UNIVERSIDAD DE MURCIA

FACULTAD DE MATEMÁTICAS

The First Stability Eigenvalue for Compact Constant Mean Curvature Surfaces

El Primer Valor Propio del Operador de Estabilidad para Superficies Compactas con Curvatura Media Constante

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Resumen

El estudio de las superficies de curvatura media constante (abreviadamente superficies CMC), y en particular las superficies minimales, es un campo de investigación muy desarrollado y fructífero, con orígenes ya clásicos, en la Geometría Diferencial que suscita actualmente gran interés. Dicho estudio se remonta a unos 250 años cuando Langrange, buscando aplicaciones del método variacional que estaba desarrollando, analizó las superficies satisfaciendo la propiedad de minimizar el área de entre todas aquellas superficies resultantes de modificar ligeramente la superficie inicial.

De este modo, las superficies de curvatura media constante aparecen como soluciones de un problema variacional. Concretamente, las superficies minimales (curvatura media cero) son los puntos críticos del funcional área mientras que las superficies de curvatura media constante son también puntos críticos del mismo funcional cuando nos restringimos a variaciones que conservan el volumen encerrado.

A pesar de ser puntos críticos de un funcional, obviamente no está garantizado que sean un mínimo para dicho funcional. Una condición que nos asegura que estamos ante un mínimo es que la segunda variación del funcional sea estrictamente positiva. Debilitar esta condición a que dicha variación sea no negativa nos lleva a la noción de estabilidad. Así, una superficie de curvatura media constante es estable si la segunda variación del área es no negativa.

La fórmula de la segunda variación del funcional área contiene un operador conocido como el operador de Jacobi u operador de estabilidad cuyo estudio está íntimamente relacionado con la estabilidad de la superficie. Para ser más específicos, el primer valor propio de este operador, denotado por λ_1, es no negativo si, y solo si, la superficie es estable. De este modo, dicho valor propio adquiere una gran importancia en el estudio de las superficies con curvatura media constante. Además, el estudio del espectro del operador de Jacobi, o más generalmente de los operadores de Schrödinger, ha sido abordado por diferentes autores puesto que es una importante línea de investigación con aplicaciones a la geometría y a la física (véase por ejemplo [BB, BGM, SI]).

En cuanto al problema que nos concierne, el cual consiste en la búsqueda de estimaciones óptimas para el primer valor propio del operador de Jacobi para superficies compactas con curvatura media constante, el primer trabajo data de 1968, en el cual Simons consideró el caso minimal (ver Lema 6.1.7 en [Si]). A este respecto, el autor encontró una estimación para dicho valor propio de cualquier hipersuperficie compacta minimal inmersa en la esfera estándar. En particular, para

superficies minimales en la 3-esfera probó que $\lambda_1 = -2$ si la superficie es un ecuador totalmente geodésico y $\lambda_1 \leq -4$ en caso contrario. Posteriormente, Wu obtuvo una caracterización en [Wu] de los toros de Clifford minimales a través del primer valor propio del operador de estabilidad, probando que $\lambda_1 = -4$ si, y solo si, Σ es un toro de Clifford minimal. Más recientemente, Perdomo dio una nueva demostración de la misma caracterización espectral en [Pe]. Resumiendo, podemos enunciar el siguiente resultado para hipersuperficies en general.

Sea Σ^n una hipersuperficie compacta, orientable y minimal inmersa en la esfera euclídea $\mathbb{S}^{n+1}$. Si λ_1 denota el primer valor propio de su operador de Jacobi, entonces

(i) o bien $\lambda_1 = -n$ y Σ es un ecuador totalmente geodésico $\mathbb{S}^n \subset \mathbb{S}^{n+1}$,

(ii) o bien $\lambda_1 \leq -2n$, con igualdad si, y solo si, Σ es un toro de Clifford minimal $\mathbb{S}^k(\sqrt{k/n}) \times \mathbb{S}^{n-k}(\sqrt{(n-k)/n}) \subset \mathbb{S}^{n+1}$.

Adicionalmente, para el caso particular de superficies, Perdomo consiguió llegar más lejos obteniendo una interesante fórmula que relaciona el primer valor propio λ_1 con el género de la superficie, su área y cierto invariante simple α. Esta relación se establece en la Proposición 3.2 en [Pe].

Sea Σ^2 una superficie compacta, orientable y minimal inmersa en $\mathbb{S}^3$. Si λ_1 denota el primer valor propio de su operador de Jacobi y f_1 es una función propia positiva asociada a λ_1, entonces

$$\lambda_1 = -4 - \frac{1}{\mathrm{Area}(\Sigma)} \left(\alpha + 8\pi(g-1)\right),$$

donde $\alpha = \int_\Sigma f_1^{-2} |\nabla f_1|^2 d\Sigma$ y g es el género de la superficie Σ.

En 2005, Alías, Barros y Brasil extendieron el resultado mencionado de Simons, Wu y Perdomo al caso de hipersuperficies compactas con curvatura media constante inmersas en la esfera estándar. Así, en este contexto, obtuvieron cotas óptimas para λ_1 y una caracterización para ciertos toros de Clifford de curvatura media constante con radios apropiados. En particular, su Teorema 2.2 en [ABB] se enuncia como sigue:

Sea Σ^n una hipersuperficie compacta, orientable con curvatura media constante H inmersa en la esfera euclídea $\mathbb{S}^{n+1}$. Si λ_1 denota el primer valor propio de su operador de Jacobi, entonces

(i) o bien $\lambda_1 = -n(1+H^2)$ y Σ es una esfera totalmente umbilical en $\mathbb{S}^{n+1}$,

(ii) o bien $\lambda_1 \leq -2n(1+H^2) + \frac{n(n-2)}{\sqrt{n(n-1)}}|H| \max_\Sigma |\phi|$, donde ϕ es el tensor de umbilicidad total de Σ, con igualdad si, y solo si,

(a) $H = 0$ *y* Σ *es un toro de Clifford minimal* $\mathbb{S}^k(\sqrt{k/n}) \times \mathbb{S}^{n-k}(\sqrt{(n-k)/n})$ *con* $k = 1, \dots, n-1$*;*

(b) $H \neq 0$*,* $n = 2$*, y* Σ^2 *es un toro de Clifford con curvatura media constante* $\mathbb{S}^1(r) \times \mathbb{S}^1(\sqrt{1-r^2})$ *con* $0 < r < 1$*,* $r \neq \sqrt{1/2}$*;*

(c) $H \neq 0$*,* $n \geq 3$*, y* Σ *es un toro de Clifford con curvatura media constante* $\mathbb{S}^{n-1}(r) \times \mathbb{S}^1(\sqrt{1-r^2})$ *con* $0 < r < \sqrt{(n-1)/n}$*;*

De esta manera, el problema para el caso de hipersuperficies compactas, orientables con curvatura media constante inmersas en la esfera quedó cerrado. Sin embargo, en 2013, Chen y Wang consideraron el mismo problema en un contexto más general que incluye el estudio de hipersuperficies con curvatura media constante. Concretamente, trabajaron con hipersuperficies compactas de Weingarten lineales inmersas también en la esfera euclídea. Así, los autores, imponiendo algunas condiciones sobre estas hipersuperficies, obtuvieron una estimación óptima para el primer valor propio del operador de Jacobi asociado al correspondiente problema variacional para hipersuperficies de Weingarten lineales. También analizaron lo que sucedía cuando la igualdad se verifica, concluyendo que esta se alcanza si, y solo si, la hipersuperficie es totalmente umbilical pero no totalmente geodésica, o bien es un toro de Clifford (ver Teorema 1.4 en [CW] para más detalles).

Llegados a este punto, nos encontramos en condiciones de explicar que nuestro principal objetivo en esta memoria es obtener cotas superiores óptimas para el primer valor propio del operador de estabilidad de superficies compactas, orientables y con curvatura media constante inmersas en diferentes variedades ambiente de dimensión 3. Además, estudiamos cuándo se alcanzan las igualdades y también deducimos restricciones sobre las superficies estables así como su completa clasificación en el caso de que sea posible.

Los ambientes naturales en los que estudiar el problema, de modo que se incluyan la esfera y los espacios de curvatura constante, son las variedades homogéneas. El siguiente paso para extender estos ambientes es considerar sumersiones de Killing riemannianas. Otra forma natural de generalizar el ambiente consiste en utilizar productos warped.

Por esta razón, los espacios ambiente que vamos a considerar son los que acabamos de mencionar: variedades homogéneas, sumersiones de Killing riemannianas y productos warped. Así, la tesis está dividida en cuatro capítulos: el primero dedicado a introducir la notación principal y enmarcar el problema; mientras que en cada uno de los otros tres capítulos se estudia el problema en las diferentes variedades ambiente citadas, recopilando toda nuestra contribución original ([AMO, MO1, MO2, MO3, MO4]).

Una vez presentado el contexto en el cual se enmarca esta tesis doctoral, pasamos a describir cómo está organizada.

El **primer capítulo** es un capítulo introductorio en el que fijamos la notación y presentamos los conceptos que necesitamos para la comprensión de la presente memoria. Además, motivamos el problema e incluimos nuestros primeros cálculos.

De esta manera, comenzamos el capítulo con un breve recordatorio sobre geometría de subvariedades. En particular, nos concentramos en la notación utilizada cuando se estudia una superficie Σ^2 compacta y orientable inmersa en una variedad riemanniana M^3 de dimensión 3. Concretamente, incluimos nociones como la de campo de vectores Killing sobre M, el tensor curvatura de Riemann R, la curvatura seccional K y la curvatura de Ricci Ric en M, el operador forma A de Σ asociado a su aplicación de Gauss N y la curvatura media H de Σ. Recordamos la expresión de la ecuación de Gauss en términos del operador forma, pero también en términos del tensor de umbilicidad total, sirviéndonos este último para definir tanto las superficies totalmente geodésicas, como las totalmente umbilicales. A continuación, recordamos cómo están definidos algunos operadores diferenciales en una variedad riemanniana; nos referimos al gradiente, divergencia, Hesiano y Laplaciano, dado que serán necesarios para llevar a cabo diferentes cálculos a lo largo de esta memoria.

Una vez fijada la notación principal, enfatizamos la relevancia de las superficies minimales y de curvatura media constante desde un punto de vista variacional. Con el fin de justificarlo, introducimos el concepto de variación normal de una superficie compacta asociada a una función diferenciable sobre la superficie, la cual no es más que modificar diferenciablemente la superficie inicial en su dirección normal modulando por la función prefijada. En consecuencia, se puede considerar una familia uniparamétrica de superficies compactas para las que se definen dos funcionales: el área de todas las superficies de la variación y el volumen encerrado entre la superficie inicial y cada una de las nuevas superficies construidas por medio de la variación. Así, incluimos las fórmulas de la primera variación del área y del volumen y, como resultado, probamos que las superficies minimales y las superficies con curvatura media constante, cuando nos restringimos a variaciones que preservan el volumen encerrado, son los puntos críticos del funcional área. El siguiente paso consiste en escribir la fórmula de la segunda variación del área, la cual nos da pie a definir el operador de Jacobi u operador de estabilidad. Así, definimos una superficie estable como aquella cuya segunda variación del área es no negativa para toda variación normal de la superficie.

En la última sección del capítulo recordamos varios hechos conocidos sobre el espectro del operador de Jacobi y, sobre todo, prestamos especial atención al primer valor propio del operador de estabilidad λ_1. Asimismo, remarcamos la relación existente entre este valor propio y la estabilidad de la superficie, la cual establece que

$$\Sigma \textit{ es estable si, y solo si, } \lambda_1 \geq 0.$$

Para finalizar, calculamos dos cotas superiores generales para λ_1 cuando consideramos cualquier superficie compacta y orientable de curvatura media constante inmersa en una variedad riemanniana de dimensión 3 arbitraria. Concretamente, tales cotas son las siguientes:

$$\begin{aligned}\lambda_1 &\leq -2H^2 - \frac{1}{\text{Area}(\Sigma)}\int_\Sigma \text{Ric}(N,N)\,d\Sigma \ \text{ y}\\ \lambda_1 &\leq -4H^2 - \frac{1}{\text{Area}(\Sigma)}\left(8\pi(g-1) + \int_\Sigma (2\,K_\Sigma + \text{Ric}(N,N))\,d\Sigma\right),\end{aligned}$$

donde K_Σ denota la curvatura seccional del plano tangente a Σ en M. Además, discutimos cuándo se alcanzan las igualdades. En este punto, nos gustaría enfatizar la importancia de estas cotas ya que serán esenciales en la consecución de nuestros resultados.

En el **segundo capítulo** consideramos superficies compactas y orientables de curvatura media constante inmersas en variedades homogéneas de dimensión 3. Recordemos que

> *M^3 es homogénea si para cada par de puntos p y q en M, existe una isometría que lleva p a q.*

Así, comenzamos observando algunos hechos generales de este tipo de variedades y exponemos su clasificación dependiendo de la dimensión del grupo de isometrías (6, 4 y 3) cuando la variedad es además simplemente conexa.

En particular, dedicamos la Sección 2.2 al estudio de variedades homogéneas simplemente conexas con grupo de isometrías de dimensión 6, es decir, formas espaciales simplemente conexas a las que denotamos por $M^3(c)$ (donde c representa su curvatura seccional constante): la esfera estándar $\mathbb{S}^3(\sqrt{1/c})$ si $c > 0$, el espacio euclídeo $\mathbb{R}^3$ si $c = 0$ y el espacio hiperbólico $\mathbb{H}^3(\sqrt{-1/c})$ si $c < 0$. Así, el principal objetivo de esta sección es generalizar el resultado de Alías, Barros y Brasil a $M^3(c)$. Para ello estudiamos un caso más general pues suponemos que M tiene curvatura seccional acotada inferiormente por c, es decir, $K \geq c$, siendo nuestras principales cotas para superficies compactas y orientables de curvatura media constante las siguientes:

$$\lambda_1 \leq -2(H^2 + c) \quad \text{ y } \quad \lambda_1 \leq -4(H^2 + c) - \frac{8\pi(g-1)}{\text{Area}(\Sigma)}.$$

Para probarlo hacemos uso de las cotas generales presentadas previamente. Además, la primera igualdad se da si, y solo si, la superficie es totalmente umbilical y $\text{Ric}(N,N) = 2c$, y la segunda igualdad se da si, y solo si, la superficie tiene curvatura de Gauss constante G, $K_\Sigma = c$ y $\text{Ric}(N,N) = 2c$. Como consecuencia de este resultado, podemos obtener cierta información para superficies compactas estables de curvatura media constante inmersas en estos ambientes. Concretamente, concluimos un resultado de no existencia de tales superficies cuando $H^2 + c > 0$ y, en caso contrario, afirmamos que si $H^2 + c = 0$ entonces Σ es topológicamente una esfera o un toro, y si $H^2 + c < 0$ entonces $\text{Area}(\Sigma)\,|H^2 + c| \geq 2\pi(g-1)$.

Por tanto, particularizando este resultado al caso $M^3(c)$, obtenemos que $\lambda_1 = -2(H^2 + c)$ y Σ es totalmente umbilical, o bien $\lambda_1 \leq -4(H^2 + c)$. Asimismo también analizamos el caso en el que la última igualdad se satisface, encontrando que tan solo es posible para la esfera. Consecuentemente, la única superficie para la que se alcanza la igualdad es el toro de Clifford con curvatura media constante inmerso en la esfera $\mathbb{S}^3(c)$. En cuanto a la estabilidad, observamos que las únicas superficies compactas y orientables de curvatura media constante inmersas en $M^3(c)$ estables son las esferas geodésicas inmersas en $\mathbb{H}^3(c)$.

En la Sección 2.3 estudiamos variedades homogéneas simplemente conexas con grupo de isometrías de dimensión 4. Con tal fin, empezamos haciendo un breve recorrido histórico sobre los principales avances relacionados con el estudio de superficies en estas variedades ambiente. Para ellas sabemos que existe una sumersión riemanniana $\Pi : M^3 \longrightarrow B^2(\kappa)$, donde $B^2(\kappa)$ es una forma espacial simplemente conexa 2-dimensional con curvatura constante κ. Además, sus fibras son las trayectorias de un campo de vectores Killing unitario ξ en M. Se define la curvatura de fibrado τ de M como la constante que satisface la ecuación $\nabla_E \xi = \tau(E \wedge \xi)$, para todo campo de vectores E sobre M, donde $\wedge$ denota el producto vectorial en M.

Con la notación anterior, usamos $\mathbb{E}^3(\kappa,\tau)$ para referirnos a tales variedades ambiente, donde las constantes κ y τ juegan un papel clave. De hecho, dependiendo de sus valores, dichos espacios están clasificados.

Clasificación de $\mathbb{E}^3(\kappa,\tau)$ dependiendo de κ y τ

	$\kappa > 0$	$\kappa = 0$	$\kappa < 0$
$\tau = 0$	$\mathbb{S}^2(\kappa) \times \mathbb{R}$		$\mathbb{H}^2(\kappa) \times \mathbb{R}$
$\tau \neq 0$	$\mathbb{S}^3_b(\kappa,\tau)$	$Nil_3(\tau)$	$\widetilde{Sl}(2,\mathbb{R})(\kappa,\tau)$

Así, una vez que hemos introducidos los espacios $\mathbb{E}^3(\kappa,\tau)$, recordamos las expresiones para su tensor curvatura de Riemann, la curvatura de Ricci y las curvaturas seccionales para cada plano tangente. Recordamos también que, dada una superficie orientable, su función ángulo $\Theta : \Sigma \longrightarrow [-1,1]$, dada por $\Theta = \langle N, \xi \rangle$, nos permite introducir dos ejemplos de superficies en $\mathbb{E}^3(\kappa,\tau)$ de forma natural, ya que aparecen cuando Θ^2 alcanza su máximo o mínimo en cada punto.

- Si $\Theta^2 \equiv 1$ la superficie es un slice horizontal, es decir, $\Sigma = B^2(\kappa) \times \{t\}$.
- Si $\Theta \equiv 0$ la superficie es un cilindro de Hopf sobre una curva regular $\gamma : I \longrightarrow B^2(\kappa)$, es decir, $\Sigma = \Pi^{-1}(\gamma)$. En particular, cuando las fibras y la curva son cerradas la superficie es un toro de Hopf.

Llegados a este punto, encontramos las cotas superiores para λ_1 para superficies compactas y orientables de curvatura media constante inmersas en $\mathbb{E}^3(\kappa,\tau)$ y buscamos caracterizaciones de las superficies citadas puesto que ellas alcanzan las estimaciones. Para proceder con ello, consideramos dichas cotas para cada una de las cinco posibilidades de espacios ambiente incluidos en la tabla recopilatoria anterior. Entre todos los resultados, merece la pena destacar los obtenidos en aquellos ambientes en los que las igualdades pueden ser alcanzadas.

Por una parte, cuando Σ está inmersa en $\mathbb{S}^2(\kappa) \times \mathbb{R}$, las estimaciones vienen dadas por

$$\lambda_1 \leq -2H^2 \quad \text{y} \quad \lambda_1 < -4H^2 - \kappa - \frac{8\pi(g-1)}{\text{Area}(\Sigma)},$$

y la primera igualdad se alcanza si, y solo si, Σ es un slice horizontal. Además, cuando consideramos como variedad ambiente el cociente $\mathbb{S}^2(\kappa) \times \mathbb{S}^1$ del producto $\mathbb{S}^2(\kappa) \times \mathbb{R}$, conseguimos las mismas estimaciones. Es más, en este caso la segunda cota se alcanza para los toros de Hopf con curvatura media constante. Por otra parte, cuando estudiamos las esferas de Berger $\mathbb{S}^3_b(\kappa, \tau)$, distinguimos dos casos dependiendo del signo de $\kappa - 4\tau^2$. En particular, si $\kappa - 4\tau^2 > 0$ obtenemos

$$\lambda_1 < -2(H^2 + \tau^2) \quad \text{y} \quad \lambda_1 \leq -4H^2 - \kappa - \frac{8\pi(g-1)}{\text{Area}(\Sigma)},$$

y la segunda igualdad se da si, y solo si, Σ es un toro de Hopf de curvatura media constante. En el resto de ambientes siempre obtenemos cotas superiores estrictas para λ_1.

Para finalizar el capítulo dedicamos una breve parte a recordar lo que sucede con la estabilidad de superficies en estos ambientes. De hecho, se conoce la clasificación completa de superficies compactas y estables con curvatura media constante en $\mathbb{E}^3(\kappa, \tau)$, siendo los slices horizontales en $\mathbb{S}^2(\kappa) \times \mathbb{R}$ las únicas posibles.

En el **tercer capítulo** extendemos los resultados del capítulo precedente a las sumersiones de Killing riemannianas, que resultan ser una generalización natural de las variedades homogéneas. Recordemos que

> *una sumersión riemanniana $\Pi : M^3 \longrightarrow B^2(\kappa)$, donde M es una variedad riemanniana de dimensión 3 y $B^2(\kappa)$ es una superficie riemanniana con curvatura de Gauss κ, es una sumersión de Killing riemanniana si sus fibras son las trayectorias de un campo de vectores Killing unitario ξ sobre M.*

Comenzamos exponiendo los progresos que podemos encontrar en la literatura sobre superficies inmersas en estos ambientes. Además, dado que las sumersiones de Killing riemannianas han sido poco estudiadas hasta la fecha, dedicamos la Sección 3.2 a introducir las principales herramientas con detalle. En primer lugar, recordamos algunos conceptos y fórmulas generales de la teoría de sumersiones riemannianas tales como los tensores naturales $\mathcal{A}$ y $\mathcal{T}$, y las expresiones para el tensor curvatura de Riemann y las curvaturas seccionales. A continuación, nos concentramos en la geometría particular de las sumersiones de Killing riemannianas. Con tal fin, incluimos la definición de la curvatura de fibrado τ de Π como la única función en M tal que $\nabla_E \xi = \tau(E \wedge \xi)$, para todo campo de vectores E sobre M. En este momento, damos su demostración que también incluye el caso homogéneo.

Seguidamente, fijamos la notación $\mathcal{M}(\kappa,\tau)$ para referirnos a una variedad riemanniana de dimensión 3 que admite una sumersión de Killing riemanniana $\Pi:\mathcal{M}(\kappa,\tau)\longrightarrow B^2(\kappa)$ con curvatura de fibrado τ. De la misma forma que hemos visto para las variedades homogéneas, en estas sumersiones las funciones κ y τ también son fundamentales para la comprensión de su geometría. Por ejemplo, cuando ambas son constantes, $\mathcal{M}(\kappa,\tau)$ se reduce precisamente a una variedad homogénea. Ya establecida la notación, calculamos en términos de κ y τ, la conexión de Levi-Civita de $\mathcal{M}(\kappa,\tau)$ para obtener la curvatura seccional de cada plano tangente a $\mathcal{M}(\kappa,\tau)$, así como la curvatura de Ricci en una dirección unitaria arbitraria.

En la Sección 3.3 prestamos atención a dos superficies especiales que no son más que la generalización natural de los mencionados slices horizontales y los toros de Hopf en variedades homogéneas. Para introducirlas, consideramos de nuevo la función ángulo $\Theta=\langle N,\xi\rangle$, de tal modo que:

- Si $\Theta^2\equiv 1$ la superficie es un slice horizontal. Entre sus propiedades cabe remarcar que son totalmente geodésicas, $\lambda_1=0$ y en consecuencia son estables.
- Si $\Theta\equiv 0$ la superficie es un toro de Hopf sobre una curva regular cerrada $\gamma: I\longrightarrow B^2(\kappa)$, es decir, $\Sigma=\Pi^{-1}(\gamma)$ con fibras cerradas. Estas superficies son llanas y su curvatura media es $H=k_\gamma/2$. Además, cuando κ es constante sobre γ, sabemos que $\lambda_1=-4H^2-\kappa$.

Asimismo, incluimos la demostración de todas estas propiedades. De este modo, estamos ya preparados para mostrar nuestra contribución.

En la Sección 3.4 recopilamos nuestros principales resultados. Para proceder a exponerlos, debemos tener en cuenta que distinguimos entre los casos en que $\kappa-4\tau^2$ es positivo y negativo. Para el primero de ellos, es decir, si consideramos una superficie compacta y orientable de curvatura media constante en $\mathcal{M}(\kappa,\tau)$ con $\kappa-4\tau^2>0$, obtenemos las siguientes cotas superiores:

$$\begin{aligned}\lambda_1 &\leq -2H^2-\frac{1}{\text{Area}(\Sigma)}\int_\Sigma(2\tau^2-|\nabla\tau|)d\Sigma \ \text{ y}\\ \lambda_1 &\leq -4H^2-\frac{8\pi(g-1)}{\text{Area}(\Sigma)}-\frac{1}{\text{Area}(\Sigma)}\int_\Sigma(\kappa-|\nabla\tau|)d\Sigma.\end{aligned}$$

Adicionalmente, comprobamos que la primera igualdad se alcanza para superficies horizontales mientras que la segunda la verifican los toros de Hopf con κ y τ constantes sobre Σ cuando la sumersión de Killing tiene fibras compactas. Para el segundo caso, es decir, si $\kappa-4\tau^2<0$, entonces

$$\begin{aligned}\lambda_1 &\leq -2H^2-\frac{1}{\text{Area}(\Sigma)}\int_\Sigma(\kappa-2\tau^2-|\nabla\tau|)d\Sigma \ \text{ y}\\ \lambda_1 &\leq -4H^2-\frac{8\pi(g-1)}{\text{Area}(\Sigma)}-\frac{1}{\text{Area}(\Sigma)}\int_\Sigma(2\kappa-4\tau^2-|\nabla\tau|)d\Sigma.\end{aligned}$$

En contraste con el caso anterior, aquí la primera igualdad se da si, y solo si, la superficie es un toro de Hopf minimal con $\tau=0$ sobre Σ y κ es constante sobre γ, y la segunda igualdad se alcanza solo para superficies horizontales con curvatura de Gauss igual a κ. Además, se estudia también el caso particular en el que la curvatura de fibrado es constante como una consecuencia interesante.

Así, después de obtener las esperadas estimaciones para λ_1, en la Sección 3.5 mostramos algunas consecuencias relacionadas con la estabilidad de las superficies compactas de curvatura media constante inmersas en $\mathcal{M}(\kappa,\tau)$. Nuestros avances consisten en dar algunas restricciones sobre la curvatura media H, esto es, obtenemos cotas superiores para H^2. Recientemente se ha demostrado que si una superficie compacta de curvatura media constante en $\mathcal{M}(\kappa,\tau)$ es estable entonces tiene que ser un grafo entero minimal o un toro de Hopf. Hemos aplicado nuestras estimaciones a estas superficies.

Para terminar con el capítulo, la intención de la última sección es proporcionar ejemplos no triviales de sumersiones de Killing riemannianas y aplicarles nuestros resultados. El primer ejemplo que merece la pena considerar aparece cuando la curvatura de fibrado se anula, y el segundo viene definido por medio de espacios de dimensión 3 que admiten un producto doblemente warped y proyectan sobre cierta superficie.

Finalmente, en el **cuarto capítulo** estudiamos resultados similares para superficies compactas y orientables de curvatura media constante inmersas en productos warped de dimensión 3. Para recordar su definición, consideremos dos variedades riemannianas $(B,\langle\ ,\ \rangle_B)$ y $(F,\langle\ ,\ \rangle_F)$ y una función diferenciable positiva $\rho : B\longrightarrow(0,+\infty)$, llamada función warping. De este modo,

> $M^3 = B\times_\rho F$ *es un producto warped si es la variedad producto* $B\times F$ *equipada con la métrica*
>
> $$\langle\ ,\ \rangle = \langle\ ,\ \rangle_B + \rho^2\langle\ ,\ \rangle_F.$$

Tras una breve introducción histórica, recordamos cómo calcular el tensor curvatura de Riemann de un producto warped, así como la curvatura de Ricci. En lo que sigue, tratamos con los dos casos posibles de forma separada: $B\times_\rho F^2$, en el que la base es de dimensión 1 y la fibra de dimensión 2; y por el contrario $B^2\times_\rho F$, donde la base es de dimensión 2 y la fibra de dimensión 1.

En la Sección 4.2 nos centramos en productos warped del primer tipo $B\times_\rho F^2$, los cuales han sido estudiados a fondo en la literatura. Así, dedicamos la primera parte a realizar un breve repaso a la bibliografía, fijamos la notación y obtenemos la curvatura seccional para cualquier plano tangente a $B\times_\rho F^2$ y la curvatura de Ricci para cada dirección unitaria.

Posteriormente, con el objetivo de dar cotas superiores óptimas para λ_1, recordamos la definición de la función ángulo en este ámbito $\Theta = \langle N,\partial_t\rangle$ para cualquier superficie orientable inmersa en el producto warped $B\times_\rho F^2$. Llegados a este punto, trabajamos con una condición de convergencia, bien establecida y utilizada en la literatura, que se formula como

$$\min_F K_F \geq \sup_B\{\rho'^2 - \rho\rho''\},$$

donde K_F denota la curvatura de Gauss de F. Dicha condición significa geométricamente que la curvatura de Ricci en M alcanza su mínimo en la dirección de ∂_t. Así, para superficies compactas y orientables de curvatura media constante inmersas en un producto warped $B \times_\rho F^2$ satisfaciendo la condición de convergencia, obtenemos las siguientes estimaciones:

$$\lambda_1 \leq -2H^2 + \frac{2}{\text{Area}(\Sigma)} \int_\Sigma \frac{\rho''}{\rho} d\Sigma \quad \text{y} \quad \lambda_1 \leq -4H^2 - \frac{8\pi(g-1)}{\text{Area}(\Sigma)} + \frac{4}{\text{Area}(\Sigma)} \int_\Sigma \frac{\rho''}{\rho} d\Sigma.$$

Como viene siendo habitual, analizamos lo que sucede cuando las igualdades se alcanzan. Por una parte, vemos que la primera igualdad se da si, y solo si, la superficie es un slice o es totalmente umbilical y está contenida en una región de M con curvatura constante $K = -\rho''/\rho$. Por otra parte, la segunda igualdad se da si, y solo si, $K = -\rho''/\rho$ sobre la superficie y $G - 2K$ es constante sobre Σ.

Hemos aplicado estas cotas a dos casos particulares que merecen una atención especial. El primero de ellos ocurre cuando la función warping es una solución de la ecuación de Jacobi $\rho'' + c\rho = 0$, para algún $c \in \mathbb{R}$, lo que significa que la curvatura de Ricci de M en la dirección ∂_t es constante. De este modo, cuando además suponemos que $B \times_\rho F^2$ satisface la condición de convergencia, concluimos un resultado de no existencia de superficies estables si $c < 0$ y en el otro caso $c \geq 0$, probamos que las únicas superficies compactas estables de curvatura media constante son algunos slices. El segundo caso aparece cuando la función warping es cóncava, es decir $\rho'' \leq 0$, y concluimos que las únicas superficies compactas y orientables de curvatura media constante estables inmersas en estos productos son las esferas topológicas totalmente geodésicas o los toros topológicos llanos y totalmente geodésicos.

Para finalizar la sección, damos como ejemplos en los que aplicar nuestros resultados los espacios pseudo-hiperbólicos $\mathbb{R} \times_{e^t} F^2$ y $\mathbb{R} \times_{\cosh t} F^2$. Así, bajo la condición de convergencia, obtenemos cotas superiores para λ_1 y además damos la clasificación completa de las superficies compactas y estables de curvatura media constante inmersas en dichos espacios:

Las únicas superficies compactas y orientables de curvatura media constante estables inmersas en un espacio pseudo-hiperbólico son los slices con fibra compacta.

En la Sección 4.3 tratamos el caso que resta, es decir, estudiamos el segundo tipo de productos warped $B^2 \times_\rho F$. Puesto que no hay prácticamente información sobre ellos en la bibliografía, comenzamos directamente estableciendo la notación necesaria y calculando la curvatura seccional de cualquier plano tangente a $B^2 \times_\rho F$ y la curvatura de Ricci en cualquier dirección unitaria.

Siguiendo el mismo esquema que en la sección previa, recordamos que dada una superficie orientable en $B^2 \times_\rho F$, su función ángulo es $\Theta = \frac{1}{\rho}\langle N, \partial_t \rangle$. Así, para este caso, buscamos hipótesis similares a la condición de convergencia anterior. De hecho, suponemos que

$$\text{Ric}_p(v, v) \leq 2K_B, \text{ para todo } v \in T_pM, \ |v| = 1, p \in M,$$

donde K_B denota la curvatura de Gauss de B. Su interpretación geométrica es que las curvaturas seccionales de M alcanzan un máximo igual a K_B en cada punto. Así, el resultado principal de la sección afirma que el primer valor propio de estabilidad para superficies compactas y orientables con curvatura media constante en $B^2 \times_\rho F$ satisfaciendo la condición de convergencia está acotado como sigue:

$$\lambda_1 \leq -2H^2 + \frac{1}{\text{Area}(\Sigma)} \int_\Sigma \frac{\Delta\rho}{\rho} d\Sigma \quad \text{y} \quad \lambda_1 \leq -4H^2 - \frac{8\pi(g-1)}{\text{Area}(\Sigma)} + \frac{2}{\text{Area}(\Sigma)} \int_\Sigma \frac{\Delta\rho}{\rho} d\Sigma.$$

Además, la primera igualdad se da si, y solo si, la superficie es totalmente umbilical con $K_\Sigma = K_B$ y $\frac{\Delta\rho}{\rho}$ es constante sobre Σ, mientras que la segunda se da si, y solo si, $\text{Ric} = 2K_B$ sobre Σ y $G - 2K_B$ es constante sobre Σ. Como caso particular, prestamos atención a las funciones warping tales que $\Delta\rho \leq 0$, es decir, a las funciones superarmónicas.

Por otra parte, es interesante también obtener cotas superiores para λ_1 cuando la función warping es convexa y $K_B \leq 0$. Su importancia reside en el hecho de que $B^2 \times_\rho F$ tiene curvatura negativa si, y solo si, ρ es estrictamente convexa y $K_B < 0$.

Para terminar, aplicando todas las estimaciones, deducimos algunas consecuencias relacionadas con la estabilidad y somos capaces de dar ejemplos interesantes que ilustran nuestros resultados. Concretamente, estudiamos tres ejemplos con la intención de mostrar uno en el cual la base sea de curvatura negativa, otro con curvatura positiva y un último ejemplo donde la base sea una superficie llana.

Summary

The study of constant mean curvature surfaces (abbreviated CMC surfaces), and in particular minimal ones, is a very developed and fruitful research field, with classical origins, in the Differential Geometry which currently arouses great interest. This study goes back some 250 years when Lagrange, looking for applications of the variational method that he was developing, analysed the surfaces with the property of minimizing area among all the resulting surfaces by slightly modifying the initial surface.

In this way, the surfaces of constant mean curvature appear as solutions of a variational problem. Specifically, the minimal surfaces (zero mean curvature) are the critical points of the area functional, whereas the constant mean curvature surfaces are also critical points of the same functional when restricted to variations preserving the enclosed volume.

Despite being critical points of a functional, it is not guaranteed that they are minimum for such a functional. A condition which ensures we are in the presence of a minimum is that the second variation of the functional is strictly positive. To weaken such a condition to be non negative yields to the notion of stability. That is, a constant mean curvature surface is stable if the second variation of the area is non negative.

The formula of the second variation of the area functional contains an operator called Jacobi operator or stability operator whose study is closely related with the stability of the surface. To be specific, the first eigenvalue of this operator, namely λ_1, is non negative if and only the surface is stable. This is how such an eigenvalue acquires a great importance in the study of constant mean curvature surfaces. Furthermore, the study of the spectrum of the Jacobi operator, or more generally the Schrödinger operators, has been approached by several authors because it is a subject with its own identity with applications to geometry and physics (see for instance [BB, BGM, SI]).

As for the problem that concerns us, which is the search of sharp estimates for the first eigenvalue of the Jacobi operator for compact surfaces with constant mean curvature, the first work in this direction dates back 1968, in which the minimal case was handled by Simons (see Lemma 6.1.7 in [Si]). In this regard, he found out an estimate for this eigenvalue on any compact minimal hypersurface in the standard sphere. In particular, for minimal surfaces in the 3-sphere he proved that $\lambda_1 = -2$ if the surface is a totally geodesic equator and $\lambda_1 \leq -4$ otherwise. Later on, Wu gave a characterization of minimal Clifford tori by means of the first stability eigenvalue, since he

proved that $\lambda_1 = -4$ if and only if Σ is a minimal Clifford torus in [Wu]. More recently, Perdomo has given a new proof of the same spectral characterization in an alternative way in [Pe]. Summing up, the following result can be formulated in general for hypersurfaces.

Let Σ^n be a compact two-sided minimal hypersurface immersed into the Euclidean sphere $\mathbb{S}^{n+1}$. If λ_1 stands for the first eigenvalue of its Jacobi operator, then

(i) either $\lambda_1 = -n$ and Σ is a totally geodesic equator $\mathbb{S}^n \subset \mathbb{S}^{n+1}$,

(ii) or $\lambda_1 \leq -2n$, with equality if and only if Σ is a minimal Clifford torus $\mathbb{S}^k(\sqrt{k/n}) \times \mathbb{S}^{n-k}(\sqrt{(n-k)/n}) \subset \mathbb{S}^{n+1}$.

Besides that, for the particular case of surfaces, Perdomo got to reach beyond by getting an interesting formula that relates the first eigenvalue λ_1 with the genus of the surface, its area and a simple invariant α. Such a relation was established in Proposition 3.2 in [Pe].

Let Σ^2 be a compact two-sided minimal surface immersed into $\mathbb{S}^3$. If λ_1 stands for the first eigenvalue of its Jacobi operator and f_1 is a positive eigenfunction associated to λ_1, then

$$\lambda_1 = -4 - \frac{1}{\mathrm{Area}(\Sigma)} \left(\alpha + 8\pi(g-1)\right),$$

where $\alpha = \int_\Sigma f_1^{-2} |\nabla f_1|^2 d\Sigma$ and g is the genus of the surface Σ.

In 2005, Alías, Barros and Brasil extended the aforementioned result by Simons, Wu and Perdomo to the case of compact constant mean curvature hypersurfaces immersed into the standard sphere. So, within this framework, they obtained sharp bounds for λ_1 as well as a characterization for Clifford tori of constant mean curvature with suitable radii. In particular, their Theorem 2.2 in [ABB] is stated as follows:

Let Σ^n be a compact two-sided hypersurface of constant mean curvature H immersed into the Euclidean sphere $\mathbb{S}^{n+1}$. If λ_1 stands for the first eigenvalue of its Jacobi operator, then

(i) either $\lambda_1 = -n(1+H^2)$ and Σ is a totally umbilic sphere in $\mathbb{S}^{n+1}$,

(ii) or $\lambda_1 \leq -2n(1+H^2) + \frac{n(n-2)}{\sqrt{n(n-1)}} |H| \max_\Sigma |\phi|$, where ϕ is the total umbilicity tensor of Σ, with equality if and only if

(a) $H = 0$ and Σ is a minimal Clifford torus $\mathbb{S}^k(\sqrt{k/n}) \times \mathbb{S}^{n-k}(\sqrt{(n-k)/n})$ with $k = 1, \ldots, n-1$;

(b) $H \neq 0$, $n = 2$, and Σ^2 is a constant mean curvature Clifford torus $\mathbb{S}^1(r) \times \mathbb{S}^1(\sqrt{1-r^2})$ with $0 < r < 1$, $r \neq \sqrt{1/2}$;

(c) $H \neq 0$, $n \geq 3$, *and* Σ *is a constant mean curvature Clifford torus* $\mathbb{S}^{n-1}(r) \times \mathbb{S}^1(\sqrt{1-r^2})$ *with* $0 < r < \sqrt{(n-1)/n}$*;*

In this way, the case of compact two-sided hypersurfaces of constant mean curvature immersed into the sphere was finished. However, in 2013, Chen and Wang considered the same problem in a more general context which contains the study of constant mean curvature hypersurfaces. In particular, they worked with compact two-sided linear Weingarten hypersurfaces immersed once again into the Euclidean sphere. So, the authors, after imposing some conditions on these hypersurfaces, got an optimal estimate for the first eigenvalue of the Jacobi operator associated to the corresponding variational problem for linear Weingarten hypersurfaces. They also analysed what occurs when the equality holds, concluding that it is attained if and only if the hypersurface is totally umbilic and non-totally geodesic, or it is a certain Clifford torus (see Theorem 1.4 in [CW] for more details).

At this point, we are in a position to explain that our main objective in this memory will be to obtain sharp upper bounds for the first stability eigenvalue of compact two-sided surfaces of constant mean curvature immersed into different 3-dimensional ambient manifolds. Moreover, we will study when the equalities are attained, and we also derive restrictions on the stable surfaces and their full classification if possible.

On the other hand, the natural ambients where studying the problem which includes the sphere and the constant curvature spaces are the homogeneous manifolds. The next step of generalization is to consider Riemannian Killing submersions. Another natural way to extend the ambient is found on warped products.

For this reason, the ambient spaces that we are going to consider are the ones mentioned above: homogeneous manifolds, Riemannian Killing submersions and warped products. So, the thesis is made up of four chapters: the first one is devoted to introduce the main notation and frame the problem; whereas each one of the other three chapters deals with the different cited ambient manifolds, respectively, and they compile our original contribution ([AMO, MO1, MO2, MO3, MO4]).

Once we have presented the context in which this PhD thesis is set, we describe how it is organized.

The **first chapter** is an introductory chapter in which we set up the notation and present the concepts that we need to the understanding of the present memory. In addition, we motivate the problem and we include our first computations.

In this way, we begin the chapter with a brief reminder about geometry of submanifolds. More particularly, we focus on the notation involved when studying a compact two-sided surface Σ^2 immersed into a 3-dimensional Riemannian manifold M^3. Specifically, we include notions such as: a Killing vector field on M; the Riemannian curvature tensor R, the sectional curvature K and the Ricci curvature Ric on M; the shape operator A of Σ associated to its Gauss map N and the mean curvature H of Σ. Furthermore, we recall the Gauss equation in terms of the shape operator, but also in terms of the total umbilicity tensor, being this last concept whereby we define both the totally geodesic surfaces, and the totally umbilic ones. Afterwards, we remind how some differential operators on a Riemannian manifold are defined, we mean: the gradient, divergence, Hessian and Laplacian, since they are needed to enable several calculations along this memory.

Once the main notation is determined, we stress the significance of minimal and constant mean curvature surfaces from a variational viewpoint. In order to justify it, we introduce the normal variation of a compact surface associated to a smooth function over the surface, which is not more that a way to modify smoothly the initial surface on its normal direction, modulated by the fixed function. Consequently, a uniparametric family of compact surfaces can be considered, and for them, two functionals are defined: the area of all the surfaces of the variation; and the enclosed volume between the initial surface and each one of the new surfaces by means of the variation. So, we include the first variation formulae for the area and the volume, and as a result, we prove that minimal surfaces and constant mean curvature surfaces when restricted to variations preserving the enclosed volume are the critical points of the area functional. As a next step, we write the second variation formula for the area which gives rise to define the so-called Jacobi operator or stability operator. Thus, we define a stable surface as that whose second variation of the area is non negative for every normal variation of the surface.

In the last section of the chapter, we remind several well known facts about the spectrum of the Jacobi operator, and above all, we pay attention to the first stability eigenvalue λ_1. We also highlight the relationship between this eigenvalue and the stability, which establishes that

$$\Sigma \textit{ is stable if and only if } \lambda_1 \geq 0.$$

To finish, we compute two general upper bounds for λ_1 when considering a compact two-sided surface of constant mean curvature immersed into an arbitrary 3-dimensional Riemannian manifold. Specifically, such bounds are written as follows:

$$\begin{aligned} \lambda_1 &\leq -2H^2 - \frac{1}{\mathrm{Area}(\Sigma)} \int_\Sigma \mathrm{Ric}(N,N)\, d\Sigma, \quad \text{and} \\ \lambda_1 &\leq -4H^2 - \frac{1}{\mathrm{Area}(\Sigma)} \left(8\pi(g-1) + \int_\Sigma (2\,K_\Sigma + \mathrm{Ric}(N,N))\, d\Sigma \right), \end{aligned}$$

where K_Σ stands for the sectional curvature of the tangent plane to Σ in M. And so, we discuss when the equalities are attained. At this point, we would like to emphasize the relevance of these bounds since they will help us to achieve our aims.

In the **second chapter** we consider compact two-sided surfaces of constant mean curvature immersed into 3-dimensional homogeneous manifolds. Let us remind that

M^3 is homogeneous if for any two points p and q in M, there exists an isometry which maps p into q.

Then, we begin by noting some general facts of these kind of manifolds, and exposing their classification depending on the dimension of the isometry group (6, 4 and 3) when the manifold is additionally simply connected.

In particular, we devote Section 2.2 to the study of simply connected homogeneous manifolds with isometry group of dimension 6, that is, simply connected space forms that we denote by $M^3(c)$ (where c stands for its constant sectional curvature): the standard sphere $\mathbb{S}^3(\sqrt{1/c})$ if $c > 0$; the Euclidean space $\mathbb{R}^3$ if $c = 0$; and the hyperbolic space $\mathbb{H}^3(\sqrt{-1/c})$ if $c < 0$. So, the main goal of this section is to generalize the result by Alías, Barros and Brasil to $M^3(c)$. To do so, we study a more general case since we assume that M has sectional curvature bounded from below by c, i.e. $K \geq c$, our main bounds for compact two-sided surfaces of constant mean curvature being the following:

$$\lambda_1 \leq -2(H^2 + c), \qquad \text{and} \qquad \lambda_1 \leq -4(H^2 + c) - \frac{8\pi(g-1)}{\text{Area}(\Sigma)}.$$

To prove it we use the general bounds presented before. Moreover, the first equality holds if and only if the surface is totally umbilic and $\text{Ric}(N, N) = 2c$, and the second equality holds if and only if the surface has constant Gaussian curvature G, $K_\Sigma = c$ and $\text{Ric}(N, N) = 2c$. From this result, we can derive information about stable compact surfaces of constant mean curvature immersed into these ambient manifolds. Specifically, we conclude the non existence of such surfaces when $H^2 + c > 0$, and otherwise we assert that if $H^2 + c = 0$ then Σ is topologically either a sphere or a torus, and if $H^2 + c < 0$ then $\text{Area}(\Sigma)\,|H^2 + c| \geq 2\pi(g-1)$.

Thus, by particularizing this result to the case $M^3(c)$, we obtain that either $\lambda_1 = -2(H^2 + c)$ and Σ is totally umbilic, or $\lambda_1 \leq -4(H^2 + c)$. Moreover, we analyse the case when the last equality holds, finding that only in the sphere is possible. And so, the surface satisfying the equality is the Clifford torus with constant mean curvature immersed into $\mathbb{S}^3(c)$. As for the stability, we observe that the only stable compact surfaces of constant mean curvature in $M^3(c)$ are the geodesic spheres immersed into $\mathbb{H}^3(c)$.

In Section 2.3 we study simply connected homogeneous manifolds with isometry group of dimension 4. To do this, we start by making a brief historical tour of the main advances related to the study of surfaces into these ambient manifolds. For them, there exists a Riemannian submersion $\Pi : M^3 \longrightarrow B^2(\kappa)$, where $B^2(\kappa)$ is a 2-dimensional simply connected space form of constant curvature κ. Moreover, its fibers are the trajectories of a unit Killing vector field ξ on M, and then the bundle curvature τ of M is the constant which satisfies the equation $\nabla_E \xi = \tau(E \wedge \xi)$ for all vector fields E on M, where $\wedge$ is the vector product in M.

With the above notation, we use $\mathbb{E}^3(\kappa, \tau)$ to refer to such ambient manifolds, where the constants κ and τ play a key role. Indeed, depending on their values, the spaces that can be found are:

Classification of $\mathbb{E}^3(\kappa,\tau)$ depending on κ and τ

	$\kappa > 0$	$\kappa = 0$	$\kappa < 0$
$\tau = 0$	$\mathbb{S}^2(\kappa)\times\mathbb{R}$		$\mathbb{H}^2(\kappa)\times\mathbb{R}$
$\tau \neq 0$	$\mathbb{S}^3_b(\kappa,\tau)$	$Nil_3(\tau)$	$\widetilde{Sl}(2,\mathbb{R})(\kappa,\tau)$

So, once we have introduced the $\mathbb{E}^3(\kappa,\tau)$-spaces, we recall the expressions for their Riemannian curvature tensor, Ricci curvature and sectional curvatures for every tangent plane. Thereafter, we remind that given a two-sided surface, its angle function $\Theta:\Sigma\longrightarrow[-1,1]$ given by $\Theta=\langle N,\xi\rangle$ allows us to introduce two examples of surfaces in $\mathbb{E}^3(\kappa,\tau)$ in a natural way, since they appear when Θ^2 attains its maximum or minimum at any point.

- If $\Theta^2\equiv 1$ the surface is a horizontal slice, that is, $\Sigma = B^2(\kappa)\times\{t\}$.
- If $\Theta\equiv 0$ the surface is a Hopf cylinder over a regular curve $\gamma: I\longrightarrow B^2(\kappa)$, that is, $\Sigma=\Pi^{-1}(\gamma)$. In particular, when the fibers and the curve are closed the surface is a Hopf torus.

Having reached this point, we are able to get upper bounds for λ_1 for compact two-sided surfaces of constant mean curvature immersed into $\mathbb{E}^3(\kappa,\tau)$ and to look for characterizations of the preceding surfaces since they attain the estimates. To proceed with, we consider such bounds for each one of the above five possibilities of spaces contained in the previous overview table. Among these results, it is worth emphasizing in which of those ambients the equalities can be attained.

On the one hand, when Σ is immersed into $\mathbb{S}^2(\kappa)\times\mathbb{R}$, the estimates are given by

$$\lambda_1 \leq -2H^2, \quad \text{and} \quad \lambda_1 < -4H^2-\kappa-\frac{8\pi(g-1)}{\text{Area}(\Sigma)},$$

and the first equality holds if and only if Σ is a horizontal slice. Furthermore, when we consider as ambient manifold the quotient $\mathbb{S}^2(\kappa)\times\mathbb{S}^1$ of the product $\mathbb{S}^2(\kappa)\times\mathbb{R}$, we get the same estimates. Even more, in this case the second equality holds for constant mean curvature Hopf tori. On the other hand, when we study the Berger spheres $\mathbb{S}^3_b(\kappa,\tau)$ we distinguish two cases depending on the sign of $\kappa-4\tau^2$. In particular, whether $\kappa-4\tau^2>0$ we obtain

$$\lambda_1 < -2(H^2+\tau^2), \quad \text{and} \quad \lambda_1 \leq -4H^2-\kappa-\frac{8\pi(g-1)}{\text{Area}(\Sigma)},$$

and the second equality holds if and only if Σ is a constant mean curvature Hopf torus. In the remaining ambient manifolds, we always obtain strict upper bounds for λ_1.

To finish the chapter, we devote a brief part to remind what occurs with the stability. In fact, the full classification of compact stable surfaces of constant mean curvature into $\mathbb{E}^3(\kappa,\tau)$ is well known, the horizontal slices in $\mathbb{S}^2(\kappa)\times\mathbb{R}$ being the only ones.

In the **third chapter** we extend the results in the preceding chapter to the Riemannian Killing submersions, which are a natural generalization of the homogeneous manifolds. Let us remember that

> *A Riemannian submersion* $\Pi : M^3 \longrightarrow B^2(\kappa)$*, where* M *is a Riemannian* 3*-manifold and* $B^2(\kappa)$ *is a Riemannian surface with Gaussian curvature* κ*, is a Riemannian Killing submersion if its fibers are the trajectories of a unit Killing vector field* ξ *on* M*.*

So, we start showing what progresses we can find about surfaces immersed into these ambient manifolds in the literature. Moreover, since the Riemannian Killing submersions are still quite unknown, we devote Section 3.2 to introduce their main tools in detail. Firstly, we remember some general facts of the theory of Riemannian submersions such as: the presentation of the natural tensors $\mathcal{A}$ and $\mathcal{T}$, and the expressions for the Riemannian curvature tensor and the sectional curvatures. After that, we focus on the particular geometry of the Riemannian Killing submersions. To do this, we include the definition of the bundle curvature τ of Π as the unique function on M so that $\nabla_E \xi = \tau(E \wedge \xi)$ for all vector fields E on M. Here, we include the proof of this fact which also generalizes the homogeneous case.

At this point, we fix the notation $\mathcal{M}(\kappa, \tau)$ for referring to a 3-dimensional Riemannian manifold which admits a Riemannian Killing submersion $\Pi : \mathcal{M}(\kappa, \tau) \longrightarrow B^2(\kappa)$ with bundle curvature τ. As is the case in homogeneous manifolds, in these submersions the functions κ and τ are indispensable to understand their geometry. For instance, when both of them are constant $\mathcal{M}(\kappa, \tau)$ reduces to a homogeneous manifold. Once we have established the notation, we compute, in terms of κ and τ, the full description of the Levi-Civita connection on $\mathcal{M}(\kappa, \tau)$ in order to get the sectional curvature of any tangent plane to $\mathcal{M}(\kappa, \tau)$, as well as the Ricci curvature in an arbitrary unit direction.

In Section 3.3, we pay attention to two special surfaces once again, which are nothing but the natural generalization of the aforementioned horizontal slices and Hopf tori in homogeneous manifolds. To introduce them, we consider again the angle function $\Theta = \langle N, \xi \rangle$ in such a way that:

- If $\Theta^2 \equiv 1$ the surface is horizontal. Among their properties it is remarkable that they are totally geodesic, $\lambda_1 = 0$ and consequently they are stable.
- If $\Theta \equiv 0$ the surface is a Hopf torus over a regular closed curve $\gamma : I \longrightarrow B^2(\kappa)$, that is, $\Sigma = \Pi^{-1}(\gamma)$ with closed fibers. This kind of surfaces satisfy being flat and their mean curvature is $H = k_\gamma/2$. Moreover, when κ is constant over γ we know that $\lambda_1 = -4H^2 - \kappa$.

We also include the proof of all these properties. So now, we are willing to show our contribution.

In Section 3.4 we compile our main results. To proceed with their exposition, we must take into account that we distinguish between the case in which $\kappa - 4\tau^2$ is positive or negative. For the former case, that is, if we consider a compact two-sided surface of constant mean curvature into $\mathcal{M}(\kappa,\tau)$ with $\kappa - 4\tau^2 > 0$ we get the following upper bounds

$$\begin{aligned}\lambda_1 &\leq -2H^2 - \frac{1}{\mathrm{Area}(\Sigma)}\int_\Sigma (2\tau^2 - |\nabla\tau|)d\Sigma, \text{ and}\\ \lambda_1 &\leq -4H^2 - \frac{8\pi(g-1)}{\mathrm{Area}(\Sigma)} - \frac{1}{\mathrm{Area}(\Sigma)}\int_\Sigma (\kappa - |\nabla\tau|)d\Sigma.\end{aligned}$$

Additionally, we check that the first equality is attained by horizontal surfaces whereas the second one is attained by Hopf tori with both κ and τ constant over Σ when the Killing submersion has compact fibers. For the latter case, i.e., if $\kappa - 4\tau^2 < 0$, then

$$\begin{aligned}\lambda_1 &\leq -2H^2 - \frac{1}{\mathrm{Area}(\Sigma)}\int_\Sigma (\kappa - 2\tau^2 - |\nabla\tau|)d\Sigma, \text{ and}\\ \lambda_1 &\leq -4H^2 - \frac{8\pi(g-1)}{\mathrm{Area}(\Sigma)} - \frac{1}{\mathrm{Area}(\Sigma)}\int_\Sigma (2\kappa - 4\tau^2 - |\nabla\tau|)d\Sigma.\end{aligned}$$

By contrast, here the first equality holds if and only if the surface is a minimal Hopf torus with $\tau = 0$ over Σ and κ is constant over γ, and the second equality holds for horizontal surfaces with Gaussian curvature equal to κ. Moreover, we have also studied the case when the bundle curvature is constant as a particular and interesting consequence.

So, after obtaining the expected estimates for λ_1, we exhibit some consequences related to the stability of compact surfaces of constant mean curvature immersed into $\mathcal{M}(\kappa,\tau)$ in Section 3.5. Our advances consist of giving some restrictions over the mean curvature H, that is, we obtain upper bounds for H^2. It has been recently proved that if a compact two-sided surface of constant mean curvature into $\mathcal{M}(\kappa,\tau)$ is stable, then it must be either an entire minimal graph or a Hopf torus. We have applied our estimates to these surfaces.

To finish with the chapter, the intention of the last section is to provide non trivial examples of Riemannian Killing submersions and apply to them our results. The first example which worths considering appears when the bundle curvature vanishes, and the second one is defined by means of 3-dimensional spaces which locally admit a doubly warped product and project over a certain surface.

Finally, in the **fourth chapter** we study similar results for compact two-sided surfaces of constant mean curvature immersed into 3-dimensional warped products. In order to remind their definition we consider two Riemannian manifolds $(B, \langle\, ,\, \rangle_B)$ and $(F, \langle\, ,\, \rangle_F)$ and a smooth positive function $\rho : B \longrightarrow (0, +\infty)$ called the warping function. In this way,

$M^3 = B \times_\rho F$ is a warped product if it is the product manifold $B \times F$ endowed with the metric tensor

$$\langle\, ,\, \rangle = \langle\, ,\, \rangle_B + \rho^2 \langle\, ,\, \rangle_F.$$

We introduce them historically and compile results to keep in mind how to calculate the Riemannian curvature tensor of a warped product as well as the Ricci curvature. In the sequel, we deal with the two admissible cases separately: $B \times_\rho F^2$ in which the base is of dimension 1 and the fiber of dimension 2; and on the contrary $B^2 \times_\rho F$ where the base is 2-dimensional and the fiber 1-dimensional.

In Section 4.2 we focus on warped products of the first type $B \times_\rho F^2$, which have been well studied in the literature. So, we devote the first part to give a short review to the bibliography, fix the notation and obtain the sectional curvature of any tangent plane to $B \times_\rho F^2$ and the Ricci curvature in any unit direction.

Afterwards, with the aim of giving sharp upper bounds for λ_1 we recall the definition of the angle function $\Theta = \langle N, \partial_t \rangle$ of a two-sided surface into the warped product. At this point, we work with a well established convergence condition formulated as

$$\min_F K_F \geq \sup_B \{\rho'^2 - \rho\rho''\},$$

where K_F denotes the Gaussian curvature of F. Its geometric interpretation is that the Ricci curvature on M attains its minimum in the direction of ∂_t, i.e., the direction ∂_t is one of the least Ricci curvature on M. Thus, for compact two-sided surfaces of constant mean curvature immersed into a warped product $B \times_\rho F^2$ satisfying the above convergence condition, we get the following estimates:

$$\lambda_1 \leq -2H^2 + \frac{2}{\text{Area}(\Sigma)} \int_\Sigma \frac{\rho''}{\rho} d\Sigma, \qquad \text{and} \qquad \lambda_1 \leq -4H^2 - \frac{8\pi(g-1)}{\text{Area}(\Sigma)} + \frac{4}{\text{Area}(\Sigma)} \int_\Sigma \frac{\rho''}{\rho} d\Sigma.$$

As usual, we analyse what happens when the equalities hold. On the one hand, we see that the first equality holds if and only if the surface is a slice, or otherwise the surface must be totally umbilic contained in a region of M which has constant curvature $K = -\rho''/\rho$. On the other hand, the second equality holds if and only if $K = -\rho''/\rho$ over the surface, and $G - 2K$ is constant over Σ.

We have applied these bounds to two cases which deserve special attention. The first one occurs when the warping function is a solution of the Jacobi equation $\rho'' + c\rho = 0$ for some $c \in \mathbb{R}$, which means that the Ricci curvature of M in the direction ∂_t is constant. So, in this case when we assume that $B \times_\rho F^2$ satisfies the convergence condition we conclude when $c < 0$ a non existence result for stable surfaces, and otherwise, i.e. $c \geq 0$, we prove that the only compact stable surfaces of constant mean curvature are the slices with some restrictions. The second one occurs when the warping function is concave, that is, $\rho'' \leq 0$ and we conclude that the only compact stable surfaces of constant mean curvature into these warped products are totally geodesic topological spheres or flat totally geodesic topological tori.

To finish the section, we give as examples of warped products where our results apply the pseudo-hyperbolic spaces $\mathbb{R} \times_{e^t} F^2$ and $\mathbb{R} \times_{\cosh t} F^2$. So for them, under the convergence condition, we obtain upper bounds for λ_1, and moreover we give the full classification of compact stable surfaces of constant mean curvature immersed into them as follows:

> *The only stable compact two-sided surfaces of constant mean curvature in a pseudo-hyperbolic space with compact fiber are the slices.*

In Section 4.3 we treat the remaining case, that is, we study the second type of warped products $B^2 \times_\rho F$. Since there is a lack of information about them, we begin by setting up the necessary notation and computing the sectional curvature of any tangent plane to $B^2 \times_\rho F$, and the Ricci curvature in any unit direction.

Following the same scheme as the previous section, we remind that given a two-sided surface in $B^2 \times_\rho F$, its angle function is $\Theta = \frac{1}{\rho}\langle N, \partial_t \rangle$. So, for this case, we look for similar assumptions to the above convergence condition. Indeed, we suppose that

$$\mathrm{Ric}_p(v,v) \leq 2K_B, \text{ for all } v \in T_pM,\ |v| = 1, p \in M,$$

where K_B denotes the Gaussian curvature of B. Its geometric interpretation is that the sectional curvatures of M attain a maximum equal to K_B at any point. So, the main result of the section claims that the first stability eigenvalue of compact surfaces of constant mean curvature in $B^2 \times_\rho F$ satisfying the convergence condition is bounded as follows:

$$\lambda_1 \leq -2H^2 + \frac{1}{\mathrm{Area}(\Sigma)} \int_\Sigma \frac{\Delta\rho}{\rho} d\Sigma, \quad \text{and} \quad \lambda_1 \leq -4H^2 - \frac{8\pi(g-1)}{\mathrm{Area}(\Sigma)} + \frac{2}{\mathrm{Area}(\Sigma)} \int_\Sigma \frac{\Delta\rho}{\rho} d\Sigma.$$

Moreover, the first equality holds if and only if the surface is totally umbilic with $K_\Sigma = K_B$ and $\frac{\Delta\rho}{\rho}$ is constant over Σ, whereas the second one holds if and only if $\mathrm{Ric} = 2K_B$ over Σ and $G - 2K_B$ is constant over Σ. As a particular case, we pay attention to the warping functions such as $\Delta\rho \leq 0$, that is, ρ superharmonic.

On the other hand, it is also interesting to achieve upper bounds for λ_1 when the warping function is convex and $K_B \leq 0$, since its significance lies in the fact that $B^2 \times_\rho F$ has negative curvature if and only if ρ is strictly convex and $K_B < 0$.

To finish, by applying all the estimates we derive some consequences related to the stability, and we are able to give interesting examples that illustrate our results. Specifically, we study three examples with the intention of showing one where the base is of negative curvature, another one with positive curvature, and the last one where the base is a flat surface.

Chapter 1

Preliminaries

Abstract. *This first chapter is devoted to introduce the notation, the basic concepts and formulae that will be used throughout this work. In particular, we see that both minimal and constant mean curvature surfaces when restricted to variations preserving the enclosed volume are critical points of the area functional. After that, we define the concept of stability as naturally appears from the second variation of the area functional and, later on it is studied that such a notion can be characterized in terms of the first eigenvalue of the so-called Jacobi operator. To finish, we obtain two general bounds for this eigenvalue for compact two-sided surfaces of constant mean curvature immersed into an arbitrary* 3*-dimensional Riemannian manifold which become very useful to develop this thesis.*

1.1 Notation

In this first part of the chapter, we set up the notation and remind several well known notions and formulae, which will be useful lately, in order to make the memory easier to understand. To proceed with, our intention is to compile it as self-contained as possible.

Let M^3 be a 3-dimensional oriented Riemannian manifold and let Σ^2 be a surface immersed into M. This means that there exists an isometric immersion $\psi : \Sigma \longrightarrow M$, that is, an immersion which preserves the metric tensor. Here and subsequently, we assume that the surface Σ is compact and two-sided, and so a unit normal vector field N globally defined on Σ can be fixed.

We denote by $\mathfrak{X}(M)$ the set of all smooth vector fields on M, which is a module over the ring $\mathcal{C}^\infty(M)$ of all smooth real-valued functions on M. Along this memory, a special type of vector field on Riemannian manifolds will occur: a vector field ξ on M is said to be a *Killing vector field* on M if the Lie derivate of the metric tensor of M vanishes. This definition can be reformulated in terms of the Levi-Civita connection ∇ of M as

$$\langle \nabla_X \xi, Y \rangle + \langle \nabla_Y \xi, X \rangle = 0$$

for all vector fields X and Y on M.

One of the crucial elements we must take into account when we study the geometry of the Riemannian manifold M is its *Riemannian curvature tensor* R, which intuitively, permits us to measure how far the Riemannian manifold is from being Euclidean. As far as the sign of this curvature tensor concerns, it is clear the absence of unanimity in the literature. So, in what follows, the convention that we will consider is the same that the one established by [dC] and [ON2] among others, i.e., $R : \mathfrak{X}(M)^3 \longrightarrow \mathfrak{X}(M)$ is the tensor given by

$$R(X, Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z$$

for every vector fields X, Y and Z on M.

Furthermore, closely related to R it is also considered a simpler real-valued function on the set of all tangent planes to M, which completely determines R. It is the so-called *sectional curvature* K, which was originally introduced by Riemann in [Ri] generalizing the Gaussian curvature for surfaces. So, for a tangent plane P to M, the sectional curvature of P is defined by

$$K(P) = K(v, w) = \frac{\langle R(v, w)v, w \rangle}{\sqrt{|v|^2|w|^2 - \langle v, w \rangle^2}}$$

for any pair of linearly independent vectors v and w spanning the plane P. Specifically, for the tangent plane to Σ its corresponding sectional curvature will be denoted by K_Σ. Let us recall that the Riemannian manifold M is said to have *constant curvature* if its sectional curvature is constant, that is, $K(P) = c$, with $c \in \mathbb{R}$, for every tangent plane P to M.

On the other hand, since the Riemannian curvature tensor is quite complicated, it is helpful to introduce new simpler tensors associated to R, which preserve rather geometric information. In particular, we will use the *Ricci curvature tensor* of M denoted by Ric. So, for every point p in M and for every tangent vectors x and y in T_pM, $\mathrm{Ric}_p(x, y)$ is the trace of the endomorphism of T_pM given by $v \longrightarrow R_p(x, v)y$. It is a symmetric tensor and we can write it with respect to a local orthonormal frame $\{E_1, E_2, E_3\}$ on M as

$$\mathrm{Ric}(X, Y) = \langle R(X, E_1)Y, E_1 \rangle + \langle R(X, E_2)Y, E_2 \rangle + \langle R(X, E_3)Y, E_3 \rangle$$

for every vector fields X and Y on M. Consequently, at each point $p \in M$, for every unit vector $u \in T_pM$, we get

$$\mathrm{Ric}_p(u, u) = K(u, e_1) + K(u, e_2)$$

where $\{e_1, e_2, u\}$ is an orthonormal basis of T_pM.

Now, we denote by A the *shape operator* of the surface Σ, i.e., the self-adjoint operator $A : \mathfrak{X}(\Sigma) \longrightarrow \mathfrak{X}(\Sigma)$ given by

$$A(X) = -\nabla_X N.$$

It is remarkable that the eigenvalues of A, namely κ_1 and κ_2, are the principal curvatures of the surface, and associated to the shape operator, it appears two algebraic invariants: the determinant and the trace, both of them of special importance when working on Differential Geometry. In particular, the latter gives rise to define the *mean curvature* H of the surface by

$$H = \frac{1}{2}\,\mathrm{tr}(A) = \frac{1}{2}(\kappa_1 + \kappa_2).$$

Note that depending on the value of the mean curvature there are some surfaces which are special. We are referring here to the particular cases of minimal surfaces ($H = 0$) and constant mean curvature surfaces. Anyway, they will be motivated in detail along this chapter as the solutions of a variational problem.

It is well known that in this context, the Gauss equation relates the *Gaussian curvature* G of Σ, that is the sectional curvature of Σ, and the sectional curvature K_Σ of the tangent plane to Σ in M in the following way,

$$G = K_\Sigma + \det(A).$$

However, in our work, we will use the Gauss equation rewritten in a most appropriate way as follows

$$|A|^2 = 2(2H^2 + K_\Sigma - G), \tag{1.1}$$

where $|A|^2 = \mathrm{tr}(A^2) = \kappa_1^2 + \kappa_2^2$ is the square of the norm of the shape operator.

When dealing with constant mean curvature surfaces, it is sometimes more convenient to use the so-called traceless second fundamental form instead of working with the second fundamental form or shape operator. This tensor is denoted by $\phi : \mathfrak{X}(\Sigma) \longrightarrow \mathfrak{X}(\Sigma)$ and it is given by $\phi = A - HI$, where I denotes the identity operator on $\mathfrak{X}(\Sigma)$. Besides that, it satisfies $\mathrm{tr}(\phi) = 0$ and,

$$|\phi|^2 = |A|^2 - 2H^2. \tag{1.2}$$

Then, the Gauss equation (1.1) can be translated in terms of ϕ obtaining easily that

$$|\phi|^2 = 2(H^2 + K_\Sigma - G). \tag{1.3}$$

In order to finish this introductory section, we would like to point out the existence of two families of surfaces which are of special significance. They are: the *totally geodesic surfaces*, for which $A = 0$; and the *totally umbilic surfaces*, for which $A = HI$. It is clear that the first ones are a particular case of the second ones. What is more, a surface is totally geodesic if and only if it is minimal and totally umbilic.

Note that the relevance of the tensor ϕ precisely lies in the fact that $|\phi|^2 = 0$ if and only if Σ is totally umbilic. In other words, from (1.2) the total umbilicity of Σ is equivalent to $|A|^2 = 2H^2$. This is the reason why ϕ is also known as the *total umbilicity tensor* of Σ.

1.2 Some differential operators

This section is just a reminder of the definition of some differential operators on a Riemannian manifold since they will be repeatedly used in the computations of this memory. More precisely, we refer the gradient, divergence, Hessian, and Laplacian, which are the natural generalizations of the well known differential operators of vector calculus on $\mathbb{R}^3$. Furthermore, we also include the properties of such operators that we will use later.

Let $(M^n, \langle\ ,\ \rangle)$ be an arbitrary Riemannian manifold with Levi-Civita connection ∇. Within this framework, the following concepts are defined:

- The *gradient* ∇f of a smooth function $f \in \mathcal{C}^\infty(M)$ is the vector field metrically equivalent to the differential $df \in \mathfrak{X}^*(M)$. Thus

$$\langle \nabla f, X\rangle = df(X) = X(f)$$

for all $X \in \mathfrak{X}(M)$. Moreover, for a local orthonormal frame $\{E_1, \ldots, E_n\}$ on M we have

$$\nabla f = \sum_{i=1}^{n} E_i(f)E_i.$$

Observe that if $h : \mathbb{R}\longrightarrow\mathbb{R}$ and $g : M\longrightarrow\mathbb{R}$ are two smooth functions, then the gradient of the composition $h \circ g : M\longrightarrow\mathbb{R}$ is given by

$$\begin{aligned}\nabla(h\circ g) &= \sum_{i=1}^{n} E_i(h\circ g)E_i = \sum_{i=1}^{n} d(h\circ g)(E_i)E_i \\ &= \sum_{i=1}^{n}(h'\circ g)dg(E_i)E_i = (h'\circ g)\sum_{i=1}^{n} E_i(g)E_i \\ &= (h'\circ g)\nabla g. \end{aligned} \tag{1.4}$$

- The *divergence* $\operatorname{div} X$ of a vector field $X \in \mathfrak{X}(M)$ is the smooth function $\operatorname{div} X : M\longrightarrow\mathbb{R}$ defined by

$$\operatorname{div} X(p) = \operatorname{tr}(v\longrightarrow\nabla_v X).$$

In this way, for a local orthonormal frame $\{E_1, \ldots, E_n\}$ on M the divergence is written as follows

$$\operatorname{div} X = \sum_{i=1}^{n} \langle \nabla_{E_i} X, E_i \rangle.$$

It will be useful the following property. If we pick up a smooth function $f : M \longrightarrow \mathbb{R}$ and a vector field $X \in \mathfrak{X}(M)$, then

$$\begin{aligned} \operatorname{div}(fX) &= \sum_{i=1}^{n} \langle \nabla_{E_i}(fX), E_i \rangle = \sum_{i=1}^{n} \langle E_i(f)X + f\nabla_{E_i} X, E_i \rangle \\ &= \sum_{i=1}^{n} \langle E_i(f)E_i, X \rangle + f \operatorname{div} X \\ &= X(f) + f \operatorname{div} X. \end{aligned} \tag{1.5}$$

- The *Hessian* H^f of a smooth function $f \in \mathcal{C}^\infty(M)$ is its second covariant differential, and so it is the symmetric tensor $\mathrm{H}^f : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathbb{R}$ defined by

  $$\mathrm{H}^f(X, Y) = X(Y(f)) - (\nabla_X Y)f = \langle \nabla_X(\nabla f), Y \rangle$$

  for every $X, Y \in \mathfrak{X}(M)$.

- The *Laplacian* Δf of a smooth function $f \in \mathcal{C}^\infty(M)$ is the divergence of its gradient, that is, $\Delta : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M)$ is given by

  $$\Delta f = \operatorname{div}(\nabla f).$$

  Now, in terms of a local orthonormal frame $\{E_1, \ldots, E_n\}$ the Laplacian of f can be written as the trace of its Hessian, i.e.,

  $$\Delta f = \sum_{i=1}^{n} \mathrm{H}^f(E_i, E_i).$$

1.3 Stability

There is a dual purpose along this section: to motivate the relevance of constant mean curvature surfaces, and to introduce their two different notions of stability, playing the stronger one a key role in this memory. With the first aim, as we have announced previously, we study both minimal and constant mean curvature surfaces from the viewpoint of solutions of a variational problem. More particularly, such surfaces can be characterized as the critical points of the area functional. To do so, we must know the first variation of the area functional. On the other hand, we introduce the definition of stable surface by analysing the sign of the second variation of the area functional. And so, without further delay we provide a brief exposition of this variational problem.

Definition 1.3.1. A *variation* of an isometric immersion ψ is a differentiable map $\Psi : \Sigma \times (-\epsilon, \epsilon) \longrightarrow M$ such that:

(i) $\Psi_t : \Sigma \longrightarrow M$, $t \in (-\epsilon, \epsilon)$, given by $\Psi_t(p) = \Psi(p,t)$, $p \in \Sigma$, is an immersion, and

(ii) $\Psi_0 = \psi$ is the original immersion.

The *variational vector field* of Ψ is defined as $\left.\dfrac{\partial \Psi}{\partial t}\right|_{t=0}$ and its normal component is

$$f = \left\langle \left.\frac{\partial \Psi}{\partial t}\right|_{t=0}, N \right\rangle.$$

Moreover, a *normal variation* is a variation with variational vector field parallel to N. Note that every smooth function $f \in \mathcal{C}^\infty(\Sigma)$ induces a normal variation $\Psi_t : \Sigma \longrightarrow M$ of the original immersion ψ with variational normal field fN, given by

$$\Psi_t(p) = \mathrm{Exp}_{\psi(p)}(tf(p)N(p)),$$

where Exp denotes the exponential map in M.

Hence, we can study the variation of the area of Σ with respect to the metric induced on Σ by the immersion Ψ_t. That is, the *area functional* $\mathcal{A} : (-\epsilon, \epsilon) \longrightarrow \mathbb{R}$ is given by

$$\mathcal{A}(t) = \mathrm{Area}(\Sigma_\mathrm{t}) = \int_\Sigma d\Sigma_t,$$

where Σ_t stands for the surface Σ endowed with the metric g_t induced by Ψ_t, i.e., $g_t = \Psi_t^*(\langle\ ,\ \rangle)$, and $d\Sigma_t$ is the area element of that metric on Σ. As well, it can be measured the volume enclosed between Ψ_0 and Ψ_t by means of the *volume functional* $\mathcal{V} : (-\epsilon, \epsilon) \longrightarrow \mathbb{R}$ defined by

$$\mathcal{V}(t) = \int_{[0,t]\times\Sigma} \Psi^*(dV),$$

where dV is the volume element of M. It is said that a variation is a *volume-preserving variation* if $\mathcal{V}(t) = \mathcal{V}(0)$ for all t.

Then, the first variation formulae for area and volume are stated in the following way.

Proposition 1.3.2. *(First Variation Formulae for area and volume) Let $\psi : \Sigma \longrightarrow M$ be a compact two-sided surface immersed into a 3-dimensional Riemannian manifold M. If $f \in \mathcal{C}^\infty(\Sigma)$, then*

(i) $\delta_f \mathcal{A} = \mathcal{A}'(0) = -2 \int_\Sigma f H d\Sigma$.

(ii) $\delta_f \mathcal{V} = \mathcal{V}'(0) = \int_\Sigma f d\Sigma$.

Here $\delta_f \mathcal{A}$ and $\delta_f \mathcal{V}$ denote, respectively, the first variation of the area and the volume under the normal variation determined by the function f.

The first variation formula for the area is due to Lawson (see Chapter I, Theorem 4 in [La]). The general reference for more details about the first variation for the volume function can be found in [BdC, BdCE]. Indeed, the corresponding authors obtained such formulae for the more general case of hypersurfaces.

As a consequence of the above proposition, minimal surfaces are characterized as critical points of the area functional, as well as constant mean curvature surfaces when restricted to volume-preserving variations.

Lemma 1.3.3. *(Characterization of minimal and constant mean curvature surfaces)*

(i) A surface Σ immersed into a 3-dimensional Riemannian manifold M is minimal if and only if $\delta_f\mathcal{A} = 0$ for every smooth function $f \in \mathcal{C}^\infty(\Sigma)$.

(ii) A surface Σ immersed into a 3-dimensional Riemannian manifold M has constant mean curvature H if and only if $\delta_f\mathcal{A} = 0$ for every smooth function $f \in \mathcal{C}^\infty(\Sigma)$ satisfying the additional condition $\int_\Sigma f d\Sigma = 0$.

Proof. (i) It is obvious that $H = 0$ implies $\delta_f\mathcal{A} = 0$ for every $f \in \mathcal{C}^\infty(\Sigma)$.

Conversely, let us assume that $\delta_f\mathcal{A} = 0$ for every $f \in \mathcal{C}^\infty(\Sigma)$, then we must show that $H = 0$. To do so, we suppose that Σ is not minimal, that is, there exists a point $p_0 \in M$ such that $H(p_0) \neq 0$. Now we consider a step function $h \in \mathcal{C}^\infty(\Sigma)$ satisfying $0 \leq h \leq 1$, with $h(p_0) = 1$ and $h = 0$ outside a neighbourhood of p_0 and, by taking $f = hH$ we have

$$0 = \delta_f\mathcal{A} = -2\int_\Sigma fH d\Sigma = -2\int_\Sigma hH^2 d\Sigma \leq 0.$$

From here, we deduce $hH^2 = 0$ over the surface, which leads to a contradiction since $hH^2(p_0) = H^2(p_0) > 0$.

(ii) Clearly, if the surface Σ has constant mean curvature, then $\delta_f\mathcal{A} = 0$ for every $f \in \mathcal{C}^\infty(\Sigma)$ satisfying $\int_\Sigma f d\Sigma = 0$.

Conversely, if $\delta_f\mathcal{A} = 0$ for every $f \in \mathcal{C}^\infty(\Sigma)$ such that $\int_\Sigma f d\Sigma = 0$, let us write the mean curvature as $H = H_0 + (H - H_0)$, where

$$H_0 = \frac{1}{\text{Area}(\Sigma)}\int_\Sigma H d\Sigma.$$

Since $\int_\Sigma (H - H_0) d\Sigma = 0$, by putting $f = H - H_0$ we follow

$$0 = \delta_f\mathcal{A} = -2\int_\Sigma fH d\Sigma = -2\int_\Sigma (H - H_0)H d\Sigma = -2\int_\Sigma (H - H_0)^2 d\Sigma.$$

Thus, we conclude that $H = H_0$ at every point, and so the surface Σ has constant mean curvature. □

Note that the aforementioned condition $\int_\Sigma f d\Sigma = 0$ means that the variations under consideration conserve the enclosed volume.

For such critical points, the sign of the second variation of the area gives rise to introduce the notion of stability. Its expression also goes back to Lawson's work (see Chapter I, Theorem 32 in [La]).

Proposition 1.3.4. *(Second Variation Formula for area) Let $\psi : \Sigma \longrightarrow M$ be a compact two-sided surface with constant mean curvature immersed into a 3-dimensional Riemannian manifold M. If $f \in \mathcal{C}^\infty(\Sigma)$, then*

$$\delta_f^2 \mathcal{A} = \mathcal{A}''(0) = -\int_\Sigma \left(f\Delta f + (|A|^2 + \mathrm{Ric}(N,N)) f^2 \right) d\Sigma,$$

where Δ stands for the Laplacian operator on Σ and $\mathrm{Ric}(N,N)$ denotes the Ricci curvature of M along the unit normal direction to Σ.

It is interesting to rewrite the above formula in terms of the Schrödinger operator $J : \mathcal{C}^\infty(\Sigma) \longrightarrow \mathcal{C}^\infty(\Sigma)$ defined by

$$J = \Delta + |A|^2 + \mathrm{Ric}(N,N), \tag{1.6}$$

in such a way that $\delta_f^2 \mathcal{A} = -\int_\Sigma f J f d\Sigma$. The operator J is the so-called *Jacobi operator* or *stability operator*. Furthermore, the above expression for J can be rewritten in terms of the total umbilicity tensor by using (1.2) so that

$$J = \Delta + 2H^2 + |\phi|^2 + \mathrm{Ric}(N,N)\,.$$

At this point, we are in a position to define the notion of stability, both for minimal and for constant mean curvature surfaces.

Definition 1.3.5. A minimal or a constant mean curvature surface Σ is said to be *stable* if it is a local minimum for the area functional. That is, $\delta_f^2 \mathcal{A} \geq 0$ for every smooth function $f \in \mathcal{C}^\infty(\Sigma)$.

Let us observe that for the particular case of constant mean curvature surfaces the notion of stability that we have just given is also called *strong stability* for some authors. The reason why this name is received is that the condition of being stable can be weakened, giving another natural notion of statibility. To be precise, a constant mean curvature surface Σ is *weakly stable* (or *volume preserving stable*) if $\delta_f^2 \mathcal{A} \geq 0$ for every smooth function $f \in \mathcal{C}^\infty(\Sigma)$ such that $\int_\Sigma f d\Sigma = 0$. Anyway, throughout this memory we focus on the strongest.

1.4 The first stability eigenvalue

In this part of the preliminaries we are going to study the spectrum of the Jacobi operator and its relationship with the stability of the surface. After that, we will get two upper bounds for the first stability eigenvalue λ_1 in the general case of a compact two-sided surface $\psi : \Sigma \longrightarrow M$ with constant mean curvature H immersed into an arbitrary 3-dimensional Riemannian manifold M. In fact, such bounds will constitute the base of the following chapters.

Our criterion establishes that a real number λ is an eigenvalue of J if it satisfies

$$Jf + \lambda f = 0$$

for some smooth function $f \in \mathcal{C}^\infty(\Sigma)$, $f \neq 0$. Note that the spectrum of J

$$\mathrm{Spec}(J) = \{\lambda_1 < \lambda_2 < \lambda_3 < \cdots\} \tag{1.7}$$

consists of an unbounded increasing sequence of eigenvalues λ_i with finite multiplicities m_i. The first stability eigenvalue λ_1 is simple ($m_1 = 1$) and satisfies the following min-max characterization

$$\lambda_1 = \min\left\{\frac{-\int_\Sigma fJf d\Sigma}{\int_\Sigma f^2 d\Sigma} : \quad f \in \mathcal{C}^\infty(\Sigma), f \neq 0\right\}. \tag{1.8}$$

Now, we want to highlight that the first stability eigenvalue plays a key role in the study of the stability of constant mean curvature surfaces and, more generally, the stability of constant mean curvature hypersurfaces. With this aim, note that if Σ is stable, on account of the notion of stability we follow

$$0 \leq \delta^2_{f_i}\mathcal{A} = -\int_\Sigma f_i J f_i d\Sigma = \lambda_i \int_\Sigma f_i^2 d\Sigma$$

for every eigenfunction f_i of the stability operator. This clearly forces $\lambda_i \geq 0$ for every eigenvalue λ_i of J, and combining this fact with (1.7) we can reduce it to say that λ_1 must be non negative. In other words, the stability can be studied in terms of the spectrum since

$$\Sigma \text{ is stable if and only if } \lambda_1 \geq 0.$$

Thus, let us observe that if we are able to give a negative upper bound for λ_1 we will derive that the surface Σ is not stable, and so, this point becomes a useful techniche when looking for non stability.

In what remains, we are going to look for the two announced upper bounds for the first stability eigenvalue.

Let us choose, without loss of generality, a first positive eigenfunction $f_1 \in \mathcal{C}^\infty(\Sigma)$ of the stability operator J (see sections 6 and 7, chapter VI in [CH]). Then, since $Jf_1 + \lambda_1 f_1 = 0$, from the definition of the Jacobi operator (1.6) it is direct that

$$\Delta f_1 = -\left(\lambda_1 + |A|^2 + \mathrm{Ric}(N,N)\right) f_1. \tag{1.9}$$

Now, the idea is to compute the Laplacian of the logarithm of f_1 (following [Pe]) to obtain a very useful integral formula. By definition

$$\Delta \log f_1 = \operatorname{div}(\nabla(\log f_1)),$$

and from the formula (1.4) we know $\nabla(\log f_1) = (\nabla f_1)/f_1$, then substituting into the above equality we have

$$\Delta \log f_1 = \operatorname{div}\left(\frac{\nabla f_1}{f_1}\right).$$

By the formula (1.5) we get

$$\begin{aligned}\operatorname{div}\left(\frac{\nabla f_1}{f_1}\right) &= \frac{1}{f_1}\operatorname{div}(\nabla f_1) + \nabla f_1\left(\frac{1}{f_1}\right)\\ &= \frac{\Delta f_1}{f_1} + \left\langle \nabla\left(\frac{1}{f_1}\right), \nabla f_1\right\rangle\\ &= \frac{\Delta f_1}{f_1} - \frac{\langle \nabla f_1, \nabla f_1\rangle}{f_1^2}.\end{aligned}$$

Combining with (1.9) the last expressions yield

$$\Delta \log f_1 = -\big(\lambda_1 + |A|^2 + \operatorname{Ric}(N,N)\big) - \frac{|\nabla f_1|^2}{f_1^2}.$$

Taking into account Stokes' Theorem we know that $\int_\Sigma \Delta \log f_1 d\Sigma = 0$, and by integrating over Σ we obtain

$$\alpha := \int_\Sigma \frac{|\nabla f_1|^2}{f_1^2} d\Sigma = -\lambda_1 \operatorname{Area}(\Sigma) - \int_\Sigma (|A|^2 + \operatorname{Ric}(N,N)) d\Sigma. \tag{1.10}$$

This last formula will be more useful if we rewrite it in terms of the total umbilicity tensor ϕ as follows

$$\lambda_1 = -2H^2 - \frac{1}{\operatorname{Area}(\Sigma)}\left(\alpha + \int_\Sigma \big(|\phi|^2 + \operatorname{Ric}(N,N)\big)\, d\Sigma\right). \tag{1.11}$$

Taking advantage of this expression we can deduce our two expected upper bounds for λ_1:

(i) On the one hand, keeping in mind that $\alpha \geq 0$ and $|\phi|^2 \geq 0$, from (1.11) we establish the **first upper bound** for λ_1,

$$\lambda_1 \leq -2H^2 - \frac{1}{\operatorname{Area}(\Sigma)} \int_\Sigma \operatorname{Ric}(N,N)\, d\Sigma. \tag{1.12}$$

(ii) On the other hand, if we integrate the Gauss equation (1.3) over Σ, the Gauss-Bonnet Theorem enables us to write

$$\int_\Sigma |\phi|^2 d\Sigma = 2H^2 \operatorname{Area}(\Sigma) + 8\pi(g-1) + \int_\Sigma 2\, K_\Sigma\, d\Sigma,$$

and afterwards, we once again use that $\alpha \geq 0$ and the equality (1.11) immediately leads to the **second upper bound** for λ_1,

$$\lambda_1 \leq -4H^2 - \frac{1}{\operatorname{Area}(\Sigma)}\left(8\pi(g-1) + \int_\Sigma (2\, K_\Sigma + \operatorname{Ric}(N,N))\, d\Sigma\right). \tag{1.13}$$

Remark 1.4.1. Let us analyse briefly what occurs when the equalities are attained in each of the obtained upper bounds for λ_1.

(i) It is direct to check that the first bound for λ_1 holds if and only if $\alpha = 0$ and $|\phi|^2 = 0$. Note that $\alpha = 0$, due to its definition (1.10), means that $\nabla f_1 = 0$ and consequently the eigenfunction f_1 is constant. Moreover, from (1.9) we conclude that $\lambda_1 = -(|A|^2 + \mathrm{Ric}(N, N))$. On the other hand, $|\phi|^2 = 0$ is equivalent to the total umbilicity of the surface Σ. Therefore, $\lambda_1 = -2H^2 - \mathrm{Ric}(N, N)$, being the Ricci curvature on the normal direction constant.

(ii) The second bound for λ_1 holds if and only if $\alpha = 0$.

Remark 1.4.2. The first upper bound can be computed in an alternative way. In order to do this, one must use the constant function $f_1 = 1$ as a test function in the min-max characterization (1.8), and it easily gets that

$$\begin{aligned}\lambda_1 &\leq -2H^2 - \frac{1}{\mathrm{Area}(\Sigma)}\int_\Sigma \mathrm{Ric}(N, N)\, d\Sigma - \frac{1}{\mathrm{Area}(\Sigma)}\int_\Sigma |\phi|^2 d\Sigma \\ &\leq -2H^2 - \frac{1}{\mathrm{Area}(\Sigma)}\int_\Sigma \mathrm{Ric}(N, N)\, d\Sigma.\end{aligned}$$

Chapter 2

Compact CMC surfaces into homogeneous 3-manifolds

Abstract. *Along this chapter, we will obtain sharp upper bounds for the first eigenvalue of the stability operator for compact constant mean curvature surfaces immersed into certain homogeneous 3-manifolds. To proceed with, we begin by introducing these ambient manifolds. Among them, we focus our attention on two cases: the ones with isometry group of dimension* 6 *(space forms), and the ones with isometry group of dimension* 4*. So, for both of them, we give upper bounds for the first eigenvalue, and as an application we derive some consequences for stable compact surfaces of constant mean curvature in such ambient spaces. Moreover, it is remarkable that we also get a characterization of Hopf tori in certain Berger spheres.*

2.1 Introduction to homogeneous 3-manifolds

Throughout this chapter homogeneous spaces play a leading role, since they are the ambient spaces in which we are going to work. We would like to clarify that we just show the part of the theory of homogeneous spaces which is used for our results. Anyway, we give a wide list of references for the reader who is interested in going into details about these spaces.

We start establishing the definition of a homogeneous manifold.

Definition 2.1.1. A Riemannian 3-manifold M^3 is said to be *homogeneous* if for any two points p and q, there exists an isometry that maps p into q.

Therefore, the isometry group of a homogeneous manifold acts transitively on the manifold. Roughly speaking, this definition is the natural way of specifying that the manifold has plenty of isometries. Thus, if M is homogeneous, it looks the same at all the points and the geometric properties which work at one point of M can be extended at every point. For instance, every symmetric space is homogeneous.

It is well known that, among others, the simply connected space forms $\mathbb{R}^3$, $\mathbb{S}^3$ and $\mathbb{H}^3$ are examples of these kind of manifolds. What is more, they are the simplest cases of homogeneous manifolds as they are isotropic too, and as a matter of fact, they are the only simply connected manifolds satisfying to be homogeneous and isotropic at the same time.

Observe that the simply connected space forms have an isometry group of dimension 6, however, by reducing the manifold to be just homogeneous, without the constraint of being isotropic, we can find other homogeneous manifolds with isometry group of dimension 4 or 3. Such manifolds, under the assumption of being simply connected, have been fully classified in [MP], where the authors have proved that they are isometric to 3-dimensional Lie groups with left invariant metrics, apart from the product $\mathbb{S}^2 \times \mathbb{R}$. Although this fact reveals that the homogeneous manifolds are strongly linked to the theory of metric Lie groups (i.e., simply connected 3-dimensional Lie groups endowed with a left invariant metric), this chapter does not seek to cover aspects about Lie groups. In this regard, the reader can consult [MP, Mi].

As far as the full classification of 3-dimensional homogeneous Riemannian manifolds concerns, it is is worth stressing that under the assumption of being simply connected it has been completed. Indeed, several authors have been contributed in such a classification (see [Bo], [Pa], [Sc] and [Th]). In a nutshell, the comprehensive list is the following:

- The spaces with 6-dimensional isometry group, that is, the simply connected space forms: the Euclidean space $\mathbb{R}^3$, the standard sphere $\mathbb{S}^3$ and the hyperbolic space $\mathbb{H}^3$.
- The spaces with 4-dimensional isometry group: the product spaces $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$, the Berger spheres $\mathbb{S}^3_b(\kappa, \tau)$, the Heisenberg space $Nil_3(\tau)$ and the universal cover $\widetilde{Sl(2, \mathbb{R})}(\kappa, \tau)$ of the Lie group $Sl(2, \mathbb{R})(\kappa, \tau)$.
- The spaces with 3-dimensional isometry group are a certain class of Lie groups, and among them it is remarkable the Lie group Sol_3.

Along this chapter, we pay attention to the first two cases, giving much more details. In fact, in Section 2.2 we focus on the study of compact surfaces immersed into the simply connected space forms that we will denote by $M^3(c)$, and in Section 2.3 we deal with compact surfaces into the simply connected homogeneous 3-manifolds with isometry group of dimension 4 which will be denoted by $\mathbb{E}^3(\kappa, \tau)$. Although the last case is not treated in this memory, readers interested in the Lie group Sol_3 can see [DM].

2.2 Compact CMC surfaces into $M^3(c)$

In this section, we focus our attention on simply connected homogeneous 3-manifolds with isometry group of dimension 6. These ambient manifolds do not need a wide introduction, since they are very well known. Indeed, we are referring, as said before, to the simply connected space forms, that is, we are going to deal with complete simply connected 3-dimensional Riemannian manifolds of constant curvature.

Its outstanding role in the Riemannian geometry is owing to the fact that they are the only complete, simply connected Riemannian manifolds with constant curvature (by Cartan's Theorem). In the sequel, as indicated, we denote them by $M^3(c)$. Then, under these assumptions we know that $M^3(c)$ is isometric to:

- the standard sphere $\mathbb{S}^3(\sqrt{1/c})$ if $c > 0$,
- the Euclidean space $\mathbb{R}^3$ if $c = 0$,
- the hyperbolic space $\mathbb{H}^3(\sqrt{-1/c})$ if $c < 0$.

What is more, along the current section, we consider 3-dimensional Riemannian manifolds M with sectional curvature K bounded from below by a constant c, i.e., $K \geq c$, which obviously contains the case of simply connected space forms with $K \equiv c$.

2.2.1 Estimates of λ_1 for compact CMC surfaces into $M^3(c)$

Now, we are in a position to understand that our first objective is to generalize Alías, Barros and Brasil's theorem, which has been shown in the summary, for compact constant mean curvature surfaces immersed into an arbitrary simply connected space form instead of studying just the standard sphere $\mathbb{S}^3$. Even more, as mentioned, we initially consider a more general context: surfaces immersed into a 3-dimensional Riemannian manifold M with $K \geq c$.

As a first approach to achieve our aim, we study estimates for the first eigenvalue of the aforementioned surfaces.

Theorem 2.2.1. *(Theorem 2.1 in [AMO]) Let M be a 3-dimensional Riemannian space with sectional curvature K bounded from below by a constant c, and Σ a compact two-sided surface of constant mean curvature H immersed into M. If λ_1 stands for the first eigenvalue of its Jacobi operator, then*

(i) $\lambda_1 \leq -2(H^2 + c)$, with equality if and only if Σ is totally umbilic in M and the normal direction to Σ is a direction of minimum Ricci curvature of M equals $2c$; and

(ii) $\lambda_1 \leq -4(H^2+c) - \dfrac{8\pi(g-1)}{\text{Area}(\Sigma)}$, *with equality if and only if* Σ *has constant Gaussian curvature,* $K_\Sigma = c$ *and the normal direction to* Σ *is a direction of minimum Ricci curvature of* M *equals* $2c$.

Proof. (i) If $\{E_1, E_2, N\}$ is a local orthonormal frame on M, then the assumption $K \geq c$ implies that

$$\text{Ric}(N,N) = K(N,E_1) + K(N,E_2) \geq 2c.$$

So, from (1.12) one easily gets that

$$\lambda_1 \leq -2H^2 - \frac{1}{\text{Area}(\Sigma)} \int_\Sigma \text{Ric}(N,N)\, d\Sigma \leq -2(H^2+c).$$

Let us analyse the case when the equality holds. The above inequalities become into equalities. As we saw in (i) of Remark 1.4.1, the first equality implies that the surface Σ is totally umbilic, and the second one occurs if $\text{Ric}(N,N) = 2c$. Because of the assumption $K \geq c$, we know that $2c \leq \text{Ric}(X,X)$ for all $X \in \mathfrak{X}(M)$ and so the normal direction to Σ is a direction of minimum Ricci curvature of M.

Conversely, if Σ is totally umbilic, i.e. $|A|^2 = 2H^2$, and $\text{Ric}(N,N) = 2c$ then

$$J = \Delta + |A|^2 + \text{Ric}(N,N) = \Delta + 2H^2 + 2c,$$

and consequently $\lambda_1 = -2(H^2+c)$.

(ii) Because of the assumption $K \geq c$, it is clear that

$$2\,K_\Sigma + \text{Ric}(N,N) \geq 2c + 2c = 4c,$$

and combining this inequality with (1.13) we get that

$$\begin{aligned} \lambda_1 &\leq -4H^2 - \frac{1}{\text{Area}(\Sigma)}\left(8\pi(g-1) + \int_\Sigma (2\,K_\Sigma + \text{Ric}(N,N))\, d\Sigma\right) \\ &\leq -4(H^2+c) - \frac{8\pi(g-1)}{\text{Area}(\Sigma)}. \end{aligned}$$

Let us study what happens when the equality holds. The last inequalities turn into equalities and it is derived that $K_\Sigma = c$ and $\text{Ric}(N,N) = 2c$, so the normal direction to Σ is a direction of minimum Ricci curvature of M as above. Moreover, if (1.13) is an equality, because of Remark 1.4.1 we have $\alpha = 0$, fact which implies that f_1 is constant and from (1.9) we deduce that

$$\lambda_1 + |A|^2 + \text{Ric}(N,N) = \lambda_1 + |A|^2 + 2c = 0,$$

and consequently $|A|^2$ is also constant, then the Gauss equation is written as

$$|A|^2 = 4H^2 + 2\,K_\Sigma - 2G = 4H^2 + 2c - 2G$$

and so the Gaussian curvature G must be constant.

Conversely, if Σ has constant Gaussian curvature, $K_\Sigma = c$ and $\mathrm{Ric}(N,N) = 2c$, then by using once again the Gauss equation, the stability operator is given by

$$J = \Delta + 4H^2 + 2\,K_\Sigma - 2G + \mathrm{Ric}(N,N) = \Delta + 4(H^2 + c) - 2G.$$

Hence $\lambda_1 = -4(H^2 + c) + 2G$ and the Gauss-Bonnet formula

$$G\,\mathrm{Area}(\Sigma) = \int_\Sigma G d\Sigma = 4\pi(1-g)$$

yields to

$$\lambda_1 = -4(H^2 + c) - \frac{8\pi(g-1)}{\mathrm{Area}(\Sigma)}.$$

□

Now, as indicated before, we are going to particularize the result above for 3-dimensional simply connected space forms $M^3(c)$. Let us recall that the following corollary therefore constitutes a generalization of the Alías, Barros and Brasil's theorem for compact surfaces with constant mean curvature into the standard sphere $\mathbb{S}^3(1)$, which have been collected in the Summary (see Theorem 2.2 in [ABB]).

Corollary 2.2.2. *(Corollary 2.2 in [AMO]) Let $M^3(c)$ be a 3-dimensional simply connected space form and Σ a compact two-sided surface of constant mean curvature H immersed into $M^3(c)$. If λ_1 stands for the first eigenvalue of its Jacobi operator, then*

(i) either $\lambda_1 = -2(H^2 + c)$ and Σ is totally umbilic in $M^3(c)$,

(ii) or $\lambda_1 \leq -4(H^2 + c)$, with equality if and only if Σ is a Clifford torus in $\mathbb{S}^3(c)$.

Proof. Since $K \equiv c$, by taking a local orthonormal frame $\{E_1, E_2, N\}$ on $M^3(c)$ we have

$$\mathrm{Ric}(N,N) = K(N,E_1) + K(N,e_2) = 2c,$$

so the normal direction of Σ is a direction of minimum Ricci curvature of $M^3(c)$. Additionally, if Σ is totally umbilic we know from (i) of Theorem 2.2.1 that $\lambda_1 = -2(H^2 + c)$.

Otherwise, by using the fact that the genus of a constant mean curvature non totally umbilic surface in $M^3(c)$ is greater than or equal to 1 (see [Sp]), we obtain from (ii) of Theorem 2.2.1 that

$$\lambda_1 \leq -4(H^2 + c) - \frac{8\pi(g-1)}{\mathrm{Area}(\Sigma)} \leq -4(H^2 + c).$$

It remains to discuss the case when the equality is attained. So, the equality holds if and only if the last two inequalities become into equalities. From the latter it is direct that $g = 1$, whereas from the former we get Σ has constant Gaussian curvature by appling (ii) of Theorem 2.2.1. Thus, by the Gauss-Bonnet formula

$$G\,\mathrm{Area}(\Sigma) = \int_\Sigma G d\Sigma = 4\pi(1-g) = 0,$$

which means that $G = 0$. This occurs only when Σ is a Clifford torus with constant mean curvature immersed into $\mathbb{S}^3(c)$ due to the compacity assumption. □

2.2.2 Stability of compact CMC surfaces into $M^3(c)$

As for the existence of compact stable surfaces of constant mean curvature H immersed into 3-dimensional Riemannian manifolds with $K \geq c$, some consequences can be obtained from Theorem 2.2.1. To discuss about it, we are going to take into account the sign of $H^2 + c$, since this term appears in both bounds of the mentioned result.

Corollary 2.2.3. *(Corollary 2.3 in [AMO]) Let M be a 3-dimensional Riemannian space with sectional curvature K bounded from below by a constant c.*

(i) There exists no stable compact two-sided surface of constant mean curvature H with $H^2 + c > 0$.

(ii) If Σ is a stable compact two-sided surface of constant mean curvature H and $H^2+c = 0$ (that is, $c = 0$ and $H = 0$ or $c < 0$ and $H^2 = -c$), then Σ is topologically either a sphere or a torus.

(iii) If Σ is a stable compact two-sided surface of constant mean curvature H and $H^2+c < 0$ (that is, $c < 0$ and $H^2 < -c$), then

$$\mathrm{Area}(\Sigma)\,|H^2 + c| \geq 2\pi(g-1).$$

Proof. To prove (i) let us take $H^2 + c > 0$, and so as a direct application of the estimate for λ_1 given in item (i) of Theorem 2.2.1 we obtain that

$$\lambda_1 \leq -2(H^2 + c) < 0$$

that is, λ_1 is negative, or equivalently, Σ can not be stable.

Items (ii) and (iii) follow from the estimate for λ_1 given in (ii) of Theorem 2.2.1 taking into account that if Σ is stable, then

$$0 \leq \lambda_1 \leq -4(H^2 + c) - \frac{8\pi(g-1)}{\mathrm{Area}(\Sigma)}.$$

Thus, when $H^2 + c = 0$ we know that $g - 1 \leq 0$, and so the genus of the surface must be zero or one concluding (ii). On the other hand, when $H^2 + c < 0$ it is immediate that $-2\pi(g-1)/\operatorname{Area}(\Sigma) \geq H^2 + c$, or equivalently,

$$\operatorname{Area}(\Sigma)\,|H^2 + c| \geq 2\pi(g-1),$$

as it has been stated in (iii). □

Consequently, in the particular case $K \equiv c$, (i) confirms that there exists no stable compact two-sided surface of constant mean curvature immersed into $\mathbb{S}^3(c)$ and $\mathbb{R}^3$.

For the case $c < 0$, i.e. $\mathbb{H}^3(c)$, we should keep in mind the fact that the stability implies weak stability. Hence, we can look for stable surfaces among the ones which are weakly stable.

Let us remind that Barbosa, Do Carmo and Eschenburg [BdCE] proved that the only compact two-sided surfaces with constant mean curvature H immersed into $M^3(c)$ which are weakly stable are the geodesic spheres. That is, the only candidate surfaces to be stable in $\mathbb{H}^3(c)$ are precisely such spheres. Thus, it is clear that the Jacobi operator for these surfaces is simplified to

$$J = \Delta + |A|^2 + \operatorname{Ric}(N, N) = \Delta + 2c.$$

Therefore, $\lambda_1 = -2c$ for them, and so, as $c < 0$ we get $\lambda_1 > 0$. Hence, the only stable compact surfaces of constant mean curvature in $M^3(c)$ are the geodesic spheres immersed into $\mathbb{H}^3(c)$.

2.3 Compact CMC surfaces into $\mathbb{E}^3(\kappa, \tau)$

2.3.1 Introduction to $\mathbb{E}^3(\kappa, \tau)$

In the sequel, we deal with homogeneous Riemannian 3-manifolds M whose isometry group has dimension 4. Furthermore, we assume that such manifolds are also simply connected unless otherwise indicated. Our claim is to show how the research of these manifolds has developed during the last years without stating much of its results. Nevertheless, we strongly encourage the reader to familiarize with two nice papers by Daniel ([Da1, Da2]) for deepen into these manifolds.

We should outline that the theory of constant mean curvature surfaces immersed into these manifolds is very rich and it has attracted the interest of many geometers along the last decade. Its starting point dates from 2004, when Abresch and Rosenberg discovered a holomorphic quadratic differential for constant mean curvature surfaces immersed into $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ in [AbR1]. After only one year, the same authors were able to extend their results to the more general case of constant mean curvature surfaces into any homogeneous 3-manifold with 4-dimensional isometry group (see [AbR2]). These works gave rise the task of how to write such quadratic differential explicitely, problem which has been solved lately in [BT, FM1].

After the appearance on the scene of such manifolds, we find a large number of contributions enriching this broad theory. Among all the results in this framework, we emphasize only those which are related to our goals. For instance, the existence and the search of relevant surfaces from a geometric viewpoint (not necessarily with constant mean curvature) is one of the main problems to approach. In this direction, there are many results such as the compatibility equations for surfaces in 3-dimensional homogeneous manifolds from Daniel ([Da1]), or the existence of rotationally invariant surfaces of constant mean curvature (see for instance [HH, PR, To]).

Along this memory, we are specially interested in a flat surface called Hopf torus. Torralbo and Urbano have achieved a classification of compact flat surfaces and their result states that

Theorem. *(Th 3.1 in [TU1]) The only flat compact surface in a homogeneous 3-manifold with isometry group of dimension four is the Hopf torus. In particular,*

1. *In the product spaces $B^2(\kappa) \times \mathbb{R}$, the Heisenberg group $Nil_3(\tau)$ and $\widetilde{Sl(2,\mathbb{R})}(\kappa,\tau)$, there are no flat compact surfaces.*
2. *In the Berger spheres, the flat compact surfaces are the Hopf tori.*

On the other hand, the classification of totally umbilic surfaces in different ambient spaces is another classical problem in the Riemannian Geometry. For instance, for space forms we can find such a classification in [Sp]. For the homogeneous manifolds, Souam and Toubiana have been the responsible of studying the totally umbilic surfaces provided they exist. In this case, we are precisely interested in their result of non existence for homogeneous 3-manifolds with non null bundle curvature τ.

Theorem. *(Th 1 in [ST]) There exist no totally umbilic surfaces in a homogeneous 3-manifold with isometry group of dimension four with $\tau \neq 0$. In particular, there are no totally geodesic surfaces.*

With regard to the stability, we find several works in the literature for both the weak stability and the strong one (see for instance [LM, MaPR, NR, So, TU2]). Moreover, a nice survey of Meeks, Pérez and Ros compiling much information about stable constant mean curvature surfaces into homogeneous manifolds is strongly recommended (see [MePR]). Very recently, we have been informed about a new preprint by Lerma and Manzano ([LM]), where the authors study the stability in a more general context, which allows us to derive a very interesting consequence.

To finish, we would like to mention another recent work by Espinar and Trejos, where the authors obtain some upper bounds for the first stability eigenvalue of complete two-sided surfaces of constant mean curvature H with finite Abresch-Rosenberg total curvature and $H^2 + \tau^2 = 0$ (see Theorem 6 in [ET] for going into details).

To go further on the results of constant mean curvature surfaces into the aforementioned homogeneous manifolds we refer the reader to a compilation of them in a Daniel, Hauswirth and Mira's book [DHM], and in order to know some of the open problems in the theory, see the above reference and [FM2].

2.3.2 Basic tools and first formulae

Now, we intend to establish basic facts and include the first computations in the considered ambient spaces. Let us begin pointing out that if M is a simply connected homogeneous 3-manifold with isometry group of dimension 4, there exists a Riemannian submersion $\Pi : M^3 \longrightarrow B^2(\kappa)$, where $B^2(\kappa)$ is a 2-dimensional simply connected space form of constant curvature κ. Moreover, its fibers are the trajectories of a unit Killing vector field ξ on M.

Let us recall that the *fibers* of Π are the inverse image of the points of $B^2(\kappa)$ by means of Π. Moreover, they are geodesics in our case as we are going to check in the next chapter. Now, we remind that a vector field on M is *vertical* if it is always tangent to fibers and *horizontal* if it is always orthogonal to fibers. Thus, the unit Killing field ξ is a vertical vector field.

Having reached this point, we are in a position to show a well known fact for the Levi-Civita connection of the ambients involved.

Lemma 2.3.1. *(Section 2.1 in [Da1]) Let M be a homogeneous 3-manifold with isometry group of dimension 4, and $\Pi : M^3 \longrightarrow B^2(\kappa)$ a Riemannian submersion with unit Killing vector field ξ. Then, there exists a constant τ so that*

$$\nabla_E \xi = \tau (E \wedge \xi) \tag{2.1}$$

for all vector field E on M, where $\wedge$ is the vector product in M.

Notice that the last formula is going to be generalized in the following chapter, and so we include its proof there (see Lemma 3.1). Here, the constant τ is called the *bundle curvature* of M, whose definition is precisely given by the formula (2.1).

Henceforth, according to the standard notation in the literature, we set up the notation $\mathbb{E}^3(\kappa, \tau)$ for a simply connected homogeneous Riemannian 3-manifold with isometry group of dimension 4, with constant Gaussian curvature of the base κ and constant bundle curvature τ.

It is relevant to highlight the role of the constants κ and τ in these homogeneous manifolds. For instance, as the isometry group of $\mathbb{E}^3(\kappa, \tau)$ has dimension 4, it is clear that $\kappa - 4\tau^2 \neq 0$, otherwise $\mathbb{E}^3(\kappa, \tau)$ would be a space form. Also note that κ and τ classify the $\mathbb{E}^3(\kappa, \tau)$-spaces. In fact, depending on their values we can distinguish the different cases:

- When $\tau = 0$, they are the product spaces $B^2(\kappa) \times \mathbb{R}$, that is

  - $\mathbb{S}^2(\kappa) \times \mathbb{R}$ for $\kappa > 0$, and
  - $\mathbb{H}^2(\kappa) \times \mathbb{R}$ for $\kappa < 0$.

- When $\tau \neq 0$, they are:
  - the Berger spheres $\mathbb{S}^3_b(\kappa, \tau)$ for $\kappa > 0$,
  - the Heisenberg group $Nil_3(\tau)$ for $\kappa = 0$, and
  - the universal cover $\widetilde{Sl(2,\mathbb{R})}(\kappa, \tau)$ of the Lie group $Sl(2,\mathbb{R})(\kappa, \tau)$ for $\kappa < 0$.

As usual, it is important to know the Riemannian curvature tensor R. It was originally computed for $\mathbb{E}^3(\kappa, \tau)$ by Daniel and its expression is given by the following result.

Proposition 2.3.2. *(Proposition 2.1 in [Da1]) Let X, Y, Z and W be vector fields on $\mathbb{E}^3(\kappa, \tau)$. Then, the Riemannian curvature tensor R of $\mathbb{E}^3(\kappa, \tau)$ is given by*

$$\begin{aligned} \langle R(X,Y)Z, W\rangle &= (\kappa - 3\tau^2)(\langle X, Z\rangle\langle Y, W\rangle - \langle Y, Z\rangle\langle X, W\rangle) \\ &+ (\kappa - 4\tau^2)(\langle Y, \xi\rangle\langle Z, \xi\rangle\langle X, W\rangle - \langle X, \xi\rangle\langle Z, \xi\rangle\langle Y, W\rangle \\ &+ \langle Y, Z\rangle\langle X, \xi\rangle\langle \xi, W\rangle - \langle X, Z\rangle\langle Y, \xi\rangle\langle \xi, W\rangle). \end{aligned}$$

To prove it, we will make use of the following lemma.

Lemma 2.3.3. *(Lemma 2.8 in [EO]) Let $\Pi : \mathbb{E}^3(\kappa, \tau) \longrightarrow B^2(\kappa)$ be a Riemannian submersion with unit Killing vector field ξ. Let $\{X, Y\}$ be a local orthonormal frame of horizontal vector fields on $\mathbb{E}^3(\kappa, \tau)$, such that $\{X, Y, \xi\}$ is positively oriented. Then*

$$\begin{aligned} K(X,Y) &= \kappa - 3\tau^2, \text{ and} \\ K(X, \xi) &= \tau^2. \end{aligned}$$

The proof of this lemma will be postponed until the Section 3.2 since it involves to introduce specific notation of submersions and we consider that its natural location is the chapter devoted to such ambient spaces (see Lemma 3.2.6).

Proof. **(Proposition 2.3.2)** Let us decompose X, Y, Z and W in their horizontal and vertical parts, respectively

$$X = X^h + \langle X, \xi\rangle\xi, \quad Y = Y^h + \langle Y, \xi\rangle\xi, \quad Z = Z^h + \langle Z, \xi\rangle\xi, \quad W = W^h + \langle W, \xi\rangle\xi,$$

where X^h, Y^h, Z^h and W^h are horizontal. To simplify the notation we will denote by x, y, z and w the real functions defined by

$$x = \langle X, \xi\rangle, \quad y = \langle Y, \xi\rangle, \quad z = \langle Z, \xi\rangle, \quad w = \langle W, \xi\rangle.$$

Then, by using the multilinearity of R, we get that $\langle R(X,Y)Z, W\rangle$ is a sum of 16 terms. Now, note the following:

- The terms in which ξ appears three or four times vanish.
- The terms in which ξ appears twice in positions 1 and 2, or 3 and 4 vanish, that is,

$$R(\xi,\xi)X^h = \nabla_{[\xi,\xi]}X^h - [\nabla_\xi, \nabla_\xi]X^h = 0.$$

- The terms in which ξ appears once vanish (see Section 2.1 to check that the matrix of R in the basis $\{e_1, e_2, \xi\}$ is diagonal).

Hence, R is reduced to the sum of 5 terms,

$$\begin{aligned}\langle R(X,Y)Z,W\rangle &= \langle R(X^h,Y^h)Z^h,W^h\rangle + yw\langle R(X^h,\xi)Z^h,\xi\rangle - yz\langle R(X^h,\xi)W^h,\xi\rangle \\ &+ xz\langle R(Y^h,\xi)W^h,\xi\rangle - xw\langle R(Y^h,\xi)Z^h,\xi\rangle,\end{aligned}$$

and bearing in mind the lemma above we get

$$\begin{aligned}\langle R(X,Y)Z,W\rangle &= (\kappa - 3\tau^2)(\langle X^h,Z^h\rangle\langle Y^h,W^h\rangle - \langle Y^h,Z^h\rangle\langle X^h,W^h\rangle) \\ &+ \tau^2(yw\langle X^h,Z^h\rangle - yz\langle X^h,W^h\rangle + xz\langle Y^h,W^h\rangle - xw\langle Y^h,Z^h\rangle).\end{aligned}$$

To finish, since $\langle X^h, Y^h\rangle = \langle X - x\xi, Y - y\xi\rangle = \langle X, Y\rangle - xy$ we obtain from the last equality that

$$\begin{aligned}\langle R(X,Y)Z,W\rangle &= (\kappa - 3\tau^2)(\langle X,Z\rangle\langle Y,W\rangle - \langle Y,Z\rangle\langle X,W\rangle) \\ &+ (\kappa - 3\tau^2)(-yw\langle X,Z\rangle + yz\langle X,W\rangle - xz\langle Y,W\rangle + xw\langle Y,Z\rangle) \\ &- \tau^2(-yw\langle X,Z\rangle + yz\langle X,W\rangle - xz\langle Y,W\rangle + xw\langle Y,Z\rangle) \\ &= (\kappa - 3\tau^2)(\langle X,Z\rangle\langle Y,W\rangle - \langle Y,Z\rangle\langle X,W\rangle) \\ &+ (\kappa - 4\tau^2)(-yw\langle X,Z\rangle + yz\langle X,W\rangle - xz\langle Y,W\rangle + xw\langle Y,Z\rangle).\end{aligned}$$

□

As a consequence of the expression for the Riemannian curvature, the Ricci curvature of $\mathbb{E}^3(\kappa,\tau)$ and the sectional curvature K for every tangent plane can be computed as follows.

Corollary 2.3.4. *Let $\Pi : \mathbb{E}^3(\kappa,\tau) \longrightarrow B^2(\kappa)$ be a Riemannian submersion with unit Killing vector field ξ.*

(i) The Ricci curvature of $\mathbb{E}^3(\kappa,\tau)$ in a unit direction η is given by

$$\mathrm{Ric}(\eta,\eta) = \kappa - 2\tau^2 + \Theta^2(4\tau^2 - \kappa), \quad \textit{where } \Theta = \langle \eta, \xi\rangle. \tag{2.2}$$

(ii) For every tangent plane P to $\mathbb{E}^3(\kappa,\tau)$ with unit normal η, the sectional curvature K of P is given by

$$K(P) = \tau^2 + \Theta^2(\kappa - 4\tau^2). \tag{2.3}$$

Proof. (i) Let us take a unit vector η and consider an orthonormal basis $\{E_1, E_2, \eta\}$. Now, we extend the vectors E_1, E_2 and η to orthonormal vector fields denoted again by the same letters. Then,

$$\mathrm{Ric}(\eta,\eta) = K(\eta, E_1) + K(\eta, E_2)$$

and by using Proposition 2.3.2 we obtain

$$K(\eta, E_i) = \langle R(\eta, E_i)\eta, E_i\rangle = (\kappa - 3\tau^2) + (\kappa - 4\tau^2)\{-\Theta^2 - \langle E_i, \xi\rangle^2\} \tag{2.4}$$

for $i = 1, 2$. Now, by decomposing ξ in terms of the basis above we have

$$\xi = \langle E_1, \xi\rangle E_1 + \langle E_2, \xi\rangle E_2 + \Theta\eta,$$

and multiplying this expression by ξ and taking into account that $\langle \xi, \xi\rangle = 1$, we obtain that

$$\Theta^2 - 1 = -(\langle E_1, \xi\rangle^2 + \langle E_2, \xi\rangle^2).$$

Then, keeping in mind (2.4) and the last identity we get

$$\begin{aligned}
\mathrm{Ric}(\eta,\eta) &= K(\eta, E_1) + K(\eta, E_2) \\
&= 2(\kappa - 3\tau^2) + (\kappa - 4\tau^2)(-2\Theta^2 - \langle E_1, \xi\rangle^2 - \langle E_2, \xi\rangle^2) \\
&= 2(\kappa - 3\tau^2) + (\kappa - 4\tau^2)(-2\Theta^2 + \Theta^2 - 1) \\
&= 2(\kappa - 3\tau^2) - (\kappa - 4\tau^2)(\Theta^2 + 1) \\
&= \kappa - 2\tau^2 + \Theta^2(4\tau^2 - \kappa).
\end{aligned}$$

(ii) Set a tangent plane P to $\mathbb{E}^3(\kappa,\tau)$ with unit normal η, and consider an orthonormal basis $\{E_1, E_2\}$ spanning P. We extend once again the vectors E_1, E_2 and η to orthonormal vector fields denoted by the same letters. Then $K(P) = K(E_1, E_2)$, and from Proposition 2.3.2 we get

$$K(P) = \langle R(E_1, E_2)E_1, E_2\rangle = (\kappa - 3\tau^2) + (\kappa - 4\tau^2)(-\langle E_1, \xi\rangle^2 - \langle E_2, \xi\rangle^2).$$

Now, observe that we can write ξ in terms of the local orthonormal frame $\{E_1, E_2, \eta\}$ on $\mathbb{E}^3(\kappa,\tau)$ as we have done in (i). Consequently, we know again that

$$\Theta^2 - 1 = -(\langle E_1, \xi\rangle^2 + \langle E_2, \xi\rangle^2),$$

which directly yields

$$\begin{aligned}
K(P) &= (\kappa - 3\tau^2) + (\kappa - 4\tau^2)\{\Theta^2 - 1\} \\
&= \tau^2 + \Theta^2(\kappa - 4\tau^2).
\end{aligned}$$

□

2.3.3 Some examples of surfaces into $\mathbb{E}^3(\kappa,\tau)$

Let us start by introducing the angle function of a two-sided surface inmersed into $\mathbb{E}^3(\kappa,\tau)$. To do so, let us remind that given a surface Σ, its unit normal is denoted by N.

Definition 2.3.5. With the notation above, the *angle function* of a two-sided surface Σ is the smooth function $\Theta : \Sigma \longrightarrow [-1,1]$ given by

$$\Theta = \langle N, \xi \rangle.$$

Note that this function receives this name because of the angle between the normal N and the vertical direction ξ is $\arccos\Theta$.

Remark 2.3.6. When $\tau \neq 0$, we know that $\{p \in \Sigma : \Theta^2(p) = 1\} = \{p \in \Sigma : \xi(p) = \pm N(p)\}$ has empty interior because the distribution $\langle\xi\rangle^{\perp}$ on $\mathbb{E}^3(\kappa,\tau)$ is not integrable (see [TU1]).

At this point, our first purpose is to introduce two examples of surfaces in $\mathbb{E}^3(\kappa,\tau)$ which are of great significance in the remainder of the chapter. They both appear when the square of the angle function attains its maximum or minimum at any point, that is, $\Theta^2 \equiv 1$ or $\Theta \equiv 0$. Let us start with the first case referred.

Definition 2.3.7. A surface Σ immersed into $B^2 \times \mathbb{R}$, with $B^2 = \mathbb{S}^2$ or $B^2 = \mathbb{H}^2$ is said to be a *horizontal slice* when its tangent plane contains only horizontal vectors at any point. That is, $\Sigma = B^2 \times \{t\}$ with $t \in \mathbb{R}$.

Now, we list some interesting **properties of the horizontal slices**:

1. $\boldsymbol{\Theta^2 \equiv 1}$ as expected.

2. They are **totally geodesic** and hence have constant mean curvature $H = 0$.

3. They are **stable** as constant mean curvature surfaces and $\boldsymbol{\lambda_1 = 0}$.

On the other hand, the existence of a Riemannian submersion $\Pi : \mathbb{E}^3(\kappa,\tau) \longrightarrow B^2(\kappa)$ allows us to get the second relevant example of surfaces which are constructed in such a way that the square of the angle function reaches its minimum. They are the Hopf tori which appear in [Pi] for the Hopf fibration, where they play a key role, and they have been studied in several works (e.g. [Bar] and [BFLM]).

Definition 2.3.8. A surface Σ immersed into $\mathbb{E}^3(\kappa,\tau)$ is said to be a *Hopf cylinder* over a regular curve $\gamma : I \longrightarrow B^2(\kappa)$ if it is the total lift of γ, i.e., $\Sigma = \pi^{-1}(\gamma)$. In particular, the surface is a *Hopf torus* if the fibers and the curve are closed.

We also state useful **properties of a Hopf cylinder** $\Sigma = \Pi^{-1}(\gamma)$ as done with the slices.

1. $\boldsymbol{\Theta \equiv 0}$.
2. They are **flat**.
3. Its mean curvature is $\boldsymbol{H = k_\gamma/2}$, where k_γ is the geodesic curvature of γ.
4. As a particular and interesting case for us, we observe that for a Hopf torus $\Pi^{-1}(\gamma)$ such that its mean curvature H is constant, we have $\boldsymbol{\lambda_1 = -4H^2 - \kappa}$.

We delay the proofs of all the properties of slices and Hopf cylinders until the Section 3.3 in the next chapter. The reason why we prefer to include them there, rather than now, is basically that we will generalize these definitions to surfaces into a more general ambient manifold there.

2.3.4 Estimates of λ_1 for compact CMC surfaces into $\mathbb{E}^3(\kappa,\tau)$

Throughout this section, we have a twofold goal: to get upper bounds for λ_1 for compact two-sided surfaces of constant mean curvature immersed into a simply-connected homogeneous 3-manifold with isometry group of dimension 4, which will be written in terms of the geometry of the surface but also the geometry of $\mathbb{E}^3(\kappa,\tau)$; and to look for the characterization of the surfaces which attain the upper bounds, which are precisely the special ones introduced in the previous section. To that end, we need again our two general bounds (1.12) and (1.13), and so, we must know $\mathrm{Ric}(N,N)$ (integrand of the first bound), which has already been obtained in (2.2), and $2\,K_\Sigma + \mathrm{Ric}(N,N)$ (integrand of the second bound) that is easily got from (2.3),

$$2\,K_\Sigma = 2\tau^2 + 2\Theta^2(\kappa - 4\tau^2),$$

and together with (2.2) allows us to conclude

$$2\,K_\Sigma + \mathrm{Ric}(N,N) = \kappa + \Theta^2(\kappa - 4\tau^2). \tag{2.5}$$

Remark 2.3.9. Before giving the results, it is interesting to observe that by reasoning as in Section 1.4 we can obtain a formula for λ_1 which relates it with the mean curvature of the surface, its area and genus, α, and the constants κ and τ. Such an expression is written as

$$\begin{aligned}\lambda_1 &= -4H^2 - \frac{1}{\mathrm{Area}(\Sigma)}\left(\alpha + 8\pi(g-1) + \int_\Sigma (2\,K_\Sigma + \mathrm{Ric}(N,N))\,d\Sigma\right)\\ &= -4H^2 - \kappa - \frac{1}{\mathrm{Area}(\Sigma)}\left(\alpha + 8\pi(g-1) + (\kappa - 4\tau^2)\int_\Sigma \Theta^2 d\Sigma\right).\end{aligned}$$

Now, we analyse case by case the five possibilities of $\mathbb{E}^3(\kappa,\tau)$ which appear in the following overview table depending on the values of κ and τ:

	$\kappa > 0$	$\kappa = 0$	$\kappa < 0$
$\tau = 0$	$\mathbb{S}^2(\kappa) \times \mathbb{R}$		$\mathbb{H}^2(\kappa) \times \mathbb{R}$
$\tau \neq 0$	$\mathbb{S}^3_b(\kappa, \tau)$	$Nil_3(\tau)$	$\widetilde{Sl}(2, \mathbb{R})(\kappa, \tau)$

Case $\tau = 0$

We begin concentrating us on the product spaces $\mathbb{S}^2(\kappa) \times \mathbb{R}$ and $\mathbb{H}^2(\kappa) \times \mathbb{R}$, that is, the bundle curvature vanishes. For instance, when the constant κ is positive we can get the bounds for λ_1 as the following result states.

Theorem 2.3.10. *(Theorem 3.1 in [AMO]) Let Σ be a compact two-sided surface of constant mean curvature H immersed into $\mathbb{S}^2(\kappa) \times \mathbb{R}$. If λ_1 stands for the first eigenvalue of its Jacobi operator, then*

(i) $\lambda_1 \leq -2H^2$, with equality if and only if Σ is a horizontal slice $\mathbb{S}^2(\kappa) \times \{t\}$; and

(ii) $\lambda_1 < -4H^2 - \kappa - \dfrac{8\pi(g-1)}{\text{Area}(\Sigma)}$.

Proof. (i) Since $\tau = 0$, $\kappa > 0$ and $1 - \Theta^2 \geq 0$, from (2.2) we know that

$$\text{Ric}(N, N) = \kappa(1 - \Theta^2) \geq 0,$$

so our first general bound (1.12) directly yields

$$\lambda_1 \leq -2H^2 - \frac{\kappa}{\text{Area}(\Sigma)} \int_\Sigma (1 - \Theta^2) d\Sigma \leq -2H^2.$$

Let us now study the case when the equality holds. It is direct that the last inequality becomes into an equality, and so $\Theta^2 \equiv 1$ which implies that Σ is a totally geodesic horizontal slice.

Conversely, if Σ is a horizontal slice, by its properties we know that it is totally geodesic and $\lambda_1 = 0$. Thus, the equality holds.

(ii) On the other hand, since $\tau = 0$, $\kappa > 0$ and $1 + \Theta^2 \geq 1$ from (2.5) we get

$$2\,K_\Sigma + \text{Ric}(N, N) = \kappa(1 + \Theta^2) \geq \kappa,$$

and by using our second general bound (1.13) it is derived that

$$\begin{aligned} \lambda_1 &\leq -4H^2 - \frac{1}{\text{Area}(\Sigma)} \left(8\pi(g-1) + \kappa \int_\Sigma (1 + \Theta^2) d\Sigma \right) \\ &\leq -4H^2 - \kappa - \frac{8\pi(g-1)}{\text{Area}(\Sigma)}. \end{aligned}$$

Moreover, if the equality holds, then it is immediate that $\Theta \equiv 0$. This last condition implies that Σ would be a Hopf torus but that is not possible because there are no flat compact surfaces in $\mathbb{S}^2(\kappa) \times \mathbb{R}$ as we have seen in Theorem 3.1 in [TU1]. □

Exceptionally, we are going to consider a non simply connected homogeneous 3-manifold with isometry group of dimension 4. We are referring to the product space $\mathbb{S}^2(\kappa)\times\mathbb{S}^1$ which is a quotient of $\mathbb{S}^2\times\mathbb{R}$. The reason whereby this case is worthwhile to study is based on the last proof, because if we analyse it carefully, it is derived that the second equality can be held. What is more, we manage to give a spectral characterization of Clifford tori of constant mean curvature by the first stability eigenvalue as it is claimed below.

Theorem 2.3.11. *(Theorem 3.3 in [AMO]) Let Σ be a compact two-sided surface of constant mean curvature H immersed into $\mathbb{S}^2(\kappa)\times\mathbb{S}^1$. If λ_1 stands for the first eigenvalue of its Jacobi operator, then*

(i) $\lambda_1 \leq -2H^2$, *with equality if and only if Σ is a horizontal slice $\mathbb{S}^2(\kappa)\times\{p\}$; and*

(ii) $\lambda_1 \leq -4H^2-\kappa-\dfrac{8\pi(g-1)}{\text{Area}(\Sigma)}$, *with equality if and only if Σ is a Hopf torus $\gamma\times\mathbb{S}^1$ over a constant curvature closed curve γ.*

Proof. The same reasoning that has been seen in Theorem 2.3.10 works here but in this case for the equality in (ii) we do have Hopf tori with constant mean curvature.

It suffices to check that if Σ is a Hopf torus with constant mean curvature then the equality is attained. Recall that, as indicated in the fourth property of Hopf tori, $\lambda_1 = -4H^2-\kappa$ and so, due to $g=1$ the equality is satisfied. □

Now, let us assume that κ is negative, i.e., we consider $\mathbb{H}^2(\kappa)\times\mathbb{R}$. Here, we observe that the obtained bounds are strict because of the assumption of compactness.

Theorem 2.3.12. *(Theorem 3.4 in [AMO]) Let Σ be a compact two-sided surface of constant mean curvature H immersed into $\mathbb{H}^2(\kappa)\times\mathbb{R}$. If λ_1 stands for the first eigenvalue of its Jacobi operator, then*

(i) $\lambda_1 < -2H^2-\kappa$; *and*

(ii) $\lambda_1 < -4H^2-2\kappa-\dfrac{8\pi(g-1)}{\text{Area}(\Sigma)}$.

Proof. (i) Since $\tau=0$, $\kappa<0$ and $1-\Theta^2\leq 1$, from (2.2) we know that

$$\text{Ric}(N,N)=\kappa(1-\Theta^2)\geq\kappa,$$

so that (1.12) directly yields

$$\lambda_1 \leq -2H^2-\frac{\kappa}{\text{Area}(\Sigma)}\int_\Sigma(1-\Theta^2)d\Sigma \leq -2H^2-\kappa.$$

Moreover, the equality cannot hold; otherwise we would have $\Theta \equiv 0$ which implies that Σ is a cylinder. However, it is not possible because of the assumption of compactness.

(ii) On the other hand, since $\tau = 0$, $\kappa < 0$ and $1 + \Theta^2 \leq 2$ we have from (2.5) that

$$2\,K_\Sigma + \mathrm{Ric}(N, N) = \kappa(1 + \Theta^2) \geq 2\kappa,$$

and consequently (1.13) shows that

$$\begin{aligned} \lambda_1 &\leq -4H^2 - \frac{1}{\mathrm{Area}(\Sigma)}\left(8\pi(g-1) + \kappa \int_\Sigma (1+\Theta^2)d\Sigma\right) \\ &\leq -4H^2 - 2\kappa - \frac{8\pi(g-1)}{\mathrm{Area}(\Sigma)}. \end{aligned}$$

Let us analyse the case when the equality holds. In such case, $\Theta^2 \equiv 1$, so $N = \pm\xi$ which implies that the surface would be a slice $\mathbb{H}^2(\kappa) \times \{t\}$ but it is not compact. □

Case $\tau \neq 0$

We now study the cases where the bundle curvature is non null. To start with, it seems to be natural discussing the simplest case, that is, the Heisenberg group $Nil_3(\tau)$ since $\kappa = 0$.

Theorem 2.3.13. *(Theorem 3.7 in [AMO]) Let Σ be a compact two-sided surface of constant mean curvature H immersed into $Nil_3(\tau)$. If λ_1 stands for the first eigenvalue of its Jacobi operator, then*

(i) $\lambda_1 < -2(H^2 - \tau^2)$*; and*

(ii) $\lambda_1 < -4(H^2 - \tau^2) - \dfrac{8\pi(g-1)}{\mathrm{Area}(\Sigma)}$.

Proof. (i) Since $\kappa = 0$ and $2\Theta^2 - 1 \geq -1$, from (2.2) we know that

$$\mathrm{Ric}(N, N) = 2\tau^2(2\Theta^2 - 1) \geq -2\tau^2,$$

so that (1.12) directly yields

$$\lambda_1 \leq -2H^2 - \frac{2\tau^2}{\mathrm{Area}(\Sigma)}\int_\Sigma (2\Theta^2 - 1)d\Sigma \leq -2H^2 + 2\tau^2.$$

Moreover, if $\lambda_1 = -2H^2 + 2\tau^2$ then the inequality (1.12) turns into an equality, and we have seen in (i) of Remark 1.4.1 that this fact implies that Σ is totally umbilic. However, by Theorem 1 in [ST] we know that there is no totally umbilic surfaces in $Nil_3(\tau)$ and so the equality cannot hold.

(ii) On the other hand, since $\kappa = 0$ and $\Theta^2 \geq 1$, (2.5) leads to

$$2\,K_\Sigma + \mathrm{Ric}(N,N) = -4\tau^2\Theta^2 \geq -4\tau^2,$$

and by (1.13) we get

$$\begin{aligned}\lambda_1 &\leq -4H^2 - \frac{1}{\mathrm{Area}(\Sigma)}\left(8\pi(g-1) - 4\tau^2\int_\Sigma \Theta^2\right)d\Sigma \\ &\leq -4H^2 + 4\tau^2 - \frac{8\pi(g-1)}{\mathrm{Area}(\Sigma)}.\end{aligned}$$

To finish, observe that if the equality holds, then $\Theta^2 \equiv 1$ which is not possible because of Remark 2.3.6. □

Now, let κ be positive. That means $\mathbb{E}^3(\kappa,\tau) = \mathbb{S}^3_b(\kappa,\tau)$ is a Berger sphere. For these homogeneous Riemannian manifolds it is very common to consider two different cases depending on the sign of $\kappa - 4\tau^2$ since the obtained results are quite different (see Proposition 2 in [To] or Theorem 3.6 in [TU1]). In fact, observe that Berger spheres with $\kappa - 4\tau^2 > 0$ are geodesic spheres of the complex projective plane, whereas the ones with $\kappa - 4\tau^2 < 0$ are geodesic spheres of the complex hyperbolic plane. In this way, with the following result we get a characterization of Hopf tori immersed into Berger spheres with $\kappa - 4\tau^2 > 0$.

Theorem 2.3.14. *(Theorem 3.9 in [AMO]) Let Σ be a compact two-sided surface of constant mean curvature H immersed into $\mathbb{S}^3_b(\kappa,\tau)$. If λ_1 stands for the first eigenvalue of its Jacobi operator, then*

(a) if $\kappa - 4\tau^2 > 0$,

(i) $\lambda_1 < -2(H^2 + \tau^2)$; and

(ii) $\lambda_1 \leq -4H^2 - \kappa - \dfrac{8\pi(g-1)}{\mathrm{Area}(\Sigma)}$, with equality if and only if Σ is a Hopf torus over a constant curvature closed curve.

(b) if $\kappa - 4\tau^2 < 0$,

(i) $\lambda_1 < -2H^2 - \kappa + 2\tau^2$; and

(ii) $\lambda_1 < -4H^2 - 2\kappa + 4\tau^2 - \dfrac{8\pi(g-1)}{\mathrm{Area}(\Sigma)}$.

Proof. (a) The proof of (i) uses the fact that in this case we assume $4\tau^2 - \kappa < 0$, and $\Theta^2 \leq 1$, so (2.2) can be bounded as

$$\begin{aligned}\mathrm{Ric}(N,N) &= \kappa - 2\tau^2 + \Theta^2(4\tau^2 - \kappa) \\ &\geq \kappa - 2\tau^2 + 4\tau^2 - \kappa \\ &= 2\tau^2,\end{aligned}$$

which by (1.12) gives

$$\lambda_1 \leq -2(H^2 + \tau^2).$$

Furthermore, if the last equality is attained, then (1.12) becomes into an equality and on account of Remark 1.4.1 it is shown that the surface is totally umbilic. Hence, from Theorem 1 in [ST] we conclude that the equality cannot hold because of the non existence of totally umbilic surfaces in the Berger spheres.

(ii) It is proved by using (2.5) and the assumption $\kappa - 4\tau^2 > 0$ that

$$2\,K_\Sigma + \mathrm{Ric}(N, N) = \kappa + \Theta^2(\kappa - 4\tau^2) \geq \kappa,$$

and by applying (1.13) we finally obtain

$$\lambda_1 \leq -4H^2 - \kappa - \frac{8\pi(g-1)}{\mathrm{Area}(\Sigma)}.$$

Moreover, when the equality holds Σ has to be a Hopf torus because of $\Theta \equiv 0$ and reciprocally, any Hopf torus with constant mean curvature satisfies the equality as we have seen before.

(b) To prove (i), since $4\tau^2 - \kappa > 0$ from (2.2) we get

$$\begin{aligned} \mathrm{Ric}(N, N) &= \kappa - 2\tau^2 + \Theta^2(4\tau^2 - \kappa) \\ &\geq \kappa - 2\tau^2, \end{aligned}$$

and (1.12) yields

$$\lambda_1 \leq -2(H^2 - \tau^2) - \kappa.$$

Once again, equality cannot happen since there exist no totally umbilic surfaces.

(ii) Since $\kappa - 4\tau^2 < 0$, by (2.5) we have

$$2\,K_\Sigma + \mathrm{Ric}(N, N) \geq 2\kappa - 4\tau^2,$$

and the general bound (1.13) allows us to obtain

$$\lambda_1 \leq -4H^2 - 2\kappa + 4\tau^2 - \frac{8\pi(g-1)}{\mathrm{Area}(\Sigma)}.$$

To conclude, observe that if the equality holds, then $\Theta^2 \equiv 1$ but once again it is not possible due to Remark 2.3.6. □

Finally, for the case $\kappa < 0$ we have the following result which is analogous to the part (b) of the last result since $\kappa - 4\tau^2 < 0$ in both of them.

Theorem 2.3.15. *(Theorem 3.10 in [AMO]) Let Σ be a compact two-sided surface of constant mean curvature H immersed into $\widetilde{Sl}(2,\mathbb{R})(\kappa,\tau)$. If λ_1 stands for the first eigenvalue of its Jacobi operator, then*

(i) $\lambda_1 < -2H^2 - \kappa + 2\tau^2$*; and*

(ii) $\lambda_1 < -4H^2 - 2\kappa + 4\tau^2 - \dfrac{8\pi(g-1)}{\text{Area}(\Sigma)}$.

Proof. Since $\kappa < 0$ we get $\kappa - 4\tau^2 < 0$, so the proof is the same that the part (b) of the above theorem. □

In relation to the stability in $\mathbb{E}^3(\kappa,\tau)$ we have to note that firstly Meeks and Pérez [MP] in a more general ambients and secondly Lerma and Manzano [LM] have proved that the only stable compact two-sided surfaces of constant mean curvature into $\mathbb{E}^3(\kappa,\tau)$ are the slices into $\mathbb{S}^2 \times \mathbb{R}$.

Chapter 3

Compact CMC surfaces into Riemannian Killing submersions

Abstract. *In this chapter, our aim is to extend the results in the preceding chapter to Riemannian Killing submersions, which are a natural generalization of the homogeneous manifolds. For this reason, we devote the first part of the chapter to introduce them and compile some formulae. After that, we exhibit two relevant surfaces immersed into them and include a list of their properties. Finally, we expose our results, that is, we find out sharp upper bounds for the first stability eigenvalue for compact constant mean curvature surfaces immersed in a Riemannian Killing submersion. As a consequence, the stability of such surfaces is studied. We also characterize constant mean curvature Hopf tori as the only ones attaining the bound in certain cases.*

3.1 Introduction to Riemannian Killing submersions

In this chapter, we intend to go one step further, in the sense that our ambient spaces constitute a generalization of the homogeneous spaces that we have just studied. Such ambient spaces are called Riemannian Killing submersions, which are Riemannian 3-manifolds which fiber over a Riemannian surface and whose fibers are the trajectories of a unit Killing vector field. So far, they are still quite unknown, even though they are well understood from the viewpoint of the Differential Topology (see for instance [St]).

However, progressively, the study of Riemannian Killing submersions is becoming of much more interest. In particular, we can find several works in which manifolds admitting a Killing field of constant length (or more particularly unitary) have been approached during the last decades (see [ADR, BN, DdL, LR]). But, it is in 2010 when the theory of Killing submersions itself has emerged thanks to a pioneer work by Rosenberg, Souam and Toubiana [RST], and from that moment, the study of surfaces immersed into these ambient manifolds has considerably increased its number of contributions (see for example [EO, Le, MO2]). Nevertheless, there still exists a lack of examples of Riemannian Killing submersions, being remarkable that Souam and Van der Veken in [SVdV] have been able to provide non trivial examples which turn out to be important at the end of the present chapter.

To emphasize the recent establishment of this theory, it is worth pointing out that in 2014, Manzano got the first classification results for Riemannian Killing submersions in [Ma]. Here, it was proved that given a simply connected surface B^2 and a smooth function τ on B, there exists a unique Riemannian Killing submersion $\Pi : M^3 \longrightarrow B^2$, where M is a simply connected and orientable 3-manifold, with bundle curvature τ. More explicitly,

- if B is a topological disk, the submersion is isomorphic to the projection $\Pi_B : B^2 \times \mathbb{R} \longrightarrow B^2$ given by $\Pi_B(p,t) = p$, for some Riemannian metric on $B^2 \times \mathbb{R}$ such that ∂_t is a unit vertical Killing vector field.
- if $B = (\mathbb{S}^2, g)$ for some Riemannian metric g, then depending on whether the value of the total bundle curvature $T = \int_B \tau$ vanishes or not there are two cases:
  - if $T = 0$, the submersion is isomorphic to the projection $\Pi_{\mathbb{S}^2} : \mathbb{S}^2 \times \mathbb{R} \longrightarrow \mathbb{S}^2$ given by $\Pi_{\mathbb{S}^2}(p,t) = p$ for some Riemannian metric on $\mathbb{S}^2 \times \mathbb{R}$ such that ∂_t is a unit vertical Killing vector field.
  - if $T \neq 0$, the submersion is isomorphic to the Hopf fibration $\Pi_{Hopf} : \mathbb{S}^3 \longrightarrow (\mathbb{S}^2, g)$ given by $\Pi_{Hopf}(z,w) = (2z\bar{w}, |z|^2 - |w|^2)$, for some Riemannian metric on $\mathbb{S}^3 \subset \mathbb{C}^2$ such that $\frac{\Pi}{T}(iz, iw)$ is a unit vertical Killing vector field.

Additionally, the author studied some geometric interpretations of the function τ such as its constancy along the fibers of Π, and some relations between the geometry of M and B were established by means of their geodesics and their isometries. As stated previously, Riemannian Killing submersions include the spaces $\mathbb{E}^3(\kappa, \tau)$. Indeed, it is known the following characterization for them.

Theorem. *(Th 5.2 in [Ma]) Let* $\Pi : M^3 \longrightarrow B^2(\kappa)$ *be a Riemannian Killing submersion with Gaussian curvature of the base* κ*. If* M *is homogeneous, then both* κ *and* τ *are constant. In particular,* M *is a* $\mathbb{E}^3(\kappa, \tau)$*-space or its quotient by a vertical translation.*

Very recently, we have been informed about the existence of a preprint by Lerma and Manzano in which they generalize the aforementioned classification by dropping the unitary condition over the Killing vector field (see [LM]). Moreover, in this framework they devote, among other things, to the study of compact stable surfaces with constant mean curvature as we will study later.

3.2 Basic tools and first formulae

Let us start setting up the basic definitions and tools that we are going to use.

Definition 3.2.1. A Riemannian submersion $\Pi : M^3 \longrightarrow B^2$, where M is a Riemannian 3-manifold and B is a Riemannian surface, is said to be a *Riemannian Killing submersion* it its fibers are the trajectories of a unit Killing vector field ξ on M.

Now, let us recall once again the terminology that we use with respect to the horizontal and vertical components of any vector field. Firstly, we remind that a vector field on M is *vertical* if it is always tangent to fibers and *horizontal* if it is always orthogonal to fibers. Moreover, if $E \in \mathfrak{X}(M)$ we denote by E^v and E^h the projections of E onto the subspaces of vertical and horizontal vectors, respectively. Thus, every arbitrary vector field $E \in \mathfrak{X}(M)$ can be decomposed as

$$E = E^h + E^v.$$

In all that remains of the chapter, the letters X and Y stand for horizontal vector fields. From this reminder, we immediate deduce that the vector field ξ satisfies to be a vertical vector field.

At this point, before studying thoroughly the geometry of the Riemannian Killing submersions, let us point out that they are, obviously, a particular case of Riemannian submersions. For that reason, it is worth showing some tools which are indispensable when making computations in this kind of structures. Specifically, O'Neill introduced two relevant tensors which are related to a Riemannian submersion Π (see Section 2 in [ON1]). On the one hand, the second fundamental form of all fibers gives rise to a tensor field $\mathcal{T}$ on M, defined by

$$\mathcal{T}_E F = (\nabla_{E^v}(F^v))^h + (\nabla_{E^v}(F^h))^v,$$

and, on the other hand, by reversing the roles of the horizontal and vertical components the tensor $\mathcal{A}$ is defined by

$$\mathcal{A}_E F = (\nabla_{E^h}(F^h))^v + (\nabla_{E^h}(F^v))^h$$

for all vector fields $E, F \in \mathfrak{X}(M)$. They have some properties such as:

1. At each point, $\mathcal{T}_E$ and $\mathcal{A}_E$ are skew-symmetric linear operators on $\mathfrak{X}(M)$, and both of them reverse the horizontal and vertical subspaces.

2. $\mathcal{T}$ is vertical, i.e., $\mathcal{T}_E = \mathcal{T}_{E^v}$; and $\mathcal{A}$ is horizontal, i.e., $\mathcal{A}_E = \mathcal{A}_{E^h}$.

3. $\mathcal{T}_V W = \mathcal{T}_W V$ for every vertical vector fields V and W; and $\mathcal{A}_X Y = -\mathcal{A}_Y X$ for every horizontal vector fields X and Y.

Moreover, we include here a compilation of the fundamental equations for a Riemannian submersion, without their proof, which are necessary to develop our work.

For instance, in the following result the author shows that the tensor $\mathcal{A}$ is basically the integrability tensor of the horizontal distribution on M.

Lemma 3.2.2. *(Lemma 2 in [ON1]) Let X and Y be horizontal vector fields on M, then*

$$\mathcal{A}_X Y = \frac{1}{2}[X, Y]^v.$$

With respect to the Riemannian curvature tensor, we find in Section 4 of the aforementioned work that the author computes five equations which are numbered from 0 to 4 and such enumeration corresponds to the number of horizontal vector fields on M in $\langle R(E_1, E_2)E_3, E_4\rangle$, being the others vertical. In our memory, we just use the equation containing three horizontal vector fields.

Proposition 3.2.3. *(Theorem 2 in [ON1]) Let X, Y and Z be horizontal vector fields on M, and let V be a vertical vector field on M, then*

$$\langle R(X, Y)Z, V\rangle = \langle(\nabla_Z \mathcal{A})_X Y, V\rangle + \langle \mathcal{A}_X Y, \mathcal{T}_V Z\rangle - \langle \mathcal{A}_Y Z, \mathcal{T}_V X\rangle - \langle \mathcal{A}_Z X, \mathcal{T}_V Y\rangle.$$

To finish with this summary of the general theory of Riemannian submersions, we are interested in the expressions for the sectional curvature.

Proposition 3.2.4. *(Corollary 1 in [ON1]) Let $\Pi : M^3 \longrightarrow B^2(\kappa)$ be a Riemannian submersion with Gaussian curvature of the base κ. If X and Y are unit horizontal vectors at a point of M, and V is a unit vertical vector, and P_{XV} and P_{XY} stand for the tangent planes to M generated by X and V in the first case and X and Y in the second one, then*

1. $K(P_{XV}) = K(X, V) = \langle(\nabla_X \mathcal{T})_V V, X\rangle + |\mathcal{A}_X V|^2 - |\mathcal{T}_V X|^2$, *and*

2. $K(P_{XY}) = K(X, Y) = \kappa - 3|\mathcal{A}_X Y|^2$.

Once we have summarized the general results in which we are interested in for Riemannian submersions, we are going to focus on the particular geometry of Riemannian Killing submersions. With such intention, we begin by introducing a well known result for the Levi-Civita connection of a Riemannian Killing submersion M(see Proposition 2.6 in [EO], Lemma 2.1 in [Ma] and Lemma 2 in [SVdV]) which generalizes the equation (2.1) for $\mathbb{E}^3(\kappa, \tau)$, and here, we include the proof for the sake of completeness.

Lemma 3.2.5. *Let $\Pi : M^3 \longrightarrow B^2$ be a Riemannian Killing submersion with unit Killing vector field ξ. Then, there exists a smooth function $\tau : M \longrightarrow \mathbb{R}$ so that*

$$\nabla_E \xi = \tau(E \wedge \xi) \tag{3.1}$$

for all vector field E on M, where $\wedge$ is the vector product in M.

Proof. On the one hand, since ξ is a Killing field we have

$$\langle \nabla_E \xi, E \rangle = 0,$$

and, on the other hand, since ξ is a unit vector field on M we also obtain that

$$\langle \nabla_E \xi, \xi \rangle = \frac{1}{2} E(\langle \xi, \xi \rangle) = 0,$$

and so, $\nabla_E \xi$ is orthogonal to E and ξ. Hence, it is directly concluded that there exists a unique function $\tau_E : M \longrightarrow \mathbb{R}$ such that

$$\nabla_E \xi = \tau_E(E \wedge \xi).$$

Note that τ_E just depends on the horizontal part of E. Consequently, it remains to prove that τ_E does not depend on the vector field E, i.e., if we assume E to be a unit horizontal vector field, and we complete it to a positively oriented local orthonormal frame $\{E, F, \xi\}$ on M, we must check $\tau_E = \tau_F$. Since ξ is a Killing field, we then conclude

$$\tau_F = \tau_F \langle F \wedge \xi, E \rangle = \langle \nabla_F \xi, E \rangle = -\langle \nabla_E \xi, F \rangle = -\tau_E \langle E \wedge \xi, F \rangle = \tau_E.$$

□

Notice that the function τ is unique and it is called the *bundle curvature* of Π, whose definition is basically given by the formula of the lemma above. Moreover, as a consequence of this result we know that $\nabla_\xi \xi = 0$ which implies that the fibers of the submersion are geodesics of M.

From now on, we fix the notation $\mathcal{M}(\kappa, \tau)$ for referring to a 3-dimensional Riemannian Killing submersion such that if $\Pi : \mathcal{M}(\kappa, \tau) \longrightarrow B$ is a Riemannian submersion, it has a unit Killing vertical field ξ, the Gaussian curvature of the base B is κ and the bundle curvature is τ.

In particular, there are some important cases depending on the functions κ and τ. For instance, if both of these functions are constant, then $\mathcal{M}(\kappa, \tau)$ is a homogeneous Riemannian 3-manifold and when $\tau = 0$, $\mathcal{M}(\kappa, \tau)$ is a product $B^2 \times \mathbb{R}$ or $B^2 \times \mathbb{S}^1$, where B^2 is an arbitrary Riemannian surface.

Now, let us see how the Levi-Civita connection works on $\mathcal{M}(\kappa, \tau)$ as a first step to familiarize us with the computations into this ambient manifolds. To do so, let us consider a positively oriented local orthonormal frame $\{X, Y, \xi\}$ on $\mathcal{M}(\kappa, \tau)$. Then, for example to compute $\nabla_X X$ we are going to decompose it as follows

$$\nabla_X X = \langle \nabla_X X, X \rangle X + \langle \nabla_X X, Y \rangle Y + \langle \nabla_X X, \xi \rangle \xi, \tag{3.2}$$

where $\langle \nabla_X X, X\rangle = 0$ because $\langle X, X\rangle = 1$, and taking into account (3.1) we have

$$\langle \nabla_X X, \xi\rangle = -\langle X, \nabla_X \xi\rangle = -\tau\langle X, X\wedge\xi\rangle = 0.$$

So, by replacing into (3.2) we get

$$\nabla_X X = \langle \nabla_X X, Y\rangle Y.$$

Analogously we can compute $\nabla_Y X$ by

$$\begin{aligned}\nabla_Y X &= \langle \nabla_Y X, X\rangle X + \langle \nabla_Y X, Y\rangle Y + \langle \nabla_Y X, \xi\rangle \xi\\ &= -\langle X, \nabla_Y Y\rangle Y - \langle X, \nabla_Y \xi\rangle \xi\\ &= -\langle X, \nabla_Y Y\rangle Y - \tau\langle X, Y\wedge\xi\rangle \xi\\ &= -\langle X, \nabla_Y Y\rangle Y - \tau\xi.\end{aligned}$$

On the other hand, bearing in mind that $\nabla_\xi \xi = 0$, we can calculate $\nabla_\xi X$ as

$$\begin{aligned}\nabla_\xi X &= \langle \nabla_\xi X, X\rangle X + \langle \nabla_\xi X, Y\rangle Y + \langle \nabla_\xi X, \xi\rangle \xi\\ &= -\langle X, \nabla_\xi Y\rangle Y - \langle X, \nabla_\xi \xi\rangle \xi\\ &= -\langle X, \nabla_\xi Y\rangle Y.\end{aligned}$$

Following these steps, we can easily obtain $\nabla_X Y$, $\nabla_Y Y$ and $\nabla_\xi Y$. Furthermore, due to (3.1) we also obtain $\nabla_X \xi$, $\nabla_Y \xi$ and $\nabla_\xi \xi$. So, we can write

$$\begin{array}{lll}\nabla_X X = aY, & \nabla_Y X = -bY - \tau\xi, & \nabla_\xi X = -cY,\\ \nabla_X Y = -aX + \tau\xi, & \nabla_Y Y = bX, & \nabla_\xi Y = cX,\\ \nabla_X \xi = -\tau Y, & \nabla_Y \xi = \tau X, & \nabla_\xi \xi = 0,\end{array}$$

where

$$a = \langle \nabla_X X, Y\rangle, \;\; b = \langle \nabla_Y Y, X\rangle \;\text{ and }\; c = \langle \nabla_\xi Y, X\rangle.$$

Now, we point out the fact that we have no restriction to make special choices of vector fields when computing with tensor equations. Even more, as it is observed in [ON1], it is natural to assume the vector fields X and Y on M to be *basic*, that is, they are horizontal and Π-related to unit vector fields $X_\star$ and $Y_\star$ on B. In fact, reciprocally every unit vector fields $X_\star$ and $Y_\star$ have a unique basic horizontal lift X and Y, respectively. Recall that $d\Pi$ restricted to the subspace of horizontal vectors is an isometry because Π is a Riemannian submersion. So, under this additional assumption we get that the horizontal component of the Lie Brackets $[\xi, X]$ and $[\xi, Y]$ must vanish (since ξ is Π-related to the zero vector field). Thus,

$$0 = [\xi, X] = \nabla_\xi X - \nabla_X \xi = -cY + \tau Y$$

and

$$0 = [\xi, Y] = \nabla_\xi Y - \nabla_Y \xi = cX - \tau X = 0,$$

and consequently c should be equal to τ. So, we finally get the full description of the Levi-Civita connection as follows

$$\begin{aligned} \nabla_X X &= aY, & \nabla_Y X &= -bY - \tau\xi, & \nabla_\xi X &= -\tau Y, \\ \nabla_X Y &= -aX + \tau\xi, & \nabla_Y Y &= bX, & \nabla_\xi Y &= \tau X, \\ \nabla_X \xi &= -\tau Y, & \nabla_Y \xi &= \tau X, & \nabla_\xi \xi &= 0. \end{aligned} \tag{3.3}$$

Taking advantage of the above expressions, we think of obtaining the sectional curvature $K(P)$ of any tangent plane P to $\mathcal{M}(\kappa,\tau)$ as our following task. In order to proceed with, observe that $\mathcal{M}(\kappa,\tau)$ has two independent horizontal directions and one vertical direction at any point, so it is clear that any arbitrary tangent plane P always contains at least one horizontal direction. That is,

$$K(P) = K(X,E)$$

where $\{X,E\}$ is an orthonormal basis spanning the plane P with X horizontal.

Now, as a previous step we are going to compute the sectional curvature for horizontal planes which are orthogonal to ξ and vertical planes which contain the direction ξ. Notice that the following lemma was also stated in the preceding chapter for the particular case of $\mathbb{E}^3(\kappa,\tau)$ (see Lemma 2.3.3). We include here the proof.

Lemma 3.2.6. *(Lemma 2.8 in [EO]) Let $\mathcal{M}(\kappa,\tau)$ be a 3-dimensional Riemannian Killing submersion with unit Killing vector field ξ. Let $\{X,Y\}$ be a local orthonormal frame of horizontal vector fields on $\mathcal{M}(\kappa,\tau)$, such that $\{X,Y,\xi\}$ is positively oriented. Then*

$$K(X,Y) = \kappa - 3\tau^2, \text{ and} \tag{3.4}$$

$$K(X,\xi) = \tau^2. \tag{3.5}$$

Proof. On the one hand, from Proposition 3.2.4 we have

$$K(X,Y) = \kappa - 3|\mathcal{A}_X Y|^2, \tag{3.6}$$

where $\mathcal{A}_X Y = 1/2[X,Y]^v$ because of Lemma 3.2.2. Since $\mathcal{A}_X Y$ is vertical, taking in mind the expressions given in (3.3) we have

$$\begin{aligned} \langle \mathcal{A}_X Y, \xi\rangle &= \frac{1}{2}\langle [X,Y]^v, \xi\rangle \\ &= \frac{1}{2}\langle [X,Y], \xi\rangle \\ &= \frac{1}{2}(\langle \nabla_X Y, \xi\rangle - \langle \nabla_Y X, \xi\rangle) \\ &= \frac{1}{2}(\langle -aX + \tau\xi, \xi\rangle - \langle -bY - \tau\xi, \xi\rangle) = \tau, \end{aligned}$$

and so, $\mathcal{A}_X Y = \tau\xi$. Hence, from (3.6) we get $K(X,Y) = \kappa - 3\tau^2$.

On the other hand, by using again Proposition 3.2.4, we know that

$$K(X,\xi) = \langle (\nabla_X \mathcal{T})_\xi \xi, X\rangle + |\mathcal{A}_X\xi|^2 - |\mathcal{T}_\xi X|^2. \tag{3.7}$$

The definition of the fundamental tensors $\mathcal{A}$ and $\mathcal{T}$ as well as (3.3) yields

$$\mathcal{A}_X\xi = (\nabla_X\xi)^h = (-\tau Y)^h = -\tau Y,$$

and

$$\mathcal{T}_\xi X = (\nabla_\xi X)^v = (-\tau Y)^v = 0.$$

Finally, by the product rule we have

$$\begin{aligned}(\nabla_X\mathcal{T})_\xi\xi &= \nabla_X \mathcal{T}_\xi\xi - \mathcal{T}_{\nabla_X\xi}\xi - \mathcal{T}_\xi\nabla_X\xi \\ &= \nabla_X(\nabla_\xi\xi)^h + \tau\mathcal{T}_Y\xi + \mathcal{T}_\xi(\tau Y) \\ &= \tau(\nabla_\xi Y)^v = 0,\end{aligned}$$

and so, substituting the last expressions in (3.7) we deduce that $K(X,\xi) = |\mathcal{A}_X\xi|^2 = \tau^2$. □

We are now ready to get the general formula of the sectional curvature for every tangent plane P to $\mathcal{M}(\kappa,\tau)$.

Proposition 3.2.7. *([MO2]) Let $\mathcal{M}(\kappa,\tau)$ be a 3-dimensional Riemannian Killing submersion with unit Killing vector field ξ. Then, for every tangent plane P to $\mathcal{M}(\kappa,\tau)$ with unit normal η, the sectional curvature K of P is given by*

$$K(P) = \tau^2 + \Theta^2(\kappa - 4\tau^2) - 2\Theta\sqrt{1-\Theta^2}X(\tau), \tag{3.8}$$

where $\Theta = \langle \eta, \xi\rangle$ and X is a unit horizontal vector contained in P.

Proof. Let us fix a tangent plane P to $\mathcal{M}(\kappa,\tau)$ and consider an orthonormal basis $\{X, E\}$ spanning P. First, we extend the vectors X and E to orthonormal vector fields and we denote these again by X and E.

Now, if we complete $\{X\}$ to a positively oriented local orthonormal frame $\{X, Y, \xi\}$ on $\mathcal{M}(\kappa,\tau)$, where Y is a horizontal vector field, we can write E in terms of such a frame as

$$E = \langle Y, E\rangle Y + \langle \xi, E\rangle\xi. \tag{3.9}$$

Thus, by using this decomposition of E and the symmetries of R we have

$$\begin{aligned}K(X,E) &= \langle R(X,E)X, E\rangle \\ &= \langle E, Y\rangle^2 K(X,Y) + \langle E,\xi\rangle^2 K(X,\xi) + 2\langle E,Y\rangle\langle E,\xi\rangle\langle R(X,Y)X,\xi\rangle.\end{aligned}$$

Now, by multiplying by E the expression (3.9) we get $\langle E,Y\rangle^2+\langle E,\xi\rangle^2=1$ and combining this fact with (3.4) and (3.5) we have

$$K(X,E)=\kappa-3\tau^2-\langle E,\xi\rangle^2(\kappa-4\tau^2)+2\langle E,Y\rangle\langle E,\xi\rangle\langle R(X,Y)X,\xi\rangle. \tag{3.10}$$

From Proposition 3.2.3 the curvature $\langle R(X,Y)X,\xi\rangle$ can be computed by using the tensors $\mathcal{T}$ and $\mathcal{A}$ as follows

$$\langle R(X,Y)X,\xi\rangle=\langle(\nabla_X\mathcal{A})_XY,\xi\rangle+\langle\mathcal{A}_XY,\mathcal{T}_\xi X\rangle-\langle\mathcal{A}_YX,\mathcal{T}_\xi X\rangle-\langle\mathcal{A}_XX,\mathcal{T}_\xi Y\rangle,$$

where $\mathcal{T}_\xi X=(\nabla_\xi X)^v=0$ because of (3.3), and $\mathcal{A}_XX=0$ since $\mathcal{A}$ has the alternation property for horizontal vector fields. Therefore, the above expression reduces to

$$\langle R(X,Y)X,\xi\rangle=\langle(\nabla_X\mathcal{A})_XY,\xi\rangle.$$

On the other hand, by using again (3.3) and the product rule we get

$$\begin{aligned}(\nabla_X\mathcal{A})_XY &= \nabla_X(\mathcal{A}_XY)-\mathcal{A}_{\nabla_XX}Y-\mathcal{A}_X(\nabla_XY)\\ &= \nabla_X(\tau\xi)-a\mathcal{A}_YY+a\mathcal{A}_XX-\tau\mathcal{A}_X\xi\\ &= X(\tau)\xi+\tau\nabla_X\xi-\tau(-\tau Y)\\ &= X(\tau)\xi.\end{aligned}$$

Summing up, substituting into (3.10) we find that the sectional curvature is given by

$$K(X,E)=\kappa-3\tau^2-\langle E,\xi\rangle^2(\kappa-4\tau^2)+2\langle E,Y\rangle\langle E,\xi\rangle X(\tau). \tag{3.11}$$

Now, we consider the unit normal vector η to the plane P, and we also extend it to a unit vector field that we denote by the same letter η. So $\{X,E,\eta\}$ is another local orthonormal frame on $\mathcal{M}(\kappa,\tau)$ which allows us to decompose ξ as

$$\xi=\langle E,\xi\rangle E+\langle\eta,\xi\rangle\eta=\langle E,\xi\rangle E+\Theta\eta.$$

From such a decomposition, on the one hand, by multiplying by ξ we get $\langle E,\xi\rangle^2=1-\Theta^2$, and on the other hand,

$$\langle E,Y\rangle\langle E,\xi\rangle=\langle\langle E,\xi\rangle E,Y\rangle=\langle\xi-\Theta\eta,Y\rangle=-\Theta\langle\eta,Y\rangle.$$

Then, replacing these expressions in (3.11) we have

$$\begin{aligned}K(X,E) &= \kappa-3\tau^2-(1-\Theta^2)(\kappa-4\tau^2)-2\Theta\langle\eta,Y\rangle X(\tau)\\ &= \tau^2+\Theta^2(\kappa-4\tau^2)-2\Theta\langle\eta,Y\rangle X(\tau).\end{aligned} \tag{3.12}$$

Hence, it remains just to check that $\langle\eta,Y\rangle=\sqrt{1-\Theta^2}$. To do so, we can suppose that $\langle\eta,Y\rangle\geq 0$ without loss of generality. Let us put η in terms of $\{X,Y,\xi\}$ in the following way

$$\eta=\langle\eta,Y\rangle Y+\Theta\xi$$

and so,

$$\langle \eta, \eta \rangle = 1 = \langle \eta, Y \rangle^2 + \Theta^2.$$

Hence, Eq. (3.12) is reduced to

$$K(X, E) = \tau^2 + \Theta^2(\kappa - 4\tau^2) - 2\Theta\sqrt{1-\Theta^2}X(\tau).$$

□

We would like to point out that the sectional curvature was also computed by Manzano in Lemma 5.1 in [Ma] in an alternative and independent way. In particular, the formula that he obtained is given by

$$K(P) = \Theta^2(\kappa - 3\tau^2) + (1 - \Theta^2)\tau^2 - 2\Theta\langle \eta \wedge \xi, \nabla\tau \rangle.$$

To finish this section, we derive the expression for the Ricci curvature tensor in a unit direction as a consequence of the result above.

Corollary 3.2.8. *([MO2]) Let $\mathcal{M}(\kappa, \tau)$ be a 3-dimensional Riemannian Killing submersion with unit Killing vector field ξ. Then, the Ricci curvature of $\mathcal{M}(\kappa, \tau)$ in a unit direction η is given by*

$$\text{Ric}(\eta, \eta) = \kappa - 2\tau^2 - \Theta^2(\kappa - 4\tau^2) + 2\Theta\sqrt{1-\Theta^2}X(\tau), \tag{3.13}$$

where $\Theta = \langle \eta, \xi \rangle$, and X is a unit horizontal vector which is orthogonal to η.

Proof. Let us fix a unit vector η, and consider a unit horizontal vector X such that $X \perp \eta$. Now, we complete $\{X\}$ to an orthonormal basis $\{X, Y, \xi\}$, and so η can be written in terms of this basis as we did in the previous proof by

$$\eta = \langle \eta, Y \rangle Y + \Theta\xi.$$

Then, we have

$$\begin{aligned} \text{Ric}(\eta, \eta) &= \langle R(X, \eta)X, \eta \rangle + \langle R(Y, \eta)Y, \eta \rangle + \langle R(\xi, \eta)\xi, \eta \rangle \\ &= K(X, \eta) + \Theta^2\langle R(Y, \xi)Y, \xi \rangle + \langle \eta, Y \rangle^2 \langle R(\xi, Y)\xi, Y \rangle, \end{aligned}$$

and, due to $\langle \eta, \eta \rangle = 1 = \langle \eta, Y \rangle^2 + \Theta^2$ and the symmetries of R we get

$$\text{Ric}(\eta, \eta) = K(X, \eta) + K(Y, \xi).$$

From (3.5) we know $K(Y, \xi) = \tau^2$, and $K(X, \eta)$ can be calculated using (3.11) replacing E by η, that is,

$$K(X, \eta) = \kappa - 3\tau^2 - \Theta^2(\kappa - 4\tau^2) + 2\langle \eta, Y \rangle \Theta X(\tau).$$

Thus, we finally obtain

$$\begin{aligned} \text{Ric}(\eta, \eta) &= \kappa - 2\tau^2 - \Theta^2(\kappa - 4\tau^2) + 2\langle \eta, Y \rangle \Theta X(\tau) \\ &= \kappa - 2\tau^2 - \Theta^2(\kappa - 4\tau^2) + 2\Theta\sqrt{1-\Theta^2}X(\tau). \end{aligned}$$

□

3.3 Some examples of surfaces into Riemannian Killing submersions

Henceforth, we consider a two-sided surface $\psi : \Sigma \longrightarrow \mathcal{M}(\kappa, \tau)$ with mean curvature H immersed into a Riemannian Killing submersion $\mathcal{M}(\kappa, \tau)$. If N stands for the Gauss map of the surface, we can consider again the *angle function* by $\Theta = \langle N, \xi \rangle$ as we did in the previous chapter (see Definition 2.3.5).

Our main aim in this section is to show examples of surfaces in Riemannian Killing submersions whose interest lies in the fact that they will be characterized by means of the first stability eigenvalue. Specifically, let us pay attention to two very special surfaces immersed into a Riemannian Killing submersion $\mathcal{M}(\kappa, \tau)$, which appear when the square of the angle function attains its maximum or minimum at any point, respectively.

Definition 3.3.1. A surface Σ immersed into $\mathcal{M}(\kappa, \tau)$ is said to be *horizontal* when its tangent plane contains only horizontal vectors at any point.

Let us enumerate some **properties of the horizontal surfaces**:

1. $\boldsymbol{\Theta^2 \equiv 1}$, since the normal direction of the surface is vertical, i.e., $N = \pm\xi$.

2. They are **totally geodesic**. To check it, consider a local orthonormal frame $\{X, Y\}$ on Σ. Now, by using (3.3) we obtain

$$[X, Y] = \nabla_X Y - \nabla_Y X = 2\tau\xi + bY - aX,$$

   and since the surface is horizontal, the Lie bracket $[X, Y]$ must be a horizontal vector field. As a consequence τ must vanish over the surface. Let us fix $N = \xi$ without loss of generality, and consider $Z \in \mathfrak{X}(\Sigma)$ which can be written as $Z = \alpha X + \beta Y$. Then, from (3.3), the shape operator A is given by

$$A(Z) = -\nabla_Z N = -\nabla_Z \xi = -\alpha \nabla_X \xi - \beta \nabla_Y \xi = \alpha\tau Y - \beta\tau X = 0.$$

   Thus, $A \equiv 0$.

3. Their first stability eigenvalue vanishes, i.e. $\boldsymbol{\lambda_1 = 0}$, and so, they are **stable**. Indeed, since horizontal surfaces are totally geodesic we know that $|A|^2 = 0$, and from (3.13) we get that the Ricci curvature on the unit normal direction N of the surface is

$$\mathrm{Ric}(N, N) = \kappa - 2\tau^2 - \Theta^2(\kappa - 4\tau^2) + 2\Theta\sqrt{1 - \Theta^2}X(\tau).$$

   As $\Theta^2 \equiv 1$ and $\tau = 0$ over the surface, we conclude that $\mathrm{Ric}(N, N) = 0$, which yields

$$J = \Delta + |A|^2 + \mathrm{Ric}(N, N) = \Delta.$$

   Thus, the first stability eigenvalue vanishes, that is, $\lambda_1 = 0$.

Example 3.3.2. If $\mathcal{M}(\kappa,\tau)$ is a product $B^2\times\mathbb{R}$ or $B^2\times\mathbb{S}^1$, where B is any Riemannian surface of Gaussian curvature κ (case when $\tau\equiv 0$), the horizontal surfaces are the slices $B\times\{t\}$.

On the other hand, let us also introduce the other special surfaces, which are obtained as the total lift of closed curves by means of the submersion (if it has compact fibers). Recall that they were presented in the previous chapter when studying the homogeneous manifolds $\mathbb{E}^3(\kappa,\tau)$.

Definition 3.3.3. A surface Σ immersed into $\mathcal{M}(\kappa,\tau)$ is said to be a *Hopf cylinder* over a regular curve $\gamma: I\longrightarrow B^2(\kappa)$ if it is the total lift of γ, i.e., $\Sigma=\Pi^{-1}(\gamma)$. In particular, the surface is a *Hopf torus* if the fibers and the curve are closed.

We compile here some useful **properties of a Hopf cylinder** $\Sigma=\Pi^{-1}(\gamma)$ which were presented in the second chapter, but in this case we include the proofs (see Proposition 2.10 in [EO]):

1. $\boldsymbol{\Theta\equiv 0}$. Let us assume that the curve γ is parametrized by the arc length, and consider the Frenet frame along the curve $\{t=\gamma', n\}$. If T and N stand for the horizontal lifts of t and n to $\mathcal{M}(\kappa,\tau)$ respectively, then $\{T,\xi\}$ is a local orthonormal frame on Σ and N is the Gauss map of the surface. Thus, $\Theta=\langle N,\xi\rangle=0$.

2. They are **flat**, i.e., the Gaussian curvature of the surface vanishes. In order to prove this fact we use the Gauss equation $G=K_\Sigma+\det(A)$. From (3.8) we know

$$K_\Sigma=\tau^2+\Theta^2(\kappa-4\tau^2)-2\Theta\sqrt{1-\Theta^2}X(\tau),$$

as $\Theta=0$, $K_\Sigma=\tau^2$. It remains to compute $\det(A)$. To do so, observe that the matrix associated to the shape operator A with respect to the local orthonormal frame of Σ given by $\{T,\xi\}$ is

$$A=\begin{pmatrix}\langle\nabla_T T,N\rangle & \langle\nabla_T\xi,N\rangle\\ \langle\nabla_\xi T,N\rangle & \langle\nabla_\xi\xi,N\rangle\end{pmatrix}.$$

Now, we fix N so that $\{T,N,\xi\}$ is a positively oriented local orthonormal frame on $\mathcal{M}(\kappa,\tau)$, then:

- by the Frenet equations $\langle\nabla_T T,N\rangle=k_\gamma$, where k_γ is the curvature of γ,
- by using (3.1) we have $\langle\nabla_T\xi,N\rangle=\tau\langle T\wedge\xi,N\rangle=-\tau$, and $\langle\nabla_\xi T,N\rangle=-\langle T,\nabla_\xi N\rangle=-\tau\langle T,\xi\wedge N\rangle=-\tau$, and
- $\nabla_\xi\xi=0$ as studied before.
  Therefore,

$$A=\begin{pmatrix}k_\gamma & -\tau\\ -\tau & 0\end{pmatrix},$$

And so, the determinant of the matrix above is $\det(A)=-\tau^2$, which enables us to conclude that $G=K_\Sigma+\det(A)=\tau^2-\tau^2=0$.

3. Its mean curvature is $\boldsymbol{H = k_\gamma/2}$, because this is a half of the trace of the matrix above. From here, we observe that they have constant mean curvature when the curve has constant curvature, and in particular, they are minimal when lifting geodesic curves.

4. As a special case which is going to be used further on, we note that for a Hopf torus $\Pi^{-1}(\gamma)$ of constant mean curvature H, when κ is constant over γ, we know that $G = 0$ and $K_\Sigma = \tau^2$, so
$$|A|^2 = 4H^2 + 2\,K_\Sigma - 2G = 4H^2 + 2\tau^2.$$
Moreover, since $\Theta = 0$ from (3.13) we have
$$\mathrm{Ric}(N, N) = \kappa - 2\tau^2,$$
and consequently, its Jacobi operator is given by
$$J = \Delta + |A|^2 + \mathrm{Ric}(N, N) = \Delta + 4H^2 + \kappa,$$
and we immediately obtain
$$\boldsymbol{\lambda_1 = -4H^2 - \kappa}.$$
Let us observe that if κ is a positive constant over γ, then the Hopf torus $\Pi^{-1}(\gamma)$ is not stable, and when κ is non positive the stability of the torus depends on its mean curvature.

3.4 Estimates of λ_1 for compact CMC surfaces into Riemannian Killing submersions

Along this section we have a dual objetive: to find out upper bounds for λ_1 for compact two-sided surfaces of constant mean curvature immersed into a Riemannian Killing submersion; and to look for the characterization of the special surfaces in the previous section by means of λ_1. To do so, we use our two general bounds (1.12) and (1.13), and consequently we must know $\mathrm{Ric}(N, N)$, which has already been obtained in (3.13), and $2\,K_\Sigma + \mathrm{Ric}(N, N)$ that is easily got from (2.3),
$$2\,K_\Sigma = 2\tau^2 + 2\Theta^2(\kappa - 4\tau^2) - 4\Theta\sqrt{1-\Theta^2}X(\tau),$$
and together with (3.13) allows us to conclude
$$2\,K_\Sigma + \mathrm{Ric}(N, N) = \kappa + \Theta^2(\kappa - 4\tau^2) - 2\Theta\sqrt{1-\Theta^2}X(\tau). \tag{3.14}$$

Before giving the main results, we would like to motivate the assumption that we will consider over the Riemannian Killing submersions $\mathcal{M}(\kappa, \tau)$ along this section. As we have seen, the homogeneous Riemannian 3-manifolds $\mathbb{E}^3(\kappa, \tau)$ are a particular case of $\mathcal{M}(\kappa, \tau)$ with both κ and τ constant, in which the sign of $\kappa - 4\tau^2$ plays an important role. Let us remind that when

$\kappa - 4\tau^2 = 0$ the manifold corresponds to a quotient of the space forms $\mathbb{S}^3$, $\mathbb{R}^3$ or $\mathbb{H}^3$ whose isometry group has dimension 6, and otherwise the manifold has isometry group of lower dimension. In this last case, we have the Berger spheres which experiment a very different behaviour according to the aforementioned sign. Therefore it seems reasonable to distinguish between both cases in the general study of Riemannian Killing submersions.

Case $\kappa - 4\tau^2 > 0$

Let us start by assuming that $\kappa - 4\tau^2 > 0$. So, when κ and τ are constant the product spaces $\mathbb{S}^2(\kappa) \times \mathbb{R}$ and $\mathbb{S}^2(\kappa) \times \mathbb{S}^1$ and some Berger spheres $\mathbb{S}^3_b(\kappa, \tau)$ are included here. That is, with the following result we generalize Theorems 2.3.10 and 2.3.11 and (a) in Theorem 2.3.14.

Theorem 3.4.1. *(Theorem 3 in [MO2]) Let $\mathcal{M}(\kappa, \tau)$ be a Riemannian Killing submersion with $\kappa - 4\tau^2 > 0$ and Σ a compact two-sided surface of constant mean curvature H immersed into $\mathcal{M}(\kappa, \tau)$. If λ_1 stands for the first eigenvalue of its Jacobi operator, then*

(i) $\lambda_1 \leq -2H^2 - \dfrac{1}{\mathrm{Area}(\Sigma)} \displaystyle\int_\Sigma (2\tau^2 - |\nabla\tau|) d\Sigma$, *with equality if and only if Σ is a horizontal surface; and*

(ii) $\lambda_1 \leq -4H^2 - \dfrac{8\pi(g-1)}{\mathrm{Area}(\Sigma)} - \dfrac{1}{\mathrm{Area}(\Sigma)} \displaystyle\int_\Sigma (\kappa - |\nabla\tau|) d\Sigma$, *with equality if and only if Σ is a Hopf torus over a constant curvature closed curve, and both κ and τ are constant over Σ.*

Proof. (i) From (3.13), the assumption $\kappa - 4\tau^2 > 0$ and the fact that $\Theta^2 \leq 1$ one easily gets that

$$\begin{aligned} \mathrm{Ric}(N, N) &= \kappa - 2\tau^2 - \Theta^2(\kappa - 4\tau^2) + 2\Theta\sqrt{1-\Theta^2}X(\tau) \\ &\geq 2\tau^2 + 2\Theta\sqrt{1-\Theta^2}X(\tau). \end{aligned} \tag{3.15}$$

Taking the function $f(\Theta) = 2\Theta\sqrt{1-\Theta^2}$ defined for $-1 \leq \Theta \leq 1$, it is a direct computation to check that its minimum and maximum are attained when $\Theta = -1/\sqrt{2}$ and $\Theta = 1/\sqrt{2}$ respectively, with $f(1/\sqrt{2}) = -f(1/\sqrt{2}) = 1$, and so

$$|2\Theta\sqrt{1-\Theta^2}| \leq 1.$$

Moreover, since X is a unit vector field, the Cauchy-Schwarz inequality allows us to know that

$$|X(\tau)| = |\langle X, \nabla\tau\rangle| \leq |X||\nabla\tau| \leq |\nabla\tau|,$$

and consequently $|2\Theta\sqrt{1-\Theta^2}X(\tau)| \leq |\nabla\tau|$. In particular,

$$2\Theta\sqrt{1-\Theta^2}X(\tau) \geq -|\nabla\tau|.$$

In this way, by taking into account (3.15) we have

$$\mathrm{Ric}(N, N) \geq 2\tau^2 - |\nabla\tau|$$

and from our first general bound (1.12), it is obtained the inequality

$$\lambda_1 \leq 2H^2 - \frac{1}{\text{Area}(\Sigma)} \int_\Sigma (2\tau^2 - |\nabla \tau|) d\Sigma.$$

Let us now analyse the case when the equality holds. The inequality (3.15) becomes also into equality which ensures that Θ^2=1 and so Σ is a horizontal surface.

Conversely, if Σ is a horizontal surface, by its properties we know that it is totally geodesic, $\tau = 0$ over the surface and $\lambda_1 = 0$. Therefore the equality holds.

(ii) Using (3.14), we can estimate the integrand in the second general bound (1.13) as

$$\begin{aligned} 2\,K_\Sigma + \text{Ric}(N, N) &= \kappa + \Theta^2(\kappa - 4\tau^2) - 2\Theta\sqrt{1 - \Theta^2}X(\tau) \\ &\geq \kappa - |\nabla \tau|, \end{aligned} \tag{3.16}$$

where we have used again $\kappa - 4\tau^2 > 0$ and $2\Theta\sqrt{1 - \Theta^2}X(\tau) \leq |\nabla \tau|$.

So, this inequality together with (1.13) directly yield to the announced estimate, i.e.,

$$\lambda_1 \leq -4H^2 - \frac{8\pi(g-1)}{\text{Area}(\Sigma)} - \frac{1}{\text{Area}(\Sigma)} \int_\Sigma (\kappa - |\nabla \tau|) d\Sigma.$$

Let us now analyse the case when the equality holds. The inequality (3.16) turns into equality as well and so the angle function Θ vanishes and $2\Theta\sqrt{1 - \Theta^2}X(\tau) = |\nabla \tau|$. The former involves that Σ^2 is a Hopf torus over a constant curvature closed curve, the latter simplifies to $|\nabla \tau| = 0$, which means that τ is constant over Σ.

Moreover, in Remark 1.4.1 we have observed that the equality in (1.13) implies $\alpha = 0$ and then from (1.9) we get

$$\lambda_1 + |A|^2 + \text{Ric}(N, N) = 0.$$

Now, we have seen before that for a Hopf torus of constant mean curvature $\text{Ric}(N, N) = \kappa - 2\tau^2$ and $|A|^2 = 4H^2 + 2\tau^2$. Thus,

$$\lambda_1 = -4H^2 - \kappa$$

which shows that κ is also constant over Σ.

Reciprocally, if Σ is a Hopf torus over a constant curvature closed curve and both κ and τ are constant over Σ^2, then $\nabla \tau = 0$, $g = 1$ and $\lambda_1 = -4H^2 - \kappa$, as we have seen in the fourth property of Hopf tori. So we conclude that the equality is satisfied. □

Remark 3.4.2. Observe that for the existence of such tori in Theorem 3.4.1, the Killing submersion must have compact fibers, which occurs in several significant cases. For instance, in Berger spheres which were studied in the previous chapter and other examples that we will see in the last section of the present chapter.

Now, it is worth pointing out the special case when the bundle curvature is constant, since the recently studied estimates are considerably simplified. So we can set the following corollary as a consequence of the previous theorem.

Corollary 3.4.3. *(Corollary 4 in [MO2]) Let $\mathcal{M}(\kappa,\tau)$ be a Riemannian Killing submersion with constant bundle curvature τ such that $\kappa-4\tau^2>0$ and Σ a compact two-sided surface of constant mean curvature H immersed into $\mathcal{M}(\kappa,\tau)$. If λ_1 stands for the first eigenvalue of its Jacobi operator, then*

(i) $\lambda_1 \leq -2(H^2+\tau^2)$, *with equality if and only if Σ is a horizontal surface; and*

(ii) $\lambda_1 \leq -4H^2 - \dfrac{8\pi(g-1)}{\text{Area}(\Sigma)} - \dfrac{1}{\text{Area}(\Sigma)}\displaystyle\int_\Sigma \kappa d\Sigma$, *with equality if and only if Σ is a Hopf torus over a constant curvature closed curve γ and κ is constant over γ.*

Case $\kappa - 4\tau^2 < 0$

Now, we are going to analyse the other case $\kappa-4\tau^2<0$. Notice that if $\mathcal{M}(\kappa,\tau)$ satisfies this assumption, then the homogeneous spaces $\mathbb{E}^3(\kappa,\tau)$ under the same hypothesis are particular cases. We mean the product space $\mathbb{H}^2(\kappa)\times\mathbb{R}$, the Heisenberg group $Nil_3(\tau)$, certain Berger spheres $\mathbb{S}^3_b(\kappa,\tau)$ and the universal cover $\widetilde{Sl(2,\mathbb{R})}(\kappa,\tau)$ of the group $Sl(2,\mathbb{R})(\kappa,\tau)$. Therefore, we generalize Theorems 2.3.12, 2.3.13, (b) in Theorem 2.3.14 and Theorem 2.3.15 as follows.

Theorem 3.4.4. *(Theorem 7 in [MO2]) Let $\mathcal{M}(\kappa,\tau)$ be a Riemannian Killing submersion with $\kappa-4\tau^2<0$ and Σ a compact two-sided surface of constant mean curvature H immersed into $\mathcal{M}(\kappa,\tau)$. If λ_1 stands for the first eigenvalue of its Jacobi operator, then*

(i) $\lambda_1 \leq -2H^2 - \dfrac{1}{\text{Area}(\Sigma)}\displaystyle\int_\Sigma(\kappa-2\tau^2-|\nabla\tau|)d\Sigma$, *with equality if and only if Σ is a Hopf torus over a closed geodesic γ, $\tau=0$ over Σ and κ is constant over γ; and*

(ii) $\lambda_1 \leq -4H^2 - \dfrac{8\pi(g-1)}{\text{Area}(\Sigma)} - \dfrac{1}{\text{Area}(\Sigma)}\displaystyle\int_\Sigma(2\kappa-4\tau^2-|\nabla\tau|)d\Sigma$, *with equality if and only if Σ is a horizontal surface with Gaussian curvature $G=\kappa$.*

Proof. (i) As we have checked in the proof of Theorem 3.4.1 we know that

$$2\Theta\sqrt{1-\Theta^2}X(\tau) \geq -|\nabla\tau|,$$

so from (3.13), the previous inequality combined with $\Theta^2\geq 0$ and the assumption $\kappa-4\tau^2<0$ leads to

$$\begin{aligned}\text{Ric}(N,N) &= \kappa-2\tau^2-\Theta^2(\kappa-4\tau^2)+2\Theta\sqrt{1-\Theta^2}X(\tau)\\ &\geq \kappa-2\tau^2-|\nabla\tau|.\end{aligned} \tag{3.17}$$

Then, our first general bound (1.12) allows us to obtain

$$\lambda_1 \leq -2H^2 - \frac{1}{\text{Area}(\Sigma)} \int_\Sigma (\kappa - 2\tau^2 - |\nabla\tau|) d\Sigma.$$

Now, let us study what occurs when the equality is held. The inequality (3.17) turns into an equality, then

$$\text{Ric}(N, N) = \kappa - 2\tau^2 - |\nabla\tau| \tag{3.18}$$

and so $\Theta \equiv 0$, that is, Σ is a Hopf torus. Moreover, if the equality holds, (1.12) is also an equality, and because of Remark 1.4.1 we know that $\phi = 0$, and consequently Σ is a totally umbilic Hopf torus over a constant curvature closed curve. By the total umbilicity and the property 3 of Hopf tori we have

$$|A|^2 = 2H^2 = k_\gamma^2/2,$$

and, since $G = 0$ and $K_\Sigma = \tau^2$ (see property 2 of Hopf tori), the Gauss equation $|A|^2 = 4H^2 + 2\,K_\Sigma - 2G$ reduces to

$$k_\gamma^2/2 = k_\gamma^2 + 2\tau^2.$$

Therefore, we conclude $k_\gamma = 0$ and $\tau = 0$ over Σ. Now, from Remark 1.4.1 we also know that $\text{Ric}(N, N)$ is constant, and from (3.18) and $\tau = 0$ we get

$$\text{Ric}(N, N) = \kappa$$

and so, we finally deduce that κ is constant over Σ.

Conversely, if Σ is a Hopf torus over a closed geodesic γ and κ is constant over γ, we know that $\lambda_1 = -\kappa$ due to property 4 of Hopf tori. Since $\tau = 0$ over Σ and $g = 1$, the equality holds.

(ii) From (3.14) we have

$$\begin{aligned} 2\,K_\Sigma + \text{Ric}(N, N) &= \kappa + \Theta^2(\kappa - 4\tau^2) - 2\Theta\sqrt{1 - \Theta^2}X(\tau) \\ &\geq 2\kappa - 4\tau^2 - |\nabla\tau|, \end{aligned} \tag{3.19}$$

where we have used $\kappa - 4\tau^2 < 0$, $\Theta^2 \leq 1$ and $2\Theta\sqrt{1 - \Theta^2}X(\tau) \leq |\nabla\tau|$.

So, this inequality and our second general bound (1.13) immediately give

$$\lambda_1 \leq -4H^2 - \frac{8\pi(g-1)}{\text{Area}(\Sigma)} - \frac{1}{\text{Area}(\Sigma)} \int_\Sigma (2\kappa - 4\tau^2 - |\nabla\tau|) d\Sigma.$$

Now, if we suppose that the equality holds, then (3.19) becomes into an equality and

$$2\,K_\Sigma + \text{Ric}(N, N) = 2\kappa - 4\tau^2 - |\nabla\tau|,$$

which implies $\Theta^2 \equiv 1$, that is, Σ is a horizontal surface. As we have observed in property 1 of a horizontal surface, Σ is totally geodesic and $\tau = 0$ over Σ, so using (3.8) we follow that $K_\Sigma = \kappa$. Moreover, the Gauss equation simplifies to

$$0 = |A|^2 = 4H^2 + 2\,K_\Sigma - 2G = 2\kappa - 2G,$$

which means $G = \kappa$.

Reciprocally, if Σ is a horizontal surface, then $\lambda_1 = 0$. Since the Gaussian curvature of Σ is $G = \kappa$, Σ is totally geodesic and $\tau = 0$ over Σ, the right hand of the inequality is

$$-\frac{8\pi(g-1)}{\text{Area}(\Sigma)} - \frac{1}{\text{Area}(\Sigma)} \int_\Sigma 2G d\Sigma,$$

which is zero by the Gauss-Bonnet Theorem, so the equality holds. □

Now, as we did before, we study the particular case in which the bundle curvature is a constant over Σ. In this way, from the last theorem we derive the next immediate consequence.

Corollary 3.4.5. *(Corollary 9 in [MO2]) Let $\mathcal{M}(\kappa,\tau)$ be a Riemannian Killing submersion with constant bundle curvature τ such that $\kappa - 4\tau^2 < 0$ and Σ a compact two-sided surface of constant mean curvature H immersed into $\mathcal{M}(\kappa,\tau)$. If λ_1 stands for the first eigenvalue of its Jacobi operator, then*

(i) $\lambda_1 \leq -2(H^2 - \tau^2) - \dfrac{1}{\text{Area}(\Sigma)} \displaystyle\int_\Sigma \kappa d\Sigma$, *with equality if and only if Σ is a Hopf torus over a closed geodesic γ, $\tau = 0$ and κ is constant over γ; and*

(ii) $\lambda_1 \leq -4(H^2 - \tau^2) - \dfrac{8\pi(g-1)}{\text{Area}(\Sigma)} - \dfrac{2}{\text{Area}(\Sigma)} \displaystyle\int_\Sigma \kappa d\Sigma$, *with equality if and only if Σ is a horizontal surface with Gaussian curvature $G = \kappa$.*

3.5 Stability of compact CMC surfaces into Riemannian Killing submersions

Let us exhibit some interesting consequences of the results in the above section related to the stability of a compact surface Σ immersed into a Riemannian Killing submersion $\mathcal{M}(\kappa,\tau)$.

More specifically, after assuming that the surface Σ is stable, our intention is to find some restrictions over the mean curvature H. What is more, under the additional assumptions of $\kappa - 4\tau^2 > 0$ and τ constant over Σ, we go further since we are able to give a complete classification of compact stable surfaces of constant mean curvature immersed into $\mathcal{M}(\kappa,\tau)$.

So, we proceed to expose these results by differentiating again between the case in which $\mathcal{M}(\kappa,\tau)$ satisfies $\kappa - 4\tau^2 > 0$ and the opposite case.

Case $\boldsymbol{\kappa - 4\tau^2 > 0}$

Let us consider a Riemannian Killing submersion $\mathcal{M}(\kappa,\tau)$ with $\kappa - 4\tau^2 > 0$. In this framework, our first result consists of the study of compact stable surfaces with constant mean curvature. To be specific, we find upper bounds for the square of the mean curvature and we also look for surfaces satisfying the equality when possible. With this aim, we apply Theorem 3.4.1 bearing in mind that a surface Σ is stable if and only if $\lambda_1 \geq 0$.

Corollary 3.5.1. *(Corollary 4 in [MO2]) Let $\mathcal{M}(\kappa,\tau)$ be a Riemannian Killing submersion with $\kappa - 4\tau^2 > 0$. If Σ is a stable compact two-sided surface of constant mean curvature H immersed into $\mathcal{M}(\kappa,\tau)$ then*

(i) $H^2 \leq \dfrac{1}{\mathrm{Area}(\Sigma)} \displaystyle\int_\Sigma \left(\frac{|\nabla\tau|}{2} - \tau^2\right) d\Sigma$, *with equality if and only if Σ is a horizontal surface; and*

(ii) $H^2 < \dfrac{2\pi(1-g)}{\mathrm{Area}(\Sigma)} + \dfrac{1}{4\,\mathrm{Area}(\Sigma)} \displaystyle\int_\Sigma (|\nabla\tau| - \kappa) d\Sigma.$

Proof. (i) If Σ is stable, we know that $\lambda_1 \geq 0$, so (i) in Theorem 3.4.1 ensures that

$$0 \leq \lambda_1 \leq -2H^2 - \frac{1}{\mathrm{Area}(\Sigma)} \int_\Sigma (2\tau^2 - |\nabla\tau|) d\Sigma$$

and from here the first inequality is a direct consequence.

Moreover, if the equality is satisfied, then the chain of inequalities above becomes into equalities and by applying again (i) of Theorem 3.4.1 we get that Σ is a horizontal surface. The converse is direct.

(ii) With a similar argument, using (ii) of Theorem 3.4.1 we get

$$0 \leq \lambda_1 \leq -4H^2 - \frac{8\pi(g-1)}{\mathrm{Area}(\Sigma)} - \frac{1}{\mathrm{Area}(\Sigma)} \int_\Sigma (\kappa - |\nabla\tau|) d\Sigma$$

and so, we easily deduce the second inequality for H^2.

In this case equality is not possible because if it holds, then $\lambda_1 = 0$, and again by (ii) of Theorem 3.4.1 we obtain that Σ is a Hopf torus and κ is a positive constant over Σ (because of $\kappa - 4\tau^2 > 0$). Hence, we get a contradiction since $\lambda_1 = -4H^2 - \kappa$ for such tori, as we previously saw in property 4 of them. □

Remark 3.5.2. Notice that for particular values of τ and κ we can deepen a bit more.

(i) If $|\nabla\tau| \leq 2\tau^2$, then for (i) of the previous corollary we have $H^2 \leq 0$ and so the only stable compact two-sided surface of constant mean curvature immersed into $\mathcal{M}(\kappa,\tau)$ is the horizontal one with $|\nabla\tau| = 2\tau^2$ over Σ.

(ii) If $|\nabla\tau| \leq \kappa$, due to (ii) of the preceding corollary we obtain

$$H^2 < \frac{2\pi(1-g)}{\text{Area}(\Sigma)}.$$

Thus, we can assert that there do not exist stable compact two-sided surfaces of constant mean curvature immersed into $\mathcal{M}(\kappa,\tau)$ with $g \geq 1$.

In particular, when the bundle curvature τ is constant, we can study the stability of the surface as an application of the Corollary 3.5.1 and one can easily establish the following result.

Corollary 3.5.3. *(Corollary 6 in [MO2]) Let $\mathcal{M}(\kappa,\tau)$ be a Riemannian Killing submersion with constant bundle curvature τ such that $\kappa - 4\tau^2 > 0$. The only stable compact two-sided surfaces of constant mean curvature immersed into $\mathcal{M}(\kappa,\tau)$ are the horizontal ones.*

Case $\kappa - 4\tau^2 < 0$

Now, following the same schedule as for the positive case, we assume $\kappa - 4\tau^2 < 0$ and we aim to get analogous results. In this spirit, we begin by applying our Theorem 3.4.4 to stable surfaces into $\mathcal{M}(\kappa,\tau)$.

Corollary 3.5.4. *(Corollary 8 in [MO2]) Let $\mathcal{M}(\kappa,\tau)$ be a Riemannian Killing submersion with $\kappa - 4\tau^2 < 0$. If Σ is a stable compact two-sided surface of constant mean curvature H immersed into $\mathcal{M}(\kappa,\tau)$ then*

(i) $H^2 < \frac{1}{\text{Area}(\Sigma)} \int_\Sigma \left(\tau^2 + \frac{|\nabla\tau|}{2} - \frac{\kappa}{2}\right) d\Sigma$; *and*

(ii) $H^2 \leq \frac{2\pi(1-g)}{\text{Area}(\Sigma)} + \frac{1}{\text{Area}(\Sigma)} \int_\Sigma \left(\tau^2 + \frac{|\nabla\tau|}{4} - \frac{\kappa}{2}\right) d\Sigma$, *with equality if and only if Σ is a horizontal surface with Gaussian curvature $G = \kappa$.*

Proof. (i) If Σ is stable, then $\lambda_1 \geq 0$ and from Theorem 3.4.4, we derive both of these inequalities. Moreover, in the case (i) equality does not satisfy because if it does, then $\lambda_1 = 0$ and from Theorem 3.4.4, Σ is a Hopf torus over a closed geodesic γ, $\tau = 0$ over Σ and κ is constant over γ, so $\lambda_1 = -\kappa > -4\tau^2 = 0$, which is a contradiction. □

As a consequence, if $\mathcal{M}(\kappa,\tau)$ is a Riemannian Killing submersion with constant bundle curvature τ and Gaussian curvature κ such that $0 \leq \kappa < 4\tau^2$, then no compact two-sided surface of constant mean curvature $|H| > \tau$ immersed into $\mathcal{M}(\kappa,\tau)$ can be stable, and if $|H| \leq \tau$ and the surface is stable, then

$$\text{Area}(\Sigma)(\tau^2 - H^2) \geq 2\pi(g-1),$$

with equality only if the surface is a flat horizontal one with $\kappa = 0$.

Remark 3.5.5. As we have already cited in the introduction of the chapter, we know from the results of Lerma and Manzano ([LM]) that any compact two-sided stable surface of constant mean curvature H immersed into $\mathcal{M}(\kappa,\tau)$ must be:

- an entire minimal graph (B compact and $\int_\Sigma \tau d\Sigma = 0$); or
- a Hopf torus.

In this way, the last corollaries are specifically applied to such surfaces. For instance, when $\kappa - 4\tau^2 > 0$ it is clear that $\kappa > 0$. As observed in property 4 of Hopf tori, in such a case they are not stable. Therefore, when $\kappa - 4\tau^2 > 0$ the only compact two-sided stable surfaces of constant mean curvature into $\mathcal{M}(\kappa,\tau)$ must be entire minimal graphs, and by applying Corollary 3.5.1 we have that they satisfy

$$0 \leq \int_\Sigma \left(\frac{|\nabla\tau|}{2} - \tau^2 \right) d\Sigma \quad \text{and} \quad 0 < 8\pi(1-g) + \int_\Sigma (|\nabla\tau| - \kappa) d\Sigma.$$

Nevertheless, for the case $\kappa - 4\tau^2 < 0$, both minimal graphs and Hopf tori could be stable and Corollary 3.5.4 imposes strong restrictions to be stable. To be specific, a stable entire minimal graph into $\mathcal{M}(\kappa,\tau)$ satisfies

$$0 < \int_\Sigma \left(\tau^2 + \frac{|\nabla\tau|}{2} - \frac{\kappa}{2} \right) d\Sigma \quad \text{and} \quad 0 \leq 2\pi(1-g) + \int_\Sigma \left(\tau^2 + \frac{|\nabla\tau|}{4} - \frac{\kappa}{2} \right) d\Sigma,$$

and any stable Hopf torus

$$k_\gamma^2 < \frac{4}{\text{Area}(\Sigma)} \int_\Sigma \left(\tau^2 + \frac{|\nabla\tau|}{2} - \frac{\kappa}{2} \right) d\Sigma \quad \text{and} \quad k_\gamma^2 < \frac{1}{\text{Area}(\Sigma)} \int_\Sigma \left(\tau^2 + \frac{|\nabla\tau|}{4} - \frac{\kappa}{2} \right) d\Sigma.$$

Also observe that as a consequence of the cited work, the Corollary 3.5.3 is slightly improved, since τ must be zero.

3.6 Some examples of Riemannian Killing submersions

As already mentioned, a Riemannian Killing submersion $\mathcal{M}(\kappa,\tau)$ with both κ and τ constant is a homogeneous Riemannian 3-manifold. Let us see some interesting examples when κ and τ are non constant, and we are going to apply our results to them.

Example 3.6.1. A remarkable case arises when the bundle curvature τ of $\mathcal{M}(\kappa,\tau)$ is zero. Thus $\mathcal{M}(\kappa,0)$ is a product $B^2\times\mathbb{R}$ or $B^2\times\mathbb{S}^1$, where B is an arbitrary Riemannian surface.

- When $\kappa>0$, Theorem 3.4.1 establishes that for any compact two-sided surface of constant mean curvature H immersed into $\mathcal{M}(\kappa,0)$ we have

$$\lambda_1\leq -2H^2$$

and the slices $B\times\{t\}$ are the only ones satisfying the equality (if B is compact). Also observe that if the surface is stable, then it has to be minimal. On the other hand, we know

$$\lambda_1\leq -4H^2-\frac{8\pi(g-1)}{\text{Area}(\Sigma)}-\frac{1}{\text{Area}(\Sigma)}\int_\Sigma \kappa d\Sigma$$

and Hopf tori $\gamma\times\mathbb{S}^1$ are the only ones that satisfy equality, where γ is any constant curvature closed curve such that κ is constant over γ.

- When $\kappa<0$, from Theorem 3.4.4 we have that

$$\lambda_1\leq -2H^2-\frac{1}{\text{Area}(\Sigma)}\int_\Sigma \kappa d\Sigma,$$

with equality only for minimal Hopf tori $\gamma\times\mathbb{S}^1$ with κ constant over γ. And

$$\lambda_1\leq -4H^2-\frac{8\pi(g-1)}{\text{Area}(\Sigma)}-\frac{2}{\text{Area}(\Sigma)}\int_\Sigma \kappa d\Sigma,$$

with equality only for slices $B\times\{t\}$ with Gaussian curvature $G=\kappa$.

As for the general case, where both κ and τ are non constant, we can find a lot of examples in the work [SVdV]. Here the authors find out 3-dimensional spaces which locally admit a doubly warped product metric and project over a certain surface as a Riemannian Killing submersion. These spaces are very interesting because they are characterized as the only ones carrying a unit Killing field which admit totally geodesic surfaces different to a Hopf cylinder or a horizontal surface, as they prove in the same work. The following one is a particular example where we can apply the theory.

Example 3.6.2. Let M be the product $I\times\mathbb{S}^1\times\mathbb{S}^1$, for some open interval I of $\mathbb{R}$ or $I=\mathbb{S}^1$, with the doubly warped product metric

$$ds^2=dx^2+\sin^2\theta(x)dy^2+\cos^2\theta(x)dz^2,$$

where $\theta: I\longrightarrow(0,\pi/2)$ is any smooth function, and B be the warped product $I\times_f\mathbb{S}^1$, with $f(u)=\frac{1}{2}\sin(2\theta(u))$, that is,

$$ds_B=du^2+\frac{1}{4}\sin^2(2\theta(u))dv^2.$$

Here, y and z stand for the coordinates associated to the chart $\mathbb{S}^1 \times \mathbb{S}^1 \longrightarrow \mathbb{R} \times \mathbb{R}$ defined by $(\cos y, \sin y, \cos z, \sin z) \longrightarrow (y, z)$. Consider $\Pi : M \longrightarrow B$ given by

$$\Pi(x, y, z) = (u, v)$$

with $u = x$ and $v = y - z$. Let us check that $\mathcal{M}(\kappa, \tau)$ is a Riemannian Killing submersion:

- Π is a submersion since this map and its differential are clearly onto. We have just to compute its differential. Let us fix a point $p \in \mathcal{M}(\kappa, \tau)$ and a vector $v = (v_1, v_2, v_3) \in T_p\mathcal{M}(\kappa, \tau)$, and set a curve $\gamma : (-\epsilon, \epsilon) \longrightarrow \mathcal{M}(\kappa, \tau)$, $\gamma(t) = (x(t), y(t), z(t))$, such that $\gamma(0) = p$ and $\gamma'(0) = v$, so
$$d\Pi_p(v) = \frac{d}{dt}\bigg|_{t=0} \Pi(\gamma(t)) = \frac{d}{dt}\bigg|_{t=0} (x(t), y(t) - z(t)) = (v_1, v_2 - v_3).$$
- To see that Π is a Riemannian submersion it remains to show that $d\Pi$ preserves scalar products of horizontal vectors. Let us observe that the fibers $\Pi^{-1}(u, v) = \{\gamma(t) : t \in I\} = \{(u, t, t - v) : t \in I\}$ with $t \in \mathbb{S}^1$ satisfy $\gamma'(t) = (0, 1, 1)$, i.e., the fibers are the integral curves of the unit Killing field $\xi = \partial_y + \partial_z$, and
$$\begin{aligned} X &= \partial_x, \\ Y &= \cos^2\theta(x)\partial_y - \sin^2\theta(x)\partial_z \end{aligned}$$
define a basis of horizontal vector fields with $d\Pi(X) = \partial_u$ and $d\Pi(Y) = \cos^2\theta(x)\partial_v + \sin^2\theta(x)\partial_v$. Now, since $\langle Y, Y\rangle = \cos^4\theta(x)\sin^2\theta(x) + \sin^4\theta(x)\cos^2\theta(x) = \cos^2\theta(x)\sin^2\theta(x)$ and $\langle \partial_v, \partial_v\rangle = \frac{1}{4}\sin^2(2\theta(x)) = \sin^2\theta(x)\cos^2\theta(x)$, we finally conclude
$$\begin{aligned} \langle X, X\rangle &= \langle \partial_u, \partial_u\rangle = 1, \\ \langle X, Y\rangle &= \langle \partial_u, \partial_v\rangle = 0, \\ \langle Y, Y\rangle &= \langle \partial_v, \partial_v\rangle = \cos^2\theta\sin^2\theta. \end{aligned}$$

It is easy to check that the bundle curvature of Π is $\tau = -\theta'$ and the Gaussian curvature of B is $\kappa = 4(\theta')^2 - 2\cot(2\theta)\theta''$, and so $\kappa - 4\tau^2 = -2\cot(2\theta)\theta''$ (see Proposition 4 in [SVdV]). Thus, if $\theta < \pi/4$ and $\theta'' < 0$, Theorem 3.4.1 asserts that for any compact two-sided surface of constant mean curvature H immersed into M, we have

$$\lambda_1 \leq -2H^2 - \frac{1}{\text{Area}(\Sigma)}\int_\Sigma (2(\theta')^2 + \theta''),$$

and the equality holds for horizontal surfaces. And

$$\lambda_1 \leq -4H^2 - \frac{8\pi(g-1)}{\text{Area}(\Sigma)} - \frac{1}{\text{Area}(\Sigma)}\int_\Sigma (4(\theta')^2 + (1 - 2\cot(2\theta))\theta''),$$

and the equality holds for any Hopf torus $\Sigma^2 = \Pi^{-1}(\{u\} \times \mathbb{S}^1)$ over any parallel of B. An example of such a function θ is given by $\theta(x) = 1/2\arctan(x)$, with $I = (0, +\infty)$. Observe that if we take $\theta(x) = 1/2\arctan(x) + \pi/4$, then $\kappa - 4\tau^2 < 0$ and Theorem 3.4.4 applies. In this case, the equality can occur in (i) because of $\tau \neq 0$.

Chapter 4

Compact CMC surfaces into 3-dimensional warped products

Abstract. *In this chapter the ambient manifolds in which we are interested are the 3-dimensional warped products. For this reason, we start by recalling such a notion and how to calculate some formulae. We approach the study of these products when the fiber is 2-dimensional, but also we deal with the case of 2-dimensional base. So, in both cases, after imposing that the warped product satisfies a suitable and well-studied convergence condition or some restriction over the warping function, we give sharp upper bounds for the first stability eigenvalue of compact constant mean curvature surfaces in such ambient spaces. As a result, we derive some consequences related to the stability, and finally we illustrate our results with some examples.*

4.1 Introduction to 3-dimensional warped products

Henceforth, we focus our attention on the so-called warped products, which are a generalization of the standard Riemannian products. The notion of warped product appeared both in the mathematical and the physical literature, and they were introduced by Bishop and O'Neill [BON] in 1969, where they used them to provide examples of manifolds of negative curvature. These ambient spaces allow us to obtain many interesting examples of manifolds where their geometry is related to the factors and a certain positive warping function, which turns out very useful in geometry and physics, above all they play a key role in general relativity (see [Ch] and [ON2]). As usual, we introduce its definition and the formulae about them which will be useful for our aim.

Let us establish the definition of a warped product. To do that, let $(B, \langle\ ,\ \rangle_B)$ and $(F, \langle\ ,\ \rangle_F)$ be Riemannian manifolds in such a way that a warped product is merely the product manifold $B \times F$ furnished with a metric which is obtained by homothetically warping the standard product metric above on each fiber. This fact gives rise to construct a rich class of metrics on $B \times F$. Let us formalize its definition.

Definition 4.1.1. Let B and F be Riemannian manifolds, and $\rho : B \longrightarrow (0, +\infty)$ a smooth positive function. The *warped product* $M = B \times_\rho F$ is the product manifold $B \times F$ endowed with the metric tensor

$$\langle\ ,\ \rangle = \Pi_B^*(\langle\ ,\ \rangle_B) + (\rho \circ \Pi_B)^2 \Pi_F^*(\langle\ ,\ \rangle_F),$$

where Π_B and Π_F stand for the canonical projections onto B and F, respectively. The function ρ is called *warping function.*

Note that if $\rho = 1$, then the warped product reduces to a standard Riemannian product manifold. Here and subsequently, for simplicity we use the following customary notation for the warped metric

$$\langle\ ,\ \rangle = \langle\ ,\ \rangle_B + \rho^2 \langle\ ,\ \rangle_F.$$

Now, let us recall some basic notation and definitions of elements related to a warped product $(M = B \times_\rho F, \langle\ ,\ \rangle)$ (see [BON, ON2] for details):

- B is called the *base* and F the *fiber.*
- For each $p \in B$, we call $\{p\} \times F$ a *fiber.* It is well known that the fibers are totally umbilic.
- For each $q \in F$, we call $B \times \{q\}$ a *leaf.* In this case, the leaves are totally geodesic.
- A tangent vector v to M is *horizontal* if v is tangent to a leaf, whereas v is *vertical* if it is tangent to a fiber.

The Riemannian curvature tensor of a warped product was computed by Bishop and O'Neill and they found out that it can be expressed in terms of its warping function ρ and the Riemannian curvature tensors of B and F, respectively. We use the notation R_B and R_F for the lifts to M of the Riemannian curvature tensors of the base B and the fiber F, respectively.

Proposition 4.1.2. *(Lemma 7.4 in [BON]) Let X, Y and Z be horizontal vector fields on M, and let U, V and W be vertical vector fields on M. Then, the Riemannian curvature tensor R of $M = B \times_\rho F$ is given by*

1. $R(X,Y)Z = R_B(X,Y)Z.$
2. $R(V,X)Y = \dfrac{\mathrm{H}^\rho(X,Y)}{\rho} V.$

3. $R(X,Y)V = R(V,W)X = 0.$

4. $R(X,V)W = \dfrac{\langle V,W\rangle}{\rho}\nabla_X(\nabla\rho).$

5. $R(V,W)U = R_F(V,W)U - \dfrac{\langle\nabla\rho,\nabla\rho\rangle}{\rho^2}(\langle V,U\rangle W - \langle W,U\rangle V).$

Consequently, it can be also computed the Ricci curvature of a warped product. Let Ric_B and Ric_F be the lifts to M of the Ricci curvature of B and F, respectively.

Corollary 4.1.3. *(Corollary 43 in Chapter 7 in [ON2]) Let $M = B \times_\rho F$ be a warped product with $d = \dim F$. Let X and Y be horizontal vector fields on M, and let V and W be vertical vector fields on M. Then, the Ricci curvature of M is given by*

1. $\mathrm{Ric}(X,Y) = \mathrm{Ric}_B(X,Y) - \dfrac{d}{\rho}\,\mathrm{H}^\rho(X,Y).$

2. $\mathrm{Ric}(X,V) = 0.$

3. $\mathrm{Ric}(V,W) = \mathrm{Ric}_F(V,W) - \langle V,W\rangle\left(\dfrac{\Delta\rho}{\rho} + (d-1)\dfrac{\langle\nabla\rho,\nabla\rho\rangle}{\rho^2}\right).$

Before finishing this introductory part, let us remind that we just focus on 3-dimensional warped products. That means that we can clearly distinguish two cases, depending on the dimension of the base and the fiber, that is:

- $B \times_\rho F^2$, where the base B is either an open interval $I \subset \mathbb{R}$ or $\mathbb{S}^1$, and the fiber F^2 is any Riemannian surface.
- $B^2 \times_\rho F$, where the fiber F is either an open interval $I \subset \mathbb{R}$ or $\mathbb{S}^1$, and the base B^2 is any Riemannian surface.

Note that the main difference at the time of making computations in these two cases lies in the fact that the warping function depends on the base B, and so, the case in which such a base is 1-dimensional is simpler as well as much more studied in the literature. For this reason, we find reasonable to begin from simpler to more complex. Thus, we devote Section 4.2 to the study of the first type and Section 4.3 to the second one.

4.2 Compact CMC surfaces into $B \times_\rho F^2$

4.2.1 Introduction to $B \times_\rho F^2$

In this section, we pay attention to warped products whose base is of dimension 1 and fiber of dimension 2. There are many contributions which intend to study them since this research field is very fruitful. Among all of them, it is remarkable a work by Montiel [Mo2] in which the author proves that every Riemannian manifold M having a non trivial closed and conformal field is locally isometric to a warped product with a 1-dimensional base. Such a paper has given many ideas to other authors to approach different problems in these ambient spaces.

Furthermore, in recent years, several authors have studied constant mean curvature hypersurfaces (and in particular surfaces) immersed into Riemannian warped product spaces satisfying a suitable convergence condition, which is well established in the current literature (see, for instance [ACdL, AD, dLdL, Mo1, Mo2]). Such a convergence condition is expressed as follows

$$\mathrm{Ric}_F \geq \sup_B\{\rho'^2 - \rho\rho''\}\langle\ ,\ \rangle_F,$$

where Ric_F is the Ricci curvature of the fiber.

Additionally, in the bibliography there are other interesting conditions over the warping function such as to suppose that it is concave, or assume that it is a solution of the Jacobi equation $\rho''+c\rho = 0$, for some $c \in \mathbb{R}$. For the latter case, Montiel [Mo1] asserted (in general for hypersurfaces) the following statement:

Theorem. *(Corollary 6 in [Mo1]) Let $B \times_\rho F^2$ be a warped product satisfying the above convergence condition and whose warping function is a nonconstant solution of $\rho''+c\rho = 0$. Let Σ be a compact two-sided surface of constant mean curvature immersed into $B \times_\rho F^2$. If Σ is weakly stable, then each connected component of Σ is either an umbilical sphere or a slice.*

As for the stability, we can find a recent work by Aledo and Rubio (see [AlR]) in which the authors study the minimal stable surfaces in this framework.

4.2.2 Basic tools and main formulae

Let us fix the notation that we are going to use in this section.

We denote by $M^3 = B \times_\rho F^2$ the warped product of the base (B, dt^2), being an open interval $I \subset \mathbb{R}$ or $\mathbb{S}^1$, and the fiber $(F^2, \langle\ ,\ \rangle_F)$, which is any Riemannian surface, with smooth warping function $\rho : B \longrightarrow (0, +\infty)$. So, M is the product manifold $B \times F^2$ endowed with the metric tensor given by

$$\langle\ ,\ \rangle = dt^2 + \rho(t)^2\langle\ ,\ \rangle_F.$$

Note that ∂_t is a unit horizontal vector field on M. Remind that with this notation, we say that a tangent vector v to M is vertical if it orthogonal to ∂_t (that is, $\langle v, \partial_t\rangle = 0$), whereas v is horizontal if it is collinear with ∂_t.

Now, as done in the previous chapters, we obtain the sectional curvature $K(P)$ of any tangent plane P to $B \times_\rho F^2$, as well as the Ricci curvature.

Proposition 4.2.1. *([MO4]) Let $M = B \times_\rho F^2$ be a 3-dimensional warped product and let K_F be the Gaussian curvature of F.*

(i) The Ricci curvature of $B \times_\rho F^2$ in a unit direction η is given by

$$\mathrm{Ric}(\eta,\eta) = \frac{1-\Theta^2}{\rho^2}K_F + (\Theta^2-1)\frac{\rho'^2}{\rho^2} - (1+\Theta^2)\frac{\rho''}{\rho}, \quad \textit{where } \Theta = \langle \eta, \partial_t\rangle. \tag{4.1}$$

(ii) For every tangent plane P to $B \times_\rho F^2$ with unit normal η, the sectional curvature K of P is given by

$$K(P) = \frac{\Theta^2}{\rho^2}K_F - \Theta^2\frac{\rho'^2}{\rho^2} + (\Theta^2-1)\frac{\rho''}{\rho}. \tag{4.2}$$

Proof. (i) Set a unit vector η, and let $V = \eta^v$ be the vertical part of η. Now, we extend the vectors η and V to local vector fields which are denoted again by η and V, respectively. Therefore, we have η decomposed as

$$\eta = V + \Theta\partial_t.$$

So, from Corollary 4.1.3 we get

$$\begin{aligned}\mathrm{Ric}(\eta,\eta) &= \mathrm{Ric}(V,V) + \Theta^2\,\mathrm{Ric}(\partial_t,\partial_t)\\ &= \mathrm{Ric}_F(V,V) - |V|^2\left(\frac{\Delta\rho}{\rho} + \frac{|\nabla\rho|^2}{\rho^2}\right) - \frac{2\Theta^2}{\rho}\,\mathrm{H}^\rho(\partial_t,\partial_t),\end{aligned} \tag{4.3}$$

where we know that

$$\nabla\rho = \rho'\partial_t, \quad \Delta\rho = \rho'' \quad \text{and} \quad \mathrm{H}^\rho(\partial_t,\partial_t) = \rho''. \tag{4.4}$$

When $V \neq 0$, by taking a unit vertical vector field W orthogonal to V (with respect to the metric of F, i.e., $\langle W,W\rangle_F^2 = 1$), we obtain a local orthonormal frame $\{V/|V|_F, W\}$ on F, and so we can compute $\mathrm{Ric}_F(V,V)$ as follows

$$\mathrm{Ric}_F(V,V) = \langle R_F(V,W)V,W\rangle_F = |V|_F^2 K_F = \frac{|V|^2}{\rho^2}K_F,$$

and since $|V|^2 = \langle \eta - \Theta\partial_t, \eta - \Theta\partial_t\rangle = 1-\Theta^2$ the above equality is simplified to

$$\mathrm{Ric}_F(V,V) = \frac{1-\Theta^2}{\rho^2}K_F,$$

and replacing the last expression and (4.4) in (4.3) we conclude

$$\begin{aligned}\mathrm{Ric}(\eta,\eta) &= \frac{1-\Theta^2}{\rho^2}K_F - (1-\Theta^2)\left(\frac{\rho''}{\rho} + \frac{\rho'^2}{\rho^2}\right) - \frac{2\Theta^2}{\rho}\rho''\\ &= \frac{1-\Theta^2}{\rho^2}K_F + (\Theta^2-1)\frac{\rho'^2}{\rho^2} - (1+\Theta^2)\frac{\rho''}{\rho}.\end{aligned}$$

When $V=0$, since $\Theta^2=1$ we have $\operatorname{Ric}(\eta,\eta)=\operatorname{Ric}(\partial_t,\partial_t)=-2\rho''/\rho$.

(ii) By reasoning analogously, we consider the vertical part $V=\eta^v$ of the normal vector η to P, and we extend such vectors to vector fields denoted by the same letters. So we can write again

$$\eta = V + \Theta\partial_t.$$

Let us make a distinction between the points where $V\neq 0$ and those where $V=0$.

On the one hand, at points where $V\neq 0$, that is, $\eta\neq\pm\partial_t$, we choose a unit vertical vector field W orthogonal to V (with respect to the warped product metric, i.e., $\langle W,W\rangle$=1). Then

$$\begin{aligned} E_1 &= \frac{\Theta}{\sqrt{1-\Theta^2}}V-\sqrt{1-\Theta^2}\partial_t, \\ E_2 &= W, \end{aligned}$$

is a local orthonormal frame on P. Hence, we can compute $K(P)$ by using that frame

$$\begin{aligned} K(P) &= K(E_1,E_2)=\langle R(E_1,E_2)E_1,E_2\rangle \\ &= \frac{\Theta^2}{1-\Theta^2}\langle R(V,W)V,W\rangle+(1-\Theta^2)\langle R(\partial_t,W)\partial_t,W\rangle, \end{aligned} \tag{4.5}$$

where we have taken into account that $\langle R(V,W)\partial_t,W\rangle=\langle R(\partial_t,W)V,W\rangle=0$ because of the symmetries of R and (3) in Proposition 4.1.2. Now, we compute $\langle R(V,W)V,W\rangle$ by using (5) in Proposition 4.1.2 and keeping in mind that $\nabla\rho=\rho'\partial_t$. So, we get

$$\begin{aligned} \langle R(V,W)V,W\rangle &= \langle R_F(V,W)V,W\rangle-\frac{\rho'^2}{\rho^2}|V|^2 \\ &= \rho^2\langle R_F(V,W)V,W\rangle_F-\frac{\rho'^2}{\rho^2}(1-\Theta^2) \\ &= \rho^2\langle V,V\rangle_F\langle W,W\rangle_F K_F+(\Theta^2-1)\frac{\rho'^2}{\rho^2}, \end{aligned}$$

where $\langle V,V\rangle=1-\Theta^2=\rho^2\langle V,V\rangle_F$, and $\langle W,W\rangle=1=\rho^2\langle W,W\rangle_F$, and so

$$\langle R(V,W)V,W\rangle=\frac{1-\Theta^2}{\rho^2}K_F+(\Theta^2-1)\frac{\rho'^2}{\rho^2}.$$

Moreover, we calculate $\langle R(\partial_t,W)\partial_t,W\rangle$ by using (2) in Proposition 4.1.2 and bearing in mind that $\mathrm{H}^\rho(\partial_t,\partial_t)=\rho''$ it is obtained

$$\langle R(\partial_t,W)\partial_t,W\rangle=-\frac{\rho''}{\rho},$$

therefore, substituting the last two expressions in (4.5) we have

$$K(P)=\frac{\Theta^2}{\rho^2}K_F-\Theta^2\frac{\rho'^2}{\rho^2}+(\Theta^2-1)\frac{\rho''}{\rho}.$$

On the other hand, at a point with $V=0$, that is, $\eta=\pm\partial_t$ one has a local orthonormal frame formed by any pair of orthonormal vertical vector fields. A straightforward computation yields the above equation with $\Theta^2=1$. □

4.2.3 Estimates of λ_1 for compact CMC surfaces into $B \times_\rho F^2$

This section is devoted to give sharp upper bounds for the first stability eigenvalue for compact two-sided surfaces with constant mean curvature immersed into the warped products $B \times_\rho F^2$. Moreover, we also intends to characterize the equalities when possible. To do so, we recall the notion of angle function of a two-sided surface Σ with Gauss map denoted by N, as well as the basic properties of a slice.

Definition 4.2.2. With the notation above, the *angle function* of a two-sided surface Σ is the smooth function $\Theta : \Sigma \longrightarrow [-1, 1]$ given by

$$\Theta = \langle N, \partial_t \rangle.$$

In the sequel, we will refer to the fibers $\{t\} \times F$, $t \in B$ as slices. As we have previously observed they are totally umbilic, and additionally $N = \pm\partial_t$, therefore $\Theta^2 \equiv 1$.

So after this brief reminder, we are in a position to observe that in order to achieve our aim, as usual, we use the general bounds (1.12) and (1.13), and so, we must estimate $\mathrm{Ric}(N, N)$, whose expression is given by (4.1), and $2\,K_\Sigma + \mathrm{Ric}(N, N)$, where

$$K_\Sigma = \frac{\Theta^2}{\rho^2} K_F - \Theta^2 \frac{\rho'^2}{\rho^2} + (\Theta^2 - 1)\frac{\rho''}{\rho} \tag{4.6}$$

by the expression (4.2).

At this point, we look for natural assumptions over the warped products in order to be able to bound λ_1. In this case, we are going to work with warped products satisfying the convergence condition mentioned before

$$\mathrm{Ric}_F \geq \sup_B \{\rho'^2 - \rho\rho''\} \langle \, , \, \rangle_F.$$

Since F is 2-dimensional, this condition is equivalent to

$$\min_F K_F \geq \sup_B \{\rho'^2 - \rho\rho''\}. \tag{C.C.1}$$

The geometric interpretation of such assumption was given by Montiel in [Mo1], and specifically, its meaning is that the Ricci curvature on M attains its minimum in the direction of ∂_t, i.e., the direction ∂_t is one of the least Ricci curvature on M.

So, the estimates that we have found under the assumption of the convergence condition (C.C.1) are compiled in the main result of the section as follows.

Theorem 4.2.3. *(Theorem 1 in [MO4]) Let $M = B \times_\rho F^2$ be a warped product satisfying the convergence condition (C.C.1) and Σ a compact two-sided surface of constant mean curvature H immersed into M. If λ_1 stands for the first eigenvalue of its Jacobi operator, then* *(i)* $\lambda_1 \leq -2H^2 + \frac{2}{\text{Area}(\Sigma)}\int_\Sigma \frac{\rho''}{\rho}d\Sigma$, *with equality if and only if* Σ *is a slice* $\{t\}\times F$ *with* F *compact, or* Σ *is a totally umbilic surface contained in a region of* M *which has constant curvature* $K = -\rho''/\rho$*; and*

(ii) $\lambda_1 \leq -4H^2 - \frac{8\pi(g-1)}{\text{Area}(\Sigma)} + \frac{4}{\text{Area}(\Sigma)}\int_\Sigma \frac{\rho''}{\rho}d\Sigma$, *with equality if and only if* $K = -\rho''/\rho$ *over* Σ *and* $G - 2K$ *is constant over* Σ.

Proof. (i) From (4.1) the convergence condition yields

$$\begin{aligned}\text{Ric}(N,N) &= \frac{1-\Theta^2}{\rho^2}K_F + (\Theta^2-1)\frac{\rho'^2}{\rho^2} - (1+\Theta^2)\frac{\rho''}{\rho}\\ &\geq \frac{1-\Theta^2}{\rho^2}(\rho'^2-\rho\rho'') + (\Theta^2-1)\frac{\rho'^2}{\rho^2} - (1+\Theta^2)\frac{\rho''}{\rho} = -2\frac{\rho''}{\rho},\end{aligned} \tag{4.7}$$

and so, from (1.12) we get the stated estimate

$$\lambda_1 \leq -2H^2 + \frac{2}{\text{Area}(\Sigma)}\int_\Sigma \frac{\rho''}{\rho}d\Sigma.$$

Let us analyse the case when the equality holds. The inequalities (1.12) and (4.7) become into equalities. From the first equality, as we have noticed in (i) of Remark 1.4.1, we know that the surface Σ is totally umbilic, and $\text{Ric}(N,N)$ is constant over Σ, but because of the second mentioned equality we have $\text{Ric}(N,N) = -2\frac{\rho''}{\rho}$, thus $\frac{\rho''}{\rho}$ is constant over Σ. On the other hand, it is direct that the equality in (4.7) is equivalent to

$$(1-\Theta^2)(K_F - (\rho'^2 - \rho\rho'')) = 0.$$

Therefore, if the first factor in the above equality vanishes, then $\Theta^2 = 1$ and consequently Σ is a slice. Otherwise, when Σ is not a slice, let us consider the non empty subset of the surface defined by

$$\Sigma' := \{p \in \Sigma : K_F = \rho'^2 - \rho\rho''\}.$$

Then, Σ is cointained in the region $R = \Pi_B(\Sigma') \times \Pi_F(\Sigma')$ of M. Since ρ''/ρ is constant over Σ, we deduce that

$$\rho'' = a\rho$$

on $\pi_B(\Sigma)$ for a certain constant $a \in \mathbb{R}$, and by integrating we get

$$\rho'^2 = a\rho^2 + b,$$

with $b \in \mathbb{R}$. So, as a consequence of the last two identities we deduce $\rho'^2 - \rho\rho'' = b$. Thus,

$$K_F = \rho'^2 - \rho\rho'' = b$$

over the region R. Now, by the formula (4.2) we conclude that the sectional curvature of every tangent plane P to $B \times_\rho F^2$ reduces to

$$K(P) = -\frac{\rho''}{\rho},$$

that is, K does not depend on the associated angle function Θ, and in fact, $K = -\rho''/\rho = a$ is constant on the region R.

Conversely, the total umbilicity implies that $|A|^2 = 2H^2$. Moreover, let us check that in both cases $\mathrm{Ric}(N,N) = -2\frac{\rho''}{\rho}$. If Σ is a slice, that means that $\Theta^2 = 1$ and so the assertion is clear from (4.1). Otherwise, the assumption $-\rho''/\rho$ constant on R implies that $K_F = \rho'^2 - \rho\rho''$ as we have seen in this proof, and once again from (4.1) we conclude that $\mathrm{Ric}(N,N) = -2\frac{\rho''}{\rho}$. Then

$$J = \Delta + |A|^2 + \mathrm{Ric}(N,N) = \Delta + 2H^2 - 2\frac{\rho''}{\rho}.$$

Since $2H^2 - 2\frac{\rho''}{\rho}$ is constant, we get $\lambda_1 = -2H^2 + 2\frac{\rho''}{\rho}$.

(ii) From (4.6) the convergence condition implies

$$\begin{aligned} K_\Sigma &= \frac{\Theta^2}{\rho^2}K_F - \Theta^2\frac{\rho'^2}{\rho^2} + (\Theta^2 - 1)\frac{\rho''}{\rho} \\ &\geq \frac{\Theta^2}{\rho^2}(\rho'^2 - \rho\rho'') - \Theta^2\frac{\rho'^2}{\rho^2} + (\Theta^2 - 1)\frac{\rho''}{\rho} = -\frac{\rho''}{\rho} \end{aligned} \tag{4.8}$$

and therefore, from (4.7) and the last inequality we get

$$2\,K_\Sigma + \mathrm{Ric}(N,N) \geq -4\frac{\rho''}{\rho}.$$

From the second upper bound (1.13) we easily obtain

$$\lambda_1 \leq -4H^2 - \frac{8\pi(g-1)}{\mathrm{Area}(\Sigma)} + \frac{4}{\mathrm{Area}(\Sigma)}\int_\Sigma \frac{\rho''}{\rho}d\Sigma.$$

Let us study what occurs when the equality is satisfied. The inequalities (1.13), (4.7) and (4.8) also turn into equalities. Due to (ii) in Remark 1.4.1 we know that the equality in (1.13) implies that $\lambda_1 = -|A|^2 - \mathrm{Ric}(N,N)$. On the other hand, the equality in (4.7) yields

$$(1-\Theta^2)(K_F - (\rho'^2 - \rho\rho'')) = 0,$$

and the equality in (4.8) gives

$$\Theta^2(K_F - (\rho'^2 - \rho\rho'')) = 0,$$

and so, by combining both of the last identities we follow $K_F = \rho'^2 - \rho\rho''$ over Σ. Again, by (4.2) $K = -\rho''/\rho$. In this way, by using the Gauss equation we have

$$\lambda_1 = -|A|^2 - \mathrm{Ric}(N,N) = -4H^2 - 2K + 2G - \mathrm{Ric}(N,N) = -4H^2 + 2G + 4\frac{\rho''}{\rho}.$$

So, $G + 2\rho''/\rho = G - 2K$ is constant over Σ.

Conversely, $K = -\rho''/\rho$ leads to $K_F = \rho'^2 - \rho\rho''$ over Σ, then the equalities hold in (4.7) and (4.8). Therefore

$$J = \Delta + 4H^2 - 2G - 4\frac{\rho''}{\rho}.$$

As $4H^2 - 2G - 4\frac{\rho''}{\rho}$ is constant, we deduce that $\lambda_1 = -4H^2 + 2G + 4\frac{\rho''}{\rho}$ and by integrating, the Gauss-Bonnet theorem enables us to assert that the equality in (ii) follows. □

Remark 4.2.4. It is worth pointing out that, when F^2 is compact, the slices $\{t\} \times F^2$, $t \in B$, have constant mean curvature $H = -\frac{\rho'(t)}{\rho(t)}$ and $\mathrm{Ric}(N, N) = -2\frac{\rho''(t)}{\rho(t)}$, so without any other assumption (neither the convergence condition), they satisfy the equality in (i), which writes as

$$\lambda_1 = \frac{2}{\rho(t)^2}(\rho(t)\rho''(t) - \rho'(t)^2). \tag{4.9}$$

Therefore, under the convergence condition, if there exists at least one slice $\{t_0\} \times F^2$ which is not stable, then

$$K_F \geq \sup_B \{\rho'^2 - \rho\rho''\} \geq \rho'(t_0)^2 - \rho(t_0)\rho''(t_0) = -\frac{1}{2}\rho(t_0)^2\lambda_1 > 0,$$

or equivalently, if F^2 has a point with non-positive curvature, then all slices are stable.

On the other hand, from (ii) we deduce

$$\lambda_1 \geq \frac{8\pi(g-1)}{\mathrm{Area}(\Sigma)}.$$

Thus provided $g \geq 1$, all slices are also stable. For $g = 0$, let us note that $(0, +\infty) \times_{\sinh t} \mathbb{S}^2$ satisfies the convergence condition (C.C.1) and all slices are unstable.

Remark 4.2.5. We would like to note that Perdomo has pointed out in a review of our work [MO4] that the region in (i) of Theorem 4.2.3 may reduce to just the surface Σ. Let us give his illustrative example.

Take the base $B = \mathbb{S}^1$ and the fiber F^2 the compact surface immersed into $\mathbb{R}^3$ defined by

$$F := \{(x, y, z) \in \mathbb{R}^3 : (y^2 + z^2)^2 = 1 - x^4\}.$$

We are going to consider the warping function as the constant $\rho = 1$, and the surface given by $\Sigma = \mathbb{S}^1 \times \sigma$, where

$$\sigma := \{(x, y, z) \in F^2 : x = 0, y^2 + z^2 = 1\}.$$

Note that $\Theta = \langle N, \partial_t \rangle = 0$, so by using (4.1) we have

$$\mathrm{Ric}(N, N) = K_F.$$

Since Σ is totally geodesic, the Jacobi operator is

$$J = \Delta + |A|^2 + \mathrm{Ric}(N, N) = \Delta + K_F,$$

where K_F vanishes over the surface Σ. Hence, $\lambda_1 = 0$, and so it is clear that the equality is attained in (i) of Theorem 4.2.3. Since Σ is not a slice, there must exist a region R on $B \times_\rho F^2$ containing the surface, such that $K = 0$ over R.

By the construction of the region in the last proof, we know that Σ' coincides with the surface and consequently, such a region R coincides with the own surface Σ. And of course, $K_\Sigma = 0$ as we can calculate from (4.6). That is, the region R can be reduced to the own surface Σ when the warping function is constant.

On the other hand, let us observe that Σ is a torus, and so its genus is $g = 1$, which means that the equality is also satisfied in (ii).

A case that deserves special focus is when the warping function satisfies the Jacobi equation

$$\rho'' + c\rho = 0,$$

for some $c \in \mathbb{R}$. Observe that in this case, $\rho'^2 - \rho\rho''$ is constant. Furthermore, from (4.1) we get

$$\mathrm{Ric}(\partial_t, \partial_t) = -2\frac{\rho''}{\rho} = 2c,$$

since $\Theta = 0$. Hence, this assumption over the warping function means that the Ricci curvature of M in the direction ∂_t is constant.

In this way, for a warped product $B \times_\rho F^2$ verifying (C.C.1) with warping function which is a solution of the aforementioned Jacobi equation we can apply Theorem 4.2.3.

Proposition 4.2.6. *(Proposition 4 in [MO4]) Let $M = B \times_\rho F^2$ be a warped product with $\rho'' + c\rho = 0$, for some $c \in \mathbb{R}$, that satisfies the convergence condition (C.C.1), and Σ a compact two-sided surface of constant mean curvature H immersed into M. If λ_1 stands for the first eigenvalue of its Jacobi operator, then*

(i) $\lambda_1 \leq -2(H^2 + c)$, with equality if and only if Σ is a slice $\{t\} \times F$ with F compact, or Σ is a totally umbilic surface contained in a region of M with constant curvature $K = c$; and

(ii) $\lambda_1 \leq -4(H^2 + c) - \dfrac{8\pi(g-1)}{\mathrm{Area}(\Sigma)}$, with equality if and only if Σ has constant Gaussian curvature, and it is contained in a region of M with constant curvature $K = c$.

Proof. Note that $\rho''/\rho = -c =$ constant. So, Theorem 4.2.3 is simplified to the claim. □

Now, in view of the upper bounds in Theorem 4.2.3, we can easily establish the following results for concave warping functions, that is $\rho'' \leq 0$.

Proposition 4.2.7. *(Proposition* 10 *in [MO4]) Let* $M = B \times_\rho F^2$ *be a warped product with concave warping function satisfying the convergence condition (C.C.1), and* Σ *a compact two-sided surface of constant mean curvature* H *immersed into* M*. If* λ_1 *stands for the first eigenvalue of its Jacobi operator, then*

(i) $\lambda_1 \leq -2H^2$*, with equality if and only if* Σ *is a slice* $\{t\} \times F$ *with* F *compact and* $\rho''(t) = 0$*, or* Σ *is a totally umbilic surface contained in a flat region of* M *and* ρ *is linear in that region; and*

(ii) $\lambda_1 \leq -4H^2 - \dfrac{8\pi(g-1)}{\mathrm{Area}(\Sigma)}$*, with equality if and only if* Σ *is a surface with constant curvature contained in a flat region of* M *and* ρ *is linear in that region.*

Proof. The upper bounds follow immediately from Theorem 4.2.3. When the equality holds in (i) or (ii), necessarily $\rho'' = 0$. □

4.2.4 Stability of compact CMC surfaces into $B \times_\rho F^2$

When searching for stable surfaces into these ambient spaces, we find some restrictions over the square of the mean curvature as shown in previous chapters.

To start with, we assume that the warped product satisfies (C.C.1), and without any other constraint, by applying Theorem 4.2.3 we obtain the following estimates.

Corollary 4.2.8. *(Corollary* 3 *in [MO4]) Let* $M = B \times_\rho F^2$ *be a warped product satisfying the convergence condition (C.C.1). If* Σ *is a stable compact two-sided surface of constant mean curvature* H *immersed into* M *then*

(i) $H^2 \leq \dfrac{1}{\mathrm{Area}(\Sigma)} \displaystyle\int_\Sigma \frac{\rho''}{\rho} d\Sigma$*, with equality if and only if* Σ *is a slice* $\{t\} \times F$ *with* F *compact and* $\rho'(t)^2 - \rho(t)\rho''(t) = 0$*, or* Σ *is a totally umbilic surface with* $H^2 = \rho''/\rho$ *contained in a region of* M *with non-positive constant curvature* $K = -H^2$*; and*

(ii) $H^2 \leq -\dfrac{2\pi(g-1)}{\mathrm{Area}(\Sigma)} + \dfrac{1}{\mathrm{Area}(\Sigma)} \displaystyle\int_\Sigma \frac{\rho''}{\rho} d\Sigma$*, with equality if and only if* $2H^2 = G + 2\rho''/\rho$*.*

Proof. Since Σ is stable, we know that $\lambda_1 \geq 0$, so (i) in Theorem 4.2.3 gives

$$0 \leq \lambda_1 \leq -2H^2 + \frac{2}{\mathrm{Area}(\Sigma)} \int_\Sigma \frac{\rho''}{\rho} d\Sigma$$

and the expected inequality is derived.

Now, if the equality holds, then the chain of inequalities do as well, and it implies from (i) in Theorem 4.2.3 that there are two possibilities: if Σ is a slice, due to the equation (4.9) in Remark 4.2.4 we have

$$0 = \lambda_1 = \frac{2}{\rho(t)^2}(\rho(t)\rho''(t) - \rho'(t)^2),$$

concluding that $\rho'(t)^2 - \rho(t)\rho''(t) = 0$; otherwise, Σ is a totally umbilic surface contained in a region of M with constant curvature $K = -\rho''/\rho$, and so

$$0 = \lambda_1 \leq -2H^2 + \frac{2}{\text{Area}(\Sigma)} \int_\Sigma \frac{\rho''}{\rho} d\Sigma,$$

which allows us to get $H^2 = \rho''/\rho$ and $K = -H^2$. Conversely, in both cases we have $H^2 = \rho''/\rho$.

(ii) Analogously, from (ii) of Theorem 4.2.3 we know

$$0 \leq \lambda_1 \leq -4H^2 - \frac{8\pi(g-1)}{\text{Area}(\Sigma)} + \frac{4}{\text{Area}(\Sigma)} \int_\Sigma \frac{\rho''}{\rho} d\Sigma,$$

and the second inequality for H^2 follows.

If the equality is attained, (ii) of Theorem 4.2.3 ensures that $K = -\rho''/\rho$ over Σ and $G - 2K$ is constant over Σ, and so

$$0 = \lambda_1 = -4H^2 + 2G + 4\frac{\rho''}{\rho}.$$

The converse is deduced by integrating. □

By combining Proposition 4.2.6 with Corollary 6 in [Mo1] we get a result for compact stable surfaces of constant mean curvature into $B \times_\rho F^2$ by assuming that (C.C.1) is satisfied and $\rho'' + c\rho = 0$. To be specific, when $c < 0$ we conclude a non existence result, and otherwise we obtain the full classification of the possible surfaces.

Corollary 4.2.9. *(Corollary 5 in [MO4]) Let $M = B \times_\rho F^2$ be a warped product with ρ a positive non-constant solution of $\rho'' + c\rho = 0$, for some $c \in \mathbb{R}$, that satisfies the convergence condition (C.C.1).*

(i) There exists no stable CMC compact two-sided surface in M with $c \geq 0$.

(ii) For $c < 0$, Σ is a stable CMC compact two-sided surface in M if and only if Σ is a slice $\{t\} \times F$ with F compact and $\rho'(t)^2/\rho(t)^2 \leq -c$. In this case

$$|c|\rho^2 \geq |c\rho^2 + \rho'^2| \geq \frac{4(g-1)}{\text{Area}(F)}.$$

Proof. Suppose that Σ is stable. Then from Proposition 4.2.6

$$0 \leq \lambda_1 \leq -2(H^2 + c),$$

which implies that $H^2 + c \leq 0$. This forces c to be negative or zero, but if $c = 0$, then $H = 0$. In Theorem 5 and Corollary 6 of [Mo1], the author proves that there are not weakly stable surfaces when $c = H = 0$, and consequently, neither stable ones. In such results, the author also proves that a weakly stable surface Σ under our assumptions is either a totally umbilic sphere with curvature $H^2 + c > 0$ or a slice. Therefore, only the second one can occur for which

$$\frac{\rho'^2}{\rho^2} + c = H^2 + c \leq 0.$$

Conversely, if Σ is a slice with $\rho'^2/\rho^2 \leq -c$, then from Proposition 4.2.6 we have

$$\lambda_1 = -2(c + \rho'^2/\rho^2) \geq 0.$$

The inequalities follow combining (ii) with the equality in (i) of the above corollary and the fact that $\mathrm{Area}(\Sigma) = \rho^2 \mathrm{Area}(F)$. □

Let us note that in the above inequality, g is also the genus of F. So, the larger genus of F or the smaller area of F, the larger ρ will have to be to find a stable slice.

We can also derive an interesting consequence about stability for compact surfaces with constant mean curvature in warped products whose warping function is concave.

Corollary 4.2.10. *(Corollary 11 in [MO4]) Let $M = B \times_\rho F^2$ be a warped product with concave warping function satisfying the convergence condition (C.C.1).*

(i) There exists no stable non-minimal compact two-sided surface of constant mean curvature in M.

(ii) If Σ is a stable minimal compact two-sided surface in M, then Σ is either a totally geodesic topological sphere or a flat totally geodesic topological torus.

Proof. The assertion (i) is obvious. Now, if Σ is minimal and stable, then from Proposition 4.2.7(i) Σ is totally umbilic, and so totally geodesic. Moreover, from (ii) we have

$$0 = \lambda_1 \leq -\frac{8\pi(g-1)}{\mathrm{Area}(\Sigma)},$$

which implies $g \leq 1$. If $g = 1$, the equality holds in Proposition 4.2.7(ii) and then $G = K_\Sigma = 0$ by the Gauss equation. □

4.2.5 Some examples of $B \times_\rho F^2$

Let us give some interesting examples of warped products in which we can apply our results. With this aim, we must look for significant warping functions. For instance, we have highlighted the particular case of those which satisfies the Jacobi equation

$$\rho'' + c\rho = 0.$$

For these special functions, when we require the warped product $B \times_\rho F^2$ to be complete, it has to be either a Euclidean space or a sphere or a pseudo-hyperbolic space $\mathbb{R} \times_{e^t} F^2$ or $\mathbb{R} \times_{\cosh t} F^2$ (see also [Mo1, Ta]). The first two are well known, so here it is worth pointing out the consequences for the latter two by using Proposition 4.2.6.

Example 4.2.11. Let us consider a compact two-sided surface Σ of constant mean curvature H immersed into a pseudo-hyperbolic space $B \times_\rho F^2$.

- When $\rho = e^t$, i.e. $M = B \times_{e^t} F^2$, where F^2 is a surface with non-negative Gaussian curvature, we can obtain the following estimates for λ_1. On the one hand

  $$\lambda_1 \leq -2(H^2 - 1),$$

  with equality if and only if Σ is a slice $\{t\} \times F$ with F compact, or Σ is a totally umbilic surface contained in a region of M with constant curvature $K = -1$. And

  $$\lambda_1 \leq -4(H^2 - 1) - \frac{8\pi(g-1)}{\text{Area}(\Sigma)},$$

  with equality if and only if Σ is a surface with constant curvature contained in a region of M with constant curvature $K = -1$.

- When $\rho = \cosh t$, i.e. $M = B \times_{\cosh t} F^2$, where F^2 is a surface with Gaussian curvature $K_F \geq -1$, the resulting bounds are

  $$\lambda_1 \leq -2(H^2 - 1),$$

  with equality if and only if Σ is a slice $\{t\} \times F$ with F compact, or Σ is a totally umbilic surface contained in a region of M with constant curvature $K = -1$. And

  $$\lambda_1 \leq -4(H^2 - 1) - \frac{8\pi(g-1)}{\text{Area}(\Sigma)},$$

  with equality if and only Σ is a surface with constant curvature contained in a region of M with constant curvature $K = -1$.

In both cases, when F is compact and since $\rho'^2/\rho^2 \leq -c$, we can derive the following classification result.

Corollary 4.2.12. *(Corollary 9 in [MO4]) The only stable compact two-sided surfaces of constant mean curvature in a pseudo-hyperbolic space with compact fiber are the slices.*

4.3 Compact CMC surfaces into $B^2 \times_\rho F$

4.3.1 Basic tools and main formulae

Let us set up the notation that we are going to use in this section.

We denote by $M^3 = B^2 \times_\rho F$ the warped product of the base $(B^2, \langle\, ,\, \rangle_B)$, which is any Riemannian surface, and the fiber (F, dt^2), being an open interval $I \subset \mathbb{R}$ or $\mathbb{S}^1$, with smooth warping function $\rho : B^2 \longrightarrow (0, +\infty)$. So, M is the product manifold $B^2 \times F$ endowed with the metric tensor given by

$$\langle\, ,\, \rangle = \langle\, ,\, \rangle_B + \rho(q)^2 dt^2.$$

In what follows, we consider $U = \frac{1}{\rho}\partial_t$ which is a unit vertical vector field defined over M.

At this point, following the same scheme that the one in the previous section, we obtain the sectional curvature $K(P)$ of any tangent plane P to $B^2 \times_\rho F$, and its Ricci curvature.

Proposition 4.3.1. *([MO3]) Let $M = B^2 \times_\rho F$ be a 3-dimensional warped product and let K_B be the Gaussian curvature of B.*

(i) The Ricci curvature of $B^2 \times_\rho F$ in a unit direction η is given by

$$\mathrm{Ric}(\eta, \eta) = (1 - \Theta^2)K_B - \frac{1}{\rho}\,\mathrm{H}^\rho(X, X) - \Theta^2 \frac{\Delta\rho}{\rho}, \tag{4.10}$$

where $\Theta = \langle \eta, U \rangle$, and X is the horizontal part of η.

(ii) For every tangent plane P to $B^2 \times_\rho F$ with unit normal η, the sectional curvature K of P is given by

$$K(P) = \Theta^2 K_B + \frac{1}{\rho}\,\mathrm{H}^\rho(X, X) + (\Theta^2 - 1)\frac{\Delta\rho}{\rho}. \tag{4.11}$$

Proof. (i) For a unit vector η, let $X = \eta^h$ be the horizontal part of η. Now, we extend the vectors η and X to vector fields which are denoted again by the same letters η and X, respectively. Therefore, we have η decomposed as

$$\eta = X + \Theta U,$$

and by using Corollary 4.1.3 we get

$$\begin{aligned} \mathrm{Ric}(\eta, \eta) &= \mathrm{Ric}(X, X) + \Theta^2\, \mathrm{Ric}(U, U) \\ &= \mathrm{Ric}_B(X, X) - \frac{1}{\rho}\,\mathrm{H}^\rho(X, X) - \Theta^2 \frac{\Delta\rho}{\rho}. \end{aligned} \tag{4.12}$$

Now, at points where $X \neq 0$ let us take a unit horizontal vector field Y orthogonal to X, so that we obtain a local orthonormal frame $\{X/|X|, Y\}$ on B. In this way, we can compute $\mathrm{Ric}_B(X, X)$ as follows

$$\mathrm{Ric}_B(X, X) = \langle R(X, Y)X, Y \rangle = |X|^2 K_B,$$

and since $|X|^2 = \langle \eta - \Theta U, \eta - \Theta U\rangle = 1 - \Theta^2$ the above equality is simplified to

$$\mathrm{Ric}_B(X, X) = (1 - \Theta^2)K_B,$$

and substituting this expression in (4.12) we conclude

$$\mathrm{Ric}(\eta, \eta) = (1 - \Theta^2)K_B - \frac{1}{\rho}\,\mathrm{H}^\rho(X, X) - \Theta^2\frac{\Delta\rho}{\rho}.$$

This formula is also true at points where $X = 0$, as $\Theta^2 = 1$ and consequently

$$\mathrm{Ric}(\eta, \eta) = \frac{1}{\rho^2}\,\mathrm{Ric}(\partial_t, \partial_t) = -\frac{\Delta\rho}{\rho}.$$

(ii) We consider again the horizontal part $X = \eta^h$ of the normal vector η to P, and we extend such vectors to vector fields denoted by the same letters, so η is decomposed as

$$\eta = X + \Theta U.$$

Let us make a distinction between the points where $X \neq 0$ and the ones where $X = 0$.

Where $X \neq 0$, that is, $\eta \neq \pm U$, we choose a unit horizontal vector field Y orthogonal to X, then

$$\begin{aligned} E_1 &= \frac{\Theta}{\sqrt{1-\Theta^2}}X - \sqrt{1-\Theta^2}U, \\ E_2 &= Y \end{aligned}$$

is a local orthonormal frame for P. So, we are in a position to compute $K(P)$ by using that frame as follows

$$\begin{aligned} K(P) &= K(E_1, E_2) = \langle R(E_1, E_2)E_1, E_2\rangle \\ &= \frac{\Theta^2}{1-\Theta^2}\langle R(X, Y)X, Y\rangle + \frac{1-\Theta^2}{\rho^2}\langle R(\partial_t, Y)\partial_t, Y\rangle, \end{aligned} \tag{4.13}$$

where we have used that $\langle R(X, Y)U, Y\rangle = \langle R(U, Y)X, Y\rangle = 0$ because of the symmetries of R and (3) in Proposition 4.1.2. On the one hand, by using (1) in Proposition 4.1.2 we get

$$\langle R(X, Y)X, Y\rangle = \langle R_B(X, Y)X, Y\rangle = |X|^2K_B = (1 - \Theta^2)K_B,$$

and, on the other hand, because of (4) in Proposition 4.1.2 we have

$$\langle R(U, Y)U, Y\rangle = \frac{1}{\rho^2}\langle R(\partial_t, Y)\partial_t, Y\rangle = -\frac{1}{\rho}\langle\nabla_Y(\nabla\rho), Y\rangle.$$

Hence, by substituting in (4.13) the last two expressions, we derive

$$\begin{aligned} K(P) &= \Theta^2K_B + (\Theta^2 - 1)\frac{\langle\nabla_Y(\nabla\rho), Y\rangle}{\rho} \\ &= \Theta^2K_B + (\Theta^2 - 1)\frac{\mathrm{H}^\rho(Y, Y)}{\rho}. \end{aligned}$$

Since the Laplacian operator Δ on B (note that we use the same notation for the Laplacian since no confusion can arise) of the function ρ can be written in terms of the local orthonormal frame $\{X/|X|, Y\}$ on B as $\Delta\rho = \frac{1}{|X|^2}\,\mathrm{H}^\rho(X,X) + \mathrm{H}^\rho(Y,Y)$, we finally obtain

$$K(P) = \Theta^2 K_B + \frac{1}{\rho}\,\mathrm{H}^\rho(X,X) + (\Theta^2 - 1)\frac{\Delta\rho}{\rho}.$$

Where $X = 0$ this formula is satisfied. In such points, one can consider a local orthonormal frame on P formed by any pair of orthonormal horizontal vector fields, and so $K(P) = K_B$ (Note that $\Theta^2 = 1$). □

4.3.2 Estimates of λ_1 for compact CMC surfaces into $B^2 \times_\rho F$

Our intention in this section is similar to the one in Section 4.2.3, that is, we find out upper bounds for λ_1 of compact two-sided surfaces of constant mean curvature immersed into $B^2 \times_\rho F$ in terms of the warping function ρ and the geometry of B. At the same time, we study what conditions the surface must satisfy to attain the equalities. To proceed with, let us introduce the angle function of a two-sided surface Σ immersed into $B^2 \times_\rho F$, where N is its Gauss map.

Definition 4.3.2. With the above notation, the angle function Θ of the surface Σ is the smooth function $\Theta : \Sigma \longrightarrow [-1, 1]$, given by

$$\Theta = \langle N, U \rangle.$$

Now, in order to obtain the expected estimates, we have to keep in mind that once again we will use (1.12) and (1.13); so we must get $\mathrm{Ric}(N,N)$, whose expression is given by (4.10), and $2\,K_\Sigma + \mathrm{Ric}(N,N)$, but from (4.11) we easily get

$$K_\Sigma = \Theta^2 K_B + \frac{1}{\rho}\,\mathrm{H}^\rho(X,X) + (\Theta^2 - 1)\frac{\Delta\rho}{\rho},$$

where X is the horizontal part of N.

Observe that the last formula and (4.10) yield to a suitable relationship between K_Σ and $\mathrm{Ric}(N,N)$ as follows,

$$K_\Sigma = K_B - \mathrm{Ric}(N,N) - \frac{\Delta\rho}{\rho}. \tag{4.14}$$

Finally, we are interested in an explicit expression of the integrand which appears in (1.13). With the above formulae we know that it is written as

$$2\,K_\Sigma + \mathrm{Ric}(N,N) = (1+\Theta^2)K_B + \frac{1}{\rho}\,\mathrm{H}^\rho(X,X) + (\Theta^2 - 2)\frac{\Delta\rho}{\rho}. \tag{4.15}$$

In this way, with the purpose of giving the desired estimates for the first eigenvalue of the Jacobi operator, we need to have some control over the Hessian of the warping function, since the obtained expressions for $\mathrm{Ric}(N,N)$ and $2\,K_\Sigma + \mathrm{Ric}(N,N)$ depend on it. To this end, let us define a function on B as

$$\mu(q) := \max\left\{ -\frac{1}{\rho}\,\mathrm{H}^\rho_q(v,v) : v \in T_qB,\ |v| = 1 \right\}.$$

We are now ready to deduce two upper bounds for λ_1 without any additional assumption. In order to do this, we use the following inequality,

$$(\Theta^2 - 1)\mu \leq (1-\Theta^2)\frac{\Delta\rho}{\rho} - \frac{1}{\rho}\,\mathrm{H}^\rho(X,X), \tag{4.16}$$

which is always satisfied. Where $X = 0$, since $\Theta^2 = 1$, the terms found in the last inequality vanish. Otherwise, we can consider a local orthonormal frame $\{X/|X|, Y\}$ on B in such a way that the Laplacian operator Δ on B satisfies

$$(1-\Theta^2)\frac{\Delta\rho}{\rho} - \frac{1}{\rho}\,\mathrm{H}^\rho(X,X) = (1-\Theta^2)\frac{1}{\rho}\,\mathrm{H}^\rho(Y,Y),$$

and consequently, it implies trivially the above inequality because of the definition of μ.

Now, from (4.10) and (4.16) we get a lower bound for the Ricci curvature as

$$\begin{aligned} \mathrm{Ric}(N,N) &= (1-\Theta^2)K_B - \frac{1}{\rho}\,\mathrm{H}^\rho(X,X) - \Theta^2\frac{\Delta\rho}{\rho} \\ &\geq (1-\Theta^2)(K_B - \mu) - \frac{\Delta\rho}{\rho}. \end{aligned} \tag{4.17}$$

On the other hand, because of the definition of μ we have

$$\frac{1}{\rho}\,\mathrm{H}^\rho(X,X) \geq (\Theta^2 - 1)\mu,$$

and

$$\frac{\Delta\rho}{\rho} = \frac{1}{(1-\Theta^2)\rho}\,\mathrm{H}^\rho(X,X) + \frac{1}{\rho}\,\mathrm{H}^\rho(Y,Y) \geq -2\mu \tag{4.18}$$

and so combining these inequalities with (4.15) we get

$$\begin{aligned} 2\,K_\Sigma + \mathrm{Ric}(N,N) &= (1+\Theta^2)K_B + \frac{1}{\rho}\,\mathrm{H}^\rho(X,X) + (\Theta^2 - 2)\frac{\Delta\rho}{\rho} \\ &\geq (1+\Theta^2)(K_B - \mu) - 2\frac{\Delta\rho}{\rho}. \end{aligned} \tag{4.19}$$

Thus, the inequalities (4.17) and (4.19) together with (1.12) and (1.13), respectively, allow us to obtain two general upper bounds for λ_1 which are true for any compact two-sided surface Σ of constant mean curvature H immersed into any warped product $M = B^2 \times_\rho F$:

$$\lambda_1 \leq -2H^2 - \frac{1}{\mathrm{Area}(\Sigma)}\int_\Sigma \left((1-\Theta^2)\,(K_B - \mu) - \frac{\Delta\rho}{\rho} \right) d\Sigma, \tag{4.20}$$

$$\lambda_1 \leq -4H^2 - \frac{8\pi(g-1)}{\mathrm{Area}(\Sigma)} - \frac{1}{\mathrm{Area}(\Sigma)}\int_\Sigma \left((1+\Theta^2)\,(K_B - \mu) - 2\frac{\Delta\rho}{\rho} \right) d\Sigma. \tag{4.21}$$

Paying attention to the above upper bounds, it seems to be natural studying the case $K_B \geq \mu$ in order to obtain simpler ones. It is worth noting that this condition is equivalent to

$$\mathrm{Ric}_p(v,v) \leq 2K_B, \text{ for all } v \in T_pM, \ |v| = 1, p \in M. \tag{C.C.2}$$

In what follows, for simplicity of notation, we sometimes use $\mathrm{Ric} \leq 2K_B$ to indicate the condition (C.C.2).

Let us check that $K_B \geq \mu$ if and only if (C.C.2) is satisfied. Suppose that $K_B \geq \mu$. Let us take a point $p = (q,t) \in B^2 \times_\rho F$, and let $v \in T_pM$ be a unit vector, then we have

$$\mathrm{Ric}_p(v,v) = \mathrm{Ric}_p(v^h, v^h) + \langle v, U\rangle^2 \, \mathrm{Ric}_p(U,U).$$

When $v^h \neq 0$, since $K_B \geq \mu$ we get from (4.10) that

$$\begin{aligned} \mathrm{Ric}_p(v^h, v^h) &= |v^h|^2 \, \mathrm{Ric}_p\left(\frac{v^h}{|v^h|}, \frac{v^h}{|v^h|}\right) = |v^h|^2 \left(K_B - \frac{1}{\rho}\, \mathrm{H}^\rho_q\left(\frac{v^h}{|v^h|}, \frac{v^h}{|v^h|}\right)\right) \\ &\leq (1 - \langle v, U\rangle^2)(K_B + \mu) \leq 2(1 - \langle v, U\rangle^2)K_B, \end{aligned}$$

and, on the other hand, since we know that $\frac{\Delta\rho}{\rho} \geq -2\mu$, it is easy to check that

$$\mathrm{Ric}_p(U,U) = -\frac{\Delta\rho}{\rho} \leq 2\mu \leq 2K_B.$$

Therefore, by combining the last expressions we obtain

$$\mathrm{Ric}_p(v,v) \leq 2(1 - \langle v, U\rangle^2)K_B + 2\langle v, U\rangle^2 K_B = 2K_B.$$

On the contrary, when $v^h \equiv 0$ it is clear that

$$\mathrm{Ric}_p(v,v) = \mathrm{Ric}_p(U,U) \leq 2K_B.$$

Conversely, if $\mathrm{Ric} \leq 2K_B$ for any unit vector $v \in T_qB$ we have $K_B - \frac{1}{\rho}\, \mathrm{H}^\rho_q(v,v) = \mathrm{Ric}_p(v,v) \leq 2K_B$, concluding $K_B \geq \mu$.

Now, it deserves to observe that we can interpret the condition (C.C.2) as a natural convergence condition which is similar to the assumption (C.C.1) that we have considered in the previous section. Specifically, its geometric interpretation is that the sectional curvatures K of M attains a maximum equal to K_B at any point.

Let us prove, indeed, that $K_B \geq \mu$ means that $K(P) \leq K_B$ for every tangent plane P to $B^2 \times_\rho F$ with unit normal η. If $K_B \geq \mu$, from (4.17) we get

$$\mathrm{Ric}(\eta,\eta) \geq (1 - \Theta^2)(K_B - \mu) - \frac{\Delta\rho}{\rho} \geq -\frac{\Delta\rho}{\rho}.$$

Now, observe that the formula (4.14) can be used to compute any sectional curvature of M, and so, the last inequality leads to

$$K(P) = K_B - \mathrm{Ric}(\eta,\eta) - \frac{\Delta\rho}{\rho} \leq K_B + \frac{\Delta\rho}{\rho} - \frac{\Delta\rho}{\rho} = K_B,$$

as expected.

So, the estimates for λ_1 that we have got under the assumption of the convergence condition (C.C.2) are compiled in the main result of the section as follows.

Theorem 4.3.3. *(Theorem 1 in [MO3]) Let $M = B^2 \times_\rho F$ be a warped product satisfying the convergence condition (C.C.2), and Σ be a compact two-sided surface of constant mean curvature H immersed into M. If λ_1 stands for the first eigenvalue of its Jacobi operator, then*

(i) $\lambda_1 \leq -2H^2 + \frac{1}{\mathrm{Area}(\Sigma)} \int_\Sigma \frac{\Delta\rho}{\rho} d\Sigma$, *with equality if and only if Σ is totally umbilic with $K_\Sigma = K_B$ and $\frac{\Delta\rho}{\rho}$ constant over Σ; and*

(ii) $\lambda_1 \leq -4H^2 - \frac{8\pi(g-1)}{\mathrm{Area}(\Sigma)} + \frac{2}{\mathrm{Area}(\Sigma)} \int_\Sigma \frac{\Delta\rho}{\rho} d\Sigma$, *with equality if and only if* $\mathrm{Ric} = 2K_B$ *over Σ and $G - 2K_B$ is constant over Σ.*

Proof. (i) From the bound (4.20) the convergence condition $K_B \geq \mu$ yields

$$\begin{aligned} \lambda_1 &\leq -2H^2 - \frac{1}{\mathrm{Area}(\Sigma)} \int_\Sigma \left((1-\Theta^2)(K_B - \mu) - \frac{\Delta\rho}{\rho} \right) d\Sigma \\ &\leq -2H^2 + \frac{1}{\mathrm{Area}(\Sigma)} \int_\Sigma \frac{\Delta\rho}{\rho} d\Sigma, \end{aligned} \tag{4.22}$$

and so we conclude the expected inequality.

Let us analyse the case when the equality holds. The inequalities (1.12) and (4.22) turn into equalities. As observed in (i) of Remark 1.4.1, the first equality implies that Σ is totally umbilic and $\mathrm{Ric}(N,N)$ is constant over Σ, whereas from the second one we have $\mathrm{Ric}(N,N) = -\frac{\Delta\rho}{\rho}$. Therefore, $\frac{\Delta\rho}{\rho}$ is constant over Σ. In addition, (4.14) shows that $\mathrm{Ric}(N,N) = -\frac{\Delta\rho}{\rho}$ is equivalent to

$$K_\Sigma = K_B - \mathrm{Ric}(N,N) - \frac{\Delta\rho}{\rho} = K_B$$

over Σ.

Conversely, by the total umbilicity we have $|A|^2 = 2H^2$, and $\mathrm{Ric}(N,N) = -\frac{\Delta\rho}{\rho}$, and then

$$J = \Delta + 2H^2 - \frac{\Delta\rho}{\rho}.$$

Since $\frac{\Delta\rho}{\rho}$ is constant over Σ, we get $\lambda_1 = -2H^2 + \frac{\Delta\rho}{\rho}$.

(ii) Since $K_B \geq \mu$, the bound (4.21) is reduced to

$$\begin{aligned}\lambda_1 &\leq -4H^2 - \frac{8\pi(g-1)}{\mathrm{Area}(\Sigma)} - \frac{1}{\mathrm{Area}(\Sigma)}\int_\Sigma \left((1+\Theta^2)(K_B-\mu) - 2\frac{\Delta\rho}{\rho}\right) d\Sigma \\ &\leq -4H^2 - \frac{8\pi(g-1)}{\mathrm{Area}(\Sigma)} + \frac{2}{\mathrm{Area}(\Sigma)}\int_\Sigma \frac{\Delta\rho}{\rho} d\Sigma, \end{aligned} \tag{4.23}$$

which yields the announced estimate.

Let us study what occurs when the equality is attained. The inequalities (1.12), (4.18) and (4.23) turn into equalities, and so it follows that $\lambda_1 = -|A|^2 - \mathrm{Ric}(N,N)$, $\mu = K_B$ and $\frac{\Delta\rho}{\rho} = -2K_B$ at points of Σ. By definition of $\Delta\rho$ and μ, the last equation implies that $-\frac{1}{\rho}\,\mathrm{H}_q^\rho(v,v) = K_B$ for all $v \in T_qB$, $|v| = 1$, $p = (q,t) \in \Sigma$. We also have $\mathrm{Ric}_p(U,U) = 2K_B$, $p \in \Sigma$, and hence, we get

$$\mathrm{Ric}_p(v,v) = 2K_B$$

for all $v \in T_pM$, $|v| = 1$, $p \in \Sigma$. From here, $K_\Sigma = K_B$ by (4.14). Then,

$$\lambda_1 = -|A|^2 + \mathrm{Ric}(N,N) = -4H^2 + 2G - 4K_B$$

by the Gauss equation. Therefore $G - 2K_B$ is constant over Σ.

Reciprocally, under the hypotheses,

$$\mathrm{Ric}(N,N) = 2K_B = \mathrm{Ric}(U,U) = -\frac{\Delta\rho}{\rho}.$$

So, from (4.14) we have $K_\Sigma = K_B$ and then

$$J = \Delta + |A|^2 + \mathrm{Ric}(N,N) = \Delta + 4H^2 - 2G + 4K_B.$$

Since $G - 2K_B$ is constant over Σ, we have $\lambda_1 = -4H^2 + 2G - 4K_B$ which proves the equality by integrating. □

Remark 4.3.4. A special case of surface immersed into the warped product $B^2 \times_\rho F$ is given by the leaves $B^2 \times \{t\}$, $t \in F$. When B^2 is compact, they are totally geodesic compact two-sided surfaces, so $|A|^2 = 0$, $K_\Sigma = K_B$ and $\mathrm{Ric}(N,N) = -\frac{\Delta\rho}{\rho}$ because $N = \pm U$.

Moreover, they satisfy equality in (i) when $\frac{\Delta\rho}{\rho}$ is constant over Σ, which is equivalent to $\frac{\Delta\rho}{\rho}$ constant. Since $\rho > 0$, by integrating over the compact surface B the constant $\frac{\Delta\rho}{\rho}$ we conclude that ρ has to be constant. In this case, the leaves have $\lambda_1 = 0$ and so they are stable.

Now, it is worth paying attention to the special case $\Delta\rho \leq 0$, that is, ρ superharmonic when the warped product $B^2 \times_\rho F$ verifies the convergence condition (C.C.2).

Corollary 4.3.5. *(Corollary 4 in [MO3]) Let $M = B^2 \times_\rho F$ be a warped product with superharmonic warping function satisfying the convergence condition (C.C.2), and Σ a compact two-sided surface of constant mean curvature H immersed into M. If λ_1 stands for the first eigenvalue of its Jacobi operator, then* *(i)* $\lambda_1 \leq -2H^2$, *with equality if and only if* Σ *is totally umbilic with* $\mathrm{Ric}(N,N)=0$*; and*

(ii) $\lambda_1 \leq -4H^2 - \dfrac{8\pi(g-1)}{\mathrm{Area}(\Sigma)}$, *with equality if and only if* Σ *has constant Gaussian curvature, and* $\mathrm{Ric}=0$ *and* $K_B=0$ *over* Σ.

Proof. It is a straightforward consequence of the theorem. For the equalities, $\Delta\rho=0$ over Σ and the assertions hold. □

Another particular case where we can apply our general upper bounds for λ_1 and obtain simpler and useful ones is when the warping function ρ is convex, that is $\mathrm{H}^\rho_q(v,v)\geq 0$ for all $v\in T_qB$, $q\in B$, and for strictly convex functions (strict inequality). Observe that we can get many interesting valid examples of the form $\rho=e^h$, with $h: B\longrightarrow\mathbb{R}$ an arbitrary convex function, because of

$$\mathrm{H}^\rho(v,v)=\rho(\langle\nabla h,v\rangle^2+\mathrm{H}^h(v,v))\geq 0.$$

And if we do not want ρ to be constant, the volume of the surface B cannot be finite and so B must be non-compact (see [BON]). Also, in the same work, the authors prove that the warped product $B^2\times_\rho F$ has negative curvature if and only if ρ is strictly convex and $K_B<0$. They also construct many examples of such warping functions over surfaces with nonpositive curvature. The following theorem gives upper bounds for λ_1 for surfaces immersed in such warped products.

Theorem 4.3.6. *(Theorem 6 in [MO3]) Let* $M=B^2\times_\rho F$ *be a warped product where* B *is a surface of nonpositive curvature* ($K_B\leq 0$) *and the warping function* ρ *is convex. Let* Σ *be a compact two-sided surface of constant mean curvature* H *immersed into* M*. If* λ_1 *stands for the first eigenvalue of its Jacobi operator, then*

(i) $\lambda_1 \leq -2H^2 - \dfrac{1}{\mathrm{Area}(\Sigma)}\displaystyle\int_\Sigma\left(K_B-\frac{\Delta\rho}{\rho}\right)$, *with equality if and only if* Σ *is totally umbilic,* $G=H^2$ *and* $K_B-\frac{\Delta\rho}{\rho}$ *constant over* Σ*; and*

(ii) $\lambda_1 \leq -4H^2 - \dfrac{8\pi(g-1)}{\mathrm{Area}(\Sigma)} - \dfrac{2}{\mathrm{Area}(\Sigma)}\displaystyle\int_\Sigma\left(K_B-\frac{\Delta\rho}{\rho}\right)$, *with equality if and only if* $|A|$ *is constant and* $\mathrm{Ric}(N,N)=0$.

Proof. Observe that ρ convex implies $\mu\leq 0$ due to the definition of μ. Hence, this fact together with $K_B\leq 0$ allow us to assert that the integrands in (4.20) and (4.21) are bounded below, respectively, by

$$(1-\Theta^2)\,(K_B-\mu)-\frac{\Delta\rho}{\rho}\geq K_B-\frac{\Delta\rho}{\rho}, \tag{4.24}$$

$$(1+\Theta^2)\,(K_B-\mu)-2\frac{\Delta\rho}{\rho}\geq 2\left(K_B-\frac{\Delta\rho}{\rho}\right),$$

and so we get the inequalities.

If the equality holds in (i), the same occurs in (1.12), (4.17) and (4.24), and then from (i) of Remark (1.4.1) Σ is totally umbilic and $\mathrm{Ric}(N,N) = K_B - \frac{\Delta\rho}{\rho}$, which is a constant, implies that $K_\Sigma = 0$ by (4.14). By the Gauss equation it follows that

$$2H^2 = |A|^2 = 4H^2 - 2G,$$

that is, $G = H^2$. Conversely, we have

$$J = \Delta + |A|^2 + \mathrm{Ric}(N,N) = \Delta + 2H^2 + K_B - \frac{\Delta\rho}{\rho}$$

which yields $\lambda_1 = -2H^2 - K_B + \frac{\Delta\rho}{\rho}$.

Finally for the equality in (ii), as in the previous theorem, the involved inequalities turn into equalities which gives $\lambda_1 = -|A|^2 - \mathrm{Ric}(N,N)$ (from (ii) of Remark (1.4.1)), $\mu = 0$,

$$(1+\Theta^2)K_B = 2K_B \qquad \text{and} \qquad 2\,K_\Sigma + \mathrm{Ric}(N,N) = 2K_B - 2\frac{\Delta\rho}{\rho}.$$

From (4.14), we deduce that $\mathrm{Ric}(N,N) = 0$, and it implies $\lambda_1 = -|A|^2$. Conversely, we get $J = \Delta + |A|^2$ with $|A|$ constant, which implies $\lambda_1 = -|A|^2$, and by the Gauss equation

$$\lambda_1 = -4H^2 + 2G - 2\,K_\Sigma$$

that yields the equality by (4.14). □

Remark 4.3.7. Under the hypotheses of the above theorem, when ρ is supposed to be strictly convex, the inequalities in (i) and (ii) are strict. When B is supposed to have negative curvature ($K_B < 0$), the equality in (ii) is also strict.

4.3.3 Stability of compact CMC surfaces into $B^2 \times_\rho F$

As we have done up to now, we are going to mention some interesting consequences related to the stability of the surfaces.

To start with, let us suppose that the warped product $B^2 \times_\rho F$ satisfies the convergence condition (C.C.2), and by applying Theorem 4.3.3 we can find the following restrictions for the surface.

Corollary 4.3.8. *(Corollary 3 in [MO3]) Let $M = B^2 \times_\rho F$ be a warped product satisfying the convergence condition (C.C.2). If Σ is a stable compact two-sided surface of constant mean curvature H immersed into M then*

(i) $H^2 \leq \dfrac{1}{2\,\mathrm{Area}(\Sigma)} \displaystyle\int_\Sigma \frac{\Delta\rho}{\rho}$ for $g \leq 1$, with equality if and only if Σ is totally umbilic and $\frac{\Delta\rho}{\rho} = 2H^2$ over Σ; and

(ii) $H^2 \leq -\dfrac{2\pi(g-1)}{\mathrm{Area}(\Sigma)} + \dfrac{1}{2\,\mathrm{Area}(\Sigma)} \displaystyle\int_\Sigma \frac{\Delta\rho}{\rho}$ *for* $g \geq 1$, *with equality if and only if* $G + \frac{\Delta\rho}{\rho} = 2H^2$ *over* Σ.

Proof. As Σ is stable, we know that $\lambda_1 \geq 0$ and the upper bounds are a direct consequence from Theorem 4.3.3 as usual.

Moreover, if the equality holds in (i), we get $0 = \lambda_1 = -2H^2 + \frac{\Delta\rho}{\rho}$ and the assertion follows. When the equality holds in (ii), we have $0 = \lambda_1 = -4H^2 + 2G + 2\frac{\Delta\rho}{\rho}$. We get the converse by integrating. □

Besides imposing the cited convergence condition, when we assume that the warping function ρ is superharmonic, from Corollary 4.3.5 we deduce that if Σ is not minimal then it cannot be stable; and neither minimal surfaces with genus $g \geq 2$.

On the other hand, let us analyse the case in which the warped product $B^2 \times_\rho F$ satisfies that $K_B \leq 0$ and ρ is convex. So, as a direct application of Theorem 4.3.6 we can claim the following statement.

Corollary 4.3.9. *(Corollary 8 in [MO3]) Let* $M = B^2 \times_\rho F$ *be a warped product where* B^2 *is a surface of nonpositive curvature* ($K_B \leq 0$) *and the warping function* ρ *is convex. If* Σ *is a stable compact two-sided surface of constant mean curvature* H *immersed into* M*, then*

(i) $H^2 \leq -\dfrac{1}{2\,\mathrm{Area}(\Sigma)} \displaystyle\int_\Sigma \left(K_B - \frac{\Delta\rho}{\rho}\right)$ *for* $g \leq 1$, *with equality if and only if* Σ *is totally umbilic and* $K_B - \frac{\Delta\rho}{\rho} = -2H^2$*; and*

(ii) $H^2 \leq -\dfrac{2\pi(g-1)}{\mathrm{Area}(\Sigma)} - \dfrac{1}{2\,\mathrm{Area}(\Sigma)} \displaystyle\int_\Sigma \left(K_B - \frac{\Delta\rho}{\rho}\right)$ *for* $g \geq 1$, *with equality if and only if* Σ *is totally geodesic and* $\mathrm{Ric}(N,N) = 0$.

4.3.4 Some examples of $B^2 \times_\rho F$

To finish this section, let us see some interesting examples that illustrate the results presented.

Example 4.3.10. Consider the warped product $\mathbb{D}^2 \times_\rho F$ where $\mathbb{D}^2$ is the Poincaré disk model, that is, $\mathbb{D}^2 = \{x \in \mathbb{R}^2 : |x| < 1\}$ endowed with the metric

$$ds^2 = \frac{4|dx|^2}{(1-|x|^2)^2},$$

and the warping function ρ is defined as

$$\rho(x) = \frac{1+|x|^2}{1-|x|^2}$$

for any $x = (x_1, x_2) \in \mathbb{D}^2$. Observe that Nitsche's model of 3-dimensional hyperbolic space is just the warped product $\mathbb{D}^2 \times_f \mathbb{R}$ (see [Ni] for the details concerning to this model).

Since the Hessian of the function ρ can be expressed in coordinates as

$$\mathrm{H}^\rho(\partial_i, \partial_j) = \frac{\partial^2 \rho}{\partial x^i \partial x^j} - \sum_{k=1}^{2} \Gamma_{ij}^k \frac{\partial \rho}{\partial x^k},$$

it is a straightforward computation to check that

$$\mathrm{H}^\rho(\partial_1, \partial_1) = \mathrm{H}^\rho(\partial_2, \partial_2) = \frac{4(1+|x|^2)}{(1-|x|^2)^3},$$

and $\mathrm{H}^\rho(\partial_1, \partial_2) = 0$. Let us proceed to check it by following the notation in [ON2]:

$$E = G = \frac{4}{(1-|x|^2)^2}, \text{ and } F = 0$$

so $Q = EG - F^2 = 16/(1-|x|^2)^4$, and

$$E_{x_1} = G_{x_1} = \frac{16x_1}{(1-|x|^2)^3} = Qx_1(1-|x|^2),$$

and analogously

$$E_{x_2} = G_{x_2} = Qx_2(1-|x|^2).$$

Then, the Christoffel symbols are calculated as follows:

$$Q\Gamma_{11}^1 = \begin{vmatrix} \frac{1}{2}Qx_1(1-|x|^2) & 0 \\ -\frac{1}{2}Qx_2(1-|x|^2) & Q^{1/2} \end{vmatrix}, \quad Q\Gamma_{11}^2 = \begin{vmatrix} Q^{1/2} & \frac{1}{2}Qx_1(1-|x|^2) \\ 0 & -\frac{1}{2}Qx_2(1-|x|^2) \end{vmatrix},$$

$$Q\Gamma_{12}^1 = \begin{vmatrix} \frac{1}{2}Qx_2(1-|x|^2) & 0 \\ \frac{1}{2}Qx_1(1-|x|^2) & Q^{1/2} \end{vmatrix}, \quad Q\Gamma_{12}^2 = \begin{vmatrix} Q^{1/2} & \frac{1}{2}Qx_2(1-|x|^2) \\ 0 & \frac{1}{2}Qx_1(1-|x|^2) \end{vmatrix},$$

$$Q\Gamma_{22}^1 = \begin{vmatrix} -\frac{1}{2}Qx_1(1-|x|^2) & 0 \\ \frac{1}{2}Qx_1(1-|x|^2) & Q^{1/2} \end{vmatrix}, \quad Q\Gamma_{22}^2 = \begin{vmatrix} Q^{1/2} & -\frac{1}{2}Qx_1(1-|x|^2) \\ 0 & \frac{1}{2}Qx_2(1-|x|^2) \end{vmatrix},$$

that is, $\Gamma_{11}^1 = \Gamma_{12}^2 = -\Gamma_{22}^1 = 2x_1/(1-|x|^2)$, and $\Gamma_{11}^2 = -\Gamma_{12}^1 = -\Gamma_{22}^2 = -2x_2/(1-|x|^2)$.

Moreover, we also have

$$\frac{\partial \rho}{\partial x_1} = \frac{4x_1}{(1-|x|^2)^2}, \qquad \frac{\partial \rho}{\partial x_2} = \frac{4x_2}{(1-|x|^2)^2},$$

and

$$\frac{\partial^2 \rho}{\partial x_1^2} = \frac{4(1+3x_1^2-x_2^2)}{(1-|x|^2)^3}, \qquad \frac{\partial^2 \rho}{\partial x_2^2} = \frac{4(1-x_1^2+3x_2^2)}{(1-|x|^2)^3}, \qquad \frac{\partial^2 \rho}{\partial x_1 \partial x_2} = \frac{16x_1x_2}{(1-|x|^2)^3}.$$

Therefore, as announced earlier we get

$$\mathrm{H}^\rho(\partial_1, \partial_1) = \frac{4(1+3x_1^2-x_2^2)}{(1-|x|^2)^3} - \frac{8x_1^2}{(1-|x|^2)^3} + \frac{8x_2^2}{(1-|x|^2)^3} = \frac{4(1+|x|^2)}{(1-|x|^2)^3},$$

$$\mathrm{H}^\rho(\partial_2, \partial_2) = \frac{4(1-x_1^2+3x_2^2)}{(1-|x|^2)^3} + \frac{8x_1^2}{(1-|x|^2)^3} - \frac{8x_2^2}{(1-|x|^2)^3} = \frac{4(1+|x|^2)}{(1-|x|^2)^3},$$

$$\mathrm{H}^\rho(\partial_1, \partial_2) = \frac{16x_1x_2}{(1-|x|^2)^3} - \frac{8x_1x_2}{(1-|x|^2)^3} - \frac{8x_1x_2}{(1-|x|^2)^3} = 0.$$

From here we easily conclude $\mathrm{H}^\rho_x(v,v) = \rho$ for any unit vector $v \in T_x\mathbb{D}^2$, $x \in \mathbb{D}^2$.

In other words, this warped product satisfies $-\frac{1}{\rho}\mathrm{H}^\rho_x(v,v) = K_B = -1$, and consequently we are under the assumptions of Theorem 4.3.3, in fact, $\Delta\rho = 2\rho$ and so $\mathrm{Ric} = -2 = 2K_B$. Then, if Σ is a compact two-sided surface of constant mean curvature H immersed into $\mathbb{D}^2 \times_\rho F$ we have

(i) $\lambda_1 \leq -2(H^2-1)$, with equality if and only if Σ is totally umbilic with $K_\Sigma = -1$; and

(ii) $\lambda_1 \leq -4(H^2-1) - \dfrac{8\pi(g-1)}{\mathrm{Area}(\Sigma)}$, with equality if and only if G is constant.

Observe that these bounds coincide with the ones that we have got in Theorem 2.2.1.

Example 4.3.11. Let $B^2 \times_\rho F$ the warped product where the base is the hemisphere defined as

$$B^2 = \mathbb{S}^2 \cap \{(x,y,z) \in \mathbb{R}^3 : z > 0\},$$

and the warping function is given by the height function $\rho : B^2 \longrightarrow (0, +\infty)$

$$\rho(q) = \langle q, w \rangle$$

for any $q \in B$, with $w \in B$ fixed. This warped product also verifies $-\frac{1}{\rho}\mathrm{H}^\rho_q(v,v) = K_B = 1$ for any unit vector $v \in T_qB$, with $q \in B$. For this purpose, we just use the fact that if $\gamma : I \longrightarrow B^2$ is a geodesic satisfying $\gamma(0) = q \in B$ and $\gamma'(0) = v \in T_qB$, then

$$\mathrm{H}^\rho_q(v,v) = (\rho \circ \gamma)''(0),$$

and thereby, $\mathrm{H}^\rho_q(v,v) = \langle \gamma''(0), w \rangle = -|v|^2\langle q, w \rangle$. To check the last equality we just take a geodesic of the sphere, whose expression is given by

$$\gamma(t) = \cos(|v|t)q + \frac{1}{|v|}\sin(|v|t)v$$

and consequently $\gamma''(0) = -|v|^2 q$.

Hence, Theorem 4.3.3 can be applied again since $\Delta\rho = -2\rho$ and $\mathrm{Ric} = 2 = 2K_B$, obtaining the following bounds.

(i) $\lambda_1 \leq -2(H^2+1)$, with equality if and only if Σ is totally umbilic with $K_\Sigma = 1$; and

(ii) $\lambda_1 \leq -4(H^2+1) - \dfrac{8\pi(g-1)}{\text{Area}(\Sigma)}$, with equality if and only if G is constant.

From here, we deduce that there are no stable compact two-sided surfaces of constant mean curvature immersed into $B^2 \times_\rho F$.

Observe that the first example, using the isometry between the Poincaré disk and the hyperbolic space $\mathbb{H}^2$ in $\mathbb{R}^3$, reduces to consider it furnished with the height function as the present example.

Example 4.3.12. Note that a warped product $B \times_\rho F$, where B is a flat surface, satisfies the convergence condition $\text{Ric} \leq 0$ if and only if ρ is convex. Therefore, both Theorem 4.3.3 and Theorem 4.3.6 are the same.

As we have said in the previous section, it is easy to construct many convex warping functions by considering $\rho = e^h$ with $h : B \longrightarrow \mathbb{R}$ an arbitrary convex function. In particular, when $B = \mathbb{R}^2$ we only have to pick for h any standard convex function over $\mathbb{R}^2$, or directly any standard positive convex function.

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