a Boolean algebra $\mathfrak{A}$ such that $\mathcal{E}$ contains no independent family of size n. Fix $r \geq 1$ and set*

$$I(n,r) := \min\{s \in \mathbb{N} \colon \binom{rs}{0} + \binom{rs}{1} + \cdots + \binom{rs}{n-1} < 2^s\}.$$

Then, for any Boolean polynomial $p(x_1, \ldots, x_r)$ the family $p(\mathcal{E}) = \{p(a_1, \ldots, a_r) : a_1, \ldots a_r \in \mathcal{E}\}$ contains no independent sequence of length $I(n,r)$.

Proof. Suppose $p(\mathcal{E})$ contains an independent sequence of length $I(n,r)$. Then there exist

$$\begin{aligned} b_1 &= p(a_{1,1}, a_{1,2}, \ldots, a_{1,r}), \\ b_2 &= p(a_{2,1}, a_{2,2}, \ldots, a_{2,r}), \\ &\ldots \\ b_{I(n,r)} &= p(a_{I(n,r),1}, a_{I(n,r),2}, \ldots, a_{I(n,r),r}) \end{aligned}$$

such that $b_1, b_2, \ldots, b_{I(n,r)}$ is an independent family. Without loss of generality, we may suppose that $\mathcal{E} = \{a_{i,j} : 1 \leq i \leq I(n,r), 1 \leq j \leq r\}$. Let us put, for convenience, $N = rI(n,r)$ and $\mathcal{E} = \{a_1, a_2, \ldots, a_N\}$. Since $\langle \mathcal{E} \rangle$ contains an independent family of size $I(n,r)$, it must contain at least $2^{I(n,r)}$ atoms. Moreover, every atom of $\langle\mathcal{E}\rangle$ has a unique representation of the form $a_1^{f(1)} \cap a_2^{f(2)} \cap \cdots \cap a_N^{f(N)}$, where $f \in 2^{\{1,2,\ldots,N\}}$ and for each element $a \in \langle\mathcal{E}\rangle$, we denote the complement of a as a^0 and a as a^1. Set

$$\mathcal{F} = \{f \in 2^{\{1,2,\ldots,N\}} : a_1^{f(1)} \cap a_2^{f(2)} \cap \cdots \cap a_N^{f(N)} \text{ is an atom of } \langle\mathcal{E}\rangle\}.$$

We claim that $|\mathcal{F}| \leq \binom{N}{0} + \binom{N}{1} + \cdots + \binom{N}{n-1}$. If not, by the Sauer-Shelah Lemma (see Lemma B.1) there exists a set $S \subseteq \{1,2,\ldots,N\}$ with $|S| = n$ such that

$$\{f|_S : f \in \mathcal{F}\} = 2^S.$$

But this means that $\{a_i : i \in S\}$ is an independent family, since for each $f \in 2^S$, the element $\bigcap_{i\in S} a_i^{f(i)}$ is nonempty because it contains an atom. This is in contradiction with the hypothesis on $\mathcal{E}$, so

$$|\mathcal{F}| \leq \binom{N}{0} + \binom{N}{1} + \cdots + \binom{N}{n-1}.$$

Since the number of atoms of $\langle\mathcal{E}\rangle$ is exactly $|\mathcal{F}|$, we conclude that

$$2^{I(n,r)} \leq \binom{N}{0} + \binom{N}{1} + \cdots + \binom{N}{n-1} = \binom{rI(n,r)}{0} + \binom{rI(n,r)}{1} + \cdots + \binom{rI(n,r)}{n-1},$$

in contradiction with the definition of $I(n,r)$. □

Theorem 2.6.6 has the following counterpart:

Theorem 2.6.7 ([AMCP16]). *Let $\mathcal{E}$ be a family in some Boolean algebra $\mathfrak{A}$ such that $\mathcal{E}$ contains no infinite independent sequence. Let, for a fixed r, $p(x_1, x_2, \ldots, x_r)$ be any Boolean polynomial. Then the family*

$$p(\mathcal{E}) = \{p(a_1, \ldots, a_r) : a_1, \ldots, a_r \in \mathcal{E}\}$$

contains no infinite independent sequence.

Proof. Consider first the polynomial $p(x,y) = x \cap y$. Suppose that $p(\mathcal{E})$ contains $c_n = a_n \cap b_n$ with $a_n, b_n \in \mathcal{E}$ such that the sequence $(c_n)_n$ is independent. By the remark following Lemma 2.6.1 there is a probability measure μ on $\mathfrak{A}$ such that $\mu(c_n) = 1/2$ and c_n's are stochastically independent with respect to μ.

For $k < n$ we have

$$1/4 = \mu(c_k \setminus c_n) \leq \mu(c_k \setminus a_n) + \mu(c_k \setminus b_n),$$

so either $\mu(c_k \setminus a_n) \geq 1/8$ or $\mu(c_k \setminus b_n) \geq 1/8$. Say that the pair $\{k, n\}$ with $k < n$ gets the color a if the first inequality holds and the color b otherwise. By the Ramsey theorem there is an infinite $N \subseteq \mathbb{N}$ such that whenever $k, n \in N$ are different then $\{k, n\}$ has the same color; say that this is a.

It follows that for $k, n \in N$, $k < n$, we have

$$\mu(a_k \triangle a_n) \geq \mu(a_k \setminus a_n) \geq \mu(c_k \setminus a_n) \geq 1/8,$$

so the family $\{a_n : n \in N\}$ is 1/8-separated by μ. Applying Lemma 2.6.1 we get a contradiction.

We can assume that $\mathcal{E}$ is closed under taking complements. If we consider the polynomial $p'(x, y) = x \cup y$ then $p'(x, y) = (p(x^c, y^c))^c$ so the result follows for $p(\mathcal{E})$ by the argument above.

The general case follows by induction on the complexity of the Boolean polynomial in question. □

Proof of Theorem 2.6.3. *(i) ⇒ (ii):* Suppose that $\mathcal{G} \subseteq \mathfrak{A}$ is a family generating $\mathfrak{A}$ such that $\mathcal{G} = \bigcup_n \mathcal{G}_n$ where each $\mathcal{G}_n$ does not contain infinite independent sequences. Without loss of generality, we may suppose $\mathcal{G}_n \subseteq \mathcal{G}_{n+1}$ for every $n \in \mathbb{N}$. Take $\{p_n : n \in \mathbb{N}\}$ an enumeration of all Boolean polynomials. Then $\mathfrak{A} = \bigcup_{n,m\in\mathbb{N}} p_n(\mathcal{G}_m)$ and each $p_n(\mathcal{G}_m)$ does not contain independent sequences due to Theorem 2.6.7. Thus, $\mathfrak{A}$ satisfyies *(ii)*.

(ii) ⇒ (iii): Notice that each element $a \in \mathfrak{A}$ determines a continuous function $f_a \colon \mathrm{ult}(\mathfrak{A}) \to \{0, 1\}$ given by the formula $f_a(u) = 1$ if and only if $a \in u$ for every $u \in \mathrm{ult}(\mathfrak{A})$. If there is a decomposition $\mathfrak{A} = \bigcup_n \mathcal{E}_n$ such that each $\mathcal{E}_n$ contains no infinite independent sequence, then the family

$$\mathcal{F} = \{\frac{1}{n} f_a : n \in \mathbb{N},\ a \in \mathcal{E}_n\}$$

is a uniformly bounded family of continuous functions which separates the points of $\mathrm{ult}(\mathfrak{A})$ and with no independent sequences. Thus, $\mathrm{ult}(\mathfrak{A})$ is WRN.

(iii) ⇒ (i): Suppose $K = \mathrm{ult}(\mathfrak{A})$ is WRN. Then, there exists a uniformly bounded family of functions $\mathcal{F} \subseteq \mathcal{C}(K)$ which separates the points of K and with no independent sequence. Without loss of generality, we may suppose that $\mathcal{F} \subseteq [0, 1]^K$. Let $\{(p_n, q_n) : n \in \mathbb{N}\}$ be an enumeration of all pairs of rational numbers in $[0, 1]$ with $p_n < q_n$. For every function $f \in \mathcal{F}$ and every $n \in \mathbb{N}$, we fix a clopen set $C_{f,n}$ of K such that

$$\{x \in K : f(x) \leq p_n\} \subseteq C_{f,n} \subseteq \{x \in K : f(x) < q_n\}.$$

Notice that the family of clopen sets $\Gamma = \{C_{f,n} : f \in \mathcal{F}, n \in \mathbb{N}\}$ separates points of K and therefore it generates $\mathfrak{A}$. Moreover, each $\Gamma_n = \{C_{f,n} : n \in \mathbb{N}\}$ does not contain independent sequences since

$$\{x \in K : f(x) < p_n\} \subseteq C_{f,n} \text{ and } \{x \in K : f(x) > q_n\} \subseteq (C_{f,n})^c.$$

Thus, $\mathfrak{A}$ is a WRN Boolean algebra. □

Section 2.7

A minimally generated Boolean algebra which is not WRN

In Section 3.2 we will study a question of R. Haydon about the existence of nontrivial convergent sequences on WRN compact spaces. Since minimally generated Boolean algebras provide an important tool for constructing compact spaces without nontrivial convergent sequences, it is natural to study what is the relation between these classes of Boolean algebras. We prove in this section that the class of WRN Boolean algebras and the class of minimally generated Boolean algebras are incomparable.

The notion of minimal extensions of algebras was introduced by S. Koppelberg, see [Kop89]; the basic facts we mention below can be found in [Kop89] or [BN07].

If $\mathfrak{B}$ is a subalgebra of a Boolean algebra $\mathfrak{A}$ and $x \in \mathfrak{A} \setminus \mathfrak{B}$ then $\mathfrak{B}(x)$ denotes the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{B} \cup \{x\}$, that is

$$\mathfrak{B}(x) = \{(b \cap x) \cup (b' \cap x^c) : b, b' \in \mathfrak{B}\}.$$

$\mathfrak{B}(x)$ is said to be a *minimal extension* of $\mathfrak{B}$ if for any algebra $\mathfrak{C}$, if $\mathfrak{B} \subseteq \mathfrak{C} \subseteq \mathfrak{B}(x)$ then either $\mathfrak{C} = \mathfrak{B}$ or $\mathfrak{C} = \mathfrak{B}(x)$. We recall the following basic fact on minimal extensions, see Proposition 3.1 in [Kop89].

Proposition 2.7.1. *Let $\mathfrak{A}$ be a Boolean algebra, $\mathfrak{B} \leq \mathfrak{A}$ a subalgebra and $x \in \mathfrak{A}$. Then $\mathfrak{B}(x)$ is a minimal extension of $\mathfrak{B}$ if and only if for every $b \in \mathfrak{B}$, $x \cap b$ or $x \cap b^c$ is in $\mathfrak{B}$.*

Proof. Suppose that the extension $\mathfrak{B} \leq \mathfrak{B}(x)$ is minimal. Take $b \in \mathfrak{B}$. If $x \cap b$ is not in $\mathfrak{B}$, then $\mathfrak{B}(x \cap b) = \mathfrak{B}(x)$. Therefore, there exist $a_1, a_2 \in \mathfrak{B}$ such that

$$x = \big(a_1 \cap (x \cap b)\big) \cup \big(a_2 \cap (x \cap b)^c\big) = (a_1 \cap x \cap b) \cup (a_2 \cap (x^c \cup b^c)).$$

Hence $x \cap b^c = a_2 \cap b^c \in \mathfrak{B}$.

Suppose now that for every $b \in \mathfrak{B}$, either $x \cap b$ or $x \cap b^c$ is in $\mathfrak{B}$. Consider any element $y \in \mathfrak{B}(x) \setminus \mathfrak{B}$. Then there are disjoint $a_1, a_2, a_3 \in \mathfrak{B}$ such that

$$y = (a_1 \cap x) \cup (a_2 \cap x^c) \cup a_3.$$

Since $a_2 \cap x^c = a_2 \cap (a_1 \cup x^c) = a_2 \cap ((a_1^c \cap x)^c)$, it follows that either $a_1 \cap x$ or $a_2 \cap x^c$ is in $\mathfrak{B}$. By symmetry, we can assume that $a_2 \cap x^c \in \mathfrak{B}$. Then y can be written as $y = (a_1 \cap x) \cup c$ where $a_1, c \in \mathfrak{B}$ are disjoint. It follows that $a_1 \cap x \notin \mathfrak{B}$ and $a_1 \cap x = y \cap c^c \in \mathfrak{B}(y)$. Since $a_1^c \cap x$ must be in $\mathfrak{B}$, we get $x \in \mathfrak{B}(y)$ so $\mathfrak{B}(x) = \mathfrak{B}(y)$, as required. □

Corollary 2.7.2. *In the setting of Proposition 2.7.1, if for every finite* $\mathfrak{B}_0 \leq \mathfrak{B}$ *there is a finite subalgebra* $\mathfrak{B}_1$ *with* $\mathfrak{B}_0 \leq \mathfrak{B}_1 \leq \mathfrak{B}$ *such that* $\mathfrak{B}_1 \leq \mathfrak{B}_1(x)$ *is a minimal extension then* $\mathfrak{B}(x)$ *is a minimal extension of* $\mathfrak{B}$*.*

A Boolean algebra $\mathfrak{A}$ is *minimally generated over* $\mathfrak{B}$ if $\mathfrak{B} \leq \mathfrak{A}$ and $\mathfrak{A}$ can be written, for some limit ordinal γ, as a union

$$\mathfrak{A} = \bigcup_{\xi<\gamma} \mathfrak{B}_\xi,$$

where $\mathfrak{B}_0 = \mathfrak{B}$, $\mathfrak{B}_{\xi+1}$ is a minimal extension of $\mathfrak{B}_\xi$ for every $\xi < \gamma$ and $\mathfrak{B}_\eta = \bigcup_{\xi<\eta} \mathfrak{B}_\xi$ for every limit ordinal $\eta \leq \gamma$. Finally, $\mathfrak{A}$ is said to be minimally generated if it is minimally generated over the trivial algebra $\mathfrak{B} = \{0, 1\}$.

The dual concept of minimally generated Boolean algebras are continuous inverse systems of simple extensions:

Definition 2.7.3. $(K_\xi, \pi^\eta_\xi)_{\xi<\eta<\kappa}$ *is an inverse system if each* π^η_ξ *is a continuous map from* K_η *to* K_ξ *for every* $\xi < \eta < \kappa$ *and* $\pi^\eta_\xi = \pi^{\eta'}_\xi \circ \pi^\eta_{\eta'}$ *for every* $\xi < \eta' < \eta < \kappa$*. The inverse limit of the system is the subspace of* $\prod_{\xi<\kappa} K_\xi$ *consisting of all points* x *which satisfy* $\pi^\eta_\xi(x_\eta) = x_\xi$ *for every* $\xi < \eta < \kappa$*.*

$(K_\xi, \pi^\eta_\xi)_{\xi<\eta<\kappa}$ *is said to be continuous if* $K_{\kappa'}$ *is the inverse limit of* $(K_\xi, \pi^\eta_\xi)_{\xi<\eta<\kappa'}$ *for every limit ordinal* $\kappa' < \kappa$*. Moreover, it is said to be based on simple extensions if for every* $\xi < \kappa$*, the bonding map* $\pi^{\xi+1}_\xi : K_{\xi+1} \to K_\xi$ *has the property that there is a unique* $x_\xi \in K_\xi$ *such that* $|(\pi^{\xi+1}_\xi)^{-1}(x_\xi)| = 2$*, and* $|(\pi^{\xi+1}_\xi)^{-1}(x)| = 1$ *for* $x \neq x_\xi$*.*

Note that if $\mathfrak{A}$ is minimally generated then its Stone space $\mathrm{ult}(\mathfrak{A})$ can be seen as a limit of a continuous inverse system based on simple extensions $(K_\xi, \pi^\eta_\xi)_{\xi<\eta<\kappa}$, where $K_0 = \{0, 1\}$ and every K_ξ is compact.

The following result name common features of the classes of minimally generated algebras and WRN algebras.

Theorem 2.7.4 ([Kop89]).

- *Every subalgebra of an interval algebra is minimally generated.*
- *Every superatomic Boolean algebra is minimally generated.*
- *A minimally generated algebra cannot contain an uncountable independent family.*

Nevertheless, we show below that there is a WRN Boolean algebra which is not minimally generated and we give an example of a minimally generated algebra which is not WRN.

Example 2.7.5 ([AMCP16]). *There exists a WRN Boolean algebra which is not minimally generated.*

Proof. Let $\mathfrak{B}$ be the algebra of subsets of $[0,1)$ generated by the chain $\{[0,t) : 0 < t < 1\}$. Then $\mathfrak{B}$ is an interval algebra and therefore it is minimally generated, see Example 2.1 in [Kop89] (note that its Stone space is the familiar split interval).

S. Koppelberg [Kop88, Example 1] proved, in particular, that the free product $\mathfrak{A} = \mathfrak{B} \otimes \mathfrak{B}$ is not minimally generated. Such a free product is generated by

$$\mathcal{G} = \Big\{[0,a) \times [0,1) : 0 < a < 1\Big\} \cup \Big\{[0,1) \times [0,a) : 0 < a < 1\Big\}.$$

Note that no three elements of $\mathcal{G}$ are independent so $\mathfrak{A}$ is a WRN algebra. □

During the rest of this section we construct a minimally generated Boolean algebra which is not WRN. We shall work in the Cantor set 2^ω; let $\mathfrak{A}_0 = \mathrm{Clop}(2^\omega)$. For every partial function σ on ω to 2 we write

$$[\sigma] = \{x \in 2^\omega : x(i) = \sigma(i) \text{ for every } i \in \operatorname{dom}\sigma\}.$$

Let $T = \{3n : n \in \omega\}$ and let $S(T)$ be the space of all permutations of T.

Let $x \in 2^\omega$, $\varphi \in S(T)$ be given. We shall define a certain set $A(x,\varphi) \subseteq 2^\omega$. First define partial functions $\sigma_n(x,\varphi)$ on ω as follows.

(i) $\sigma_n(x,\varphi)(i) = x(i)$ if $i \in 3n \setminus T$;

(ii) $\sigma_n(x,\varphi)(\varphi(i)) = x(\varphi(i))$ if $i \in T$ and $i < 3n$;

(iii) $\sigma_n(x,\varphi)(\varphi(3n)) = (x(\varphi(3n)) + 1) \bmod 2$.

Note that every $\sigma_n(x,\varphi)$ is defined on the set

$$(3n \setminus T) \cup \varphi(\{i \in T : i \leq 3n\}),$$

so the domain of $\sigma_n(x,\varphi)$ is of size $3n+1$. We now set

$$A(x,\varphi) = \bigcup_n [\sigma_n(x,\varphi)]\,.$$

We shall say below that a sequence $(A_n)_n$ of subsets of 2^ω converges to a point $x \in 2^\omega$ if every neighborhood of x contains A_n for almost all n.

Claim 2.7.6. *For any x and φ, $([\sigma_n(x,\varphi)])_{n=1}^\infty$ is a sequence of disjoint clopen subsets of 2^ω converging to x.*

Proof. If $n < k$ then $\sigma_n(x,\varphi)(\varphi(3n)) \neq \sigma_k(x,\varphi)(\varphi(3n))$ so the clopen sets in question are disjoint. If τ is any partial function with a finite domain I and $x \in [\tau]$ then take n_0 such that

$$I \subseteq (3n_0 \setminus T) \cup \varphi(\{i \in T : i < 3n_0\}).$$

Then $\sigma_n(x,\varphi)$ extends τ so $[\sigma_n(x,\varphi)] \subseteq [\tau]$ for every $n \geq n_0$. □

Notice that $S(T)$ is a G_δ-set in T^T, which is separable and completely metrizable, so $S(T)$ is a Polish space. Let us now fix a Borel bijective map $g\colon 2^\omega \to S(T)$ (recall that between any two uncountable Polish spaces there is always a Borel isomorphism; the fact that g is Borel will be needed for the proof of Theorem 2.7.10). For every $x \in 2^\omega$ take $A_x = A(x, g(x))$. We define the desired algebra $\mathfrak{A}$ of subsets of 2^ω as the one generated by $\mathfrak{A}_0$ together with the family $\{A_x : x \in 2^\omega\}$. Notice that the compact space $\mathrm{ult}(\mathfrak{A})$ is just the split Cantor space induced by the family of continuous characteristic functions $\{1_{A_x}\colon 2^\omega \setminus \{x\} \to \{0,1\} : x \in 2^\omega\}$.

Claim 2.7.7. *The algebra $\mathfrak{A}$ is minimally generated.*

Proof. Note that for any distinct $x, y \in 2^\omega$, $[\sigma_n(x, g(x))]$ and $[\sigma_n(y, g(y))]$ are sequences of clopen sets converging to x and y, respectively. It follows that either $x \notin A_y$ and then $A_x \cap A_y$ is clopen or, $x \in A_y$ and then $A_x \setminus A_y$ is clopen. Therefore, $\mathfrak{A}$ is minimally generated over $\mathfrak{A}_0$ by Proposition 2.7.1 and hence $\mathfrak{A}$ is minimally generated (since $\mathfrak{A}_0$ is minimally generated because it is countable). □

Claim 2.7.8. *The algebra $\mathfrak{A}$ is not WRN.*

Proof. Take any decomposition $\mathfrak{A} = \bigcup_{n<\omega} \mathcal{A}_n$. We shall prove that there is $n < \omega$ such that $\mathcal{A}_n$ contains an independent sequence.

Define $\Phi_n = \{g(x) : A_x \in \mathcal{A}_n\}$ for every $n < \omega$. Since $\bigcup_n \Phi_n = g(2^\omega) = S(T)$ and $S(T)$ is a Polish space, the Baire Category Theorem asserts that there exists $n_0 < \omega$ and a partial function ψ from ω to ω such that $\Phi_{n_0} \cap [\psi]$ is dense in $S(T) \cap [\psi]$. We can assume that the domain of ψ is $\{0, 3, ..., 3(p-1)\}$ for some p; fix also $i_0 \in \omega$ such that the range of ψ is included in $\{0, 3, ..., 3(i_0 - 1)\}$.

Note that, by density, for any $i \geq i_0$ there is $x_i \in 2^\omega$ such that $A_{x_i} \in \mathcal{A}_{n_0}$, $g(x_i)(3p) = 3i$ and $g(x_i)(3j) = \psi(3j)$ for every $j < p$. Passing to a subsequence of i's, we can additionally assume that x_i have constant values for all $n < 3p$. Then the following are satisfied:

(a) for every $n < p$ and for every i, j,

$$[\sigma_n(x_i, g(x_i))] = [\sigma_n(x_j, g(x_j))]\,.$$

(b) there is a partial function σ from ω to 2 with domain of size $3p$ such that for every $i \geq i_0$ we have
$$[\sigma_p(x_i, g(x_i))] = [\sigma] \cap C_i^{\epsilon_i},$$
where we write $C_i^{\epsilon_i} = \{x \in 2^\omega : x(3i) = \epsilon_i\}$ for the corresponding one-dimensional cylinder in 2^ω.

Let μ be the canonical product measure on 2^ω. We shall prove that $\{A_{x_i}\}_{i \geq i_0}$ is ε-separated for some $\varepsilon > 0$.

Note that for every x_i and every $n < \omega$,
$$\mu([\sigma_n(x_i, g(x_i))]) = \frac{1}{2^{3n+1}}.$$
Using (a)–(b) above, for distinct $i, j \geq i_0$ we get
$$\mu(A_{x_i} \setminus A_{x_j}) \geq \mu\Big(\Big(([\sigma] \cap C_i^{\epsilon_i}) \setminus ([\sigma] \cap C_j^{\epsilon_j})\Big) \setminus \Big(\bigcup_{n>p} [\sigma_n(x_j, g(x_j))]\Big)\Big) \geq$$
$$\geq \frac{1}{2^{3p+2}} - \sum_{n>p} \frac{1}{2^{3n+1}} = \frac{1}{2^{3p+2}} - \frac{1}{2^{3p+4}} \frac{1}{1 - 2^{-3}} = \frac{5}{7} \frac{1}{2^{3p+2}}.$$
It follows that the sets A_{x_i} for $i \geq i_0$ are ε-separated with $\varepsilon > 0$ so by Lemma 2.6.1 there is an independent subsequence in $\mathcal{A}_{n_0}$ and we are done. □

The following result summarizes our considerations and gives another property of the Boolean algebra we have constructed. Recall that a function $f\colon L \to \mathbb{R}$ over a topological space L is a Baire-one function if it is the pointwise limit of a sequence of continuous functions on L. By $B_1(L)$ we denote the space of Baire-one functions on L equipped with the topology of pointwise convergence. A compact space K is said to be *Rosenthal* if it can be embedded into $B_1(L)$ with L a Polish space, i.e. L a separable completely metrizable topological space. We shall use the following result.

Theorem 2.7.9. *[Deb14, Corollary 4.9] Every separable compact space consisting of Borel functions over a Polish space is Rosenthal.*

Theorem 2.7.10 ([AMCP16])**.** *There is a minimally generated algebra $\mathfrak{A}$ such that its Stone space $K = \mathrm{ult}(\mathfrak{A})$ is a separable Rosenthal compact space which is not WRN.*

Proof. By Claim 2.7.8 the algebra $\mathfrak{A}$ is not WRN so $K = \mathrm{ult}(\mathfrak{A})$ is not WRN. It follows easily from Claim 2.7.7 that $\mathfrak{A}_0$ is a dense subalgebra of $\mathfrak{A}$. Hence K has a countable π-base so is, in particular, separable. We prove below that K is indeed Rosenthal.

Given an ultrafilter $u \in K$, let z_u be the unique point in 2^ω such that

$$\bigcap\{C \in \mathrm{Clop}(2^\omega) : C \in u\} = \{z_u\}.$$

Claim A. For every $u \in K$ we have

$$\{y \in 2^\omega : z_u \in A_y\} \subseteq \{y \in 2^\omega : A_y \in u\} \subseteq \{y \in 2^\omega : z_u \in A_y\} \cup \{z_u\}.$$

The first inclusion is clear. To check the latter, note first that $\overline{A_y} = A_y \cup \{y\}$ for every $y \in 2^\omega$ since A_y is the union of clopen sets converging to y. Hence if $A_y \in u$ and $z_u \notin A_y$ then $y = z_u$ (otherwise, $z_u \notin \overline{A_y}$ which contradicts the definition of z_u).

Claim B. For every $u \in K$, $\{y \in 2^\omega : A_y \in u\}$ is a Borel subset of 2^ω.

By Claim A, it is sufficient to check that for any $z \in 2^\omega$ the set $\{y \in 2^\omega : z \in A_y\}$ is Borel. But

$$\{y \in 2^\omega : z \in A_y\} = \bigcup_n \{y \in 2^\omega : z \in [\sigma_n(y, g(y))]\},$$

and every set $\{y \in 2^\omega : z \in [\sigma_n(y, g(y))]\}$ is Borel because the function g is Borel.

Consider now the following mapping $f \colon K \to 2^\omega \times 2^{2^\omega}$

$$f(u) = \left(z_u, \chi_{\{y \in 2^\omega : A_y \in u\}}\right).$$

Then f is injective since every ultrafilter $u \in K$ is uniquely determined by the family of generators of the algebra $\mathfrak{A}$ that are in u. It is clear that f is continuous. It follows from Claim B that K is homeomorphic to a pointwise-compact set of Borel functions on a Polish space. Since K is separable, K is Rosenthal compact by Theorem 2.7.9. □

Section 2.8
Uniformly WRN and Strongly WRN Boolean algebras

In this section we introduce two subclasses of WRN algebras; they are defined by natural conditions that are slightly stronger than that of Definition 2.6.2.

Definition 2.8.1. *A Boolean algebra $\mathfrak{A}$ is in the class $\mathcal{I}(n)$, where $n \geq 1$, if $\mathfrak{A}$ is generated by a family $\mathcal{G} \subseteq \mathfrak{A}$ such that $\mathcal{G}$ contains no $n+1$ independent elements.*

Definition 2.8.2. *A Boolean algebra* $\mathfrak{A}$ *is* uniformly weakly Radon-Nikodým *(UWRN) if* $\mathfrak{A}$ *is generated by a family* $\mathcal{G} = \bigcup_{n\in\mathbb{N}} \mathcal{G}_n$ *such that no* $\mathcal{G}_n$ *contains an independent sequence of length* n*.*

Notice that Theorem 2.6.6 provides an analogue result of Proposition 2.6.3 for UWRN Boolean algebras.

Corollary 2.8.3 ([AMCP16])**.** *A Boolean algebra* $\mathfrak{A}$ *is UWRN if and only if there is a decomposition* $\mathfrak{A} = \bigcup_{n\in\mathbb{N}} \mathcal{E}_n$ *such that no* $\mathcal{E}_n$ *contains an independent sequence of length* n*.*

Consequently, the class of UWRN algebras is stable under taking subalgebras.

Note that every interval algebra is in $\mathcal{I}(1)$. In turn, the following holds.

Theorem 2.8.4 ([AMCP16])**.** *Every Boolean algebra from* $\mathcal{I}(1)$ *is minimally generated.*

Proof. Take a Boolean algebra $\mathfrak{A} \in \mathcal{I}(1)$ and a family $\mathcal{G}$ generating $\mathfrak{A}$ containing no independent pairs of elements. We shall check the following.

Claim. For every finite $\mathcal{J} \subseteq \mathcal{G}$ and every $x \in \mathcal{G}$, the extension $\langle\mathcal{J}\rangle \leq \langle\mathcal{J} \cup \{x\}\rangle$ is minimal.

It is clear that Claim holds if $\mathcal{J} = \{y\}$, since x, y are not independent. We argue by induction on $|\mathcal{J}|$.

Suppose that every extension $\langle\mathcal{J}\rangle \leq \langle\mathcal{J} \cup \{x\}\rangle$ is minimal whenever $|\mathcal{J}| = n$. Take $x \in \mathcal{G}$ and $\mathcal{J} \subseteq \mathcal{G}$ with $|\mathcal{J}| = n+1$. We prove that the extension $\langle\mathcal{J}\rangle \leq \langle\mathcal{J} \cup \{x\}\rangle$ is also minimal. Choose $y \in \mathcal{J}$ and set $\mathcal{S} = \mathcal{J} \setminus \{y\}$. We are going to prove that for every $z \in \langle\mathcal{J}\rangle$, $z \cap x$ or $z^c \cap x$ is in $\langle\mathcal{J}\rangle$. Since $z \in \langle\mathcal{J}\rangle = \langle\mathcal{S} \cup \{y\}\rangle$, we know that

$$z = (a \cap y) \cup (b \cap y^c) \text{ for some } a, b \in \langle\mathcal{S}\rangle.$$

Since $|\mathcal{S}| = n$, we know that $a \cap x$ or $a^c \cap x$ is in $\langle\mathcal{S}\rangle$, and $b \cap x$ or $b^c \cap x$ is in $\langle\mathcal{S}\rangle$.

Without loss of generality, suppose that $z \cap x \notin \langle\mathcal{J}\rangle$. Since

$$z \cap x = (a \cap y \cap x) \cup (b \cap y^c \cap x),$$

then either $a \cap y \cap x \notin \langle\mathcal{J}\rangle$ or $b \cap y^c \cap x \notin \langle\mathcal{J}\rangle$.

Consider the case when $a \cap y \cap x \notin \langle\mathcal{J}\rangle$. Then $a \cap x \notin \langle\mathcal{S}\rangle$ so $a^c \cap x \in \langle\mathcal{S}\rangle$. Moreover, it follows that $x \cap y \neq 0$ and $y \not\subseteq x$. Since y and x are not independent, this leaves us two possibilities: either $x \subseteq y$ or $x \cup y = 1$. Hence $y^c \cap x = 0$ or $y^c \cap x = y^c$ so $y^c \cap x \in \langle\mathcal{J}\rangle$ in both cases. By easy calculation we get

$$z^c = (a \cap y)^c \cap (b \cap y^c)^c = (a^c \cup y^c) \cap (b^c \cup y) = (a^c \cap b^c) \cup (a^c \cap y) \cup (b^c \cap y^c) =$$

$$= (a^c \cap y) \cup (b^c \cap y^c),$$

and it follows that

$$z^c \cap x = \big((a^c \cap x) \cap y\big) \cup \big((y^c \cap x) \cap b^c\big) \in \langle \mathcal{J} \rangle.$$

If $b \cap y^c \cap x \notin \langle \mathcal{J} \rangle$ then in a similar way we get $b^c \cap x \in \langle \mathcal{S} \rangle$ and $x \cap y \in \langle \mathcal{J} \rangle$, giving $z^c \cap x \in \langle \mathcal{J} \rangle$. This finishes the proof of Claim.

Now we conclude the proof of the theorem applying Claim and Corollary 2.7.2. □

Note that Example 2.7.5 in fact gives the following.

Corollary 2.8.5. *There exists a Boolean algebra in $\mathcal{I}(2)$ which is not minimally generated.*

We can also strengthen the condition of Definition 2.6.2 in the following way.

Definition 2.8.6. *Let us say that a Boolean algebra is strongly WRN (SWRN) if it is generated by a family containing no infinite independent sequence.*

The classes of UWRN and SWRN Boolean algebras are incomparable, see Corollary 3.2.6 and the following result.

Proposition 2.8.7. *There exists a UWRN algebra which is not SWRN.*

Proof. Let $\mathfrak{A}$ be the algebra of clopen sets of a countable product of one point compactifications of a discrete set of cardinality ω_1. Let $\mathcal{F} = \{e^n_\alpha : n \in \mathbb{N}, \alpha < \omega_1\}$ be the canonical generators of $\mathfrak{A}$ which are independent except for the relation $e^n_\alpha \cap e^n_\beta = 0$ whenever $\alpha \neq \beta$. Clearly the algebra $\mathfrak{A}$ is UWRN. We prove below that it is not SWRN.

Suppose that $\mathcal{G}$ is a system of generators. It is enough to check that the image of $\mathcal{G}$ under some quotient contains an infinite independent sequence. Express each e^n_α as a Boolean polynomial of generators from $\mathcal{G}$ and in turn each such generator as a Boolean polynomial of generators from $\mathcal{F}$. Let $F_k(e^n_\alpha)$ be the set of all $\beta < \omega_1$ such that e^k_β appears in such expression of e^n_α. Notice that for every $\alpha < \omega_1$, $n \in \mathbb{N}$ each set $F_k(e^n_\alpha)$ is finite and, moreover, $F_k(e^n_\alpha) = \emptyset$ for all except finitely many $k \in \mathbb{N}$. By passing, for each n, to an uncountable subset $A_n \subseteq \omega_1$ (by this we mean, making a quotient that makes each e^n_α, $\alpha \notin A_n$ vanish), we can suppose that for every n there is $m_n \in \mathbb{N}$ such that $F_k(e^n_\alpha) = \emptyset$ if $k \geq m_n$ and $|F_k(e^n_\alpha)| < m_n$ if $k < m_n$. Moreover, we can also suppose that each family $\{F_k(e^n_\alpha) : \alpha \in A_n\}$ is a Δ-system. By removing all roots (that form just a countable set), we can suppose that the family $\{F_k(e^n_\alpha) : \alpha \in A_n\}$ is always pairwise disjoint. Now it is easy to get $\alpha_n \in A_n$ such that $F_k(e^n_{\alpha_n}) \cap F_q(e^m_{\alpha_m}) = \emptyset$ for all k, q, n, m with $n \neq m$. If we make vanish all generators of $\mathcal{F}$ except the $e^n_{\alpha_n}$'s, we will find that one of the generators from $\mathcal{G}$ (call it g_n) in the expression of $e^n_{\alpha_n}$ is $e^n_{\alpha_n}$ itself. Thus, there is an infinite independent sequence in $\mathcal{G}$. □

Section 2.9
Measures on WRN compact spaces

Recall that by a measure μ on a Boolean algebra we mean a finitely additive nonnegative probability functional $\mathfrak{A} \to [0,1]$.

Definition 2.9.1. *Let μ be a measure on an algebra $\mathfrak{A}$. We say that the measure μ*

(i) is nonatomic *if for every $\varepsilon > 0$ there are $n \geq 1$ and a finite partition $\{a_1, \ldots, a_n\}$ of $\mathbf{1}_{\mathfrak{A}}$ such that $\mu(a_i) < \varepsilon$ for every $i \leq n$.*

(ii) has countable type (or is separable) *if there is a countable subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ such that for every $a \in \mathfrak{A}$ we have*

$$\inf\{\mu(a \triangle b) : b \in \mathfrak{B}\} = 0.$$

(iii) is strongly countably determined (or uniformly regular) *if there is a countable subalgebra $\mathfrak{C} \subseteq \mathfrak{A}$ such that for every $a \in \mathfrak{A}$ we have*

$$\mu(a) = \sup\{\mu(c) : c \in \mathfrak{C}, c \subseteq a\}.$$

Clearly a strongly countably determined measure has countable type; recall that the reverse implication does not hold in general.

Definition 2.9.2. *Let μ be a regular Borel probability measure on a compact space K.*

(i) A zero set *in K is a set of the form $f^{-1}(\{0\})$, where $f \in \mathcal{C}(K)$.*

(ii) The measure μ is uniformly regular *if there is a countable family $\mathcal{Z}$ of zero subsets of K such that for every open set U in K*

$$\mu(U) = \sup\{\mu(C) : C \in \mathcal{Z}, C \subseteq U\}.$$

(iii) We say that μ is concentrated on a Borel set $C \subseteq K$ if $\mu(C) = 1$.

Notice that every measure μ in a Boolean algebra $\mathfrak{A}$ determines a measure $\widehat{\mu}$ in $\mathrm{ult}(\mathfrak{A})$ such that $\widehat{\mu}(\widehat{a}) = \mu(a)$ for every $a \in \mathfrak{A}$. If μ is a measure with countable type, then $\widehat{\mu}$ is separable, i.e. $L_1(\mu)$ is separable as a Banach space. Moreover, if μ is strongly countable determined then $\widehat{\mu}$ is uniformly regular.

Every uniformly regular measure on a compact space is separable and is concentrated on a separable subspace. We prove in this section that every measure on a WRN compact space is separable and study the problem of the existence of measures on WRN compact spaces which are not concentrated on a separable set.

Let us start by studying the case of WRN Boolean algebras. P. Borodulin-Nadzieja [BN07] proved the following result concerning minimally generated algebras:

Theorem 2.9.3. *Let $\mathfrak{A}$ be a minimally generated algebra. Then every measure μ on $\mathfrak{A}$ has countable type.*

Proposition 2.9.4 ([AMCP16]). *If μ is a measure on a WRN algebra $\mathfrak{A}$ then μ has countable type.*

Proof. Suppose otherwise; note that then there is $\varepsilon > 0$ and an uncountable family $\mathcal{F}$ such that $\mu(a \triangle b) \geq \varepsilon$ for any distinct $a, b \in \mathcal{F}$.

Since $\mathfrak{A}$ is WRN, we have a decomposition $\mathfrak{A} = \bigcup_n \mathcal{E}_n$ as in Proposition 2.6.3(ii). But then $\mathcal{F} \cap \mathcal{E}_n$ is uncountable for some n and we arrive at a contradiction with Lemma 2.6.1. □

Remark 2.9.5. *Notice that if every measure on a Boolean algebra $\mathfrak{A}$ has countable type, then $\mathfrak{A}$ contains no uncountable independent sequence and, consequently,* ult$(\mathfrak{A})$ *cannot be mapped onto 2^{ω_1}. In particular, the Stone space of any minimally generated algebra or WRN algebra cannot contain a copy of $\beta\mathbb{N}$.*

Proposition 2.9.6 ([AMCP16]). *If μ is a nonatomic measure on $\mathfrak{A}$ and $\mathfrak{A} \in \mathcal{I}(1)$ then μ is strongly countably determined.*

Proof. By the assumption, $\mathfrak{A} = \langle \mathcal{G} \rangle$ where $\mathcal{G}$ contains no independent pair.

Fix $\varepsilon > 0$. There is a finite $\mathcal{G}_0 \subseteq \mathcal{G}$ such that $\mathfrak{B} = \langle \mathcal{G}_0 \rangle$ has all atoms of measure $< \varepsilon$. Take any $g \in \mathcal{G}$ and consider $b_0, b_1 \in \mathfrak{B}$, where b_0 is the maximal element of $\mathfrak{B}$ contained in g, while b_1 is the minimal element of $\mathfrak{B}$ containing g.

Claim. $b_1 \setminus b_0$ is an atom of $\mathfrak{B}$.

Indeed, for any $h \in \mathcal{G}_0$, either $h \subseteq g$ which implies $h \subseteq b_0$, or $h \cap g = \mathbf{0}$ which gives $h \cap b_1 = \mathbf{0}$, or $g \subseteq h$ which implies $b_1 \subseteq h$, or else $h \cup g = \mathbf{1}$ and in this case $h \supseteq b_1 \setminus b_0$. So $b_1 \setminus b_0$ is split by no $h \in \mathcal{G}_0$ and hence it is an atom of $\mathfrak{B}$.

It follows from Claim that $\mu(b_1 \setminus b_0) < \varepsilon$, so

$$\mu(b_0) = \mu(b_1) - \mu(b_1 \setminus b_0) \geq \mu(g) - \varepsilon,$$

so b_0 approximates g from inside; likewise, $b_1^c \subseteq g^c$ and $\mu(g^c \setminus b_1^c) \leq \varepsilon$. Now, taking a countable $\mathcal{G}' \subseteq \mathcal{G}$ such that μ is nonatomic on $\mathfrak{C} = \langle \mathcal{G}' \rangle$, it follows that for every $g \in \mathcal{G}$, we have

$$\mu(g) = \sup\{\mu(c) : c \in \mathfrak{C}, c \subseteq g\} \text{ and } \mu(g^c) = \sup\{\mu(c) : c \in \mathfrak{C}, c \subseteq g^c\}.$$

By a standard argument we conclude that $\mu(a) = \sup\{\mu(c) : c \in \mathfrak{C}, c \subseteq a\}$ for every $a \in \mathfrak{A}$, so μ is strongly countably determined. □

P. Borodulin-Nadzieja [BN07, Corollary 4.11 and Example 4.12] proved that a nonatomic measure on an algebra that is minimally generated by a sequence of order type ω_1 is strongly countably determined but this is no longer true for arbitrary minimally generated algebras.

Problem 2.9.7. *Is it true that for every n and every algebra $\mathfrak{A} \in \mathcal{I}(n)$, every nonatomic measure on $\mathfrak{A}$ is strongly countably determined?*

Note that if the answer to the above problem is positive then every nonatomic measure on a UWRN algebra is strongly countably determined. In turn, this would imply that if K is a zero-dimensional compact space with $\mathrm{Clop}(K)$ being a UWRN algebra then every regular Borel measure on K is concentrated on a separable subspace.

Problem 2.9.8 (J. Rodríguez)**.** *Is it true that every regular Borel probability measure on a WRN compact space K is concentrated on a separable subspace of K?*

This question is motivated by the following results due to J. Rodríguez:

Proposition 2.9.9 ([AMCP16])**.** *If K is WRN compact and if μ is a regular Borel probability measure on K then μ has countable type (i.e. $L_1(\mu)$ is separable).*

Proof. Since K is WRN compact, $\mathcal{C}(K)$ is spanned by some weakly precompact set $W \subseteq \mathcal{C}(K)$. If we consider the natural embedding $\mathcal{C}(K) \hookrightarrow L_1(\mu)$ then the image of W is norm-separable. Indeed, otherwise for some $\varepsilon > 0$ we could find functions $f_n \in W$ such that $\int_K |f_n - f_k| \, \mathrm{d}\mu \geq \varepsilon$ for $n \neq k$. But then $(f_n)_n$ admits no weakly Cauchy subsequence, a contradiction.

Since W is norm-separable in $L_1(\mu)$, a standard argument gives that $\mathcal{C}(K) \subseteq \overline{\mathrm{span}}(W)$ is also norm-separable in $L_1(\mu)$. But $\mathcal{C}(K)$ is dense in $L_1(\mu)$ so $L_1(\mu)$ is separable itself. □

Let X be a Banach space and let K be a weak*-compact subset of the dual unit ball B_{X^*}. Let μ be a regular Borel probability measure on K; denote by $f : K \to X^*$ the identity function. Then for every $B \in \mathrm{Borel}(K)$ there is a vector $\nu(B) = \int_B f \, \mathrm{d}\mu \in X^*$ which is the Gelfand integral of f on B, that is

$$\langle \nu(B), x \rangle = \int_B x \, \mathrm{d}\mu \tag{2.1}$$

for every $x \in X$, see [DU77, page 53]. Here every $x \in X$ is seen as a continuous function $K \ni x^* \to x^*(x)$ on K. In other language, $\nu(B)$ is the barycenter of a measure $1/\mu(B) \cdot \mu_B$ which is the normalized restriction of μ to B.

Theorem 2.9.10. *Let X be a Banach space and $K \subseteq B_{X^*}$ a weak*-compact and convex subset. Then for every probability regular Borel measure μ on K with countable type there is a weak*-closed and weak*-separable set $L \subseteq K$ such that $\mu(L) = 1$, i.e. μ is concentrated on a separable subspace of K.*

Proof. Consider the set

$$S := \left\{ \frac{1}{\mu(B)} \int_B f \, \mathrm{d}\mu : \, B \in \mathrm{Borel}(K), \, \mu(B) > 0 \right\}.$$

As above, we write $\nu(B) = \int_B f \, \mathrm{d}\mu$ for simplicity.

Claim. The set S is norm-separable.

Since μ is separable there is a countable family $\mathcal{A}$ of Borel subsets of K of positive measure such that $\inf\{\mu(A \triangle B) : A \in \mathcal{A}\} = 0$ for every Borel set $B \subseteq K$. Note that

$$\|\nu(B) - \nu(A)\| = \sup_{x \in B_X} \left| \int_B x \, \mathrm{d}\mu - \int_A x \, \mathrm{d}\mu \right| \le \mu(B \triangle A). \tag{2.2}$$

Fix $\varepsilon > 0$ and a Borel set B of positive measure; take $A \in \mathcal{A}$ such that $\mu(B \triangle A) < \varepsilon \cdot \mu(B)$ and $|1/\mu(B) - 1/\mu(A)| < \varepsilon$. Then, using (2.2) we get

$$\left\| \frac{1}{\mu(B)} \nu(B) - \frac{1}{\mu(A)} \nu(A) \right\| \le \left\| \frac{1}{\mu(B)} \nu(B) - \frac{1}{\mu(B)} \nu(A) \right\| + \left\| \frac{1}{\mu(B)} \nu(A) - \frac{1}{\mu(A)} \nu(A) \right\|$$

$$\le \frac{1}{\mu(B)} \|\nu(B) - \nu(A)\| + \|\nu(A)\| \left| \frac{1}{\mu(B)} - \frac{1}{\mu(A)} \right| \le 2\varepsilon.$$

and the claim is proved.

Since S is norm-separable in X^*, the weak*-closed convex hull $L := \overline{\mathrm{co}(S)}^{w^*}$ is weak*-separable. Let us check that L fulfills the required properties.

First note that $L \subseteq K$. To verify this it suffices to check that $S \subseteq K$. Take any $x^* \in X^* \setminus K$. By the Hahn-Banach theorem, there is $x \in X$ such that $x^*(x) > \alpha := \sup\{y^*(x) : y^* \in K\}$, therefore

$$\left\langle \frac{1}{\mu(B)} \int_B f \, \mathrm{d}\mu, x \right\rangle \overset{(2.1)}{=} \frac{1}{\mu(B)} \int_B x \, \mathrm{d}\mu \le \alpha < x^*(x),$$

for every $B \in \mathrm{Borel}(K)$ with $\mu(B) > 0$. Hence $x^* \notin S$.

It remains to prove that $\mu(L) = 1$; we achieve it by checking that for every $x^* \in K \setminus L$ there is a weak*-open set $U \subseteq K$ such that $x^* \in U$ and $\mu(U) = 0$. Again, the Hahn-Banach theorem ensures the existence of $x \in X$ such that

$$x^*(x) > \beta := \sup_{y^* \in S} y^*(x) \overset{(2.1)}{=} \sup \left\{ \frac{1}{\mu(B)} \int_B x \, \mathrm{d}\mu : \, B \in \mathrm{Borel}(K), \, \mu(B) > 0 \right\}.$$

Fix $\beta < \gamma < x^*(x)$. Then x^* belongs to the weak*-open set $U := \{y^* \in K : y^*(x) > \gamma\}$ and $\int_U x \, \mathrm{d}\mu \ge \gamma\mu(U)$. On the other hand, by the very definition of β we also have $\beta\mu(U) \ge \int_U x \, \mathrm{d}\mu$. Then $\beta\mu(U) \ge \gamma\mu(U)$ and so $\mu(U) = 0$. □

As a consequence of Proposition 2.9.9 and Theorem 2.9.10 we get the following:

Corollary 2.9.11. *Suppose that X is a Banach space not containing ℓ^1 and that the set $K \subseteq B_{X^*}$ is weak*-compact and convex. Then every probability regular Borel measure μ on K is concentrated on a separable subspace of K.*

Chapter 3

Sequential properties

Section 3.1

Introduction

In this chapter we study sequential topological properties. In particular, we focus in Banach spaces whose dual ball (with the weak*-topology) is sequentially compact or sequential. A topological space T is said to be sequentially compact if every sequence in T contains a convergent subsequence. Moreover, T is said to be Fréchet-Urysohn (FU for short) if the closure of any subset F of T coincides with the set of limits of convergent sequences in F. Every FU compact space is sequentially compact. A Banach space is said to have weak*-angelic dual if its dual ball with the weak*-topology is FU.

Sequential spaces generalize FU spaces. If T is a topological space and F is a subspace of T, the sequential closure of F, which we denote by $S(F)$, is the set of limits of convergent sequences in F. F is said to be sequentially closed if it coincides with its sequential closure. A topological space is said to be sequential if any sequentially closed subspace is closed. Thus, every FU space is sequential. Another natural generalization of the FU property is countable tightness. A topological space T is said to have countable tightness if the closure of any subspace F of T coincides with the union of closures of countable subsets of F. Obviously, every FU space has countable tightness. Moreover, the following holds:

Lemma 3.1.1. *Every sequential topological space T has countable tightness.*

Proof. Take F a subset of T and put

$$\hat{F} = \{x \in T : \text{There exists a countable set } C \subseteq F \text{ with } x \in \overline{C}\}.$$

We have to prove that $\hat{F} = \overline{F}$. It is immediate that $\hat{F}$ is sequentially closed, so the conclusion follows from the sequentiality of T. □

Example 3.1.2. *[Fra69] Let $T = \beta\mathbb{N}$ be the topological space whose topology is generated by the open sets of $\beta\mathbb{N}$ and the sets of the form $\{x\} \cup \mathbb{N}$ with $x \in \beta\mathbb{N} \setminus \mathbb{N}$. Since $\beta\mathbb{N}$ does not contain convergent sequences, it is immediate that T is not sequential. On the other hand, if $F \subseteq T$ and $x \in \overline{F}$, then $(\{x\} \cup \mathbb{N}) \cap F \neq \emptyset$, so $x \in F$ or $x \in \overline{F \cap \mathbb{N}}$. Thus, $\overline{F} = F \cup (\overline{F \cap \mathbb{N}})$ and T has countable tightness.*

Example 3.1.2 provides a topogical space with countable tightness which is not sequential. Nevertheless, whether every compact space with countable tightness is sequential is known as the Moore-Mrowka Problem and it is undecidable in ZFC [Bal88]. Moreover, Fedorchuk [Fed77] provided a consistent example of a compact space with countable tightness with no convergent subsequences, so which is non-sequentially compact. Therefore, for a compact space K, we have the following implications:

K is FU $\Rightarrow$ K is sequential $\Rightarrow$ K is sequentially compact

$\Downarrow$

K has countable tightness

In [Pli15] A. Plichko asked the following:

Question 3.1.3. *Does every Banach space with weak*-sequential dual ball have weak*-angelic dual?*

In Section 3.3 we provide sufficient conditions for a Banach space to have weak*-sequential dual ball. Finally, we provide a negative answer to Plichko's question in Section 3.4, showing several examples of Banach spaces with weak*-sequential dual ball and no weak*-angelic dual.

Another property studied by A. Plichko in [Pli15] is property $\mathcal{E}$ of Efremov. A Banach space X is said to have property $\mathcal{E}$ if the weak*-closure of any convex subset C of B_{X^*} coincides with the weak*-sequential closure of C. Moreover, we say that X has property $\mathcal{E}'$ if every weak*-sequentially closed convex subset of B_{X^*} is weak*-closed. Thus, if X has weak*-angelic dual then it has property $\mathcal{E}$ and if X has weak*-sequential dual ball then X has property $\mathcal{E}'$.

Another related Banach space properties are Mazur property and property (C) of Corson. A Banach space X has Mazur property if every weak*-sequentially continuous functional $x^{**} \in X^{**}$ is weak*-continuous and, therefore, $x^{**} \in X$. The relation between Mazur property and Banach spaces with weak*-sequential dual ball is given by the following Lemma:

Lemma 3.1.4. *A topological space T is sequential if and only if every sequentially continuous function $f\colon T \to T'$ is continuous for every topological space T'.*

Proof. If T is sequential and $C \subseteq T'$ is closed, then $f^{-1}(C)$ is sequentially closed by the sequential continuity of f. Thus, $f^{-1}(C)$ is closed for every closed set $C \subseteq T'$ and f is continuous.

If T is not sequential, then we can take T' the topological space with T as underlying set and such that a set U is open in T' if and only if its complementary is sequentially closed in T. It can be easily seen that the identity function $i\colon T \to T'$ is sequentially continuous but not continuous. □

Thus, it follows from the Banach-Dieudonné Theorem that every Banach space with weak*-sequential dual ball has the Mazur property. Moreover, the following holds:

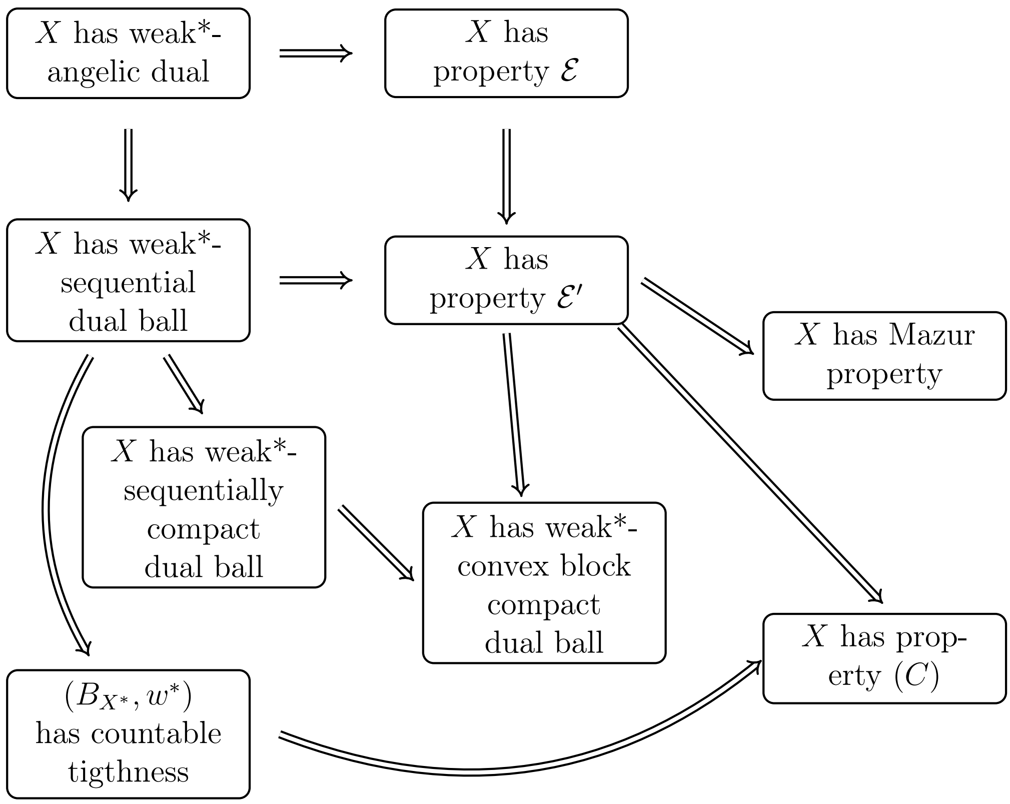
Lemma 3.1.5. *If X has property $\mathcal{E}'$ then it has Mazur property.*

Proof. Take any weak*-sequentially continuous functional $x^{**}\colon X^* \to \mathbb{R}$. It is enough to prove that x^{**} is weak*-continuous over B_{X^*}. Without loss of generality, $\|x^{**}\| \leq 1$. Take any closed interval $C = [a,b] \subseteq [-1,1]$. Then, $\{x^* \in B_{X^*} : x^{**}(x^*) \in C\}$ is a weak*-sequentially closed convex subset of B_{X^*}, so it is weak*-closed by property $\mathcal{E}'$. Thus, $\{x^* \in B_{X^*} : x^{**}(x^*) \in C\}$ is weak*-closed for every closed interval $C \subseteq [-1,1]$, which implies that x^{**} is weak*-continuous. □

A Banach space X has property (C) of Corson if every family of closed convex subsets of X with empty intersection contains a countable subfamily with empty intersection. R. Pol discovered that property (C) is a convex analogue to countable tightness:

Theorem 3.1.6 ([Pol80])**.** *A Banach space X has property (C) if and only if every point in the weak*-closure of any convex subspace $C \subseteq B_{X^*}$ is in the weak*-closure of a countable subset of C.*

Thus, we have the following implications among these Banach space properties (for the definition of a Banach space with weak*-convex block compact dual ball see Definition 3.2.9):



Notice that $\mathcal{C}([0,\omega_1])$ has weak*-sequentially compact dual ball but it does not have property (C). Moreover, $\ell_1(\omega_1)$ has the Mazur property [Edg79, Section 5] but it does not have property (C).

In [PY00] it is asked whether property (C) implies property $\mathcal{E}$. J.T. Moore in an unpublished paper and C. Brech in her PhD Thesis [Bre08] provided a negative answer under some additional consistent axioms, but the question is still open in ZFC. Notice that the convex version of Plichko's question is whether property $\mathcal{E}'$ implies property $\mathcal{E}$. A negative answer to this question would provide an example of a Banach space with property (C) not having property $\mathcal{E}$.

Section 3.2
Sequential compactness

In Chapter 2 we studied several classes of compact spaces. Notice that every separable subspace in a Σ-product $\Sigma(\Gamma)$ is metrizable, so every Eberlein and every Corson compact space is sequentially compact. Moreover, every RN and every QRN compact space is sequentially compact due to the following result [Fab97, Theorem 5.1.12]:

Lemma 3.2.1. *Every compact space fragmented by a metric is sequentially compact.*

Proof. Let K be a compact space fragmented by a metric d and take t_n a sequence in K. Set $F = \bigcap_{n\in\mathbb{N}} \overline{\{t_i : i \geq n\}}$, which is a nonempty compact space in K. Since d fragments K, we can construct by induction a sequence U_n of open sets in K with $U_n \cap (F \cap \bigcap_{i<n} U_i) \neq \emptyset$, $\mathrm{diam}_d(U_n \cap (F \cap \bigcap_{i<n} U_i)) < \frac{1}{n}$ and $\overline{U_{n+1}} \subseteq U_n$ for every $n \in \mathbb{N}$. Notice that

$$C := \bigcap_{n\in\mathbb{N}} \overline{U_n} \cap F = \bigcap_{n\in\mathbb{N}} U_n \cap F$$

has diameter zero, so $C = \{t\}$ for some $t \in F$. Take $t_{n_k} \in U_k$ for every $k \in \mathbb{N}$. We claim that the subsequence t_{n_k} converges to t. Suppose that s is a cluster point of t_{n_k}. Then, $s \in \bigcap_{n\in\mathbb{N}} \overline{U_n} \cap F = C$, so $s = t$ and we conclude that K is sequentially compact. □

Moreover, H. Rosenthal proved that every Rosenthal compact space is sequentially compact [Ros78, Lemma 3.8].

Nevertheless, R. Haydon provided an example of a WRN compact space which is not sequentially compact. Let us recall the construction of D. H. Fremlin used by Haydon [Hay81]. In particular, this construction provides an example of an SWRN Boolean algebra $\mathfrak{F}$ such that $\mathrm{ult}(\mathfrak{F})$ is not sequentially compact.

Example 3.2.2. *Let $\mathcal{G}$ be a family of subsets of $\mathbb{N}$ maximal with respect to the condition that for every $A, B \in \mathcal{G}$ there exists $\varepsilon_1, \varepsilon_2 \in \{0,1\}$ such that $A^{\varepsilon_1} \cap B^{\varepsilon_2}$ is finite. Let $\mathfrak{F}$ be the subalgebra of subsets of $\mathbb{N}$ generated by $\mathcal{G}$ (note that $\mathcal{G}$ contains all finite subsets of $\mathbb{N}$). It is clear that $\mathcal{G}$ does not contain an infinite independent sequence, so $\mathfrak{F}$ is an SWRN algebra.*

Notice that $\mathrm{ult}(\mathfrak{F})$ *contains a natural copy of $\mathbb{N}$ which consists of principal ultrafilters of $\mathfrak{F}$. By the maximality of $\mathcal{G}$ every infinite $A \subseteq \mathbb{N}$ is split into two infinite parts by some $G \in \mathcal{G}$. Consequently, the sequence of natural numbers in* $\mathrm{ult}(\mathfrak{F})$ *does not contain a converging subsequence, and therefore* $\mathrm{ult}(\mathfrak{F})$ *is not sequentially compact.*

Note that $\mathrm{ult}(\mathfrak{F}) \setminus \mathbb{N}$ *is a compact space which is homeomorphic to the Stone space of the quotient Boolean algebra $\mathfrak{A} = \mathfrak{F}/fin$. Then $\mathfrak{A}$ is generated by $\mathcal{G}^\bullet = \{G^\bullet : G \in \mathcal{G}\}$, where by $G^\bullet$ we denote the class of $G \in \mathcal{G}$ in $\mathfrak{A} = \mathfrak{F}/fin$. Since no pair from $\mathcal{G}^\bullet$ is independent, $\mathfrak{A}$ is in $\mathcal{I}(1)$. J. Bourgain proved that every sequence of nonprincipal ultrafilters in* $\mathrm{ult}(\mathfrak{F})$ *contains a convergent subsequence, cf. [HS80]. Thus,* $\mathrm{ult}(\mathfrak{A})$ *is sequentially compact.*

We shall now generalize Bourgain's idea mentioned above.

Theorem 3.2.3. *Let K be a compact space, $n \in \mathbb{N}$ and $\mathcal{F} \subseteq \mathcal{C}(K)$ a uniformly bounded family of continuous functions separating the points of K and such that $\mathcal{F}$ does not contain independent families of size $n+1$. Then K is sequentially compact.*

Proof. Fix a sequence $t_k \in K$ and define

$$\hat{t_k} \colon \mathcal{F} \to \mathbb{R}$$

by the formula $\hat{t_k}(f) = f(t_k)$ for every $f \in \mathcal{F}$. Since $\mathcal{F}$ is uniformly bounded, the sequence $\hat{t_k}$ is also uniformly bounded.

We claim that $\hat{t_k}$ has a pointwise convergent subsequence. Suppose this is not the case. By Theorem 2.2.8, passing to a subsequence if necessary, we may suppose that $\hat{t_k}$ is an independent sequence of functions. Thus, there exists $p < q$ such that the sequence $(A_k^0, A_k^1)_{k\in\mathbb{N}}$ is independent, where $A_k^0 = \{f \in \mathcal{F} : \hat{t_k}(f) = f(t_k) < p\}$ and $A_k^1 = \{f \in \mathcal{F} : \hat{t_k}(f) = f(t_k) > q\}$. Write $\{0,1\}^{n+1} = \{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{2^{n+1}}\}$ and take

$$f_k \in \bigcap_{i=1}^{2^{n+1}} A_i^{\varepsilon_i(k)} \text{ for every } k = 1, 2, \dots, n+1.$$

Notice that for any $\varepsilon_i \in \{0,1\}^{n+1}$ we have that $f_k(t_i) > q$ if $\varepsilon_i(k) = 1$ and $f_k(t_i) < q$ if $\varepsilon_i(k) = 0$ for every $k = 1, 2, ..., n+1$. This is a contradiction since then $\{f_1, f_2, \dots, f_{n+1}\} \subseteq \mathcal{F}$ is an independent family of size n+1.

Thus $\hat{t_k}$ has a pointwise convergent subsequence. Let $\hat{t_{n_k}}$ be a subsequence converging to a function $\hat{t} \colon \mathcal{F} \to \mathbb{R}$. Then,

$$\lim_k f(t_{n_k}) = \lim_k \hat{t_{n_k}}(f) = \hat{t}(f) \text{ for every } f \in \mathcal{F}.$$

In particular, for any cluster points t, s of the sequence t_{n_k}, we have

$$f(t) = \lim_k f(t_{n_k}) = \hat{t}(f) = f(s) \text{ for every } f \in \mathcal{F}.$$

Since $\mathcal{F}$ separates points, we conclude that $t = s$ and t_{n_k} is a convergent subsequence. □

Corollary 3.2.4 ([AMCP16])**.** *If $\mathfrak{A}$ is a Boolean algebra in $\mathcal{I}(n)$ for some $n \in \mathbb{N}$, then* $\mathrm{ult}(\mathfrak{A})$ *is sequentially compact.*

Proof. Let $\mathfrak{A}$ be a Boolean algebra in $\mathcal{I}(n)$ for some $n \in \mathbb{N}$ and take $\mathcal{G} \subseteq \mathfrak{A}$ such that $\mathcal{G}$ does not contain independent families of size n+1. Then, $\mathcal{F} = \{\chi_g \colon \mathrm{ult}(\mathfrak{A}) \to \mathbb{R} : g \in \mathcal{G}\}$ satisfies the conditions of Theorem 3.2.3. □

Corollary 3.2.5 ([AMCP16])**.** *If $\mathfrak{A}$ is a UWRN Boolean algebra, then* $\mathrm{ult}(\mathfrak{A})$ *is sequentially compact.*

Proof. Since $\mathfrak{A}$ is UWRN we have $\mathfrak{A} = \langle \mathcal{G} \rangle$, and the decomposition $\mathcal{G} = \bigcup_n \mathcal{G}_n$ as in Definition 2.8.1. If we let $\mathfrak{A}_n = \langle \mathcal{G}_n \rangle$ for every n we have an obvious embedding

$$\mathrm{ult}(\mathfrak{A}) \to \prod_{n=1}^{\infty} \mathrm{ult}(\mathfrak{A}_n).$$

We conclude the proof applying Corollary 3.2.5 and the fact that the class of sequentially compact spaces is stable under closed subspaces and countable products. □

Corollary 3.2.6 ([AMCP16]). *The Boolean algebra $\mathfrak{F}$ from Example 3.2.2 is SWRN but not UWRN.*

We remark that another example of an SWRN Boolean algebra which is not UWRN is given by the well-known example of an Eberlein compact space which is not uniformly Eberlein constructed by Y. Benyamini and T. Starbird [BS76].

As far as we are concerned, the following problem is open:

Problem 3.2.7. *(Haydon, [Hay81]) Does every infinite WRN compact space contain a nontrivial convergent sequence?*

As we noted above in Remark 2.9.5, typical spaces without nontrivial convergent sequences, such as $\beta\mathbb{N}$, are not WRN. Problem 3.2.7 is related with a well-known problem posed by Efimov:

Problem 3.2.8. *(Efimov) Does every infinite compact space contain a nontrivial convergent sequence or a homeomorphic copy of $\beta\mathbb{N}$?*

An infinite compact space not containing nontrivial convergent sequences neither a homeomorphic copy of $\beta\mathbb{N}$ is said to be an Efimov space. Although there are several consistent examples of Efimov spaces, it is unknown whether an Efimov space can be constructed in ZFC. Notice that Haydon's problem is equivalent to ask whether there exists a WRN Efimov space. We highlight the consistency of the existence of Efimov compact spaces in the class MS [BN07].

Although there are WRN algebras which are not SWRN, in order to give an answer to Problem 3.2.7 for zero-dimensional compact spaces, it is enough to consider SWRN algebras because we can use a similar argument as in the proof of Corollary 3.2.5.

We finish this section with a convex version of sequential compactness:

Definition 3.2.9. *If $(x_n)_{n\in\mathbb{N}}$ is a sequence in a Banach space, we say that $(y_k)_{k\in\mathbb{N}}$ is a convex block subsequence of $(x_n)_{n\in\mathbb{N}}$ if there is a sequence $(I_k)_{k\in\mathbb{N}}$ of finite* *subsets of* $\mathbb{N}$ *with* $\max(I_k) < \min(I_{k+1})$ *and a sequence* $a_n \in [0,1]$ *with* $\sum_{n\in I_k} a_n = 1$ *for every* $k \in \mathbb{N}$ *such that* $y_k = \sum_{n\in I_k} a_n x_n$.

A Banach space X *is said to have weak*-convex block compact dual ball if every bounded sequence in* X^* *has a weak*-convergent convex block subsequence.*

Every Banach space containing no isomorphic copies of ℓ_1 has weak*-convex block compact dual ball [Bou79]. Therefore, every WPG Banach space also has weak*-convex block compact dual ball. Moreover, if a Banach space X does not have weak*-convex block compact dual ball then X^* contains an isometric copy of $L_1(\{0,1\}^{\omega_1})$, cf. [HLO87, Corollary 3c] and [Sch89]. Therefore, $\mathcal{C}(K)$ has weak*-convex block compact dual ball for every compact space K in the class MS.

We finish the section proving that every Banach space with property $\mathcal{E}'$ has weak*-convex block compact dual ball. First we need the following lemma:

Lemma 3.2.10. *Let* X *be a Banach space with property* $\mathcal{E}'$ *and* $C \subseteq B_{X^*}$ *a convex subset. If the weak*-sequential closure of* C *is equal to* $\overline{C}^{\|\cdot\|}$, *then* $\overline{C}^{w^*} = \overline{C}^{\|\cdot\|}$.

Proof. Obviously $\overline{C}^{\|\cdot\|}$ is contained in $S(C)$ (the weak*-sequential closure of C). Suppose $S(C) = \overline{C}^{\|\cdot\|}$ but $\overline{C}^{w^*} \neq \overline{C}^{\|\cdot\|}$. Then $S(C)$ is not weak*-closed. Since X has property $\mathcal{E}'$, there exists a sequence $x_n^* \in S(C)$ which converges to a point $x^* \notin S(C)$. It follows from the equality $S(C) = \overline{C}^{\|\cdot\|}$ that there exists $y_n^* \in C$ such that $\|y_n^* - x_n^*\| < \frac{1}{2^n}$ for every $n \in \mathbb{N}$. Then, $y_n^* = x_n^* + (y_n^* - x_n^*)$ is weak*-convergent to x^*. Thus $x^* \in S(C)$ in contradiction with our assumption. □

Theorem 3.2.11. *Let* X *be a Banach space with property* $\mathcal{E}'$. *Then* X *has weak*-convex block compact dual ball.*

Proof. Let $(x_n^*)_{n=1}^\infty$ be a sequence in B_{X^*}. Set C the convex hull of $\{x_n^* : n \in \mathbb{N}\}$. We divide the proof in two cases.

Consider first the case $S(C) = \overline{C}^{\|\cdot\|}$. Then $S(C) = \overline{C}^{w^*} = \overline{C}^{\|\cdot\|}$ by Lemma 3.2.10. In particular, $\overline{C}^{w^*}$ is a weak*-closed set which is norm-separable. We prove that this implies that $\overline{C}^{w^*}$ with the weak*-topology is metrizable. Let $D \subseteq C$ be a countable norm-dense set in C and fix a set $D' = \{x_{d,d'} \in B_X : d, d' \in D\}$, where $\frac{\|d-d'\|}{2} \leq d(x_{d,d'}) - d'(x_{d,d'})$ for every $d, d' \in D$. If $x^*, y^* \in S(C)$, then we can take $d_n, d_n' \in D$ with $\|x^* - d_n\| < \frac{1}{2^n}$ and $\|y^* - d_n'\| < \frac{1}{2^n}$ for every $n \in \mathbb{N}$. Notice that

$$\|d_n - d_n'\| \leq 2(d_n(x_{d_n,d_n'}) - d_n'(x_{d_n,d_n'})) \leq 2(x^*(x_{d_n,d_n'}) - y^*(x_{d_n,d_n'})) + 2(\frac{1}{2^n} + \frac{1}{2^n})$$

$$= 2(x^*(x_{d_n,d_n'}) - y^*(x_{d_n,d_n'})) + \frac{1}{2^{n-2}}$$

for every $n \in \mathbb{N}$. Thus, if $x^*(x) = y^*(x)$ for every $x \in D'$, then $\|d_n - d'_n\|$ converges to zero and therefore $x^* = \lim d_n = \lim d'_n = y^*$. This implies that D' determines a countable family of weak*-continuous functions on $\overline{C}^{w^*}$ which separates points. Thus, $\overline{C}^{w^*}$ is metrizable.

Therefore, if $S(C) = \overline{C}^{\|\cdot\|} = \overline{C}^{w^*}$, then $\overline{C}^{w^*}$ is weak*-metrizable and, in particular, $(x_n^*)_{n=1}^\infty$ contains a weak*-convergent subsequence.

We consider now the second case. Suppose that there exists $x^* \in S(C) \setminus \overline{C}^{\|\cdot\|}$. Let $(y_n^*)_{n=1}^\infty$ be a sequence in C weak*-convergent to x^*. Write

$$y_n^* = \sum_{k=1}^\infty \lambda_k^n x_k^*,$$

with

$$\sum_{k=1}^\infty \lambda_k^n = 1,\ 0 \le \lambda_k^n \le 1 \text{ and } \lambda_k^n = 0 \text{ for all except finitely many } k \in \mathbb{N}$$

for every $n \in \mathbb{N}$. Without loss of generality, we may suppose that $(\lambda_k^n)_{n=1}^\infty$ converges to some point λ_k for every $k \in \mathbb{N}$. Moreover, a standard diagonal argument proves that we can assume that each sequence $(\lambda_k^n)_{n=1}^\infty$ is eventually monotone. Notice that $0 \le \sum_{k=1}^\infty \lambda_k \le 1$. We claim that $\sum_{k=1}^\infty \lambda_k < 1$. If $\sum_{k=1}^\infty \lambda_k = 1$ then $\sum_{k=1}^\infty \lambda_k x_k^* \in \overline{C}^{\|\cdot\|}$ and it can be easily seen that $x^* = \sum_{k=1}^\infty \lambda_k x_k^*$, in contradiction with $x^* \notin \overline{C}^{\|\cdot\|}$.

Thus, $0 \le \lambda := \sum_{k=1}^\infty \lambda_k < 1$. Set $N_n = \{k \in \mathbb{N} : \lambda_k^n > \lambda_k\}$ for every $n \in \mathbb{N}$. Notice that $N_n \neq \emptyset$ and that

$$\sum_{k \in N_n} (\lambda_k^n - \lambda_k) = 1 - \sum_{k \in N_n} \lambda_k - \sum_{k \notin N_n} \lambda_k^n \ge 1 - \sum_{k=1}^\infty \lambda_k = 1 - \lambda > 0. \tag{3.1}$$

Passing to a subsequence if necessary, we suppose that there exists

$$\lambda' = \lim_n \sum_{k \in N_n} (\lambda_k^n - \lambda_k) \ge 1 - \lambda > 0. \tag{3.2}$$

Set $y_n^+ = \sum_{k \in N_n} \lambda_k^n x_k^*$ and $y_n^- = \sum_{k \notin N_n} \lambda_k^n x_k^*$. We claim that y_n^- is Cauchy and therefore norm-convergent. Fix $\varepsilon > 0$ and take $N_0, N_1 \in \mathbb{N}$ such that $\sum_{k > N_0} \lambda_k < \frac{\varepsilon}{4}$, $(\lambda_k^n)_{n \ge N_1}$ is monotone for every $k \le N_0$ and

$$\sum_{k=1}^{N_0} |\lambda_k^n - \lambda_k^m| < \frac{\varepsilon}{2}$$

for every $n, m \geq N_1$. Then,

$$\|y_n^- - y_m^-\| \leq \sum_{k=1}^{N_0} |\lambda_k^n - \lambda_k^m| + \sum_{k>N_0,\ k\notin N_n} \lambda_k^n + \sum_{k>N_0,\ k\notin N_m} \lambda_k^m \leq \frac{\varepsilon}{2} + 2\sum_{k>N_0} \lambda_k < \varepsilon$$

for every $n, m \geq N_1$. Thus, y_n^- is norm convergent and, since $y_n^+ = y_n^* - y_n^-$, the sequence $(y_n^+)_{n=1}^\infty$ is weak*-convergent to a point y^*.

Set

$$z_n^* = \frac{1}{\sum_{k\in N_n}(\lambda_k^n - \lambda_k)} \sum_{k\in N_n} (\lambda_k^n - \lambda_k)x_k^* = \frac{1}{\sum_{k\in N_n}(\lambda_k^n - \lambda_k)}(y_n^+ - \sum_{k\in N_n} \lambda_k x_k^*)$$

for every $n \in \mathbb{N}$. It follows from (3.1) and (3.2) that z_n^* is well-defined, $z_n^* \in C$ and it converges to $\frac{1}{\lambda'}(y^* - \sum_{k\in M} \lambda_k x_k^*)$, where

$$M = \{k \in \mathbb{N} : (\lambda_k^n)_{n=1}^\infty \text{ is eventually decreasing}\}.$$

For each $n, k \in \mathbb{N}$, write $\beta_k^n = \frac{\lambda_k^n - \lambda_k}{\sum_{k'\in N_n}(\lambda_{k'}^n - \lambda_{k'})}$ if $k \in N_n$ and $\beta_k^n = 0$ if $k \notin N_n$. Then, $(\beta_k^n)_{n=1}^\infty$ converges to zero and $z_n^* = \sum_{k=1}^\infty \beta_k^n x_k^*$. By taking small perturbations of z_n^*, we are going to construct a convex block subsequence $(u_n^*)_{n=1}^\infty$ of $(x_n^*)_{n=1}^\infty$ with the same limit than $(z_n^*)_{n=1}^\infty$.

Fix $u_1^* = z_1^*$ and take a finite subset I_1 of $\mathbb{N}$ such that $u_1^* = \sum_{k\in I_1} \beta_k^1 x_k^*$. Fix $n_1 \in \mathbb{N}$ such that $\sum_{k\in I_1} \beta_k^n < \frac{1}{2}$ for every $n \geq n_1$ and take a finite set I_2 with $\max(I_1) < \min(I_2)$ such that $z_{n_1}^* = \sum_{k\in I_1\cup I_2} \beta_k^{n_1} x_k^*$. Take $u_2^* = \frac{1}{\sum_{k\in I_2}\beta_k^{n_1}} \sum_{k\in I_2} \beta_k^{n_1} x_k^*$. Notice that $u_2^* \in C$ and

$$\|z_{n_1}^* - u_2^*\| < \frac{1}{2} + \sum_{k\in I_2} \beta_k^{n_1}\left(\frac{1}{\sum_{k'\in I_2}\beta_{k'}^{n_1}} - 1\right) \leq \frac{1}{2} + \left(\frac{1}{\sum_{k\in I_2}\beta_k^{n_1}} - 1\right).$$

Repeating this argument we construct a sequence $(u_n^*)_{n=1}^\infty$ in C, an increasing sequence $(n_k)_{k=1}^\infty$ in $\mathbb{N}$ and a sequence of finite sets $(I_k)_{k=1}^\infty$ of $\mathbb{N}$ with $\max(I_k) < \min(I_{k+1})$ such that

$$\sum_{k\in I_1\cup I_2\cup\cdots\cup I_r} \beta_k^m < \frac{1}{2^r}$$

for every $m \geq n_r$,

$$z_{n_r}^* = \sum_{k\in I_1\cup I_2\cup\cdots\cup I_{r+1}} \beta_k^{n_r} x_k^*,$$

$$u_{r+1}^* = \frac{1}{\sum_{k\in I_{r+1}}\beta_k^{n_r}} \sum_{k\in I_{r+1}} \beta_k^{n_r} x_k^*$$

and

$$\|z^*_{n_r} - u^*_{r+1}\| < \frac{1}{2^r} + \sum_{k\in I_{r+1}} \beta_k^{n_r}\left(\frac{1}{\sum_{k'\in I_{r+1}}\beta_{k'}^{n_r}} - 1\right) \leq \frac{1}{2^r} + \left(\frac{1}{\sum_{k\in I_{r+1}}\beta_k^{n_r}} - 1\right)$$

$$\leq \frac{1}{2^r} + \left(\frac{2^r}{2^r-1} - 1\right) = \frac{1}{2^r} + \frac{1}{2^r-1},$$

where the last inequality follows from

$$\sum_{k\in I_{r+1}} \beta_k^{n_r} = 1 - \sum_{k\in I_1\cup I_2\cup\cdots\cup I_r} \beta_k^{n_r} \geq 1 - \frac{1}{2^r} = \frac{2^r-1}{2^r}.$$

Since $\|z^*_{n_r} - u^*_{r+1}\|$ converges to zero, we conclude that $(u^*_r)_{r=1}^\infty$ is a convex block subsequence of $(x^*_n)_{n=1}^\infty$ which is weak*-convergent with the same limit than $(z^*_n)_{n=1}^\infty$. □

Section 3.3

Sequential topological spaces

Definition 3.3.1. *Let T be a topological space and F a subspace of T. For any $\alpha \leq \omega_1$ we define $S_\alpha(F)$ the αth sequential closure of F by induction on α:*

- *$S_0(F) = F$;*
- *$S_{\alpha+1}(F)$ is the sequential closure of $S_\alpha(F)$ for every $\alpha < \omega_1$;*
- *$S_\alpha(F) = \bigcup_{\beta<\alpha} S_\beta(F)$ if α is a limit ordinal.*

Notice that $S_{\omega_1}(F)$ is sequentially closed for every subspace F. Thus, a topological space T is sequential if and only if $S_{\omega_1}(F) = \overline{F}$ for every subspace F of T. We say that T has *sequential order* α if $S_\alpha(F) = \overline{F}$ for every subspace F of T and for every $\beta < \alpha$ there exists F with $S_\beta(F) \neq \overline{F}$. Therefore, a topological space T is sequential with sequential order ≤ 1 if and only if it is FU.

Every Eberlein and, in general, every Corson compact space is FU.

Lemma 3.3.2. *Every Corson compact space $K \subseteq \Sigma(\Gamma)$ is FU.*

Proof. Let $F \subseteq K$ and $t \in \overline{F}$. Set $\mathrm{supp}(t) = \{\gamma \in \Gamma : t(\gamma) \neq 0\}$. Since $\mathrm{supp}(t)$ is countable, we can take F_n an increasing family of finite subsets of $\mathrm{supp}(t)$ such that $\mathrm{supp}(t) = \bigcup_{n\in\mathbb{N}} F_n$. Fix $t_1 \in F$ such that $|t_1(\gamma) - t(\gamma)| < \frac{1}{2}$ for every $\gamma \in F_1$. Again, we can write $\mathrm{supp}(t_1) = \bigcup_{n\in\mathbb{N}} F_n^1$, where F_n^1 is an increasing family of finite subsets. Repeating this argument, we can construct a sequence $(t_k)_{k=1}^\infty$ in F such that $\mathrm{supp}(t_k) = \bigcup_{n\in\mathbb{N}} F_n^k$ with F_n^k an increasing family of finite subsets and $|t_k(\gamma) - t(\gamma)| < \frac{1}{2^k}$ for every $\gamma \in F_k \cup F_k^1 \cup F_k^2 \cup \cdots \cup F_k^{k-1}$ for every $k \geq 2$. Then, it is clear that $(t_k)_{k=1}^\infty$ converges to t. □

Moreover, J. Bourgain, D. Fremlin and M. Talagrand proved that every Rosenthal compact space is FU [BFT78]. Nevertheless, RN compact spaces might not be sequential; $[0, \omega_1]$ is an RN compact space which does not have countable tightness and, therefore, it is not sequential.

We provide in this section some sufficient conditions for a Banach space to have weak*-sequential dual ball. We need first the following Lemma:

Lemma 3.3.3. *Let $f\colon K \to L$ be a continuous function, where K, L are topological spaces and K is sequentially compact. Then, $f(S_\alpha(F)) = S_\alpha(f(F))$ for every $F \subseteq K$ and every ordinal α.*

Proof. The inclusion $f(S_\alpha(F)) \subseteq S_\alpha(f(F))$ follows from the continuity of f.

We prove the other inclusion by induction on α. The case $\alpha = 0$ is immediate. Suppose $\alpha = 1$. Take $s \in S_1(f(F))$. Then, there exists a sequence t_n in F such that $f(t_n)$ converges to s. Since K is sequentially compact, without loss of generality we may suppose t_n is converging to some point t. Then, it follows from the continuity of f that $f(t) = s$. Thus, $s \in f(S_1(F))$.

Now suppose the result true for every $\beta < \alpha$ and $\alpha \geq 2$. If α is a limit ordinal then

$$f(S_\alpha(F)) = f\left(\bigcup_{\beta<\alpha} S_\beta(F)\right) = \bigcup_{\beta<\alpha} f(S_\beta(F)) = \bigcup_{\beta<\alpha} S_\beta(f(F)) = S_\alpha(f(F)).$$

If $\alpha = \beta + 1$ is a successor ordinal then

$$f(S_\alpha(F)) = f(S_1(S_\beta(F))) = S_1(f(S_\beta(F))) = S_1(S_\beta(f(F))) = S_\alpha(f(F)).$$

□

Theorem 3.3.4. *Let X be a Banach space with weak*-sequentially compact dual ball. Let $Y \subseteq X$ be a subspace with weak*-sequential dual ball with sequential order $\leq \gamma_1$ and such that X/Y has weak*-sequential dual ball with sequential order $\leq \gamma_2$. Then X has weak*-sequential dual ball with sequential order $\leq \gamma_1 + \gamma_2$.*

Proof. We have to prove that if $F \subseteq B_{X^*}$ and $x^* \in \overline{F}^{w^*}$ then $x^* \in S_{\gamma_1+\gamma_2}(F)$. Without loss of generality, we suppose $x^* = 0$. Let $R\colon X^* \to Y^*$ be the restriction operator. For each finite set $A \subseteq B_X$ and each $\varepsilon > 0$, define

$$F_{A,\varepsilon} = \{x^* \in F : |x^*(x)| \leq \varepsilon \text{ for all } x \in A\}.$$

Since R is weak*-weak* continuous and $0 \in \overline{F_{A,\varepsilon}}^{w^*}$, it follows that

$$0 \in \overline{R(F_{A,\varepsilon})}^{w^*} = S_{\gamma_1}(R(F_{A,\varepsilon})) = R(S_{\gamma_1}(F_{A,\varepsilon})),$$

where the last equality follows from Lemma 3.3.3.

Thus, for every finite set $A \subseteq B_X$ and every $\varepsilon > 0$ we can take $x^*_{A,\varepsilon} \in S_{\gamma_1}(F_{A,\varepsilon})$ such that $R(x^*_{A,\varepsilon}) = 0$. Notice that every weak*-open neighborhood of 0 contains a point of the form $x^*_{A,\varepsilon}$. Therefore, $0 \in \overline{G}^{w^*}$, where

$$G := \{x^*_{A,\varepsilon} : \ A \subseteq B_X \text{ finite}, \varepsilon > 0\} \subseteq Y^\perp \cap B_{X^*}.$$

Note that $(Y^\perp \cap B_{X^*},\ w^*)$ is homeomorphic to the dual ball of $(X/Y)^*$ with the weak*-topology. Hence it is sequential with sequential order $\leq \gamma_2$, so

$$0 \in S_{\gamma_2}(G) \subseteq S_{\gamma_2}(S_{\gamma_1}(F)) = S_{\gamma_1+\gamma_2}(F).$$

□

For any ordinal $\gamma \leq \omega_1$, we say that X has *property* $\mathcal{E}(\gamma)$ if $S_\gamma(C) = C$ for every convex subset C in (B_{X^*}, w^*). Thus, property $\mathcal{E}$ is property $\mathcal{E}(1)$ and property $\mathcal{E}'$ is property $\mathcal{E}(w_1)$.

Lemma 3.3.5. *Let X and Y be Banach spaces. Set $R\colon X^* \to Y^*$ a weak*-weak* linear continuous function. If X has weak*-convex block compact dual ball, then $R(S_\alpha(F)) = S_\alpha(R(F))$ for every convex set $F \subseteq B_{X^*}$ and every ordinal α.*

Proof. We only prove $S_1(R(F)) \subseteq R(S_1(F))$, since the rest of the proof is analogous to the proof of Lemma 3.3.3. Let $s \in S_1(R(F))$. Fix a sequence t_n in F such that $R(t_n)$ converges to s. Since X has weak*-convex block compact dual ball, there exists a weak*-convergent convex block subsequence t'_n of t_n. Let t be the weak*-limit of t'_n. Since F is convex, t'_n is a sequence in F and, therefore, $t \in S_1(F)$. Since every convex block subsequence of $R(t_n)$ converges to s, it follows from the linearity and continuity of R that $R(t) = \lim R(t'_n) = s$, so $s \in R(S_1(F))$. □

Using Lemma 3.3.5 instead of Lemma 3.3.3, the proof of Theorem 3.3.4 transfers verbatim to give a proof of the following Theorem:

Theorem 3.3.6. *Let X be a Banach space with weak*-convex block compact dual ball. Let $Y \subseteq X$ be a subspace with property $\mathcal{E}(\gamma_1)$ such that X/Y has property $\mathcal{E}(\gamma_2)$. Then X has property $\mathcal{E}(\gamma_1 + \gamma_2)$.*

Theorem 3.3.7. *Let X be a Banach space and $(X_n)_{n\in\mathbb{N}}$ an increasing sequence of subspaces with $X = \overline{\bigcup_{n\in\mathbb{N}} X_n}$. Suppose that each X_n has weak*-sequential dual ball with sequential order α_n. Then X has weak*-sequential dual ball with sequential order $\leq \alpha + 1$, where $\alpha := \sup\{\alpha_n : n \in \mathbb{N}\}$.*

Proof. Set $R_n\colon X^* \to X_n^*$ the restriction operator for every $n \in \mathbb{N}$. Since the countable product of sequentially compact spaces is sequentially compact and (B_{X^*}, w^*) is homeomorphic to a subspace of $\prod(B_{X_n^*}, w^*)$, it follows that X has weak*-sequentially compact dual ball.

In order to prove the theorem, it is enough to prove that if $F \subseteq B_{X^*}$ and $0 \in \overline{F}^{w^*}$ then $0 \in S_{\alpha+1}(F)$. Since B_{X^*} is weak*-sequentially compact, we have that $0 \in \overline{R_n(F)}^{w^*} = S_\alpha(R_n(F)) = R_n(S_\alpha(F))$ for every $n \in \mathbb{N}$, where the last equality follows from Lemma 3.3.3. Thus, we can take a sequence $x_n^* \in S_\alpha(F)$ such that $R_n(x_n^*) = 0$. Now there exists some subsequence of x_n^* converging to a point $x^* \in S_{\alpha+1}(F)$. Since $R_n(x^*) = 0$ for every $n \in \mathbb{N}$, we conclude that $x^* = 0$. □

Corollary 3.3.8. *Let X be a Banach space and $(X_\alpha)_{\alpha<\gamma}$ an increasing sequence of subspaces with $X = \overline{\bigcup_{\alpha<\gamma} X_\alpha}$, where γ is a countable limit ordinal. Suppose that each X_α has weak*-sequential dual ball with sequential order $\leq \theta_\alpha$. Then X has weak*-sequential dual ball with sequential order $\leq \theta+1$ where $\theta := \sup\{\theta_\alpha : \alpha < \gamma\}$.*

Proof. Take γ_n an increasing sequence of ordinals with $\sup_n \gamma_n = \gamma$ an apply Theorem 3.3.7 to $(X_{\gamma_n})_{n\in\mathbb{N}}$. □

The next theorem follows from combining Theorem 3.3.4 and Corollary 3.3.8:

Theorem 3.3.9. *Let γ be a countable ordinal, X_γ a Banach space and $(X_\alpha)_{\alpha\leq\gamma}$ an increasing sequence of subspaces of X_γ such that:*

(1) X_0 has weak-sequential dual ball with sequential order $\leq \theta$;*

(2) each quotient $X_{\alpha+1}/X_\alpha$ has weak-angelic dual;*

(3) $X_\alpha = \overline{\bigcup_{\beta<\alpha} X_\beta}$ for every limit ordinal $\alpha \leq \gamma$;

(4) X_γ has weak-sequentially compact dual ball.*

Then each X_α has weak-sequential dual ball with sequential order $\leq \theta+\alpha$ if $\alpha < \omega$ and sequential order $\leq \theta + \alpha + 1$ if $\alpha \geq \omega$.*

Proof. It follows from (4) that every X_α has weak*-sequentially compact dual ball. Thus, the result for $\alpha < \omega$ follows by applying inductively Theorem 3.3.4.

Suppose $\alpha \geq \omega$ and X_β has weak*-sequential dual ball with sequential order $\leq \theta + \beta + 1$ for every $\beta < \alpha$. If α is a limit ordinal then it follows from (3) and from Corollary 3.3.8 that X_α has weak*-sequential dual ball with sequential order $\leq \sup_{\beta<\alpha}\{\theta+\beta+1\}+1 = \theta+\alpha+1$. If α is a successor ordinal then the result is a consequence of Theorem 3.3.4. □

We prove now a convex version of Corollary 3.3.8:

Lemma 3.3.10. *Let X be a Banach space and $(X_\alpha)_{\alpha<\gamma}$ an increasing sequence of subspaces with $X = \overline{\bigcup_{\alpha<\gamma} X_\alpha}$, where γ is a countable limit ordinal. Suppose that each $X_\alpha \in \mathcal{E}(\theta_\alpha)$ for some ordinal θ_α. Then $X \in \mathcal{E}(\theta+1)$ where $\theta := \sup\{\theta_\alpha : \alpha < \gamma\}$.*

Proof. It is enough to prove the result for $(X_n)_{n\in\mathbb{N}}$ and apply the argument of Corollary 3.3.8. By Theorem 3.2.11, each X_n has weak*-convex block compact dual ball. Set $R_n : X^* \to X_n^*$ the restriction operator for every $n \in \mathbb{N}$. We prove first that X^* also has weak*-convex block compact dual ball. Let $(x_n^*)_{n\in\mathbb{N}}$ be a sequence in B_{X^*}. Then, we can construct inductively sequences $(x_{m,n}^*)_{n\in\mathbb{N}}$ for every $m \in \mathbb{N}$ such that:

(1) $x_{1,n}^* = x_n^*$ for every $n \in \mathbb{N}$;

(2) each sequence $(x_{m+1,n}^*)_{n\in\mathbb{N}}$ is a convex block subsequence of $(x_{m,n}^*)_{n\in\mathbb{N}}$;

(3) each sequence $(R_{m+1}(x_{m+1,n}^*))_{n\in\mathbb{N}}$ is weak*-convergent.

Notice that a convex block subsequence of a convex block subsequence of $(x_n^*)_{n\in\mathbb{N}}$ is also a convex block subsequence of $(x_n^*)_{n\in\mathbb{N}}$. Thus, each $(x_{m,n}^*)_{n\in\mathbb{N}}$ is a convex block subsequence of $(x_n^*)_{n\in\mathbb{N}}$ by (1) and (2). Moreover, the sequence $(x_{n,n}^*)_{n\in\mathbb{N}}$ is also a convex block subsequence of $(x_n^*)_{n\in\mathbb{N}}$. We claim that $(x_{n,n}^*)_{n\in\mathbb{N}}$ is weak*-convergent. Let $x^* \in X^*$ be the unique functional which satisfy $x^*(x) = \lim_n R_{m+1}(x_{m+1,n}^*(x))$ for every $x \in X_{m+1}$ and every $m \in \mathbb{N}$. Then, $\lim_n R_{m+1}(x_{n,n}^*(x)) = x^*(x)$ for every $x \in X_{m+1}$ and every $m \in \mathbb{N}$, i.e. $\lim_n x_{n,n}^*(x) = x^*(x)$ for every $x \in \bigcup_{n\in\mathbb{N}} X_n$. Since $X = \overline{\bigcup_{n\in\mathbb{N}} X_n}$, we conclude that $(x_{n,n}^*)_{n\in\mathbb{N}}$ is weak*-convergent to x. Therefore, X has weak*-convex block compact dual ball.

Take $F \subseteq B_{X^*}$ a convex set with $0 \in \overline{F}^{w^*}$. In order to finish the proof it is enough to check that $0 \in S_{\theta+1}(F)$. Since X has weak*-convex block compact dual ball, it follows from Lemma 3.3.5 that

$$0 \in \overline{R_n(F)}^{w^*} = S_\theta(R_n(F)) = R_n(S_\theta(F)).$$

Thus, we can take a sequence $x_n^* \in S_\theta(F)$ with $R_n(x_n^*) = 0$ for every $n \in \mathbb{N}$. Let y_n^* be a weak*-convergent convex block subsequence of x_n^*. Then, y_n^* is weak*-convergent to zero since $\lim_n y_n^*(x) = \lim_n x_n^*(x) = 0$ for every $x \in \bigcup_{n\in\mathbb{N}} X_n$. Thus, $0 \in S_{\theta+1}(F)$ and the proof is complete. □

We also have the following convex equivalent version of Theorem 3.3.9.

Theorem 3.3.11. *Let γ be a countable ordinal, X_γ a Banach space and $(X_\alpha)_{\alpha\leq\gamma}$ an increasing sequence of subspaces of X_γ such that:*

1. *X_0 has property $\mathcal{E}(\theta)$;*
2. *each quotient $X_{\alpha+1}/X_\alpha$ has $\mathcal{E}$;*
3. *$X_\alpha = \overline{\bigcup_{\beta<\alpha} X_\beta}$ if α is a limit ordinal;*
4. *X_γ has weak*-convex block compact dual ball.*

Then each X_α has property $\mathcal{E}(\theta+\alpha)$ if $\alpha<\omega$ and property $\mathcal{E}(\theta+\alpha+1)$ if $\alpha\geq\omega$.

Proof. The proof is verbatim the same as in Theorem 3.3.9, using Lemma 3.3.10 instead of Corollary 3.3.8 and Theorem 3.3.6 instead of Theorem 3.3.4. □

Section 3.4
Banach spaces with weak*-sequential dual ball nonFU

In this section we apply the results of Section 3.3 in order to provide a negative answer to Plichko's question 3.1.3. We see first an example of a sequential compact space which is not FU:

Example 3.4.1. *Let $\{N_r : r\in\Gamma\}$ be an uncountable maximal almost disjoint family of infinite subsets of $\mathbb{N}$, i.e. an uncountable maximal family with respect to the condition that each N_r is an infinite subset of $\mathbb{N}$ and $N_r\cap N_{r'}$ is finite for every $r\neq r'$. Consider $\mathfrak{A}$ the Boolean algebra of subsets of $\mathbb{N}$ generated by $\{N_r : r\in\Gamma\}\bigcup\{F : F\subseteq\mathbb{N}$ is finite$\}$. The compact space $K=\mathrm{ult}(\mathfrak{A})$ contains a natural copy of $\mathbb{N}$ which consists of principal ultrafilters of $\mathfrak{A}$. It can be easily seen that $K=\mathbb{N}\cup\{u_r : r\in\Gamma\}\cup\{\infty\}$, where each ultrafilter u_r corresponds to the unique ultrafilter containing N_r and not containing any $N_{r'}$ with $r'\neq r$ neither any finite set of $\mathbb{N}$ and where ∞ denotes the unique ultrafilter not containing finite sets neither sets of the form N_r. If $(n_k)_{k\in\mathbb{N}}$ is a sequence in $\mathbb{N}\subseteq K$ consisting of different elements, then, by the maximality of the family $\{N_r : r\in\Gamma\}$, there exists N_r such that the set $N_r\cap\{n_k : k\in\mathbb{N}\} =: \{m_k : k\in\mathbb{N}\}$ is infinite. Thus, the sequence $(m_k)_{k\in\mathbb{N}}$ is a subsequence of $(n_k)_{k\in\mathbb{N}}$ which converges to u_r. On the other hand, if $u_{r_n}\neq u_{r_m}$ whenever $n\neq m$, then the sequence $(u_{r_n})_{n\in\mathbb{N}}$ converges to ∞. Thus, it is clear that for every set $F\subseteq K$ we have $S_2(F)=\overline{F}$ and that $S_1(\mathbb{N})=\mathbb{N}\cup\{u_r : r\in\Gamma\}$, $S_2(\mathbb{N})=S_1(\mathbb{N})\cup\{\infty\}$. Therefore, we conclude that K is a sequential compact space with sequential order 2.*

We recall the definition of Johnson-Lindenstrauss space JL_2. Let $\{N_r : r \in \Gamma\}$ be an uncountable maximal almost disjoint family of infinite subsets of $\mathbb{N}$. For each N_r, we denote by χ_{N_r} the characteristic function of N_r in ℓ_∞. The Johnson-Lindenstrauss space JL_2 is defined as the completion of span $(c_0 \cup \{\chi_{N_r} : r \in \Gamma\}) \subseteq \ell_\infty$ with respect to the norm

$$\Big\| x + \sum_{1\le i\le k} a_i \chi_{N_{r_i}} \Big\| = \max \Big\{ \Big\| x + \sum_{1\le i\le k} a_i \chi_{N_{r_i}} \Big\|_\infty, \Big(\sum_{1\le i\le k} |a_i|^2 \Big)^{\frac{1}{2}} \Big\},$$

where $x \in c_0$ and $\|\cdot\|_\infty$ is the supremum norm in ℓ_∞.

Notice that each vector $y \in$ span $(c_0 \cup \{\chi_{N_r} : r \in \Gamma\})$ has a unique expression of the form $y = x + \sum_{1\le i\le k} a_i \chi_{N_{r_i}}$, where $x \in c_0$ and each $a_i \neq 0$. Moreover, each a_i is determined by $a_i = \lim_{n\in N_{r_i}} y(n)$ for every $i = 1, 2, \dots k$. Thus, the norm $\|\cdot\|$ is well-defined.

If we just consider the supremum norm in the definition of JL_2 then we obtain the space JL_0. Notice that JL_0 is isomorphic to $\mathcal{C}(K)$, where K is the compact space in Example 3.4.1; the hyperplane $H = \{f \in \mathcal{C}(K) : f(\infty) = 0\}$ is isomorphic to $\mathcal{C}(K)$ (see Remark 3.4.4) and the operator $T\colon H \to JL_0$ given by the formula $T(f) = f|_{\mathbb{N}}$ for every $f \in H$ defines an isometry between H and JL_0.

Moreover, we can also define a bounded linear operator $S\colon JL_2 \to JL_0$ such that

$$S(x + \sum_{1\le i\le k} a_i \chi_{N_{r_i}}) = x + \sum_{1\le i\le k} a_i \chi_{N_{r_i}}$$

for every $x \in c_0$ and every scalars a_i, $1 \le i \le k$. Since S has dense range and $\|S\| \le 1$, we conclude that S^* determines a weak*-embedding from $B_{JL_0^*}$ into $B_{JL_2^*}$. In particular, K is homeomorphic to a subset of $B_{JL_2^*}$ with the weak*-topology, so JL_2 does not have weak*-angelic dual.

We refer the reader to [JL74] for more details about these spaces.

Theorem 3.4.2. *The Johnson-Lindenstrauss space JL_2 has weak*-sequential dual ball with sequential order* 2.

Proof. Using the definition of the norm in JL_2, it can be easily seen that JL_2 has a natural copy of c_0 such that JL_2/c_0 is isometric to $\ell_2(\Gamma)$ [JL74].

Since c_0 and $\ell_2(\Gamma)$ are Asplund spaces and being Asplund is a three-space property [CG97, Theorem 4.11.a], we have that JL_2 is Asplund and, in particular, $B_{JL_2^*}$ with the weak*-topology is Radon-Nikodým, so it is sequentially compact due to Lemma 3.2.1.

Moreover, c_0 and $\ell_2(\Gamma)$ have weak*-angelic dual. It follows from Theorem 3.3.7 that JL_2 has weak*-sequential dual ball with sequential order ≤ 2. Since JL_2 does not have weak*-angelic dual, we conclude that JL_2 has weak*-sequential dual ball with sequential order 2. $\square$

Thus, Johnson-Lindenstrauss space JL_2 provides a negative answer to Plichko's Question 3.1.3. Since $B_{JL_0^*}$ is weak*-homeomorphic to a subspace of $B_{JL_2^*}$ and $B_{JL_0^*}$ is not FU because it contains a copy of K, where K is the compact space in Example 3.4.1, the space JL_0 is another example of a Banach space with weak*-sequential dual ball which does not have weak*-angelic dual (its dual ball has sequential order 2).

We are going to prove that, in general, the dual ball of a $\mathcal{C}(K)$ space with K a scattered compact space of countable height has weak*-sequential dual ball.

Definition 3.4.3. *Let K be a scattered compact space. We define the γth Cantor-Bendixson derivative $K^{(\gamma)}$ by induction in γ:*

1. $K^{(0)} = K$;
2. $K^{(\alpha+1)}$ *is the set of nonisolated points of* $K^{(\alpha)}$;
3. $K^{(\alpha)} = \bigcap_{\beta<\alpha} K^{(\beta)}$ *if α is a limit ordinal.*

We denote by $ht(K)$ the height of K, i.e. the minimal ordinal γ such that the γth Cantor-Bendixson derivative $K^{(\gamma)}$ is finite.

Remark 3.4.4. *Recall that, as a consequence of the Hahn-Banach Theorem, every two subspaces of a Banach space with the same finite codimension are isomorphic. Since c_0 is isomorphic to every finite-codimensional subspace of c_0, if X is a Banach space containing a complemented copy of c_0 then*

$$X \approx Z \oplus c_0 \approx Z \oplus (c_0 \oplus \mathbb{R}^n) \approx (Z \oplus c_0) \oplus \mathbb{R}^n \approx X \oplus \mathbb{R}^n$$

for every $n \in \mathbb{N}$, where $\oplus$ denotes the direct sum and $\approx$ denotes that the corresponding spaces are isomorphic.

Thus, if X is a Banach space containing a complemented copy of c_0, then X is isomorphic to every subspace of X of finite codimension.

Since every Banach space with weak*-sequential dual ball has the Mazur property (see Lemma 3.1.5), the following theorem improves [Kap86, Theorem 4.1]:

Theorem 3.4.5. *Let K be an infinite scattered compact space. If $ht(K) < \omega$ then $\mathcal{C}(K)$ has weak*-sequential dual ball with sequential order $\leq ht(K)$. If $\omega \leq ht(K) < \omega_1$ then $\mathcal{C}(K)$ has weak*-sequential dual ball with sequential order $\leq ht(K) + 1$.*

Proof. Since K is scattered, it is Radon-Nikodým. Thus $\mathcal{C}(K)$ is Asplund and $B_{\mathcal{C}(K)^*}$ is Radon-Nikodým. In particular, $\mathcal{C}(K)$ has weak*-sequentially compact dual ball due to Lemma 3.2.1.

Denote by $\{K^{(\alpha)} : \alpha \leq \gamma\}$ the Cantor-Bendixson derivatives of K, where $\gamma = ht(K)$ is the height of K. For every $\alpha \leq \gamma$, set

$$X_\alpha = \{f \in \mathcal{C}(K) : f(t) = 0 \text{ for every } t \in K^{(\alpha)}\}.$$

Since K is infinite and sequentially compact, $\mathcal{C}(K)$ contains a complemented copy of c_0 and every finite-codimensional subspace of $\mathcal{C}(K)$ is isomorphic to $\mathcal{C}(K)$ due to Remark 3.4.4. Therefore, since X_γ is a finite-codimensional subspace of $\mathcal{C}(K)$, it is isomorphic to $\mathcal{C}(K)$.

For every $\alpha \leq \gamma$ and every $t \in K^{(\alpha)} \setminus K^{(\alpha+1)}$ fix a clopen set C_t in K with $\{t\} = C_t \cap K^{(\alpha)}$. Notice that $X_\alpha = \overline{\text{span}}\{\chi_{C_t} : t \in K \setminus K^{(\alpha)}\}$ for every $\alpha \leq \gamma$. For every $0 \leq \alpha < \gamma$ the family

$$\{\chi_{C_t} + X_\alpha : t \in K^{(\alpha)} \setminus K^{(\alpha+1)}\} \subseteq X_{\alpha+1}/X_\alpha$$

is equivalent to the canonical basis of $c_0(K^{(\alpha)} \setminus K^{(\alpha+1)})$. Thus for every $0 \leq \alpha < \gamma$ we have that $X_{\alpha+1}/X_\alpha$ is isomorphic to $c_0(K^{(\alpha)} \setminus K^{(\alpha+1)})$. Moreover, if $\alpha \leq \gamma$ is a limit ordinal then $\bigcap_{\beta<\alpha} K^{(\beta)} = K^{(\alpha)}$ and therefore

$$\overline{\bigcup_{\beta<\alpha} X_\beta} = \overline{\text{span}}\{\chi_{C_t} : t \in K \setminus K^{(\beta)} \text{ for some } \beta < \alpha\} = \overline{\text{span}}\{\chi_{C_t} : t \in K \setminus K^{(\alpha)}\} = X_\alpha.$$

Now the conclusion follows from Theorem 3.3.9. □

Notice that the split interval SI is FU since it is Rosenthal. Nevertheless, R. Pol proved that $\mathcal{C}(SI)$ does not have property (C) [Pol80]. Thus, it is not true for a general compact space K that if K is sequential then $B_{\mathcal{C}(K)^*}$ is weak*-sequential. We refer the reader to [FPRN00] for a discussion on this topic.

Since JL_0 is isomorphic to $\mathcal{C}(K)$ with K the scattered compact space of Example 3.4.1, which has $ht(K) = 2$ and sequential order 2, it follows from Theorem 3.4.5 that JL_0 has weak*-sequential dual ball with sequential order 2.

The known examples in ZFC of sequential compact spaces are all of sequential order ≤ 2. Nevertheless, A.I. Baškirov constructed sequential compact spaces of any sequential order $\leq \omega_1$ under the Continuum Hypothesis [Baš74]. A different construction was also given by V. Kannan in [Kan79]. Baškirov's construction is studied in detail in [Bal10] and, as C. Baldovino highlights in [Bal10, Remark 6.8], these constructions are scattered compact spaces such that the sequential order and the scattering height coincide whenever the sequential order is a successor ordinal.

Moreover, A. Dow constructed under the assumption $\mathfrak{b} = \mathfrak{c}$ a scattered compact space K of sequential order 4 such that the sequential order and the scattering height coincide [Dow05].

Corollary 3.4.6. *Under the Continuum Hypothesis, there exist Banach spaces with weak*-sequential dual ball with arbitrarily large countable sequential order. Moreover, for any $\alpha < \omega$ there exists a Banach space with weak*-sequential dual ball with sequential order α.*

On the other hand, under $\mathfrak{b} = \mathfrak{c}$, there exist Banach spaces with weak-sequential dual ball of any sequential order ≤ 4.*

Appendix

Appendix A

Todorcevic Tree

S. Argyros pointed out that the family of all segments of the Todorcevic tree satisfy the conditions of Lemma 2.4.7. We include a complete proof of this fact in this Appendix.

Definition A.1. *Let A be a subset of ω_1 with A and A^c stationary sets, i.e. sets whose intersection with every closed and unbounded subset of ω_1 is nonempty. Consider T the set of closed subsets of ω_1 which are contained in A and define an order $\leq$ in T by the formula $s \leq t$ if $t \cap \alpha = s$ for some $\alpha < \omega_1$, i.e. if s is an initial segment of t. The set T equipped with the order $\leq$ is called the Todorcevic tree.*

The existence of a set A such that A and A^c are stationary follows from [Kun77, Theorem 3.2]. Let $\mathcal{S}$ be the family of all segments of the Todorcevic tree, where by a segment we mean a linearly ordered subset S of T such that if $s_1, s_2 \in S$ and $s_1 \leq s_2$ then the set $[s_1, s_2] = \{s \in T : s_1 \leq s \leq s_2\}$ is contained in T.

Proposition A.2. *$\mathcal{S}$ satisfies conditions (1), (2) and (3) of Lemma 2.4.7.*

Proof. Take $K = \{1_S : S \in \mathcal{S}\} \subseteq \{0,1\}^\Gamma$, where $\Gamma = T$. If $S \notin \mathcal{S}$, then S is not linearly ordered and therefore there exist $s_1, s_2 \in \mathcal{S}$ with $s_1 \not\leq s_2$ and $s_2 \not\leq s_1$ or there exist $s_1, s_2 \in S$ and $s_3 \notin S$ with $s_1 \leq s_3 \leq s_2$. In both cases it is clear that we can find an open neighborhood of 1_S in $\{0,1\}^\Gamma$ whose intersection with K is empty, so K is compact.

Suppose now that there exists $S \in \mathcal{S}$ uncountable. Then, the set $B = \bigcup_{s \in S} s$ is uncountable and since S is a segment in T, B is closed in ω_1. This is a contradiction since $A^c \cap B = \emptyset$ and A^c is stationary. Thus property (2) is also satisfied.

In order to finish the proof, we have to show that if $S \in \mathcal{S}$ then the set $L_S = \{S \cap S' : S' \in \mathcal{S}\}$ is countable. Since S is countable, there are at most countably many segments $S' \subseteq S$. Since L_S consists of segments contained in S, we conclude that L_S is countable and the proof is complete. □

In fact, if we fix a tree T' with no uncountable branch, then the family of all segments of the tree T' also satisfies conditions (1), (2) and (3) of Lemma 2.4.7.

It follows from [Tod84, Lemma 9.12] that $\mathcal{S}$ satisfies condition (4) of Lemma 2.4.7. Todorcevic's proof is based on metamathematical arguments. R. Haydon provided another proof using games [Hay95, Theorem 3.2] and A. Avilés gave another combinatorial proof in [Avi07, Theorem 12]. We include here a proof based on the one given in [Avi07]. Recall that a set $D \subseteq T$ is dense if for every $t \in T$ there exists $t' \in D$ with $t \leq t'$.

Lemma A.3. *Let D be a dense subset of T and $R \subseteq D$ a countable set with $\gamma = \sup\{\max(t) : t \in R\}$. Then there exists a countable subset $R' \subseteq D$ with $R \subseteq R'$, $\gamma < \gamma' = \sup\{\max(t) : t \in R'\}$, $\gamma' \neq \max(t)$ for every $t \in R'$ and such that for every $t \in R'$ and every $\eta \in A$ with $\max(t) < \eta < \gamma'$ there exists $t' \in R'$ with $t \cup \{\eta\} \leq t'$.*

Proof. Take $R_0 = R$, $\gamma_0 = \gamma$ and $\eta_0 \in A$ with $\gamma_0 < \eta_0$. Since D is dense, for every $t \in R_0$ and every $\eta \in A$ with $\max(t) < \eta \leq \eta_0$ we can fix an element $s_{t,\eta} \in D$ with $t \cup \{\eta\} \leq s_{t,\eta}$. Set $R_1 = R_0 \cup \{s_{t,\eta} : t \in R_0,\ \eta \in A,\ \max(t) < \eta \leq \eta_0\}$. Since R_0 and η_0 are countable, R_1 is also countable. Repeating this argument we can construct a increasing sequence R_n of countable subsets of D and an increasing sequence η_n in A such that $\eta_n > \gamma_n := \sup\{\max(t) : t \in R_n\}$ and for every $t \in R_n$ and $\eta \in A$ with $\max(t) < \eta \leq \eta_n$ there exists $t' \in R_{n+1}$ with $t \cup \{\eta\} \leq t'$. Thus, $R' = \bigcup_{n\in\mathbb{N}} R_n$ satisfies the condition of the Lemma. □

Proposition A.4. *$\mathcal{S}$ satisfies condition (4) of Lemma 2.4.7.*

Proof. We have to prove that if $T = \bigcup_{n\in\mathbb{N}} \Gamma_n$ then there exist $S \in \mathcal{S}$ and $n_0 \in \mathbb{N}$ such that $\Gamma_{n_0} \cap S$ is infinite. Suppose by contradiction that there exists a countable decomposition $T = \bigcup_{n\in\mathbb{N}} \Gamma_n$ with $\Gamma_n \cap S$ finite for every $n \in \mathbb{N}$ and every $S \in \mathcal{S}$. This implies that each Γ_n does not contain infinite increasing sequences of T. Without loss of generality, we may suppose that Γ_n is an increasing sequence of subsets of T. Notice that the set $D = \{t \in T : t \text{ is maximal for some } \Gamma_n\}$ is dense in T.

Iterating Lemma A.3, we can construct an increasing family $\{R_\xi : \xi < \omega_1\}$ of countable subsets of D such that if ξ is a limit ordinal then $R_\xi = \bigcup_{\xi'<\xi} R_{\xi'}$ and such that if $\gamma_\xi = \sup\{\max(t) : t \in R_\xi\}$ then $(\gamma_\xi)_{\xi<\omega_1}$ is strictly increasing, $\max(t) < \gamma_\xi$ for every $t \in R_\xi$ and for every $\eta \in A$ and $t \in R_\xi$ with $\max(t) < \eta < \gamma_\xi$ there exists $t' \in R_\xi$ with $t \cup \{\eta\} \leq t'$. Notice that the set $\{\gamma_\xi : \xi < \omega_1\}$ is a closed uncountable subset of ω_1. Since A is stationary, there exists $\gamma_\xi \in A$.

By definition of γ_ξ, we can take an increasing sequence η_n in A converging to γ_ξ. We construct an increasing sequence t_n in R_ξ with $\max(t_n)$ converging to γ_ξ. Fix $t_1 \in R_\xi$. If t_n is already defined, take i with $\max(t_n) < \eta_i$ and set $t_{n+1} \in R_\xi$

such that $t_n \cup \{\eta_i\} \leq t_{n+1}$. It is clear that $\max(t_n)$ converges to γ_ξ. Since $\gamma_\xi \in A$, we have that $t = \bigcup_{n\in\mathbb{N}} t_n \cup \{\gamma_\xi\}$ is a closed subset of A, so $t \in T$. Since D is dense, there is $t' \in D$ with $t \leq t'$. Then, there exists $n_0 \in \mathbb{N}$ with t' maximal in Γ_{n_0}. Since $t_n \leq t'$ and $t_n \in D$ for every $n \in \mathbb{N}$, we conclude that t_n is an increasing sequence in Γ_{n_0}, contradicting our initial assumption. □

The reader can find in [Arg92] and [Fab97] the construction of Recničenko's family and the proof of the fact that this family of sets also satisfy the conditions of Lemma 2.4.7.

Appendix B

Sauer-Shelah Lemma

The following proof of the Sauer-Shelah Lemma is based on the proof contained in Gil Kalai's blog [Kal].

Lemma B.1 (Sauer-Shelah)**.** *Let N, n be natural numbers with $1 \leq n \leq N$ and let $T = \{1, 2, \ldots, N\}$. Then for every family $C \subseteq 2^T$ with*

$$|C| > \binom{N}{0} + \binom{N}{1} + \cdots + \binom{N}{n-1},$$

there exists a set $S \subseteq T$ with $|S| = n$ such that $\{f|_S : f \in C\} = 2^S$.

Proof. We first prove the following stronger result:

Claim. For every family $C \subseteq 2^T$ there exists a family of sets $\mathcal{F} \subseteq \mathcal{P}(T)$ such that $|\mathcal{F}| = |C|$ and

$$\{f|_S : f \in C\} = 2^S \text{ for any } S \in \mathcal{F}.$$

We check the claim by induction on $|C|$. If $|C| = 1$ then take $\mathcal{F} = \{\emptyset\}$. Suppose $|C| \geq 2$. Without loss of generality, we may suppose that both the families

$$C_0 = \{f \in C : f(1) = 0\} \text{ and } C_1 = \{f \in C : f(1) = 1\},$$

are nonempty. Put $T' = T \setminus \{1\}$. By induction, there exists $\mathcal{F}_0 \subseteq \mathcal{P}(T')$ with $|\mathcal{F}_0| = |C_0'|$ such that

$$\{f|_S : f \in C_0'\} = 2^S \text{ for any } S \in \mathcal{F}_0,$$

where $C_0' = \{f|_{T'} : f \in C_0\}$. Now take $C_1' = \{f|_{T'} : f \in C_1\}$. Again by induction, there exists $\mathcal{F}_1 \subseteq \mathcal{P}(T')$ with $|\mathcal{F}_1| = |C_1'|$ such that

$$\{f|_S : f \in C_1'\} = 2^S \text{ for any } S \in \mathcal{F}_1.$$

Set

$$\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1 \cup \{S \cup \{1\} : S \in \mathcal{F}_0 \cap \mathcal{F}_1\},$$

and note that

$$|\mathcal{F}| = |\mathcal{F}_0| + |\mathcal{F}_1| = |C_0'| + |C_1'| = |C_0| + |C_1| = |C|.$$

Therefore it is enough to prove that $\{f|_S : f \in C\} = 2^S$ for any $S \in \mathcal{F}$, but this is a consequence of the properties of $\mathcal{F}_0$ and $\mathcal{F}_1$. Thus the claim is proved.

Now the lemma follows from the fact that T has exactly $\binom{N}{0} + \binom{N}{1} + \cdots + \binom{N}{n-1}$ subsets of cardinality smaller than n, so by the assumption on $|C|$ there exists a set $S \subseteq T$ with $|S| \geq n$ such that $\{f|_S : f \in C\} = 2^S$. $\square$

Bibliography

[AGL⁺16] A. Avilés, A.J. Guirao, S. Lajara, J. Rodríguez, and P. Tradacete, *Weakly compactly generated Banach lattices*, Studia Math. **234** (2016), no. 2, 165–183.

[AK06] F. Albiac and N.J. Kalton, *Topics in Banach Space Theory*, Graduate texts in Mathematics, Springer, New York, London, 2006.

[AK13] A. Avilés and P. Koszmider, *A continuous image of a Radon-Nikodým compact space which is not Radon-Nikodým*, Duke Math. J. **162** (2013), no. 12, 2285–2299.

[AMCP16] A. Avilés, G. Martínez-Cervantes, and G. Plebanek, *Weakly Radon-Nikodým Boolean algebras and independent sequences*, ArXiv e-prints **arXiv:1610.04257** (2016).

[AO51] A. Alexiewicz and W. Orlicz, *Remarks on Riemann-integration of vector-valued functions*, Studia Math. **12** (1951), no. 1, 125–132.

[Arg92] S.A. Argyros, *Weakly Lindelöf determined Banach spaces not containing* $\ell_1(\mathbb{N})$, ArXiv e-prints **arXiv:math/9210210** (1992).

[Arv02] A.D. Arvanitakis, *Some remarks on Radon–Nikodým compact spaces.*, Fundam. Math. **172** (2002), no. 1, 41–60.

[Avi05] A. Avilés, *Radon-Nikodým compact spaces of low weight and Banach spaces*, Studia Math. **166** (2005), no. 1, 71–82.

[Avi07] ———, *Renormings of the dual of James tree spaces*, Bull. London. Math. Soc. **39** (2007), no. 2, 221–231.

[Bal88] Z. Balogh, *On compact Hausdorff spaces of countable tightness*, Proc. Amer. Math. Soc. **105** (1988), no. 3, 755–764.

[Bal10] C. Baldovino, *Sequential order under CH*, Math. Pannonica **21** (2010), no. 2, 177–206.

[Baš74] A.I. Baškirov, *On the classification of quotient mappings and compact sequential spaces*, Sov. Math. Dokl. **15** (1974), no. 4, 1104–1109.

[BFT78] J. Bourgain, D. Fremlin, and M. Talagrand, *Pointwise compact sets of Baire-measurable functions*, Amer. J. Math. **100** (1978), no. 4, 845–886.

[BJ95] T. Bartoszynski and H. Judah, *Set theory: On the structure of the real line*, Ak Peters Series, Taylor & Francis, 1995.

[BN07] P. Borodulin-Nadzieja, *Measures on minimally generated Boolean algebras*, Topology Appl. **154** (2007), 3107–3124.

[Bou79] J. Bourgain, *La proprieté de Radon-Nikodým*, Publ. Math. Univ. Pierre et Marie Curie **36** (1979).

[BR80] J. Bourgain and H.P. Rosenthal, *Martingales valued in certain subspaces of L^1*, Israel J. Math. **37** (1980), no. 1–2, 171–183.

[Bre08] C. Brech, *Construções genéricas de espaços de Asplund $\mathcal{C}(K)$*, Ph.D. thesis, Universidade de São Paulo and Université Paris 7, 2008.

[BRW77] Y. Benyamini, M.E. Rudin, and M. Wage, *Continuous images of weakly compact subsets of Banach spaces*, Pacific J. Math. **70** (1977), no. 2, 309–324.

[BS76] Y. Benyamini and T. Starbird, *Embedding weakly compact sets into Hilbert space*, Israel J. Math. **23** (1976), no. 2, 137–141.

[BS92] T. Bartoszynski and S. Shelah, *Closed measure zero sets*, Ann. Pure Appl. Logic **58** (1992), no. 2, 93–110.

[CG97] J. Castillo and M. González, *Three-space Problems in Banach Space Theory*, Springer LNM 1667, 1997.

[CRSP10] J.M. Calabuig, J. Rodríguez, and E.A. Sánchez-Pérez, *Weak continuity of Riemann integrable functions in Lebesgue-Bochner spaces*, Acta Math. Sin. **26** (2010), no. 2, 241–248.

[Deb14] G. Debs, *Descriptive aspects of Rosenthal compacta*, in Recent Progress in General Topology III (2014), 205–227.

[Dew02] N. Dew, *Asymptotic structure of Banach spaces*, Ph.D. thesis, University of Oxford, 2002.

[DFJP74] W.J. Davis, T. Figiel, W.B. Johnson, and A. Pełczyński, *Factoring weakly compact operators*, J. Funct. Anal. **17** (1974), no. 3, 311–327.

[Die84] J. Diestel, *Sequences and series in Banach spaces*, Graduate texts in mathematics, Springer-Verlag, 1984.

[DK95] M. Džamonja and K. Kunen, *Properties of the class of measure separable compact spaces*, Fundam. Math. **147** (1995), no. 3, 261–277.

[Dow05] A. Dow, *Sequential order under MA*, Topology Appl. **146/147** (2005), 501–510.

[dRF79] G.C. da Rocha Filho, *Integral de Riemann Vetoral e Geometria de Espaços de Banach*, Ph.D. thesis, Universidade de São Paulo, 1979.

[DU77] J. Diestel and J.J. Uhl, *Vector measures*, Mathematical surveys, vol. 15, Amer. Math. Soc., Prov., R.I., 1977.

[Edg79] G.A. Edgar, *Measurability in a Banach space, II*, Ind. Univ. Math. J. **28** (1979), 559–579.

[EGS14] P. Enflo, V.I. Gurariy, and J.B. Seoane-Sepúlveda, *Some results and open questions on spaceability in function spaces*, Trans. Am. Math. Soc. **366** (2014), no. 2, 611–625.

[Fab97] M. Fabian, *Gâteaux differentiability of convex functions and topology. Weak Asplund spaces*, Canadian Mathematical Society Series of Monographs and Advanced Texts, John Wiley and Sons, New York, 1997.

[Far87] V. Farmaki, *The structure of Eberlein, uniformly Eberlein and Talagrand compact spaces in* $\Sigma(\mathbb{R}^\Gamma)$, Fundam. Math. **128** (1987), no. 1, 15–28.

[Fed77] V.V. Fedorchuk, *A compact space having the cardinality of the continuum with no convergent sequences*, Math. Proc. Camb. Phil. Soc. **81** (1977), 177–181.

[FHH+11] M. Fabian, P. Habala, P. Hájek, V. Montesinos, and V. Zizler, *Banach space theory: The basis for linear and nonlinear analysis*, CMS Books in Mathematics, Springer New York, 2011.

[FHM98] M. Fabian, M. Heisler, and E. Matoušková, *Remarks on continuous images of Radon-Nikodým compacta*, Commentat. Math. Univ. Carol. **39** (1998), no. 1, 59–69.

[FP04] D.H. Fremlin and G. Plebanek, *Independence-precalibers of measure algebras*, Preprint (2004).

[FPRN00] R. Frankiewicz, G. Plebanek, and C. Ryll-Nardzewski, *Between the Lindelöf property and countable tightness*, Proc. Amer. Math. Soc. **129** (2000), no. 1, 97–103.

[Fra69] S.P. Franklin, *On two questions of Moore and Mrowka*, Proc. Amer. Math. Soc. **21** (1969), no. 3, 597–599.

[GM12] E. Glasner and M. Megrelishvili, *Representations of dynamical systems on Banach spaces not containing l_1*, Trans. Am. Math. Soc. **364** (2012), no. 12, 6395–6424.

[GM14] ______, *Eventual nonsensitivity and tame dynamical systems*, ArXiv e-prints **arXiv:1405.2588** (2014).

[GMA10] M. González and A. Martínez-Abejón, *Tauberian operators*, Operator Theory: Advances and Applications, Birkhäuser Basel, 2010.

[Gor91] R. Gordon, *Riemann Integration in Banach Spaces*, Rocky Mountain J. Math. **21** (1991), no. 3, 923–949.

[Gra27] L.M. Graves, *Riemann integration and Taylor's theorem in General Analysis*, Trans. Amer. Math. Soc. **29** (1927), no. 1, 163–177.

[Gul77] S.P. Gul'ko, *On properties of subsets of Σ-products*, Sov. Math. Dokl. **18** (1977), no. 1, 1438–1442.

[Hay81] R. Haydon, *Nonseparable Banach spaces*, Functional analysis: surveys and recent results II (1981), 19–30.

[Hay84] ______, *Darboux integrability and separability of types in stable Banach spaces.*, Sémin. Analyse Fonctionnelle, Paris 1983-84, Publ. Math. Univ. Paris VII 20, 95–115, 1984.

[Hay95] ______, *Baire trees, bad norms and the Namioka property*, Mathematika **42** (1995), 30–42.

[Hei97] L. Heindorf, *On subalgebras of Boolean interval algebras*, Proc. Amer. Math. Soc. **125** (1997), no. 8, 2265–2274.

[HLO87] R. Haydon, M. Levy, and E. Odell, *On sequences without weak* convergent convex block subsequences*, Proc. Amer. Math. Soc. **100** (1987), no. 1, 94–98.

[HS80] J. Hagler and F. Sullivan, *Smoothness and weak* sequential compactness*, Proc. Amer. Math. Soc. **78** (1980), no. 4, 497–503.

[JL74] W.B. Johnson and J. Lindenstrauss, *Some remarks on weakly compactly generated Banach spaces*, Israel J. Math. **17** (1974), no. 2, 219–230.

[JL79] ______, *Correction to "Some remarks on weakly compactly generated Banach spaces" [Israel J. Math. 17 (1974), 219–230; MR 54 #5808]*, Israel J. Math. **32** (1979), no. 4, 382–383.

[Kad94] V.M. Kadets, *On the Riemann integrability of weakly continuous functions.*, Quaest. Math. **17** (1994), no. 1, 33–35.

[Kal] G. Kalai, *Gil Kalai's blog*, `https://gilkalai.wordpress.com/2008/09/28/extremal-combinatorics-iii-some-basic-theorems/`.

[Kan79] V. Kannan, *Ordinal invariants in topology II. Sequential order of compactifications*, Compositio Math. **39** (1979), no. 2, 247–262.

[Kap86] T. Kappeler, *Banach spaces with the condition of Mazur*, Math. Z. **191** (1986), 623–631.

[Kop88] S. Koppelberg, *Counterexamples in minimally generated Boolean algebras*, Acta Univ. Carolin. Math. Phys. **29** (1988), 27–36.

[Kop89] ______, *Minimally generated Boolean algebras*, Order **5** (1989), 393–406.

[Kos15a] P. Koszmider, *Uncountable equilateral sets in Banach spaces of the form $C(K)$*, ArXiv e-prints **arXiv:1503.06356** (2015).

[Kos15b] ______, *Universal objects and associations between classes of Banach spaces and classes of compact spaces*, Selected topics in combinatorial analysis, no. 17(25), Matematički institut SANU, Beograd, 2015, pp. 93–115.

[Kos16] ______, *On the problem of compact totally disconnected reflection of nonmetrizability*, Topology Appl. **213** (2016), 154–166.

[Kun77] K. Kunen, *Combinatorics*, Handbook of Mathematical Logic, North-Holland, Amsterdam, 1977, pp. 371–401.

[Lac12] H.E. Lacey, *The Isometric Theory of Classical Banach Spaces*, Grundlehren der mathematischen Wissenschaften, Springer Berlin Heidelberg, 2012.

[Lin04] Pei-Kee Lin, *Köthe-Bochner Function Spaces*, Birkhäuser, Springer-Science+Business Media New York, 2004.

[MC15] G. Martínez-Cervantes, *On weakly Radon-Nikodým compact spaces*, accepted in Israel J. Math. (2015).

[MC16a] ______, *Banach spaces with weak*-sequential dual ball*, ArXiv e-prints **arXiv:1612.05948** (2016).

[MC16b] ______, *Riemann integrability versus weak continuity*, J. Math. Anal. Appl. **438** (2016), no. 2, 840 – 855.

[MdRF84] F. Miraglia and G.C. da Rocha Filho, *The measurability of Riemann integrable functions with values in Banach spaces and applications*, São Paulo Ime/usp (1984), 1–27.

[MP12] W. Marciszewski and G. Plebanek, *On measures on Rosenthal compacta*, J. Math. Anal. Appl. **385** (2012), no. 1, 185–193.

[Nac65] L. Nachbin, *Topology and order*, Van Nostrand Math. Studies, Princeton, New Jersey, 1965.

[Nam87] I. Namioka, *Radon-Nikodým compact spaces and fragmentability*, Mathematika **34** (1987), 258–281.

[Nam02] ______, *On generalizations of Radon-Nikodým compact spaces*, Topology Proc. **26** (2002), 741–750.

[Nar07] K.M. Naralenkov, *Asymptotic Structure of Banach Spaces and Riemann Integration*, Real Anal. Exchange **33** (2007), no. 1, 113–126.

[NOR73] A.S. Nemirovski, M.Ju. Ochan, and R. Redjouani, *Conditions for Riemann integrability of functions with values in a Banach space*, Mosc. Univ. Math. Bull. **27** (1973), no. 3-4, 124–126.

[Ode85] E. Odell, *A nonseparable Banach space not containing a subsymmetric basic sequence*, Israel J. Math. **52** (1985), no. 1–2, 97–109.

[OSV91] J. Orihuela, W. Schachermayer, and M. Valdivia, *Every Radon-Nikodým Corson compact space is Eberlein compact*, Studia Math. **2** (1991), no. 98, 157–174.

[Paw96] J. Pawlikowski, *A characterization of strong measure zero sets*, Israel J. Math. **93** (1996), no. 1, 171–183.

[PdRF80] A. Pełczyński and G.C. da Rocha Filho, *Operadores de Darboux*, Seminario Brasileiro de Analise, 12, Sao Jose dos Campos, 1980. Trabalhos Apresentados. Rio de Janeiro, SBM, 1980, pp. 293–296.

[Piz89] M. Pizzotti, *Darboux-integrabilidade e mensurabilidade de funções Riemann-integráveis definidas em compactos*, Ph.D. thesis, Universidade de São Paulo, 1989.

[Pli15] A. Plichko, *Three sequential properties of dual Banach spaces in the weak* topology*, Topology Appl. **190** (2015), 93–98.

[Pol80] R. Pol, *On a question of H.H. Corson and some related problems*, Fundam. Math. **109** (1980), no. 2, 143–154.

[PS15] G. Plebanek and D. Sobota, *Countable tightness in the spaces of regular probability measures.*, Fundam. Math. **229** (2015), no. 2, 159–169.

[PY00] A. Plichko and D. Yost, *Complemented and uncomplemented subspaces of Banach spaces*, Ext. Math. **15** (2000), 335–371.

[Ros74a] H.P. Rosenthal, *A characterization of Banach spaces containing* l^1, Proc. Natl. Acad. Sci. USA **71** (1974), 2411–2413.

[Ros74b] ______, *The heredity problem for weakly compactly generated Banach spaces*, Compositio Math. **18** (1974), no. 1, 83–111.

[Ros78] ______, *Some recent discoveries in the isomorphic theory of Banach spaces*, Bull. Amer. Math. Soc. **84** (1978), no. 5, 803–831.

[Sch89] T. Schlumprecht, *On Dual Spaces with Bounded Sequences Without Weak* Convergent Convex Blocks*, Proc. Amer. Math. Soc. **107** (1989), no. 2, 395–408.

[Sof12] M.A. Sofi, *Weaker forms of continuity and vector-valued Riemann integration*, Colloq. Math. **129** (2012), no. 1, 1–6.

[Ste90] C. Stegall, *More facts about conjugate Banach spaces with the Radon-Nikodým property*, Acta Univ. Carolin. Math. Phys. **31** (1990), 107–117.

[Ste94] ______, *Spaces of Lipschitz functions on Banach spaces*, Functional analysis (Essen, 1991), Lecture Notes in Pure and Appl. Math. **150** (1994), 265–278.

[Tal79] M. Talagrand, *Espaces de Banach Faiblement* κ*-Analytiques*, Annals of Mathematics **110** (1979), no. 3, pp. 407–438.

[Tal81] ______, *Sur les espaces de Banach contenant* $\ell^1(\tau)$, Israel J. Math. **40** (1981), no. 3, 324–330.

[Tal83] ———, *La proprieté de Dunford-Pettis dans* $\mathcal{C}(K, E)$ *et* $L^1(E)$, Israel J. Math. **44** (1983), no. 4, 317–321.

[Tal84] ———, *Pettis integral and measure theory*, Memoirs of the Amer. Math. Soc., no. 307, Amer. Math. Soc., 1984.

[Tod84] S. Todorcevic, *Trees and linearly ordered sets*, Handbook of set-theoretic topology, North-Holland, Amsterdam, 1984, pp. 235–293.

[Wan96] C. Wang, *On the weak property of Lebesgue of Banach spaces*, Journal of Nanjing University Mathematical Biquarterly **13** (1996), no. 2, 150–155.

[WW01] C. Wang and K. Wan, *On the Weak Property of Lebesgue of* $L^1(\Omega, \Sigma, \mu)$, Rocky Mountain J. Math. **31** (2001), no. 2, 697–703.

[WY00] C. Wang and Z. Yang, *Some Topological Properties of Banach Spaces and Riemann Integration*, Rocky Mountain J. Math. **30** (2000), no. 1, 393–400.

Index