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# Analysis and Design of Reset Control Systems

# Análisis y Diseño de Sistemas de Control Reseteados

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# Análisis y Diseño de Sistemas de Control Reseteados

Tesis Doctoral


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Ph.D. Thesis


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# Abstract

This thesis is focused on the field of reset control, which emerged more than 50 years ago with the main goal of overcoming the fundamental limitations of the linear compensators. In the last years, a multitude of works have shown the potential benefits of the reset compensation. Nevertheless, there are only a few works dealing with two of the most common limitations in industrial process control: the time-delay and the saturation. It is well-known that both limitations may lead into a detriment of the performance, and even the destabilization of the closed-loop system. Therefore, this thesis aim at analyzing the stability of the reset control systems under these limitations.

First, we address the stability analysis of time-delay reset control systems. We develop stability criteria for impulsive delay dynamical systems, based on the Lyapunov-Krasovskii method. The applicability of the results to time-delay reset control systems arises naturally, since they are a particular class of impulsive systems. We first focus our attention on impulsive delay systems with state-dependent resetting law. The application of the criteria to reset control systems supposes an improvement of the results in the literature, guaranteeing the stability of the system for larger values of the time-delay. In addition, in order to overcome the limitations of the base system, we consider the stabilization of impulsive delay systems by imposing time-dependent conditions on the reset intervals. As a result, stability criterion is developed for impulsive systems, and in particular time-delay reset control systems, with unstable base system. Finally, the new criterion is used to establish conditions for the stability of a reset control system containing a proportional-integral plus Clegg (PI+CI) integrator compensator.

On the other hand, we study the stability of the reset control systems in presence of plant input saturation. In particular, we propose a method to obtain an approximation of the region of attraction based on a representation of the behavior of the reset control

system by polytopes and directed graphs.

From the application point of view, it is considered the PI+CI compensator. We study several modifications of the compensator which are proposed in the literature, for instance variable band resetting law and variable reset ratio. In addition, we propose new design improvements such as switching reset ratio, in order to enhance the performance in real applications. On the other hand, it is developed a systematic method for PI+CI tuning, based on the analysis and optimization of the impulse response of a set of linear and time invariant systems. Therefore, simple tuning rules have been developed for processes modeled by first and second order systems, and integrator plus dead time systems. Finally, the superior performance of the PI+CI compensator with the proposed tuning method is shown in real experiments of process control. First, the control of an in-line pH process is performed, showing that the tuning rules for first and second order systems can be applied to more complex processes. Second, we accomplish the control of water level in a tank. In both cases, the PI+CI supposes an improvement of the performance in comparison to its linear counterpart PI. In particular, the reset compensation provides a faster response with smaller overshoot and settling time.

# Resumen

Esta tesis se enmarca principalmente en el area de los sistemas de control reseteados, con especial énfasis en el análisis de estabilidad y diseño de los sistemas de control compuestos de controladores reseteados. El control mediante controladores reseteados surgió hace 60 años con el claro objetivo de superar las limitaciones fundamentales de los controladores lineales. H. W. Bode demostró en 1940 que existe una relación entre la fase y la magnitud de la función de respuesta en frecuencia de un sistema lineal e invariante en el tiempo. Esta y otras propiedades han sido utilizadas desde entonces para caracterizar que es posible conseguir mediante un control lineal, es decir cuales son las limitaciones fundamentales de los controladores lineales. Un forma de superar estas limitaciones es buscar controladores no lineales que no estén sometidos a ellas. Esta idea es explotada por primera vez en 1958, cuando J. C. Clegg introduce un nuevo controlador no lineal, el cual consiste en un integrador cuyo estado es puesto a cero siempre que su entrada es igual a cero. Este integrador no lineal es la primera propuesta de controlador reseteado y es conocido como Integrador de Clegg. Un análisis del Integrador de Clegg mediante la función descriptiva revela un aumento de fase de 52° con respecto al integrador lineal, a consta de un pequeño aumento de la magnitud. Esta característica del Integrador de Clegg lo convierte en un potencial elemento para la superación de las limitaciones fundamentales de los controladores lineales.

En los últimos años una multitud de trabajos han mostrado los potenciales beneficios de la compensación mediante acciones de reset. Sin embargo, solo unos pocos trabajos abordan dos de las limitaciones mas comunes en las aplicaciones de control de procesos como es el retardo y la saturación, los cuales imponen fuertes restricciones sobre el sistema de control y producen un deterioro del rendimiento. En particular, el retardo disminuye la fase del sistema a altas frecuencias, siendo necesario aumentarla para mantener la estabilidad del sistema, lo que conlleva a su vez un aumento de la magnitud y por lo tanto de la sensibilidad al ruido. Por otra parte, la saturación limita la actuación del controlador sobre la planta, rompiendo la interconexión entre el controlador y la

planta cuando los limites de la saturación son alcanzados. Este hecho puede suponer la aparición de ciclos limites, inestabilidad, etc. Por lo tanto, es imprescindible el desarrollo de herramientas de análisis de los sistemas de control reseteados en presencia de retardo y saturación. Por otra parte, hay dos factores importantes para que los controladores reseteados puedan ser satisfactoriamente aplicados en procesos industriales. En primer lugar su diseño debe ser sencillo, de manera que puedan ser fácilmente comprendidos por los ingenieros de control e implementados en los sistemas de automatización industriales. En segundo lugar, es necesario desarrollar métodos sencillos que puedan ser utilizados para el ajuste de los parámetros de diseño de los controladores.

En este contexto, los objetivos principales de esta tesis son: el desarrollo de criterios que permitan garantizar la estabilidad de los sistemas de control recetados con retardo, caracterizar la región de convergencia asintótica de los sistemas de control reseteados en presencia de saturación a la entrada de la planta, desarrollar un método sistemático para el diseño de controladores reseteados, que permita cuando sea posible la obtención de reglas de sintonización sencillas, y finalmente mostrar los beneficios de los controladores reseteados y las reglas de ajuste propuestas mediante experimentos de control en varios procesos industriales.

Los sistemas de control reseteados implican la combinación de dinámicas continuas y discretas lo que conlleva un compartimiento híbrido complejo. Como consecuencia de la dinámica híbrida, el sistema puede experimentar varios comportamientos patológicos, como son *beating*, *deadlock* y soluciones *Zeno*. Desde el punto de vista del control, este tipo de comportamientos son indeseables y deben ser evitados. Por ello es necesario una correcta descripción matemática de los sistemas de control reseteados. En esta tesis se adopta el marco de los sistemas dinámicos impulsivos, siendo los sistemas descritos mediante ecuaciones diferenciales impulsivas. En trabajos recientes se ha demostrado que este marco es adecuado para la definición de los sistemas de control reseteados, garantizando propiedades fundamentales como la existencia y unicidad de las soluciones, y continuidad de las soluciones ante condiciones iniciales. Por otra parte, los sistemas de control reseteados con retardo son descritos mediante ecuaciones diferenciales funcionales e impulsivas, adoptando la técnica de regularización temporal para garantizar la correcta definición del sistema. La utilización del marco de los sistemas impulsivos proporciona un formalismo matemático solido y además permite que resultados específicos para sistemas de control reseteados puedan ser generalizados y aplicados a otros tipos de sistemas impulsivos.

En relación a la estabilidad de los sistemas de control reseteados con retardo, en primer lugar mediante el método de Lyapunov-Krasovskii se desarrollan condiciones generales que permiten garantizar la estabilidad de sistemas impulsivos con retardo cuyo sistema base es no lineal y variable en el tiempo. El criterio general es particularizado a un conjunto de funcionales de Lyapunov-Krasovskii (LK), obteniendo condiciones en forma de desigualdades matriciales lineales, que pueden ser fácilmente comprobadas mediante algoritmos de optimización convexa. Para el caso de sistemas impulsivos con sistema base lineal e invariante en el tiempo se propone una extensión de los criterios para garantizar la estabilidad del sistema cuando el retardo es desconocido pero pertenece a un intervalo prefijado. Los resultados son directamente obtenidos mediante argumentos de convexidad. Por otra parte, un conjunto de ejemplos muestran que los nuevos criterios de estabilidad aplicados a sistemas de control reseteados con retardo proporcionan resultados menos conservadores que los trabajos anteriores. En particular, es posible garantizar la estabilidad del sistema para retardos mas grandes. A pesar de la mejora obtenida, los nuevos criterios comparten una fuerte limitación con los resultados anteriores. En todos los casos es necesario que el sistema base sea estable. Por lo tanto, aunque en la práctica el sistema reseteado pueda ser estable para retardos mayores, su estabilidad no puede ser garantizada. Con el objetivo de superar las limitaciones del sistema base, se propone la estabilización de sistemas impulsivos mediante condiciones temporales en los intervalos de reset. Además, inspirados por los nuevos resultados en el campo de los sistemas muestreados, se contempla la combinación del funcional LK con un funcional auxiliar, de manera que imponiendo condiciones a la suma de los dos funcionales se consiguen criterios de estabilidad menos conservadores. La principal ventaja de este método es que permite que el funcional LK aumente durante los saltos y durante la dinámica de flujo, siempre y cuando disminuya entre un instante de reset y otro. De esta forma se permite que el sistema base sea inestable. Varios ejemplos de sistemas impulsivos y sistemas de control reseteados muestran una clara mejora con respecto a los resultados previos, garantizando por primera vez la estabilidad de un sistema de control reseteado con retardo cuyo sistema base es inestable. Finalmente, se considera el caso de un sistema de control reseteado compuesto por un controlador proporcional-integral mas un Integrador de Clegg (PI+CI). La realización en el espacio de estados del controlador PI+CI conlleva una serie de problemas. En particular, el sistema base resultante no es asintóticamente estable y por lo tanto los criterios de estabilidad propuestos en trabajos previos no son aplicables. Sin embargo, el criterio de estabilidad propuesto en esta tesis carece de dicha limitación, lo que permite su aplicación a este tipo de sistema de control reseteados.

Otro objetivo de la tesis es la caracterización de la región de atracción de los sistemas de control reseteados en presencia de saturación a la entrada de la planta. En los resultados previos se propone una aproximación elipsoidal de la región de atracción, la cual es obtenida mediante el segundo método de Lyapunov. Sin embargo, en esta tesis se propone una aproximación mediante la unión de politopos, que siendo no convexa proporciona mayor flexibilidad. La idea principal del método propuesto es la división del espacio de estados mediante politopos, y la utilización de un grafo dirigido cuyos nodos son los politopos y los enlaces entre los nodos representan las trayectorias del sistema fluyendo a través de los politopos. La representación mediante un grafo directo permite caracterizar de forma compacta el comportamiento del sistema de control reseteado. Además, se desarrollan un conjunto de resultados teóricos que garantizan la existencia y unicidad de las soluciones, y permiten comprobar la existencia de conjuntos invariantes en el interior de los politopos. Mediante estos resultados teóricos es posible determinar si las trayectorias del sistema convergen al origen para un determinado conjunto de condiciones iniciales, es decir una aproximación de la región de atracción. Con el objetivo de obtener estimaciones mas grandes se propone un algoritmo iterativo de modificación del grafo mediante particiones de los politopos. Finalmente, varios ejemplos muestran que la estimación de la región de atracción obtenida mediante el procedimiento propuesto es significativamente mayor que la obtenida con los resultados previos.

Desde el punto de vista práctico, se considera la utilización de un controlador PI+CI. Al igual que el controlador proporcional-integral (PI), la principal ventaja del controlador PI+CI es el balance entre rendimiento y sencillez, proporcionado un buen rendimiento en una amplia variedad de ampliaciones con una estructura muy sencilla. Además, su gran similitud con el bien conocido controlador PI hace que su aplicación en la industria sea mas sencilla. Por otro lado, en trabajos previos se han propuesto múltiples mejoras de diseño del controlador, por ejemplo el uso de una banda de reset variable y un porcentaje de reset variable. En esta tesis se realiza un análisis de las mejoras y se proponen nuevas modificaciones del controlador, por ejemplo el uso de un porcentaje de reset conmutado que permite utilizar las acciones de reset para mejorar tanto el seguimiento de referencias como el rechazo de perturbaciones. Por otra parte, se desarrolla un método sistemático para el diseño del controlador PI+CI, basado en la representación de la respuesta del sistema de control reseteado mediante la respuesta impulsional de un conjunto de sistemas lineales e invariantes en el tiempo. El análisis y optimización del conjunto de respuestas impulsiones permite obtener un conjunto de reglas de sintonización sencillas aplicables a procesos modelados como sistemas de primer y segundo orden. Por otra parte, se contempla el caso de procesos modelados como integradores con retardo. En este caso, se desarrolla una regla de ajuste para el sistema base que permite garantizar una respuesta oscilatoria, necesaria para que se produzcan acciones de reset. Además, se obtiene una regla de ajuste para el porcentaje de reset que permite mejorar del rechazo de perturbaciones.

Finalmente, los beneficios del compensador PI+CI y de las reglas de ajuste propuestas son comprobados sobre sistemas reales de control de procesos. En primer lugar, se realiza el control de un proceso de pH en línea en una planta piloto destinada al tratamiento de productos en la industria conservera. El proceso es modelado como un sistema de primer orden con retardo para el cual se realiza un ajuste de un controlador PI mediante el método SIMC. Además, se obtiene un modelo del proceso mediante un sistema de segundo orden sin retardo, de manera que es posible aplicar las reglas de ajuste previamente obtenidas para el controlador PI+CI y sistemas de segundo orden. A pesar de que el proceso no es realmente un sistema de segundo orden, los experimentos muestran que el controlador PI+CI tiene un rendimiento superior al controlador PI. En segundo lugar, se considera el control de nivel de agua en un tanque cilíndrico. En este caso, se calcula el caudal de salida para un punto de operación y considerando la diferencia de caudales como señal de control se modela el proceso mediante un integrador con retardo. Con el modelo obtenido se sintoniza un controlador PI mediante el método SIMC y un controlador PI+CI mediante las reglas de ajuste propuestas. Nuevamente los resultados experimentales muestran un claro beneficio del control reseteado tanto para el seguimiento de referencias como el rechazo de perturbaciones. En particular, el controlador PI+CI proporciona una respuesta mas rápida con menor *overshoot*, y tiempo de establecimiento.

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# Chapter 1

# Introduction

In this chapter, we give an introduction to reset control systems. First, the reset control is motivated by the overcoming of the fundamental limitations of the linear control. Second, it is provided a survey of reset control raging from the beginning in 1958 to 2015. In addition, we perform an analysis of the bibliography in the field of reset control systems, providing a complete list of the published works, and interesting information such as the number of published works by year and the distribution by countries. Finally, we expose the motivations and the main objectives of the thesis.

## 1.1 Preliminaries

Modern engineering dynamical systems usually demand the control of the evolution with time of one or more of the system variables. In this context, control strategies are required to ensure transient and steady-state behavior for the systems, guarantying performance specification in presence of uncertainties (partial ignorance) of the medium (external disturbances, model uncertainties, etc.). When the uncertainty of the system and disturbances exceed specified tolerances, such that an adequate control design cannot be developed without sensing the system variables, the concept of feedback arises. Feedback, which may be considered at the core of the control theory, refers to the interconnection of two (or more) dynamical systems, such that each system affects the dynamic of the others. This principle has many interesting properties that have been exploited in designing feedback control systems. For instance, feedback provides a system with robustness to uncertainty and also insensitivity to external disturbances. Nevertheless, feedback intrinsically entails the need of a sensor to measure the system variable, and consequently noise in the measurement. This imposes a fundamental

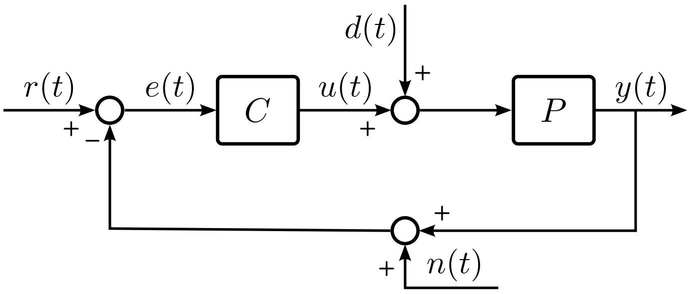


Figure 1.1 Standard feedback control system formed by a plant P and a compensator C.

limitation on the feedback control.

A standard feedback control system is showed in Fig. 1.1 and consists of the feedback interconnection of a system P (normally called plant) and a compensator[1] C, that implements the control strategy. The compensator senses the operation of the system by a *measured signal* $y$ (probably affected by a *sensor noise* $n$), compares it against the desired behavior $r$ (*reference signal*), computes corrective actions $u$ (*control signal*) based on a model of the system, and actuates the system to effect the desired change. Here, the system P is also influenced by an exogenous signal $d$ (*disturbance signal*). The control system can be simply modeled using Laplace transform, when both the plant and the compensator are linear and time-invariant (LTI) systems. The function $L(s) = P(s)C(s)$ is the (open) loop transfer function, where $P(s)$ and $C(s)$ are the transfer functions of the plant and the compensator respectively. Since the system is LTI, the relations between the inputs and the exogenous signals can be expressed in terms of transfer functions. The feedback system in Fig. 1.1 is completely characterized by four transfer functions, called the *Gang of four* ([9]):

- Complementary sensitivity function: $T(s) = \frac{L(s)}{1+L(s)}$.
- Load disturbance sensitivity function[2]: $\frac{P(s)}{1+L(s)}$.
- Noise sensitivity function[3]: $\frac{C(s)}{1+L(s)}$.

[1]The difference between compensator and controller is fuzzy. Both terms can be found in the literature without any distinction between them. However, some authors reserve the term compensator to those controllers that can be defined in the form of rational transfer functions. In this way, the term is linked to the classic control design of lead and lag phase compensation. In this thesis, both terms will be used interchangeably, although compensator will be used preferentially.

[2]The load disturbance sensitivity function is also called the input sensitivity function.

[3]The noise sensitivity function is sometimes called the output sensitivity function.

- Sensitivity function: $S(s) = \frac{1}{1+L(s)}$.

The feedback control system can be asked to satisfy many performance specifications, which clearly lead into a trade-off between the multiple performance objectives. This constitutes a classical and important issue in control theory, which is the topic of fundamental limitations in feedback systems. This line of work for linear systems traces back to the seminal work of H. W. Bode [51] in 1940.

For a large class of practical feedback control systems (with no open-loop zeros and poles in the right half plane, and enough roll-off), Bode showed that the logarithmic sensitivity integral is zero, that is

$$\int_0^\infty \log |S(j\omega)| d\omega = 0, \tag{1.1}$$

which means that making the sensitivity magnitude small at some frequency range necessarily makes the sensitive magnitude larger at other frequency range. This principle is known as Bode's integral formula, and establishes the bases of the fundamental limitations of the LTI systems (see, e.g., [248]). For instance, a consequence of the Bode's integral formula is that if disturbance attenuation is improved in one frequency range, typically at low frequencies, then the appearance of disturbance amplification is unavoidable at higher frequencies. I. M. Horowitz ([148]) used Bode's results in feedback control design, given also some preliminary results about the logarithmic sensitivity integral for open-loop unstable systems. Several other works investigated fundamental limitations in the presence of open-loop systems with poles and zeros in the right half plane, and also with time-delays ([98, 100, 101]). Some fundamental limitations are inherently linked to the plant ([93]). However, other limitations are consequence of using LTI control, and thus, in principle it might be possible that more complex control strategies, such as nonlinear control, overcome these limitations and outperform all the linear control design solutions. This is the main motivation of using non-LTI compensation.

In 1958, J. C. Clegg devised a nonlinear integrator (subsequently called Clegg integrator, CI) in an attempt to overcome the fundamental limitations of LTI compensation [79]. The Clegg integrator consists of a linear integrator whose state is brought to zero instantaneously whenever its input is zero. The behavior of the Clegg integrator can be observed in Fig. 1.2, where its response to a sinusoidal input is represented. Clegg based the superiority of his nonlinear integrator in comparison with its linear

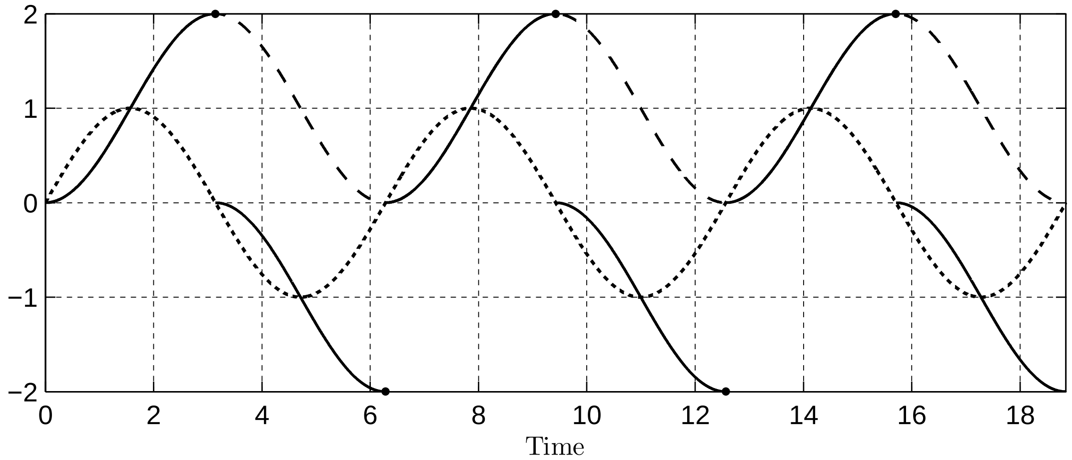


Figure 1.2 Response of the Clegg integrator and the linear integrator to a sinusoidal input. Input (dotted), linear integrator (dashed), and Clegg integrator (solid).

counterpart, using a describing function analysis (see, e.g., [107]). The CI describing function is given by

$$CI(j\omega) = \frac{1.62}{j\omega} e^{-j38.1^{\circ}} \tag{1.2}$$

and has the distinctive characteristic of having only 38.1 degrees of phase lag compared to the 90 degrees of an integrator, which gives an extra phase lead of 51.9 degrees at every frequency. An, in addition, the magnitude is only increasing by a factor of 1.62, that is 4.19 dB. Although this analysis should be taken with some care, due to the fact that the describing function analysis can only be used in a qualitative way, this extra phase lead of 51.9 degrees at all frequencies supposes a breakdown of the stringent relationship (1.1), making possible to obtain values $\int_0^\infty \log |S(j\omega)| d\omega < 0$. For example, for a control system with plant $P(s) = \frac{1}{s+1}$ and a CI as compensator C, by using (1.2) it is obtained that $\int_0^\infty \log |S(j\omega)| d\omega \approx -0.85$, thus CI overcomes the performance of any LTI compensator in terms of the describing function. This is related with the suggestion of Clegg showing the CI capability in reducing the overshoot without sacrificing velocity of response, by using an illustrative example of an electric servomechanism. Even though Clegg did not explicitly mentioned in his work, the idea behind his example is the adverse situation of having two integrators in the feedback loop, that unavoidably implies an overshoot for a step reference ([93]).

Besides the CI, several other nonlinear compensators appeared in the following years, aim at overcoming fundamental limitations of LTI control; for instance the driven limiter [11], the split-path nonlinear (SPAN) filter [97], and the nonlinear integrator with no phase-shift [165]. It turns out that these nonlinear compensators, including CI, were proposed in a rather heuristic way, waiting for rigorous control theory to come. It was not until impulsive and hybrid systems theory became mature that these control design techniques started to be seen as a serious alternative to more traditional LTI compensation.

## 1.2 Survey of Reset Control

Reset control began undoubtedly with the appearance of the Clegg integrator in the work of Clegg [79]. However, it was not until the works of Horowitz and coworkers that reset control was quantitatively incorporated into control design. First, Krishnan and Horowitz [168] proposed a complete and quantitative control design procedure based on the Clegg integrator. They provided the feedback control system with a reset structure consisting of a Clegg integrator in parallel with a linear integrator. The favorable properties of the describing function of the Clegg integrator permit a faster reduction of the loop magnitude, and thus, the sensor noise amplification[4]. Subsequently, Horowitz and Rosenbaum [151] extended the analysis and design procedure to a first order reset element (FORE). The key point in the design was a two step procedure: first, a linear compensator is designed with the aim of meeting all the design specification except for the overshoot, then the pole of the FORE is selected to meet the overshoot specification.

In spite of the fact that the design procedure in the works of Horowitz did not have recourse to the describing function or any other approximation analysis, their research was restricted to the CI and FORE elements, and no general theory was developed. Most likely for this reason, research on reset control systems was not further developed during two decades, until impulsive and hybrid dynamical systems theory became a mainstream research line. The conclusion of the seminal work [151] ends up with the sentence:

[4]The sensor noise amplification is one of the fundamental limitation of the feedback. The sensor amplification is referred to as *Cost of feedback*, mainly in the Quantitative Feedback Theory (QFT) community ([149])

> "This problem will be solved more satisfactorily when stability criteria are developed for feedback loops containing non-linear elements such a FORE".

In 1995, J. E. Bobrow, F. Jabbari and K. Thai proposed in [50] the use of a resetting action to suppress mechanical vibrations of flexible structures. The work does not seems to be influenced by the previous works due to the lack of references to them. In addition, it can be considered as the first experimental application of reset control. Two years later, two new works on reset control appeared, [160] and [58], and they supposed the new dawn of reset control research. The work of H. Hu, Y. Zheng, Y. Chait and C. V. Hollot [160] is the first of a large number of journals, conference proceedings, and several PhD thesis developed by the authors and coworkers. The development of a theoretical framework for general reset control (disseminated through their works, see for instance [47]) can be considered as one of their main contribution on the field of reset control. They also produced a series of papers exploring stability analysis of reset control systems [45, 48, 74, 75], performance analysis [72, 73, 144], experimental application of reset control [306], and demonstration of reset control overcoming fundamental limitations of linear feedback [46]. Their intensive research culminated in one of the most important stability results on reset control systems, the so-called $H_\beta$*-condition* [47]. Despite these works consider a more general class of reset control systems with higher order compensators, they keep intact the original ideas of resetting the compensator states to zero when its input is zero. This characteristic is the hallmark that has made the reset control different from other related approaches, such as impulsive/hybrid control systems.

On the other hand, the work [58] and its journal version [59] follow a completely different line that the works of Y. Chait, C. V. Hollot, and coworkers. These works are focused on the design of energy-based (resetting) impulsive/hybrid controllers to enhance the overall energy dissipation of the closed-loop system. In comparison to the other works, this line of research considers a more general class of systems, that involves: firstly, reset instants which depend on time conditions and state conditions; and secondly, the reset of the states to non-zero values. In this way, the results are mainly supported by the framework of impulsive differential equations ([12–14, 172, 245]). These features make this line to be far from the original idea of reset control. Further development has been achieved in this line, leading in the extension of classical dissipativity theory to impulsive dynamical systems and also the development of a complete framework for the design of energy-dissipating hybrid controllers ([138]).

Although there were different research lines regarding reset control during the 1990s and mid 2000s, in general, they all adopted the mathematical formulation of impulsive differential equations, mainly developed during the decade of the 1980s in Russia (see, e.g., [12, 172], and the references therein). However, L. Zaccarian, D. Nešić and A. R. Teel exposed in [288] that the zero crossing formulation for reset control systems developed by Y. Chait, C. V. Hollot, and coworkers is prone to problems. They asserted for instance that:

- It does not represent the behavior of the circuit implementation of the Clegg integrator ([79]) for all the initial conditions.
- It does not provide robust results, for example robust stability, mainly since the reset condition is defined by a hyperplane, and thus, it is possible to generate arbitrarily small noise that will prevent the state from resetting.
- The trajectory of the system can flow overall the state-space except in a set of zero measure (a hyperplane). Consequently, the system is required to satisfy suitable conditions on the whole state-space for establishing useful stability results (see [47]).

As an alternative, the authors proposed a new class of models for reset control systems containing the classic CI and FORE elements. In order to mathematically describe the new models they adopted a mathematical framework for hybrid system previously introduced in [110]. Under this framework, the authors claimed that a more appropriate definition of the reset control system is to consider that the reset actions occur whenever the sign of the output of the CI and FORE is different to the sing of its input. In terms of the hybrid systems framework, this definition is understood as the existence of two dynamics: the “flow” dynamics and the “jump” dynamics, and two associated sets: the “flow” set and the “jump” set. In this way, the system evolves by flowing or jumping (reset action) depending on whether the state of the system at that time belongs to the jump set or to the flow set, respectively. As a result, the advantages are twofold: the hybrid systems framework provides with useful robustness properties, and the new reset condition confines the system to the sector of the state-space in which the sign of the input and the output of the CI and FORE are the same, allowing to establish less conservative stability results based on piecewise quadratic Lyapunov functions. During the last ten years a multitude of results have been developed in the field of reset control systems based on this new hybrid model approach: stability analysis in [143, 209, 211, 236, 237, 289, 291, 299], performance analysis in [1, 2, 190, 276, 290], saturated reset control systems in [189, 262], experimental applications in [222, 223], reset passivation in [96], etc.

In mid 2000s, another line of research, essentially practical, emerged in the field of reset control (see, for example, [108, 109, 130, 174, 177, 303, 304]). This line is mainly characterized by the following two features:

- The reset actions occur at some fixed instants that are previously design, for instance the reset instants can be periodic.
- The states are not simply reset to zero. Then the problem is how to design the sequence of reset actions, which is normally design to minimize some cost function.

This approach has been successfully applied to the control of hard disk drive (HDD) servo systems in [130, 174, 177, 181] and to the control of a piezoelectric (PZT) microactuator positioning state in [304].

Alongside the above approaches, the initial representation of reset control system as impulsive dynamical systems is further developed since 2006, mainly by the works of A. Baños, A. Barreiro, and coworkers. By the first time, reset control systems were applied to systems with time-delays; since past experience shows that one of key instruments in reset compensation is the extra phase lead obtained with respect to its base[5] LTI compensator, application to systems with time-delays seems to be natural field since, in principle, phase lead can be used to alleviate the phase lag due to the time-delay in a feedback loop. Several research lines related with reset control has been developed, not only related to time-delay systems (most of the work is summarized in the recent monograph [20]):

- A delay-dependent stability analysis is accomplished in [37], providing delay-dependent conditions in both the time domain and frequency domain; in addition, delay-independent stability analysis and a generalized $H_\beta$-condition for time-delay reset systems, based on a generalized Kalman-Yakubovich-Popov (KYP) lemma, is developed in [19]. This research line has been subsequently followed in several works ([82, 133, 196, 197, 236]), being one the main roots of this thesis work.

[5] In absence of reset actions, the resulting linear system is called the base linear system, or in general, base system.

- Formalization of the impulsive dynamical systems framework, avoiding the use of time regularization, basically searching for well-posedness and stability criteria that gives a solid foundation to reset control systems. It has been shown that for exogenous signals with some smoothness properties (Bohl functions), reset control systems are well-posed (existence and uniqueness of solutions and continuous dependence on initial conditions) for most of the reset compensators that has been used in practice ([28, 29]). Stability results for stable and unstable base systems by time-dependent conditions on the reset instants are given [22, 23], improving $H_\beta$-condition (only usable for some stable base systems); in addition, input-output stability has been also approached in [66], based on application of passivity results; successfully applied to passive teleoperation in [91, 92].

- Development of PI-based reset compensators, with the goal of obtaining a simple and easy-to-tune reset compensator, with special application to process control. The result has been the PI+CI compensator, with different resetting laws: zero crossing, fixed reset band, variable reset band, anticipative reset ([34, 35, 270, 271]), that has been successfully applied in a number of practical cases ([25, 83, 271, 272]).

Other research lines in the field of reset control systems, directly or indirectly related with the above impulsive dynamical systems approach, are:

- **Reset observers:** This line of research began in 2010 with the work [219], motivated by the advantageous properties that reset elements have shown in control application. In that work, the authors proposed a new adaptative observer called *reset adaptative observer*, where a linear integrator is substituted for a reset element. This strategy supposed a breakdown of the limitation of linear adaptative observer to reduce simultaneously the overshoot and settling time of the estimation process. Further developments in this line can be found in: optimal observer design [216], MIMO systems [220], nonlinear systems [119, 221] and time-delay systems [217, 294]. It is interesting to mention that reset adaptative observer has been successfully used to control the cooking pot temperature in induction hobs (see [215]).

- **Discrete-time reset control systems:** Nowadays, analog controllers have been replaced by digital controllers (digital circuits, digital computers, or microprocessors) whose inputs and outputs are defined at discrete time instances. Therefore, control strategies should be transformed into discrete-time versions that are adequate to computer-based implementation. Discrete time-models of reset control systems were used in [174, 176, 177] to applied reset compensation to control servomechanisms in hard disk drives. However, it is not until the work [31] that a formal representation of time-discrete reset control systems, coherent with the existing continuous-time results, is developed. Discrete-time reset control under this formulation has been applied to networked control systems (NCS) in [30, 32, 228, 229]. The works [123, 124, 284] are concerned with the stability analysis and design of reset control systems when the reset condition is replaced by a discrete-time counterpart. Other works in the line of discrete-time reset control systems are [67, 134].

- **Fractional-order reset control systems:** In this line, fractional dynamics of hybrid systems is studied [156]. In particular, properties of reset control systems with fractional-order are analyzed as a subclass of the fractional hybrid systems [153, 154]. The applicability of fractional-order reset control is illustrated in [155] with an example of speed control of a servomotor. Most of the results in this line can be found in [152, 157, 158].

Throughout reset control ages, a multitude of experimental application have shown the potentials benefits of reset compensation, starting from the control of mechanical elements until the process control application in the last years. A sample of the many applications is:

- Vibration suppression for flexible structures: [50].
- Tape-speed control system: [306].
- Speed control of rotational flexible mechanical system: [144].
- Hard disk drive: [125, 126, 130, 131, 146, 174, 176, 177, 181, 279].
- Marine thruster control: [15].
- Piezoelectric actuator positioning: [303, 304].
- Teleoperation: [89, 91, 92].
- Solar collector fields: [272].
- In-line pH process: [25, 62, 230].
- Heat exchanger temperature control: [202, 271].

- Liquid level control: [83].
- Exhaust gas recirculation valve: [222, 223].
- Induction hobs: [215].
- Gantry crane: [240].
- Servomotor-speed control: [155].
- Synchronous motor with permanent magnet excitation: [86] (simulation).

### 1.2.1 Analysis of the Bibliography

In spite of the sound previous works and the increasing interest of the control community in the last years, reset control system is still in infancy. If a bibliographic search is performed, after some treatment it can be observed that there are no much more than a hundred published works (including all the conferences and journal papers that usually have some similar contents), which is a clear indicator of the initial development of the subject. It is interesting to see some figures about the state of the art in reset control systems[6]. In Table 1.1, it is shown authors, date an type of publication; Fig. 1.3 shows a historical view of the number of published works by year, and Fig. 1.4 a distribution by countries (thesis and books have been excluded). The H-index of the topic “reset control” is 20 (for the provided list of works).

[6]The data have been gathered from Scopus and Google Scholar.

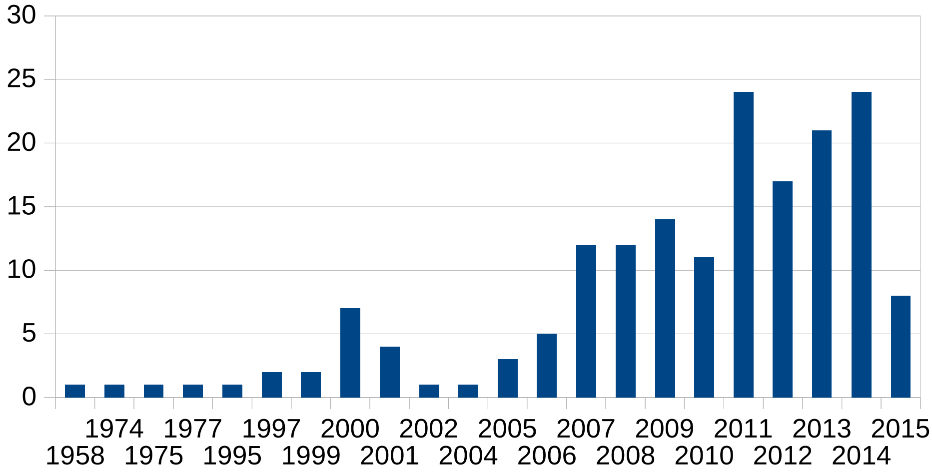


Figure 1.3 Historical view of the number of published works by year.

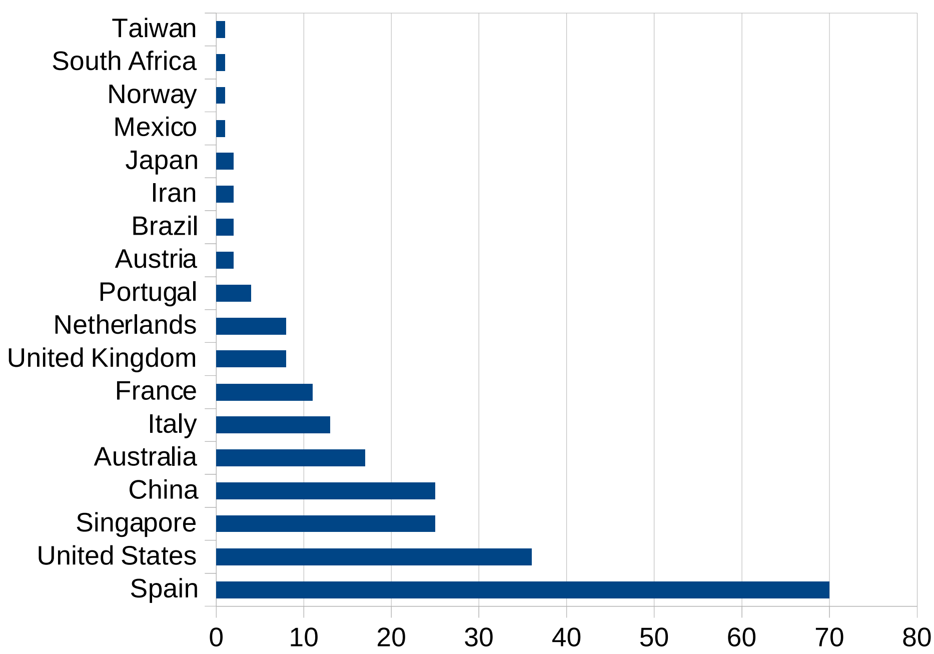


Figure 1.4 Distribution of the number of published works by countries.

| Authors | Date | | Ref. |
|---|---|---|---|
| Clegg | 1958 | J | [79] |
| Khishnan, Horowitz | 1974 | J | [168] |
| Horowitz, et al. | 1975 | J | [151] |
| Karybakas | 1977 | J | [165] |
| Bobrow, Jabbari, et al. | 1995 | J | [50] |
| Hu, Zheng, et al. | 1997 | C | [160] |
| Hollot, Zheng, et al. | 1997 | C | [145] |
| Bupp, Bernstein, et al. | 1997 | C | [58] |
| Zheng | 1998 | T | [305] |
| Beker, Hollot, et al. | 1999 | C | [48] |
| Beker, Hollot, et al. | 1999 | C | [43] |
| Bupp, Bernstein, et al. | 2000 | J | [59] |
| Zheng, Chait, et al. | 2000 | J | [306] |
| Chen, Hollot, et al. | 2000 | C | [74] |
| Haddad, et al. | 2000 | C | [137] |
| Chen, Hollot, et al. | 2000 | C | [75] |
| Chen, Chait, et al. | 2000 | C | [72] |
| Beker, Hollot, et al. | 2000 | C | [44] |
| Chen | 2000 | T | [71] |
| Chen, Chait, et al. | 2001 | J | [73] |
| Beker, Hollot, et al. | 2001 | J | [46] |
| Beker, Hollot, et al. | 2001 | C | [45] |
| Hollot, Beker, et al. | 2001 | B | [144] |
| Beker | 2001 | T | [42] |
| Chait, Hollot | 2002 | J | [68] |
| Beker, Hollot, et al. | 2004 | J | [47] |
| Zaccarian, Nešić, et al. | 2005 | C | [288] |
| Nešić, Zaccarian, et al. | 2005 | C | [210] |
| Li, G. Guo, Wang | 2005 | C | [180] |
| G. Guo, Yu, Ma | 2006 | J | [121] |
| Hong, Wong | 2006 | C | [146] |
| Baños, Barreiro | 2006 | C | [17] |
| Zaccarian, Nešić, et al. | 2006 | C | [289] |
| Li, G. Guo, Wang | 2006 | C | [181] |
| Wu, G. Guo, Wang | 2007 | J | [279] |
| Baños, Vidal | 2007 | C | [35] |
| Baños, Carrasco, et al. | 2007 | C | [22] |
| Baños, Vidal | 2007 | C | [34] |
| Baños, Barreiro | 2007 | C | [18] |
| Baños, Carrasco, et al. | 2007 | C | [21] |
| Y. Guo, Wang, Xie | 2007 | C | [125] |
| Zaccarian, Nešić, et al. | 2007 | C | [290] |
| J. Zheng, et al. | 2007 | C | [303] |
| Witvoet, et al. | 2007 | C | [276] |
| Loquen, et al. | 2007 | C | [189] |
| Y. Guo, Wang, et al. | 2007 | C | [131] |
| Lahee | 2007 | C | [171] |
| Bakkeheim, et al. | 2008 | J | [15] |
| J. Zheng, et al. | 2008 | J | [304] |
| Nešić, Zaccarian, et al. | 2008 | J | [211] |
| Aangenent, et al. | 2008 | C | [2] |
| Nešić, Teel, et al. | 2008 | C | [208] |
| Vidal, Baños | 2008 | C | [269] |
| Carrasco, Baños | 2008 | C | [64] |
| Carrasco, Baños, et al. | 2008 | C | [65] |
| Vidal, Baños, et al. | 2008 | C | [272] |
| Fernández, et al. | 2008 | C | [91] |
| Loquen, et al. | 2008 | C | [190] |
| Y. Guo, et al. | 2009 | J | [130] |
| Baños, Barreiro | 2009 | J | [19] |
| Y. Guo, Wang, Xie | 2009 | J | [126] |
| Li, Du, et al. | 2009 | J | [176] |

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| Authors | Date | | Ref. |
|---|---|---|---|
| Li, Du, Wang | 2009 | C | [174] |
| Li, Du, et al. | 2009 | C | [177] |
| Vidal, Baños | 2009 | C | [270] |
| Baños, Dormido, et al. | 2009 | C | [26] |
| Baños, Perez, et al. | 2009 | C | [31] |
| Li, Du, et al. | 2009 | C | [178] |
| K. El Rifai, O. El Rifai | 2009 | C | [242] |
| Raimundez, et al. | 2009 | C | [239] |
| Carrasco, Baños, et al. | 2009 | C | [63] |
| Y. Guo, Wang, et al. | 2009 | C | [128] |
| Carrasco | 2009 | T | [61] |
| Vidal | 2009 | T | [268] |
| Aangenent, et al. | 2010 | J | [1] |
| Zheng, Fu | 2010 | J | [302] |
| Carrasco, Baños, et al. | 2010 | J | [66] |
| Barreiro, Baños | 2010 | J | [37] |
| Vidal, Baños | 2010 | J | [271] |
| G. Guo | 2010 | J | [120] |
| Paesa, Franco, et al. | 2010 | C | [219] |
| Polenkova, et al. | 2010 | C | [231] |
| Loquen, Nešić, et al. | 2010 | C | [188] |
| Baños, Perez, et al. | 2010 | C | [32] |
| Li, Wang | 2010 | C | [179] |
| Loquen | 2010 | T | [187] |
| Zaccarian, Nešić, Teel | 2011 | J | [291] |
| Baños, Carrasco, et al. | 2011 | J | [23] |
| Fernández, Blas, et al. | 2011 | J | [92] |
| Paesa, Franco, et al. | 2011 | J | [220] |
| Baños, Dormido, et al. | 2011 | J | [27] |
| Tarbouriech, et al. | 2011 | J | [262] |
| Nešić, Teel, et al. | 2011 | J | [209] |
| Forni, Nešić, et al. | 2011 | J | [96] |
| Paesa, Baños, Sagues | 2011 | J | [216] |
| Li, G. Guo, Wang | 2011 | J | [182] |
| Li, Du, Wang | 2011 | J | [175] |
| Y. Guo, Wang, et al. | 2011 | J | [129] |
| Perez, Baños, et al. | 2011 | C | [228] |
| Barreiro, Baños, et al. | 2011 | C | [39] |
| Y. Guo, Wei | 2011 | C | [132] |
| Y. Guo, Gui, et al. | 2011 | C | [123] |
| Y. Guo, Gui, Yang | 2011 | C | [122] |
| Perez, Baños, et al. | 2011 | C | [230] |
| Perez, Baños, et al. | 2011 | C | [229] |
| Paesa, Carrasco, et al. | 2011 | C | [218] |
| Hetel, Daafouz, et al. | 2011 | C | [142] |
| Satoh | 2011 | C | [246] |
| Dormido, Baños, et al. | 2011 | C | [88] |
| Paesa, Baños, et al. | 2011 | C | [217] |
| Paesa | 2011 | T | [215] |
| Raimundez, et al. | 2012 | J | [240] |
| Baños, Vidal | 2012 | J | [36] |
| Suyama, Kosugi | 2012 | J | [259] |
| Paesa, Franco, et al. | 2012 | J | [221] |
| Carrasco, Baños | 2012 | J | [62] |
| Y. Guo, Wang, Xie | 2012 | J | [127] |
| Barreiro, Baños | 2012 | J | [38] |
| Y. Guo, Gui, et al. | 2012 | J | [124] |
| Baños, Mulero | 2012 | J | [28] |
| Valério, S. da Costa | 2012 | J | [264] |
| Baños, Davó | 2012 | C | [24] |
| Moreno, et al. | 2012 | C | [201] |
| Prieto, Barreiro, et al. | 2012 | C | [236] |

continue in the next column

continue in the next page

| Authors | Date | | Ref. |
|---|---|---|---|
| Panni, Alberger, et al. | 2012 | C | [222] |
| Y. Guo, Xie | 2012 | C | [133] |
| Davó, Baños | 2012 | C | [81] |
| Polenkova, et al. | 2012 | C | [232] |
| Baños, Barreiro | 2012 | B | [20] |
| Moreno, et al. | 2013 | J | [202] |
| Guillen-Flores, et al. | 2013 | J | [119] |
| Nešić, Teel, et al. | 2013 | J | [207] |
| HosseinNia, et al. | 2013 | J | [155] |
| Prieto, Barreiro, et al. | 2013 | J | [235] |
| Bras, Carapito, et al. | 2013 | J | [55] |
| Hetel, Daafouz, et al. | 2013 | J | [143] |
| Carrasco, et al. | 2013 | J | [67] |
| Liu, Xum Wang | 2013 | J | [185] |
| Lu, Lee | 2013 | J | [191] |
| Li, Fen, Xinmin | 2013 | C | [173] |
| HosseinNia, Tejado | 2013 | C | [156] |
| Falcon, Barreiro, et al. | 2013 | C | [89] |
| HosseinNia, et al. | 2013 | C | [154] |
| Wu, Guo, Gui, Jiang | 2013 | C | [280] |
| Davó, Baños | 2013 | C | [83] |
| Mercader, Davó, et al. | 2013 | C | [197] |
| Suyama, Kosugi | 2013 | C | [260] |
| Davó, Baños | 2013 | C | [82] |
| Mercader, et al. | 2013 | C | [196] |
| Zhao, Nešić, et al. | 2013 | C | [292] |
| Perez | 2013 | T | [227] |
| HosseinNia | 2013 | T | [152] |
| Yuan, Wu | 2014 | J | [285] |
| Ghaffari, et al. | 2014 | J | [108] |
| Yuan, Wu | 2014 | J | [286] |
| continue in the next column | | | |
| HosseinNia, et al. | 2014 | J | [157] |
| Shakibjoo, Vasegh | 2014 | J | [253] |
| Panni, Waschl, et al. | 2014 | J | [223] |
| Ghaffari, et al. | 2014 | J | [109] |
| Zhao, Wang | 2014 | J | [294] |
| Ogura, Martin | 2014 | J | [214] |
| Baños, Davó | 2014 | J | [25] |
| Barreiro, Baños, et al. | 2014 | J | [40] |
| Baños, Perez, et al. | 2014 | J | [30] |
| HosseinNia, et al. | 2014 | C | [158] |
| Delgado, Cacho, et al. | 2014 | C | [86] |
| Y. Guo, Zhu | 2014 | C | [134] |
| Bragagnolo, et al. | 2014 | C | [54] |
| Zhao, Wang, Li | 2014 | C | [299] |
| HosseinNia, et al. | 2014 | C | [153] |
| Zhao, Nešić, et al. | 2014 | C | [293] |
| Davó, Baños, Moreno | 2014 | C | [84] |
| Acho | 2014 | C | [3] |
| Zhao, Wang, Li | 2014 | C | [298] |
| Vettori, et al. | 2014 | C | [267] |
| Delgado, et al. | 2014 | C | [85] |
| Prieto | 2014 | T | [234] |
| Baños, Perez, et al. | 2014 | B | [33] |
| Zhao, Wang | 2015 | J | [297] |
| Zhao, Wang | 2015 | J | [295] |
| Zhao, Wang | 2015 | J | [296] |
| Zhao, Wang, et al. | 2015 | J | [300] |
| Zhao, Yin, Shen | 2015 | J | [301] |
| Yu, Y. Guo, Zhao | 2015 | J | [284] |
| Heertjes, et al. | 2015 | C | [141] |
| Mercader, Davó, et al. | 2015 | C | [198] |

Table 1.1 Publications in the research field of reset control. Journal (J), conference (C), thesis (T), and book (B).

## 1.3 Motivation and Objectives

### 1.3.1 Motivation

Since the work of Bode in 1940 [51], a multitude of researchers have brought to light the factors that fundamentally limits the achievable performance of linear control systems (see, e.g., [99, 248]). For instance, Bode's gain-phase relationship assumes stable, minimum phase loop transfer function and links loop gain and phase. This relationship implies that if the loop phase is increased to provide large phase margin, and thus, good tolerance to plant uncertainty near the crossover frequency, then it will result in a slow gain attenuation. As a consequence, more measurement noise is injected into the system, producing a deterioration of the performance. Fundamental limitations have been also studied for non-minimum phase systems, poles and zeros in the right half plane and time-delay ([6, 100]), showing that there are more restrictive limitations in those cases. Unfortunately, time-delay is present in most of the industrial processes, so that the limitation of the LTI compensation is important beyond the theoretical realm. This motivates the challenging task of finding nonlinear compensator to be able to overcome the fundamental limitations of the linear control. Reset control systems appeared in 1958 with the invention of the Clegg integrator. Its advantageous properties suggested by the describing function made the CI a potential element to overcome the linear compensation. During the last 20 years a multitude of works have shown real benefits of the reset compensation over linear feedback control schemes, overcoming in some cases the fundamental limitations of linear control without resorting to approximations [46]; from theoretical results dealing with stability [23], asymptotic tracking [1], disturbance rejection [96, 291, 293], etc, to experimental applications [15, 202, 223, 272, 306].

It should be emphasized that many nonlinear compensators have been suggested in the literature, which exhibit similar advantages as reset compensation. However, as mentioned in [150] by the pioneer in the field of reset control Prof. Horowitz:

> "The difficult challenge is how to integrate such nonlinear elements into a systematic quantitative design technique".

Unlike other approaches, reset compensator are simple to understand and to implement; they are essentially linear systems that undergo jumps at certain times, where these times are usually determined by their input and their own dynamic. Therefore, roughly speaking, the behavior of the reset control system is mainly determined by the base linear system, and thus, easier to understand. On the other hand, conditional expressions and reset action of the linear integrator are, in general, needed for the implementation of reset compensators, and those features are common in most of the simulation softwares and industrial *programable logic controllers* (PLC). In this way, reset compensation could be seen as another tool available in the toolbox of the control engineers, such as two-degree of freedom scheme and anti-windup strategies, whose appropriate application on a linear feedback control scheme can lead into substantial benefits (the PI+CI compensator and the Smith predictor plus Clegg integrator compensator in [202] are a clear illustration of this idea).

Finally, it should be emphasized, the importance of the theoretical insight into the reset control systems to the more general field of impulsive and hybrid systems. Since the reset control systems are a subclass of the impulsive/hybrid systems, any particular result for reset control systems has the potential to be generalized and applied to those more general systems.

### 1.3.2 Objetives

Our target in this thesis is that providing advance in the field of reset control systems; both from the theoretical and engineering applications point of views. The general objectives of the thesis are exposed in the following subsections.

#### Stability Analysis of Time-delay Reset Control Systems

As it is commented in [42], a misconception regarding reset control systems, is thinking that since reset actions zero the compensator states, then the "energy" of the system decreases. Consequently, one may wrongly think that the stability of the base linear system must imply the stability of the reset control system. Several examples in [42, 160, 161] and Example 3.4 of this thesis have been proved that the intuition fails, and reset actions can destabilize a stable base linear system with (Example 3.4) and without time-delay. Therefore, stability needs to be guaranteed in advance for a proper practical application of reset compensation. Even though the interest on reset control has increased during last years (see Fig. 1.3), most of the research effort has been dedicated to systems without time-delay. Time-delay systems are straightforwardly defined and studied in the frequency domain, while its time domain formulation is awkward. On the contrary, by its nonlinear nature, reset control systems are necessarily defined in a time domain (usually by impulsive differential equations). This is the main reason why the stability analysis of time-delay reset control systems is more challenging. Only few works[7] have dealt with the stability analysis of time-delay reset control systems ([19, 37, 133, 196, 236]), being the results conservative. The main limitation of these results is the need of a stable base system. The stabilization of unstable systems by impulsive actions has been addressed in several works in the field of impulsive systems (see, e.g., [256] and the references therein). However, most of the results are not focused on control problem, and they do not regard with the limitations imposed by the fixed structure of a control closed-loop system. Consequently, to the best of the author knowledge, they are not applicable, at least directly, to time-delay reset control systems.

In this context, one of the main objective of this thesis is to developed new stability criteria that: in first place, can guarantee the stability of the time-delay reset control system for time-delays greater than those obtained by the results in the literature, and second, completely remove the restrictive condition of a stable base system.

### Stability Analysis of Reset Control Systems with Saturation

Saturation is one of the most common nonlinearities encountered in control practice, since many physical systems are subject to magnitude-limitation in its input. This may cause performance deterioration, limit cycles, and in some cases unstable behavior. Investigation of how saturation may impact control systems stability and performance has attracted research efforts from several decades (see, e.g., [163, 261], and the references therein). A key topic, that has received considerable attention, is the estimation of the region of attraction. The analysis of reset control systems in presence of saturation has been tackled for the first time in [189], considering the FORE compensator, and using Lyapunov-like approaches and a hybrid systems framework. Some extensions, in the same context, have been proposed in [94, 262]. Another goal of this thesis is the analysis of how reset compensation can alleviate the effect of saturation on the closed-loop stability, focusing on the development of a procedure to estimate the region of attraction.

### New Tuning Methods for the PI+CI Compensator

Proportional integral (PI) and proportional integral derivative (PID) compensators are by far the most adopted control strategies in industry. It is mainly owing to is

[7]The analysis of discrete-time time-delay reset control systems has been accomplished in [30]. Nevertheless, the discrete-time makes the problem more manageable, since it is a finite-dimensional system.

simplicity and capability of solving a wide range of control problems. In addition, a great effort has been made to fulfill the need of methods for finding good values of the compensator for different processes, making the PID compensator accessible to control engineering practitioners. The PI+CI compensator is devised in [35] as a reset counterpart of the PI, showing its advantageous over the PI compensator in a multitude of works. In the attempt to provide a higher improvement of the reset compensation, several modifications of the PI+CI has been proposed in the literature. However, the lack of systematic procedure for the tuning of the PI+CI compensator complicates its applicability. In this context, another objective of the thesis is to develop a systematic method to successfully design a PI+CI compensator, resulting whenever it is possible, into simple tuning rules for its design parameters.

#### Experimental Application of the Proposed Tuning Rules

As a final objective, it is considered the application of the PI+CI with the developed tuning rules to real processes, in order to illustrate the industrial applicability of the results. Two industrial control processes are considered as a target: in-line control pH processes and liquid level control processes. In the first case, the experiments will be carried out in an industrial pilot plant, design for the experimentation of thermal treatments common in the tinned food industry. In addition, the pilot plant will be provided with a water tank in order to perform the liquid level experiments.

## 1.4 List of Publications

Most of the results exposed in this thesis have been published (or submitted) to several international conferences and journals. In this section we provide the list of publications:

- *Tuning rules for a reset PI compensator with variable reset.* A. Baños and M. A. Davó. Proceedings of the IFAC Conference on Advances in PID Control (2012), [24].

- *PI+CI tuning for integrating plus deadtime systems.* M. A. Davó and A. Baños. Proceedings of the $17^{th}$ Conference on Emerging Technologies & Factory Automation (2012), [81].

- *Reset control of a liquid level process.* M. A. Davó and A. Baños. Proceedings of the $18^{th}$ Conference on Emerging Technologies & Factory Automation (2013), [83].

- *$H_\infty/H_2$ analysis for time-delay reset control systems.* P. Mercader, M. A. Davó and A. Baños. Proceedings of the $3^{rd}$ International Conference on Systems and Control (2013), [197].

- *Delay-dependent stability of reset control systems with input/output delays.* M. A. Davó and A. Baños. Proceedings of the $52^{nd}$ Conference on Decision and Control (2013), [82].

- *Tuning of reset proportional integral compensation with a variable reset ratio and reset band.* A. Baños and M. A. Davó. *IET Control Theory and Applications* (2014), [25].

- *Region of attraction estimation for saturated reset control systems* . M. A. Davó, A. Baños and J. C. Moreno. Proceedings of the $14^{th}$ International Conference on Systems, Control, and Automation (2014), [84].

- *An impulsive dynamical systems framework for reset control systems.* A. Baños, J. I. Mulero, A. Barreiro and M. A. Davó. ePrint arXiv: 1505.07673 (2015), [29] (submitted to *International Journal of Control*).

- *Performance analysis of PI and PI+CI compensation for an IPDT process.* P. Mercader, M. A. Davó and A. Baños. Proceedings of the $23^{rd}$ Mediterranean Conference on Control and Automation (2015), [198].

- *Stability of time-delay reset systems with a nonlinear and time-varying.* A. Baños and M. A. Davó. *International Journal of Robust and Nonlinear Control* (submitted).

- *Reset control of integrating plus dead time processes.* M. A. Davó and A. Baños. *Journal of Process Control* (second revision).

- *Stability of time-delay reset control systems with time-dependent resetting law.* M. A. Davó, F. Gouaisbaut, A. Baños, S. Tarbouriech and A. Seuret. 5th IFAC Conference on Analysis and Design of Hybrid Systems (accepted).

## 1.5 Overview of Contents

This thesis is organized as follows:

- In Chapter 2, the description of the framework of impulsive dynamical systems (IDS) and its extension to impulsive delay dynamical systems (IDDS) is given. It is also provided a definition of reset control systems and time-delay reset control systems under the IDS (IDDS) framework. Several results on the well-posedness of the reset control systems are also given. The PI+CI compensator is introduced as a particular reset compensators, and a multitude of design improvements are analyzed. A framework for hybrid systems and its extension for hybrid systems with time-delay are also presented.

- In Chapter 3, we deal with the stability analysis of impulsive delay dynamical systems. Firstly, a general Lyapunov-Krasovskii-based criterion is devised for the stability of state-dependent IDDSs with nonlinear and time-varying base systems. The general result is applied to several LK functionals to developed stability criteria in the form of LMIs for checking the stability of an IDDS with nonlinear and time-varying base system and a particular structure, and an IDDS with LTI base system. In the second part of the chapter, two general Lyapunov-Krasovskii-based criteria are obtained based on time-dependent conditions. The first result provides less conservative conditions for IDDSs with stable base LTI systems and the second one allows the analysis of IDDSs with unstable LTI base systems. Finally, it is obtained LMI conditions for the stability of a time-delay reset control system comprises a LTI plant and a PI+CI compensator.

- In Chapter 4, we focus on the influence of the saturation on the reset control system. In particular how saturation may impact the region of attraction of the control system. The main idea is to represent the set of trajectories of the closed-loop reset control system as a directed graph of polytopes and arcs between them. Several conditions of positive invariance and reachability of the origin for the polytopes are used to analyze the graph and compute an estimate of the region of attraction. The proposed technique is compared with the method proposed in the literature.

- In Chapter 5, we address the tuning of the PI+CI compensator. First, a systematic method for PI+CI tuning is developed based on the analysis of the reset control system as the impulse response of a set of LTI systems. As a result, simple tuning rules are provided for first/second order plants and integrating plus dead time plants. Finally, it is presented the experimental results obtained when applying the PI+CI with the proposed tuning rules to an in-line pH control process and a liquid level control process.

- In Chapter 6, the most relevant contributions of the thesis are summarized, and we give a few potential research directions in the line of this thesis.

# Chapter 2

# Reset Control Systems

In this chapter, we give basics of reset control systems. The framework of impulsive dynamical systems and its extension of impulsive delay dynamical systems are used for the description of the (time-delay) reset control systems. Fundamental results regarding the well-posedness (existence, uniqueness and continuous dependence on the initial condition) of the reset control systems are provided. In addition, other frameworks for the modeling of reset control systems are presented. Finally, a proportional-integral plus Clegg integrator (PI+CI) compensator is introduced, considering several design improvements proposed in the literature. In addition, new modifications with potential benefits in practical applications are proposed.

## 2.1 Preliminaries

The complexity of the world involves the natural interaction of continuous and discrete behaviors. Many dynamical systems such as ecological systems, geological systems, neural systems, physical systems, etc, exhibit an interacting mixture of continuous and discrete dynamics. The increasingly complex nature of engineering systems involving software and physical aspects of systems, controller architectures with real-time embedded software, multiple modes of operation, communication, etc, also give rise to the complex interaction of continuous and discrete dynamics. To capture the evolution of these systems, mathematical models are needed that combine in one way or another the dynamics of the continuous parts of the system with the dynamics of the logic and discrete parts. The analysis and synthesis of these models lay on the research area of hybrid systems theory. A sample of the possible models for hybrid systems is provided in the survey [247]: hybrid automata, switched systems, piecewise linear or piecewise affine models, hybrid Petri nets, hybrid inclusions, etc. Models for continuous systems and for discrete systems have been originally studied by different disciplines, such as control theory (continuous dynamic) and computer science (discrete dynamic). Consequently, the different frameworks for hybrid systems analysis and modeling place greater emphasis on the continuous or discrete dynamic depending on their discipline origin. An important class of hybrid systems are the impulsive systems, which are characterized by a really simple discrete part and a complex continuous dynamics. The discrete dynamic of the impulsive systems arises from instantaneous changes of the state variables that occur when a simple condition is satisfied, for example some prefixed instants are reached, or the trajectory hits a predefined hypersurface. It is clear that reset control systems lay into the class of impulsive systems, and in general hybrid systems. The rich continuous behavior of the impulsive systems, and in particular reset control systems, makes some frameworks more adequate to model and analyze this kind of systems. Therefore, in spite of the wide variety of frameworks for hybrid systems, in this chapter, it will be only introduced those frameworks[1] that have been successfully used for modeling reset control systems.

An important issue in the study of impulsive/hybrid systems, directly related with their solution concept, is *well-posedness*. Historically, the term well-posedness dates back to 1902 when Jacques Hadamard ([136]) formulated three conditions that mathematical models of physical phenomena should satisfy, namely that:

1. A solution exists.
2. The solution is unique.
3. The solution depends continuously on the problem data.

An impulsive/hybrid system with the above properties will be denoted as a *well-posed* system. The posible refinements of the notion of well-posedness to be made throughout the document will be explicitly mentioned.

Impulsive/hybrid systems involve an interacting mixture of continuous and discrete dynamics, which provides the systems with a very rich dynamical behavior. In particular, the trajectories of impulsive/hybrid systems can exhibit multiple complex phenomena such as:

[1]Reset control systems have been also modeled as a hybrid automata (see, e.g., [194] for details on hybrid automata). However, there are only three works ([231, 232, 267]) concern with the stability of reset control systems by the framework of hybrid automata.

- *Deadlock:* This situation corresponds to a dynamical system state from which no continuation, continuous or discrete, is possible.

- *Beating*[2]*.* An impulsive/hybrid system experiences beating when the system trajectory encounters the same resetting surface a finite or infinite number of times in zero time.

- Zeno[3] solutions: An impulsive/hybrid system possesses a *Zeno solution* if the system experiences an infinite number of discrete transitions in a finite amount of time.

- Confluence: Finally, this situation involves system solutions that coincide after some time and behave as a single solution thereafter.

These phenomena, along with other difficulties such as discontinuous solutions, make the analysis of the well-posedness of impulsive/hybrid systems extremely challenging.

## 2.2 Impulsive Dynamical Systems

One of the main frameworks for reset control systems modeling is the framework of *Impulsive Dynamical Systems*[4] (IDS) [138], which is mainly based on the theory of impulsive differential equations ([12, 172]). In this way, the description of the impulsive system is accomplished by an state-space representation, which is mainly composed by the following three elements:

- *Differential equation.* The continuous-time dynamic of the system between reset events is governed by a differential equation, as follows:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t)). \tag{2.1}$$

  In general, the function $\mathbf{f}$ can be both nonlinear and time-varying.

[2] Beating is also known as *Pulse Phenomenon* (especially in impulsive systems) and also as *Livelock*, mainly in the field of computation science.

[3] The name Zeno refers to the Greek philosopher Zeno of Elea (ca. 490-430 BC), who is best known for his paradoxes, designed to support that the belief in plurality and change is mistaken, and in particular that motion is nothing but an illusion. An example is Zeno's Second Paradox of Motion, in which Achilles is racing against a tortoise.

[4] The notation, definitions, and in general the results in [138] are adapted for a better correspondence with the scope of the thesis, which is the reset control systems.

- *Difference equation.* The way the states are instantaneously changed, when a reset event occurs, is described by the difference equation

$$\dot{\mathbf{x}}(t^+) = \mathbf{I}(t, \mathbf{x}(t)). \tag{2.2}$$

  Normally, no conditions are imposed on the function $\mathbf{I}$.

- *Resetting law.* Finally, to fill out the description of an impulsive system, it is necessary to provide a criterion for determining when the states of the system are to be reset. Based on [138], the resetting law is described by a set $\mathcal{S} \subset \mathbb{R}_+ \times \Omega$, referred to as *resetting set*. In this way, a reset event occurs at instant $t$ in which $(t, \mathbf{x}(t)) \in \mathcal{S}$. In the particular case wherein the resetting set is a hypersurface, the term *reset surface* is also used to denote the resetting set.

An impulsive system with the above three elements is described by the following impulsive differential equation

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t)), & (t, \mathbf{x}(t)) \notin \mathcal{S}, \\ \mathbf{x}(t^+) = \mathbf{I}(t, \mathbf{x}(t)), & (t, \mathbf{x}(t)) \in \mathcal{S}, \end{cases} \tag{2.3}$$

where $t \geq t_0 \geq 0$, $\mathbf{f} : \mathbb{R}_+ \times \Omega \to \mathbb{R}^n$ and $\mathbf{I} : \mathcal{S} \to \Omega$ are continuous functions on their both parameters and $\Omega \subseteq \mathbb{R}^n$ is an open set. $\mathbf{x}(t) \in \mathbb{R}^n$ is the system state at the instant $t \in [t_0, \infty)$ and $\mathbf{x}(t^+) = \lim_{\epsilon \to 0, \epsilon \geq 0} \mathbf{x}(t+\epsilon)$ is referred to as *after-reset state*.

The freedom in the choice of the elements of an impulsive system gives rise to several types of systems. A possible classification of the impulsive systems is that provided in [172], which is based on the choice of the resetting law. Hence, three types of impulsive systems are:

- *Systems with impulses at fixed times.* Let $(t_k)$, $k = 1, 2, \ldots$ be a sequence of *prescribed reset times*, such that $t_k \to \infty$ as $k \to \infty$, then the resetting set is define by $\mathcal{S} = \{(t, \mathbf{x}) \in \mathbb{R}_+ \times \mathbb{R}^n : \exists\ k > 0,\ t = t_k\}$.

- *Systems with impulses at variable times.* In this case, the resetting set is defined by a sequence of surfaces in the space $(t, \mathbf{x})$, that is $\mathcal{S} = \{(t, \mathbf{x}) \in \mathbb{R}_+ \times \mathbb{R}^n : \exists\ k > 0,\ t = \tau_k(\mathbf{x})\}$. The sequence of functions $\tau_k(\mathbf{x})$ satisfy

$$\tau_k(\mathbf{x}) < \tau_{k+1}(\mathbf{x}), \qquad \lim_{k \to \infty} \tau_k(\mathbf{x}) = \infty \tag{2.4}$$

for all $\mathbf{x}$ and $k = 1, 2, \ldots$. This type of systems offer more complex behavior compared to the systems with fixed reset instants. For instance, solutions may hit the same surface $t = \tau_k(\mathbf{x})$ several times (behavior called *pulse phenomenon*), and different solutions may coincide after some time and behave as a single solution thereafter (*confluence phenomenon*). A generalization of this class of systems is to consider the resetting set $\mathcal{S} = \{(t, \mathbf{x}) \in \mathbb{R}_+ \times \mathbb{R}^n : h(t, \mathbf{x}) = 0\}$, where $h(t, \mathbf{x})$ might represent more general surfaces.

- *Autonomous systems with impulses.* In this case, the reset instants only depend on the state $\mathbf{x}(t)$, that is $\mathcal{S} = \{(t, \mathbf{x}) \in \mathbb{R}_+ \times \mathbb{R}^n : \mathbf{x} \in \mathcal{M}\}$, where $\mathcal{M} \subset \mathbb{R}^n$ is an arbitrary subset. The notation $\mathcal{M}$ will be reserved for those cases in which the IDS can be classified as an autonomous system with impulses. In addition, we will also refer to $\mathcal{M}$ as the resetting set[5] in those cases.

The above classification is also considered in [138], but in that case the classes are renamed as follows: time-dependent impulsive dynamical systems (fixed instants), state-dependent impulsive dynamical systems (autonomous systems), and both time/state-dependent impulsive dynamical systems.

A possible extension of the IDS (2.3) is to consider input and output signals. Based on the monograph [282], an impulsive dynamical system with input and output may be represented by

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t), \mathbf{u}(t)), & (t, \mathbf{x}(t), \mathbf{u}(t)) \notin \mathcal{S}, \\ \mathbf{x}(t^+) = \mathbf{I}(t, \mathbf{x}(t), \mathbf{u}(t)), & (t, \mathbf{x}(t), \mathbf{u}(t)) \in \mathcal{S}, \\ \mathbf{y}(t) = \mathbf{h}(t, \mathbf{x}(t), \mathbf{u}(t)), & \end{cases} \tag{2.5}$$

where $\mathbf{u}(t) \in \mathcal{U} \subseteq \mathbb{R}^{m_u}$ is the input signal, and the output signal $\mathbf{y}(t) \in \mathcal{Y} \subseteq \mathbb{R}^{m_y}$ is obtained by the function $\mathbf{h} : \mathbb{R}_+ \times \Omega \times \mathcal{U} \to \mathbb{R}^{m_y}$, which has not effect on the dynamic of the system. The functions $\mathbf{f}$ and $\mathbf{I}$ are defined similarly. In this case, the resetting set is $\mathcal{S} \subset \mathbb{R}_+ \times \Omega \times \mathcal{U}$. Since the resetting law may depend on the input, another classification of the impulsive dynamical systems would be adequate. Following the idea behind [138, 172, 282], throughout the thesis, the impulsive dynamical systems are classified depending on the resetting set $\mathcal{S}$:

- *Time-dependent impulsive dynamical systems.*
- *State-dependent impulsive dynamical systems.*

[5] The notation $\mathcal{M}$ is usually used to denote the resetting set in the field of reset control systems.

- *Input-dependent impulsive dynamical systems.*
- Any combination of the above classes (time/state, time/input, state/input,...).

### 2.2.1 Solution of the Impulsive Dynamical System

The system described by the first equation in (2.3) will be referred to as the *base system*[6], and we refer to the second equation as the *resetting law*. In addition, the notation $\mathbf{x}(t, t_0, \mathbf{x}_0)$ is used to denote the solution $\mathbf{x}(t)$ of the IDS (2.3) at time $t \geq t_0$ with initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$. For a given pair $(t_0, \mathbf{x}_0)$, there may exist instants $t$ at which $(t, \mathbf{x}(t))$ intersects $\mathcal{S}$, and those instants will be referred to as *reset instants*. At the reset instants the state $\mathbf{x}(t)$ jumps to $\mathbf{x}(t^+) = \mathbf{I}(t, \mathbf{x}(t)) \in \mathcal{S}_{\mathcal{R}}$ where $\mathcal{S}_{\mathcal{R}}$ is the *after-reset set*, that is, the set of all the states that can be reached with a reset action. Similarly as the notation $\mathcal{M}$, we will also use $\mathcal{M}_{\mathcal{R}}$ to denote the after-reset set for those IDS classified as autonomous systems with impulses (state-dependent IDS). The sequence of reset instants is denoted by $\mathbb{T}_{t_0,\mathbf{x}_0} = (t_1, t_2, \cdots)$ (the subscript is added to emphasise the dependence on the initial instant and the initial condition). Note that the sequence of reset instants may be finite, so that $\infty$ is considered as an element of the sequence. In this way, there is three possible cases: i) $\mathbb{T}_{t_0,\mathbf{x}_0} = (\infty)$ (there is no reset actions), ii) $\mathbb{T}_{t_0,\mathbf{x}_0} = (t_i)_{i=1}^{N} = (t_1, t_2, \cdots, t_{N-1}, \infty)$ (a finite number of reset actions), iii) $\mathbb{T}_{t_0,\mathbf{x}_0} = (t_i)_{i=1}^{\infty}$ (an infinite number of reset actions).

Following [138], a formal definition of a solution to the IDS (2.3) is given as follows:

**Definition 2.1 (Solution of IDS).** For a given initial instant $t_0 \in \mathbb{R}_+$ and a initial condition $\mathbf{x}_0 \in \Omega$. A function $\mathbf{x} : \mathbb{I}_{t_0,\mathbf{x}_0} \to \Omega$ is a *solution* of the IDS (2.3) on the interval $\mathbb{I}_{t_0,\mathbf{x}_0}$, with initial condition $\mathbf{x}_0$, if:

- $\mathbf{x}(t_0) = \mathbf{x}_0$.
- $\mathbf{x}(t)$ is absolutely continuous on every interval $(t_{k-1}, t_k] \subseteq \mathbb{I}_{t_0,\mathbf{x}_0}$, $k = 1, 2, \ldots$.
- $\mathbf{x}(t)$ satisfies the differential equation $\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t))$ for all $t \in \mathbb{I}_{t_0,\mathbf{x}_0}$, $t \notin \mathbb{T}_{t_0,\mathbf{x}_0}$.
- $\mathbf{x}(t)$ is left-continuous on $\mathbb{I}_{t_0,\mathbf{x}_0}$ and $\mathbf{x}(t^+) = \mathbf{I}(t, \mathbf{x}(t))$ for all $t \in \mathbb{T}_{t_0,\mathbf{x}_0}$.

An important issue, in relation to the solution of the IDS, is that the reset instants are well-defined and are distinct for any initial time $t_0$ and any initial condition $\mathbf{x}_0$, this property is referred to as *well-posedness of the reset instants* ([28, 29, 138]). In

[6]The base system is referred to as *continuous-time dynamic* in [138].

the following, a formal definition of *well-posed reset instants* will be given based on the solution of the base system. For the IDS (2.3), let $\psi(t, t_0, \psi_0)$ (or simply $\psi(t)$) denotes the solution of the (continuous) base system where $t_0 \in \mathbb{R}_+$, $t \geq t_0$ and $\psi(t_0) = \psi_0 \in \Omega$ is the initial condition. It is further assumed that $\mathbf{f}$ satisfies enough conditions such that the base system has a unique solution over the interval $[t_0, \infty)$. This is the case, for example, if $\mathbf{f}$ is globally Lipschitz continuous (see, e.g., [273]).

**Definition 2.2** (**Well-posed reset instants).** An IDS given by (2.3) has *well-posed reset instants* if for any initial time $t_0 \in \mathbb{R}_+$ and any initial condition $\mathbf{x}_0 \in \Omega$, there exists a sequence $\mathbb{T}_{t_0,\mathbf{x}_0} : \{1, 2, \cdots\} \to \mathbb{R}_+ \cup \{\infty\}$, denoted by $\mathbb{T}_{t_0,\mathbf{x}_0} = (t_1, t_2, \cdots)$, and given by the following procedure:

**if** $(t_0, \mathbf{x}_0) \in \mathcal{S}$ **then**
  $t_1 \leftarrow t_0$ ;
**else**
  $t_1 \leftarrow \min\{\Delta > t_0 : (\Delta, \psi(\Delta, t_0, \mathbf{x}_0)) \in \mathcal{S}\}$;
**end**
**while** $t_i \neq \infty$ **do**
  $\mathbf{x}_i \leftarrow \mathbf{I}(t_i, \psi(t_i, t_{i-1}, \mathbf{x}_{i-1}))$ ;
  $t_{i+1} \leftarrow t_i + \min\{\Delta \in \mathbb{R}_+ : \Delta > 0 \wedge (t_i + \Delta, \psi(t_i + \Delta, t_i, \mathbf{x}_i) \in \mathcal{S}\}$;
  $i \leftarrow i + 1$ ;
  $N \leftarrow i$;
**end**

A system that does not have well-posed reset instants is said to have *ill-posed reset instants*. In the following, two examples of IDS with ill-posed reset instants are exposed.

**Example 2.1.** The following example is adapted from [29]. Consider the following state-dependent IDS

$$\begin{cases} \dot{\mathbf{x}}(t) = A\mathbf{x}(t), & \mathbf{x}(t) \notin \mathcal{M}, \\ \mathbf{x}(t^+) = A_R\mathbf{x}(t), & \mathbf{x}(t) \in \mathcal{M}, \end{cases} \tag{2.6}$$

for some constants $\alpha$, $\omega > 0$, and matrices

$$A = \begin{bmatrix} -\alpha & 0 & 0 \\ 0 & 0 & -\omega \\ 0 & \omega & 0 \end{bmatrix}, \qquad A_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{2.7}$$

In addition, the resetting set $\mathcal{M} = \{\mathbf{x} \in \mathbb{R}^3 : \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \mathbf{x} = 0\}$. Now, consider for instance the initial condition $\mathbf{x}_0 = (1, 1, 0) \notin \mathcal{M}$, then the first reset instant is well-defined and is given by $t_1 = \frac{\pi}{2\omega}$, since $\mathbf{x}(t_1) = (e^{-\alpha \frac{\pi}{2\omega}}, 0, 1) \in \mathcal{M}$. The after-reset state is given by $\mathbf{x}(t_1^+) = (e^{-\alpha \frac{\pi}{2\omega}}, 0, 0)$. It is clear that the reset condition is again satisfied, implying formally an infinite sequence of resettings to be performed instantaneously (beating phenomenon). Therefore, the IDS has ill-posed reset instants.

**Example 2.2.** The system in this example is used in [211] for bringing to light some weak points in the definition of reset control systems in [47]. Consider the IDS (2.6) with matrix $A_R$ given in (2.7) and matrix $A$ given by

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & -1 \end{bmatrix}. \tag{2.8}$$

Now the resetting set is defined as follows:

$$\mathcal{M} = \{\mathbf{x} \in \mathbb{R}^3 : \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x} = 0 \wedge \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \mathbf{x} \neq 0\} \tag{2.9}$$

and the after-reset set is given by $\mathcal{M}_{\mathcal{R}} = \text{span}\{(0, 1, 0)\}$. Let the initial condition be $\mathbf{x}_0 = (0, x_{20}, 0)$, then the first reset instant is given by

$$t_1 = \min\{\Delta > 0 : e^{A\Delta}\mathbf{x}_0 \in \mathcal{M}\}. \tag{2.10}$$

Note that $e^{A\Delta}\mathbf{x}_0 \in \mathcal{M}$ for all $\Delta \in (0, T_1)$ with $T_1$ some instant prior to $t^* \approx 3.11$, the first non-zero instant in which $e^{At^*}\mathbf{x}_0 \in \mathcal{M}_{\mathcal{R}}$. Since $\Delta$ belongs to an open set, the minimum does not exist, and thus, the IDS has ill-posed reset instants.

Note that the IDS in the Example 2.1 fails since after a reset action the state still belongs to the resetting set. Therefore, it seems natural that the after-reset set should be removed from the resetting set for an adequate definition of the system. This solution is adopted in [47] for the definition of reset control systems. Regarding the Example 2.2, in this case the IDS fails since the evolution of the system after the reset instant is necessarily through the resetting set, leading into an ill-defined system.

In order to assure the well-posed of the reset instants two assumptions can be made (see, e.g., [138]), such that, it is only allowed Zeno solutions and confluence phenomenon. Roughly speaking, the assumptions consist in ensuring that immediately after a reset event the system evolves by the (continuous) base system and not by the resetting law. Formally, the assumptions stand as follows:

1. If $(t, \mathbf{x}(t)) \in \bar{\mathcal{S}} \setminus \mathcal{S}$, then there exists $\epsilon > 0$ such that, for all $\delta$ with $0 < \delta < \epsilon$, it is satisfied
$$(t+\delta, \psi(t+\delta, t, \mathbf{x}(t))) \notin \mathcal{S}. \tag{2.11}$$

2. If $(t_k, \mathbf{x}(t_k)) \in \partial\mathcal{S} \cap \mathcal{S}$, then $\mathbf{x}(t_k^+) \notin \mathcal{S}$ and there exists $\epsilon > 0$ such that, for all $\delta$ with $0 < \delta < \epsilon$, it is satisfied
$$(t_k+\delta, \psi(t_k+\delta, t_k, \mathbf{I}(t_k, \mathbf{x}(t_k)))) \notin \mathcal{S}. \tag{2.12}$$

Therefore, the resetting set of an IDS should be properly defined so that the above assumptions are satisfied. However, there is a simple modification of the system such that the resulting IDS has well-posed reset instants, and in addition, it also prevents of Zeno solutions. This technique is referred to as *time regularization* and was devised for hybrid systems in [162] with the main goal of avoiding Zeno solutions. An IDS with time regularization can be represented by the following impulsive differential equation

$$\left\{ \begin{array}{l} \left. \begin{array}{l} \dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t)), \\ \dot{\Delta}(t) = 1, \end{array} \right\} \quad (t, \mathbf{x}(t), \Delta(t)) \notin \mathcal{S}, \\ \left. \begin{array}{l} \mathbf{x}(t^+) = \mathbf{I}(t, \mathbf{x}(t)), \\ \Delta(t^+) = 0, \end{array} \right\} \quad (t, \mathbf{x}(t), \Delta(t)) \in \mathcal{S}, \end{array} \right. \tag{2.13}$$

where a new state $\Delta(t) \in \mathbb{R}_+$ has been introduced. This state represents the time between two consecutive reset actions. The regularization of the system comes from the definition of a new resetting set in which it is imposed a lower bound on the reset interval through $\Delta(t)$. Let define the set $\mathcal{Z} \subset \mathbb{R}_+ \times \Omega$ and a real value $\Delta_m > 0$, then the resetting set is defined by

$$\mathcal{S} = \{(t, \mathbf{x}, \Delta) \in \mathbb{R}_+ \times \Omega \times \mathbb{R}_+ : (t, \mathbf{x}) \in \mathcal{Z} \wedge \Delta \geq \Delta_m\}. \tag{2.14}$$

Note that $\Delta_m > 0$ ensures that Assumption 2 is guaranteed, that is, when a trajectory intersects the resetting set $\mathcal{S}$, it instantaneously exits $\mathcal{S}$. As a result, no infinite number of resets will be performed after a reset instant. In addition, every reset event will occur at least $\Delta_m$ seconds after the previous reset event, that is the reset instants satisfy $t_k - t_{k-1} \geq \Delta_m$, $k = 1, 2, \ldots$, and Assumption 1 is also guaranteed. Therefore, it is clear that for any initial time $t_0$ and any initial condition $\mathbf{x}_0$ there exists a finite or infinite sequence of well-posed reset instants $\mathbb{T}_{t_0,\mathbf{x}_0} = (t_1, t_2, \ldots)$ such that they are distinct and satisfy $t_0 \leq t_1 < t_2 < \cdots$. It should be pointed out that from the point of view of the well-posedness of the reset instants, $\Delta_m$ can be set arbitrarily small, and thus the regularized system will be a good model of the real system for many practical applications.

Now, suppose that the IDS (2.3) has well-posed reset instants, then the trajectory $\mathbf{x}(t)$ of the impulsive system from $\mathbf{x}_0$ is given by $\psi(t, t_0, \mathbf{x}_0)$ for $t_0 < t \leq t_1$, where $t_1$ is the instant at which $(t_1, \mathbf{x}(t_1)) \in \mathcal{S}$ with $\mathbf{x}(t_1) = \psi(t_1)$. At that instant the state is instantaneously transferred to $\mathbf{x}(t_1^+) = \mathbf{I}(t_1, \mathbf{x}(t_1))$, according to the resetting law. The trajectory $\mathbf{x}(t)$, $t_1 < t \leq t_2$ is given by $\psi(t, t_1, \mathbf{x}(t_1^+))$, where $t_2$ is again such that $(t_2, \psi(t_2)) \in \mathcal{S}$. The same procedure is applied to the rest of the intervals. In this way, the solution $\mathbf{x}(t)$ to (2.3) is made up of a initial condition $\mathbf{x}_0$ and a sequence of continuous solution segments $\psi(t, t_k, \mathbf{x}(t_k^+))$, that is

$$\mathbf{x}(t, t_0, \mathbf{x}_0) = \begin{cases} \mathbf{x}_0, & t = t_0, \\ \psi(t, t_0, \mathbf{x}_0^+), & t \in (t_0, t_1], \\ \psi(t, t_1, \mathbf{x}(t_1^+)), & t \in (t_1, t_2], \\ & \vdots \\ \psi(t, t_k, \mathbf{x}(t_k^+)), & t \in (t_k, t_{k+1}]. \end{cases} \tag{2.15}$$

In the case that the sequence of reset instants is finite then $\mathbf{x}(t, t_0, \mathbf{x}_0) = \psi(t, t_N, \mathbf{x}(t_{N-1}^+))$, $t \in (t_{N-1}, \infty)$, where $N-1$ is the number of reset instants. On the other hand, if there is not reset instants, then $\mathbf{x}(t, t_0, \mathbf{x}_0) = \psi(t, t_0, \mathbf{x}_0)$, $t \in [t_0, \infty)$. Note that the solution $\mathbf{x}(t)$ is left-continuous with right limit, that is, it is continuous everywhere except at the reset instants $t_k$, and

$$\begin{aligned} \mathbf{x}(t_k) &= \lim_{\substack{\epsilon \to 0 \\ \epsilon \geq 0}} \mathbf{x}(t_k - \epsilon), \\ \mathbf{x}(t_k^+) &= \mathbf{I}(t_k, \mathbf{x}(t_k)) = \lim_{\substack{\epsilon \to 0 \\ \epsilon \geq 0}} \mathbf{x}(t_k + \epsilon). \end{aligned} \tag{2.16}$$

### 2.2.2 Impulsive Control Systems

In spite of the fact that impulsive control systems are not under study in the thesis, it is important to briefly introduce them due to two main reasons: first, some of the developed results can be applied to impulsive control systems as long as the results applied to general impulsive systems; second, it is important to clearly distinguish impulsive control systems to reset control systems, which are indeed the main scope of the thesis.

Impulsive control is a control paradigm based on impulsive differential equations, in which a nonimpulsive plant should have at least one *impulsively* changeable state variable (impulsive plants may be also considered). The impulse control consists in controlling a plant by applying instantaneous changes on its state variables (the plant may also subject to nonimpulsive control in parallel). Although, some plants can be clearly classified as impulsive plants, for instance plants involving impacts or collisions, the concept of impulsive plant is fuzzy, and there is not crisp standard to distinguish impulsive plants from conventional plants. If the control actions are taken in a much shorter time period comparing with the time constant, or natural period of the plant, then the plant may be studied under the impulsive control paradigm. Of course, control systems with impulsive control are impulsive systems, and thus, they can be modeled by the IDS framework presented before. It is out the scope of the thesis to report the results in impulsive control systems, so that the reader is referred to the monograph [282]. Note that in contrast to the impulsive control systems, the plant is nonimpulsive in the reset control systems. This fact is the main limitation in the application of the existing stability results for impulsive delay systems to reset control systems with time-delay. Finally, besides the aforementioned classification of impulsive systems, it is interesting to considered a parallel classification[7] suggested in [282], based on the type of the control actions:

- *Type-I: Impulsive control law.* In this case, a nonimpulsive plant is controlled by a pure impulsive control law, that is, the control law consists in changing instantaneously the state of the plant.

- *Type-II: Impulsive/continuous control law.* The plant is controlled by both an impulsive and a continuous control law.

- *Type-II: Continuous control law.* In this kind of systems, the plant itself is impulsive and the control is continuous.

[7] In [282], the classes of impulsive control systems are simply named *Type-I*, *Type-II* and *Type-III*. We have included an extra name for a better association with their definition.

### 2.2.3 Impulsive Delay Dynamical Systems

Impulsive delay systems and its mathematical foundation called *Impulsive functional differential equations* (or *Impulsive delay differential equations*) arises from the necessity to model processes which are characterized by the fact that their states not only change by jumps but also by the dependence of the process on its history at each moment of time. They are a natural generalization of impulsive systems (impulsive differential equations) and time-delay systems (functional differential equations). A recent monograph in the field of impulsive delay systems is [256], where the author presents the main contribution on the stability theory for impulsive functional differential equations based on Lyapunov method. On the other hand, throughout the thesis we will refer to the impulsive delay systems as impulsive delay dynamical systems (IDDS) for the sake of the homogeneity, since it will be used the same formulation as in the IDS case.

The study of IDDS is much more challenging. For instance, the analysis of the well-posedness of the system is a hard problem, in particular for state-dependent systems. To the author knowledge, there is not results to guarantee the well-posed of the reset instants and to avoid Zeno solutions neither for general IDDSs nor for reset control systems with time-delay. Therefore, the technique of time regularization will be used. An IDDS with time regularization may be described by the following impulsive functional differential equation

$$\left\{ \begin{array}{ll} \left. \begin{array}{l} \dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}_t), \\ \dot{\Delta}(t) = 1, \end{array} \right\} & (t, \mathbf{x}_t, \Delta(t)) \notin \mathcal{S}, \\ \left. \begin{array}{l} \mathbf{x}(t^+) = \mathbf{I}(t, \mathbf{x}_t), \\ \Delta(t^+) = 0, \end{array} \right\} & (t, \mathbf{x}_t, \Delta(t)) \in \mathcal{S}, \\ \left. \begin{array}{l} \mathbf{x}(t) = \phi(t - t_0), \\ \Delta(t) = 0, \end{array} \right\} & t \in [t_0 - h, 0], \end{array} \right. \tag{2.17}$$

where $t \geq t_0 \in \mathbb{R}_+$ and $\Delta(t) \in \mathbb{R}_+$ is the *reset interval.* For the sake of simplicity, it is assumed that the evolution of the system at instant $t$ only depends on the system state at prior instants in the interval $[t-h, t]$, where the constant $h > 0$ is the maximum *time-delay*[8]. In comparison with impulses systems without time-delay, the initial condition of these systems is a left-continuous function $\phi : [-h, 0] \to \Omega$ with right limits and $\Omega \subseteq \mathbb{R}^n$. Note that it is important to distinguish between the *lumped state* $\mathbf{x}(t) \in \mathbb{R}^n$

[8] In the most general case, the evolution of the system may depend on the whole interval $[t_0 - h, t]$.

and the true *distributed state* given by some function $\mathbf{x}_t : [-h, 0] \to \mathbb{R}^n$, defined by $\mathbf{x}_t(\theta) = \mathbf{x}(t + \theta)$, $\theta \in [-h, 0]$. In this case, the (functional) differential equation is governed by a continuous (on both parameters) function $\mathbf{f} : \mathbb{R}_+ \times \mathcal{PC}([-h, 0], \Omega) \to \mathbb{R}^n$. In addition, the resetting law is defined by the function $\mathbf{I} : \mathbb{R}_+ \times \mathcal{PC}([-h, 0], \Omega) \to \mathbb{R}^n$. For notational simplicity it is assumed $\Omega = \mathbb{R}^n$ (unless other set is explicitly specified), since throughout the thesis the systems under study are defined on the whole state-space.

In this case, the set $\mathcal{Z}$ is defined by $\mathcal{Z} \subset \mathbb{R}_+ \times \mathcal{PC}([-h, 0], \mathbb{R}^n)$, and the resetting set is given by

$$\mathcal{S} = \{(t, \varphi, \Delta) \in \mathbb{R}_+ \times \mathcal{PC}([-h, 0], \mathbb{R}^n) \times \mathbb{R}_+ : (t, \varphi) \in \mathcal{Z} \wedge \Delta \geq \Delta_m\}. \tag{2.18}$$

### Solution to the IDDS

In the following, the definition of solution to the IDS given before will be extended to IDDS. In addition, it is shown that the solution to an IDDS with well-posed reset instants is made up of continuous segments from the base system.

**Definition 2.3 (Solution of IDDS).** For a given initial instant $t_0 \in \mathbb{R}_+$ and a initial condition $\phi \in \mathcal{PC}([-h, 0], \mathbb{R}^n)$. A function $\mathbf{x} : \mathbb{I}_{t_0,\phi} \to \Omega$ is a *solution* of the IDDS (2.17) on the interval $\mathbb{I}_{t_0,\phi}$, with initial condition $\phi$, if:

- $\mathbf{x}(t) = \phi(t)$ for $t \in [t_0 - h, 0]$.
- $\mathbf{x}(t)$ is absolutely continuous on every interval $(t_{k-1}, t_k] \subseteq \mathbb{I}_{t_0,\mathbf{x}_0}$, $k = 1, 2, \cdots$.
- $\mathbf{x}(t)$ satisfies the functional differential equation $\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}_t)$ for all $t \in \mathbb{I}_{t_0,\phi}$, $t \notin \mathbb{T}_{t_0,\phi}$.
- $\mathbf{x}(t)$ is left-continuous in $\mathbb{I}_{t_0,\phi}$ and $\mathbf{x}(t^+) = \mathbf{I}(t, \mathbf{x}_t)$ for all $t \in \mathbb{T}_{t_0,\phi}$.

For the IDDS (2.17), let $\psi(t, t_0, \phi)$ denotes the solution of the base system with initial condition $\phi$. Again, it is assumed that $\mathbf{f}$ fulfills enough conditions to guarantee that the solution $\psi$ exists on the interval $[t_0, \infty)$ and is unique. In this way, the definition of well-posed reset instants for the IDDS is identical to Definition 2.2, where now the reset instants are formed by considering the base system of the IDDS. Of course, since the time regularization has been directly applied to the IDDS under study, it is clear that the reset instants are well-posed. For an IDDS with well-posed reset instants, the problem of the existence and uniqueness of the solutions has been tackled for instance in [16], by imposing essentially Lipschitz continuity on $\mathbf{f}$. Finally, the solution $\mathbf{x}(t, t_0, \phi)$ of (2.17) is made up of a initial condition $\phi$ and a sequence of continuous solution segments $\psi(t, t_k, \mathbf{x}_{t_k^+})$, that is

$$\mathbf{x}(t, t_0, \phi) = \begin{cases} \phi(0), & t = [t_0 - h, t_0], \\ \psi(t, t_0, \phi), & t \in (t_0, t_1], \\ \psi(t, t_1, \mathbf{x}_{t_1^+}), & t \in (t_1, t_2], \\ & \vdots \\ \psi(t, t_k, \mathbf{x}_{t_k^+}), & t \in (t_k, t_{k+1}]. \end{cases} \tag{2.19}$$

**Shifted-distributed State Representation**

As remarked earlier, an important fact in IDDS is that the evolution of the system does not only depend on the current state $\mathbf{x}(t)$ but also on the prior states. For this reason, it was defined the distributed state $\mathbf{x}_t$, that is, a function that recovers the state of the system in the interval $[t - h, t]$ for a given $t \geq t_0$ (see Fig. 2.1 for a better understanding). However, this notation it turns out to be tedious for the study of the stability by the approach proposed in this thesis. Therefore, for the sake of the simplicity of analysis in future chapters, we will define several terms. First, the sequence of reset interval will be denoted by $(\Delta_k)$, where $\Delta_k = t_{k+1} - t_k$, $k = 0, 1, \ldots$. Note that $\Delta_k \geq \Delta_m$ by the definition of the resetting set. In addition, if the set of reset instants is finite then $\Delta_N = \infty$. On the other hand, for all reset instant $t_k \in \mathbb{T}_{t_0,\phi}$ and also $t_0$, it is also defined the *shifted-distributed state*, which is a function $\chi_k \in \mathcal{PC}([0, \Delta_k] \times [-h, 0], \mathbb{R}^n)$, such that $\chi_k(\tau, \theta)$ is left-continuous on both parameters and is defined by $\chi_k(\tau, \theta) = \mathbf{x}(t_k + \tau + \theta)$. It is clear that the distributed state may be recovered by the shifted-distributed state, that is $\mathbf{x}_t = \chi_k(t - t_k, \cdot)$ where $t_k$ is the greater reset instant such that $t_k \leq t$. Fig. 2.2 shows the function $\chi_k$ when its parameters change.

## 2.3 Hybrid Systems

In this section, it is briefly presented a framework for modeling hybrid systems referred to as Hybrid Inclusion framework (HI), which is based on a generalization of the solutions and the time domain of the hybrid system. This modeling framework was proposed in [110] and more formally stated in [113]. One of the main motivation of the framework is to provide tools to study robust stability, that in words of the authors, it is important for predicting the behavior of a hybrid control system in the presence of

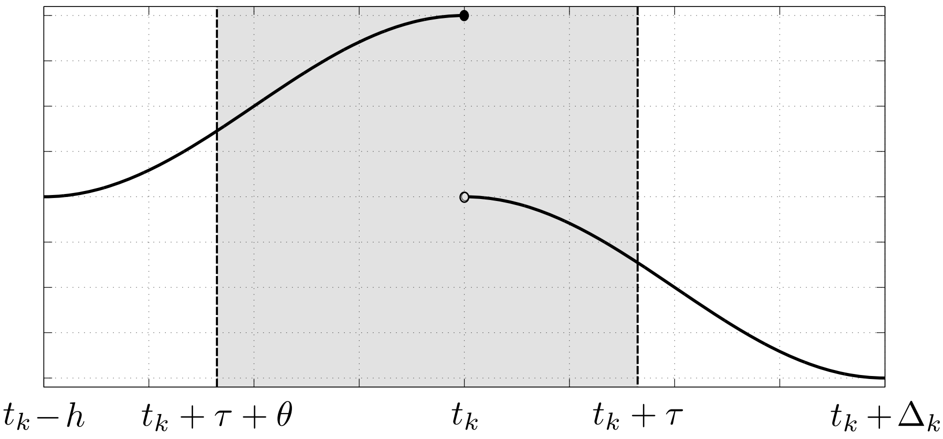



Figure 2.1 Illustration of the distributed state. The shaded area is the domain of $\mathbf{x}_{t_k+\tau}$.

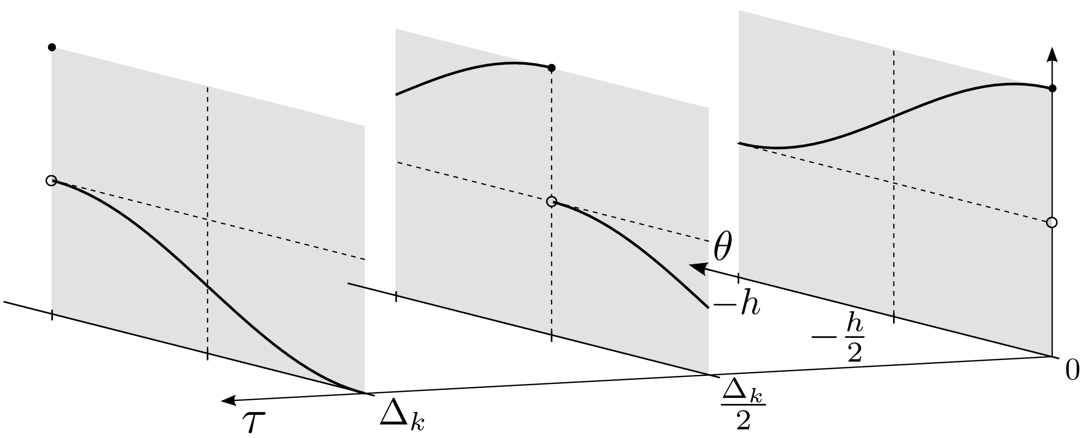



Figure 2.2 Illustration of the shifted-distributed state representation. The shaded areas are the domains of $\chi_k(0,\cdot)$, $\chi_k(\frac{\Delta_k}{2},\cdot)$, and $\chi_k(\Delta_k,\cdot)$.

measurement noise and other modeling uncertainty. In the last years, there has been a considerable number of results ([1, 15, 206, 211, 223, 288]) where this framework has been successfully applied. For a detailed exposition of the framework, the reader is referred to the tutorial [111] and the monograph [112].

In this framework, a model of a hybrid system is represented in the following form

$$\begin{cases} \dot{\mathbf{x}} \in F(\mathbf{x}), & \mathbf{x} \in C, \\ \mathbf{x}^+ \in G(\mathbf{x}), & \mathbf{x} \in D, \end{cases} \tag{2.20}$$

where $\mathbf{x}^+$ is the after-reset state and the motion of the system is governed by a constrained differential inclusion and a constrained difference inclusion. A hybrid system in form (2.20) is defined by four elements called *Data of a hybrid system*:

- *Flow set.* A set $C \subset \mathbb{R}^n$.
- *Flow map.* A set-value mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ with $C \subset \text{dom } F$.
- *Jump set.* A set $D \subset \mathbb{R}^n$.
- *Jump map.* A set-value mapping $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ with $D \subset \text{dom } G$.

In comparison with the impulsive systems framework, here the solutions are parameterized by both the time $t$, and the number of jumps that have occurred, $j$. In principle, this provides a more natural treatment of Zeno solutions and solutions with multiple consecutive jumps (no continuous evolution between then). The solution of the hybrid system are what it is called *hybrid arcs*, that is, a function $\phi : E \rightarrow \mathbb{R}^n$ where $E \subset \mathbb{R}_+ \times \mathbb{N}$, is a *hybrid time domain* (union of a finite of infinite sequence of intervals $[t_j, t_{j+1}] \times \{j\}$). In addition, a smoothness property is required on $\phi$, the function $t \rightarrow \phi(t, j)$ for a given $j$ should be locally absolutely continuous on the interval $I^j = \{t : (t, j) \in E\}$. Finally, a hybrid arc $\phi$ has to satisfy certain conditions to be a solution of the hybrid system:

**Definition 2.4** (**Solution to a hybrid system**). A hybrid arc $\phi$ is a solution to the hybrid system $(C, F, D, G)$, if $\phi(0, 0) \in \bar{C} \cup D$, and

- for all $j \in \mathbb{N}$ such that $I^j = \{t : (t, j) \in \text{dom } \phi\}$ has nonempty interior

$$\begin{array}{lll} \phi(t, j) \in C & \text{for all} & t \in \text{int}(I^j), \\ \dot{\phi}(t, j) \in F(\phi(t, j)) & \text{for almost all} & t \in I^j, \end{array} \tag{2.21}$$

- for all $(t, j) \in \text{dom } \phi$ such that $(t, j+1) \in \text{dom } \phi$

$$\begin{aligned} &\phi(t, j) \in D, \\ &\phi(t, j+1) \in G(\phi(t, j)). \end{aligned} \tag{2.22}$$

As mentioned before, an important issue in the formulation of impulsive/hybrid systems is the well-posed of the system (existence, uniqueness and continuous dependence). In [112], the authors considered restrictive to insist on the uniqueness of solutions and continuous dependence on the initial condition. In this way, the problem of well-posedness of the hybrid system is reduced to the existence of solution and a relaxed sense of continuous dependence on the initial condition (*Nominal Well-posedness*). Despite the lesser restrictive notion of well-posedness, it is still necessary to impose conditions on the data of the hybrid system to guarantee its well-posed. For instance, among other basic assumptions, it is assumed that $C$ and $D$ are closed sets in $\mathbb{R}^n$.

Regarding reset control systems, the framework HI was first applied to model the Clegg integrator and FORE in [288].

### 2.3.1 Hybrid Systems with Memory

In [183], it is accomplished a first step to extend the framework HI to hybrid systems that are subjected to time-delays. In this first work, the authors gave a generalized formulation of *hybrid systems with memory* and established some basic existence and nominal well-posedness results. The case considered in [183] assumes that the flow and jump sets are subsets of $\mathbb{R}^n$. Later on, in [184] the same authors considered a more general case of hybrid systems where the flow and jump sets are subsets of a function space. In addition, it is analyzed the stability of the solutions for hybrid systems with memory, providing conditions formulated using Lyapunov-Razumikhin functions.

As well as in its non delayed counterpart, the state of the system are considered as a function of both the time and the number of jumps that occurred. In this case, the domain of time is called *hybrid time domain with memory*, and it is formed as a union of a finite or infinite sequence of sets of the form $[t_j, t_{j+1}] \times \{j\}$ and of a finite or infinite sequence of sets of the form $[s_k, s_{k-1}] \times \{-k+1\}$. In this way, the hybrid arcs are extended to *Hybrid arc with memory* , that are composed by a hybrid time domain with memory $E$ and a function $\phi : E \to \mathbb{R}^n$. In particular, the hybrid arc with memory is called *hybrid memory arc*, when the domain of $\phi$ is restricted to $\mathbb{R}_- \times \mathbb{Z}_-$. A *hybrid* *memory arc* may be understood as the history of the system due to the time-delay. Again, the hybrid system (with memory) consists of four elements:

- *Flow set.* A set $C \subset \mathcal{M}$.
- *Flow map.* A set-value functional $F : \mathcal{M} \rightrightarrows \mathbb{R}^n$ with $C \subset \text{dom } F$.
- *Jump set.* A set $D \subset \mathcal{M}$.
- *Jump map.* A set-value mapping $G : \mathcal{M} \rightrightarrows \mathbb{R}^n$ with $D \subset \text{dom } G$.

where $\mathcal{M}$ is the collection of all hybrid memory arcs. It should not be confused with the resetting set of state-dependent IDSs. Finally, a solution to the hybrid system with memory is defined as follows:

**Definition 2.5 (Solution to a hybrid system with memory).** A hybrid arc $\phi$ is a solution to the hybrid system $(C, F, D, G)$, if $\mathcal{A}_{[0,0]} \in C \cup D$, and

- for all $j \in \mathbb{Z}_+$ and almost all $t \in I_j$,

$$\mathcal{A}_{[t,j]}\phi \in C, \quad \dot{\phi}(t,j) \in F(\mathcal{A}_{[t,j]}\phi), \tag{2.23}$$

- for all $j \in \mathbb{Z}_+$ and $(t,j) \in \text{dom } \phi$ such that $(t, j+1) \in \text{dom } \phi$,

$$\mathcal{A}_{[t,j]}\phi \in D, \quad \phi(t,j+1) \in G(\mathcal{A}_{[t,j]}\phi), \tag{2.24}$$

where $I^j = \{t : (t,j) \in \text{dom } \phi\}$ and for a given hybrid arc with memory $\phi$ and $(t,j) \in \text{dom } \phi$, $\mathcal{A}_{[t,j]}$ is an operator defines as follows:

$$\mathcal{A}_{[t,j]}\phi(s,k) = \phi(t+s, j+k) \tag{2.25}$$

for $(s,k)$ such that $(t+s, j+k) \in \text{dom } \phi$.

The operator $\mathcal{A}_{[t,j]}$ plays the same role as the the distributed state $\mathbf{x}_t$ used, for instance, in the IDDS framework presented in Section 2.2.3. That is, just like $\mathbf{x}_t$ allows to recover the state of the system at a past time $t - \theta \in [t-h, t]$, the operator $\mathcal{A}_{[t,j]}$ allows to recover the state of the hybrid system at the generalized time $(t+s, j+k)$. Therefore, it is clear that $\mathcal{A}_{[0,0]}$ is the initial condition of the hybrid system with memory.

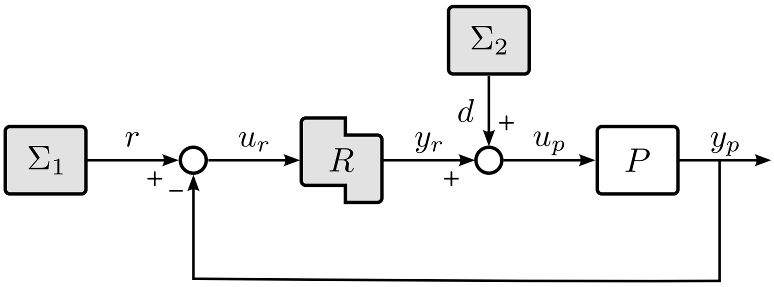



Figure 2.3 Standard reset control system formed by a LTI plant P, a reset compensator R and two exosystems $\Sigma_1$ and $\Sigma_2$.

## 2.4 Reset Control Systems

Consider a feedback control system as shown in Fig. 2.3, given by the feedback interconnection of a LTI system P (referred to as *plant* and also *process*) and a compensator R (both single-input-single-output). P is given by the state-space system

$$P : \begin{cases} \dot{\mathbf{x}}_p(t) = A_p\mathbf{x}_p(t) + B_p u_p(t), \\ y_p(t) = C_p\mathbf{x}_p(t), \end{cases} \tag{2.26}$$

where $\mathbf{x}_p \in \mathbb{R}^{n_p}$ and $A_p \in \mathbb{R}^{n_p \times n_p}$, $B_p \in \mathbb{R}^{n_p \times 1}$, $C_p \in \mathbb{R}^{1 \times n_p}$. The compensator R is a *reset compensator*, that is, a LTI system provided with resets actions, and thus, a reset compensator is an impulsive system itself (a state/input-dependent IDS). Let $\mathbf{x}_r$ be the $n_r$-dimensional state of the reset compensator then R is described by

$$R : \begin{cases} \dot{\mathbf{x}}_r(t) = A_r\mathbf{x}_r(t) + B_r u_r(t), & (\mathbf{x}_r(t), u_r(t)) \notin \mathcal{R}, \\ \mathbf{x}_r(t^+) = A_\rho\mathbf{x}_r(t), & (\mathbf{x}_r(t), u_r(t)) \in \mathcal{R}, \\ y_r(t) = C_r\mathbf{x}_r(t) + D_r u_r(t), \end{cases} \tag{2.27}$$

where $\mathbf{x}_r \in \mathbb{R}^{n_r}$, $A_r \in \mathbb{R}^{n_r \times n_r}$, $B_r \in \mathbb{R}^{n_r \times 1}$, $C_r \in \mathbb{R}^{1 \times n_r}$, $D_r \in \mathbb{R}$. The instantaneous changes of the reset state are governed by a linear map $A_\rho \in \mathbb{R}^{n_r \times n_r}$. The input of the compensator $u_r$ is considered into the resetting set[9] for the sake of generalization.

[9] The resetting set is denoted by $\mathcal{R}$ instead of $\mathcal{S}$ in order to facilitate the association with the reset compensator.

Now suppose that the last $n_\rho$ compensator states are set to zero at the reset instants, then the matrix $A_\rho$ can be defined as follows:

$$A_\rho = \begin{bmatrix} I & 0 \\ 0 & 0_{n_\rho} \end{bmatrix}. \tag{2.28}$$

In addition, consider that the matrix $A_r$ is partitioned in blocks as follows:

$$A_r = \begin{bmatrix} A_{r_{11}} & A_{r_{12}} \\ A_{r_{21}} & A_{r_{22}} \end{bmatrix} \tag{2.29}$$

with $A_{r_{22}} \in \mathbb{R}^{n_\rho \times n_\rho}$. Following the definitions in [29], a reset compensator with the above matrices is called a *full reset compensator* if all the elements of $A_\rho$ are zero, otherwise it is called *partial reset compensator*. On the other hand, if $A_{r_{21}} = 0$ ($A_{r_{12}} = 0$), then the compensator is referred to as *right reset compensator* (*left reset compensator*). On the other hand, in the particular case of $\mathcal{R} = \{(\mathbf{x}_r, u_r) \in \mathbb{R}^{n_r+1} : u_r = 0\}$ the resetting law is referred to as *zero crossing resetting law*.

The considered feedback structure has two inputs: the reference $r \in \mathcal{C}(\mathbb{R}_+, \mathbb{R})$ and the disturbance $d \in \mathcal{C}(\mathbb{R}_+, \mathbb{R})$. The plant input is $u_p$ and the compensator input is the error signal, formed by the difference between the plant output $y_p$ and the reference $w$. As a result, the closed-loop is obtained by making the connections

$$u_r(t) = r(t) - y_p(t), \qquad u_p(t) = d(t) + y_r(t). \tag{2.30}$$

Defining the closed-loop state $\mathbf{x} = (\mathbf{x}_p, \mathbf{x}_r) \in \mathbb{R}^n$ with $n = n_p + n_r$ and the the input $\mathbf{u} = (w, d)$, the result is the closed-loop system

$$\begin{cases} \dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t), & (\mathbf{x}(t), \mathbf{u}(t)) \notin \mathcal{S}, \\ \mathbf{x}(t^+) = A_R\mathbf{x}(t), & (\mathbf{x}(t), \mathbf{u}(t)) \in \mathcal{S}, \\ y(t) = C\mathbf{x}(t), \end{cases} \tag{2.31}$$

where matrices $A$, $B$ and $C$ are given by

$$A = \begin{bmatrix} A_p - B_pD_rC_p & B_pC_r \\ -B_rC_p & A_r \end{bmatrix}, \quad B = \begin{bmatrix} B_pD_r & B_p \\ B_r & 0 \end{bmatrix}, \quad C = \begin{bmatrix} C_p & 0 \end{bmatrix}. \tag{2.32}$$

According to the resetting set of the reset compensator R, the resetting set of the impulsive system (2.31) is defined by

$$\mathcal{S} = \left\{ (\mathbf{x}, \mathbf{u}) \in \mathbb{R}^{n+2} : C_M \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} \in \mathcal{R} \right\} \tag{2.33}$$

with $C_M$ given by

$$C_M = \left[ \begin{array}{cc|cc} 0 & I & 0 & 0 \\ -C_p & 0 & 1 & 0 \end{array} \right]. \tag{2.34}$$

The instantaneous changes of the state are governed by the linear map $A_R \in \mathbb{R}^{n \times n}$, given by

$$A_R = \begin{bmatrix} I & 0 \\ 0 & A_\rho \end{bmatrix}. \tag{2.35}$$

A natural and widely spread assumption in control theory is to consider that the exogenous signals of the system are Bohl functions (functions having stricly proper rational Laplace transforms, see, e.g., [203]). In this way, the reference and the disturbance of the closed-loop system (2.31) can be generated by exosystems $\Sigma_1$ and $\Sigma_2$, respectively, as it can be observed in Fig. 2.3. The exosystems are described in the state-space form as follows:

$$\begin{cases} \dot{\mathbf{w}}_1(t) = A_1 \mathbf{w}_1(t), & \mathbf{w}_1(0) = \mathbf{w}_{10}, \\ \dot{\mathbf{w}}_2(t) = A_2 \mathbf{w}_2(t), & \mathbf{w}_2(0) = \mathbf{w}_{20}, \\ r(t) = C_1 \mathbf{w}_1(t), \\ d(t) = C_2 \mathbf{w}_2(t), \end{cases} \tag{2.36}$$

where $\mathbf{w}_1 \in \mathbb{R}^{n_1}$, $\mathbf{w}_2 \in \mathbb{R}^{n_2}$, and $A_1 \in \mathbb{R}^{n_1 \times n_1}$, $A_2 \in \mathbb{R}^{n_2 \times n_2}$, $C_1 \in \mathbb{R}^{1 \times n_1}$ and $C_2 \in \mathbb{R}^{1 \times n_2}$. $\mathbf{w}_{10}$ and $\mathbf{w}_{20}$ are the initial condition of the reference and the disturbance, respectively. Defining the augmented state $\mathbf{x} = (\mathbf{x}_p, \mathbf{x}_r, \mathbf{w}_1, \mathbf{w}_2) \in \mathbb{R}^n$, with dimension $n = n_p + n_r + n_1 + n_2$, a state-space realization of the reset control system is

$$\begin{cases} \dot{\mathbf{x}}(t) = A\mathbf{x}(t), & \mathbf{x}(t) \notin \mathcal{M}, \\ \mathbf{x}(t^+) = A_R \mathbf{x}(t), & \mathbf{x}(t) \in \mathcal{M}, \\ y(t) = C\mathbf{x}(t), \end{cases} \tag{2.37}$$

where now the matrices $A$, $C$, and $A_R$ are given by

$$A = \begin{bmatrix} A_p - B_p D_r C_p & B_p C_r & B_p D_r C_1 & B_p C_2 \\ -B_r C_p & A_r & B_r C_1 & 0 \\ 0 & 0 & A_1 & 0 \\ 0 & 0 & 0 & A_2 \end{bmatrix}, \tag{2.38}$$

$$C = \begin{bmatrix} C_p & 0 & 0 & 0 \end{bmatrix}, \tag{2.39}$$

$$A_R = \begin{bmatrix} I & 0 & 0 \\ 0 & A_\rho & 0 \\ 0 & 0 & I \end{bmatrix}. \tag{2.40}$$

The resulting system is autonomous since the exogenous signals have been included in the state-space representation of the system, and thus, the system is classified as a state-dependent IDS. In this case, the resetting set is simply defined by

$$\mathcal{M} = \{\mathbf{x} \in \mathbb{R}^n : C_M \mathbf{x} \in \mathcal{R}\}. \tag{2.41}$$

### 2.4.1 Well-posedness of Reset Control Systems

As mentioned before, the time regularization strategy guarantees that the IDS (IDDS) has well-posed reset instants with no Zeno solutions. However, it implies a modification of the resetting law definition. In addition, it is known that time regularization poses intrinsic difficulties in analysis and implementation of reset control systems ([211, 287]), and thus, it is desirable to avoid that restriction. In this section, we expose some of the results concerning the well-posedness of the reset control systems without the time regularization strategy. The following results are a sampled of those provided in [28, 29]. The proofs of the results will be omitted, and the reader is referred to those works for a deep analysis of the results.

Consider the reset control system (2.37) where the reset matrix $A_\rho$ of the compensator is given by (2.28). In addition, let the reset control system be provided with a zero crossing resetting law. For an adequate definition of the reset control system, the definition of the resetting set for reset control systems proposed in [47] is used, that is

$$\mathcal{M} = \left\{\mathbf{x} \in \mathbb{R}^n : \begin{bmatrix} -C_p & 0 & C_1 & 0 \end{bmatrix} \mathbf{x} = 0 \wedge (A_R - I)\mathbf{x} \neq \mathbf{0}\right\}. \tag{2.42}$$

Note that $\mathcal{R}(A_R) = \mathcal{N}(A_R - I)$, and thus, condition $(A_R - I)\mathbf{x} \neq \mathbf{0}$ guarantees that immediately after a reset action the system state does not belong to the resetting set.

The first result provides a necessary and sufficient condition for the existence and uniqueness of solutions to the reset control system presented in this section.

**Proposition 2.1.** *The reset control system with $A_\rho$ and $\mathcal{M}$ given by (2.28) and (2.42), respectively, has a unique solution over the interval $[0, \infty)$, for any initial condition $\mathbf{x}_0 \in \mathbb{R}^n$, if and only if it has well-posed reset instants.*

From the above proposition it is concluded that the well-posedness of the reset instants implies that the reset control system do not have Zeno solutions (the solution is defined on $[0, \infty)$).

The following proposition guarantees that a large amount of the reset control systems have well-posed reset instants.

**Proposition 2.2.** *The reset control system with $A_\rho$ and $\mathcal{M}$ given by (2.28) and (2.42), respectively, has well-posed reset instants if the reset compensator is full reset or partial reset with right reset.*

It should be mentioned the importance of the above proposition. Note that most of the reset compensator proposed in the literature are full reset, for instance Clegg integrator and FORE compensator, or partial reset with right reset, for instance the Horowitz reset compensator ([168]) and the PI+CI compensator ([20, 271]).

As it is proved in [29], the previous results guarantee the existence and uniqueness of the solution of the reset control system over the interval $[0, \infty)$. Another important issue concert with the well-posedness of the system is the continuous dependence on the initial conditions. The following result is based on the Hausdorff[10] distance between the set of points defining the reset control system trajectories. Given two nonempty subsets $\mathcal{A}$, $\mathcal{B} \subset \mathbb{R}^n$, the Hausdorff distance between them is

$$d_H(\mathcal{A}, \mathcal{B}) = \max\{h(\mathcal{A}, \mathcal{B}, h(\mathcal{B}, \mathcal{A})\}, \tag{2.43}$$

where $h(\mathcal{A}, \mathcal{B}) = \sup\{\inf\{\|a - b\|, a \in \mathcal{A}\}, b \in \mathcal{B}\}$. With some abuse of notation, let $\mathbf{x}([0, T], \mathbf{x}_0)$ be the set of points corresponding to the trajectory of the system for

[10] Several convenient metrics has been successfully developed in the literature to represent distances between impulsive/hybrid systems solutions; for example the Skorokhod distance [57], and the graphical distance in the HI framework [60, 112].

$t \in [0, T]$, that is

$$\mathbf{x}([0, T], \mathbf{x}_0) = \{\mathbf{x}(t, \mathbf{x}_0) \in \mathbb{R}^n : t \in [0, T]\}. \tag{2.44}$$

Then, the definition of continuous dependence based on the Hausdorff distance is given as follows:

**Definition 2.6 (Continuous dependence).** For a reset control system with $A_\rho$ and $\mathcal{M}$ given by (2.28) and (2.42), respectively, and with well-posed reset instants, the solution depends continuously on the initial condition at $\mathbf{x}_0 \in \mathbb{R}^n$ if for any $\epsilon > 0$, $T \in \mathbb{R}_+ \setminus \mathbb{T}_{\mathbf{x}_0}$, there exist $\delta > 0$ such that for any $\mathbf{x}_0^* \in \mathbb{R}^n$, with $\|\mathbf{x}_0 - \mathbf{x}_0^*\| < \delta$, it is true that

$$d_H(\mathbf{x}([0, T], \mathbf{x}_0^*), \mathbf{x}([0, T], \mathbf{x}_0)) < \epsilon. \tag{2.45}$$

From the definition, it is clear that continuous dependence fails on those initials conditions such that $\mathbf{x}_0 \in \mathcal{M}$. In order to check the continuous dependence to other initial conditions, the following proposition is provided.

**Proposition 2.3.** *The reset control system with $A_\rho$ and $\mathcal{M}$ given by (2.28) and (2.42), respectively, and well-posed reset instants, the solution depends continuously on the initial condition at $\mathbf{x}_0 \in \mathbb{R}^n \setminus \mathcal{M}$ if $CA\mathbf{x}(\hat{t}_k, \mathbf{x}_0) \neq 0$, $k = 1, 2, \ldots$, where $\hat{t}_k$ are those instants such that $\mathbf{x}(\hat{t}_k, \mathbf{x}_0) \in \bar{\mathcal{M}}$.*

Roughly speaking, the proposition guarantees the continuous dependence on the initial condition at $\mathbf{x}_0$ by preventing the trajectory of the system to reach the reset surface $\bar{\mathcal{M}}$ tangentially. One of the most importance consequences of the above proposition is that it provides a mathematical tool to analyze the sensitivity of the reset control system to sensor noise (see [29]).

### 2.4.2 Time-delay Reset Control Systems

Consider the feedback interconnection of Fig. 2.4, composed by a LTI system P and a reset compensator R given by (2.26) and (2.27), respectively. The reference, $r$, and the disturbance, $d$, are generated by the exosystems $\Sigma_1$ and $\Sigma_2$ defined by (2.36). It is considered that the interconnection of R and P is affected by a time-delay $h$, which is given by an input time-delay $h = h_i$ ($h_o = 0$) or an output time-delay $h = h_o$ ($h_i = 0$). Note that from the point of view of the input/output behavior of the system, the time-delay $h$ can be moved to the plant input or output. However, the evolution of the system for the same initial condition may be different from the input time-delay to

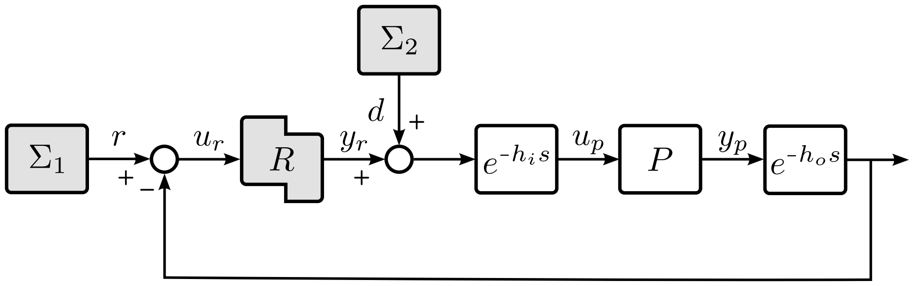



Figure 2.4 Standard time-delay reset control system formed by a LTI plant P, a reset compensator R, exosystems $\Sigma_1$ and $\Sigma_2$, and input/output time-delay.

output time-delay. In addition, as it will be seen in Chapter 3, the stability results may change depending on where the time-delay is placed, mainly since the developed conditions are sufficient but not necessary.

Consider one of the following cases: $h_i > 0$, $h_o = 0$ or $h_i = 0$, $h_o > 0$. Note that the system is assumed to be affected by only one constant time-delay. Now let make the following connections

$$u_r(t) = r(t) - y_p(t - h_o), \qquad u_p(t) = d(t - h_i) - y_r(t - h_i), \tag{2.46}$$

then the time-delay reset control system is defined by

$$\begin{cases} \left.\begin{array}{l} \dot{\mathbf{x}}(t) = A\mathbf{x}(t) + A_d\mathbf{x}(t-h), \\ \dot{\Delta}(t) = 1, \end{array}\right\} & (\mathbf{x}(t), \mathbf{x}(t-h), \Delta(t)) \notin \mathcal{S}, \\ \left.\begin{array}{l} \mathbf{x}(t^+) = A_R\mathbf{x}(t), \\ \Delta(t^+) = 0, \end{array}\right\} & (\mathbf{x}(t), \mathbf{x}(t-h), \Delta(t)) \in \mathcal{S}, \\ y(t) = C\mathbf{x}(t) + C_h\mathbf{x}(t-h). & \end{cases} \tag{2.47}$$

Note that the time has been removed from the functions, since the resulting system is time-invariant. The states $\mathbf{x}_p$, $\mathbf{x}_r$, $\mathbf{w}_1$, and $\mathbf{w}_2$ have been collected in $\mathbf{x} = (\mathbf{x}_p, \mathbf{x}_r, \mathbf{w}_1, \mathbf{w}_2) \in \mathbb{R}^n$ with $n = n_p + n_r + n_1 + n_2$. The matrix $A_R$ is given by (2.40), and the matrices $A$, $A_d$, $C$, and $C_h$ depend on the position of the time-delay, see Table 2.1. Regarding the reset actions, they are applied when the input and the state of the compensator belong to $\mathcal{R}$, so that the resetting set is defined as follows:

$$\mathcal{S} = \left\{ (\mathbf{x}, \mathbf{y}, \Delta) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ : C_M \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \in \mathcal{R} \wedge \Delta \geq \Delta_m \right\}, \tag{2.48}$$

| Matrix | Input time-delay | Output time-delay |
|---|---|---|
| $A$ | $\begin{bmatrix} A_p & 0 & 0 & 0 \\ -B_rC_p & A_r & B_rC_1 & 0 \\ 0 & 0 & A_1 & 0 \\ 0 & 0 & 0 & A_2 \end{bmatrix}$ | $\begin{bmatrix} A_p & B_pC_r & B_pD_rC_1 & B_pC_2 \\ 0 & A_r & B_rC_1 & 0 \\ 0 & 0 & A_1 & 0 \\ 0 & 0 & 0 & A_2 \end{bmatrix}$ |
| $A_d$ | $\begin{bmatrix} -B_pD_rC_p & B_pC_r & B_pD_rC_1 & B_pC_2 \\ 0 & 0 & 0 & -B_rB_pC_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ | $\begin{bmatrix} -B_pD_rC_p & 0 & 0 & 0 \\ -B_rC_p & 0 & 0 & -B_rB_pC_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ |
| $C$ | $\begin{bmatrix} C_p & 0 & 0 & 0 \end{bmatrix}$ | $\begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$ |
| $C_h$ | $\begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$ | $\begin{bmatrix} C_p & 0 & 0 & 0 \end{bmatrix}$ |

Table 2.1 Matrices $A$, $A_d$, $C$, and $C_h$ of a time-delay reset control system for input and output time-delay.

where now the matrix $C_M$ is given by

$$C_M = \left[\begin{array}{cccc|c} 0 & I & 0 & 0 & 0 \\ 0 & 0 & C_1 & 0 & 0 \end{array}\right] - \left[\begin{array}{c|c} 0 & 0 \\ C & C_h \end{array}\right]. \tag{2.49}$$

Note that the resetting set for input time-delay is slightly different to the case of output time-delay, since in that case the input of the compensator is delayed.

### 2.4.3 PI+CI Compensator

It is well known that PID compensators are the most applied control strategy in industrial control practice. As it was shown in some studies, more than 95% of the control loops in industry utilize the PID as feedback compensator. In addition, the derivative action is not frequently applied mainly due to its implications on the cost of feedback (noise amplification), so that most of the compensators are PI. This predominance of PID compensation can be attributed to its simplicity (only three tuning parameters), and its effectiveness in a wide range of applications: motor drive, automotive industry, flight control, process control, etc. The reader is referred to the

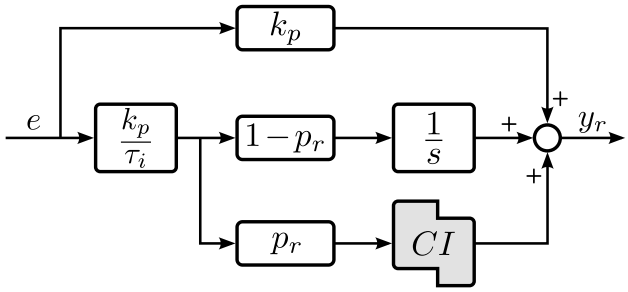



Figure 2.5 Structure of the PI+CI compensator.

monographs [8, 274] for a detailed exposition of the recent developments in the field of PID control.

Based on the simplicity of the PI and the benefits of the reset compensation A. Baños and A. Vidal devised in [35] a new reset compensator referred to as PI+CI. This compensator is defined by means of a base PI compensator, that will be referred to as $\mathrm{PI}_{base}$, whose integral term is partly reset. The PI+CI can be simply formed by a parallel connection of PI compensator and a Clegg integrator, that is reset to zero when its input is zero. As a result, it has three terms: a proportional term P, an integral term I, and a Clegg integrator CI. Fig. 2.5 shows the block diagram structure of a PI+CI compensator, where the input is the error signal, the output is the control signal, $k_p$ is the proportional gain, and $\tau_i$ is the integral time constant. Beside the tuning parameters of its linear counterpart, the PI+CI possesses a new dimensionless parameter, $p_r \in [0, 1]$, referred to as the *reset ratio*. The new parameter represents the part of the whole integral term (I+CI) over which the reset action is applied. Note that for $p_r = 0$ a PI compensator is obtained, and that for $p_r = 1$ the result is a full reset P+CI compensator. In the state-space, the PI+CI compensator can be expressed by using a state $\mathbf{x}_r = (x_i, x_{ci}) \in \mathbb{R}^2$, where $x_i$ is the integral term state and $x_{ci}$ corresponds to the Clegg integrator state. A state-space realization is given by the following impulsive differential equation[11]

$$PI + CI : \begin{cases} \dot{\mathbf{x}}_r(t) = A_r \mathbf{x}_r(t) + B_r e(t), & e(t) \neq 0, \\ \mathbf{x}_r(t^+) = A_\rho \mathbf{x}_r(t), & e(t) = 0, \\ y_r(t) = C_r \mathbf{x}_r(t) + D_r e(t), & \end{cases} \tag{2.50}$$

[11]The input of the compensator has been denoted by $e(t)$ instead of $u_r(t)$ for a better association with the error signal.

where the matrices $A_r$, $B_r$, $C_r$, $D_r$ and $A_\rho$ are given by

$$A_r = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad B_r = \begin{bmatrix} 1 - p_r \\ p_r \end{bmatrix}, \qquad C_r = \frac{k_p}{\tau_i}\,[1 \;\; 1]\,,$$
$$D_r = k_p, \qquad A_\rho = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{2.51}$$

Throughout the thesis, it will be also used the parameter $k_i = \frac{k_p}{\tau_i}$. The above state-space realization is a non-minimal realization of the PI+CI. This fact implies several consequences regarding the asymptotically stability of the closed-loop system:

- Consider the reset control system (2.37) with the above PI+CI as a compensator, then it is clear that the matrix A is not Hurwitz, and thus, the base system is not asymptotically stable. As a result, the $H_\beta$-*condition* developed in [47] cannot be applied. On the other hand, the result in [23] does not require the matrix A to be Hurwitz, and thus, the result can be applied. However, it should be mentioned that [23] provides an stabilization criterion by imposing time-dependent condition on the reset intervals. So the resulting reset control system is a time/state-dependent IDS. Finally, if the reset control system is modeled by the HI framework, then an stability criterion in the form of LMIs can be obtained by applying the results in [211] to a piecewise Lyapunov function. The procedure would follow similarly to the one used in [188] for developing stability criterion of reset control system composed by a FORE.

- Now consider the time-delay reset control system (2.47) with the PI+CI (2.50). It can be deduced that the base system is again not asymptotically stable. In order to make clear this fact, let consider the initial condition given by $\phi(\theta) = (\mathbf{0}, x_i(\theta), x_{ci}(\theta))$, $\theta \in [-h, 0]$, with $x_{ci}(\theta) = \frac{p_r-1}{p_r} x_i(\theta)$. For this initial condition, the base system does not evolve, and thus, the state does not approaches the origin. Unlike the non-delay case, to the knowledge of the author, there is not stability criteria that can guarantee the asymptotically stability of such time-delay reset control system with unstable base system, even though time-dependent conditions are imposed on the reset intervals. There are several stability results regarding the stability of IDDSs with unstable base systems. However, they impose hard restriction on the reset actions that hamper the application of the results. In Chapter 3, it will be developed an adequate stability criterion for testing the asymptotical stability of time-delay reset control systems with a PI+CI compensator by means of time-dependent condition on the reset interval.

Most likely for the stability issue, an alternative minimal realization of the PI+CI is developed in [268], which only uses one state:

$$PI + CI : \begin{cases} \dot{\mathbf{x}}_r(t) = e(t), & e(t) \neq 0, \\ \mathbf{x}_r(t^+) = (1 - r_k)\mathbf{x}_r(t), & e(t) = 0, \\ y_r(t) = \frac{k_p}{\tau_i}\mathbf{x}_r(t) + k_p e(t), \end{cases} \tag{2.52}$$

where $r_k$ changes every reset instant and it is given by

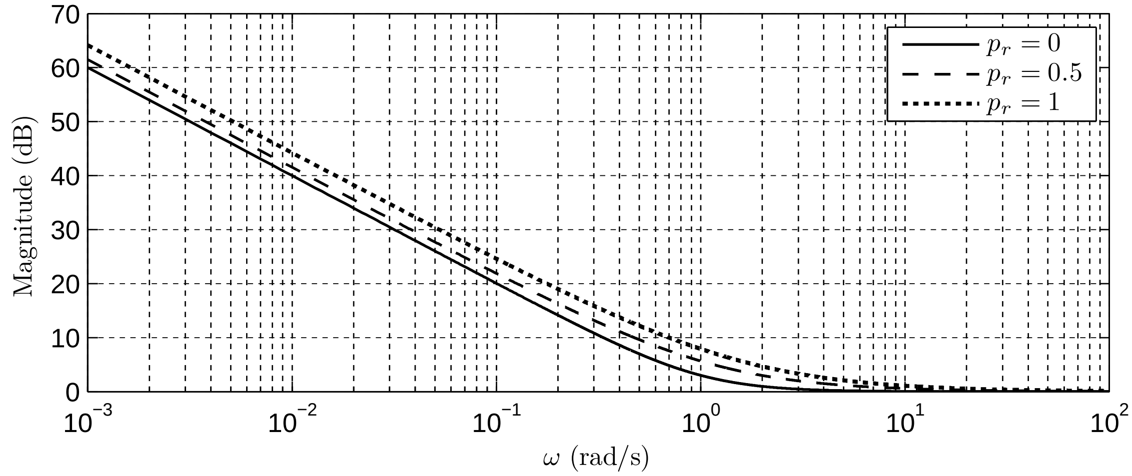
$$r_k = \frac{p_r x_{ci}(t_k)}{(1 - p_r)x_i(t_k) + p_r x_{ci}(t_k)}. \tag{2.53}$$

This realization of the PI+CI poses several problems; first, the definition is confused since $r_k$ is given as a function of the states $x_i$ and $x_{ci}$ from the state-space realization (2.50), and in addition, $A_\rho$ will be time-varying, and thus, no stability results are available to guarantee the (asymptotical) stability of the reset control system.

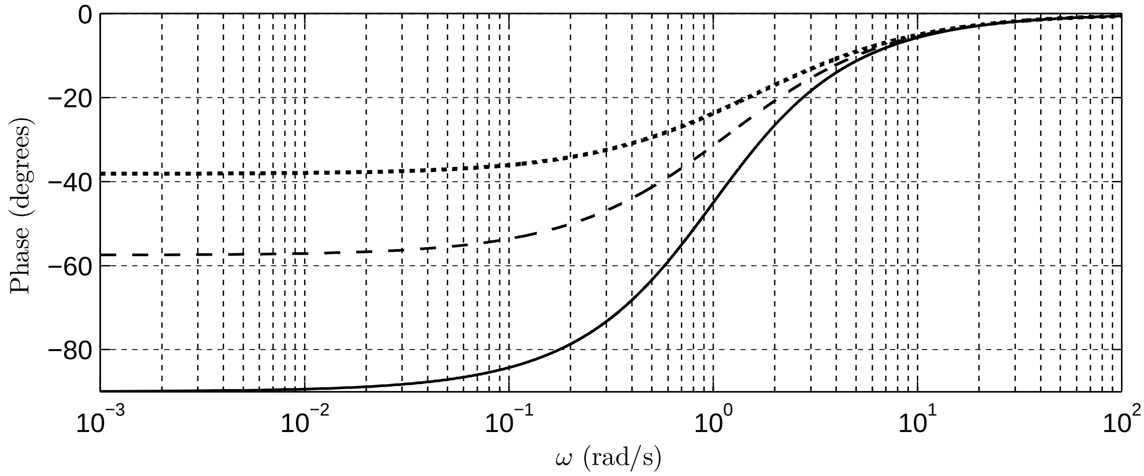
On the other hand, since the beginning of the reset compensation, its benefits has been intuited by the frequency domain linearization (describing function), showing how it provides a gain/phase relation much more favorable than its linear counterpart in terms of phase margins and robustness. Therefore, it is of interest to consider the describing function of the PI+CI, which is given by

$$PI + CI(\omega) = k_p \left( \frac{j\left(\omega\tau_i + \frac{4}{\pi}p_r\right) + 1}{j\omega\tau_i} \right). \tag{2.54}$$

Similarly to the CI, the describing function of the PI+CI does not depend on the amplitude, which facilitates the analysis. Fig. 2.6 shows the magnitude and phase of the describing function (2.54) for several values of the reset ratio. It can be seen that the reset action may introduce a phase lead of up to 50 degrees and a relatively small increment of the magnitude for frequencies lower than $1/\tau_i$. In addition, the describing function of the Clegg integrator has favorable properties that permit a faster reduction of the loop magnitude, that is, the cost of feedback. It is thus of great interest to analyze whether the Clegg integrator properties (in the sense of the describing function) are still presented in the PI+CI compensator. An analysis of the reset control system sensibility with respect to noise may be performed using the describing function of the PI+CI compensator, $PI + CI(\omega)$. Assuming an additive noise $n$ added to the




(a) Magnitude in dB.




(b) Phase in degrees.

Figure 2.6 PI+CI describing function for $k_p = 1$, $\tau_i = 1$ and several values of the reset ratio: $p_r = 0$ (solid), $p_r = 0.5$ (dashed), and $p_r = 1$ (dotted).

closed-loop output $y$ of the reset control system, the frequency response from the noise $n$ to the control input $y_r$ is simply

$$\frac{Y_r(\omega)}{N(\omega)} = -\frac{PI + CI(\omega)}{1 + P(j\omega)(PI + CI(\omega))}, \tag{2.55}$$

where $P(jw)$ is the frequency response of the plant. Considering the describing function (2.54), it follows

$$\frac{Y_r(\omega)}{N(\omega)} = -\frac{PI(j\omega) + \dfrac{4p_r}{\pi\tau_i\omega}}{1 + P(j\omega)PI(j\omega) + \dfrac{4p_r}{\pi\tau_i\omega}}, \tag{2.56}$$

where $PI(j\omega)$ is the frequency response of the $\text{PI}_{base}$. Therefore, it is clear that for hight frequencies the frequency response of the PI+CI is similar to the response of the PI, and thus, the sensitivity with respect to the noise of both compensator will be similar.

### Design Improvements

In practice, a number of simple modifications of the original PI+CI compensator definition have been found useful in improving performance and robustness of the closed-loop system. In this section, several modifications will be analyzed, including both changes over the resetting law, that regulates the instants at which reset actions are performed, and in the compensator design.

*Variable band resetting law:* The improvement of the reset compensation has been shown to reduce in processes with dominant time-delay. The problem in these processes is that the reset action is done at the reset instants but the process suffers the reset action $h$ units time later, being $h$ the time-delay of the process. In [268], a fixed reset band is proposed as a first attempt to overcome this lack of coordination. In this case, the reset action is applied when the error signal enters into a fixed band, that is $t_k$ is a reset instant if $e(t_k) = -\delta \wedge \dot{e}(t_k) > 0$ or $e(t_k) = \delta \wedge \dot{e}(t_k) < 0$, with $\delta$ some non-negative real number. The main drawback of this solution is that the parameter $\delta$ should be properly chosen depending on the level of the reference and/or disturbance. A solution to this problem is to consider a variable band (variable band resetting law). This means that at every reset instant the band value may be different, and in general this value will be determined by a combination of the error value and its derivative. In this way, in [271] the authors proposed to use a simple first order approximation of the error, that is, $t_k$ is a reset instant if $h\dot{e}(t_k) + e(t_k) = 0$, with $h$ the time-delay of the process. As it will be shown in Chapter 5, a variable band resetting law can be also used to improve the performance in processes without time-delay, and thus, it is considered the variable band $\theta\dot{e}(t_k) + e(t_k) = 0$ where $\theta$ is a tuning parameter. The advantages of the fixed/variable band has been also studied in the frequency domain by means of the describing function (see [20] and the references therein). Consider a Clegg integrator with a variable band resetting law, its describing function results in

$$CI_\theta(\omega) = \frac{1}{j\omega}\left(1 + \frac{2j}{\pi}\left(1 + e^{2j\tan^{-1}(\omega\theta)}\right)\right). \tag{2.57}$$

Fig. 2.7 (dotted line) shows the phase of $CI_\theta$ where the frequency has been normalized with $\theta$. It is clear from the figure that $CI_\theta$ gives an extra phase lead over a CI up to $27°$ in the frequency interval $[\frac{0.1}{\theta}, \frac{1}{\theta}]$( up to $75°$ over an integrator), which is especially useful for processes with time-delay. On the other hand, the idea of variable band was further developed in [234, 235], where it is proposed a multiple anticipation resetting law. In this way, the reset actions are applied at instant $t_k$ such that[12]

$$(\theta_1\dot{e}(t_k) + e(t_k) = 0) \vee (\theta_2\dot{e}(t_k) + e(t_k) = 0) \vee \ldots \vee (\theta_N\dot{e}(t_k) + e(t_k) = 0), \tag{2.58}$$

where $\theta_i > 0$, $i = 1, 2, \ldots, N$, are design to improve the closed-loop performance.

In practice, sensors introduce noise in the closed-loop which affects the behavior of the compensator. In particular, the PI+CI compensator has three terms, the proportional, the integral and the Clegg integrator. The two first terms are the same as the PI compensator and it is well known that they do not pose problem at high frequency. On the other hand, PI+CI has been shown to have a similar cost of feedback to the PI (see Fig. 2.6). However, the derivative term used in the variable band resetting law is much sensitive to noise. This term cannot amplify directly the noise in the closed-loop, but it may produce undesirable reset actions. Therefore, it is recommended to provide the variable band with a low pass filter, such that the sensitivity to noise of the derivative is reduced. For simplicity, we consider the following first order low pass filter

$$F(s) = \frac{1}{\tau_f s + 1}, \tag{2.59}$$

[12]In [235], the multiple anticipation resetting law is analyzed by the describing function, and thus, the authors considered an alternative implementation of the resetting law, based on anticipation reset angles.

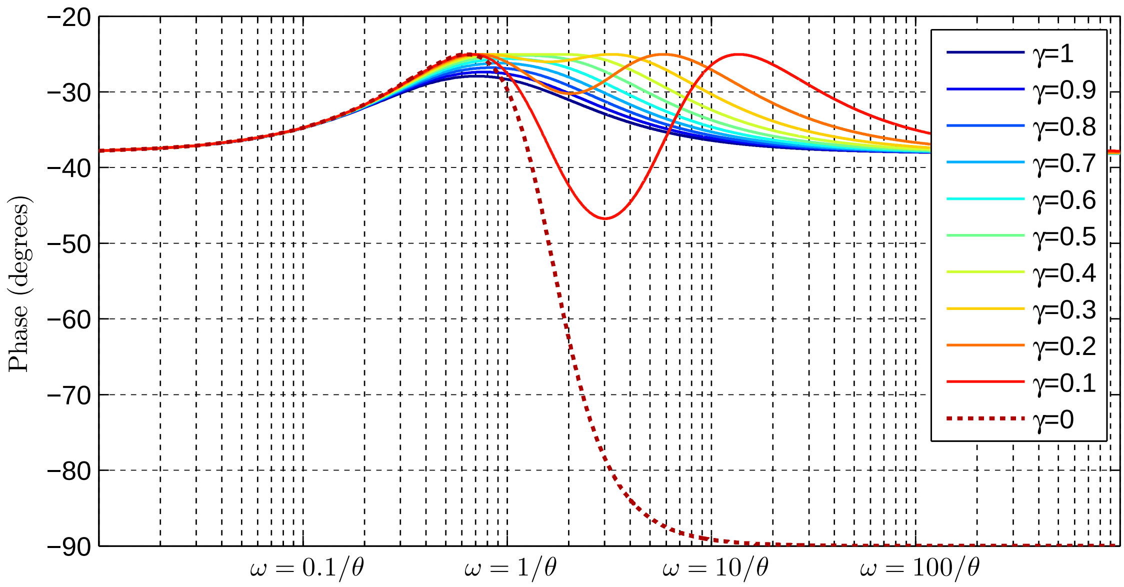



Figure 2.7 PI+CI describing function of the Clegg integrator with variable band resetting law and a first order low pass filter. Frequency is normalized by the parameter $\theta$ of the variable band.

where $\tau_f$ is a design parameter, which should be set depending on the frequency of the noise. Thus, a reset action occur at instant $t_k$, such that $\theta \dot{e}_f(t_k) + e(t_k) = 0$ where $\dot{e}_f(t)$ is the filtered derivative, that is obtained by the inverse Laplace transform of the function $sF(s)E(s) = \frac{s}{\tau_f s+1}E(s)$ where $E(s)$ is the Laplace transform of the error signal.

In addition, it is necessary to limit the parameter $\tau_f$ in order to avoid the influence of the filter over the behavior of the variable reset band. The main advantage of the variable reset band is that it gives an extra phase lead over a CI to 27° in the frequency interval $[\frac{0.1}{\theta}, \frac{1}{\theta}]$ and up to 75° over an integrator. The describing function of the Clegg integrator with variable band and first order low pass filter may be computed (see Appendix F), resulting in

$$CI_{\theta,\gamma}(\omega) = \frac{1}{j\omega}\left(1 + \frac{2j}{\pi}\left(1 + e^{2j\tan^{-1}\left(\frac{\omega\theta}{1+\gamma(1+\gamma)(\theta\omega)^2}\right)}\right)\right), \tag{2.60}$$

where $\theta$ is the design parameter of the variable band resetting law. The parameter $\tau_f$ has been set as percentage of $\theta$, given by $\tau_f = \gamma\theta$ with $\gamma \in [0, 1]$. Fig. 2.7 shows the phase of $CI_{\theta,\gamma}$ for different values of $\gamma$. It is clear from the figure that if $\gamma$ increases the phase will be more similar to the CI, losing extra phase added by the variable band. It can be easily computed that the maximum extra phase is approximately given for $\gamma \leq 0.4$. Hence, if $\tau_f \leq 0.4\theta$, the noise will be filtered and the influence of the filter over the reset actions will be insignificant.

*Variable reset ratio:* In [268], it is proposed to use a variable reset ratio instead of a fixed value. The main reason is that the reset action produces an undesirable undershoot in processes with non-pure integrator ([168]). Therefore, it arises a trade-off between the overshoot and the undershoot, that is, large values of the reset ratio are adequate to reduce the overshoot, but it produces a large undershoot. The solution is to set the reset ratio as a function of the error derivative through the tuning of a new parameter, the derivative time constant, $\tau_d$. In this way, the variable reset ratio $p_r$ is given by the following expression

$$p_r(t) = \bar{p}_r - \tau_d \dot{e}(t), \tag{2.61}$$

where $\bar{p}_r$ is a constant base value of reset ratio. In practice, a low-pass filter $F(s) = 1/(\tau_f s + 1)$ is used with the pure derivative term to avoid the amplification of the sensor noise. Hence, three parameters must be fixed to tune the PI+CI with variable reset ratio. There is not tuning rule for the three parameters, but a design guideline is suggested in [268]. On the other hand, a first attempt to provide a systematic method for tuning the PI+CI with variable reset ratio for first and second order plants is accomplished in [24]. Further developments of these ideas are made in [25], providing a tuning rule for the PI+CI with variable reset ratio. The results in that work are studied in Chapter 5.

*Switching reset ratio:* In control practice, two common problems are reference tracking (servo mode) and disturbance rejection (regulatory mode), and it is well known that there exits a trade-off between both problems in some processes (see, e.g., [106]). Similarly, this trade-off appears when the PI+CI compensator is applied. In this case, beside the trade-off in the tuning of parameter of its PI counterpart ($k_p$ and $\tau_i$), a proper election of the reset ratio for reference tracking, it does not imply a good performance in disturbance rejection. Simple solutions have been used in the literature (see [20]), for example: tuning the $\text{PI}_{base}$ to avoid reset actions or directly disable the reset action for step disturbance rejection. However, this method does not take advantages of the reset actions for disturbance rejection. In this thesis, it is proposed to use a switching reset ratio method. This strategy consists in setting the reset ratio to an adequate value for reference tracking when a change of the reference is detected, and switch to an adequate reset ratio for disturbance rejection. The switching signal $\sigma : \mathbb{R}_+ \to \{1, 2\}$ is used to switch between both values of the reset ratio. Several switching signals $\sigma$ can be used in practice. For simplicity, it will be assumed that reference changes only occur when the control system is in its steady state; thus, when at the instant $s_k$, $k = 1, 3, 5 \ldots$, a reference change is detected then $\sigma(s_k) = 1$. In the following, a simple strategy is proposed to switch between the two modes. At some instant $s \geq 0$, the two modes are: ($\sigma(s) = 1$) setpoint tracking mode, ($\sigma(s) = 2$) load disturbance rejection mode. Initially, $\sigma(0) = 2$. If a setpoint change is detected at the instant $s_1$ then $\sigma(s_1) = 1$, and the designer has to choose when the mode is changed to $\sigma(s_2) = 2$ at the instant $s_2$ (two strategies will be shown bellow); again, after a new setpoint change at $s_3$, with $\sigma(s_3) = 1$, the mode it changed to $\sigma(s_4) = 2$ (depending of the switching strategy), $\cdots$. The proposed two switching strategies that have been shown to be effective in practice are (see Fig. 2.8):

- *Time-triggered switching:* $s_{k+1} = s_k + \rho$, for some constant $\rho > 0$ and $k = 1, 3, 5, \ldots$. It is clear that necessarily $\rho > \theta$ due to the process time-delay, and in general a good choice is $\rho$ bigger that the settling time.

- *Triggered-by-counter switching:* The PI+CI compensator modeled as given by (2.50) can be refined by using a counter $q : \mathbb{R}_+ \to \{1, 2, \cdots\}$ that counts the number of reset actions that are performed after a setpoint change. Hence, if $\sigma(s_k) = 1$ at some instant $s_k$, and at some instant $s_{k+1} > s_k$ the counter satisfies $q(s_{k+1}) = N_s$ then $\sigma(s_{k+1}) = 2$, where $N_s \geq 1$ is a design parameter. In practice, the control system is usually affected by the sensor noise, that may produce several reset actions in short time, and thus, if $N_s$ is not sufficiently large then the reset ratio could be switched too early within the transient stage. This effect of the noise on the counter $q$ makes this strategy trickier. In practice, some trial and error is needed for a proper election of $N_s$, but the use of some improvements of the PI+CI, (for example time regularization and noise filtering in the resetting law) may help to minimize the effect of the noise on the counter.

It should be pointed out the difference between the variable reset ratio and the switching reset ratio. In the first case, the reset ratio changes every reset instant through some function. On the contrary, the switching reset ratio implies that the reset ratio changes once after every change of the reference. In addition, both strategies can be used at the same time. Suppose that $\mathcal{P}_1$ is the function that defines the adequate changes of the reset ratio for reference tracking and similarly $\mathcal{P}_2$ for disturbance

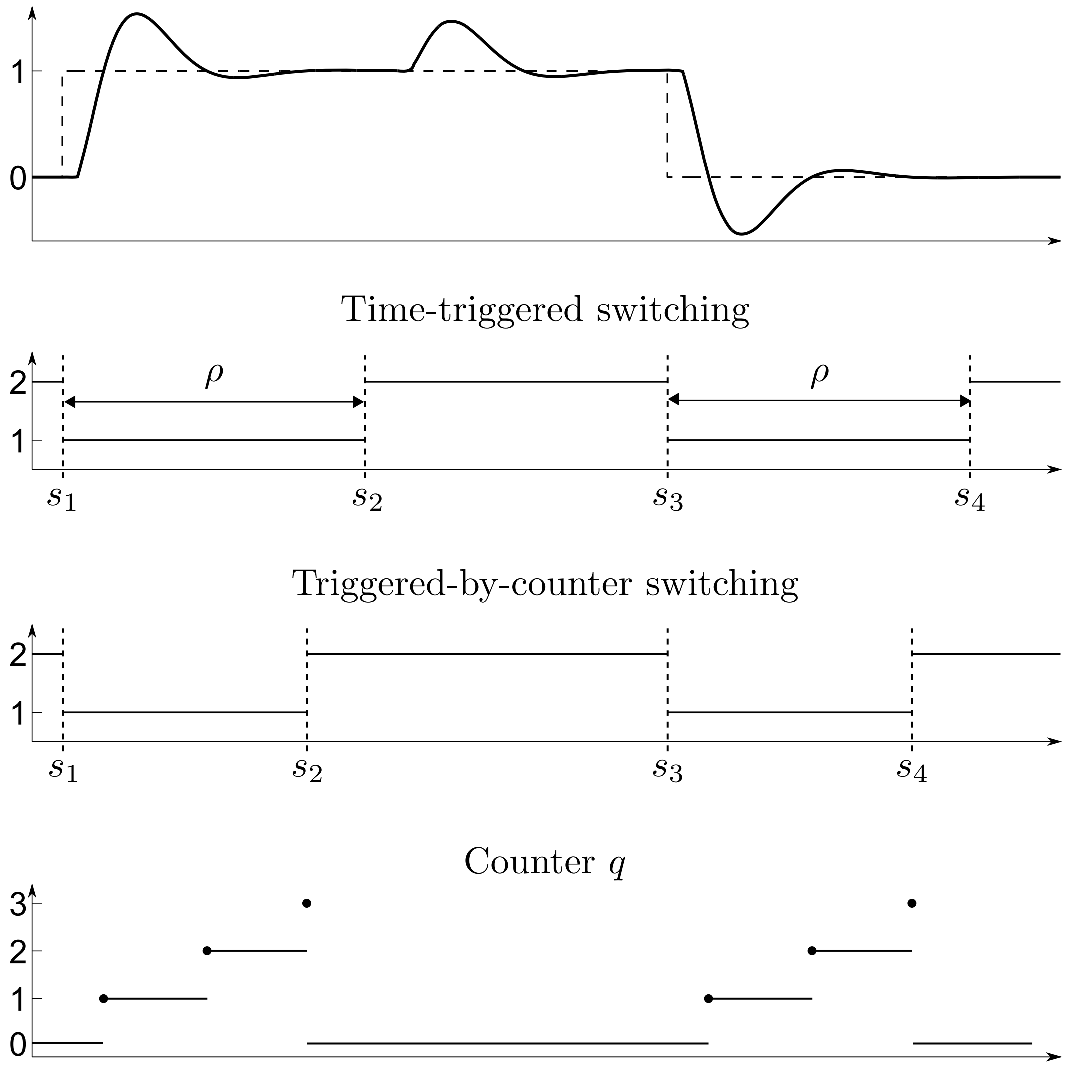



Figure 2.8 Switching signals: time-triggered with $\rho \approx t_s$ and triggered-by-counter with $N_s = 3$.

rejection. Then, the switching reset ratio is simply applied by $\mathcal{P}_\sigma(t)$, so that the evolution of the reset ratio is different for references and disturbances, and the change is accomplished by the signal $\sigma(t)$.

Finally, note that this idea can be generalized to multiple types of references (step, ramp, etc). Suppose that it can be detected the type of reference then a different function $\mathcal{P}$ can be designed for every type of reference. In this way, $\sigma : \mathbb{R}_+ \to 1, 2, \ldots, N$ where $\mathcal{P}_i$, $i = 1, \ldots, N-1$ are the functions for the references and $\mathcal{P}_N$ for disturbance rejection. The same approach can be applied for multiples types of disturbances, but the detectability of the type of disturbance could be a hard problem for real applications.

*Time-dependent stabilization:* There is not applicable methods to check the stability of a reset control system composed by a PI+CI with the above modifications and processes with time-delay. However, as it will be studied in Chapter 5, the stability of the PI+CI can be guaranteed by imposing time-dependent conditions to the reset instants. In practice, these conditions means that some reset actions have to be inhibited and in some instants it is necessary to force reset actions.

The PI+CI compensator with the above improvements may be represented in the state-space as follows:

$$\begin{cases} \left.\begin{array}{l} \dot{\mathbf{x}}_r(t) = A_r \mathbf{x}_r(t) + B_r e(t), \\ \dot{p}_r(t) = 0, \\ \dot{\Delta}(t) = 1, \end{array}\right\} & (\mathbf{x}_r(t), e(t), \dot{e}(t)) \notin \mathcal{R} \vee \Delta(t) < \Delta_m, \\ \left.\begin{array}{l} \mathbf{x}_r(t^+) = A_\rho \mathbf{x}_r(t), \\ p_r(t^+) = \mathcal{P}_{\sigma(t)}(\mathbf{x}_r(t), e(t)), \\ \Delta(t^+) = 0, \end{array}\right\} & ((\mathbf{x}_r(t), e(t), \dot{e}(t)) \in \mathcal{R} \wedge \Delta(t) \geq \Delta_m) \vee \Delta(t) = \Delta_M, \\ v(t) = C_r \mathbf{x}_r(t) + D_r e(t), & \end{cases} \tag{2.62}$$

where matrices $A_r$ and $D_r$ are in (2.51). However, there are two different options for matrices $B_r$ and $C_r$. First, the reset ratio is set at the input of the integrator and the Clegg integrator, that is

$$B_r = \begin{bmatrix} 1 - p_r(t) \\ p_r(t) \end{bmatrix}, \qquad C_r = \frac{k_p}{\tau_i} \begin{bmatrix} 1 & 1 \end{bmatrix}. \tag{2.63}$$

The other option is to set the reset ratio at the output, and thus the matrices are given by

$$B_r = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad C_r = \frac{k_p}{\tau_i} \left[\, 1 - p_r(t) \;\; p_r(t) \,\right]. \tag{2.64}$$

Note that there is not differences between both implementation if the reset ratio is constant, but on the contrary, if the reset ratio is variable the options are completely different. In both cases, a change on the reset ratio only affects the evolution of the integrator and the Clegg integrator. However, in the second case, it also produces a jump on the output of the integrator (not on the state).

On the other hand, the variable reset ratio has been implemented by functions $\mathcal{P}_\sigma()$, where $\sigma(t)$ is the switching signal for the switching reset ratio. Note that the parameters of $\mathcal{P}_\sigma()$ are the state and the error signal, but eventually it may depend on more parameters, for instance, it will be considered the reference and disturbance signals as parameters in Chapter 5. Finally, it has been added a new state $\Delta(t)$, the *reset interval state*, that is used to impose conditions on the interval between consecutive reset instants. So that, reset actions are deactivated during $\Delta_m$ units time, and if there is not reset events after $\Delta_M$ units time, then a reset action is forced.

**Example 2.3.** Consider the reset control system (2.37) composed by a first order plus dead time (FOPDT) process given by

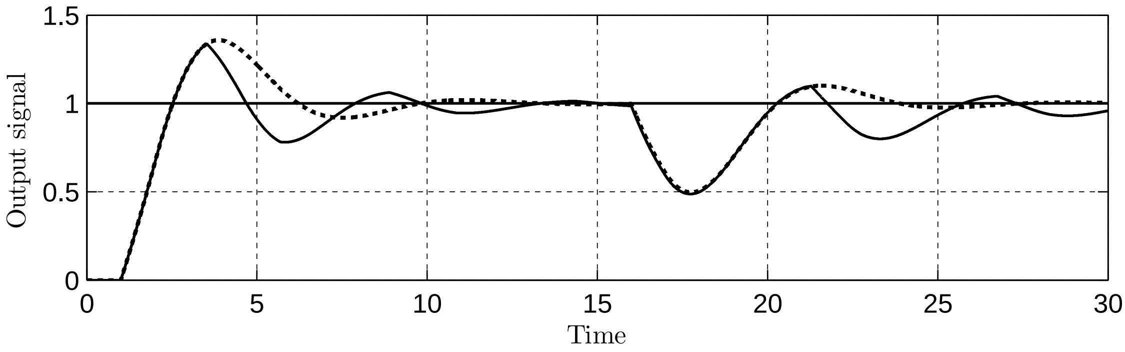
$$P(s) = \frac{1}{2s+1} e^{-s} \tag{2.65}$$

and the PI+CI compensator (2.62) with $k_p = 1.2$ and $\tau_i = 0.9$. The parameters have been chosen with the aim of illustrating the effect of the improvements, and thus, they are not necessarily the parameters of a "well-tuned" PI. Consider the configurations of the PI+CI shown in Table 2.2: PI+CI - 1 is the original PI+CI, PI+CI - 2 uses a variable band resetting law, PI+CI - 3 considers a variable reset ratio beside the variable band, and finally, PI+CI - 4 is also provided with a switching reset ratio. The strategy and tuning proposed in [268] is used for the variable reset ratio. Fig. 2.9 shows the step setpoint and step disturbance responses of the four proposed configurations for the PI+CI. Firstly, it can be clearly seen that the PI+CI without improvements increases the undershoot (both setpoint and disturbances) and barely reduces the overshoot. This is mainly due to the lack of coordination of the reset instants and the zero crossing instants, cause by the time-delay. The variable band resetting law

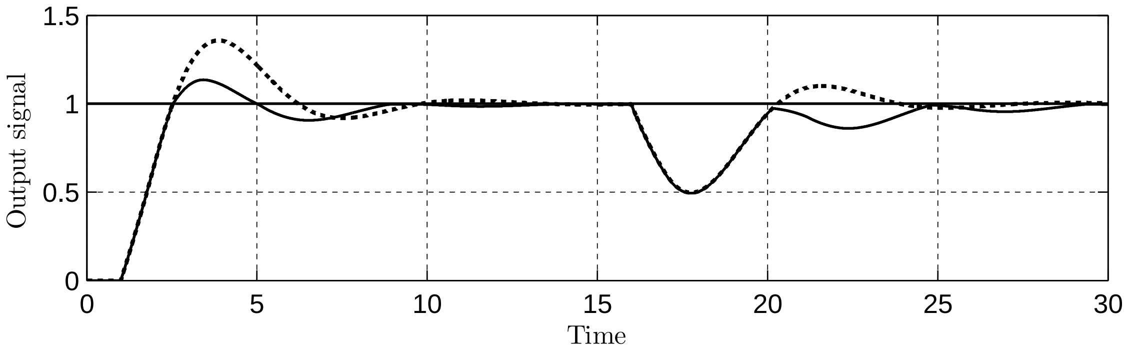
| | Reset ratio | Resetting law | Switching reset ratio |
|---|---|---|---|
| PI+CI - 1 | Constant<br>$\mathcal{P}_1 = \mathcal{P}_2 = 0.3$ | Zero crossing | - |
| PI+CI - 2 | Constant<br>$\mathcal{P}_1 = \mathcal{P}_2 = 0.3$ | Variable band<br>$\theta = 1$ | - |
| PI+CI - 3 | Variable<br>$\mathcal{P}_1 = \mathcal{P}_2 = 0.3 + 0.72\dot{e}(t)$ | Variable band<br>$\theta = 1$ | - |
| PI+CI - 4 | Variable<br>$\mathcal{P}_1 = 0.3 + 0.72\dot{e}(t)$<br>$\mathcal{P}_2 = 0.1$ | Variable band<br>$\theta = 1$ | Time-triggered<br>$\rho = 20$ |

Table 2.2 PI+CI configurations for the Example 2.3.

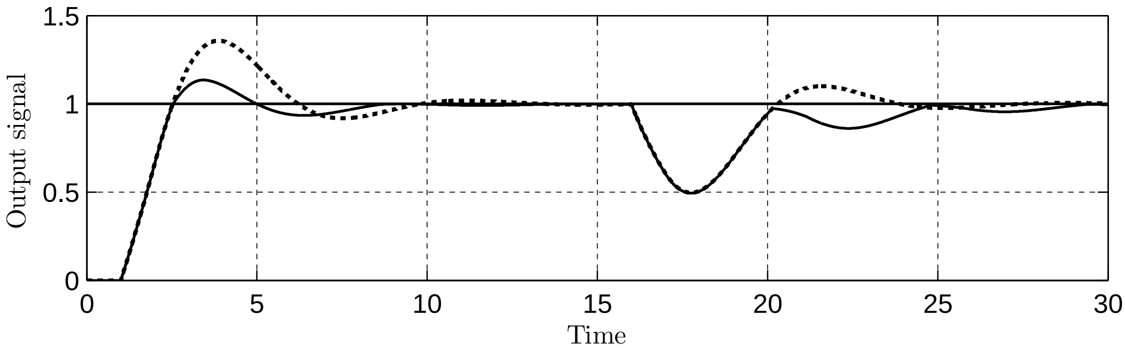
of the PI+CI - 2 greatly solves the coordination problem, resulting in a reduction of both the undershoot and the overshoot. However, the undershoot of both setpoint and disturbance response is still greater than the base PI case. The variable reset ratio of the third configuration reduces the undershoot of the setpoint response, but does not affect the disturbance response. Finally, the undershoot of the disturbance response is improved by the switching reset ratio of the configuration 4.



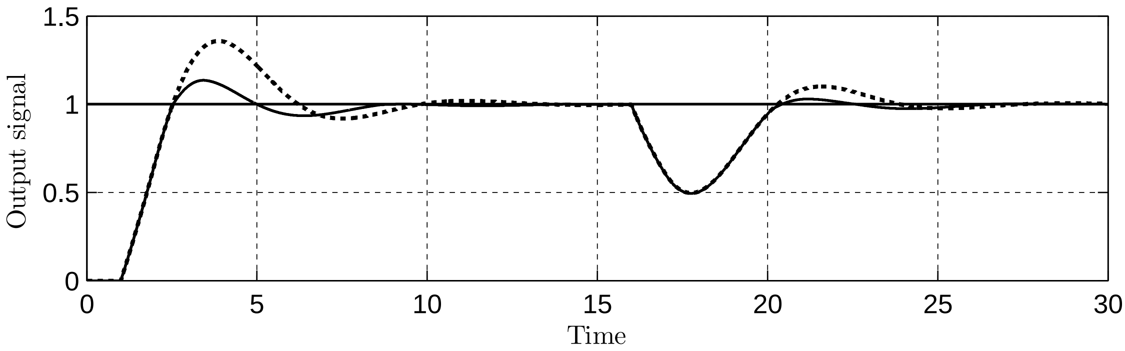

(a) PI+CI -1: No improvements.




(b) PI+CI - 2: Variable band.




(c) PI+CI - 3: Variable reset ratio and variable band.




(d) PI+CI - 4: Variable reset ratio, variable band, and switching reset ratio.

Figure 2.9 Output signals for the four PI+CI configurations in the Example 2.3. PI+CI (solid) and $\mathrm{PI}_{base}$ (dotted)

# Chapter 3

# Stability Analysis of IDDSs

This chapter concerns the stability analysis of impulsive delay dynamical systems, based on the Lyapunov-Krasovskii method. Time-delay reset control systems are considered as the main application of the general results. In the first part of the chapter, we focus on providing stability criteria for IDDSs with state-dependent resetting law. By simple convexity arguments the criteria are extended to cope unknown time-delays belonging to a prescribed interval. Next, we develop theoretical results for the stabilization of IDDSs with stable and unstable base systems by imposing time-dependent conditions on the reset intervals. Finally, we tackle the stabilization of time-delay reset control systems, comprising LTI plants and the PI+CI compensator.

For the sake of the applicability, the stability criteria are given in the form of linear matrix inequalities (LMI).

## 3.1 Introduction

Throughout this chapter, the object of study is the internal stability of the impulsive delay dynamical system introduced in Chapter 2 which is described by

$$\begin{cases} \left.\begin{array}{l} \dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}_t), \\ \dot{\Delta}(t) = 1, \end{array}\right\} & (t, \mathbf{x}_t, \Delta(t)) \notin \mathcal{S}, \\ \left.\begin{array}{l} \mathbf{x}(t^+) = \mathbf{I}(t, \mathbf{x}_t), \\ \Delta(t^+) = 0, \end{array}\right\} & (t, \mathbf{x}_t, \Delta(t)) \in \mathcal{S}, \\ \left.\begin{array}{l} \mathbf{x}(t) = \phi(t - t_0), \\ \Delta(t) = 0, \end{array}\right\} & t \in [t_0 - h, 0]. \end{cases} \tag{3.1}$$

The functions $\mathbf{f}$ and $\mathbf{I}$ will be supposed to be continuous on both parameters, and to satisfy enough additional conditions to ensure that there exists a unique solution $\mathbf{x}(t, t_0, \phi)$ over the interval $[t_0, \infty)$, corresponding to any initial condition $\phi$. In addition, the function $\mathbf{f}$ is assumed to be locally Lipschitz continuous in its second variable, which is usually enough to guarantee local existence and uniqueness of solutions (see, e.g., [16]). The theory below for the state-space restricted to the open set $\Omega \subseteq \mathbb{R}^n$ is valid and it is only for notational purposes that the domain of definition of $\mathbf{f}$ and $\mathbf{I}$ is taken $\mathbb{R}_+ \times \mathcal{PC}([-h, 0], \mathbb{R}^n)$. In addition, for the sake of simplicity, the initial condition $\phi$ is assumed to be piecewise absolutely continuous, that is, $\phi \in \mathcal{PAC}([-h, 0], \mathbb{R}^n)$. The solution $\mathbf{x}(t, t_0, \phi)$ of the system is also piecewise absolutely continuous, and thus, $\mathbf{x}_t \in \mathcal{PAC}([-h, 0], \mathbb{R}^n)$. The smoothness property of $\mathbf{x}_t$ allows to apply on it the norm $\|\mathbf{x}_t\|_W$ given by

$$\|\mathbf{x}_t\|_W = \max_{\theta \in [-h,0]} \|\mathbf{x}_t(\theta)\| + \left( \int_{-h}^{0} \|\dot{\mathbf{x}}_t(\theta)\|^2 d\theta \right)^{\frac{1}{2}}. \tag{3.2}$$

Let remind that $\dot{\mathbf{x}}_t$ is the upper right-hand derivative of $\mathbf{x}(t)$, and thus, the above integral is a simple Riemann integral.

On the other hand, for the stability analysis, it is assumed that $\mathbf{f}(t, \phi) = \mathbf{0}$ and $\mathbf{I}(t, \phi) = \mathbf{0}$ for all $t \in \mathbb{R}_+$ and $\phi(\theta) = \mathbf{0}$, $\theta \in [-h, 0]$. Therefore, the IDDS (3.1) possesses a *trivial solution* $\mathbf{x}(t) = \mathbf{0}$ for $t \in [t_0 - h, \infty)$. This means that if the system starts at the *zero initial condition* $\phi(t) = \mathbf{0}$, $t \in [t_0 - h, t_0]$, then the system state will remain zero thereafter.

In the following, it is defined the *internal stability* or stability in the *sense of Lyapunov* of the trivial solution, which is concerned with the behavior of the solution $\mathbf{x}(t, t_0, \phi)$ when the initial condition is close to the zero initial condition, in absent of exogenous signals. Moreover, related concepts such as *(global, uniform) asymptotic stability* well be also defined.

**Definition 3.1** (***global/uniform/asymptotic stability).***

1. If for any $t_0 \in \mathbb{R}_+$ and $\epsilon > 0$, there exists a $\delta = \delta(t_0, \epsilon)$ such that

$$\|\phi\|_W < \delta \Longrightarrow \|\mathbf{x}(t, t_0, \phi)\| < \epsilon, \quad \forall t \geq t_0, \tag{3.3}$$

then the trivial solution of the system (3.1) is *stable*.

2. If the trivial solution is stable and, in addition, there exists a $\delta = \delta(t_0, \epsilon) > 0$ such that

$$\|\phi\|_W < \delta \Longrightarrow \lim_{t\to\infty} \|\mathbf{x}(t, t_0, \phi)\| = 0, \tag{3.4}$$

then the trivial solution is *asymptotically stable*.

3. If the trivial solution is stable and $\delta$ can be chosen independently of $t_0$, then the trivial solution is *uniformly stable*.

4. If the trivial solution is uniformly stable and there exists a $\delta > 0$ such that, for every $\eta > 0$, there is a $T = T(\eta)$ such that

$$\|\phi\|_W < \delta \Longrightarrow \|\mathbf{x}(t, t_0, \phi)\| < \eta, \quad \forall t \geq t_0 + T, \tag{3.5}$$

then the trivial solution is uniformly asymptotically stable.

5. If the trivial solution is (uniformly) asymptotically stable and $\delta$ can be an arbitrarily large, finite number, then the trivial solution is *globally (uniformly) asymptotically stable*.

**Remark 3.1.** *Most of the stability criteria developed in this chapter will be based on a Lyapunov-Krasovskii (LK) functional that depends on both the solution* $\mathbf{x}(t)$ *and its derivative* $\dot{\mathbf{x}}(t)$*. Thus, the initial condition* $\phi$ *is necessarily restricted to be piecewise absolutely continuous with a square-integrable right-hand derivative (see, e.g., [102, 104, 169]). Note that the solution* $\mathbf{x}(t, t_0, \phi)$ *to the IDDS (3.1) with piecewise continuous initial conditions* $\phi$ *becomes differentiable for* $t \geq t_0+h$ *and for* $t \in [t_0, t_0+h]$ *and it also satisfy the uniform bound* $\|\mathbf{x}(t, t_0, \phi)\| \leq m\|\phi\|$ *with some constant* $m > 0$ *([140, 258]). Therefore, the uniform (asymptotic) stability of the IDDS (3.1) with piecewise absolutely continuous initial conditions implies the uniform (asymptotic) stability of the IDDS with a wider class of piecewise continuous initial conditions.*

The main result that concerts with the stability of nonlinear differential equation (or nonlinear systems) is the well known method of Lyapunov[1] functions. This theory has been extended to functional differential equations (including nonlinear time-delays systems) ([117, 140, 170]). In addition, later on it has been used to study the stability of impulsive delay dynamical systems (see, e.g., [256] and the references therein). There are two main approaches for studying the stability; the Lyapunov-Razumikhim technique and the Lyapunov-Krasovskii method. This thesis is focused on the stability analysis by the Lyapunov-Krasovskii method, which is roughly speaking based on

[1] It is named in honor of the Russian mathematician Aleksandr Mikhilovich Lyapunov ([193]).

the existence of a functional $V$ restricted to several boundedness conditions, and the negativity of its derivative along the solutions of the system.

Note that the solution of the IDDS (3.1) is piecewise left-continuous with jumps discontinuities. Therefore, the derivative of the functional along such trajectories should be defined carefully. Let $V : \mathbb{R}_+ \times \mathcal{PAC}([-h, 0], \mathbb{R}^n) \to \mathbb{R}_+$ be continuously differentiable with respect to all of its arguments, and let $\mathbf{x}(t, t_0, \phi)$ be the solution of the system (3.1). Thus, $V(t, \mathbf{x}_t)$ only has (jump) discontinuities at $t \in \mathbb{T}_{t_0,\phi}$. In addition, the upper right-hand derivative of $V$ along the solution $\mathbf{x}(t, t_0, \phi)$ is defined by

$$\dot{V}(t, \mathbf{x}_t) = \limsup_{\epsilon \to 0,\, \epsilon > 0} \frac{V(t+\epsilon, \mathbf{x}_{t+\epsilon}) - V(t, \mathbf{x}_t)}{\epsilon} \tag{3.6}$$

for all $t \in [t_0, \infty) \setminus \mathbb{T}_{t_0,\phi}$. In addition, it is also defined the jumps of the functional along the solution $\mathbf{x}(t, t_0, \phi)$ by

$$\Delta V(t, \mathbf{x}_t) = V(t, \mathbf{I}(t, \mathbf{x}_t)) - V(t, \mathbf{x}_t) \tag{3.7}$$

for any $t \in \mathbb{T}_{t_0,\phi}$.

## 3.2 State-dependent IDDS

This section is devoted to the stability of the IDDS (3.1), when its resetting law is restricted to state-dependent conditions, that is, the resetting set $\mathcal{S}$ is assumed to be defined by

$$\mathcal{S} = \{(\psi, \Delta) \in \mathcal{PC}([-h, 0], \mathbb{R}^n) \times \mathbb{R}_+ : \psi \in \mathcal{Z} \wedge \Delta > \Delta_m\}. \tag{3.8}$$

The IDDS with the above resetting set will be referred to as *state-dependent*[2] *IDDS*. The main result in this section consists of a set of sufficient conditions for the global/uniform/asymptotic stability of the state-dependent IDDS, in the form of a Lyapunov-Krasovskii-like proposition. The application of the main proposition to several LK functionals will lead into asymptotic stability criteria for a class of state-dependent IDDS in the form of LMI conditions.

[2]In spite of the fact that the resetting set includes a time-dependent condition, we still keep the name of state-dependent IDDS. This is due to the time-condition does not force reset actions, and in addition, it can be set as small as it is needed to avoid its effect on most of the trajectories. Therefore, the reset actions are mainly due to a state-dependent condition.

**Proposition 3.1.** *Assume that* $u, v, w : \mathbb{R}_+ \to \mathbb{R}_+$ *are continuous nondecreasing functions, and in addition,* $u, v \in \mathcal{K}$. *If there exists a Lyapunov-Krasovskii functional* $V : \mathbb{R}_+ \times \mathcal{PAC}([-h, 0], \mathbb{R}^n) \to \mathbb{R}_+$ *such that*

$$u(\|\psi(0)\|) \leq V(t, \psi) \leq v(\|\psi\|_W) \tag{3.9a}$$

*for any* $\psi \in \Upsilon = \{\psi \in \mathcal{PAC}([-h, 0], \mathbb{R}^n) : \|\psi\|_W < \gamma\}$ *for some* $\gamma > 0$, *and that for every solution* $\mathbf{x}(t, t_0, \phi)$ *of the IDDS (3.1),* $V(t, \mathbf{x}_t)$ *is continuous for all* $t \geq t_0$ *and* $t_0 \in \mathbb{R}_+$ *except on the set* $\mathbb{T}_{t_0,\phi}$, *and in addition, it is satisfied*

$$\dot{V}(t, \mathbf{x}_t) \leq -w(\|\mathbf{x}_t(0)\|), \qquad (\mathbf{x}_t, \Delta(t)) \notin \mathcal{S}, \tag{3.9b}$$

$$\Delta V(t, \mathbf{x}_t) \leq 0, \qquad (\mathbf{x}_t, \Delta(t)) \in \mathcal{S}, \tag{3.9c}$$

*where* $\dot{V}$ *and* $\Delta V$ *are evaluated along the trajectories of (3.1) with* $\mathbf{x}_t \in \Upsilon$, *then the trivial solution of the IDDS (3.1) is uniformly stable. If* $w(s) > 0$ *for* $s > 0$ *then the solution is uniformly asymptotically stable. In addition, if* $\lim_{s\to\infty} u(s) = \infty$ *and* $\Upsilon = \mathcal{PAC}([-h, 0], \mathbb{R}^n)$, *then it is globally uniformly asymptotically stable.*

*Proof.* For a given $\epsilon > 0$, let set $\epsilon_1 < \min(\epsilon, \gamma)$, then it can be found some $\delta = \delta(\epsilon_1)$, $0 < \delta < \epsilon_1$, such that $v(\delta) < u(\epsilon_1)$. Suppose that $\mathbf{x}(t, t_0, \phi)$ is the solution of (3.1) for $(t_0, \phi) \in \mathbb{R}_+ \times \mathcal{PAC}([-h, 0], \mathbb{R}^n)$, where $\|\phi\|_W < \delta$. Therefore, $\mathbf{x}(t, t_0, \phi)$ is locally absolutely continuous on $[t_0 - h, \infty) \setminus \mathbb{T}_{t_0,\phi}$, where $\mathbb{T}_{t_0,\phi} = \{t_1, t_2, \cdots\}$ is the set of reset instants corresponding to the initial time $t_0$ and the initial condition $\phi$. In addition, the time regularization implies $t_0 < t_1 < t_2 < \cdots$. Now, we will prove that $\|\mathbf{x}(t)\| < \epsilon_1 < \epsilon$ for $t \geq t_0$. Suppose that the previous inequality is violated for some $t$. Let $T \geq t_0$ be the smallest value such that $\lim_{s\to T, s>T} \|\mathbf{x}(s)\| \geq \epsilon_1$. This quantity is well defined since the solution $\mathbf{x}(t, t_0, \phi)$ is left-continuous. Note that if $T \in \mathbb{T}_{t_0,\phi}$ then $\|\mathbf{I}(T, \mathbf{x}_T)\| = \epsilon_1$, otherwise $\|\mathbf{x}(T)\| = \epsilon_1$. Thus, by definition, we have $\|\mathbf{x}(t)\| < \epsilon_1$ for all $t \in [t_0, T)$. Since $\delta < \epsilon_1$, conditions (3.9b) and (3.9c) imply

$$V(t, \mathbf{x}_t) \leq V(t_k, I(t_k, \mathbf{x}_{t_k})) \leq V(t_k, \mathbf{x}_{t_k}) \tag{3.10}$$

for $t \in (t_k, t_{k+1}] \cap [t_0, T]$, $k = 1, 2, \ldots$, and

$$V(t, \mathbf{x}_t) \leq V(t_0, \phi) \tag{3.11}$$

for $t \in [t_0, t_1] \cap [t_0, T]$. Hence, since $\|\phi\|_W < \delta < \epsilon_1 < \gamma$, using (3.9a) it follows

$$V(T, \mathbf{x}_T) \leq V(t_0, \phi) \leq v(\|\phi\|_W) < v(\delta) < u(\epsilon_1). \tag{3.12}$$

But, from (3.9a) and (3.9c), it follows

$$u(\epsilon_1) = u(\|\mathbf{x}(T)\|) \leq V(T, \mathbf{x}_T), \quad T \notin \mathbb{T}_{t_0,\phi} \tag{3.13}$$

and

$$u(\epsilon_1) = u(\|\mathbf{I}(T, \mathbf{x}_T)\|) \leq V(T, \mathbf{I}(T, \mathbf{x}_T)) \leq V(T, \mathbf{x}_T), \quad T \in \mathbb{T}_{t_0,\phi}, \tag{3.14}$$

which is a contradiction in both cases. Therefore, the original assumption is false, and $\|\mathbf{x}(t)\| < \epsilon_1 < \epsilon$ for all $t \geq t_0$. Finally, since $\|\phi\|_W < \delta < \epsilon$, it is proved that the trivial solution is uniformly stable.

To prove uniform asymptotic stability is a bit more involved. For $\epsilon > 0$ choose $\delta_a > 0$ as the constant of uniform stability, then it is true that $\|\phi\|_W < \delta_a$ implies $\|\mathbf{x}(t)\| < \min(\epsilon, \gamma)$ for all $t \geq t_0$. Now, it has to be shown that for any $\eta > 0$ there exists some $T(\delta_a, \eta)$ such that $\|\mathbf{x}(t)\| < \eta$ for $t \geq t_0 + T$. This is equivalent to prove that $\|\mathbf{x}_{t_0+T}\|_W < \delta_b$ for $\delta_b = \delta(\eta)$ the constant of uniform stability. By contradiction, suppose that there not exists such $T$, then $\|\mathbf{x}_t\|_W \geq \delta_b$ for $\|\phi\|_W < \delta_a$ and for all $t \geq t_0$. Thus, since $\mathbf{f}$ is locally Lipschitz continuous, there exist $\delta_c > 0$ and a sequence $(\tau_j)$, $j = 1, 2, \ldots$ such that

$$t_0 + (2j - 1)2h \leq \tau_j \leq t_0 + 4jh, \tag{3.15}$$

where $\tau_j \notin \mathbb{T}_{t_0,\phi}$ and $\|\mathbf{x}(\tau_j)\| \geq \delta_c$. Since the system is time regularized then $t_{k+1} - t_k \geq \Delta_m$ for $t_k \in \mathbb{T}_{t_0,\phi}$, $k = 1, 2, \ldots$. In addition, by the stability and the Lipschitz continuity assumption on $\mathbf{f}$, there exits a constant $L > 0$ such that $\|\dot{\mathbf{x}}(t)\| < L$ for all $t \in [t_0, \infty) \setminus \mathbb{T}_{t_0,\phi}$. Therefore, it is possible to build a set of intervals $I_j = [\tau_j - \frac{\delta_c}{2L}\check{\alpha}_j, \tau_j + \frac{\delta_c}{2L}\hat{\alpha}_j]$ with $\check{\alpha}_j$, $\hat{\alpha}_j \in \{0, 1\}$, $\check{\alpha}_j + \hat{\alpha}_j > 0$, that do not contain reset instants and do not overlap (by using a number $L > 0$ large enough and proper values of $\check{\alpha}_j$, $\hat{\alpha}_j$), that is $I_j \cap \mathbb{T}_{t_0,\phi} = \emptyset$, $j = 1, 2, \ldots$, and then by using the mean-value theorem on the intervals $[\tau_j, t] \subset I_j$ and $[t, \tau_j] \subset I_j$

$$\|\mathbf{x}(t)\| = \|\mathbf{x}(\tau_j) + \dot{\mathbf{x}}(\tau_j + \theta_1(t - \tau_j))(t - \tau_j)\|, \tag{3.16}$$

$$\|\mathbf{x}(t)\| = \|\mathbf{x}(\tau_j) - \dot{\mathbf{x}}(\tau_j + \theta_2(\tau_j - t))(\tau_j - t)\| \tag{3.17}$$

for some $\theta_1, \theta_2 \in (0,1)$, and then

$$\|\mathbf{x}(t)\| \geq \|\mathbf{x}(\tau_j)\| - \|\dot{\mathbf{x}}(\tau_j + \theta(t-\tau_j))\| |t-\tau_j| \geq \frac{\delta_c}{2} \tag{3.18}$$

for some $\theta \in (-1,1)$ and for all $t \in I_j$. In addition, from (3.9b) it is true that $\dot{V}(t,\mathbf{x}_t) \leq -w(\|\mathbf{x}(t)\|) \leq -w(\frac{\delta_c}{2}) < 0$, for any $t \in I_j$, this means that $V(t,\mathbf{x}_t)$ is decreasing with at least a ratio $-w(\frac{\delta_c}{2})$ in each interval $I_j$, $j = 1,2,\ldots$. On the other hand, the reset instants in the sequence $\mathbb{T}_{t_0,\phi} = \{t_1, t_2, \cdots\}$ may be renamed as $\mathbb{T}_\phi = (t_{j,l})$, where the reset instant $t_{j,l}$ corresponds to the $l^{th}$-instant prior to $\tau_j \in I_j$, that is

$$t_0 < t_{1,1} < t_{1,2} < \cdots t_{1,N_1} < \check{\tau}_1 = \tau_1 - \frac{\delta_c}{2L}\check{\alpha}_1 < \hat{\tau}_1 = \tau_1 + \frac{\delta_c}{2L}\hat{\alpha}_1 < t_{2,1} < \cdots t_{2,N_2} < \check{\tau}_2 \tag{3.19}$$

for some integers $N_j \geq 0$. Therefore, by integrating $\dot{V}(t,\mathbf{x}_t)$ over the interval $[\check{\tau}_1, \hat{\tau}_1]$ and using also (3.9c), it is obtained

$$V(\check{\tau}_1, \mathbf{x}_{\check{\tau}_1}) \leq V(t_{1,N_1}, \mathbf{I}(t_{1,N_1}, \mathbf{x}_{t_{1,N_1}})) \leq V(t_{1,N_1}, \mathbf{x}_{t_{1,N_1}}) \leq \cdots \leq V(t_0, \phi), \tag{3.20}$$

$$\begin{aligned} V(\tau_2, \mathbf{x}_{\tau_2}) &\leq V(\hat{\tau}_1, \mathbf{x}_{\hat{\tau}_1}) \leq V(\check{\tau}_1, \mathbf{x}_{\check{\tau}_1}) - w(\frac{\delta_c}{2})\frac{\delta_c}{2L}(\check{\alpha}_1 + \hat{\alpha}_1) \\ &\leq \cdots \leq V(t_0, \phi) - w(\frac{\delta_c}{2})\frac{\delta_c}{2L}(\check{\alpha}_1 + \hat{\alpha}_1). \end{aligned} \tag{3.21}$$

Since $\check{\alpha}_j + \hat{\alpha}_j \geq 1$ for $j = 1,2,\ldots$, repeating the reasoning for any $\tau_j$, it follows

$$V(\tau_j, \mathbf{x}_{\tau_j}) \leq V(t_0, \phi) - w(\frac{\delta_c}{2})(j-1)\frac{\delta_c}{2L}, \tag{3.22}$$

then for a large enough $j$, it gives $V(\tau_j, \mathbf{x}_{\tau_j}) < 0$ which is a contradiction. Finally, if $\lim_{s\to\infty} u(s) = \infty$ and $\Upsilon = \mathcal{PAC}([-h,0], \mathbb{R}^n)$, then $\delta_a$ above may be chosen arbitrarily large, and $\epsilon$ can be set after $\delta_a$ to satisfy $v(\delta_a) < u(\epsilon)$. Therefore, global uniform asymptotic stability can be concluded.

□

**Remark 3.2.** *Proposition 3.1 implies an extension of the previous results in two lines: first, to the best of the author knowledge the results in the literature for impulse delay differential equations via Lyapunov functionals impose on the LK functional a condition* *similar to (see, e.g., [16, 256], and the references therein)*

$$V(t_k, \mathbf{I}(t_k, \mathbf{x}_{t_k})) \leq \beta_k V(t_k, \mathbf{x}(t_k)) \tag{3.23}$$

*with* $\beta_k \geq 0$ *and* $\sum_{k=1}^{\infty} \beta_k < \infty$. *Note that the above condition is more restrictive than condition (3.9c) of Proposition 3.1. For instance, condition (3.23) requires the value of the functional* $V$ *to decrease at the reset instants for a large enough time, which is not necessary in Proposition 3.1. On the other hand, Proposition 3.1 is an extension of the stability result in the field of time-delay reset control systems ([19, 37, 133]), since it is applicable to IDDSs with nonlinear and time-varying base systems.*

### 3.2.1 IDDS with a Nonlinear and Time-varying Base System

In general, for an impulsive (delay) system with nonlinear base system without a particular structure, there is not a systematic procedure for generating LK functionals candidates[3]. Therefore, in this section the applicability of the above result is illustrated by means of an example. It will be derived stability conditions for an IDDS with a specific structure by applying Proposition 3.1 to a particular LK functional.

**Example 3.1.** Consider an IDDS given by

$$\begin{cases} \left.\begin{cases} \dot{x}_1(t) = a_1(t)x_1^3(t) + b_1(t)x_1^3(t-h) + c_1(t)x_2(t), \\ \dot{x}_2(t) = a_2(t)x_2(t) + b_2(t)x_2(t-h) + c_2(t)x_1^3(t), \end{cases}\right\} & x_1(t) \neq 0, \\ \\ \left.\begin{cases} x_1(t^+) = x_1(t), \\ x_2(t^+) = 0, \end{cases}\right\} & x_1(t) = 0, \end{cases} \tag{3.24}$$

where $a_1(t)$, $b_1(t)$, $c_1(t)$, $a_2(t)$, $b_2(t)$, and $c_2(t)$ are arbitrary continuous bounded functions with $a_1(t), a_2(t) \leq -\delta$, $|b_1(t)|, |b_2(t)| \leq \frac{\delta}{2}$, and $|c_1(t)|, |c_2(t)| \leq \frac{\delta}{4}$ for some given $\delta > 0$. Consider the neighbourhood $\|\psi\|_W \leq \gamma = \frac{1}{\sqrt{5}}$ of the trivial solution, then it is easy to prove that the resulting function $\mathbf{f}$ of the IDDS (3.1) is Lipschitz continuous in this neighbourhood.

[3]The systematic search of Lyapunov functions (functionals) is a challenging problem in nonlinear systems, there are only some procedures to construct some kind of Lyapunov functions (see, e.g., [224]). In some cases, the Lyapunov function arises from energy functions, for example in electrical or mechanical systems.

For stability purpose, consider the candidate LK functional $V$ defined by

$$V(\mathbf{x}_t) = \frac{x_1^4(t)}{4} + \frac{x_2^2(t)}{2} + \frac{\delta}{2}\int_{-h}^{0}(x_1^6(t+\alpha) + x_2^2(t+\alpha))d\alpha, \tag{3.25}$$

where $\mathbf{x}(t) = (x_1(t), x_2(t)) \in \mathbb{R}^2$. Let define the continuous nondecreasing functions $u(s) = \frac{1}{4}s^4$ and $v(s) = 2s^2$, then the above functional satisfies condition (3.9a) as follows:

$$u(\|\mathbf{x}_t(0)\|) = \frac{1}{4}(x_1^2(t) + x_2^2(t))^2 \leq \frac{1}{4}(x_1^4(t) + x_2^2(t)) \leq V(\mathbf{x}_t) \tag{3.26}$$

and

$$V(\mathbf{x}_t) \leq x_1^2(t) + x_2^2(t) + \frac{\delta}{2}\int_{-h}^{0}(x_1^2(t+\alpha) + x_2^2(t+\alpha))d\alpha \leq 2\|\mathbf{x}_t\|^2 \leq 2\|\mathbf{x}_t\|_W^2. \tag{3.27}$$

The last inequality is obvious from the definition of the norms. It is easy to check that the above conditions are satisfied for any $\|\mathbf{x}_t\|_W \leq \gamma = \frac{1}{\sqrt{5}}$.

The derivative of the functional along the solutions of the system (3.24) is given by

$$\dot{V}(\mathbf{x}_t) = x_1^3(t)\dot{x}_1(t) + x_2(t)\dot{x}_2(t) + \frac{\delta}{2}(x_1^6(t) - x_1^6(t-h)) + \frac{\delta}{2}(x_2^2(t) - x_2^2(t-h)). \tag{3.28}$$

After some manipulations, the derivative of the functional is bounded by

$$\dot{V}(\mathbf{x}_t) \leq \xi^\top(t)\Gamma\xi(t), \tag{3.29}$$

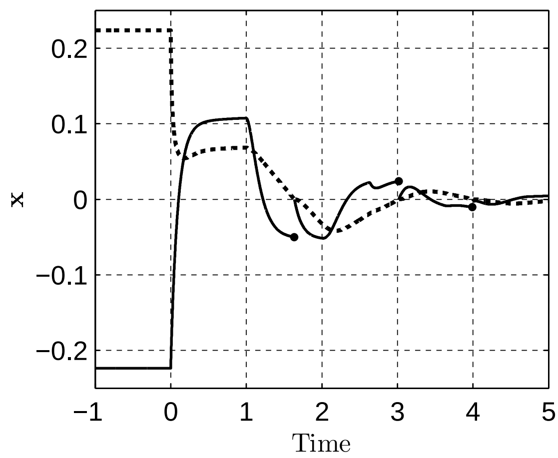
where $\xi(t) = (|x_1(t)|^3, |x_1(t-h)|^3, |x_2(t)|, |x_2(t-h)|)$ and

$$\Gamma = \begin{bmatrix} -\frac{\delta}{2} & \frac{\delta}{4} & \frac{\delta}{4} & 0 \\ \frac{\delta}{4} & -\frac{\delta}{2} & 0 & 0 \\ \frac{\delta}{4} & 0 & -\frac{\delta}{2} & \frac{\delta}{4} \\ 0 & 0 & \frac{\delta}{4} & -\frac{\delta}{2} \end{bmatrix}. \tag{3.30}$$

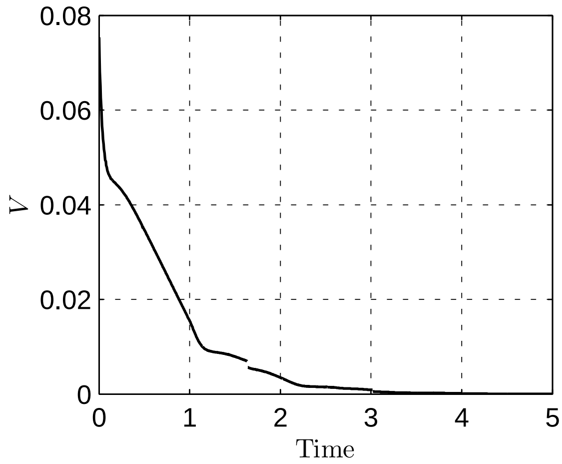
Since $\Gamma < 0$ for any $\delta > 0$, then defining $w(s) = \lambda_m(-\Gamma)s^6$ it follows

$$-\dot{V}(\mathbf{x}_t) \geq \lambda_m(-\Gamma)\left(|x_1(t)|^6 + |x_1(t-h)|^6 + |x_2(t)|^2 + |x_2(t-h)|^2\right) \geq w(\|\mathbf{x}_t(0)\|) \tag{3.31}$$

for any $\|\mathbf{x}_t\|_W \leq \gamma$, and thus condition (3.9b) is satisfied. Finally, $\Delta V(\mathbf{x}_t) = -\frac{x_2^2(t)}{2}$, which is negative for any reset instant, then the trivial solution of the system (3.24) is uniformly asymptotically stable.




(a) Trajectory: $x_1$ (dotted) and $x_2$ (solid).




(b) LK functional.

Figure 3.1 Trajectory and value of the LK functional for the IDDS with nonlinear and time-varying base system of the Example 3.1.

Now, let particularize the system (3.24) with some specific functions $a_1(t)$, $b_1(t)$, $c_1(t)$, $a_2(t)$, $b_2(t)$, and $c_2(t)$ given by

$$\begin{array}{lll} a_1(t) = 50e^{-t} - 100\delta, & b_1(t) = \frac{\delta}{2}\sin(t), & c_1(t) = \frac{\delta}{4}, \\ a_2(t) = -e^{-t} - \delta, & b_2(t) = -\frac{\delta}{2}, & c_2(t) = \frac{\delta}{4}\sin(t). \end{array} \tag{3.32}$$

It can be easily seen that the above functions satisfy the bounds for a given $\delta > 0$. In addition, consider the initial condition $\phi(t) = (\frac{1}{2\sqrt{5}}, -\frac{1}{2\sqrt{5}})$, $t \in [t_0 - h, 0]$ such that $\|\phi\|_W = \frac{1}{\sqrt{10}} < \frac{1}{\sqrt{5}}$. The evolution of the system with $h = 1$ and $t_0 = 0$ is plotted in Fig. 3.1a. In addition, Fig. 3.1b shows the value of the functional (3.25) along this trajectory. Observe how the functional always decreases, during both the continuous dynamic and the impulsive events.

### 3.2.2 IDDS with a LTI Base System

In this section, Proposition 3.1 is used to derive stability criteria in the form of LMIs for a particular class of IDDSs. Consider the system (3.1) with a LTI base system, a single constant time-delay and a linear resetting law, that is, an IDDS described by

$$\begin{cases} \left.\begin{array}{l} \dot{\mathbf{x}}(t) = A\mathbf{x}(t) + A_d\mathbf{x}(t-h), \\ \dot{\Delta}(t) = 1, \end{array}\right\} & (\mathbf{x}(t), \mathbf{x}(t-h), \Delta(t)) \notin \mathcal{S}, \\ \left.\begin{array}{l} \mathbf{x}(t^+) = A_R\mathbf{x}(t), \\ \Delta(t^+) = 0, \end{array}\right\} & (\mathbf{x}(t), \mathbf{x}(t-h), \Delta(t)) \in \mathcal{S}, \\ \left.\begin{array}{l} \mathbf{x}(t) = \phi(t), \\ \Delta(t) = 0, \end{array}\right\} & t \in [-h, 0], \end{cases} \tag{3.33}$$

where $h > 0$ is the time-delay, $A$, $A_d$, $A_R \in \mathbb{R}^{n\times n}$, the initial time is $t_0 = 0$, and the resetting set is given by

$$\mathcal{S} = \{(\mathbf{x}, \mathbf{y}, \Delta) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ : (\mathbf{x}, \mathbf{y}) \in \mathcal{Z} \wedge \Delta \geq \Delta_m\} \tag{3.34}$$

for some open subset $\mathcal{Z} \subset \mathbb{R}^{2n}$.

In general, the choice of the LK functional is crucial for deriving stability criteria. Special forms of the functional lead into different classes of conditions. A simple classification can be made based on the dependency on the length of the time-delay; one class is independent of the time-delay, and its members are called *delay-independent* conditions. The other class makes use of information of the length of the time-delay, and its members are called *delay-dependent* conditions.

Following, we will apply the same methodology as in previous works in the field of reset control systems, which mainly consists in adapting the results for the LTI base system to the reset control system. In this case, the LK functional candidates are taken from those that have been proposed to LTI time-delay systems. The main reasons of this approach are: first, both cases share the condition imposed on the derivative of the functional, and thus, it is expected that similar LMI conditions can be obtained without major difficulties; second, the LK functionals usually consist of the sum of a quadratic term and integrals of quadratic terms. Hence, the reset actions does not affect the integral terms (due to the integral), so that an LMI condition for the reset actions is almost obtained directly from the non-integral term.

### Delay-independent Conditions

The delay-independent stability criterion presented in this section is a slightly modified version of the one proposed in [19] for time-delay reset control systems with zero crossing resetting law. The proof of the proposition arises from the application of Proprosition 3.1 to a particular functional. Probably the simplest stability criterion can be obtained by using the following LK functional

$$V(\psi) = \psi(0)^\top P\psi(0) \ + \int_{-h}^{0} \psi^\top(\alpha)Q\psi(\alpha)\, d\alpha, \tag{3.35}$$

where the matrices $P$ and $Q$ are symmetric and positive definite.

**Proposition 3.2.** *If there exist symmetric matrices $P$, $Q \in \mathbb{S}^n_+$ such that*

$$\Gamma = \begin{bmatrix} A^\top P + PA + Q & \frac{1}{2}(A_d^\top P + PA_d) \\ \star & -Q \end{bmatrix} < 0 \tag{3.36a}$$

*and*

$$\Theta^\top (A_R^\top P A_R - P)\Theta \leq 0 \tag{3.36b}$$

*for some $\Theta \in \mathbb{R}^{n\times m}$, $m \geq 0$, such that $\mathrm{proj}_{[1,n]}(\mathcal{Z}) \subseteq \mathcal{R}(\Theta)$, then the trivial solution of the system (3.33) is globally asymptotically stable.*

*Proof.* Consider the LK functional (3.35) with $P$, $Q$ the matrices of the proposition. For this functional, since $P$, $Q > 0$, it is true that

$$\begin{aligned} V(\psi) \leq & \lambda_M(P)\|\psi(0)\|^2 + h\lambda_M(Q)\|\psi\|^2 \leq (\lambda_M(P) + h\lambda_M(Q))\|\psi\|^2 \\ \leq & (\lambda_M(P) + h\lambda_M(Q))\|\psi\|_W^2 = v(\|\psi\|_W) \end{aligned} \tag{3.37}$$

and

$$V(\psi) \geq \lambda_m(P)\|\psi(0)\|^2 + h\lambda_m(Q)\|\psi\|^2 \geq \lambda_m(P)\|\psi(0)\|^2 = u(\|\psi(0)\|), \tag{3.38}$$

where $u, v : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous nondecreasing functions and $u, v \in \mathcal{K}$. The derivative of $V$ along the solution of (3.33), after some manipulation, is given by

$$\dot{V}(\mathbf{x}_t) = \begin{bmatrix} \mathbf{x}^\top(t) & \mathbf{x}^\top(t-h) \end{bmatrix} \Gamma \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{x}(t-h) \end{bmatrix}. \tag{3.39}$$

Therefore, condition (3.36a) is obtained, and it implies

$$-\dot{V}(\mathbf{x}_t) > \lambda_m(-\Gamma)(\|\mathbf{x}(t)\|^2 + \|\mathbf{x}(t-h)\|^2) \geq \lambda_m(-\Gamma)(\|\mathbf{x}(t)\|^2) = w(\|\mathbf{x}(t)\|) \quad (3.40)$$

for all $(\mathbf{x}(t), \mathbf{x}(t-h), \Delta(t)) \notin \mathcal{S}$, where $w : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous nondecreasing function and $w \in \mathcal{K}$. For the functional (3.35) the condition (3.9c) is given by

$$\Delta V(\mathbf{x}_t) = \mathbf{x}^\top(t)(A_R^\top P A_R - P)\mathbf{x}(t) \leq 0, \quad (3.41)$$

where $(\mathbf{x}(t), \mathbf{x}(t-h), \Delta(t)) \in \mathcal{S}$. Suppose that $\text{proj}_{[1,n]}(\mathcal{Z}) \subseteq \mathcal{R}(\Theta)$ for some $\theta \in \mathbb{R}^{n \times m}$, then for every $\mathbf{x}(t)$ there exists $\bar{\mathbf{x}}$ such that $\mathbf{x}(t) = \Theta\bar{\mathbf{x}}$. Therefore, condition (3.41) is rewritten as follows:

$$\bar{\mathbf{x}}^\top \Theta^\top (A_R^\top P A_R - P)\Theta\bar{\mathbf{x}} \leq 0. \quad (3.42)$$

Finally, the above condition is satisfied by the LMI condition (3.36b). As a result, all the conditions of Proposition 3.1 are satisfied, and thus, the trivial solution of the system (3.33) is globally asymptotically stable.

□

**Remark 3.3.** *Note that the election of the matrix $\Theta$ may be crucial for the success of the stability criterion. The simplest choice is $\Theta = I$, and thus, $\mathcal{R}(\Theta) = \mathbb{R}^n$ and* $\text{proj}_{[1,n]}(\mathcal{Z}) \subseteq \mathcal{R}(\Theta)$ *is satisfied for all $\mathcal{Z}$. However, it is easy to see that in some cases a proper election of $\Theta$ leads into a reduction of the dimension of the LMI (3.36b), which may produce less conservative results. This fact has been used in [47] among other, for the stability analysis of reset control systems and in [19, 37] for time-delay reset control systems.*

**Example 3.2.** This example illustrates the fact commented in Remark 3.3. Consider a two-dimensional system with some matrices $A$ and $A_d$. In addition, let set $A_R = \text{diag}(1, 0)$ and $\mathcal{Z} = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^4 : [1 \;\; 0]\mathbf{x} = 0\}$. For some positive definite matrix

$$P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}, \quad (3.43)$$

the simple election of $\Theta = I$ implies that the following condition

$$\begin{bmatrix} 0 & -p_2 \\ -p_2 & -p_3 \end{bmatrix} < 0 \quad (3.44)$$

has to be satisfied in order to guarantee (3.36b), which necessarily makes $p_2 = 0$. However, if $\Theta = [0 \;\; 1]^\top$ then $\mathrm{proj}_{[1,2]}(\mathcal{Z}) = \mathcal{R}(\Theta)$ and in addition, condition (3.36b) is directly satisfied, since $p_3 > 0$. Therefore, the second choice of $\Theta$ is clearly less restrictive, and thus, less conservative results can be expected. In fact, it is directly derived that all the IDDS with the above features are asymptotically stable if their base systems satisfy condition (3.36a).

**Example 3.3.** Consider the numerical example used in [19] to illustrate the delay-independent stability results. The example consists of an IDDS defined by (3.33), whose matrices are given by

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \qquad A_d = \begin{bmatrix} -0.25 & 0 \\ -0.25 & -0.25 \end{bmatrix}. \tag{3.45}$$

The base system has been shown to be asymptotically stable independently of the time-delay ([117]), and in addition, there exist matrices $P$ and $Q$ such that condition (3.36a) is satisfied. Now, consider the resetting set $\mathcal{Z} = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^4 : [1 \;\; 0]\mathbf{x} = 0\}$, such that $\Theta = [0 \;\; 1]^\top$, and the matrix

$$A_R = \begin{bmatrix} a_1 & 0 \\ a_2 & a_3 \end{bmatrix}. \tag{3.46}$$

Then condition (3.36b) is simply reduce to $(a_3^2 - 1)p_3 \leq 0$. Hence, the IDDS is asymptotically stable independently of the time-delay for all $a_1$, $a_2$ and $|a_3| \leq 1$.

### Delay-dependent Conditions

This section is focused on delay-dependent stability criteria for the IDDS (3.33). The key point is to search for adequate Lyapunov-Krasovskii functionals, such that, from an application of the Proposition 3.1, it derives conditions that depend explicitly on the time-delay.

Again the idea is to directly use those LK functionals that have been successfully applied to LTI time-delay systems. The main approach to develope delay-dependent stability criterion for LTI time-delay systems has involved the addition of a quadratic double integral term to the LK functional (3.35). Nevertheless, the double integral leads to an integral term in the derivative of the functional and the problem is how to deal with this integral in order to obtain computational conditions, such as LMI conditions. There are two important methods in the literature[4]: the discretized LK functional method, and the model transformation method. The first method uses the idea of a complete quadratic LK functional (see [167] for a exhaustive treatment of the complete Lyapunov functionals), whose existence gives sufficient and necessary conditions for the stability. However, it is a distributed functional and to numerically check the existence of such a functional is not an easy task. Therefore, a discretization of the LK functional is proposed in [116], allowing to write the results in the form of LMIs. The advantage of this method is that the estimate of the maximum allowable time-delay that guarantees the stability of the system is very close to the actual value.

On the other hand, in the model transformation method, the system is transformed into a system with a distributed time-delay. The purpose of this transformation is to bring the integral term into the system equation and produce both cross terms and quadratic integral terms in the derivative of the LK functional. Then, the integral terms are removed by using an inequality, for example Park's inequality or Moon et al.'s inequality (see Appendix B). However, the model transformation introduces spurious poles which are not present in the original system ([118]), and usually introduces conservatism.

More recently, a new complete LK functional is proposed in [102], which results in an augmented LK functional. In [251], the augmented functional and the Wirtinger-based integral inequality was used to provide adequate stability conditions for systems with the nominal value of the time-delay different to zero, that is, systems that are unstable without time-delay. On the other hand, the delay-partitioning approach proposed in [114] improved the results based on a model transformation (as conservative as the discretization method) at the expense of increased complexity of the resulting LMIs. Finally, the work [252] provides a generic set of integral inequalities based on Legendre polynomials, which encloses the well known and widely used Jensen's inequality. These inequalities are the source of a new LK functional which provides stability conditions in a form of LMIs that are as conservative and efficient as the discretization method and partitioning-delay method.

In the following, three different delay-dependent stability criteria will be developed for the IDDS (3.33), based on the application of Proposition 3.1 to three LK functionals.

[4]The reader is referred to [104, 117], and the references therein for a deeper treatment.

The first result will be obtained by applying the model transformation method and considering the following LK functional

$$V(\mathbf{x}_t) = \mathbf{x}^\top(t)P\mathbf{x}(t) + \int_{t-h}^{t} \mathbf{x}^\top(\alpha)Q\mathbf{x}(\alpha)d\alpha + \int_{-h}^{0}\int_{t+\beta}^{t} \dot{\mathbf{x}}^\top(\alpha)R\dot{\mathbf{x}}(\alpha)d\alpha d\beta. \tag{3.47}$$

**Proposition 3.3.** *If there exist matrices $P$, $Q$, $R \in \mathbb{S}_+^n$, $X \in \mathbb{S}^n$, and $Y \in \mathbb{R}^n$ such that*

$$\Gamma(h) = \begin{bmatrix} PA + A^\top P + hX + Y + Y^\top + hA^\top RA + Q & PA_d - Y + hA^\top RA_d \\ \star & -Q + hA_d^\top RA_d \end{bmatrix} < 0, \tag{3.48a}$$

$$\begin{bmatrix} X & Y \\ Y^\top & R \end{bmatrix} \geq 0, \tag{3.48b}$$

$$Y(A_R - I) = 0, \tag{3.48c}$$

$$\Theta^\top(A_R^\top PA_R - P)\Theta \leq 0, \tag{3.48d}$$

*for some $\Theta \in \mathbb{R}^{n\times m}$, $m \geq 0$, such that $\text{proj}_{[1,n]}(\mathcal{Z}) \subseteq \mathcal{R}(\Theta)$, then the trivial solution of the system (3.33) is globally asymptotically stable.*

*Proof.* Consider the LK functional (3.47), then from (3.38) there exists a function $u \in \mathcal{K}$ such that $V(\mathbf{x}_t) \geq u(\|\mathbf{x}_t(0)\|$. In addition, by the inequality (3.37) it follows

$$\begin{aligned} V(\mathbf{x}_t) &\leq (\lambda_M(P) + h\lambda_M(Q))\,\|\mathbf{x}_t\|^2 + \int_{-h}^{0}\int_{\beta}^{0} \dot{\mathbf{x}}_t^\top(\alpha)R\dot{\mathbf{x}}_t(\alpha)d\alpha d\beta \\ &\leq (\lambda_M(P) + h\lambda_M(Q))\|\mathbf{x}_t\|^2 + h\lambda_M(R)\int_{-h}^{0} \|\dot{\mathbf{x}}_t(\alpha)\|^2 d\alpha \leq v(\|\mathbf{x}_t\|_W) \end{aligned} \tag{3.49}$$

for some function $v \in \mathcal{K}$.

By the model transformation method and considering the fundamental theorem of calculus (see Appendix C), the upper right-hand derivative of the system (3.33) can be written as follows:

$$\dot{\mathbf{x}}(t) = (A + A_d)\mathbf{x}(t) - A_d\int_{t-h}^{t} \dot{\mathbf{x}}(\alpha)d\alpha - A_d(A_R - I)\sum_{t_k\in[t-h,t)} \mathbf{x}(t_k) \tag{3.50}$$

Therefore, the upper right-hand derivative of the functional along the solutions of the system (3.33) is given by

$$\begin{aligned}\dot{V}(\mathbf{x}_t) &= He\left(\mathbf{x}^\top(t)P(A+A_d)\mathbf{x}(t)\right) - He\Big(\mathbf{x}^\top(t)PA_d\int_{t-h}^{t}\dot{\mathbf{x}}(\alpha)d\alpha\Big) + \mathbf{x}^\top(t)Q\mathbf{x}(t)\\ &\quad - \mathbf{x}^\top(t-h)Q\mathbf{x}(t-h) + h\dot{\mathbf{x}}^\top(t)R\dot{\mathbf{x}}(t) - \int_{t-h}^{t}\dot{\mathbf{x}}^\top(\alpha)R\dot{\mathbf{x}}(\alpha)d\alpha\\ &\quad - He\Big(\mathbf{x}^\top(t)PA_d(A_R-I)\sum_{t_k\in[t-h,t)}\mathbf{x}(t_k)\Big).\end{aligned} \tag{3.51}$$

Applying Park's inequality (see Appendix B) into the above equation, it is obtained

$$\dot{V}(\mathbf{x}_t) \le \begin{bmatrix}\mathbf{x}(t)\\ \mathbf{x}(t-h)\end{bmatrix}^\top \Gamma(h)\begin{bmatrix}\mathbf{x}(t)\\ \mathbf{x}(t-h)\end{bmatrix} - He\left(\mathbf{x}^\top(t)Y(A_R-I)\sum_{t_k\in[t-h,t)}\mathbf{x}(t_k)\right) \tag{3.52}$$

subject to condition (3.48b), where $\Gamma(h)$ is given[5] in (3.48a). Note that the last term of the upper bound is null by (3.48c), and thus, condition (3.48a) implies

$$-\dot{V}(\mathbf{x}_t) \ge \lambda_m(-\Gamma(h))(\|\mathbf{x}(t)\|^2 + \|\mathbf{x}(t-h)\|^2) \ge \lambda_m(-\Gamma(h))(\|\mathbf{x}(t)\|^2) = w(\|\mathbf{x}(t)\|) \tag{3.53}$$

and then, condition (3.9b) of Proposition 3.1 is satisfied. Finally, as it has been proved in the delay-independent case, condition (3.48d) implies (3.9c).

□

**Remark 3.4.** *In comparison with the result in [37] (stability for time-delay reset control systems), the main difference is that a more general LK functional has been used in the proposed result, in particular the LK functional used in [37] can be obtained from (3.47) by making* $R = A_d^\top Z A_d$, *where* $Z > 0$. *On the other hand, choosing the matrices* $X$ *and* $Y$ *as follows:*

$$X = (W^\top + P)Z^{-1}(W+P), \tag{3.54}$$

$$Y = (W^\top + P)A_d, \tag{3.55}$$

*where* $W$ *is an appropriate matrix, the Proposition 3.3 applied to a time-delay reset control system is equivalent to the result in [37] as long as the reset control system is*

[5]The time-delay $h$ has been added to $\Gamma(h)$ as a parameter, in order to emphasize the dependence on the time-delay. In addition, it will allow to simplify the subsequent results.

*only affected by an output time-delay, since the structure of $A_d$ and $A_R$ for that case implies $A_d(A_R - I) = 0$, and thus, (3.48c) is directly satisfied.*

**Remark 3.5.** *The LK functional (3.47) has already been used in [236], where the stability of time-delay reset control systems with anticipative resetting law is analyzed. However, as it has been shown in the proof of the Proposition 3.3, it is necessary to impose a condition on the matrix $Y$, which has been omitted in [236]. It should be pointed out that there are several published works where the effect of the discontinuities into the derivative of the functional have not been taken into account. As a consequence, it is necessary extra assumptions for the strict correctness of the results.*

As mentioned before, the delay-partitioning approach provides less conservative results for the case of LTI time-delay systems. Likewise, we can expect that this approach will also improve the results for impulsive delay dynamical systems. In this case, we obtain a new stability criterion based on the LK functional devised in [114], and given by

$$V(\mathbf{x}_t) = \mathbf{x}^\top(t)P\mathbf{x}(t) + \int_{t-\frac{h}{r}}^{t} \mathbf{x}_e(\alpha)^\top Q\mathbf{x}_e(\alpha)d\alpha + \int_{-\frac{h}{r}}^{0}\int_{t+\beta}^{t} \dot{\mathbf{x}}^\top(\alpha)R\dot{\mathbf{x}}(\alpha)d\alpha d\beta, \quad (3.56)$$

$$\mathbf{x}_e(t) = \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{x}(t-\frac{1}{r}h) \\ \vdots \\ \mathbf{x}(t-\frac{r-1}{r}h) \end{bmatrix}. \quad (3.57)$$

The integer $r > 0$ sets the number of parts in which the interval $[t-h, t]$ is divided into. In the case of LTI time-delay systems, an increment of the parameter $r$ implies a reduction of the conservativeness, so that the maximum time-delay for which the stability of the system is guaranteed is closer to the actual value. The improvement of the results is due to the reduction of the gap in the Jensen's inequality by the reduction of the interval of integration (see [56] for a detailed analysis). The parameter $r$ plays the same role in the case of the IDDS, but the extra conditions added to deal with the reset actions make less obvious its effect.

**Proposition 3.4.** *The trivial solution of the system (3.33) is globally asymptotically stable, if there exists an integer $r > 0$ and matrices $P$, $R \in \mathbb{S}_+^n$ and $Q \in \mathbb{S}_+^{nr}$ such that*

$$\Gamma(h) = M_P^\top \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} M_P + M_R^\top \begin{bmatrix} \frac{h}{r}R & 0 \\ 0 & -\frac{r}{h}R \end{bmatrix} M_R + M_Q^\top \begin{bmatrix} Q & 0 \\ 0 & -Q \end{bmatrix} M_Q < 0, \tag{3.58a}$$

$$R(A_R - I) = 0, \tag{3.58b}$$

$$\Theta^\top (A_R^\top P A_R - P)\Theta \leq 0, \tag{3.58c}$$

*for some $\Theta \in \mathbb{R}^{n\times m}$, $m \geq 0$, such that* $\mathrm{proj}_{[1,n]}(\mathcal{Z}) \subseteq \mathcal{R}(\Theta)$, *where*

$$M_P = \left[\begin{array}{ccc} A & 0 & A_d \\ \hline I_n & & 0 \end{array}\right], \quad M_R = \left[\begin{array}{ccc} A & 0 & A_d \\ \hline -I_n & I_n & 0 \end{array}\right], \quad M_Q = \left[\begin{array}{cc} I_{rn} & 0 \\ \hline 0 & I_{rn} \end{array}\right]. \tag{3.59}$$

*Proof.* First, note that

$$\int_{t-\frac{h}{r}}^{t} \|\mathbf{x}_e(\alpha)\|^2 d\alpha = \int_{t-h}^{t} \|\mathbf{x}(\alpha)\|^2 d\alpha \tag{3.60}$$

then using similar arguments to the proof of Proposition 3.3 it can be easily proved that the LK functional (3.56) satisfies condition (3.9a). In addition, the upper right-hand derivative of the functional along the solutions of the system is given by

$$\begin{aligned} \dot{V}(\mathbf{x}_t) = {} & He(\dot{\mathbf{x}}(t)P\mathbf{x}(t)) + \mathbf{x}_e(t)^\top Q\mathbf{x}_e(t) - \mathbf{x}_e(t-\tfrac{h}{r})^\top Q\mathbf{x}_e(t-\tfrac{h}{r}) \\ & + \tfrac{h}{r}\dot{\mathbf{x}}(t)R\dot{\mathbf{x}}(t) - \int_{t-\frac{h}{r}}^{t} \dot{\mathbf{x}}^\top(\alpha)R\dot{\mathbf{x}}(\alpha)d\alpha \end{aligned} \tag{3.61}$$

Applying the Jensen's inequality and the fundamental theorem of calculus (Appendix C) to the integral term, it follows

$$\begin{aligned} \dot{V}(\mathbf{x}_t) \leq {} & He(\dot{\mathbf{x}}(t)P\mathbf{x}(t)) + \mathbf{x}_e(t)^\top Q\mathbf{x}_e(t) - \mathbf{x}_e(t-\tfrac{h}{r})^\top Q\mathbf{x}_e(t-\tfrac{h}{r}) + \frac{h}{r}\dot{\mathbf{x}}(t)R\dot{\mathbf{x}}(t) \\ & - \frac{r}{h}\Big(\mathbf{x}(t) - \mathbf{x}(t-\tfrac{h}{r}) - (A_R - I)\sum_{t_k\in[t-h,t)}\mathbf{x}(t_k)\Big)^\top R\Big(\mathbf{x}(t) - \mathbf{x}(t-\tfrac{h}{r}) - (A_R - I)\sum_{t_k\in[t-h,t)}\mathbf{x}(t_k)\Big). \end{aligned} \tag{3.62}$$

Condition (3.58b) guarantees that the term related to the jumps is canceled[6]. Hence, the above equation can be simply rewritten as follows:

$$\dot{V}(\mathbf{x}_t) \leq \mathbf{x}_e^\top(t)\Gamma(h)\mathbf{x}_e(t). \tag{3.63}$$

Following the procedure of the proof of Proposition 3.3, there exists a function $w$ such that condition (3.9b) is guaranteed. Finally, condition (3.58c) is the same as in Proposition 3.3, and thus, condition (3.9c) is satisfied.

□

**Example 3.4.** This example shows that condition (3.58b) of the above proposition is key for the correctness of the result. Consider a time-delay reset control system that consists of a pure integrator plant, $P(s) = 1/s$, controlled by a P+CI (proportional plus Clegg integrator) compensator with a zero crossing resetting law. The closed-loop matrices are given by

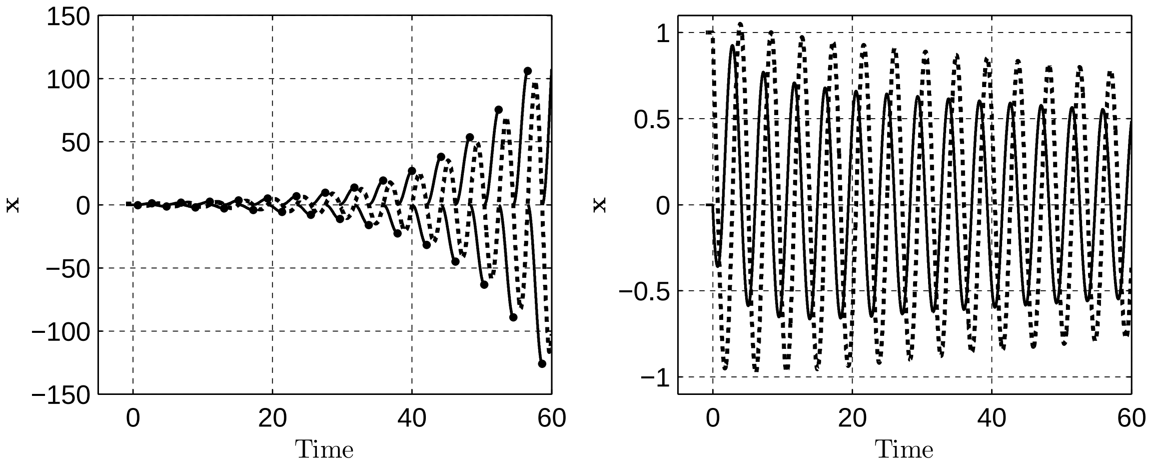
$$A = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, \qquad A_d = \begin{bmatrix} -1.4 & 0.3 \\ 0 & 0 \end{bmatrix}, \qquad A_R = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{3.64}$$

The base system has been proved to be stable for a time-delay $h = 1$ by using the results in [114]. The asymptotic stability of the time-delay reset control system is not guaranteed by Proposition 3.4. Now, let remove (3.58b) from Proposition 3.4, then the rest of LMI conditions hold. However, Fig. 3.2 shows the solution of the reset control system and its base system for a particular absolutely continuous initial condition, and it can be seen that the trajectory of the reset control system is divergent, while the trajectory of the base system converges to the origin. On the other hand, it should be emphasized that this example illustrates how the reset actions can destabilizes a stable base system, and thus the importance of adequate stability analysis of the reset control systems.

Finally, we will apply the approach of integral inequalities based on Legendre polynomials (see Appendix D), to obtain delay-dependent stability criterion. The result is derived by using the following LK functional (proposed in [252])

$$V(\mathbf{x}_t) = \mathbf{x}_N^\top(t)P_N\mathbf{x}_N(t) + \int_{t-h}^{t} \mathbf{x}^\top(\alpha)S\mathbf{x}(\alpha)d\alpha + h\int_{-h}^{0}\int_{t+\beta}^{t} \dot{\mathbf{x}}(\alpha)^\top R\dot{\mathbf{x}}(\alpha)d\alpha d\beta. \tag{3.65}$$

[6]Note that $(A_R - I)^\top R = 0$ is equivalent to $R(A_R - I) = 0$, since $R$ is symmetric. Hence, it is only necessary to require one of the equalities.




(a) Trajectory of the reset control system. (b) Trajectory of the base system.

Figure 3.2 Trajectory of the reset control system and its base system from Example 3.4, with initial condition $\phi(\theta) = (1, 0)$, $\theta \in [-1, 0]$. $x_p$ (dotted) and $x_r$ (solid).

This functional introduces the extra-states function

$$\mathbf{x}_N(t) = \begin{bmatrix} \mathbf{x}_t(0) \\ \int_{-h}^{0} L_0(\alpha)\mathbf{x}_t(\alpha)d\alpha \\ \vdots \\ \int_{-h}^{0} L_{N-1}(\alpha)\mathbf{x}_t(\alpha)d\alpha \end{bmatrix} \tag{3.66}$$

that represents the projection of the distributed state $\mathbf{x}_t$ on the set of Legendre polynomials of degree less than a prescribed upper-bound $N$. The main advantage of this approach is not the LK functional, but rather a set of integral inequalities obtained by the properties of the Legendre polynomials, that together with the functional (3.65) leads into a set of LMI criteria which are efficient in terms of conservatism and complexity. Before giving the stability criterion, it is necessary to introduce an integral inequality, which is a modification of the one proposed in [252] (see Appendix D ) to deal with the discontinuities of the solution of the IDDS (3.33).

**Corollary 3.1.** *Let $\psi \in \mathcal{PAC}([-h,0],\mathbb{R}^n)$ with $h>0$, and $\dot{\psi} \in \mathcal{PC}$ the upper right-hand derivative of $\psi$. The function $\psi$ is assumed to have $M \geq 0$ jump discontinuities at $(\theta_j)_{j=1}^{M}$ with $-h \leq \theta_1 < \theta_2 < \cdots < \theta_M < 0$. Then, for a given $R \in \mathbb{S}_+^n$ the integral inequality*

$$\int_{-h}^{0} \dot{\psi}^\top(\alpha) R \dot{\psi}(\alpha) d\alpha \geq \frac{1}{h}\xi^\top \left(\sum_{k=0}^{N}(2k+1)\Gamma_N^\top(k) R \Gamma_N(k)\right)\xi + \frac{1}{h}\sum_{k=0}^{N}(2k+1){\textstyle\sum_k}^\top R {\textstyle\sum_k} - He\left(\frac{1}{h}\xi^\top\left(\sum_{k=0}^{N}(2k+1)\Gamma_N^\top(k) R {\textstyle\sum_k}\right)\right) \tag{3.67}$$

*holds for all integer $N>0$, where*

$$\xi = \begin{bmatrix}\psi^\top(0) & \psi^\top(-h) & \frac{1}{h}\Omega_0^\top & \ldots & \frac{1}{h}\Omega_N^\top\end{bmatrix}^\top, \tag{3.68}$$

$$\Gamma_N(k) = \begin{bmatrix} I & (-1)^{k+1} I & \gamma_{Nk}^0 I & \ldots & \gamma_{Nk}^{N-1} I\end{bmatrix}, \tag{3.69}$$

$$\gamma_{Nk}^i = \begin{cases} -(2i+1)(1-(-1)^{k+i}), & i \leq k \\ 0, & i > k \end{cases}, \tag{3.70}$$

$${\textstyle\sum_k} = \sum_{j=1}^{M} L_k(\theta_j)(\psi(\theta_j^+) - \psi(\theta_j)). \tag{3.71}$$

*Proof.* The application of Lemma D.1 yields to the following inequality

$$\int_{-h}^{0} \dot{\psi}^\top(\alpha) R \dot{\psi}(\alpha) d\alpha \geq \frac{1}{h}\left(\sum_{k=0}^{N}(2k+1)\tilde{\Omega}_k^\top R \tilde{\Omega}_k\right), \tag{3.72}$$

where

$$\tilde{\Omega}_k = \int_{-h}^{0} L_k(\alpha)\dot{\psi}(\alpha) d\alpha \tag{3.73}$$

for all $k = 0, 1, \ldots, N$, and $L_k$ is the $k$-th Legendre polynomial. An integration by parts (see Appendix C) ensures that it is satisfied

$$\tilde{\Omega}_k = L_k(0)\psi(0) - L_k(-h)\psi(-h) - \sum_{k=j}^{M} L_k(\theta_j)(\psi(\theta_j^+) - \psi(\theta_j)) - \int_{-h}^{0} \dot{L}_k(\alpha)\psi(\alpha) d\alpha \tag{3.74}$$

for all $k \geq 0$. Applying properties (D.3) and (D.4) of the Legendre polynomials, the following expression is derived

$$\tilde{\Omega}_k = \psi(0) - (-1)^k \psi(-h) - \sum_{j=1}^{M} L_k(\theta_j)(\psi(\theta_j^+) - \psi(\theta_j)) + \sum_{i=0}^{k-1} \gamma_{Nk}^i \Omega_i = \Gamma_N(k)\xi - \Sigma_k. \tag{3.75}$$

Replacing $\tilde{\Omega}_k$ in (3.72) with the above expression, we get the inequality (3.67), which concludes the proof.

□

**Proposition 3.5.** *The trivial solution of the system (3.33) is globally asymptotically stable, if there exist an integer $N > 0$, a matrix $P_N \in \mathbb{S}^{(N+1)n}$, and matrices $S$, $R \in \mathbb{S}_+^n$ such that*

$$P_N + \frac{1}{h} diag(0_n, S_{N-1}) > 0, \tag{3.76a}$$

$$\Gamma(h) = \begin{bmatrix} \Phi_N(h) & \left(\Gamma_{N,N}^\top R_N - G_N(h)^\top P_N\right)(A_R - I)_N \\ \star & (A_R - I)_N R_N (A_R - I)_N \end{bmatrix} < 0, \tag{3.76b}$$

$$\Theta_N^\top \left(A_{R_N} P_N A_{R_N} - P_N\right) \Theta_N < 0, \tag{3.76c}$$

*for some $\Theta \in \mathbb{R}^{n \times m}$, $m \geq 0$, such that* $\text{proj}_{[1,n]}(\mathcal{Z}) \subseteq \mathcal{R}(\Theta)$, *where*

$$\Phi_N(h) = He\left(G_N^\top(h) P_N H_N\right) + \tilde{S}_N + h^2 F_N^\top R F_N - \Gamma_{N,N}^\top R_N \Gamma_{N,N}, \tag{3.77}$$

$$\tilde{S}_N = diag(S, -S, 0_{Nn}), \tag{3.78}$$

$$R_N = diag(R, 3R, \ldots, (2N+1)R), \tag{3.79}$$

$$S_N = diag(S, 3S, \ldots, (2N+1)S), \tag{3.80}$$

$$F_N = \begin{bmatrix} A & A_d & 0_{n,n(N+1)} \end{bmatrix}, \tag{3.81}$$

$$G_N(h) = \begin{bmatrix} I & 0_n & 0_{n,nN} \\ 0_{nN,n} & 0_{nN,n} & hI_{nN} \end{bmatrix}, \tag{3.82}$$

$$H_N = \begin{bmatrix} F_N^\top & \Gamma_N^\top(0) & \Gamma_N^\top(1) & \cdots & \Gamma_N^\top(N-1) \end{bmatrix}, \tag{3.83}$$

$$(A_R - I)_N = diag((A_R - I), (A_R - I), \ldots, (A_R - I)), \tag{3.84}$$

$$\Gamma_{N,N} = \begin{bmatrix} \Gamma_0 \\ \vdots \\ \Gamma_N \end{bmatrix}, \tag{3.85}$$

$$A_{R_N} = diag(A_R, I, \ldots, I), \tag{3.86}$$

$$\Theta_N = diag(\Theta, I, \ldots, I). \tag{3.87}$$

*Proof.* Consider the extra-states function $\mathbf{x}_N(t)$ given by (3.66), then applying an integration by parts and following the proof of Corollary (3.1), the upper right-hand derivative of the extra-states function can be expressed as follows:

$$\dot{\mathbf{x}}_N(t) = H_N \xi(t) - (A_R - I)_N \sum_{NN}, \tag{3.88}$$

where $H_N$ and $(A_R - I)_N$ are defined in the proposition. In addition, $\xi_N(t)$ and $\sum_{NN}$ are given by

$$\xi(t) = \begin{bmatrix} \mathbf{x}_t(0) \\ \mathbf{x}_t(-h) \\ \frac{1}{h}\int_{-h}^{0} L_0(\alpha)\mathbf{x}_t(\alpha)d\alpha \\ \vdots \\ \frac{1}{h}\int_{-h}^{0} L_{N-1}(\alpha)\mathbf{x}_t(\alpha)d\alpha \end{bmatrix}, \tag{3.89}$$

$$\sum_{NN} = \sum_{t_k \in [t-h,t)} \begin{bmatrix} 0 \\ L_0(t-t_k)\mathbf{x}_t(t-t_k) \\ \vdots \\ L_{N-1}(t-t_k)\mathbf{x}_t(t-t_k) \end{bmatrix}. \tag{3.90}$$

Following the proof provided in [252], it will be proved that the LK functional (3.65) satisfies condition (3.36a). First, since $S > 0$, the Lemma D.1 can be applied to the second term of the functional, resulting in

$$V(\mathbf{x}_t) \geq \mathbf{x}_N^\top(t) \left(P_N + \frac{1}{h} diag(0_n, S_{N-1})\right) \mathbf{x}_N(t) + h \int_{-h}^{0} \int_{\beta}^{0} \dot{\mathbf{x}}_t(\alpha)^\top R \dot{\mathbf{x}}_t(\alpha) d\alpha d\beta. \tag{3.91}$$

Then, condition (3.76a) guarantees

$$V(\mathbf{x}_t) \geq \lambda_m \left( P_N + \frac{1}{h} diag(0_n, S_{N-1}) \right) \|\mathbf{x}_t(0)\|^2 = u(\|\mathbf{x}_t(0)\|). \tag{3.92}$$

In addition, there exists a sufficiently large scalar $\lambda > 0$ such that

$$P_N < \lambda \mathrm{diag}(I, I, 3I, 5I, \ldots, (2N-1)I). \tag{3.93}$$

Thus, it holds

$$V(\mathbf{x}_t) \leq \lambda \|\mathbf{x}_t\|^2 + \lambda \sum_{i=0}^{N-1} (2i+1) \Omega_i^\top \Omega_i + \int_{-h}^{0} \mathbf{x}_t^\top(\alpha) S \mathbf{x}_t(\alpha) d\alpha + h \int_{-h}^{0} \int_{\beta}^{0} \dot{\mathbf{x}}_t(\alpha)^\top R \dot{\mathbf{x}}_t(\alpha) d\alpha d\beta \tag{3.94}$$

with $\Omega_i = \int_{-h}^{0} L_i(\alpha) \mathbf{x}_t(\alpha) d\alpha$. Applying Lemma D.1, it is obtained

$$V(\mathbf{x}_t) \leq \lambda \|\mathbf{x}_t\|^2 + \int_{-h}^{0} \mathbf{x}_t^\top(\alpha)(\lambda h I + S) \mathbf{x}_t(\alpha) d\alpha + h \int_{-h}^{0} \int_{\beta}^{0} \dot{\mathbf{x}}_t(\alpha)^\top R \dot{\mathbf{x}}_t(\alpha) d\alpha d\beta. \tag{3.95}$$

Since $(\lambda h I + S) > 0$ and $R > 0$, there exists a function $v \in \mathcal{K}$ such that $V(\mathbf{x}_t) \leq v(\|\mathbf{x}_t\|_W)$, and thus, condition (3.36a) is satisfied.

Consider now the derivative of the LK functional along the solutions of the system, given by

$$\begin{aligned} \dot{V}(\mathbf{x}_t) = & He\left(\mathbf{x}_N^\top(t) P_N \dot{\mathbf{x}}_N(t)\right) + \mathbf{x}_t^\top(0) S \mathbf{x}_t(0) - \mathbf{x}_t^\top(-h) S \mathbf{x}_t(-h) + h^2 \dot{\mathbf{x}}_t^\top(0) R \dot{\mathbf{x}}_t(0) \\ & - h \int_{-h}^{0} \dot{\mathbf{x}}_t^\top(\alpha) R \dot{\mathbf{x}}_t(\alpha) d\alpha. \end{aligned} \tag{3.96}$$

Note that $\mathbf{x}_N(t) = G_N(h)\xi(t)$, then equation (3.88) yields to

$$\begin{aligned} \dot{V}(\mathbf{x}_t) = & \xi^\top(t) \left( He\left(G_N^\top(h) P_N H_N\right) + \tilde{S}_N + h^2 F_N^\top R F_N \right) \xi(t) - h \int_{-h}^{0} \dot{\mathbf{x}}_t^\top(\alpha) R \dot{\mathbf{x}}_t(\alpha) d\alpha \\ & - He\left(\xi^\top(t) G_N(h) P_N (A_R - I)_N \Sigma_{NN}\right). \end{aligned} \tag{3.97}$$

Applying Corollary 3.1 to the above equation, it is obtained the following bound

$$\dot{V}(\mathbf{x}_t) \leq \begin{bmatrix} \xi(t) \\ \Sigma_{NN} \end{bmatrix}^\top \Gamma(h) \begin{bmatrix} \xi(t) \\ \Sigma_{NN} \end{bmatrix} \tag{3.98}$$

with $\Gamma(h)$ given in (3.76a). The existence of a function $w$, such that condition (3.9b) is satisfied, follows as in the proof of the previous delay-dependent criteria.

Finally, condition (3.9c) with the LK functional (3.65) is written as follows:

$$\Delta V(\mathbf{x}_t) = \xi^\top(t) \left(A_{R_N} P_N A_{R_N} - P_N\right) \xi(t) \leq 0, \tag{3.99}$$

which is rewritten as condition (3.76c), considering the matrix $\Theta_N$ and the arguments in the proof of Proposition 3.4.

□

The length of the time-delay has been assumed to be perfectly known in Propositions 3.3, 3.4, and 3.5, that is, these criteria only ensure the stability of the IDDS (3.33) for the pointwise time-delay $h$. Nevertheless, it could be of interest to known whether the IDDS is stable when the time-delay is unknown but belongs to a prescribed interval $h \in [h_m, h_M]$. Subsequently, Propositions 3.3, 3.4, and 3.5 are extended to consider the case of unknown time-delay. The results follow mainly from the application of convexity arguments with respect to the time-delay in the LMI conditions.

**Corollary 3.2.** *Suppose there exist matrices $P$, $Q$, $R \in \mathbb{S}^n_+$, $X \in \mathbb{S}^n$, and $Y \in \mathbb{R}^n$ such that the LMIs (3.48b), (3.48c), (3.48d), and*

$$\Gamma(h_m) < 0, \qquad \Gamma(h_M) < 0 \tag{3.100}$$

*hold, where $\Gamma(h)$ is defined in Proposition 3.3. Then the trivial solution of the system (3.33) is globally asymptotically stable for all constant time-delay $h \in [h_m, h_M]$.*

*Proof.* Note that (3.48b), (3.48c), and (3.48d) are independent of the time-delay, then the proof is directly obtained by the convexity of $\Gamma(h)$ with respect to the time-delay.

□

**Corollary 3.3.** *Suppose there exist matrices $P$, $R \in \mathbb{S}^n_+$, and $Q \in \mathbb{S}^{nr}_+$, such that the LMIs (3.58b), (3.58c), and*

$$\Gamma(h_m) + \frac{h_m - h_M}{h_m h_M}\Phi < 0, \qquad \Gamma(h_M) < 0 \tag{3.101}$$

*hold, where $\Gamma(h)$ is defined in Proposition 3.4 and $\Phi$ is given by*

$$\Phi = M_R^\top \begin{bmatrix} 0 & 0 \\ 0 & -rR \end{bmatrix} M_R. \tag{3.102}$$

*Then the trivial solution of the system (3.33) is globally asymptotically stable for all constant time-delay $h \in [h_m, h_M]$.*

*Proof.* First, note that conditions (3.58b) and (3.58c) are independent of the time-delay. Now, let $\Gamma(h)$ be written as follows:

$$\Gamma(h) = M_P^\top \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} M_P + M_R^\top \begin{bmatrix} \frac{h}{r}R & 0 \\ 0 & 0 \end{bmatrix} M_R + M_Q^\top \begin{bmatrix} Q & 0 \\ 0 & -Q \end{bmatrix} M_Q + \frac{1}{h}\Phi \quad (3.103)$$

The matrix $\Phi$ is negative semidefinite due to the structure of $M_R$ and the positive definiteness of $R$. Therefore, it is satisfied

$$\Gamma(h) \leq \Gamma(h) + \frac{h - h_M}{h_M h}\Phi. \quad (3.104)$$

Note that the term on the right-hand side is convex with respect to $h$. Therefore, it is only necessary to check the negative definiteness at the edges of the interval $[h_m, h_M]$, that results into the conditions (3.101).

□

**Corollary 3.4.** *Suppose there exists an integer $N > 0$, a matrix $P_N \in \mathbb{S}^{(N+1)n}$ and matrices $S$, $R \in \mathbb{S}_+^n$ such that the LMI (3.76c), and*

$$P_N + \frac{1}{h_M} diag(0, S_{N-1}) > 0, \quad (3.105)$$

$$\Gamma(h_m) < 0, \qquad \Gamma(h_M) < 0 \quad (3.106)$$

*hold, where $\Gamma(h)$ is defined in Proposition 3.5. Then the trivial solution of the system (3.33) is globally asymptotically stable for all constant time-delay $h \in [h_m, h_M]$.*

*Proof.* The proof is similar to the one provided in [252] for LTI time-delay systems. First, the positive definiteness of $S$ directly implies

$$P_N + \frac{1}{h} diag(0, S_{N-1}) > P_N + \frac{1}{h_M} diag(0, S_{N-1}) > 0. \quad (3.107)$$

Hence, condition (3.76a) is satisfied for the time-delay $h$.

On the other hand, the Schur complement is applied to the term $h^2 F_N^\top R F_N$ in $\Gamma(h)$ with the decomposition $(hF_N^\top R)R^{-1}(RF_N h)$, which leads to a convex LMI with respect to $h$. Therefore, the positive definiteness of $\Gamma(h_m)$ and $\Gamma(h_M)$ implies $\Gamma(h) < 0$, and thus, condition (3.76b) . The proof is complete by noting that condition (3.76c) is independent of the time-delay.

□

**Example 3.5.** Consider an autonomous time-delay reset control system (this example is taken from [37]) that is formed by a pure integrator plant $P(s) = 1/s$ and a PFORE (proportional plus first order reset element) reset compensator with a zero crossing resetting law, and the following base system

$$R_{base}(s) = \frac{s+2}{s+1}. \tag{3.108}$$

As a result, the closed-loop matrices are given by

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \quad A_d = \begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix}, \quad A_R = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{3.109}$$

It can be seen that the base system is stable for all constant time-delay $h \leq 0.87$. Table 3.1 shows the maximum allowable time-delay $h_M$ obtained by the proposed criteria for both input and output time-delay. The minimum $h_m$ in all cases is lower than $10^{-10}$. In the case of Corollary 3.2, the minimum time-delay is exactly $h_m = 0$. It is shown that the less conservative results, in terms of the maximum allowable time-delay, are those given by Corollary 3.4. The result in [37] is only applicable to the case of output time-delay. In that case, the maximum time-delay is the same as the maximum from Corollary 3.2, since the criterion in [37] is a particular case (see Remark 3.4).

| Method | Input time-delay | Output time-delay |
|---|---|---|
| Base system | 0.87 | 0.87 |
| Corollary. 3.2 | 0.442 | 0.675 |
| Corollary. 3.3 | 0.767 | 0.731 |
| Corollary. 3.4 | 0.816 | 0.803 |

Table 3.1 Maximum allowable input/output time-delay, $h_M$, for the Example 3.5 obtained by Corollaries 3.2, 3.3, and 3.4.

**Example 3.6.** This example, retrieved from [37], aims at illustrating that the stability results can be absolutely different for input and output time-delay. Consider the autonomous time-delay reset control system composed by a plant P given by

$$P(s) = \frac{s+1}{s(s+0.2)}, \tag{3.110}$$

and a FORE reset compensator (zero crossing resetting law), with base system

$$R_{base}(s) = \frac{1}{s+1}. \tag{3.111}$$

The closed-loop matrices are given by

$$A = \begin{bmatrix} -0.2 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{3.112}$$

In Table 3.2, it can be seen the results obtained from the application of Corollaries 3.2, 3.3, and 3.4 to the IDDS. The minimum $h_m$ in all the cases is lower than $10^{-10}$, and exactly zero in the case of Corollary 3.2. Again, the less conservative results are obtained by Corollary 3.4, that provides the same maximum time-delay (with a precision of 3 decimal digits) as the base system. However, the key point in this example is that all the criteria fail when the output time-delay is considered. Roughly speaking, this fact is mainly due to the matrix Θ, since the resetting set forces the matrix Θ to be the identity matrix, and as a result, conditions (3.48d), (3.58c), and (3.76c) are more restrictive.

| Method | Input time-delay | Output time-delay |
|---|---|---|
| Base system | 0.201 | 0.201 |
| Corollary 3.2 | 0.179 | - |
| Corollary 3.3 | 0.184 | - |
| Corollary 3.4 | 0.201 | - |

Table 3.2 Maximum allowable input/output time-delay, $h_M$, for the Example 3.6 obtained by Corollaries 3.2, 3.3, and 3.4.

## 3.3 Stabilization by Time-dependent Conditions

In the previous section, the (asymptotic) stability of an IDDS with state-dependent resetting law was analyzed. The unique temporal condition upon the reset instants is the time regularization. However, it does not play any role in the stability criteria. In this section, the IDDS (3.1) is provided with time-dependent condition, that is, besides the time regularization, additional constraints are imposed on the reset intervals. These constraints consist on a lower and an upper bound of the reset intervals. In this way, the IDDS is forced to evolve by the continuous dynamic at least during a prefixed amount of time (similar to the time regularization). In addition, if there is no reset action after another prefixed amount of time, then the IDDS is forced to perform a reset action. From this point of view the results in this section can be understood as criteria for the stabilization of the IDDS. A base system, potentially unstable, can be stabilized by adequate reset actions. However, another point of view is to consider a general IDDS with time/state-dependent resetting law such that its reset instants naturally satisfy the restrictions on the reset intervals. In this case, the results could be understood as an stability criteria. The problem is that, in general, it is not possible to know the sequence of reset instants for all the initial conditions[7].

The IDDS with time/state-dependent conditions under study in this section is defined by

$$\begin{cases} \left.\begin{array}{l} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}_t), \\ \dot{\Delta}(t) = 1, \end{array}\right\} & (t, \mathbf{x}_t, \Delta(t)) \notin \mathcal{S}, \\ \left.\begin{array}{l} \mathbf{x}(t^+) = \mathbf{I}(\mathbf{x}_t), \\ \Delta(t^+) = 0, \end{array}\right\} & (t, \mathbf{x}_t, \Delta(t)) \in \mathcal{S}, \\ \left.\begin{array}{l} \mathbf{x}(t) = \phi(t), \\ \Delta(t) = 0, \end{array}\right\} & t \in [-h, 0], \end{cases} \tag{3.113}$$

where function $\mathbf{f}$ is assumed to be Lipschitz continuous. In addition, for the sake of the simplicity, the system is assumed to be time-invariant, and thus, without loss of generality, the initial time is $t_0 = 0$. In this case, the resetting set is given by

$$\mathcal{S} = \{(t, \psi, \Delta) \in \mathbb{R}_+ \times \mathcal{PC}([-h, 0], \mathbb{R}^n) \times \mathbb{R}_+ : ((t, \psi) \in \mathcal{Z} \wedge \Delta \geq \Delta_m) \vee \Delta = \Delta_M\}, \tag{3.114}$$

[7] It is clearly possible in the case of a IDDS with fixed reset instants.

where $\Delta_m > 0$ and $\Delta_M \geq \Delta_m$ are the lower and the upper bound of the reset intervals, respectively. Note that the lower bound imposed for the stability results implies a time regularization. Therefore, the parameter $\Delta_m$, previously used for the time regularization, is now used for the stability analysis. Moreover, it should be point out that the upper bound of the reset intervals is not considered when the parameter $\Delta_M$ is set to infinity, that is condition $\Delta = \Delta_M$ is removed from the definition of the resetting set.

As in the previous section, the general results will be applied to the class of IDDS with LTI base systems. In these cases, the functions $\mathbf{f}$ and $\mathbf{I}$ of the IDDS (3.113) are particularized as follows: $\mathbf{f}(\mathbf{x}_t) = A\mathbf{x}(t) + A_d\mathbf{x}(t-h)$ and $\mathbf{I}(\mathbf{x}_t) = A_R\mathbf{x}(t)$ with $A$, $A_d$, $A_R \in \mathbb{R}^{n\times n}$. The reset actions occur when $(t, \mathbf{x}(t), \mathbf{x}(t-h), \Delta(t)) \in \mathcal{S}$, where the resetting set is given by

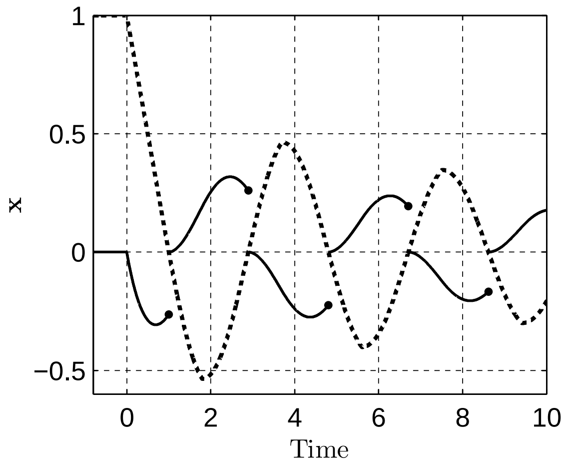
$$\mathcal{S} = \{(t, \mathbf{x}, \mathbf{y}, \Delta) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ : ((t, \mathbf{x}, \mathbf{y}) \in \mathcal{Z} \wedge \Delta \geq \Delta_m) \vee \Delta = \Delta_M\} \tag{3.115}$$

for some subset $\mathcal{Z} \subset \mathbb{R}^{2n}$.

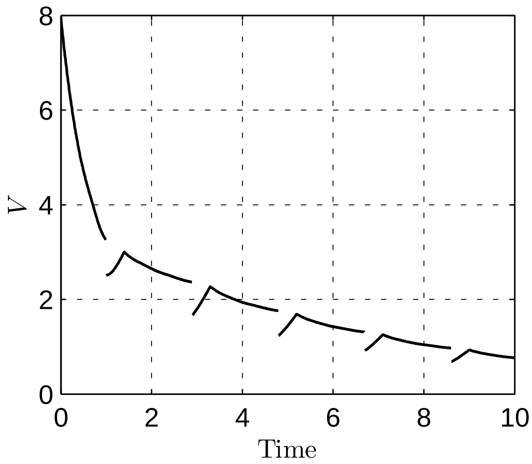
### 3.3.1 Stable Base System

The results in the previous section can be considered as a strict application of the direct Lyapunov (Krasovskii) method, in the sense that the LK functional is always forced to decrease (both in the continuous and the impulsive dynamic). However, a simple inspection of Propositions 3.3, 3.4, and 3.5 reveals that the reset actions have a potential effect on the derivative of the LK, that has been suppressed by restrictive conditions on the matrices of the functionals.

**Example 3.7.** Consider the time-delay reset control system of the Example 3.5 with input time-delay $h = 0.8$. The application of the Proposition 3.4 does not guarantee the asymptotic stability of the system for any value of $r$. Let apply again the Proposition 3.4 for instance with $r = 2$ but removing the condition $R(A_R - I) = 0$. In this case, the rest of the LMI conditions hold. However, the asymptotic stability is not guaranteed since all the conditions of the proposition are no fulfilled. In Fig. 3.3a, it can be seen the evolution of the system for the initial condition $\phi(\theta) = (1, 0)$, $\theta \in [-0.8, 0]$. Fig. 3.3b shows the value of the functional along the solution of the system. It can be seen that the functional always decreases except after a reset action, that increases during exactly $h/r = 0.4$ units of time. This fact is due to the effect of the reset actions on the derivative of the LK functional through the double integral term.




(a) Trajectory: $x_p$ (dotted) and $x_r$ (solid).




(b) LK functional.

Figure 3.3 Trajectory and value of the LK functional (matrices obtained without condition (3.58b)) for the time-delay reset control system of the Example 3.7 with initial condition $\phi(\theta) = (1, 0)$, $\theta \in [-0.8, 0]$.

As the above example suggests, we may obtain less conservative results by allowing bounded increments of the functional after the reset actions. The main idea is to move the classical condition of a strict decrement of the LK functional to a new functional $W$, which is composed by the LK functional $V$ and an auxiliary functional $\mathcal{V}$. The advantage is that the functional $\mathcal{V}$ is subjected to less restrictive conditions (mainly boundedness conditions), and that lightens the requirements imposed on the LK functional. In addition, the conditions on the functional $W$ will be imposed locally (an interval time after a reset action), reason why the shifted-distributed state representation, $\chi_k$, is more convenient. In the following, a formal development of this idea is accomplished, providing a stability criterion in the form of a Lypunov-Krasovskii-like proposition.

**Proposition 3.6.** *Suppose $u$, $v$, $w_1$, $w_2 : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous nondecreasing functions, $u$, $v \in \mathcal{K}$, and in addition, $\Delta_M = \infty$. If there exist a real number $\rho$ satisfying $0 < \rho < \Delta_m$, real numbers $\eta_0, \eta_1, \eta_2 \geq 0$, a functional $V : \mathcal{PAC}([-h,0],\mathbb{R}^n) \to \mathbb{R}_+$ that satisfies*

$$u(\|\psi(0))\| \leq V(\psi) \leq v(\|\psi\|_W) \tag{3.116a}$$

*and*

$$V(\mathbf{I}(\psi)) \leq \eta_0 V(\psi) \tag{3.116b}$$

*for any function $\psi \in \mathcal{PAC}([-h,0],\mathbb{R}^n)$ and there exists a continuous functional $\mathcal{V} : \mathbb{R}_+ \times \mathcal{PAC}([0,\rho] \times [-h,0],\mathbb{R}^n) \to \mathbb{R}$, which satisfies*

$$\mathcal{V}(\rho,\chi_k) - \mathcal{V}(0,\chi_k) = V(\chi_k(0^+,\cdot)) - V(\chi_k(0,\cdot)), \tag{3.116c}$$

$$\mathcal{V}(0,\chi_k) \leq \eta_1 V(\chi_k(0,\cdot)), \tag{3.116d}$$

$$-\eta_2 V(\chi_k(0,\cdot)) \leq \mathcal{V}(\tau,\chi_k) \tag{3.116e}$$

*for all $\tau \in [0,\rho]$ and $k > 0$. In addition, the following inequalities hold*

$$\dot{W}(\sigma,\chi_k) \leq -w_1(\|\chi_k(\sigma^+,0)\|), \quad \sigma \in [0,\rho], \quad k > 0, \tag{3.116f}$$

$$\dot{V}(\chi_k(\sigma,\cdot)) \leq -w_2(\|\chi_k(\sigma,0)\|), \quad \sigma \in [\rho,\Delta_k), \quad k \geq 0 \tag{3.116g}$$

*with*

$$W(\sigma,\chi_k) = V(\chi_k(\sigma^+,\cdot)) + \mathcal{V}(\sigma,\chi_k), \tag{3.117}$$

*where $\dot{W}$ and $\dot{V}$ are evaluated along the solutions of the system (3.113), then the trivial solution of the system is stable. In addition, if $\lim_{s\to\infty} u(s) = \infty$ and $w(s) > 0$ for $s > 0$ then the trivial solution is globally asymptotically stable.*

*Proof.* First, let define $\eta = \max(1, \eta_0 + \eta_1 + \eta_2)$. Integrating (3.116f) with respect to $\sigma$ over $[0,\tau]$, the following inequality is obtained

$$V(\chi_k(\tau^+,\cdot)) - V(\chi_k(0^+,\cdot)) + \mathcal{V}(\tau,\chi_k) - \mathcal{V}(0,\chi_k) \leq 0 \tag{3.118}$$

for all $\tau \in [0,\rho]$ and $k > 0$. Inequalities (3.116b), (3.116d) and (3.116e) yield to

$$V(\chi_k(\tau^+,\cdot)) \leq \eta V(\chi_k(0,\cdot)). \tag{3.119}$$

In addition, by condition (3.116c) it follows

$$V(\chi_k(\rho,\cdot)) \leq V(\chi_k(0,\cdot)). \tag{3.120}$$

On the other hand, for simplicity it is assumed without loss of generality that there exists an infinite number of reset actions, then integration (3.116g) with respect to $\sigma$ over $[\tau, \Delta_k]$ leads to

$$V(\chi_k(\tau,\cdot)) \leq V(\chi_k(\rho,\cdot)) \tag{3.121}$$

for all $\tau \in [\rho, \Delta_k]$ and $k \geq 0$. If the sequence of reset instants is finite, then there exists $N > 0$ such that $\Delta_N = \infty$, and thus, condition (3.121) is retrieved with $\tau \in [\rho, \Delta_N)$.

Consider equations (3.119) and (3.120) then

$$V(\chi_k(\tau,\cdot)) \leq \eta V(\chi_k(0,\cdot)) \tag{3.122}$$

for all $\tau \in [0, \Delta_k]$, and

$$V(\chi_{k+1}(0,\cdot)) \leq V(\chi_k(0,\cdot)) \tag{3.123}$$

with $k > 0$. Considering all the previous reset instants, the following inequality is obtained

$$V(\chi_k(\tau,\cdot)) \leq \eta V(\chi_1(0,\cdot)) \tag{3.124}$$

for $k = 2, 3, \ldots$ and $\tau \in [0, \Delta_k]$. In addition, the Lipschitz continuity of $\mathbf{f}$ and $v \in \mathcal{K}$ imply that there exists $\mu = \mu(K_L\rho)$ such that it is satisfied

$$V(\chi_0(\tau,\cdot)) \leq \mu v(\|\phi\|_W), \tag{3.125}$$

where $\tau \in [0, \Delta_0]$ and $K_L$ is the Lipschitz constant of the function $\mathbf{f}$ (see for instance [258]). Inequalities (3.116a), (3.124), (3.125), and $V(\chi_1(0,\cdot)) = V(\chi_0(\Delta_0,\cdot))$ imply

$$u(\|\chi_k(\tau,0)\|) \leq V(\chi_k(\tau,\cdot)) \leq \eta V(\chi_1(0,\cdot)) \leq \eta\mu v(\|\phi\|_W). \tag{3.126}$$

For a given $\epsilon > 0$, there exists a $\delta = \delta(\epsilon)$, $0 < \delta < \epsilon$, such that $\eta\mu v(\delta) < u(\epsilon)$. Therefore, for all $\|\phi\|_W < \delta$ it is satisfied

$$u(\|\chi_k(\tau,0)\|) < \eta\mu v(\delta) < u(\epsilon) \tag{3.127}$$

for all $\tau \in [0, \Delta_k]$, and $k = 0, 1, \ldots$. The above condition implies $\|\mathbf{x}(t)\| < \epsilon$, $t \geq 0$, and thus the trivial solution of the system (2.13) is stable.

In order to prove asymptotic stability, for $\epsilon > 0$ choose $\delta_a$ as the constant for uniform stability, then it is true that $\|\phi\|_W < \delta_a$ implies $\|\mathbf{x}(t)\| < \epsilon$ for $t \geq 0$. Therefore, it has to be shown that for any $\eta > 0$ there exists some $T = T(\delta_a, \eta)$ such that $\|\mathbf{x}(t)\| < \eta$ for $t \geq T$. Let $\delta_b = \delta(\eta)$ be the constant for uniform stability, then it is sufficient to prove that $\|\mathbf{x}_t\| < \delta_b$ for $t \geq T$. By contradiction, suppose that the solution $\mathbf{x}(t, \phi)$ with $\|\phi\|_W < \delta_a$ satisfies $\|\mathbf{x}_t\|_W \geq \delta_b$ for all $t \geq 0$. Thus, since $\mathbf{f}$ is Lipschitz continuous, there exist $\delta_c > 0$ and a sequence $(\tau_j)$, $j = 1, 2, \ldots$ such that

$$(2j-1)2h \leq \tau_j \leq 4jh, \tag{3.128}$$

where $\tau_j \notin \mathbb{T}_\phi$ and $\|\mathbf{x}(\tau_j)\| \geq \delta_c$. Since $t_{k+1} - t_k \geq \Delta_m$ for $t_k \in \mathbb{T}_\phi$, $k = 1, 2, \ldots$, and in addition, by the stability and the assumption on $\mathbf{f}$, there exists a constant $L > 0$ such that $\|\dot{\mathbf{x}}(t)\| < L$, for any $t \in [0, \infty) \setminus \mathbb{T}_\phi$, then it is possible to build a set of intervals $I_j = [\tau_j - \frac{\delta_c}{2L}\check{\alpha}_j, \tau_j + \frac{\delta_c}{2L}\hat{\alpha}_j]$ with $\check{\alpha}_j$, $\hat{\alpha}_j \in \{0, 1\}$, $\check{\alpha}_j + \hat{\alpha}_j > 0$, that do not contain reset instants and do not overlap (by using $L$ large enough), that is $I_j \cap \mathbb{T}_\phi = \emptyset$, $j = 1, 2, \ldots$.

On the other hand, the reset instants in the sequence $\mathcal{T}_\phi = \{t_1, t_2, \cdots\}$ may be renamed as $\mathbb{T}_\phi = \{t_{j,l}\}$, where the reset instant $t_{j,l}$ corresponds to the $l^{th}$-reset instant prior to $\tau_j \in I_j$, that is

$$0 < t_{1,1} < t_{1,2} < \cdots t_{1,N_1} < \check{\tau}_1 = \tau_1 - \tfrac{\delta_c}{2L}\check{\alpha}_1 < \hat{\tau}_1 = \tau_1 + \tfrac{\delta_c}{2L}\hat{\alpha}_1 < t_{2,1} < \cdots t_{2,N_2} < \check{\tau}_2 \cdots \tag{3.129}$$

for some integers $N_j \geq 0$. In addition, we define (see Fig. 3.4)

$$r(j) = \max_{i \leq j, N_i > 0} i, \qquad R(j) = \sum_{i=1}^{i \leq j} N_j$$
$$\Lambda_j = \tau_j - t_{r(j),N_{r(j)}}, \qquad \check{\Lambda}_j = \check{\tau}_j - t_{r(j),N_{r(j)}}, \qquad \hat{\Lambda}_j = \hat{\tau}_j - t_{r(j),N_{r(j)}}. \tag{3.130}$$

Then by using the mean-value theorem on the intervals $[\tau_j, t] \subset I_j$ and $[t, \tau_j] \subset I_j$

$$\|\mathbf{x}(t)\| = \|\mathbf{x}(\tau_j) + \dot{\mathbf{x}}(\tau_j + \theta_1(t - \tau_j))(t - \tau_j)\|, \tag{3.131}$$

$$\|\mathbf{x}(t)\| = \|\mathbf{x}(\tau_j) - \dot{\mathbf{x}}(\tau_j + \theta_2(\tau_j - t))(\tau_j - t)\| \tag{3.132}$$

for some $\theta_1, \theta_2 \in (0, 1)$, and then

$$\|\mathbf{x}(t)\| \geq \|\mathbf{x}(\tau_j)\| - \|\dot{\mathbf{x}}(\tau_j + \theta(t - \tau_j))\||t - \tau_j| \geq \delta_c - L\frac{\delta_c}{2L} = \frac{\delta_c}{2} \tag{3.133}$$

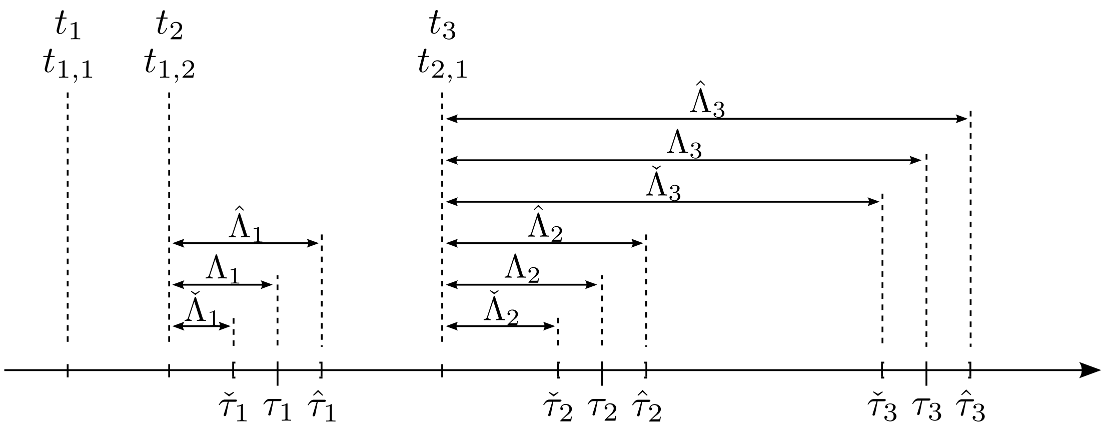



Figure 3.4 Notation example: $r(1) = 1$, $r(2) = 2$, $r(3) = 2$, $N_1 = 2$, $N_2 = 1$, $N_3 = 0$, $R(1) = 2$, $R(2) = 3$, $R(3) = 3$.

for some $\theta \in (-1, 1)$ and any $t \in I_j$. Therefore, from (3.116f) and (3.116g) it follows

$$\dot{W}(\sigma, \chi_{R(j)}) \leq -w_1(\|\chi_{R(j)}(\sigma^+, 0)\|) \leq -w_1\left(\frac{\delta_c}{2}\right) < 0, \quad \hat{\Lambda}_j < \rho \tag{3.134}$$

or

$$\dot{V}(\chi_{R(j)}(\sigma, \cdot)) \leq -w_2(\| \chi_{R(j)}(\sigma, 0)\|) \leq -w_2\left(\frac{\delta_c}{2}\right) < 0, \quad \check{\Lambda}_j > \rho \tag{3.135}$$

for all $\sigma \in [\check{\Lambda}_j, \hat{\Lambda}_j]$.

Let the integer $i$ be the smallest value such that $N_i > 0$, and consider the following two cases:

*Case 1:* Suppose $\hat{\Lambda}_i < \rho$ , then integrating $\dot{W}(\sigma, \chi_{R(i)})$ over the interval $[\check{\Lambda}_i, \hat{\Lambda}_i]$, it is obtained

$$W(\hat{\Lambda}_i, \chi_{R(i)}) - W(\check{\Lambda}_i, \chi_{R(i)}) \leq -w_1\left(\frac{\delta_c}{2}\right)\frac{\delta_c}{2L} \tag{3.136}$$

From (3.116f), $W$ is nonincreasing, hence it follows

$$W(\Delta_{R(i)}, \chi_{R(i)}) - W(0, \chi_{R(i)}) \leq -w_1\left(\frac{\delta_c}{2}\right)\frac{\delta_c}{2L}. \tag{3.137}$$

Applying condition (3.116c) to the above equation, it is obtained

$$V(\chi_{R(i)}(\rho, \cdot)) \leq V(\chi_{R(i)}(0, \cdot)) - w_1\left(\frac{\delta_c}{2}\right)\frac{\delta_c}{2L}. \tag{3.138}$$

*Case 2:* Suppose $\check{\Lambda}_i > \rho$, then integrating $\dot{V}(\chi_{R(i)}(\sigma, \cdot))$ respect to $\sigma$ over the interval $[\check{\Lambda}_i, \hat{\Lambda}_i]$, it is obtained

$$V(\chi_{R(i)}(\hat{\Lambda}_i, \cdot)) - V(\chi_{R(i)}(\check{\Lambda}_i, \cdot)) \leq -w_2 \left(\frac{\delta_c}{2}\right) \frac{\delta_c}{2L}. \tag{3.139}$$

Since $V$ is nonincreasing by (3.116g), then

$$V(\chi_{R(i)}(\Delta_{R(i)}, \cdot)) \leq V(\chi_{R(i)}(\rho, \cdot)) - w_2 \left(\frac{\delta_c}{2}\right) \frac{\delta_c}{2L}. \tag{3.140}$$

Now, assume again that there exists an infinite sequence of reset instants and consider all the intervals $I_j$, then it is satisfied

$$V(\chi_{R(j)}(\Delta_{R(j)}, \cdot)) \leq V(\chi_{R(i)}(0, \cdot)) - (j - i + 1)\frac{\delta_c}{2L}\omega, \tag{3.141}$$

where $\omega = \min(w_1(\frac{\delta_c}{2}), w_2(\frac{\delta_c}{2}))$. Hence, for a large enough $j > i$, it is obtained $V(\chi_{R(j)}(\Delta_{R(j)}, \cdot)) < 0$ which is a contradiction. If the sequence of reset instants is finite the same contradiction is derived, taking into account only the Case 2 after the last reset instant. Therefore, the original assumption is false, and there exists $T$ such that $\|\mathbf{x}_t\|_W < \delta_b$ for all $t \geq T$. Finally, since $\lim_{s\to\infty} u(s) = \infty$, then $\delta_a$ above may be chosen arbitrarily large, and $\epsilon$ can be set after $\delta_a$ to satisfy $v(\delta_a) < u(\epsilon)$. Thus, the trivial solution is globally asymptotically stable.

□

**Remark 3.6.** *Proposition 3.6 is strongly based on the results for sampled-data systems of the works [250] and [249], which are an extension of the results in [103, 204]. In the first case, the author developed stability criterion by the addition of an auxiliary functional $\mathcal{V}$ to a Lyapunov function $V$ and imposing a negativity condition on the derivative of the mixture of both. This yields into less conservative constraints on the Lyapunov function $V$, that only has to decrease at the sampled-instants. In the second work, the author applied the same method to Networked Control Systems (NCS), where a LK functional is used, instead of a Lyapunov function.*

*In spite of the fact that a similar approach has been applied to sampled-data systems, these results are not directly applicable to IDDSs, where the problem is much more challenging. The main reason is that the solution of the system is piecewise-continuous which usually makes the LK functional also discontinuous. In addition, in comparison*

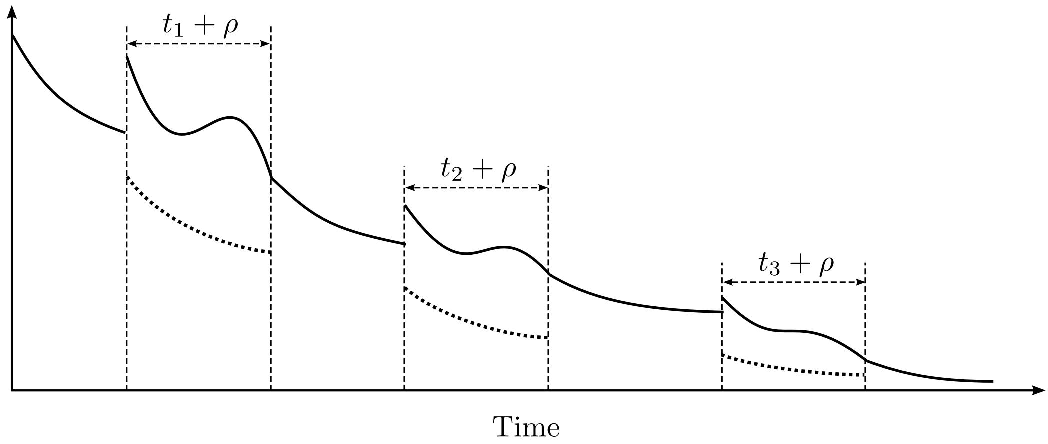



Figure 3.5 Illustration of the evolution of a LK functional satisfying the conditions of Proposition 3.6. $V$ (solid) and $W$ (dotted).

*with [250], here the time-delay affects the continuous dynamic of the system while in [250], the time-delay only affects the sampling.*

The main idea in Proposition 3.6 is depicted in Fig. 3.5. First, the LK functional is not forced to decrease due to the impulsive actions, since $\eta_0$ in the condition (3.116b) can be greater than one. In addition, the LK functional is allowed to increase during $\rho$ units time after a reset event, whenever its value at the instant $t_k + \rho$ is strictly lower than the value at $t_k$ (before the jump). This will allow a less restrictive treatment of the effect of the impulse actions into the derivative of the LK functional. Finally, it should be pointed out that, despite the modifications, the stability of the base system is still required by condition (3.116g).

### IDDS with a LTI Base System

This section is focused on the application of the Proposition 3.6 to develop stability criterion for the IDDS (3.113) with LTI base system. In Section 3.2.2, several LK functionals have been resulted useful to obtain LMI conditions. The Examples 3.5 and 3.6 showed that the Legendre polynomials approach and its corresponding LK functional provide the less conservative results. However, the delay-partitioning approach and the LK functional (3.56) are used in this section. The main reason of this choice will be exposed after the results for a better understanding.

The LK functional (3.56) is reformulated with the notation of the shifted-distributed state representation as follows:

$$\begin{aligned} V(\chi_k(\tau,\cdot)) &= \chi_k^\top(\tau,0)P\chi_k(\tau,0) + \int_{-\frac{h}{r}}^{0} \chi_k^{e\top}(\tau,\alpha)Q\chi_k^e(\tau,\alpha)d\alpha \\ &\quad + \int_{-\frac{h}{r}}^{0}\int_{\beta}^{0} \dot{\chi}_k^\top(\tau^+,\alpha)R\dot{\chi}_k(\tau^+,\alpha)d\alpha d\beta, \end{aligned} \tag{3.142}$$

where $\dot{\chi}_k(\tau^+,\theta)$ is the upper right-hand derivative of $\mathbf{x}(t_k+\tau+\theta)$, and

$$\chi_k^e(\tau,\theta) = \begin{bmatrix} \chi_k(\tau,\theta) \\ \chi_k(\tau,\theta-\frac{1}{r}h) \\ \vdots \\ \chi_k(\tau,\theta-\frac{r-1}{r}h) \end{bmatrix}. \tag{3.143}$$

**Proposition 3.7.** *The trivial solution of the system (3.113) with LTI base system, $\Delta_M=\infty$, and $\Delta_m > \frac{h}{r}$ for a given integer $r \geq 1$, is globally asymptotically stable, if there exist matrices $P$, $R$, $U$, $S_1$, $X_2 \in \mathbb{S}_+^n$, matrix $Q \in \mathbb{S}_+^{nr}$, matrix $X_1 \in \mathbb{S}^n$, matrix $S_2 \in \mathbb{R}^n$, and matrix $Y \in \mathbb{R}^{(r+2)n\times n}$ such that the LMIs*

$$\Pi_0 < 0, \tag{3.144a}$$

$$\begin{bmatrix} S_1 & S_2 \\ S_2^\top & \frac{r}{h}X_2 \end{bmatrix} \geq 0, \tag{3.144b}$$

$$\Theta^\top\left(M_0^\top\Pi_0M_0 + \Pi_1 + N_R + \frac{h}{r}(\Pi_2+\Pi_3) + \frac{r}{h}\Pi_4\right)\Theta < 0, \tag{3.144c}$$

$$\begin{bmatrix} \Theta^\top\left(M_0^\top\Pi_0M_0 + \Pi_1 + N_R - \frac{h}{r}\Pi_3 + \frac{r}{h}\Pi_4\right)\Theta & \frac{h}{r}\Theta^\top Y \\ \star & -\frac{h}{r}U \end{bmatrix} < 0, \tag{3.144d}$$

*where*

$$\Pi_0 = M_P^\top\begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix}M_P + M_R^\top\begin{bmatrix} \frac{h}{r}R & 0 \\ 0 & -\frac{r}{h}R \end{bmatrix}M_R + M_Q^\top\begin{bmatrix} Q & 0 \\ 0 & -Q \end{bmatrix}M_Q, \tag{3.145}$$

$$\Pi_1 = -N_{13}^\top S_1N_{13} - He(N_{13}^\top S_2N_3) + He(YN_{13}), \tag{3.146}$$

$$\Pi_2 = He(N_0^\top S_1 N_{13}) + He(N_0^\top S_2 N_3) + N_0^\top U N_0, \tag{3.147}$$

$$\Pi_3 = N_3^\top X_1 N_3, \tag{3.148}$$

$$\Pi_4 = N_3^\top (A_R^\top P A_R - P) N_3, \tag{3.149}$$

$$\begin{aligned}
&M_0 = \begin{bmatrix} I_{(r+1)n} & 0_{(r+1)n\times n} \end{bmatrix}, && N_0 = \begin{bmatrix} A & 0_{n,n(r-1)} & A_d & 0_n \end{bmatrix},\\
&M_P = \left[\begin{array}{ccc} A & 0_{n,(r-1)n} & A_d \\ \hline I_n & \multicolumn{2}{c}{0_{n,rn}} \end{array}\right], && N_1 = \begin{bmatrix} I_n & 0_{n,n(r+1)} \end{bmatrix},\\
&M_R = \left[\begin{array}{ccc} A & 0_{n,(r-1)n} & A_d \\ \hline -I_n & I_n & 0_{n,(r-1)n} \end{array}\right], && N_2 = \begin{bmatrix} 0_n & I_n & 0_{n,nr} \end{bmatrix},\\
&M_Q = \left[\begin{array}{cc} I_{rn} & 0_{rn\times n} \\ \hline 0_{rn\times n} & I_{rn} \end{array}\right], && N_3 = \begin{bmatrix} 0_{n,n(r+1)} & I_n \end{bmatrix},\\
&N_{13} = N_1 - A_R N_3, && N_4 = (A_R - I) N_3,
\end{aligned}$$
$$N_R = \tfrac{r}{h} \begin{bmatrix} N_1 \\ N_2 \\ N_4 \end{bmatrix}^\top \begin{bmatrix} 0 & 0 & R \\ 0 & 0 & -R \\ R & -R & -R \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ N_4 \end{bmatrix}. \tag{3.150}$$

*Proof.* As it has been proved in the proof of Proposition 3.4, the LK functional (3.142) satisfies condition (3.116a). In addition, for any function $\chi_k$ the following equation is obtained

$$\begin{aligned}
V(\mathbf{I}(\chi_k(\tau,\cdot))) - \eta_0 V(\chi_k(\tau,\cdot)) &= \chi_k^\top(\tau,0)(A_R^\top P A_R - \eta_0 P)\chi_k(\tau,0) \\
&+ (1-\eta_0)\left(\int_{-\frac{h}{r}}^{0} \chi_k^{e\top}(\tau,\alpha) Q \chi_k^e(\tau,\alpha) d\alpha\right) \\
&+ (1-\eta_0)\left(\int_{-\frac{h}{r}}^{0}\int_{\beta}^{0} \dot{\chi}_k^\top(\tau^+,\alpha) R \dot{\chi}_k(\tau^+,\alpha) d\alpha d\beta\right).
\end{aligned} \tag{3.151}$$

Hence, for given matrices $P > 0$, $Q > 0$, and $R > 0$, it is always possible to find $\eta_0 \geq 1$ such that condition (3.116b) is satisfied.

On the other hand, the upper right-hand derivative of the functional along the solutions of the system is given by

$$\begin{aligned}\dot{V}(\chi_k(\tau^+,\cdot)) = {} & He(\dot{\chi}_k^\top(\tau^+,0)P\chi_k(\tau^+,0)) + \chi_k^{e\top}(\tau^+,0)Q\chi_k^e(\tau^+,0) \\ & - \chi_k^{e\top}(\tau^+,-\tfrac{h}{r})Q\chi_k^e(\tau^+,-\tfrac{h}{r}) + \tfrac{h}{r}\dot{\chi}_k^\top(\tau^+,0)R\dot{\chi}_k(\tau^+,0) \\ & - \int_{-\frac{h}{r}}^{0} \dot{\chi}_k^\top(\tau^+,\alpha)R\dot{\chi}_k^\top(\tau^+,\alpha)d\alpha.\end{aligned} \tag{3.152}$$

Since $\Delta_k \geq \Delta_m > \frac{h}{r}$, the Jensen's inequality yields to

$$\begin{aligned}& - \int_{-\frac{h}{r}}^{0} \dot{\chi}_k^\top(\tau^+,\alpha)R\dot{\chi}_k(\tau^+,\alpha)d\alpha \leq -\tfrac{r}{h}\Big(\chi_k(\tau^+,0) - \chi_k(\tau^+,-\tfrac{h}{r}) \\ & - \nu(\tau)(A_R - I)\chi_k(0,0)\Big)^\top R\Big(\chi_k(\tau^+,0) - \chi_k(\tau^+,-\tfrac{h}{r}) - \nu(\tau)(A_R - I)\chi_k(0,0)\Big),\end{aligned} \tag{3.153}$$

where

$$\nu(\tau) = \begin{cases} 1, & 0 \leq \tau < \frac{h}{r}, \\ 0, & \frac{h}{r} \leq \tau \leq \Delta_k. \end{cases} \tag{3.154}$$

Replacing (3.153) in (3.152), a bound of the derivative is given by

$$\begin{aligned}\dot{V}(\chi_k(\tau^+,\cdot)) \leq {} & He(\dot{\chi}_k^\top(\tau^+,0)P\chi_k(\tau^+,0)) + \chi_k^{e\top}(\tau^+,0)Q\chi_k^e(\tau^+,0) \\ & - \chi_k^{e\top}(\tau^+,-\tfrac{h}{r})Q\chi_k^e(\tau^+,-\tfrac{h}{r}) + \tfrac{h}{r}\dot{\chi}_k^\top(\tau^+,0)R\dot{\chi}_k(\tau^+,0) \\ & - \tfrac{r}{h}\Big(\chi_k(\tau^+,0) - \chi_k(\tau^+,-\tfrac{h}{r}) - \nu(\tau)(A_R - I)\chi_k(0,0)\Big)^\top R \\ & \Big(\chi_k(\tau^+,0) - \chi_k(\tau^+,-\tfrac{h}{r}) - \nu(\tau)(A_R - I)\chi_k(0,0)\Big).\end{aligned} \tag{3.155}$$

Let $\rho$ in the Proposition 3.6 be given by $\rho = \frac{h}{r}$, then $v(\tau) = 0$ for $\tau \in [\frac{h}{r}, \Delta_k]$. Hence, $\Pi_0 < 0$ guarantees that there exists a function $w_2$ such that condition (3.116g) is guaranteed (the proof follows as in the proof of Proposition 3.4). Now consider the following functional

$$\begin{aligned}\mathcal{V}(\tau,\chi_k) = {} & \tau\tfrac{r}{h}\chi_k^\top(0,0)A_R^\top PA_R\chi_k(0,0) + (1 - \frac{r}{h}\tau)\chi_k^\top(0,0)P\chi_k(0,0) \\ & + (\tfrac{h}{r} - \tau)\left(\zeta_k^\top(\tau)S_1\zeta_k(\tau) + He(\zeta_k^\top(\tau)S_2\chi_k(0,0))\right) \\ & + (\tfrac{h}{r} - \tau)\int_0^\tau \dot{\chi}_k^\top(s^+,0)U\dot{\chi}_k(s^+,0)ds \\ & + \tau(\tfrac{h}{r} - \tau)\chi_k^\top(0,0)X_1\chi_k(0,0) + \chi_k^\top(0,0)X_2\chi_k(0,0),\end{aligned} \tag{3.156}$$

where $\zeta_k(\tau) = \chi_k(\tau^+, 0) - \chi_k(0^+, 0)$ and $\tau \in [0, \frac{h}{r}]$. Taking the above functional with $\tau = 0$ and $\tau = \frac{h}{r}$, it is obtained

$$\mathcal{V}(\tfrac{h}{r}, \chi_k) - \mathcal{V}(0, \chi_k) = \chi_k^\top(0,0)(A_R P A_R - P)\chi_k(0,0) = \Delta, V(\chi_k(0,\cdot)) \tag{3.157}$$

and thus, condition (3.116c) is satisfied. Moreover, from the functional $V$ and $\mathcal{V}$ it follows

$$\lambda_m(P)\|\chi_k(0,0)\|^2 \le \chi_k^\top(0,0)P\chi_k(0,0) \le V(\chi_k(0,\cdot)) \tag{3.158}$$

and

$$\mathcal{V}(0, \chi_k) = \chi_k^\top(0,0)(P + X_2)\chi_k(0,0) \le \lambda_M(P + X_2)\|\chi_k(0,0)\|^2. \tag{3.159}$$

Therefore, by choosing $\eta_1 = \frac{\lambda_M(P+X_2)}{\lambda_m(P)}$, condition (3.116d) is satisfied. Consider now the following terms of $\mathcal{V}$

$$(\tfrac{h}{r} - \tau)\left(\zeta_k^\top(\tau)S_1\zeta_k(\tau) + He(\zeta_k^\top(\tau)S_2\chi_k(0,0))\right) + \chi_k^\top(0,0)X_2\chi_k(0,0). \tag{3.160}$$

These terms can be rewritten in a matrix form as follows:

$$\begin{bmatrix} \zeta_k^\top(\tau) \\ \chi_k^\top(0,0) \end{bmatrix}^\top \begin{bmatrix} (\frac{h}{r} - \tau)S_1 & (\frac{h}{r} - \tau)S_2 \\ (\frac{h}{r} - \tau)S_2^\top & X_2 \end{bmatrix} \begin{bmatrix} \zeta_k(\tau) \\ \chi_k(0,0) \end{bmatrix}. \tag{3.161}$$

Note that if the above matrix is positive semidefinite then (3.160) is equal or greater than zero. In addition, it is convex with respect to $\tau$, and thus, we only must assure the positive semidefiniteness at the edges of the interval $[0, \frac{h}{r}]$. Consider the two possible cases for the matrix in (3.161): first, $\tau = \frac{h}{r}$ then $X_2 > 0$ guarantees that the matrix is positive semidefinite, second, if $\tau = 0$ then the positive semidefiniteness of the matrix is assured by condition (3.144b). In addition, $U > 0$ implies that the integral term in $\mathcal{V}$ is positive definite. Therefore, from the above arguments it is obtained

$$\begin{aligned} -\mathcal{V}(\tau, \chi_k) \le {} & -\tau\tfrac{r}{h}\chi_k^\top(0,0)A_R P A_R\chi_k(0,0) - (1 - \tfrac{h}{r}\tau)\chi_k^\top(0,0)P\chi_k(0,0) \\ & - \tau(\tfrac{h}{r} - \tau)\chi_k^\top(0,0)X_1\chi_k(0,0). \end{aligned} \tag{3.162}$$

Considering the maximums of the following functions: $\tau\frac{r}{h}$, $1 - \frac{r}{h}\tau$ and $\tau(\frac{h}{r} - \tau)$ for $\tau \in [0, \frac{h}{r}]$, then it follows

$$-\left(\lambda_M(P) + \tfrac{h}{2r}\|X_1\| + \|A_R^\top P A_R\|\right)\|\chi_k(0,0)\|^2 \le \mathcal{V}(\tau, \chi_k). \tag{3.163}$$

Finally, the above equation and (3.158) imply that condition (3.116e) is satisfied for

$$\eta_2 = \frac{1}{\lambda_m(P)}\left(\lambda_M(P) + \left(\frac{h}{2r}\right)^2 \|X_1\| + \|A_R^\top P A_R\|\right). \tag{3.164}$$

Now consider the derivative of $W$ for all $k = 1, 2, \ldots$, using the upper bound (3.155) of the LK functional and $\nu(\tau) = 1$ for $\tau \in [0, \frac{h}{r})$, it is obtained

$$\begin{aligned}
\dot{W} \leq\ & He\left(\dot{\chi}_k^\top(\tau^+, 0)P\chi_k(\tau^+, 0)\right) + \tfrac{h}{r}\dot{\chi}_k^\top(\tau^+, 0)R\dot{\chi}_k(\tau^+, 0) - \tfrac{r}{h}\Big(\chi_k(\tau^+, 0) - \chi_k(\tau^+, -\tfrac{h}{r})\\
&- (A_R - I)\chi_k(0, 0)\Big)^\top R\Big(\chi_k(\tau^+, 0) - \chi_k(\tau^+, -\tfrac{h}{r}) - (A_R - I)\chi_k(0, 0)\Big)\\
&+ \chi_k^{e\top}(\tau^+, 0)Q\chi_k^e(\tau^+, 0) - \chi_k^{e\top}(\tau^+, -\tfrac{h}{r})Q\chi_k^e(\tau^+, -\tfrac{h}{r})\\
&+ \tfrac{r}{h}\chi_k^\top(0, 0)(A_R P A_R - P)\chi_k(0, 0) + \left(\tfrac{h}{r} - \tau\right)\Big(He(\dot{\chi}_k^\top(\tau^+, 0)S_1\zeta_k(\tau))\\
&+ He(\dot{\chi}_k^\top(\tau^+, 0)S_2\chi_k(0, 0)) + \dot{\chi}_k^\top(\tau^+, 0)U\dot{\chi}_k(\tau^+, 0)\Big)\\
&- \zeta_k^\top(\tau)S_1\zeta_k(\tau) - He(\zeta_k^\top(\tau)S_2\chi_k(0, 0))\\
&- \int_0^\tau \dot{\chi}_k^\top(s^+, 0)U\dot{\chi}_k(s^+, 0)ds + (\tfrac{h}{r} - 2\tau)\chi_k^\top(0, 0)X_1\chi_k(0, 0).
\end{aligned} \tag{3.165}$$

Let define an augmented vector $\xi_k(\tau) = (\chi_k^e(\tau, 0), \chi_k(\tau, -h), \chi_k(0, 0))$, then since $U$ is positive definite, the following inequality holds (the inequality follows directly from the Basic inequality in Appendix B)

$$\begin{aligned}
-\int_0^\tau \dot{\chi}_k^\top(s^+, 0)U\dot{\chi}_k(s^+, 0)ds \leq\ & \tau\xi_k^\top(\tau^+)YU^{-1}Y^\top\xi_k(\tau^+)\\
&+ He(\xi_k^\top(\tau^+)Y(\chi_k(\tau^+, 0) - \chi_k(0^+, 0)))
\end{aligned} \tag{3.166}$$

for some matrix $Y \in \mathbb{R}^{(r+2)n\times n}$.

Using the definition of matrices (3.145)-(3.150), it is obtained the following inequality, $\dot{W} \leq \xi_k(\tau^+)^\top\Gamma(\tau)\xi_k(\tau^+)$ for $\tau \in [0, \frac{h}{r})$ where

$$\Gamma(\tau) = M_0^\top\Pi_0 M_0 + \Pi_1 + N_R + (\tfrac{h}{r} - \tau)\Pi_2 + (\tfrac{h}{r} - 2\tau)\Pi_3 + \tau YU^{-1}Y^\top + \tfrac{r}{h}\Pi_4 \tag{3.167}$$

for $\tau$ in the closed set $[0, \frac{h}{r}]$. Note that $\Gamma(\tau)$ is convex on $\tau$. Hence, it is necessary and sufficient to ensure that is negative definite at the edges. If $\tau = 0$ then it is obtained condition (3.144c). The case $\tau = \frac{h}{r}$ leads to condition (3.144d) after appliying the Schur complement to deal with the term $U^{-1}$. Therefore, by conditions (3.144c) and (3.144d), $\Gamma(\tau)$ is negative definite for all $\tau \in [0, \frac{h}{r}]$. In addition, it is obtained the inequality

$$-\dot{W} \geq \xi_k(\tau^+)^\top \Gamma(\tau)\xi_k(\tau^+) \geq \min\{\lambda_m(\Gamma(\tau)) : \tau \in [0, \tfrac{h}{r}]\}\|\chi_k(\tau^+, 0)\|^2 = w_1(\|\chi_k(\tau^+, 0)\|) \quad (3.168)$$

for $\tau \in [0, \frac{h}{r})$. Since the right limits of $\dot{W}$ and $w_1$ exist, the above inequality is also true in the closed interval $[0, \frac{h}{r}]$, and thus, condition (3.116f) is satisfied. All the conditions of Proposition 3.6 are fulfilled and the proof is complete.

□

**Remark 3.7.** *The application of the approach in Proposition 3.7 to the others LK functionals would yield to similar stability criteria, and maybe less conservative, specially for the LK functional based on Legendre polynomials. Note that the methodology applied to obtain the LMI conditions implies that $\Delta_m$ has to be greater than the interval of integration in the double integral term. For the delay-partitioning LK functional it is obtained $\Delta_m > h/r$, that is, the parameter $r$ provides flexibility in the election of $\Delta_m$. On the contrary, the lower bound of $\Delta_m$ would be fixed in the other two cases, $\Delta_m > h$. This fact is the main motivation for the election of the delay-partitioning LK functional.*

**Remark 3.8.** *In comparison with Proposition 3.4, the parameter $r$ plays several roles in the stability criterion of Proposition 3.7: first, it is used to reduce the conservatism (gap of the Jensen's inequality) and second, it establishes the maximum time that the LK functional can increase and also a lower bound of the reset interval (time regularization). Therefore, we cannot expect that an increment of $r$ will always produce a greater allowable time-delay.*

**Example 3.8.** Consider the time-delay reset control system of the Example 3.5. Table 3.3 shows the maximum time-delay for which Proposition 3.7 guarantees the asymptotic stability. Unlike Corollaries 3.2, 3.3, and 3.4, Proposition 3.7 does not guaranteed the stability for an uncertain time-delay belonging to a known interval. However, the search of the maximums time-delay, $h_M$, has been accomplish by an iterative search with a step of 0.001. Therefore, the system is asymptotically stable for all constant time-delay $h \in \{0.001, 0.002, 0.003, \ldots, h_M\}$.

Consider the initial condition $\phi(\theta) = (1, 0)$, $\theta \in [-h, 0]$ and the parameter $r = 2$. Fig. 3.6a shows the evolution of the system with input time-delay $h = 0.844$ for this initial condition. In addition, the value of the LK functional and the functional $W$

| Method | Input time-delay | Output time-delay |
|---|---|---|
| Base system | 0.87 | 0.87 |
| Corollary 3.4 | 0.816 | 0.803 |
| Proposition 3.7, r=1 | 0.816 | 0.768 |
| Proposition 3.7, r=2 | 0.844 | 0.788 |
| Proposition 3.7, r=3 | 0.843 | 0.783 |
| Proposition 3.7, r=5 | 0.832 | 0.771 |
| Proposition 3.7, r=10 | 0.814 | 0.756 |

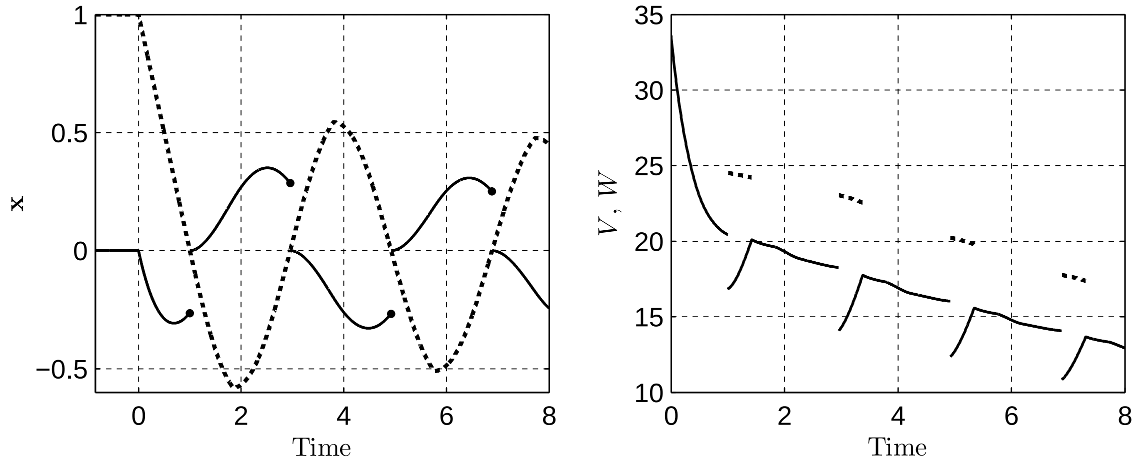
Table 3.3 Maximum input/output time-delay, for the Example 3.6 obtained by Proposition 3.7. for several values of the parameter $r$.

along the trajectory are shown in Fig. 3.6b. Fig. 3.7 shows the evolution of the system, the LK functional and the functional $W$ for an output time-delay $h = 0.788$. It can be seen in both cases that the LK functional increases after every reset instant, while the functional $W$ always decreases. Note that $W$ is only defined on the interval $(t_k, t_k + \rho]$ for every reset instant $t_k$.

**Example 3.9.** The maximum input time-delay in the Example 3.6 is 0.201, which is provided by Corollary 3.4. Since the maximum time-delay is the same as the analytical maximum time-delay of the base system (with a precision of three decimal digits), the application of the Proposition 3.7 to this reset control system cannot overcome this value. In fact, its application results into the same maximum input time-delay. However, Corollary 3.4 guarantees the stability of the system for smaller values of the time-delay, but it is not possible by Proposition 3.7. Regarding the output time-delay, there is not time-delay for which the stability of the system can be guaranteed by Proposition 3.7.

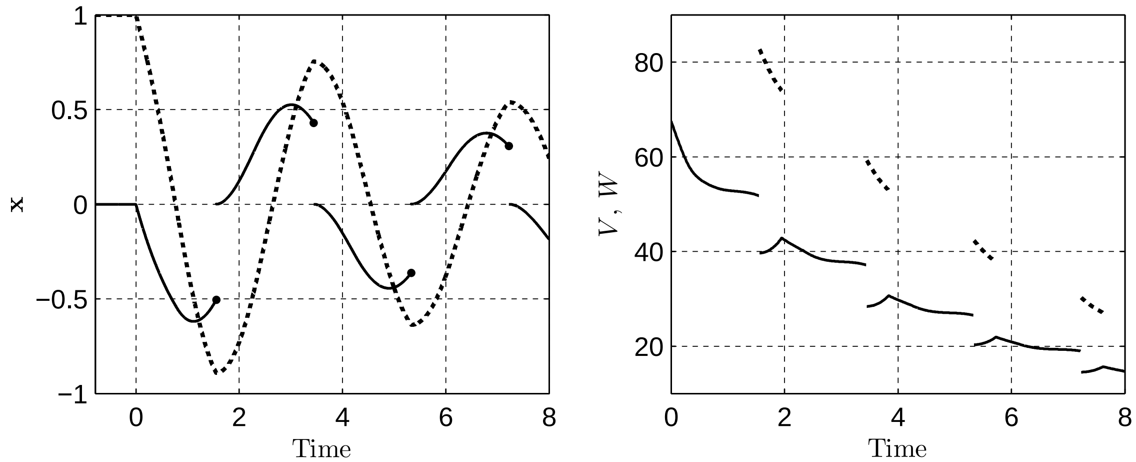
### 3.3.2 Unstable Base System

Proposition 3.6 supposes an improvement over the results in Section 3.2 , since the LK functional is not force to always decrease. Nevertheless, the stability of the base system is still required, and thus, the maximum time-delay for which Proposition 3.7 guarantees the asymptotic stability is upper bounded by the maximum allowable time-delay of the base system. Note that the time-dependent condition imposed on




(a) Trajectory: $x_p$ (dotted) and $x_r$ (solid). (b) LK functional $V$ (solid) and $W$ (dotted).

Figure 3.6 Input time-delay: Trajectory and value of the LK functional $V$ and functional $W$ (it is only defined after the reset instants) for the time-delay reset control system of the Example 3.8 with input time-delay $h = 0.844$, $r = 2$ and initial condition $\phi(\theta) = (1, 0)$, $\theta \in [-0.844, 0]$.




(a) Trajectory: $x_p$ (dotted) and $x_r$ (solid). (b) LK functional $V$ (solid) and $W$ (dotted).

Figure 3.7 Output time-delay: Trajectory and value of the LK functional $V$ and functional $W$ (it is only defined after the reset instants) for the time-delay reset control system of the Example 3.8 with output time-delay $h = 0.788$, $r = 2$ and initial condition $\phi(\theta) = (1, 0)$, $\theta \in [-0.788, 0]$.

the reset interval does not force reset actions, then there may exist an initial condition that makes the system to produce a finite set of reset instants. Therefore, the stability of the system after the last reset instant is subjected to the stability of the base system. In this section, an upper bound of the reset intervals is imposed, in order to avoid such trajectories. In addition, the idea of bounded increments of the functional is developed further, allowing the LK functional to have increments during the whole reset interval, instead of only $\rho$ units time after a reset action (Proposition 3.6). This approach can be applied since the upper bound of the reset intervals guarantees that the LK functional increases during a finite amount of time.

**Proposition 3.8.** *Suppose $u$, $v$, $w$: $\mathbb{R}_+ \to \mathbb{R}_+$ are continuous nondecreasing functions and in addition $u$, $v \in \mathcal{K}$. If there exist real numbers $\eta_0, \eta_1, \eta_2 \geq 0$, a functional $V$ : $\mathcal{PAC}([-h,0],\mathbb{R}^n) \to \mathbb{R}_+$ that satisfies (3.116a), (3.116b) for all $\psi \in \mathcal{PAC}([-h,0],\mathbb{R}^n)$ and there exists a continuous functional $\mathcal{V} : \mathbb{R}_+ \times \mathcal{PAC}([0,\Delta_k] \times [-h,0],\mathbb{R}^n) \to \mathbb{R}$, which satisfies (3.116d), (3.116e), and*

$$\mathcal{V}(\Delta_k, \chi_k) - \mathcal{V}(0, \chi_k) = V(\chi_k(0^+, \cdot)) - V(\chi_k(0, \cdot)) \tag{3.169a}$$

*for all $\tau \in [0, \Delta_k]$ and $k > 0$. In addition, the following inequality hold*

$$\dot{W}(\sigma, \chi_k) \leq -w(\|\chi_k(\sigma^+, 0)\|), \quad \sigma \in [0, \Delta_k], \quad k > 0 \tag{3.169b}$$

*with*

$$W(\sigma, \chi_k) = V(\chi_k(\sigma^+, \cdot)) + \mathcal{V}(\sigma, \chi_k) \tag{3.170}$$

*where $\dot{W}$ is evaluated along the solutions of the system (3.113), then the trivial solution of the system is stable. In addition, if $\lim_{s\to\infty} u(s) = \infty$ and $w(s) > 0$ for $s > 0$ then the trivial solution is globally asymptotically stable.*

*Proof.* Considering the integral of (3.169b) over the interval $\tau \in [0, \Delta_k]$, the proof follows similarly as the proof of Proposition 3.6. Note that, in this case the sequence of reset instants is infinite, since $t_{k+1} - t_k \leq \Delta_M$. Regarding the asymptotic stability, the proof is directly obtained considering only Case 1 in the proof of Proposition 3.6. □

### IDDS with a LTI Base System

Consider the IDDS (3.113) with a LTI base system, then the following asymptotic stability criterion is obtained by the application of the Proposition 3.8 to the LK functional (3.142).

**Proposition 3.9.** *The trivial solution of the system (3.113) with LTI base system and constants $\Delta_m$ and $\Delta_M$, $\frac{h}{r} < \Delta_m \leq \Delta_M$ for a given integer $r \geq 1$, is globally asymptotically stable, if there exist matrices $P$, $R$, $U$, $S_1$, $X_2 \in \mathbb{S}^n_+$, matrix $Q \in \mathbb{S}^{nr}_+$, matrix $X_1 \in \mathbb{S}^n$, matrix $S_2 \in \mathbb{R}^n$, and matrix $Y \in \mathbb{R}^{(r+2)n\times n}$ such that*

$$\Pi_4 \leq 0 \qquad \text{if} \;\; \Delta_m < \Delta_M, \tag{3.171}$$

$$\begin{bmatrix} S_1 & S_2 \\ S_2^\top & \frac{1}{\Delta_M}X_2 \end{bmatrix} \geq 0, \tag{3.172}$$

$$M_0^\top \Pi_0 M_0 + \Pi_1 + N_R + \alpha(\Pi_2 + \Pi_3) + \frac{1}{\Delta_M}\Pi_4 < 0, \tag{3.173}$$

$$\begin{bmatrix} \Upsilon(\alpha,\beta) & \frac{h}{r}Y \\ \frac{h}{r}Y^\top & -\frac{h}{r}U \end{bmatrix} < 0, \tag{3.174}$$

$$\begin{bmatrix} M_0^\top \Pi_0 M_0 + \Pi_1 - \alpha\Pi_3 + \frac{1}{\Delta_M}\Pi_4 & \alpha Y \\ \alpha Y^\top & -\alpha U \end{bmatrix} < 0 \tag{3.175}$$

*for $\beta \in \{0,1\}$, $\alpha = \{\Delta_m, \Delta_M\}$, where $\Pi_0$, $\Pi_1$, $\Pi_2$, $\Pi_3$, $\Pi_4$, $M_0$, $M_P$, $M_R$, $M_Q$, $N_0$, $N_1$, $N_2$, $N_3$, $N_4$, $N_{13}$, and $N_R$ are provided in Proposition 3.7, and $\Upsilon(\alpha,\beta)$ is defined by*

$$\Upsilon(\alpha,\beta) = M_0^\top \Pi_0 M_0 + \Pi_1 + \beta N_R + (\alpha - \tfrac{h}{r})\Pi_2 + (\alpha - 2\tfrac{h}{r})\Pi_3 + \tfrac{1}{\Delta_M}\Pi_4 \tag{3.176}$$

*Proof.* Most of the proof follows as the proof of Proposition 3.7, then it will be omitted some steps for brevity. It has already proved that the LK functional (3.142) satisfies conditions (3.116a) and (3.116b). On the other hand, the functional $\mathcal{V}$ is modified as follows:

$$\begin{aligned} \mathcal{V}(\tau,\chi_k) = {} & \tau\tfrac{1}{\Delta_k}\chi_k^\top(0,0)A_R^\top P A_R \chi_k(0,0) + (1 - \tfrac{1}{\Delta_k}\tau)\chi_k^\top(0,0)P\chi_k(0,0) \\ & + (\Delta_k - \tau)\left(\zeta_k^\top(\tau)S_1\zeta_k(\tau) + He(\zeta_k^\top(\tau)S_2\chi_k(0,0))\right) \\ & + (\Delta_k - \tau)\int_0^\tau \dot{\chi}_k^\top(s^+,0)U\dot{\chi}_k(s^+,0)ds \\ & + \tau(\Delta_k - \tau)\chi_k^\top(0,0)X_1\chi_k(0,0) + \chi_k^\top(0,0)X_2\chi_k(0,0), \end{aligned} \tag{3.177}$$

where $\zeta_k(\tau) = \chi_k(\tau^+,0) - \chi_k(0^+,0)$ and $\tau \in [0,\Delta_k]$. It can be easily proved that the above functional satisfies conditions (3.169a) and (3.116d) (considering $\tau \in [0,\Delta_k]$ in the proof of Proposition 3.7 ). In order to prove condition (3.116e), consider the

| $\Delta_k$ | $\tau$ | Condition |
|---|---|---|
| $\Delta_m$ | 0 | $\begin{pmatrix} \Delta_m S_1 & \Delta_m S_2 \\ \Delta_m S_2^\top & X_2 \end{pmatrix} \geq 0$ |
| $\Delta_m$ | $\Delta_m$ | $X_2 \geq 0$ |
| $\Delta_M$ | 0 | $\begin{pmatrix} \Delta_M S_1 & \Delta_M S_2 \\ \Delta_M S_2^\top & X_2 \end{pmatrix} \geq 0$ |
| $\Delta_M$ | $\Delta_M$ | $X_2 \geq 0$ |

Table 3.4 Possible combinations in equation (3.178).

following term of the functional $\mathcal{V}$ (it has been rewritten in a matrix form)

$$\begin{bmatrix} \zeta_k^\top(\tau) \\ \chi_k^\top(0,0) \end{bmatrix}^\top \begin{bmatrix} (\Delta_k - \tau)S_1 & (\Delta_k - \tau)S_2 \\ (\Delta_k - \tau)S_2^\top & X_2 \end{bmatrix} \begin{bmatrix} \zeta_k(\tau) \\ \chi_k(0,0) \end{bmatrix} \tag{3.178}$$

By convexity argument, the above term is equal or greater than zero, if the matrix in the middle is positive semidefinite at the edges of the arguments $\tau$ and $\Delta_k$. In Table 3.4, it can be seen the four possible combinations and their corresponding conditions. Note that all the conditions are satisfied by $X_2 > 0$ and (3.172), since $\Delta_M \geq \Delta_m$.

In addition, $U > 0$ implies that the integral term in $\mathcal{V}$ is positive definite. Therefore, from the above arguments, it is obtained

$$\begin{aligned} &\tau \tfrac{1}{\Delta_k} \chi_k^\top(0,0) A_R P A_R \chi_k(0,0) + (1 - \tfrac{1}{\Delta_k}\tau)\chi_k^\top(0,0) P \chi_k(0,0) \\ &+\tau(\Delta_k - \tau)\chi_k^\top(0,0) X_1 \chi_k(0,0) \leq \mathcal{V}(\tau, \chi_k). \end{aligned} \tag{3.179}$$

Consider the maximums of following functions: $\tau\frac{1}{\Delta_k}$, $1 - \frac{1}{\Delta_k}\tau$ and $\tau(\Delta_k - \tau)$ for $\tau \in [0, \Delta_k]$ and $\Delta_k \in [\Delta_m, \Delta_M]$, then condition (3.116e) is satisfied with $\eta_2$ given by

$$\eta_2 = \frac{1}{\lambda_m(P)} \left( \lambda_M(P) + \frac{\Delta_M^2}{4} \|X_1\| + \|A_R^\top P A_R\| \right). \tag{3.180}$$

On the other hand, considering the upper bound (3.155) of the derivative of $V$, the following bound of the derivative of $W$ is obtained for all $k = 1, 2, \ldots$

$$\begin{aligned}\dot{W} \leq\ & He(\dot{\chi}_k^\top(\tau^+,0)P\chi_k(\tau^+,0)) + \tfrac{h}{r}\dot{\chi}_k^\top(\tau^+,0)R\dot{\chi}_k(\tau^+,0) - \tfrac{r}{h}\Big(\chi_k(\tau^+,0) - \chi_k(\tau^+,-\tfrac{h}{r}) \\ & - \nu(\tau)(A_R - I)\chi_k(0,0)\Big)^\top R\Big(\chi_k(\tau^+,0) - \chi_k(\tau^+,-\tfrac{h}{r}) - \nu(\tau)(A_R - I)\chi_k(0,0)\Big) \\ & + \chi_k^{e\top}(\tau^+,0)Q\chi_k^e(\tau^+,0) - \chi_k^{e\top}(\tau^+,-\tfrac{h}{r})Q\chi_k^e(\tau^+,-\tfrac{h}{r}) \\ & + \tfrac{1}{\Delta_k}\chi_k^\top(0,0)(A_R P A_R - P)\chi_k(0,0) + (\Delta_k - \tau)\Big(He(\dot{\chi}_k^\top(\tau^+,0)S_1\zeta_k(\tau)) \\ & + He(\dot{\chi}_k^\top(\tau^+,0)S_2\chi_k(0,0)) + \dot{\chi}_k^\top(\tau^+,0)U\dot{\chi}_k(\tau^+,0)\Big) \\ & - \zeta_k^\top(\tau)S_1\zeta_k(\tau) - He(\zeta_k^\top(\tau)S_2\chi_k(0,0)) - \int_0^\tau \dot{\chi}_k^\top(s^+,0)U\dot{\chi}_k(s^+,0)ds \\ & + (\Delta_k - 2\tau)\chi_k^\top(0,0)X_1\chi_k(0,0).\end{aligned} \tag{3.181}$$

The integral inequality (3.166) is used to deal with the integral term. Therefore, by using the definition of matrices (3.145)-(3.150), and (3.176), it is obtained $\dot{W} \leq \xi_k(\tau^+)^\top\Gamma(\tau)\xi_k(\tau^+)$ for all $k > 0$, where

$$\Gamma(\tau) = M_0^\top\Pi_0 M_0 + \Pi_1 + \nu(\tau)N_R + (\Delta_k - \tau)\Pi_2 + (\Delta_k - 2\tau)\Pi_3 + \tau Y U^{-1} Y^\top + \tfrac{1}{\Delta_k}\Pi_4 \tag{3.182}$$

and $\xi_k(\tau) = (\chi_k^e(\tau,0), \chi_k(\tau,-h), \chi_k(0,0))$. Consider $\Delta_m < \Delta_M$ then $\Pi_4 \leq 0$ by condition (3.171). Hence, it is satisfied

$$\frac{1}{\Delta_k}\Pi_4 \leq \frac{1}{\Delta_M}\Pi_4. \tag{3.183}$$

Substituting (3.183) into (3.182), the resulting expression is convex on $\tau$ and $\Delta_k$, and thus, it is necessary and sufficient to ensure that it is negative definite only at the edges of the intervals $\tau \in [0, \Delta_k]$ and $\Delta_k \in [\Delta_m, \Delta_M]$. Note that there is a discontinuity at $\tau = \frac{h}{r}$ due to $\nu(\tau)$, but it is sufficient to check the negativity in both sides of the discontinuity (Table 3.5 shows the possible combinations and their respective conditions). On the other hand, if $\Delta_m = \Delta_M$ then the inequality (3.183) results in a equality, and condition (3.171) is not necessary. Finally, the Schur complement is used to deal with the term $U^{-1}$ and write the conditions in a LMI form. The rest of the proof follows as the proof of Proposition 3.7.

□

| $\Delta_k$ | $\tau$ | $\nu(\tau)$ | Condition |
|---|---|---|---|
| $\Delta_m$ | 0 | 1 | (3.173) |
| $\Delta_m$ | $\frac{h}{r}$ | 1 | (3.174) |
| $\Delta_m$ | $\frac{h}{r}^+$ | 0 | (3.174) |
| $\Delta_m$ | $\Delta_m$ | 0 | (3.175) |
| $\Delta_M$ | 0 | 1 | (3.173) |
| $\Delta_M$ | $\frac{h}{r}$ | 1 | (3.174) |
| $\Delta_M$ | $\frac{h}{r}^+$ | 0 | (3.174) |
| $\Delta_M$ | $\Delta_M$ | 0 | (3.175) |

Table 3.5 Possible combinations in equation (3.167).

| Method | Input time-delay | Output time-delay |
|---|---|---|
| Base system | 0.87 | 0.87 |
| Proposition 3.3 | 0.442 | 0.675 |
| Proposition 3.4, $r = 10$ | 0.767 | 0.731 |
| Proposition 3.5, $N = 3$ | 0.816 | 0.803 |
| Proposition 3.7, $r = 2$ | 0.844 | 0.788 |
| Proposition 3.9, $r = 10$ | 1.433 | 1.458 |

Table 3.6 Maximum allowable input/output time-delay for the time-delay reset control system of the Example 3.10 obtained by Proposition 3.9.

**Example 3.10.** Consider the time-delay reset control system of the Example 3.5. Table 3.6 shows the maximum time-delay for which Proposition 3.9 guarantees the asymptotic stability. It is also shown the maximum time-delay obtained by Propositions 3.3, 3.4, 3.5, and 3.7. Note that the results do not guarantee the stability for time-delay smaller than the maximum obtained. Fig. 3.8 shows the lower and upper bound of $\Delta_k$ as a function of the time-delay for input and output time-delay. Given a time-delay, if the reset interval is between the lines (solid for input time-delay and dashed for output time-delay) then the reset control system is globally asymptotically stable by Proposition 3.9 with $r = 10$. Since the results are only valid for a pointwise time-delay, the plots have been obtained with a time-delay step of 0.001. Note that Example 3.5

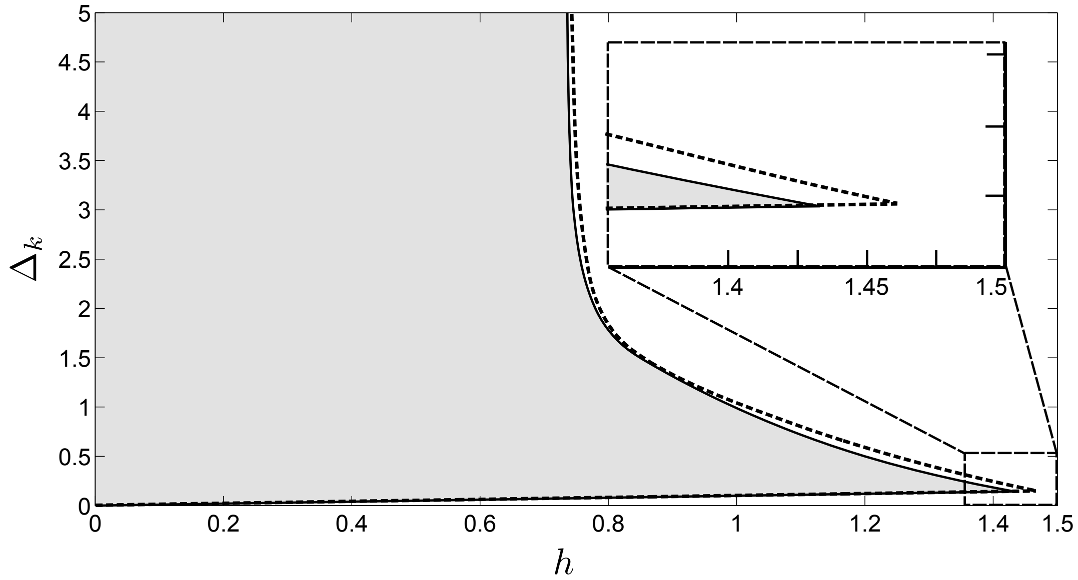
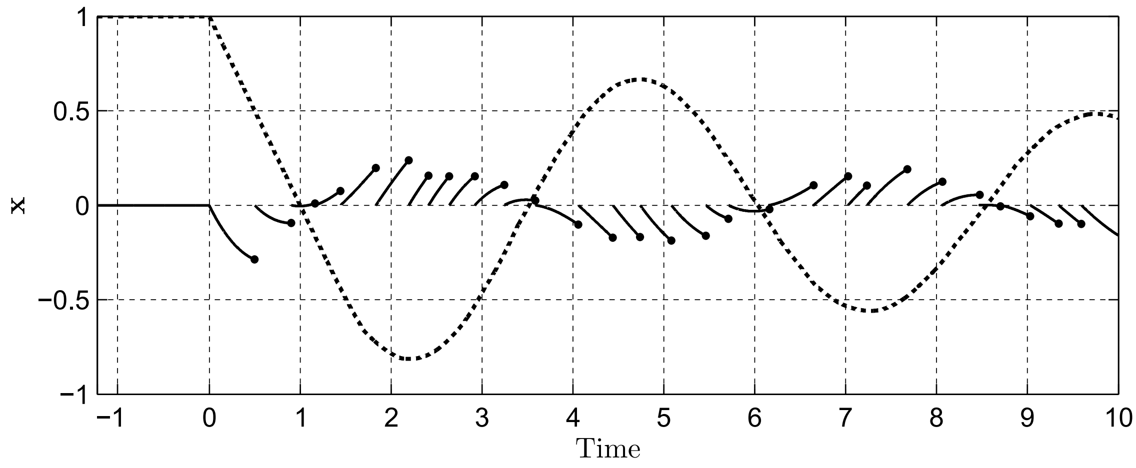


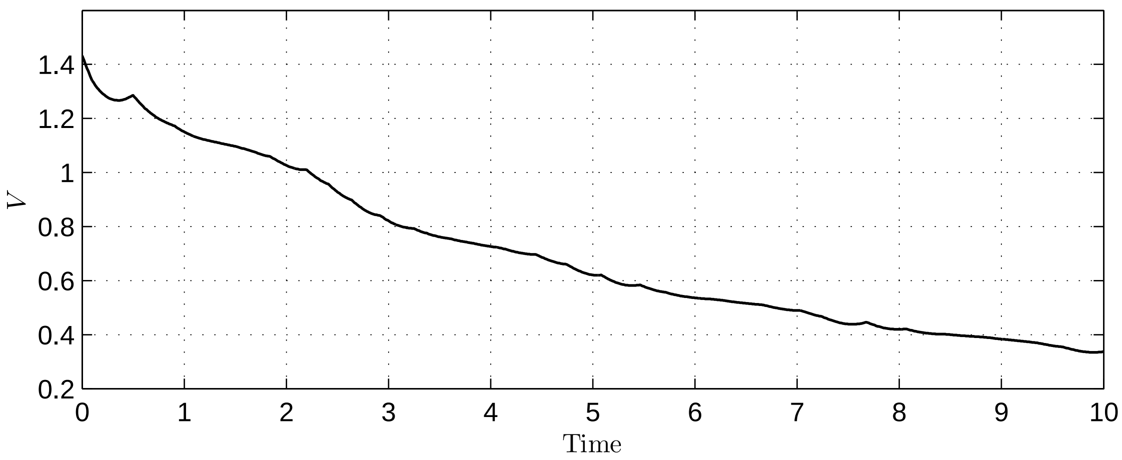

Figure 3.8 Allowable reset interval as a function of the time-delay for the time-delay reset control system of the Example 3.10 obtained by Proposition 3.9 with $r = 10$. Input time-delay (solid) and output time-delay (dashed).

shows that the time-delay reset control system is asymptotically stable for a time-delay smaller than 0.816 for input time-delay and 0.803 for output time-delay. Therefore, the importance of the results in this example is that the time-dependent conditions of Proposition 3.9 allows to guarantee the stability further than those bounds, and also to overcome the stability of the base system. Showing that an unstable LTI control system can be stabilized by reset actions on the compensator.

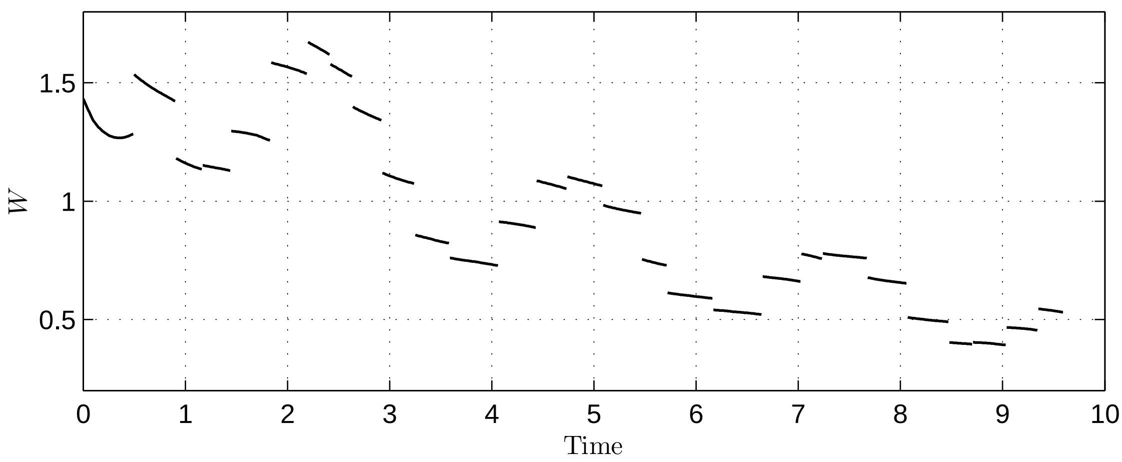
Consider the input/output time-delay $h = 1.2$, then $\Delta_m = 0.121$ and $\Delta_M = 0.5$ are adequate values for both input and output time-delay. In addition, let the reset instants be specified randomly satisfying $\Delta_m \leq t_{k+1} - t_k \leq \Delta_M$, then Fig. 3.9 and 3.10 show the evolution of the system for the initial condition $\phi(\theta) = (1, 0)$, $\theta \in [-1.2, 0]$. It can be also seen the value of the LK functional (with $r = 10$) and the value of the functional $W$ along the trajectories. In order to clearly show how the LK functional is allowed to increase between reset instants, Fig. 3.11 and 3.12 show part of the evolution of the LK functional (both input and output time-delay) where it clearly increases. On the contrary, it can be appreciated in the figures how the functional $W$ always decreases between the reset instants.




(a) Trajectory: $x_p$ (dotted) and $x_r$ (black).

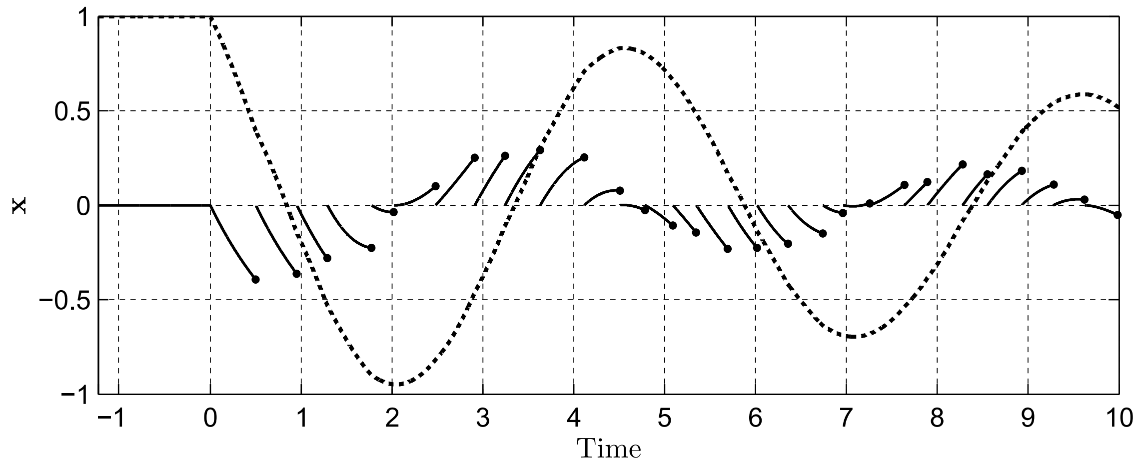



(b) LK functional V.

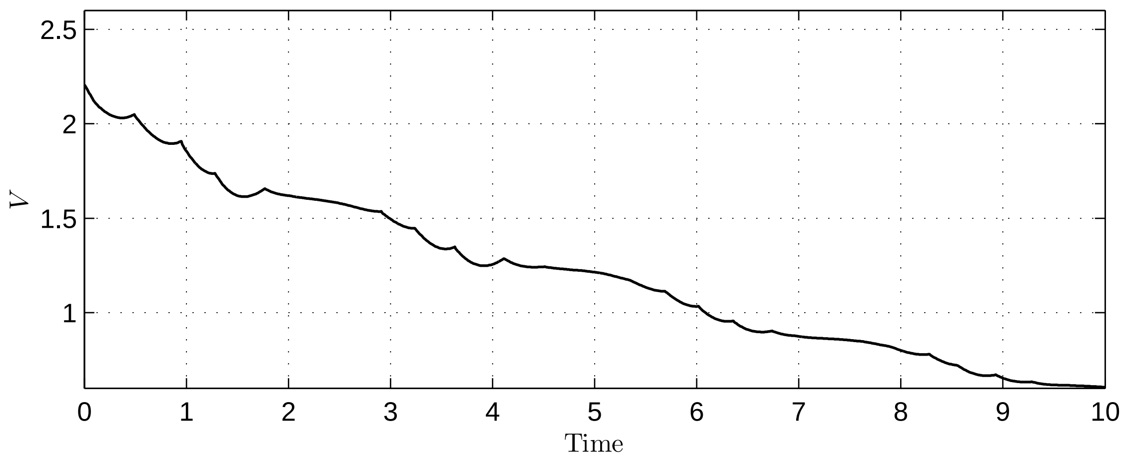



(c) Functional $W$.

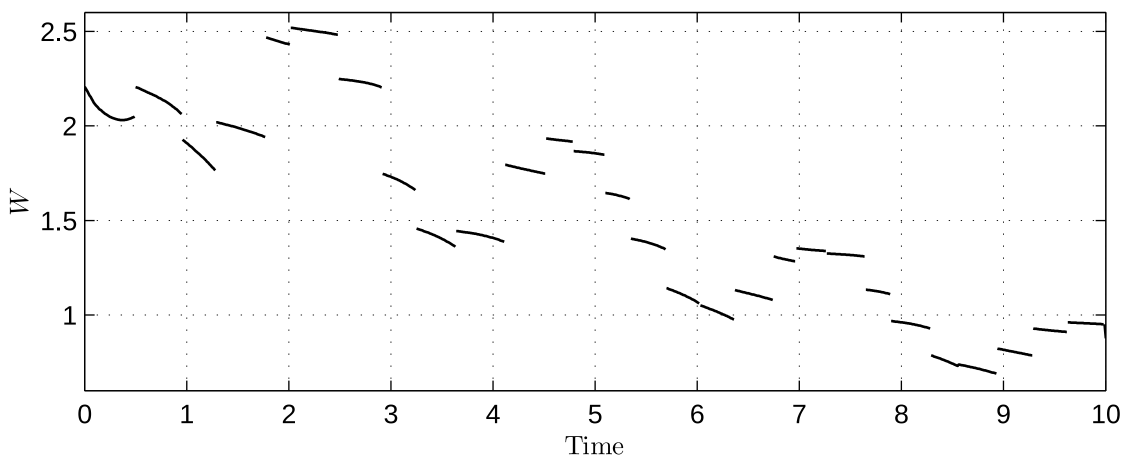
Figure 3.9 Input time-delay: Trajectory and value of the LK functional and the functional $W$ for the time-delay reset control system of the Example 3.10 with input time-delay $h = 1.2$, $r = 10$, $\Delta_m = 0.121$, $\Delta_M = 0.5$, and initial condition $\phi(\theta) = (1, 0)$, $\theta \in [-1.2, 0]$.




(a) Trajectory: $x_p$ (dotted) and $x_r$ (black).




(b) LK functional V.




(c) Functional $W$.

Figure 3.10 Output time-delay: Trajectory and value of the LK functional and the functional $W$ for the time-delay reset control system of the Example 3.10 with output time-delay $h = 1.2$, $r = 10$, $\Delta_m = 0.121$, $\Delta_M = 0.5$, and initial condition $\phi(\theta) = (1, 0)$, $\theta \in [-1.2, 0]$.

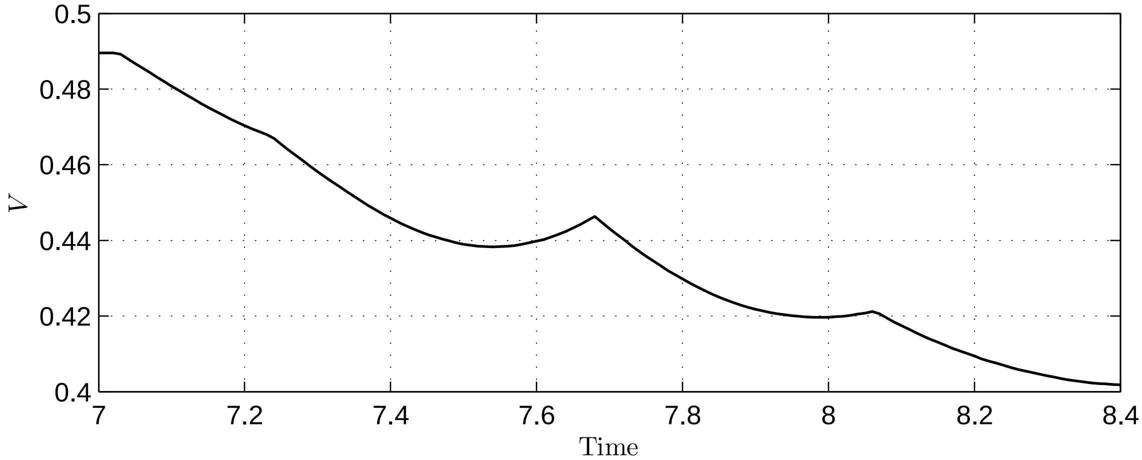



Figure 3.11 Input time-delay: Zoom of the value of the LK functional for the time-delay reset control system of the Example 3.10.

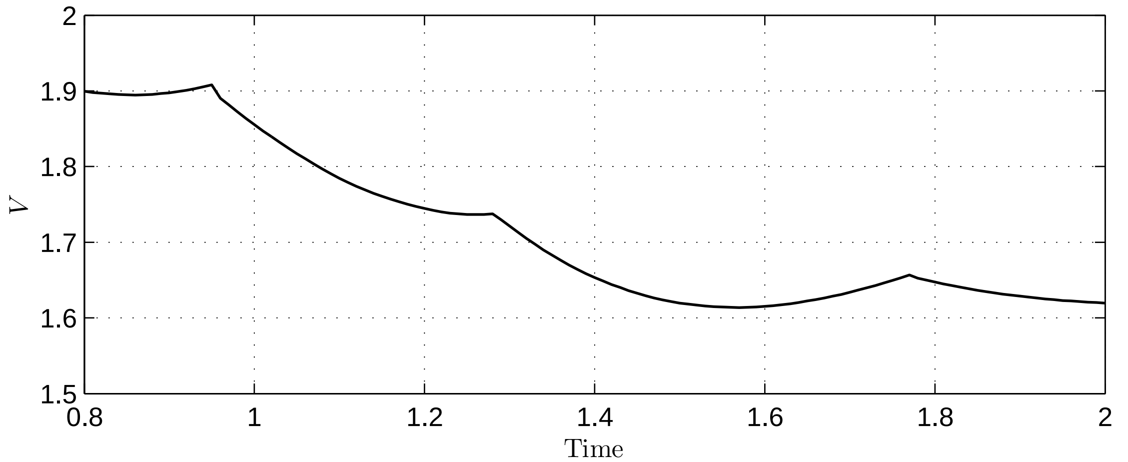



Figure 3.12 Output time-delay: Zoom of the value of the LK functional for the time-delay reset control system of the Example 3.10.

**Example 3.11.** Consider the IDDS (3.113) with a LTI base system defined by the following matrices (the example is taken from [307])

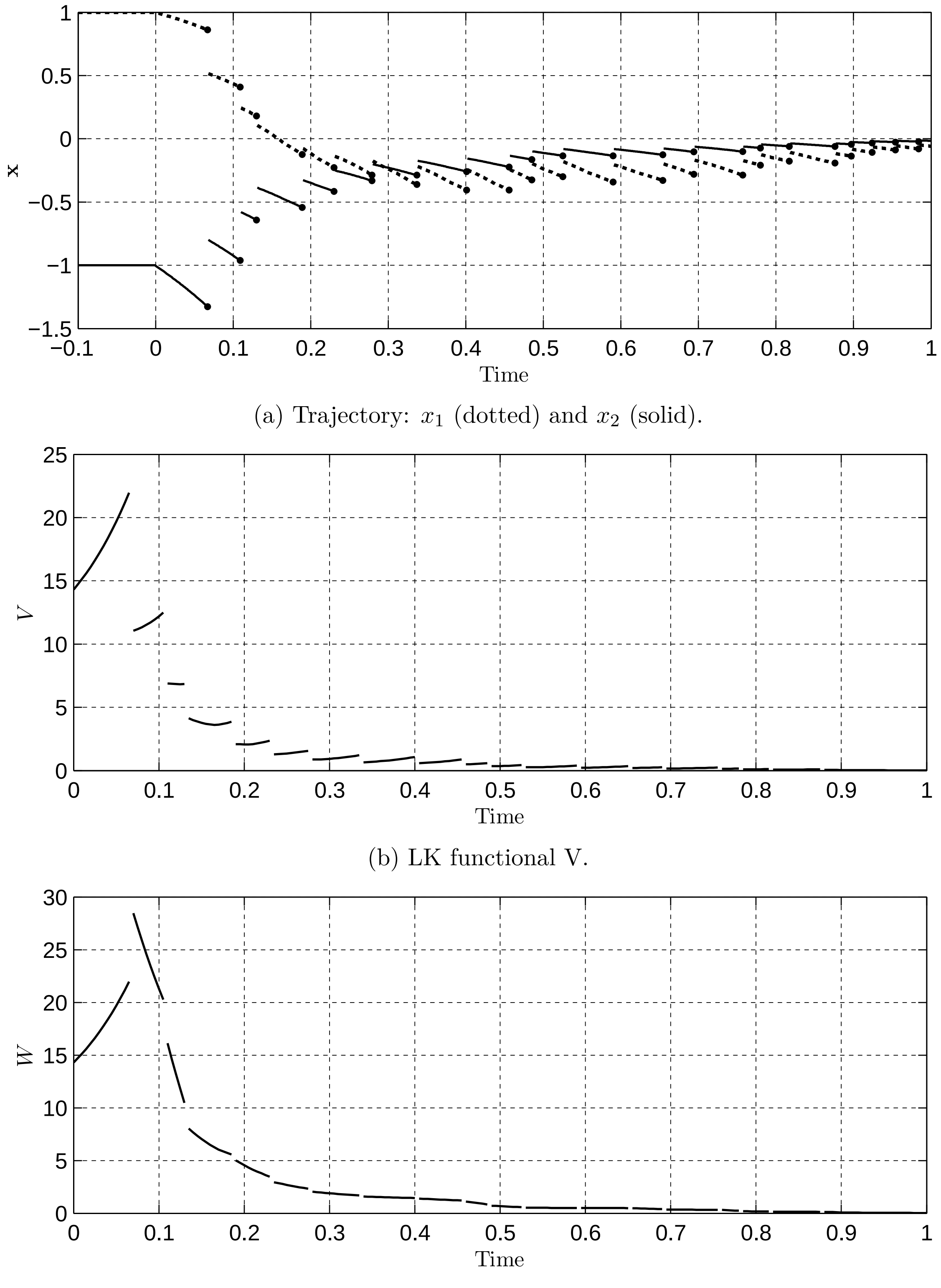
$$A = \begin{bmatrix} 2 & \frac{5}{3} \\ \frac{1}{2} & 4 \end{bmatrix}, \quad A_d = \begin{bmatrix} 3 & 5 \\ \frac{1}{3} & 1 \end{bmatrix}, \quad A_R = \begin{bmatrix} \frac{3}{5} & 0 \\ 0 & \frac{3}{5} \end{bmatrix}. \tag{3.184}$$

It is known that the base system is unstable for the time-delay $h = 0.1$. In addition, [307] shows that if $\Delta_M < 0.0095$ then the trivial solution of the system is globally exponentially stable for any fixed time-delay $h \in (0, \infty)$. Since the proposed method is delay-dependent, we consider for instance the interval $[0.001, 1]$. Applying Proposition 3.9 with $r = 40$ and time-delay step of 0.001, it follows that the impulsive system is globally asymptotically stable for a given $h \in \{0.001, 0.002, \cdots, 1\}$, $\Delta_m > h/40$ and $\Delta_M = 0.0671$, that is $h/40 < t_{k+1} - t_k \leq 0.0671$. It can be seen that the bound 0.0095 obtained in [307] is conservative in this interval of time-delay. Finally, note that $\Delta_m$ strongly depends on $r$, hence if it is desired to reduce $\Delta_m$ or increase $h$ then $r$ should be increased, which results in a longer time of computation. Fig. 3.13 shows the evolution of the system for the time-delay $h = 0.1$, the initial condition $\phi(\theta) = (1, -1)$, $\theta \in [-0.1, 0]$ and random reset instants. It is also shown the value of the LK functional and the functional $W$ along the trajectory. Again, it can be seen that after some reset actions the functional $V$ increases, but on the contrary, the functional $W$ always decreases.

**Example 3.12.** The IDDS with LTI base system under study in this example is defined by the following matrices

$$A = \begin{bmatrix} 0.1 & 2 \\ -1 & 0.1 \end{bmatrix}, \quad A_d = \begin{bmatrix} 1 & 1 \\ 0.1 & 0 \end{bmatrix}, \quad A_R = \begin{bmatrix} 0 & 0 \\ 0 & 1.1 \end{bmatrix}. \tag{3.185}$$

This system without impulses has been proved to be unstable for $h \in [0, 4]$ by using the Lambert W function ([283]). Note that $\|A_R\| = 1.1$, which means that both the continuous dynamic and the discrete dynamic are unstable, in the sense that both may move away the system state from the origin. To the author knowledge, there is not previous method that can be applied to this kind of systems. In this example, we consider a periodic resetting law, that is $\Delta_m = \Delta_M$. Fig. 3.14 shows the minimum and the maximum period of reset, that stabilizes the IDDS, as a function of the time-delay. That is, for a given time-delay $h$, if the reset actions are applied periodically with period inside the shaded area, then the impulsive system is globally asymptotically stable. The plot has been obtained with a time-delay step




(a) Trajectory: $x_1$ (dotted) and $x_2$ (solid).

(b) LK functional V.

(c) Functional $W$.

Figure 3.13 Trajectory and value of the LK functional and the functional $W$ for the IDDS of the Example 3.11 with time-delay $h = 0.1$, $r = 40$, $\Delta_m = 0.0026$, $\Delta_M = 0.0671$, and initial condition $\phi(\theta) = (1, -1)$, $\theta \in [-0.1, 0]$.

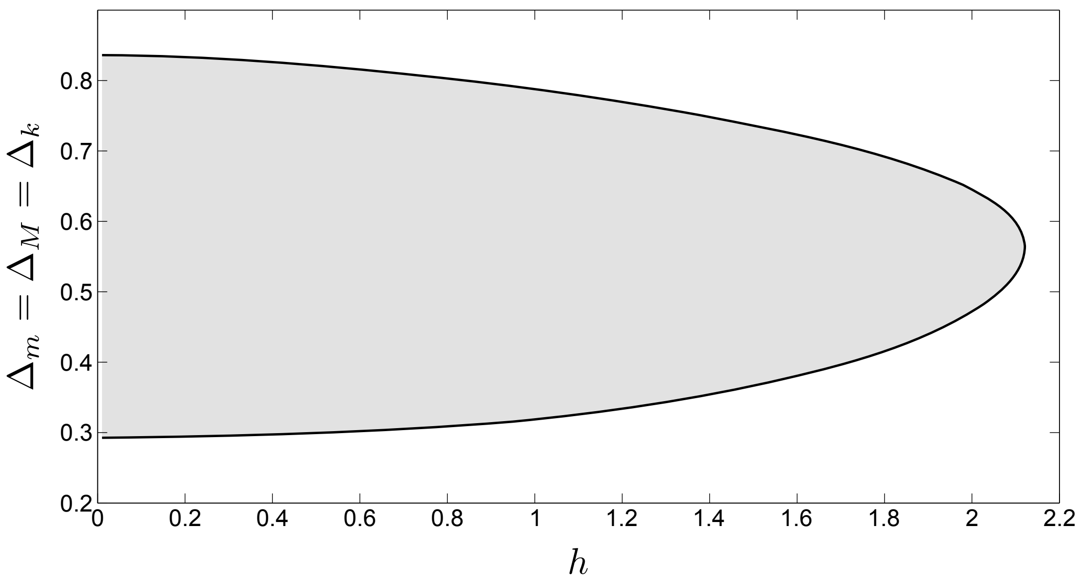


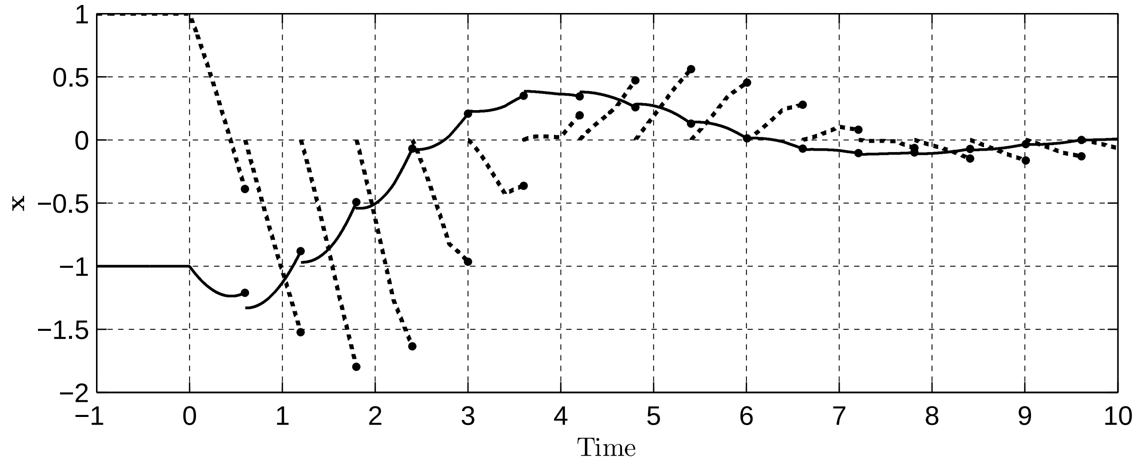

Figure 3.14 Allowable reset interval (periodic reset) as a function of the time-delay for the IDDS of the Example 3.12 obtained by Proposition 3.9 with $r = 10$.

of 0.001, and thus, strictly speaking the asymptotic stability is only guaranteed for a constant time-delay $h \in \{0.001, 0.002, \ldots\}$ and the reset period inside the shaded area.

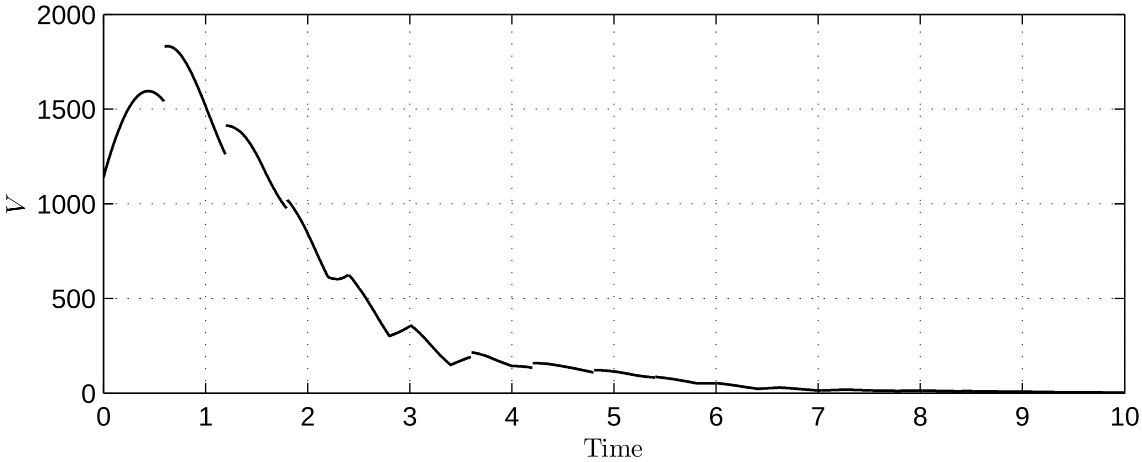
Finally, consider the time-delay $h = 1$, the parameter $r = 10$, periodic reset instants with period 0.6, and the initial condition $\phi(\theta) = (1, -1)$, $\theta \in [-1, 0]$. The evolution of the system and the functionals $V$ and $W$ are shown in Fig. 3.15. Note that the functional $V$ increases instantaneously due to both reset actions (for instance at the second reset instant) and continuous dynamic (for instance in the time interval $t \in [2, 3]$).

### 3.3.3 Time-delay Reset Control Systems with PI+CI Compensator

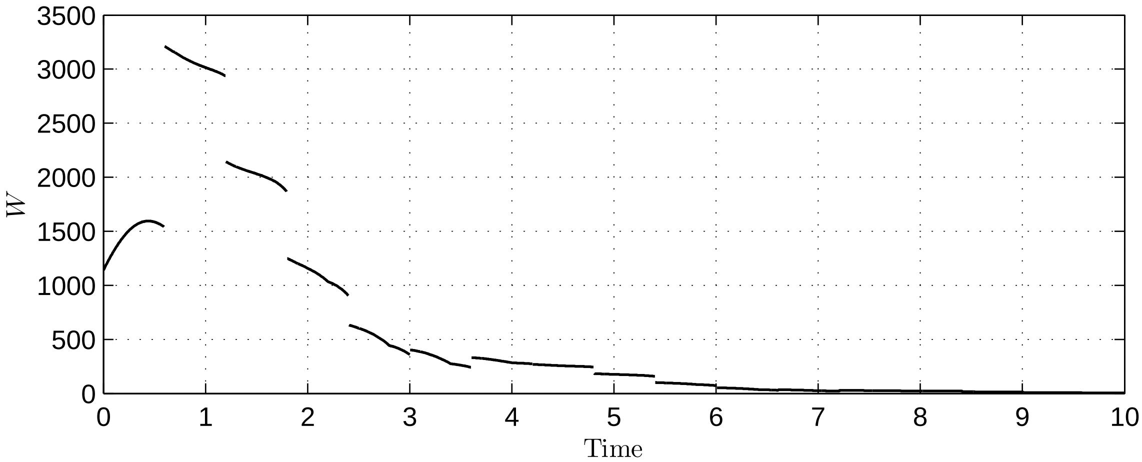
Consider an autonomous time-delay reset control system with zero reference and disturbance, and formed by a LTI system P and a PI+CI compensator. As it was discussed in Chapter 2, the non-minimal realization of the PI+CI implies that the trivial solution of the base closed-loop system is not asymptotically stable. In addition, the minimal realization (2.52) of the PI+CI leads into a variable resetting matrix $A_R$. As a result, there is not stability criteria applicable to the time-delay reset control system.




(a) Trajectory: $x_1$ (dotted) and $x_2$ (solid).




(b) LK functional V.




(c) Functional $W$.

Figure 3.15 Trajectory and value of the LK functional and the functional $W$ for the IDDS of the Example 3.12 with time-delay $h = 1$, $r = 10$, $\Delta_m = \Delta_M = 0.6$, and initial condition $\phi(\theta) = (1, -1)$, $\theta \in [-1, 0]$.

In order to developed stability results, we proposed the following new realization[8] of the PI+CI

$$\begin{cases} \left.\begin{array}{l} \dot{\mathbf{x}}_r(t) = A_r\mathbf{x}_r(t) + B_r e(t), \\ \dot{\Delta}(t) = 1, \end{array}\right\} & (\mathbf{x}_r(t), e(t), \dot{e}(t)) \notin \mathcal{R} \vee \Delta(t) < \Delta_m, \\ \left.\begin{array}{l} \mathbf{x}_r(t^+) = A_\rho \mathbf{x}_r(t), \\ \Delta(t^+) = 0, \end{array}\right\} & ((\mathbf{x}_r(t), e(t), \dot{e}(t)) \in \mathcal{R} \wedge \Delta(t) \geq \Delta_m) \vee \Delta(t) = \Delta_M, \\ v(t) = C_r\mathbf{x}_r(t) + D_r e(t), & \end{cases} \tag{3.186}$$

where $\mathbf{x}_r(t) = (x_i, x_{ci})$ and the matrices $A_r$, $B_r$, $C_r$, $D_r$ and $A_\rho$ are given by

$$A_r = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_r = \begin{bmatrix} \frac{k_p}{\tau_i} \\ 0 \end{bmatrix}, \quad C_r = [\ 1 \quad 1\ ], \quad D_r = k_p, \quad A_\rho = \begin{bmatrix} 1 & 0 \\ -p_r & 0 \end{bmatrix}, \tag{3.187}$$

where $p_r \in [0, 1)$ (nonzero equilibrium points appear for $p_r = 1$, and thus, the trivial solution is not globally asymptotically stable). Note that the main difference to the other realization is that in this case the last state of the PI+CI compensator ($x_{ci}$) evolves by the reset actions, but not by the continuous dynamic. Let $\mathbf{x}(t) = (\mathbf{x}_p(t), x_i(t), x_{ci}(t)) = (\mathbf{x}_C(t), x_{ci}(t))$ be the closed-loop state, then the time-delay reset control system is defined by the following IDDS

$$\begin{cases} \left.\begin{array}{l} \dot{\mathbf{x}}(t) = A\mathbf{x}(t) + A_d\mathbf{x}(t-h), \\ \dot{\Delta}(t) = 1, \end{array}\right\} & (\mathbf{x}(t), \mathbf{x}(t-h), \Delta(t)) \notin \mathcal{S}, \\ \left.\begin{array}{l} \mathbf{x}(t^+) = A_R\mathbf{x}(t), \\ \Delta(t^+) = 0, \end{array}\right\} & (\mathbf{x}(t), \mathbf{x}(t-h), \Delta(t)) \in \mathcal{S}, \end{cases} \tag{3.188}$$

where matrices $A$, $A_d$, and $A_R$ are given by

$$A = \left[\begin{array}{c|c} A_1 & A_2 \\ \hline 0 & 0 \end{array}\right] = \left[\begin{array}{cc|c} A_p & B_p & B_p \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right], \tag{3.189}$$

[8] We use the notation of the original PI+CI, in spite of the fact that now the states have not a direct match with the Clegg integrator.

$$A_d = \left[\begin{array}{c|c} A_{d_1} & 0 \\ \hline 0 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -B_p k_p C_p & 0 & 0 \\ -\frac{k_p}{\tau_i} C_p & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right], \tag{3.190}$$

$$A_R = \left[\begin{array}{c|c} I & 0 \\ \hline A_{R_1} & 0 \end{array}\right] = \left[\begin{array}{cc|c} & I & 0 \\ \hline 0 & -p_r & 0 \end{array}\right]. \tag{3.191}$$

In the following, it will be proved that the global asymptotic stability of the time-delay reset control system can be derived from the application of Proposition 3.8 to only the partial solution $\mathbf{x}_C(t)$.

**Corollary 3.5.** *If all the conditions of Proposition 3.8 for the global asymptotic stability are satisfied for the partial solution $\mathbf{x}_C(t)$ of the system (3.188), then the trivial solution of the time-delay reset control system (3.188) is globally asymptotically stable.*

*Proof.* Consider the solution $\mathbf{x}(t, \phi)$ of the IDDS (3.188) with initial condition $\phi = (\phi_C, \phi_{ci}) \in \mathcal{PAC}([-h, 0], \mathbb{R}^n)$. Proposition 3.8 guarantees that for all $\epsilon_c > 0$, there exists $\delta = \delta(\epsilon_c)$ such that $\|\phi_C\|_W < \delta$ implies $\|\mathbf{x}_C(t)\| < \epsilon_c$ for all $t \geq 0$. In addition, it is also guaranteed that for all $\eta_c > 0$, there exists $T = T(\eta_c)$ such that $\|\mathbf{x}_C(t)\| < \eta_c$ for all $t \geq T$. Let the initial condition $\phi$ satisfies $\|\phi\|_W \leq \delta$ and $\epsilon_c = \epsilon/\sqrt{2}$ for a given $\epsilon > 0$. In addition, note that $x_{ci}(t) = [0 \quad \cdots \quad 0 \quad -p_r]\mathbf{x}_C(t_k)$ for all $t \in (t_k, t_{k+1}]$ and $k = 1, 2, \ldots$. Hence, it is satisfied

$$\begin{aligned} &\|x_{ci}(t)\| < \epsilon_c, && t \in [0, t_1] \\ &\|x_{ci}(t)\| = p_r \|\mathbf{x}_C(t_k)\| < p_r \epsilon_c < \epsilon_c, && t \in (t_k, t_{k+1}], \end{aligned} \tag{3.192}$$

that yields into

$$\|x_{ci}(t)\| < \epsilon_c, \qquad t \in [0, \infty). \tag{3.193}$$

Considering the solution $\mathbf{x}(t)$, it is obtained

$$\|\mathbf{x}(t)\|^2 = \|\mathbf{x}_C(t)\|^2 + \|x_{ci}(t)\|^2 \leq 2\epsilon_c^2 \tag{3.194}$$

and

$$\|\mathbf{x}(t)\| < \sqrt{2}\epsilon_c = \epsilon \tag{3.195}$$

for all $t \in [0, \infty)$. Therefore, the time-delay reset control system (3.188) is stable with constants $\epsilon$ and $\delta$. The global asymptotic stability is proved by considering $\eta_c = \eta/\sqrt{2}$ and the same procedure.

□

Delay-dependent asymptotic stability conditions for the time-delay reset control system (3.188) is provided in the sequel.

**Proposition 3.10.** *The trivial solution of the time-delay reset control system (3.188) is asymptotically stable, if for a given integer $N > 0$ and scalars $0 < \Delta_m \leq \Delta_M$, there exist a matrix $P_N \in \mathbb{S}^{(N+1)n}$, matrices $S$, $R$, $U$, $X_2 \in \mathbb{S}_+^n$, matrices $S_1$, $X_1 \in \mathbb{S}^n$, matrix $S_2 \in \mathbb{R}^n$ and $Y \in \mathbb{R}^{(N+3)n\times n}$ such that the LMIs*

$$P_N + \frac{1}{h} diag(0_n, S_{N-1}) > 0, \tag{3.196a}$$

$$\begin{bmatrix} S_1 & S_2 \\ S_2^\top & \frac{1}{\Delta_M} X_2 \end{bmatrix} > 0, \tag{3.196b}$$

$$\Pi_1 + \alpha\left(\Pi_2 + \Pi_3\right) < 0, \tag{3.196c}$$

$$\begin{bmatrix} \Pi_1 - \alpha\Pi_3 & \alpha Y \\ \alpha Y^\top & -\alpha U \end{bmatrix} < 0 \tag{3.196d}$$

*for $\alpha \in \{\Delta_m, \Delta_M\}$, where $G_N(h)$, $\Gamma_N(k)$, and $\tilde{S}_N$ are given in Proposition 3.5 and*

$$\begin{aligned} \Pi_1 = {} & He\left(\left[G_N(h) \quad 0\right]^\top P_N H_N\right) + \tilde{S}_N + h^2 F_N^\top R F_N - \Gamma_{N,N}^\top R_N \Gamma_{N,N} \\ & - N_{12}^\top S_1 N_{12} - He\left(N_{12}^\top S_2 N_2\right) + He\left(Y N_{12}\right), \end{aligned} \tag{3.197}$$

$$\Pi_2 = He\left(F_N^\top S_1 N_{12}\right) + He\left(F_N^\top S_2 N_2\right) + F_N^\top U F_N, \tag{3.198}$$

$$\Pi_3 = N_2^\top X_1 N_2, \tag{3.199}$$

$$N_1 = \left[I_n \quad 0\right], \qquad N_2 = \left[0 \quad I_n\right], \qquad N_{12} = N_1 - N_2, \tag{3.200}$$

$$F_N = \begin{bmatrix} A_1 & A_{d_1} & 0_{n,n(N+1)} \end{bmatrix}, \tag{3.201}$$

$$H_N = \begin{bmatrix} F_N^\top & \Gamma_N^\top(0) & \Gamma_N^\top(1) & \dots & \Gamma_N^\top(N-1) \\ A_{R_1}^\top A_2^\top & 0 & 0 & \dots & 0 \end{bmatrix}^\top, \tag{3.202}$$

$$\Gamma_{N,N} = \begin{bmatrix} \Gamma_0 & 0 \\ \vdots & \vdots \\ \Gamma_N & 0 \end{bmatrix}. \tag{3.203}$$

*Proof.* First, the shifted-distributed state representation is applied on the partial solution $\mathbf{x}_C(t)$, such that $\chi_k(\tau,\theta) = \mathbf{x}_C(t_k + \tau + \theta)$. Consider now the LK functional (3.65) and the extra-states function (3.66) with the notation of the shifted-distributed state representation

$$\begin{aligned} V(\chi_k(\tau,\cdot)) = \chi_{N,k}^\top(\tau)(t)P_N\chi_{N,k}(\tau) + \int_{t-h}^{t} \chi_k^\top(\tau,\alpha)S\chi_k(\tau,\alpha)d\alpha \\ + h\int_{-h}^{0}\int_{t+\beta}^{t} \dot{\chi}_k^\top(\tau,\alpha)R\dot{\chi}_k(\tau,\alpha)d\alpha d\beta, \end{aligned} \tag{3.204}$$

$$\chi_{N,k}(\tau) = \begin{bmatrix} \chi_k(\tau,0) \\ \int_{-h}^{0} L_0(\alpha)\chi_k(\tau,\alpha)d\alpha \\ \vdots \\ \int_{-h}^{0} L_{N-1}(\alpha)\chi_k(\tau,\alpha)d\alpha \end{bmatrix}. \tag{3.205}$$

This LK functional was proved to satisfy condition (3.116a) in the proof of Proposition 3.6. In addition, since $\mathbf{x}_C(t)$ is continuous, condition (3.116b) is directly satisfied, and condition (3.169a) is simply given by

$$\mathcal{V}(\Delta_k,\chi_k) - \mathcal{V}(0,\chi_k) = 0. \tag{3.206}$$

On the other hand, note that $x_{ci}(t) = A_{R_1}\mathbf{x}_C(t) = A_{R_1}\chi_k(0,0)$ for all $t \in (t_k, t_{k+1}]$, $k = 1, 2, \ldots$. Hence, the continuity of $\mathbf{x}_C(t)$ leads into the following expression of the derivative of the extra-states function (see proof of Proposition 3.5)

$$\dot{\chi}_{N,k}(\tau) = H_N\xi_k(\tau), \tag{3.207}$$

where

$$\xi_k(\tau) = \begin{bmatrix} \chi_k(\tau,0) \\ \chi_k(\tau,-h) \\ \frac{1}{h}\int_{-h}^{0} L_0(\alpha)\chi_k(\tau,\alpha)d\alpha \\ \vdots \\ \frac{1}{h}\int_{-h}^{0} L_{N-1}(\alpha)\chi_k(\tau,\alpha)d\alpha \\ \chi_k(0,0) \end{bmatrix}. \tag{3.208}$$

Following the proof of Proposition 3.5, the derivative of the LK functional along the partial solution $\mathbf{x}_C(t)$ is given by

$$\begin{aligned}\dot{V}(\chi_k(\tau,\cdot)) = {} & \xi_k^\top(t)\left(He\left([G_N(h)\ \ 0]^\top P_N H_N\right) + \tilde{S}_N + h^2 F_N^\top R F_N\right)\xi_k(t) \\ & - h\int_{-h}^{0}\dot{\chi}_k^\top(\tau^+,\alpha)R\dot{\chi}_k(\tau^+,\alpha)d\alpha.\end{aligned} \tag{3.209}$$

Now, consider the following functional

$$\begin{aligned}\mathcal{V}(\tau,\chi_k) = {} & (\Delta_k-\tau)\left(\zeta_k^\top(\tau)S_1\zeta_k(\tau) + He(\zeta_k^\top(\tau)S_2\chi_k(0,0))\right) \\ & + (\Delta_k-\tau)\int_0^\tau \dot{\chi}_k^\top(s^+,0)U\dot{\chi}_k(s^+,0)ds \\ & + \tau(\Delta_k-\tau)\chi_k^\top(0,0)X_1\chi_k(0,0) + \chi_k^\top(0,0)X_2\chi_k(0,0),\end{aligned} \tag{3.210}$$

where $\zeta_k(\tau) = \chi_k(\tau,0) - \chi_k(0,0)$. This functional satisfies condition (3.206) since

$$\mathcal{V}(\Delta_k,\chi_k) = \mathcal{V}(0,\chi_k) = \chi_k^\top(0,0)X_2\chi_k(0,0). \tag{3.211}$$

Following the procedure in the proof of Propositions 3.7 and 3.9, it is guaranteed that the LK functional and the functional $\mathcal{V}$ satisfy conditions (3.116d) and (3.116e) for the following $\eta_1$ and $\eta_2$

$$\eta_1 = \frac{\lambda_M(X_2)}{\lambda_m\left(P_N + \frac{1}{h}\text{diag}(0,S_{N-1})\right)}, \qquad \eta_2 = \tfrac{\Delta_M^2}{4}\|X_1\|. \tag{3.212}$$

The derivative of $V$ and the derivative of $\mathcal{V}$ lead into the following bound of the derivative of the functional $W$

$$\begin{aligned}\dot{W} \le {} & \xi_k^\top(t)\left(He\left([G_N(h)\ \ 0]^\top P_N H_N\right) + \tilde{S}_N + h^2 F_N^\top R F_N\right)\xi_k(t) \\ & - h\int_{-h}^{0}\dot{\chi}_k^\top(\tau^+,\alpha)R\dot{\chi}_k(\tau^+,\alpha)d\alpha + (\Delta_k-\tau)\Big(He(\dot{\chi}_k^\top(\tau^+,0)S_1\zeta_k(\tau)) \\ & + He(\dot{\chi}_k^\top(\tau^+,0)S_2\chi_k(0,0)) + \dot{\chi}_k^\top(\tau^+,0)U\dot{\chi}_k(\tau^+,0)\Big) \\ & - \zeta_k^\top(\tau)S_1\zeta_k(\tau) - He(\zeta_k^\top(\tau)S_2\chi_k(0,0)) - \int_0^\tau \dot{\chi}_k^\top(s^+,0)U\dot{\chi}_k(s^+,0)ds \\ & + (\Delta_k-2\tau)\chi_k^\top(0,0)X_1\chi_k(0,0).\end{aligned} \tag{3.213}$$

Applying Corollary D.1, the inequality (3.166), and using the matrices in the proposition, it is obtained $\dot{W} \le \xi_k(\tau^+)\Gamma(\tau)\xi_k(\tau^+)$ for all $k > 0$, where

$$\Gamma(\tau) = \Pi_1 + (\Delta_k-\tau)\Pi_2 + (\Delta_k-2\tau)\Pi_3 + \tau Y U^{-1} Y^\top. \tag{3.214}$$

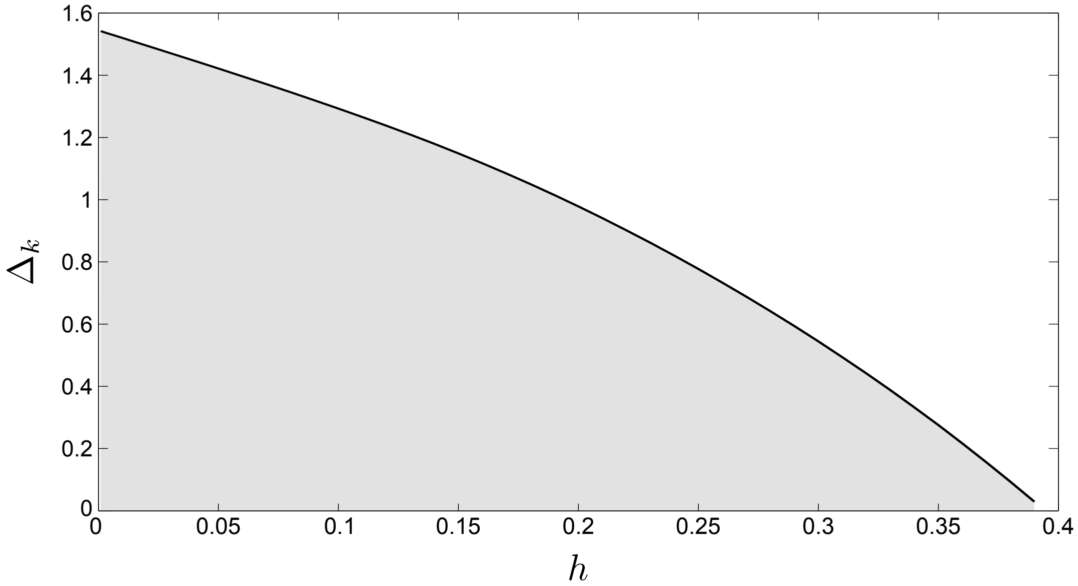



Figure 3.16 Allowable reset interval as a function of the time-delay for the time-delay reset control system of the Example 3.13 with $\beta = 0.5$, obtained by Proposition 3.10 with $r = 10$.

The proof ends by applying the Schur complement and convexity arguments on $\tau$ and $\Delta_k$.

□

**Example 3.13.** Consider the time-delay reset control system (3.188) composed by the following LTI system

$$P : \begin{cases} \dot{\mathbf{x}}_p(t) = \begin{bmatrix} \beta & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x}_p(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u_p(t), \\ y_p(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}_p(t) \end{cases} \tag{3.215}$$

and the PI+CI compensator (3.186) with $k_p = 1$, $\tau_i = 1$, $p_r = 0.99$. The resetting set $\mathcal{R}$ of the compensator is not specified, since the stability results are independent of $\mathcal{R}$. Consider the parameter $\beta \in \{0.5, 1\}$, then the base system is not asymptotically stable for all time-delay. Subsequently, the stability of the system is studied by the application of Proposition 3.10 for the two values of $\beta$:

*Case $\beta = 0.5$:* Fig. 3.16 shows the results obtained from the application of Proposition 3.10 to the time-delay reset control system. The figure shows $\Delta_m$ and $\Delta_M$ as a function of the time-delay (results obtained with a time-delay step of 0.001).

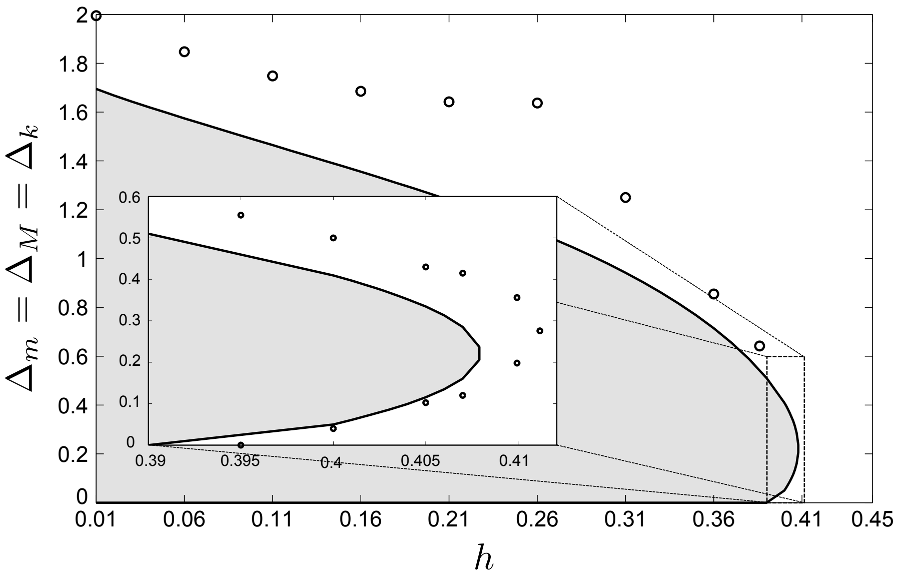
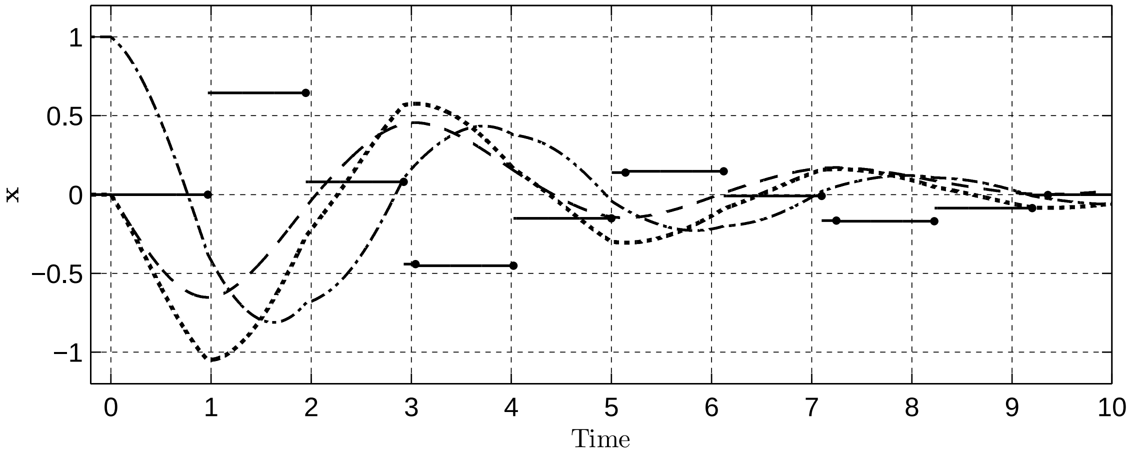



Figure 3.17 Allowable reset period as a function of the time-delay for the time-delay reset control system of the Example 3.13 obtained by Proposition 3.10 with $r = 10$.

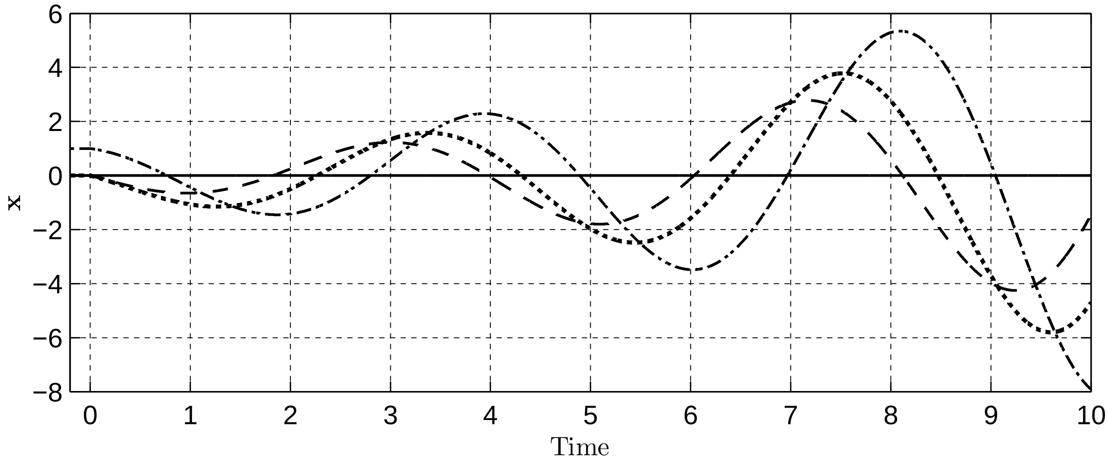
The system is asymptotically stable if for a given time-delay $h$, the reset intervals are inside the shaded area. Suppose now that the reset actions are periodic, that is $\Delta_m = \Delta_M = \Delta_k$. Fig. 3.17 shows the allowable reset period, obtained by Proposition 3.10, as a function of the time-delay (step time-delay of 0.001). In addition, it is also shown the maximum and minimum period for which the instability of the time-delay reset control system is guaranteed by simulation.

In order to show an example of a trajectory, consider a PI+CI provided with a zero crossing resetting law, and in addition, consider the time-delay $h = 0.2$, the parameter $r = 10$, $\Delta_m = 0.001$, $\Delta_M = 0.978$, and the initial condition $\phi(\theta) = (1, 0, 0, 0)$, $\theta \in [-0.2, 0]$. The evolution of the time-delay reset control system and its base system are shown in Fig. 3.18. Note that the trajectory of the base system diverges while the trajectory of the reset control system converges to zero. In addition, it can be seen that the first three states of the system evolve by the continuous dynamic, but the last state evolves only by the reset actions.

*Case* $\beta = 1$*:* The system is stable for the time-delay $h = 0.1$ by Proposition 3.10 for $r = 10$, $\Delta_m = 0.5$, and $\Delta_M = 0, 63$. Consider the PI+CI with a zero crossing resetting law, and the initial condition $\phi(\theta) = (1, 0, 0, 0)$, $\theta \in [-0.1, 0]$. The evolution of the time-delay reset control system and its base system can be shown in Fig. 3.19. Again




(a) Trajectory: $x_{p_1}$ (dashed-dotted), $x_{p_2}$ (dashed), $x_i$ (dotted) and $x_{ci}$ (solid).

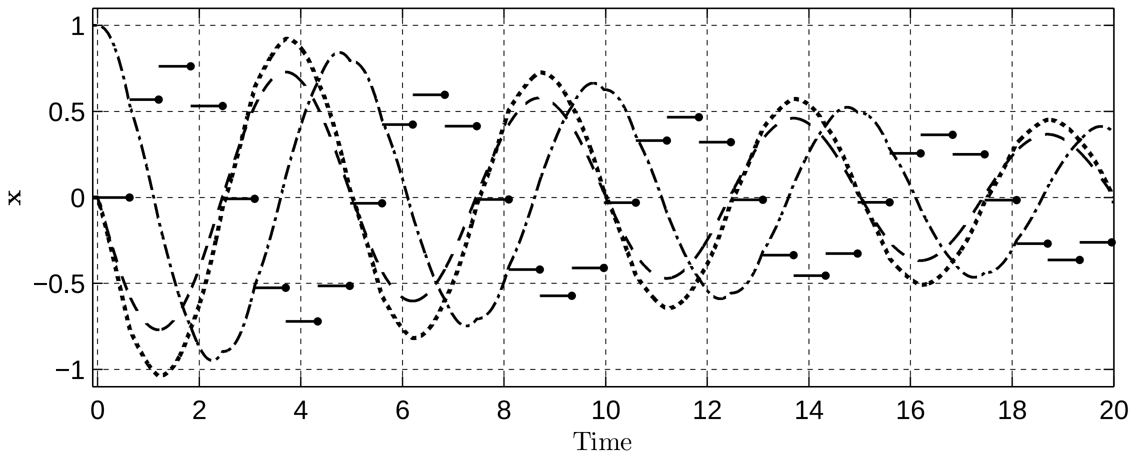



(b) Trajectory: $x_{p_1}$ (dashed-dotted), $x_{p_2}$ (dashed), $x_i$ (dotted) and $x_{ci}$ (solid).

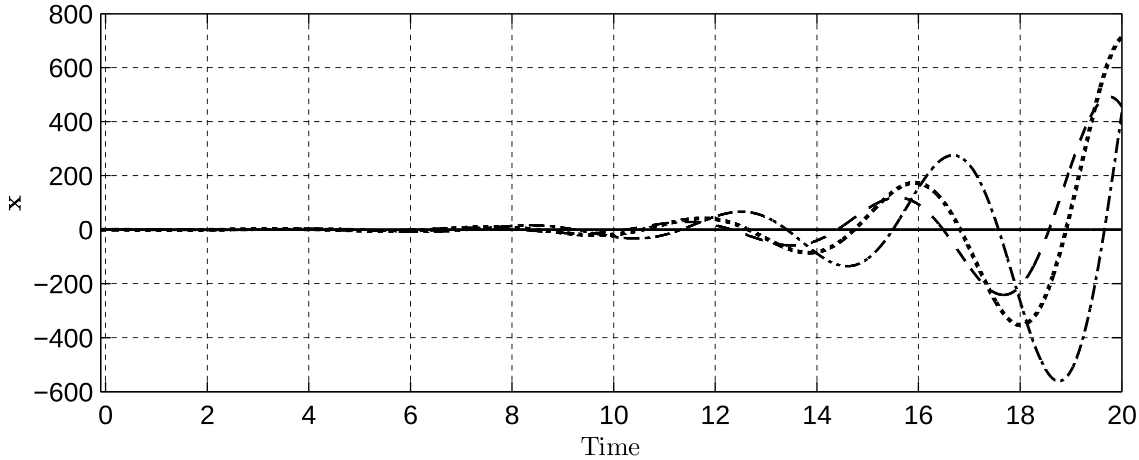
Figure 3.18 Trajectory and value of the LK functional and the functional $W$ for the time-delay reset control system of the Example 3.13 with $\beta = 0.5$, time-delay $h = 0.2$, $r = 10$, $\Delta_m = 0.001$, $\Delta_M = 0.978$, and initial condition $\phi(\theta) = (1, 0)$, $\theta \in [-0.2, 0]$.

the trajectory of the base system diverges while the trajectory of the reset control system converges to zero.

Now consider that the reset actions are periodic with period 0.1, then Fig. 3.20a shows the evolution of the system. It can be seen that the system is unstable, showing that a very frequent reset actions make the system unstable. Let go further, and suppose that the states of the compensator does not evolves (the limit when the reset intervals tend to zero). Therefore, the compensator is only a proportional compensator. In this case, the evolution of the system is plotted in Fig. 3.20b. It is clear that



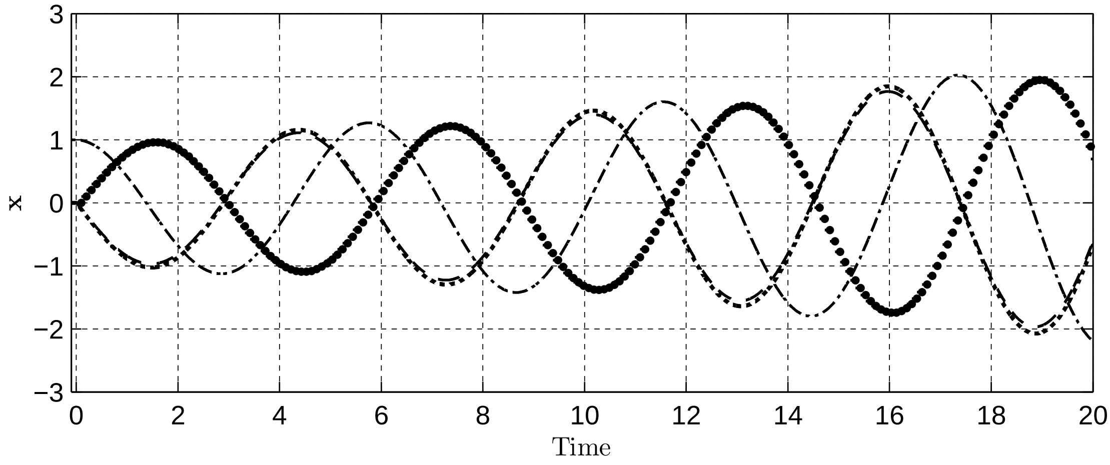

(a) Trajectory: $x_{p_1}$ (dashed-dotted), $x_{p_2}$ (dashed), $x_i$ (dotted) and $x_{ci}$ (solid).



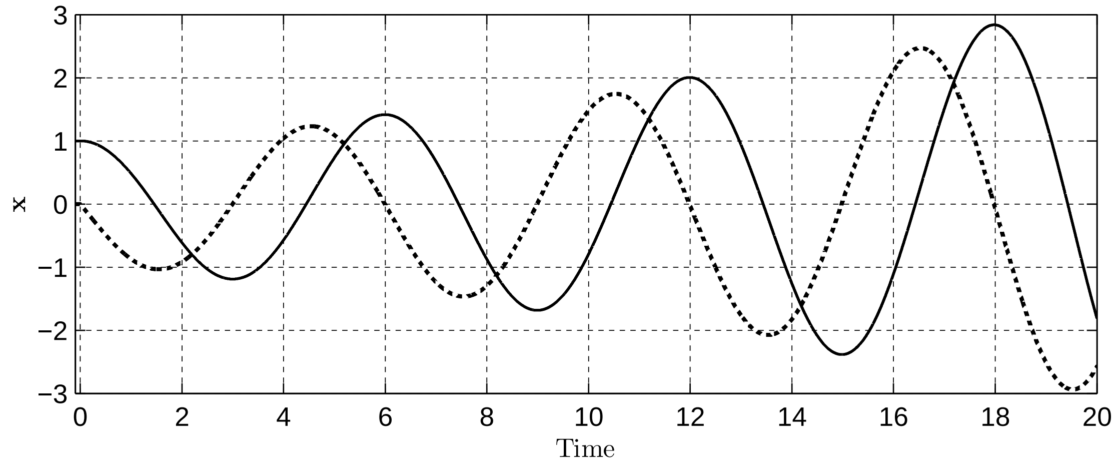

(b) Trajectory: $x_{p_1}$ (dashed-dotted), $x_{p_2}$ (dashed), $x_i$ (dotted) and $x_{ci}$ (solid).

Figure 3.19 Trajectory of the time-delay reset control system of the Example 3.13 and its base system with time-delay $h = 0.1$, $\Delta_m = 0.121$, $\Delta_M = 0.5$, and the initial condition $\phi(\theta) = (1, 0)$, $\theta \in [-0.1, 0]$.

the trajectory diverges, and the system is unstable. This example is importance, since it shows that the instability of the system does not arise from the dynamic of the compensator. Otherwise, we can expect that applying reset actions with enough frequency the states of the compensator will have practically no effect on the dynamic of the system, and thus, the system could be stable.




(a) Trajectory: $x_{p_1}$ (dashed-dotted), $x_{p_2}$ (dashed), $x_i$ (dotted) and $x_{ci}$ (solid).




(b) Trajectory: $x_{p_1}$ (solid) and $x_{p_2}$ (dotted).

Figure 3.20 Trajectory of the time-delay reset control system of the Example 3.13 with time-delay $h = 0.1$ and initial condition $\phi(\theta) = (1, 0)$, $\theta \in [-0.1, 0]$. Trajectory with reset period 0.1 (Fig. a), and trajectory with no evolution of the compensator states (Fig. b).

# Chapter 4

# Reset Control Systems with Saturation

This chapter is devoted to the stability analysis of reset control systems subject to plant input saturation. In particular, we address the estimation of the region of attraction. First, several theoretical results concerning the solutions of the saturated reset control system are provided. Second, we develop a procedure to obtain an estimation of the region of attraction as the union of polytopes. The main idea is the representation of the behavior of the system by a directed graph whose nodes are regions of the state-space represented by polytopes. The analysis of the arcs in the graph allows to establish the convergence of the trajectories for a set of initial conditions.

## 4.1 Preliminaries

### 4.1.1 Background about Polytopes

This section provides some definitions regarding polytopes that will be used throughout the chapter.

**Definition 4.1 (Hyperplane).** A *hyperplane* in $\mathbb{R}^n$ is a $(n-1)-$dimensional affine subspace. A hyperplane is denoted by $\mathcal{H}(N,\gamma)$ and parameterized by a row vector $N \in \mathbb{R}^{1\times n}$ and a real number $\gamma \in \mathbb{R}$. The hyperplane is defined by $\mathcal{H}(N,\gamma) = \{\mathbf{x} \in \mathbb{R}^n : N\mathbf{x} = \gamma\}$.

A hyperplane $\mathcal{H}(N,\gamma)$ separates $\mathbb{R}^n$ into two half-spaces, given by the inequalities $N\mathbf{x} < \gamma$ and $N\mathbf{x} > \gamma$. In the following, the half-space representation will be used to defined a closed convex polytope; the name polytope is used by simplicity.

**Definition 4.2 (Polytope).** A $n$-dimensional *polytope* $\mathcal{P}$ in $\mathbb{R}^n$ is the intersection of $m$ half-spaces defined the hyperplanes $\{\mathcal{H}(N_1, \gamma_1), \ldots, \mathcal{H}(N_m, \gamma_m)\}$ (called *supporting hyperplanes*):

$$\mathcal{P} = \bigcap_{i=1}^{m} \{\mathbf{x} \in \mathbb{R}^n : N_i \mathbf{x} \leq \gamma_i\}. \tag{4.1}$$

Parameterizing the polytope by a matrix $G \in \mathbb{R}^{m \times n}$ and a vector $\mathbf{g} \in \mathbb{R}^m$, the polytope is defined by

$$\mathcal{P}(G, \mathbf{g}) = \{\mathbf{x} \in \mathbb{R}^n : G\mathbf{x} \preceq \mathbf{g}\}, \tag{4.2}$$

where the rows of $G$ and $\mathbf{g}$ are given by $N_i$ and $\gamma_i$, respectively. In addition, we denote $\mathcal{H}_i \mathcal{P}$ the $i-$th supporting hyperplane.

Note that a polytope is not necessarily a bounded set, and that it comprises elements of different dimensionality like *vertices*, *edges*, ..., *facets*. These elements are usually referred to as $k$-faces; that is, a vertex is a 0-face, an edge is a 1-face, $\cdots$, and a facet is a $(n-1)$-face.

**Definition 4.3 ($k-$face).** For a given polytope $\mathcal{P}$, a $k - face$ is an intersection of the polytope $\mathcal{P}$ with $n-k$ supporting hyperplanes.

A *facet* of a polytope $\mathcal{P}$ is the intersection of the polytope with one of its supporting hyperplanes. The facet associated with the $i-$th supporting hyperplane is denoted by $\partial_i \mathcal{P}$.

Note that a facet is a $(n-1)-$face, and a $k-$face is a $k-$dimensional polytope.

**Definition 4.4 (Partition).** A set of polytopes $\{\mathcal{P}_1, \mathcal{P}_2, \ldots, P_N\}$ is a *partition* of a region $\mathcal{X} \subseteq \mathbb{R}^n$ if

$$\bigcup_{i=1}^{N} \mathcal{P}_i = \mathcal{X}, \quad \mathrm{int}(\mathcal{P}_i) \cap \mathrm{int}(\mathcal{P}_j) = \emptyset, \; i \neq j. \tag{4.3}$$

### 4.1.2 Reset Control System with Plant Input Saturation

Consider a reset control system as shown in Fig. 4.1, given by the feedback interconnection of a LTI system P with input saturation, and a reset compensator R (both single-input-single-output). The state-space representation of P is given by

$$P : \begin{cases} \dot{\mathbf{x}}_p(t) = A_p \mathbf{x}_p(t) + B_p sat(u_p(t)), \\ y_p(t) = C_p \mathbf{x}_p(t), \end{cases} \tag{4.4}$$

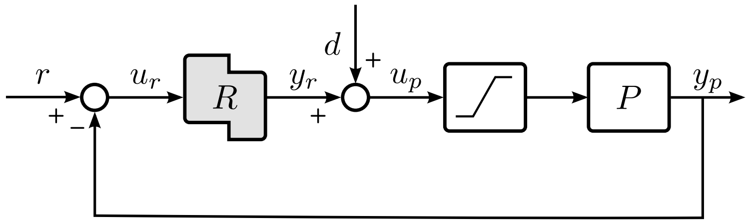



Figure 4.1 Reset control system with plant input saturation.

where $\mathbf{x}_p \in \mathbb{R}^{n_p}$ and $A_p \in \mathbb{R}^{n_p \times n_p}$, $B_p \in \mathbb{R}^{n_p \times 1}$, $C_p \in \mathbb{R}^{1 \times n_p}$. The input of P is limited in magnitude by the following saturation function

$$sat(u_p) = \begin{cases} u_0 & \text{if} \quad u_p > u_0, \\ u_p & \text{if} \quad -u_0 \leq u_p \leq u_0, \\ -u_0 & \text{if} \quad u_p < -u_0, \end{cases} \tag{4.5}$$

where $u_0 > 0$ denotes the level of saturation. It has been considered the case of symmetric saturation for the sake of the simplicity. The compensator R is a zero crossing reset compensator given by

$$R : \begin{cases} \dot{\mathbf{x}}_r(t) = A_r \mathbf{x}_r(t) + B_r u_r(t), & u_r(t) \neq 0, \\ \mathbf{x}_r(t^+) = A_\rho \mathbf{x}_r(t), & u_r(t) = 0, \\ y_r(t) = C_r \mathbf{x}_r(t) + D_r u_r(t), & \end{cases} \tag{4.6}$$

where $\mathbf{x}_r \in \mathbb{R}^{n_r}$, $A_r \in \mathbb{R}^{n_r \times n_r}$, $B_r \in \mathbb{R}^{n_r \times 1}$, $C_r \in \mathbb{R}^{1 \times n_r}$, $D_r \in \mathbb{R}$. It is assumed that the last $n_\rho$ compensator states are set to zero at the reset instants, then the reset matrix $A_\rho$ is partitioned in blocks as follows:

$$A_\rho = \begin{bmatrix} I & 0 \\ 0 & 0_{n_\rho} \end{bmatrix}. \tag{4.7}$$

The feedback interconnection with zero exogenous signals is obtained by making $u_r(t) = -y_p(t)$ and $u_p(t) = y_r(t)$. Hence, defining the closed-loop state as $\mathbf{x} = (\mathbf{x}_p, \mathbf{x}_r) \in \mathbb{R}^n$, with $n = n_p + n_r$, and considering the saturation function (4.5), the state-space is divided into three *regions of saturation* (see, e.g., [261]) (see Fig. 4.2), which are defined by unbounded polytopes. We will denote de regions of saturation by

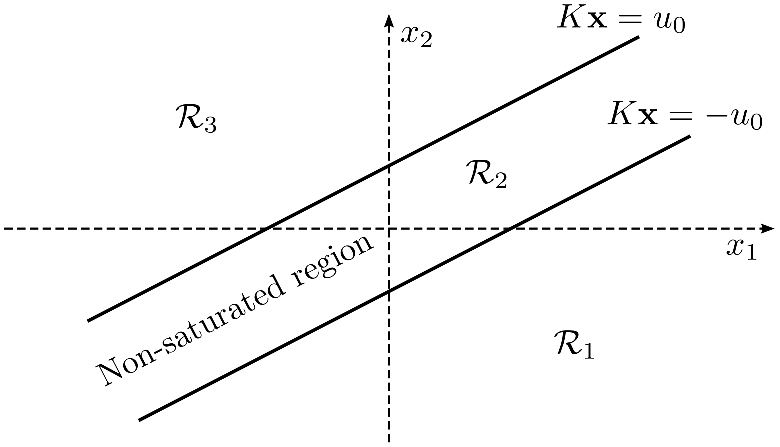



Figure 4.2 Illustration of the regions of saturation. $\mathcal{R}_j$, $j \in \{1, 2, 3\}$.

$\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_3$, where

$$\begin{aligned} \mathcal{R}_1 &= \mathcal{P}\left(K, -u_0\right), \\ \mathcal{R}_2 &= \mathcal{P}\left(\begin{bmatrix} K \\ -K \end{bmatrix}, (u_0, u_0)\right), \\ \mathcal{R}_3 &= \mathcal{P}\left(-K, -u_0\right), \end{aligned} \tag{4.8}$$

and

$$K = \begin{bmatrix} -D_r C_p & C_r \end{bmatrix}. \tag{4.9}$$

The notation of the regions of saturation should not be confused with the resetting set of the reset compensators as it was defined in Chapter 2.

The region $\mathcal{R}_2$ will be referred to as *non-saturated region*[1], since the saturation is not active inside that region. As a result, the *saturated reset control system* can be represented as a switched and impulsive system with affine dynamic as follows:

$$\begin{cases} \dot{\mathbf{x}}(t) = A_j \mathbf{x}(t) + \mathbf{a}_j, & \mathbf{x}(t) \in \mathcal{R}_j \setminus \mathcal{M}, \quad j \in \{1, 2, 3\}, \\ \mathbf{x}(t^+) = A_R \mathbf{x}(t), & \mathbf{x}(t) \in \mathcal{M}, \\ y(t) = C\mathbf{x}(t), \end{cases} \tag{4.10}$$

[1]The region of the state-space where the saturation is not active is usually called *region of linearity* ([261]). However, the name *non-saturated region* has been considered more appropriate, since the system under study is a reset control system; that is, the system is nonlinear even thought the saturation is not active.

with $A_1 = A_3 = A$, $A_2 = A + BK$, $\mathbf{a}_1 = -\mathbf{a}_3 = -Bu_0$ and $\mathbf{a}_2 = \mathbf{0}$. Matrices $A$, $B$, $C$ and $A_R$ are given by

$$A = \begin{bmatrix} A_p & 0 \\ -B_r C_p & A_r \end{bmatrix}, \qquad B = \begin{bmatrix} B_p \\ 0 \end{bmatrix},$$
$$C = \begin{bmatrix} C_p & 0 \end{bmatrix}, \qquad A_R = \begin{bmatrix} I & 0 \\ 0 & A_\rho \end{bmatrix}. \tag{4.11}$$

The resetting set $\mathcal{M} \subset \mathbb{R}^n$ and the after-reset set $\mathcal{M}_{\mathcal{R}}$ are defined as follows:

$$\mathcal{M} = \mathcal{N}(C) \setminus \mathcal{M}_{\mathcal{R}}, \quad \mathcal{M}_{\mathcal{R}} = \mathcal{N}\left(\begin{bmatrix} I - A_R \\ C \end{bmatrix}\right). \tag{4.12}$$

Note that, in general, a reset control system with plant input saturation may have limit cycles (this is not the case for a saturation-free reset control system with zero crossing resetting law). For the analysis of the attraction region to be developed in the following, it will be assumed that no limit cycles exist. In addition, for the sake of the simplicity, we consider the following assumptions: the matrix $A_2 = A + BK$ is Hurwitz, that is, the origin is locally asymptotically stable for the base system, the pair $(A, B)$ is controllable, and finally the plant and the base compensator have no complex poles, that is all the eigenvalues of the matrix $A$ are real.

## 4.2 Theoretical Results

In this section, we tackle some fundamental properties regarding the solutions of the reset control system and its base system.

### 4.2.1 Existence and Uniqueness of the Solutions

This section addresses the existence and uniqueness of the solutions of the saturated reset control system (4.10). To do this, we provide first some properties of the solutions of its base system.

As usual, the base system of the saturated reset control system (4.10) is defined as the system resulting from removing the resetting law; in this case, it is given by

$$\begin{cases} \dot{\mathbf{x}}_b(t) = A_j \mathbf{x}_b(t) + \mathbf{a}_j, \quad \mathbf{x}_b(t) \in \mathcal{R}_j, \quad j \in \{1,2,3\}, \\ y(t) = C\mathbf{x}_b(t). \end{cases} \tag{4.13}$$

In the next Proposition, the existence/uniqueness of solutions as well as the continuous dependence on the initial condition are investigated for the base saturated control system.

**Proposition 4.1.** *The base system (4.13) has exactly one solution over* $[0,\infty)$ *for every initial condition* $\mathbf{x}_0 \in \mathbb{R}^n$*. In addition, for a given initial condition* $\mathbf{x}_0$*, let* $T \in [0,\infty)$ *be specified, then for any* $\epsilon > 0$*, there exists a* $\delta = \delta(\epsilon, T) > 0$ *such that for every initial condition* $\mathbf{x}_0^*$ *with* $\|\mathbf{x}_0 - \mathbf{x}_0^*\| < \delta$*, it is true that* $\|\mathbf{x}_b(t,\mathbf{x}_0) - \mathbf{x}_b(t,\mathbf{x}_0^*)\| \leq \epsilon$*.*

*Proof.* The basic idea of the proof consists in showing that the right hand-side of the differential equation (4.13) is globally Lipchitz continuous, so that the direct application of Theorems 25 and 57 of Section 2.4 in [273] concludes the proof. To proof the Lipchitz continuity, let $\eta \geq \max_{j\in\{1,2,3\}}(\|A_j\|)$, then it is clear that for any given $j \in \{1,2,3\}$ it is satisfied the Lipchitz condition with constant $\eta$ for all $\mathbf{x}$, $\mathbf{y} \in \mathcal{R}_j$.

Now suppose that $\mathbf{x} \in \mathcal{R}_1$ and $\mathbf{y} \in \mathcal{R}_3$ then there exists $\alpha_1$, $\alpha_2 \in [0,1]$ such that $\mathbf{u}_1 = \alpha_1\mathbf{x} + (1-\alpha_1)\mathbf{y} \in \mathcal{R}_1 \cap \mathcal{R}_2$ and $\mathbf{u}_2 = \alpha_2\mathbf{x} + (1-\alpha_2)\mathbf{y} \in \mathcal{R}_2 \cap \mathcal{R}_3$. Since $\mathbf{x}$, $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{y}$ are collinear then $\|\mathbf{x}-\mathbf{y}\| = \|\mathbf{x}-\mathbf{u}_1\| + \|\mathbf{u}_1-\mathbf{u}_2\| + \|\mathbf{u}_2-\mathbf{y}\|$, and it follows:

$$\begin{aligned} \|(A_1\mathbf{x}+\mathbf{a}_1) - (A_3\mathbf{y}+\mathbf{a}_3)\| \leq{} & \|(A_1\mathbf{x}+\mathbf{a}_1) - (A_1\mathbf{u}_1+\mathbf{a}_1)\| \\ & + \|(A_2\mathbf{u}_1+\mathbf{a}_2) - (A_2\mathbf{u}_2+\mathbf{a}_2)\| \\ & + \|(A_2\mathbf{u}_2+\mathbf{a}_2) - (A_3\mathbf{y}+\mathbf{a}_3)\| \leq \eta\|\mathbf{x}-\mathbf{y}\|. \end{aligned} \tag{4.14}$$

The same inequality can be obtained for the cases $\mathbf{x} \in \mathcal{R}_1$ and $\mathbf{y} \in \mathcal{R}_2$, and $\mathbf{x} \in \mathcal{R}_3$ and $\mathbf{y} \in \mathcal{R}_2$. Therefore, the Lipchitz condition is guaranteed for every $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$, and the proof is complete.

□

The above proposition guarantees the well-posedness of the base system (4.13); there exists a unique solution over $[0,\infty)$ for every initial condition, and in addition, the solution depends continuously on the initial condition. However, the well-posedness of the base system does not imply the well-posedness of the saturated reset control system (4.10). In the sequel, sufficient conditions for the existence and uniqueness of the solution of the saturated reset control system are developed.

**Proposition 4.2.** *If the reset control system (4.10) without saturation* ($u_0 = \infty$) *has a unique solution on* $[0, \infty)$*, for all* $\mathbf{x}_0 \in \mathbb{R}^n$*, and in addition,* $\mathcal{M}_R \subset \mathrm{int}(\mathcal{R}_2)$*, then the saturated reset control system (4.10) has a unique solution on* $[0, \infty)$*, for any initial condition* $\mathbf{x}_0 \in \mathbb{R}^n$*.*

*Proof.* Consider an initial condition for which the solution of the system does not perform reset actions, then Proposition 4.1 guarantees that the solution is unique and exists along the interval $[0, \infty)$. Now suppose that reset actions occur. The condition $\mathcal{M}_R \subset \mathrm{int}(\mathcal{R}_2)$ guarantees that there exists an $\epsilon > 0$ such that $\|\mathbf{x} - \mathbf{y}\| > \epsilon$ for all $\mathbf{x} \in \mathcal{M}_R$ and $\mathbf{y} \notin \mathrm{int}(\mathcal{R}_2)$, and in addition, it is guaranteed that after a reset instant the system evolves by the non-saturated dynamic. Therefore, the existence and uniqueness of the solution of system (4.10) are guaranteed by the existence and uniqueness of the solution of the reset control system without saturation. To prove that the solution exists on the interval $[0, \infty)$, suppose that there exists a Zeno solution, and that $t_k \to t_\infty < \infty$ as $k \to \infty$. Since the reset control system without saturation has not Zeno solution, then there exists an infinite sequence of reset instants $(t_i)$, $i = 1, 2, \ldots$ (it is a subset of the set of reset instants) and $T_i \in (t_i, t_{i+1}]$ such that $\mathbf{x}_i = \mathbf{x}(t_i^+, \mathbf{x}_0) \in \mathcal{M}_R$ and $e^{A_2(T_i - t_i)}\mathbf{x}_i \notin \mathrm{int}(\mathcal{R}_2)$. Otherwise, the saturated reset system evolves through the non-saturated region and no Zeno solution is possible. In addition, it is satisfied

$$\|e^{A_2(T_i - t_i)}\mathbf{x}_i - \mathbf{x}_i\| \to 0, \quad i \to \infty. \tag{4.15}$$

However, it is a contradiction, since $\|\mathbf{x} - \mathbf{y}\| > \epsilon$ for all $\mathbf{x} \in \mathcal{M}_R$ and $\mathbf{y} \notin \mathrm{int}(\mathcal{R}_2)$. Therefore, no Zeno solutions are possible, and the proof is complete.

□

### 4.2.2 Equilibrium Points, Region of Attraction, and Region of Convergence

A point $\mathbf{x}_e$ is an *equilibrium point* of the system (4.10) if for some $j \in \{1, 2, 3\}$ $\mathbf{x}_e \in \mathcal{R}_j \setminus \mathcal{M}$ and it satisfies $A_j\mathbf{x}_e + \mathbf{a}_j = \mathbf{0}$. It is clear that $\mathbf{0}$ is an equiliium point, and it is the only equilibrium point in $\mathcal{R}_2$ if $\mathcal{N}(A_2) = \{\mathbf{0}\}$ (which is satisfied for example if $A_2$ is Hurwitz). Therefore, besides the origin due to the saturation there are only two more points which are *candidate* equilibrium points, $\mathbf{x}_{e+} = A^{-1}Bu_0$ and $\mathbf{x}_{e-} = -A^{-1}Bu_0$ (remind that $A_1 = A_3 = A$). Note that $\mathbf{x}_{e+}$ and $\mathbf{x}_{e-}$ are only defined when the matrix $A$ is nonsingular (by the controllability assumption, there is not an infinity number of equilibrium points when $A$ is singular, see, e.g., [261]). In addition, if $\mathbf{x}_{e+} \in \mathcal{R}_1$ and $\mathbf{x}_{e-} \in \mathcal{R}_3$, then they are equilibrium points of the saturated reset control system, once it is shown that the candidates $\mathbf{x}_{e+}$ and $\mathbf{x}_{e-}$ does not belong to the resetting set.

**Proposition 4.3.** *The origin $\mathbf{0} \in \mathcal{R}_2$ is the unique equilibrium point in the non-saturated region. In addition, if $\mathbf{x}_{e+} \in \mathcal{R}_1$ and $\mathbf{x}_{e-} \in \mathcal{R}_3$ are the unique points that satisfies $A\mathbf{x}_{e+} - Bu_0 = \mathbf{0}$ and $A\mathbf{x}_{e-} + Bu_0 = \mathbf{0}$, then $\mathbf{x}_{e+}$ and $\mathbf{x}_{e-}$ are the unique equilibrium points of the saturated reset control system (4.10) in their respective regions.*

*Proof.* Since $A_2$ is Hurwitz, $\mathbf{0} \in \mathcal{R}_2$ is the unique equilibrium point of the base system. In addition, since by definition of the resetting set $\mathcal{M}$ it is clear that $\mathbf{0} \notin \mathcal{M}$, then $\mathbf{0}$ is the unique equilibrium point of the saturated reset control system in the non-saturated region.

For $\mathbf{x}_{e+} \in \mathcal{R}_1$, use $\mathbf{x}_{e+} = (\mathbf{x}_{pe+}, \mathbf{x}_{re+})$. Assume that $\mathbf{x}_{e+} \in \mathcal{M}$, then $\mathbf{x}_{e+} \in \mathcal{N}(C)$, and thus, by using $A\mathbf{x}_{e+} - Bu_0 = \mathbf{0}$ we obtain

$$\begin{aligned} A_p\mathbf{x}_{pe+} - B_p u_0 &= \mathbf{0}, \\ A_r\mathbf{x}_{re+} &= \mathbf{0}. \end{aligned} \tag{4.16}$$

Therefore, it directly follows that $\mathbf{x}_{re+} = \mathbf{0}$, otherwise $A_r$ would be singular, and then $A$ would be also singular which implies that $A\mathbf{x} - Bu_0 = \mathbf{0}$ has no solution for $\mathbf{x}$. Finally, since $\mathbf{x}_{re+} = \mathbf{0}$ then $\mathbf{x}_{e+} \in \mathcal{M}_{\mathcal{R}}$, which is a contradiction (note that $\mathcal{M}_{\mathcal{R}} \cap \mathcal{M} = \emptyset$). As a result, $\mathbf{x}_{e+} \notin \mathcal{M}$, and thus, $\mathbf{x}_{e+}$ is an equilibrium point of the saturated reset control system. The same procedure is applied to the case $\mathbf{x}_{e-}$.

□

**Definition 4.5 (Region of attraction and Region of convergence).** For the saturated reset control system (4.10), the *region of attraction* $\mathcal{R}_A$ is the set of all states $\mathbf{x}_0 \in \mathbb{R}^n$ for which the solution $\mathbf{x}(t, \mathbf{x}_0)$ of the saturated reset control system converges asymptotically to the origin, that is

$$\mathcal{R}_A = \{\mathbf{x}_0 \in \mathbb{R}^n : \mathbf{x}(t, \mathbf{x}_0) \text{ is a solution of (4.10), and } \lim_{t\to\infty} \mathbf{x}(t, \mathbf{x}_0) = \mathbf{0}\} \tag{4.17}$$

Any subset of $\mathcal{R}_A$ is referred to as a *region of convergence*, and denoted by $\mathcal{R}_C$. In addition, if $\mathbf{0} \in \text{int}(\mathcal{R}_C)$ then $\mathcal{R}_C$ is called a *region of asymptotic stability* ($RAS$).

### 4.2.3 Positive Invariant Sets

This section provides sufficient conditions to check if a subset of the state-space, defined by a polytope, has trajectories that remains inside the subset forever. A formal definition of these subsets is given as follows:

**Definition 4.6 (Positive invariant set).** A set $\mathcal{X} \subset \mathbb{R}^n$ is a positive invariant set if for every initial condition $\mathbf{x}_0 \in \mathcal{X}$ the solution $\mathbf{x}(t, \mathbf{x}_o)$ of the reset control system (4.10) remains in $\mathcal{X}$, that is, if $\mathbf{x}_0 \in \mathcal{X}$ then $\mathbf{x}(t, \mathbf{x}_0) \in \mathcal{X}$ $\forall t \geq 0$.

In order to make us focus on the systematic analysis instead of putting too much efforts on some complicated arguments for trivial cases, it will be assumed that the matrix $A$ is diagonalizable (consequently, $A_1$ and $A_3$ are diagonalizable), and in addition, the algebraic multiplicity of its eigenvalues is 1.

Let $\sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be the set of eigenvalues of $A$. The eigenvalues are assumed to be ordered, that is $\lambda_1 < \lambda_2 < \ldots < \lambda_n$. $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is the set of eigenvectors of $A$, where $\mathbf{v}_i$ is the eigenvector corresponding to $\lambda_i$, $i = 1, .., n$. For a real eigenvalue $\lambda_i$, we define $\mathcal{D}_{\lambda_i}$ as the subspace spanned by the first $i$ eigenvectors, that is $\mathcal{D}_{\lambda_i} = span\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_i\} \subseteq \mathbb{R}^n$.

In the following, a technical result will be developed in order to avoid the existence of positive invariant sets in the interior of a polytope. For the sake of the simplicity, the result is focused on polytopes whose interior does not intersect with the resetting set, $\text{int}(\mathcal{P}) \cap \mathcal{M} = \emptyset$, and in addition, they are completely inside of one of the regions of saturation, that is $\mathcal{P} \subseteq \mathcal{R}_j$ for some $j \in \{1, 3\}$. Therefore, the evolution of the system throughout the polytope is governed by the simple differential equation $\dot{\mathbf{x}}(t) = A_j \mathbf{x}(t) + \mathbf{a}_j$. Here, if there exists, $\mathbf{x}_e$ is the solution of $A_j \mathbf{x}_e + \mathbf{a}_j = \mathbf{0}$.

**Proposition 4.4.** *If for a bounded polytope $\mathcal{P} = \mathcal{P}(G, \mathbf{g}) \subseteq \mathcal{R}_j$ for some $j \in \{1, 3\}$, the following statements hold:*

- $\text{int}(\mathcal{P}) \cap \mathcal{M} = \emptyset$,
- *$int(\mathcal{P})$ does not contain any equilibrium point,*
- *$\partial\mathcal{P}$ contains an equilibrium point, and*
- *$\mathcal{D}_{\lambda_j} \cap \text{int}(\mathcal{P}) = \emptyset$, with $\lambda_j$ (if there exists) the largest negative real eigenvalue.*

*Then there is not positive invariant sets in the interior of $\mathcal{P}$.*

*Proof.* By contradiction, assume that $\mathcal{P}$ has some positive invariant set. First note that $\text{int}(\mathcal{P}) \cap \mathcal{M} = \emptyset$ implies that if a trajectory remains in the interior of the polytope then will never reach the reset surface. Hence, it is only necessary to analyze the dynamic of the base system. In addition, since $\mathcal{P} \subseteq \mathcal{R}_j$ for some $j \in \{1, 3\}$, the dynamic of the base system is given by the differential equation $\dot{\mathbf{x}}(t) = A_j\mathbf{x}(t) + \mathbf{a}_j$, whose solution is given by

$$\mathbf{x}(t) = Ve^{Dt}V^{-1}\mathbf{x}_0 + \int_0^t Ve^{D(t-\tau)}V^{-1}\mathbf{a}_j d\tau \tag{4.18}$$

where $V = [\mathbf{v}_1|\mathbf{v}_2|\cdots|\mathbf{v}_n]$ and $D = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Define the vectors $\alpha = V^{-1}\mathbf{x}_0$ and $\beta = V^{-1}\mathbf{a}_j$, then the solution can be rewritten as (see, e.g., [199])

$$\mathbf{x}(t) = V \begin{pmatrix} \gamma_1(t) \\ \gamma_2(t) \\ \vdots \\ \gamma_n(t) \end{pmatrix} = \sum_{i=1}^{n} \gamma_i(t)\mathbf{v}_i, \tag{4.19}$$

where

$$\gamma_i(t) = \begin{cases} \alpha_i e^{\lambda_i t} + \frac{\beta_i}{\lambda_i}(e^{\lambda_i t} - 1), & \lambda_i \neq 0, \\ \alpha_i + \beta_i t, & \lambda_i = 0. \end{cases} \tag{4.20}$$

Let $\lambda_s$ be the largest eigenvalue such that $\alpha_s \neq 0$ and/or $\beta_s \neq 0$, thus $\gamma_i(t) = 0$ for any $t \geq 0$ and $i \geq s+1$. Since $\mathbf{x}(t, \mathbf{x}_0) \in \text{int}(\mathcal{P})$ for all $t \geq 0$, then it is satisfied

$$GV \begin{pmatrix} \gamma_1(t) \\ \vdots \\ \gamma_s(t) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \prec \mathbf{g} \tag{4.21}$$

for all $t \geq 0$. Suppose $\lambda_s < 0$, then $\mathbf{x}(t) \to \mathbf{x}_e = \sum_{i=1}^{s} -\frac{\beta_i}{\lambda_i}$. If $\mathbf{x}_e \notin \mathcal{P}$ then the trajectory always leaves the polytope in a finite time. If $\mathbf{x}_e \in \mathcal{P}$, then equation (4.21) implies that $D_{\lambda_s} \cap \text{int}(\mathcal{P}) \neq \emptyset$, which is a contradiction, since $D_{\lambda_s} \subseteq D_{\lambda_j}$, with $\lambda_j$ the largest negative real eigenvalue. Now, suppose $\lambda_s \geq 0$, then $\gamma_s(t) \to \pm\infty$ as time increases. Since the polytope is bounded, the trajectory will leave the polytope for some finite instant. Therefore, the proof is complete.

□

**Proposition 4.5.** *If for an unbounded polytope $\mathcal{P} = \mathcal{P}(G, \mathbf{g}) \subseteq \mathcal{R}_j$ for some $j \in \{1, 3\}$, the statements in Proposition 4.4 hold, and in addition, $D_{\lambda_j} \cap \operatorname{int}(\mathcal{P})$ is bounded, where $\lambda_j$ is defined as the largest positive real eigenvalue such that span$\{\mathbf{v}_j\} \cap \mathcal{P}$ is unbounded, then there is not positive invariant sets in the interior of $\mathcal{P}$.*

*Proof.* The proof is equal to the proof of Proposition 4.4. However, when $\lambda_s \geq 0$, the bounded argument cannot be applied. In this case, equation (4.21) implies that $\operatorname{span}(\mathbf{v}_s) \cap \mathcal{P}$ and $D_{\lambda_s} \cap \operatorname{int}(\mathcal{P})$ are unbounded. Therefore, $\operatorname{span}(\mathbf{v}_j) \cap \mathcal{P}$ and $D_{\lambda_j} \cap \operatorname{int}(\mathcal{P})$ are also unbounded by the definition of $\lambda_j$ and $D_{\lambda_j}$. By the assumption in the proposition the last statement is not possible, and the proof is complete.

□

### 4.2.4 Ellipsoidal Estimate of the Region of Attraction

The first work dealing with the characterization of the region of attraction of a reset control system is [189], where the results are limited to the CI and FORE compensators. These results are generalized in [187] to the study of systems controlled with a reset compensator of higher dimension. The authors proposed a criterion for the local asymptotic stability of the reset control system, based on the second Lyapunov method. The region of attraction is approximated by an ellipsoid given by $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}^\top E \mathbf{x} \leq 1\}$, where the symmetric matrix $E$ is related with a quadratic Lyapunov function, $V(\mathbf{x}) = \mathbf{x}^\top P \mathbf{x}$. The results are developed under the HI framework. However, they can be easily adapted to the IDS framework. The following corollary is directly obtained from the results in [187] considering the formulation in the IDS framework.

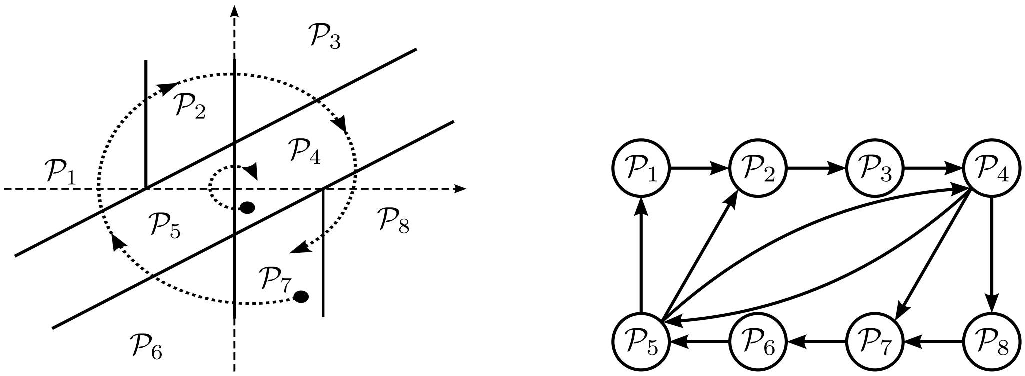
**Corollary 4.1.** *If there exist a matrix $P \in \mathbb{S}_+^{n \times n}$, a matrix $G \in \mathbb{R}^{1 \times n}$, and positive scalars $U$ satisfying:*

$$\begin{bmatrix} A^\top P + PA & PB - G^\top \\ \star & -2U \end{bmatrix} < 0, \tag{4.22}$$

$$A_R^\top P A_R - P \leq 0, \tag{4.23}$$

$$\begin{bmatrix} P & K^\top U - G^\top \\ \star & u_0^2 \end{bmatrix} \geq 0, \tag{4.24}$$

*then the ellipsoid given by $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}^\top \frac{P}{U^2} \mathbf{x} \leq 1\}$ is a region of asymptotic stability of the system (4.10).*




(a) State-space partition. (b) Directed graph.

Figure 4.3 Illustrative example of a state-space partition and its associated graph.

## 4.3 Estimate of the Region of Attraction

### 4.3.1 Directed Graph Representation

In this section, we define two directed graphs denoted by $D_{base}$ and $D_{reset}$ which are associated with the saturated base system and the reset control system, respectively. Consider a partition of the state-space given by a set of polytopes, then the nodes of the graphs are the polytopes. Roughly speaking, there will be an arc between two nodes $P_I$ and $P_F$, if there is a trajectory that starting inside $P_I$ the trajectory reaches $P_F$ immediately after leaving $P_I$ (see Fig. 4.3). Therefore, the graphs can be understood as a reduced representation of the trajectories of the system.

#### Base System

The directed graph associated with the base system is given by $D_{base} = (\mathcal{N}_b, E_b)$, where the set of polytopes $\mathcal{N}_b$ is obtained by the following procedure:

**begin**
    **for** $j = 1$ **to** 3 **do**
        Find a partition $\Sigma = \{\mathcal{P}_1, \mathcal{P}_2, \ldots\}$ of $\mathcal{R}_j$ ;
        **forall the** $\mathcal{P}$ *in* $\Sigma$ **do**
            Divide $\mathcal{P}$ by the hyperplane $\mathcal{H}(C, 0)$ ;
            Add the resulting two polytopes to $\mathcal{N}_b$;
        **end**
    **end**
**end**

Consider the solution of the base system given by $\mathbf{x}_b(t, \mathbf{x}_0)$; thus, two polytopes $\mathcal{P}_I$ and $\mathcal{P}_F$ are said to be *connected*, denoted by $\mathcal{P}_I \rightarrow \mathcal{P}_F$, if there exists some $T > 0$ and an initial condition $\mathbf{x}_0 \in \text{int}(\partial_i \mathcal{P}_I)$ for some $i$, such that $\mathbf{x}_b(t, \mathbf{x}_0) \in \text{int}(\mathcal{P}_F)$ for all $t \in (0, T]$ (see Fig. 4.4). We will use $\mathcal{P}_I \xrightarrow{\partial_i} \mathcal{P}_F$ whenever the information of the facet is needed. Therefore, the set of arcs $E_b$ of the graph is formed by pairs $(\mathcal{P}_I, \mathcal{P}_F)$ such that $\mathcal{P}_I \rightarrow \mathcal{P}_F$.

**Corollary 4.2.** *Two polytopes $\mathcal{P}_I$ and $\mathcal{P}_F$, with $\mathcal{P}_I \subseteq \mathcal{R}_j$ are connected if and only if there exists $i$ and $k$ such that for some $\mathbf{x} \in \text{int}(\partial_i \mathcal{P}_I) \cap \text{int}(\partial_k \mathcal{P}_F)$ it is satisfied $N(A_j \mathbf{x} + \mathbf{a}_j) > 0$, with $N$ the normal row vector of the supporting hyperplane $\mathcal{H}_i \mathcal{P}_I$.*

### Reset System

The reset control system has clearly a richer dynamical behavior, which makes challenging to capture the behavior of all the trajectories of the reset control system with a simple directed graph. In order to simplify the analysis, we design the directed graph to capture the behavior of the discontinuous trajectories until the first reset instant, and then, this information will be used to conclude where the trajectories approach the origin or not. For this purpose, consider a previously known region of asymptotic stability, denoted by $\mathcal{R}_C^0$, such that $\mathcal{R}_C^0 \cap \mathcal{M} \neq \emptyset$. For instance, $\mathcal{R}_C^0$ can be obtained by Corollary 4.1. Now, suppose that a trajectory reaches $\mathcal{R}_C^0$ after the first reset instant, then it can be concluded that the trajectory approaches the origin. In order to exploit this idea, it is defined two special polytopes $\mathcal{P}^{\mathbf{0}}$ and $\mathcal{P}^{\infty}$. The first polytope $\mathcal{P}^{\mathbf{0}}$ is defined in such a way that it only contains the origin and consequently is a region of convergence. This polytope will be used to represent trajectories that reach $\mathcal{R}_C^0$ after the first reset instant. The second polytope $\mathcal{P}^{\infty}$ can be understood as an empty

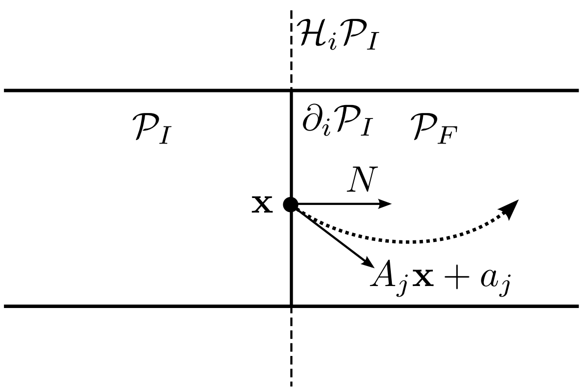
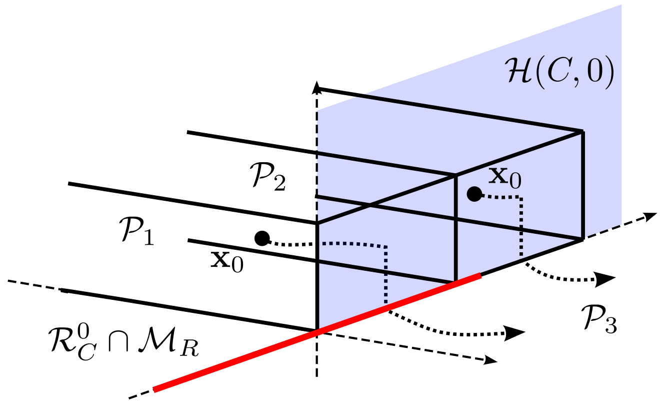
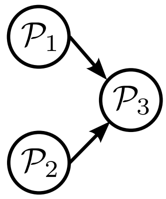


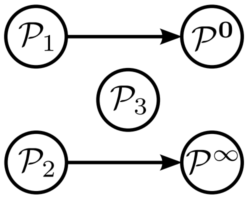

Figure 4.4 Connected polytopes: $\mathcal{P}_I \xrightarrow{\partial_i} \mathcal{P}_F$.




(a) State-space partition.




(b) Graph $D_{base}$.




(c) Graph $D_{reset}$.

Figure 4.5 Illustrative example of the construction of the graph $D_{reset}$. All the trajectories from $\mathcal{P}_1$ trough the reset surface reach the subset $\mathcal{R}_C^0 \cap \mathcal{M}_R$ (red line).

polytope. This polytope represents the trajectories that not reach $\mathcal{R}_C^0$ after the first reset instant, and then, they are assumed not to approach the origin. Note that these trajectories may converge to the origin, for instance reaching $\mathcal{R}_C^0$ after several reset actions.

The directed graph associated with the reset control system is $D_{reset}(\mathcal{N}_r, E_r)$, where $\mathcal{N}_r = \mathcal{N}_b \cup \{\mathcal{P}^{\mathbf{0}}, \mathcal{P}^{\infty}\}$. Note that $\text{int}(\mathcal{P}^{\mathbf{0}}) = \emptyset$ and $\text{int}(\mathcal{P}^{\infty}) = \emptyset$, and thus, $\mathcal{N}_r$ is coherent with the definition of partition. The set of arcs $E_r$ is obtained by the following procedure (the procedure is illustrated in Fig. 4.5):

**begin**
  **forall the** $(\mathcal{P}_I, \mathcal{P}_F)$ *in* $E_b$ *with* $\mathcal{P}_I \xrightarrow{\partial_i} \mathcal{P}_F$ **do**
    **if** $\mathcal{H}_i \mathcal{P}_I \neq \mathcal{H}(C,0)$ **then**
      Add $(\mathcal{P}_I, \mathcal{P}_F)$ to $E_b$;
    **else if** $\forall \mathbf{x} \in \partial_i P,\ A_R \mathbf{x} \in \mathcal{R}_C^0$ **then**
      Add $(\mathcal{P}_I, \mathcal{P}^{\mathbf{0}})$ to $E_b$;
    **else**
      Add $(\mathcal{P}_I, \mathcal{P}^{\infty})$ to $E_b$;
    **end**
  **end**
**end**

### 4.3.2 Checking Polytopes as Regions of Convergence

This section provides some definitions and results which are necessary for the analysis of the directed graphs, in order to conclude whether the interior of a polytope is a region of convergence or not.

**Corollary 4.3.** *Consider the solution of the base system of (4.10),* $\mathbf{x}_b(t, \mathbf{x}_0)$*, with initial condition* $\mathbf{x}_0$*, and let* $T \geq 0$ *be specified, then for each* $\epsilon > 0$*, there exists an initial condition* $\mathbf{x}_0^*$ *such that* $\|\mathbf{x}_b(t, \mathbf{x}_0) - \mathbf{x}_b(t, \mathbf{x}_0^*)\| \leq \epsilon$ *for all* $t \in [0, T]$*, and, in addition the trajectory* $\mathbf{x}_b(t, \mathbf{x}_0^*)$ *does not cross any* $k - face$ *with* $k < n - 1$ *and has only transversal crossings with the facets.*

*Proof (sketch).* The proof follows from the following two facts:

- Proposition 4.1 guarantees that for a given $T \geq 0$ and for each $\epsilon > 0$, there exits a $\delta = \delta(T, \epsilon)$ such that for all initial condition $\mathbf{x}_0^*$ with $\|\mathbf{x}_0 - \mathbf{x}_0^*\| < \delta$, it is true that $\|\mathbf{x}_b(t, \mathbf{x}_0) - \mathbf{x}_b(t, \mathbf{x}_0^*)\| \leq \epsilon$ for all $t \in [0, T]$.

- The number of polytopes in $\mathcal{N}_b$ is finite, and thus, the number of supporting hyperplanes and $k-$faces is also finite.

□

**Definition 4.7 (Convergence region candidate (CRC)).** A polytope $\mathcal{P} \in \mathcal{N}_r$ is said to be a *convergence region candidate* if one of the following statement is true:

- $\mathcal{P} \subseteq \mathcal{R}_c^0$,

- $\mathcal{P}$ has not a positive invariant set whenever $P \nsubseteq \mathcal{R}_2$, and, in addition for every arc $\mathcal{P} \to \bar{\mathcal{P}}$, $\bar{\mathcal{P}}$ is a CRC.

The special polytope $\mathcal{P}^{\mathbf{0}}$ and $\mathcal{P}^{\infty}$ are also defined such that $\mathcal{P}^{\mathbf{0}}$ is a CRC and $\mathcal{P}^{\infty}$ will never be a CRC.

Note that the above definition is recursive, and thus, if a polytope $\mathcal{P} \nsubseteq \mathcal{R}_c^0$ is a CRC, then there exists a sequence $\mathcal{P} \to \mathcal{P}_1 \to \mathcal{P}_2 \to \cdots \to \mathcal{P}_N$ such that $\mathcal{P}_N \subseteq \mathcal{R}_c^0$ or $\mathcal{P}_N \to \mathcal{P}^0$

**Proposition 4.6.** *Consider a system described by* $\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + \mathbf{a}$, $t \geq 0$ *with the matrix* $A$ *diagonalizable and a hyperplane* $\mathcal{H}(N, \gamma)$. *Define* $W$ *the set of solutions of*

$$\begin{bmatrix} N \\ NA \\ \vdots \\ NA^n \end{bmatrix} \mathbf{x} = \begin{bmatrix} \gamma \\ -N\mathbf{a} \\ \vdots \\ -NA^{n-1}\mathbf{a} \end{bmatrix}, \tag{4.25}$$

*Then* $W$ *is the unique positive invariant set of* $\mathcal{H}(N, \gamma)$.

*Proof.* First, we prove that $W$ is a positive invariant set. To do this, it is necessary to prove that for every initial condition $\mathbf{x}_0 \in W$, it is satisfied that $\mathbf{x}(t, \mathbf{x}_0) \in \mathcal{H}(N, \gamma)$ for $t \in [0, \infty)$, which can be written by

$$N(e^{At}\mathbf{x}_0 + \int_0^t e^{A(t-\tau)} a d\tau) = \gamma. \tag{4.26}$$

Since $\mathbf{x}_0 \in W$, then $NA^{k+1}\mathbf{x}_0 = -NA^k\mathbf{a}$ for $k = 0, \ldots, n-1$. Therefore, using the finite series expansion

$$e^{At} = \sum_{k=0}^{n-1} \alpha_k(t) A^k \tag{4.27}$$

for some scalar functions $\alpha_0(t)$, $\alpha_1(t)$, …, $\alpha_{n-1}(t)$, the equation (4.26) is rewritten as follows:

$$N(e^{At} + A\int_0^t e^{A(t-\tau)} d\tau)\mathbf{x}_0 = \gamma. \tag{4.28}$$

Since $A$ is diagonalizable, it follows $A = \sum_{k=1}^{n} \lambda_k G_k$ and

$$e^{At} = \sum_{k=1}^{n} e^{\lambda_k t} G_k, \tag{4.29}$$

where $G_k$ is the $k^{th}$ spectral projector associated with the eigenvalue $\lambda_k$. Applying (4.29) to (4.28) it is obtained

$$N(G_1 + \cdots + G_n)\mathbf{x}_0 = NI\mathbf{x}_0 = \gamma. \tag{4.30}$$

Hence, condition (4.26) is satisfied, and the trajectory belongs to the hyperplane. Since $W$ is an affine subspace, condition (4.26) is satisfied for all hyperplane such that $W \subseteq \mathcal{H}(\bar{N}, \bar{\gamma})$. Therefore, $\mathbf{x}(t, \mathbf{x}_0)$ does not leave the set $W$. Finally, to prove that $W$ is unique, note that for every initial condition $\mathbf{x}_0 \in \mathcal{H}(N, \gamma) \setminus W$, there exists $i \in \{1, \ldots, n\}$ such that $N(A^i\mathbf{x}_0 + A^{i-1}\mathbf{a}) \neq 0$. □

In order to guarantee that the propounded procedure provides an estimation of the region of attraction, the graph $D_{reset}$ has to satisfied the following condition:

**Condition 1:** For every unbounded $k$-face in $\mathcal{R}_j$ with $j \in \{1, 3\}$ that has a positive invariant set, there exists an unbounded polytope $\mathcal{P}$ that contains the $k$-face and is not a CRC.

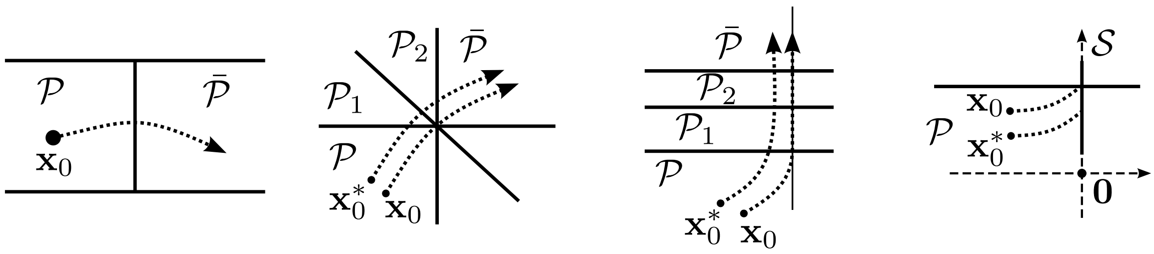
**Proposition 4.7.** *Suppose that the graph $D_{reset}(\mathcal{N}_r, E_r)$ satisfied Condition 1, then every polytope $\mathcal{P} \in \mathcal{N}_r$ is a region of convergence if it is a CRC.*

*Proof (sketch).* Suppose that $\mathcal{P} \not\subseteq \mathcal{R}_2$, then $\mathcal{P}$ has not a positive invariant set by the definition of CRC, and thus, for all $\mathbf{x}_0 \in \text{int}(\mathcal{P})$ the trajectory $\mathbf{x}(t, \mathbf{x}_0)$ leaves the polytope in a finite time. Four cases can be distinguished:

*Case i) (see Fig. 4.6a):* $\mathbf{x}(t, \mathbf{x}_0)$ leaves the polytope through the interior of a facet. Suppose that the facet does not belong to the resetting set $\mathcal{M}$, then there exists a polytope $\bar{\mathcal{P}}$ such that the trajectory $\mathbf{x}(t, \mathbf{x}_0)$ reaches the interior of $\bar{\mathcal{P}}$ after leaving $\mathcal{P}$, and thus there exists an arc $\mathcal{P} \to \bar{\mathcal{P}}$. In addition, $\bar{\mathcal{P}}$ must be a CRC.

*Case ii) (see Fig. 4.6b):* $\mathbf{x}(t, \mathbf{x}_0)$ leaves the polytope through a $k-$face and reach the interior of another polytope $\bar{\mathcal{P}}$ immediately. Suppose that $\mathbf{x}(t, \mathbf{x}_0) \in \text{int}(\mathcal{P})$ for $t \in [0, T_1)$, $\mathbf{x}(T_1, \mathbf{x}_0) \in \mathcal{P} \cap \bar{\mathcal{P}}$, and $\mathbf{x}(t, \mathbf{x}_0) \in \text{int}(\bar{\mathcal{P}})$ for $t \in (T_1, T_2]$. Corollary 4.3 guarantees that for all $\epsilon > 0$ there exists $\mathbf{x}_0^*$ such that $\|\mathbf{x}(t, \mathbf{x}_0) - \mathbf{x}(t, \mathbf{x}_0^*)\| \leq \epsilon$ for all $t \in [0, T_2]$. For a small enough $\epsilon$, it is satisfied that $\mathbf{x}_0^* \in \text{int}(\mathcal{P})$ and $\mathbf{x}(T_2, \mathbf{x}_0^*) \in \text{int}(\bar{\mathcal{P}})$. In addition, the trajectory only flows through the interior of polytopes, producing transversal crossings with their facets. Therefore, there exists a set of polytopes $\{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_N\}$ such that $\mathcal{P} \to \mathcal{P}_1 \to \cdots \to \mathcal{P}_N \to \bar{\mathcal{P}}$. Finally, the polytope $\bar{\mathcal{P}}$ is a CRC, since the polytope $\mathcal{P}$ is a CRC.

*Case iii) (see Fig. 4.6c):* $\mathbf{x}(t, \mathbf{x}_0)$ leaves the polytope through a $k-$face and remains inside the $k-$face for a while. Suppose that the trajectory reaches the interior of some polytope at some instant, then the proof follows as in Case i). On the contrary, consider the case in which the trajectory flows indefinitely through the supporting hyperplane of the $k-$face, and suppose that the trajectory does not approach the origin. Since the trajectory does not reach the interior of a polytope, then the hyperplane is also the supporting hyperplane of at least one unbounded polytope, $\bar{\mathcal{P}}$, that by Condition 1, it is not a CRC. Finally, Corollary 4.3 guarantees that there exists an



(a) Case i). (b) Case ii). (c) Case iii). (d) Case iv).

Figure 4.6 Illustration of the four cases studied in the proof of Proposition 4.7.

initial condition inside the polytope $\mathcal{P}$ that reaches the interior of $\bar{\mathcal{P}}$ and also a set of polytopes $\{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_N\}$ such that $\mathcal{P} \to \mathcal{P}_1 \to \cdots \to \mathcal{P}_N \to \bar{\mathcal{P}}$. Hence, since $\bar{\mathcal{P}}$ is not a CRC, it can be concluded that $\mathcal{P}$ is not a CRC, which is not possible. Consequently, by contradiction the trajectory approaches the origin or reaches the interior of a polytope $\bar{\mathcal{P}}$, that is a CRC.

*Case iv) (see Fig. 4.6d):* $\mathbf{x}(t, \mathbf{x}_0)$ leaves the polytope through a facet $\partial_i\mathcal{P}$ that belongs to the resetting set. Corollary 4.3 guarantees that there exits an initial condition inside the polytope that reaches the reset surface, and thus, there exists and arc $\mathcal{P} \to \mathcal{P}^{\mathbf{0}}$ ( $\mathcal{P} \to \mathcal{P}^{\infty}$ would be a contradiction). Therefore, it is satisfied that $A_R\mathbf{x} \in \mathcal{R}_C^0$ for all $\mathbf{x} \in \partial_i\mathcal{P}$ and thus the trajectory approaches the origin.

Note that all the trajectories $\mathbf{x}(t, \mathbf{x}_0)$ with $\mathbf{x}_0 \in \text{int}(\mathcal{P})$ approach the origin or reach the interior of a polytope $\bar{\mathcal{P}}$, that is a CRC. Repeating the reasoning for $\bar{\mathcal{P}}$, then all the trajectories $\mathbf{x}(t, \mathbf{x}_0)$ with $\mathbf{x}_0 \in \text{int}(\bar{\mathcal{P}})$ approach the origin or reach the interior of a polytope $\bar{\mathcal{P}}_2$. Repeating the reasoning for every $\bar{\mathcal{P}}_i$, it follows that the trajectories approach the origin, due to $\bar{\mathcal{P}}_i \to \mathcal{P}^{\mathbf{0}}$, or $\bar{\mathcal{P}}_i \to \bar{\mathcal{P}}_N$ such that $\bar{\mathcal{P}}_N \in \mathcal{R}_C^0$. Therefore, all the trajectories approach the origin.

The proof of the case $\mathcal{P} \subseteq \mathcal{R}_2$ follows similarly. It should be note that if the polytope has a positive invariant set, then the trajectory does not leave the non-saturated region and also does not perform reset actions, since $\text{int}(\mathcal{P}) \cap \mathcal{M} = \emptyset$ by the procedure to get $D_{reset}$. Therefore, the trajectory approaches the origin, since the base system is asymptotically stable.

□

## 4.4 A Method for Attraction Region Estimation

This section provides an algorithm to obtain an estimation of the region of attraction, based on the results developed in the previous sections. Before exposing the main algorithm, we introduce two problems and two possible solutions to the problems, that will be parts of the main algorithm.

**Problem 1:** Consider a pair $(\mathcal{P}_I, \mathcal{P}_F) \in E_r$. The problem consists in finding a partition of $\mathcal{P}_I$ such that there exists a polytope $\mathcal{P}$ in the partition that satisfies $\mathcal{P} \xrightarrow{\partial_i} \mathcal{P}_F$ for some $i$ and there is not $\bar{\mathcal{P}} \neq \mathcal{P}_F$ such that $\mathcal{P} \xrightarrow{\partial_i} \bar{\mathcal{P}}$.

**Solution 1**: Let $L$ be the number of supporting hyperplanes of $\mathcal{P}_F$. A possible solution to Problem 1 is given by the following procedure:

```
Data: (P_I, P_F)
Result: Partition Σ = {P_1, P_2, ...}
begin
    P ← P_I;
    Σ ← {P};
    for k = 1 to L do
        H ← the common supporting hyperplane of P and P_F;
        if int(H ∩ P) ∩ H_k P_F ≠ ∅ then
            Divide P by H_k P_F;
            Add the resulting polytopes to Σ (replacing P);
            P ← P̄ ∈ Σ such that P̄ → P_F;
        end
    end
end
```

**Problem 2:** Consider a pair $(\mathcal{P}_I, \mathcal{P}_F) \in E_r$, such that $\mathcal{P}_I \subseteq \mathcal{R}_j$ is not a CRC and $\mathcal{P}_F$ is a CRC, and $\{\mathcal{P}_I\}$ the partition resulting of the Problem 1 for the pair $(\mathcal{P}_I, \mathcal{P}_F)$. The problem consists in finding a partition $\{\mathcal{P}_1, \mathcal{P}_2, \ldots\}$ of $\mathcal{P}_I$ such that for some polytope $\mathcal{P}$ belonging to the partition, it is satisfied that $\mathcal{P} \to P_F$ and there is not $\bar{\mathcal{P}} \neq \mathcal{P}_F$ such that $\mathcal{P} \to \bar{\mathcal{P}}$.

**Solution 2**: Consider the index $k$ such that $\mathcal{P}_I \xrightarrow{\partial_k} P_F$, then we define the set $\Gamma = \{\mathcal{H}_i\mathcal{P}_I\}$, $i = 1, \ldots, L$ as the set of supporting hyperplanes such that $\mathcal{H}_i\mathcal{P}_I \cap \partial_k P_I \neq \emptyset$ and there exists $\bar{\mathcal{P}} \neq \mathcal{P}_F$ such that $\mathcal{P}_I \xrightarrow{\partial_i} \bar{\mathcal{P}}$. A possible solution to Problem 2 is given by the following procedure:

**Data**: $(\mathcal{P}_I, \mathcal{P}_F)$
**Result**: $\Sigma = \{\mathcal{P}_1, \mathcal{P}_2, \ldots\}$
**begin**
  Compute $\Gamma$;
  $\mathcal{P} \leftarrow \mathcal{P}_I$;
  $\Sigma \leftarrow \{P\}$;
  **forall the** $\mathcal{H} \in \Gamma$ **do**
    $\mathcal{H}_R(N, \gamma) \leftarrow \mathcal{H}$;
    Find a rotation of $\mathcal{H}_R(N, \gamma)$ such that:

$$\begin{cases} \text{int}(\mathcal{P}) \cap \mathcal{H}_R(N, \gamma) \neq \emptyset \\ \mathcal{H}_R(N, \gamma) \cap \mathcal{H} = \mathcal{H} \cap \mathcal{H}_k P_I \\ N(A_j\mathbf{x} + \mathbf{a}_j) \leq 0, \quad \forall \mathbf{x} \in \text{int}(\mathcal{P}) \cap \mathcal{H}_R(N, \gamma) \end{cases}$$

    **if** *there exists solution* **then**
      Divide $\mathcal{P}$ by $\mathcal{H}_R(N, \gamma)$ ;
      Add the resulting polytopes to $\Sigma$ (replacing $\mathcal{P}$);
      $\mathcal{P} \leftarrow \bar{\mathcal{P}} \in \Sigma$ such that $\bar{\mathcal{P}} \rightarrow \mathcal{P}_F$;
    **else**
      No solution has been found.
    **end**
  **end**
**end**

The intersection of two $n-$dimensional hyperplanes is a $(n-2)-$dimensional affine subspace, and thus, its orthogonal complement is an affine subspace of dimension 2. Then the rotation of the hyperplanes, keeping the same intersection, can be performed by the rotation of the projections of their normal vectors into the orthogonal complement. Therefore, there is only one degree of freedom, which is the angle of rotation in the orthogonal complement. Due to this fact, it is possible to find the rotation of the hyperplane $\mathcal{H}_R(N, \gamma)$ in the above procedure by a simple iterative search over one parameter.

Finally, an estimate of the region of attraction of the saturated reset control system can be obtained with the following procedure:

```
begin
    Initialize D_base(N_b, E_r);
    Initialize D_reset(N_r, E_r);
    Compute R_C^0;
    Find an unbounded polytope P such that if x ∈ P ∩ S then A_R x ∈ R_C^0;
    Divide the polytopes in N_r with the supporting hyperplanes of P;
    Add the resulting polytopes to N_r (Replacing the others);
    Update the arcs of the graph D_reset;
    while not stop condition do
        Pick a pair (P_I, P_F) from E_r such that P_F is a CRC;
        Solve Problem 1 with data (P_I, P_F) and result Σ_P1;
        if there is solution 1 then
            Add the resulting polytopes to N_r (Replacing the others);
            Update the arcs of the graph D_reset;
            P ← P̄ ∈ Σ_P1 such that P̄ → P_F ;
            Solve Problem 2 with data (P, P_F) and result Σ_P2;
            if there is solution 2 then
                Add the resulting polytopes to N_r ;
                Update the arcs of the graph D_reset;
            end
        end
    end
end
```

Different conditions may be used to stop the algorithm. For instance, the arcs can be marked as *visited* when they have been picked, then the algorithm will finish if all the arcs are *visited*. In order to obtain a larger region of convergence the algorithm can be applied iterativelty, using the obtained graph in each iteration as the initial graph of the next iteration. In addition, to avoid numerical problems it is recommendable to set a minimum size of polytope for which the polytope cannot be divided.

The result of the algorithm is a partition of the state-space where some polytopes are CRC.

## 4.5 Cases of Study

This section is devoted to the application of the proposed procedure and the comparison with the previous published works. Two cases are under study: an unstable first order plant controlled by a P+CI compensator, and a higher order system composed by a second order plant and a FORE compensator.

### 4.5.1 P+CI Compensator and First Order Plant

Consider the saturated reset control system (4.10) where the plant P is an unstable first order process given by

$$P : \begin{cases} \dot{x}_p(t) = 0.5x_p(t) + sat(u_p(t)), \\ y_p(t) = x_p(t), \end{cases} \tag{4.31}$$

where the level of saturation is $u_0 = 1$, and the reset compensator is a full reset P+CI compensator given as follows:

$$R : \begin{cases} \dot{x}_r(t) = u_r(t), & u_r(t) \neq 0, \\ x_r(t^+) = 0, & u_r(t) = 0, \\ y_r(t) = x_r(t) + 2u_r(t). \end{cases} \tag{4.32}$$

The matrices of the saturated reset control system are given by $\mathbf{a}_1 = -\mathbf{a}_3 = (-1, 0)$, $\mathbf{a}_2 = \mathbf{0}$, and

$$A_1 = A_3 = \begin{bmatrix} 0.5 & 0 \\ -1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1.5 & 1 \\ -1 & 0 \end{bmatrix}. \tag{4.33}$$

The resetting set is given by $\mathcal{M} = \text{span}\{(0,1)\} \backslash \{\mathbf{0}\}$ and the after-reset set is $\mathcal{M}_R = \{\mathbf{0}\}$. The regions of saturation can be seen in Fig. 4.7, and they are defined by the following polytopes:

$$\begin{aligned} \mathcal{R}_1 &= \mathcal{P}\left(\begin{bmatrix} -2 & 1 \end{bmatrix}, -1\right), \\ \mathcal{R}_2 &= \mathcal{P}\left(\begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix}, (1,1)\right), \\ \mathcal{R}_3 &= \mathcal{P}\left(\begin{bmatrix} 2 & -1 \end{bmatrix}, -1\right). \end{aligned} \tag{4.34}$$

Since the reset compensator is full reset, then by Proposition 2.2, it follows that the reset control system without saturation has well-posed reset instants, and in addition,

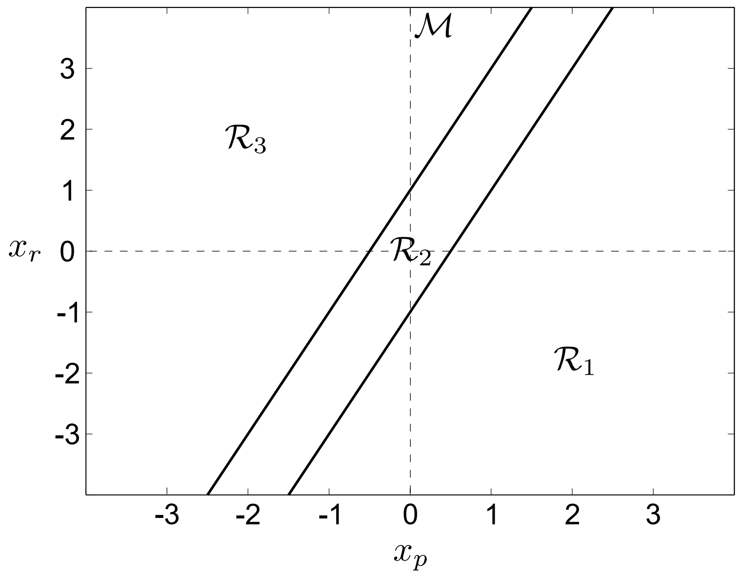
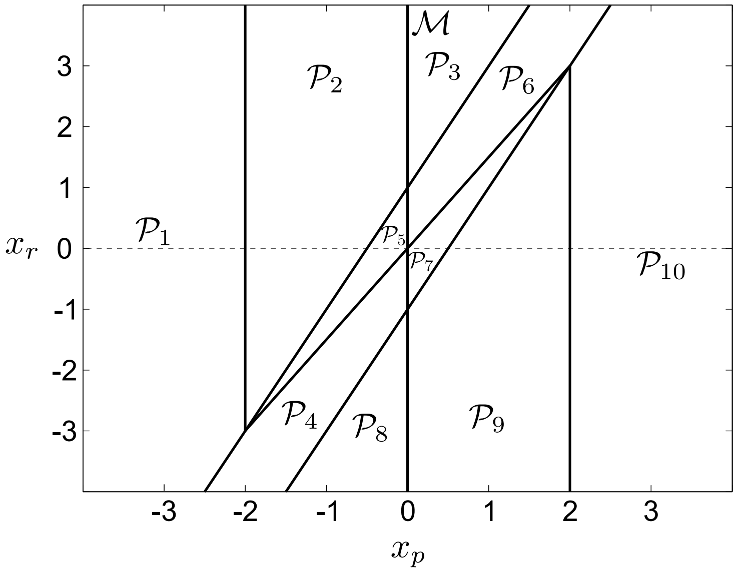


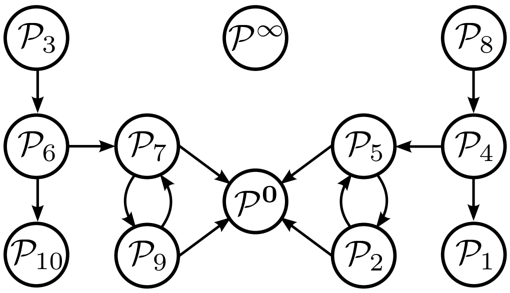

Figure 4.7 Regions of saturation, $\mathcal{R}_j$, $j \in \{1, 2, 3\}$.

there exists a unique solution over the interval $[0, \infty)$. As a result, Proposition 4.2 guarantees that the saturated reset control system has a unique solution on $[0, \infty)$, for any initial condition $\mathbf{x}_0 \in \mathbb{R}^2$. On the other hand, system under study satisfied all the assumptions made throughout the chapter. Therefore, $\mathbf{0}$ is the unique equilibrium point of the saturated reset control system (note that $A_1$ and $A_3$ are singular).

Now, the procedure proposed in Section 4.4 is applied. First, it is proposed the partition of the state-space shown in Fig. 4.8a. For this partition, the directed graph associated with the saturated reset control system is obtained (see Fig. 4.8b). Regarding the initial region of asymptotic stability $\mathcal{R}_C^0$, it can be simply set $\mathcal{R}_C^0 = \{\mathbf{0}\}$, since $\mathcal{M}_R = \{\mathbf{0}\}$. Note that polytopes $\mathcal{P}_2$, $\mathcal{P}_5$, $\mathcal{P}_7$, and $\mathcal{P}_9$ are directly CRCs. The result of applying the algorithm in Section 4.4 can be seen in Fig. 4.9, where the CRCs polytopes are shown in gray. The facets of the polytopes $\mathcal{P}_1$ and $\mathcal{P}_{10}$ defined by the supporting hyperplanes $\mathcal{H}([1\ 0], -1)$ and $\mathcal{H}([-1\ 0], -1)$, respectively, are the only unbounded k-faces which have positive invariant sets (Proposition 4.6). However, the polytopes $\mathcal{P}_1$ and $\mathcal{P}_{10}$ are not CRCs, which implies Condition 1, and thus, all the CRCs in the obtained directed graph are regions of convergence. In Fig. 4.9, it is also shown in dark gray the region of convergence given by the results proposed in [187] (Corollary 4.1). A direct conclusion from the results, it is that the proposed procedure provides a larger region of convergence in comparison to the region obtained by the previous results. Note that some polytopes are unbounded, that is, the region of convergence obtained is unbounded in some directions.

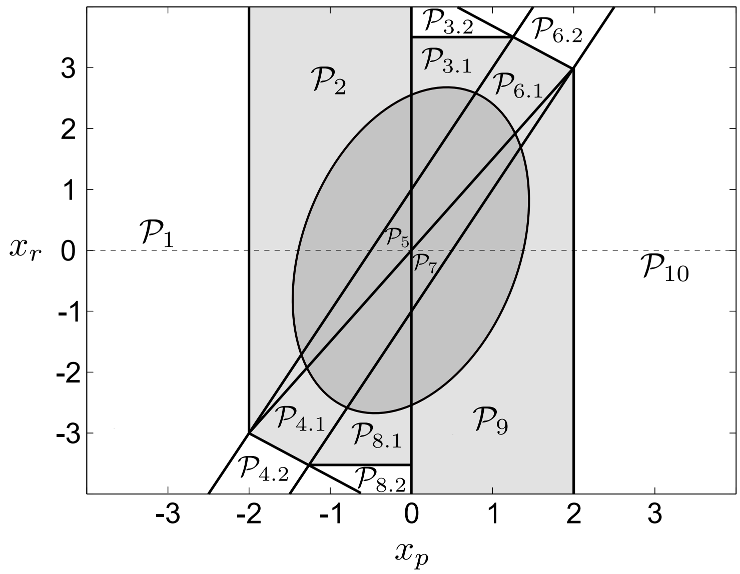



(a) Initial state-space partition.

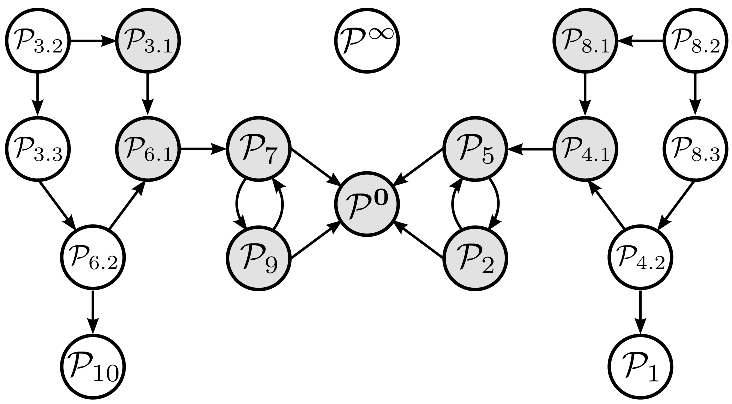



(b) Initial directed graph.

Figure 4.8 Initial state-space partition and initial directed graph associated with the saturated reset control system.




(a) Obtained state-space partition. Region of convergence: proposed method (gray and dark-gray), and Corollary 4.1 (dark-gray).




(b) Obtained directed graph. CRCs polytopes (gray).

Figure 4.9 Obtained state-space partition and directed graph associated with the saturated reset control system.

### 4.5.2 FORE Compensator and Second Order Plant

In this case, a region of convergence of a third order closed-loop system will be obtained. The plant P is a second order plant and the compensator is a FORE compensator:

$$P : \begin{cases} \dot{\mathbf{x}}_p(t) = \begin{bmatrix} 0 & 0.1 \\ 1 & -0.9 \end{bmatrix} \mathbf{x}_p(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} sat(u_p(t)), \\ y_p(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{x}_p(t), \end{cases} \tag{4.35}$$

$$R : \begin{cases} \dot{x}_r(t) = -2x_r(t) + u_r(t), & u_r(t) \neq 0, \\ x_r(t^+) = 0, & u_r(t) = 0, \\ y_r(t) = x_r(t). \end{cases} \tag{4.36}$$

The level of saturation is $u_0 = 1$, and the matrices of the saturated reset control system are given by $\mathbf{a}_1 = -\mathbf{a}_3 = (-1, 0)$, $\mathbf{a}_2 = \mathbf{0}$, and

$$A_1 = A_3 = \begin{bmatrix} 0 & 0.1 & 0 \\ 1 & -0.9 & 0 \\ 0 & -1 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0.1 & 1 \\ 1 & -0.9 & 0 \\ 0 & -1 & -2 \end{bmatrix}. \tag{4.37}$$

The resetting set is given by $\mathcal{M} = \text{span}\{(1,0,0),(0,0,1)\} \setminus \text{span}\{(1,0,0)\}$ and the after-reset set is $\mathcal{M}_R = \text{span}\{(1,0,0)\}$. The regions of saturation are given by the following polytopes (see Fig. 4.10):

$$\begin{aligned} \mathcal{R}_1 &= \mathcal{P}\left(\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, -1\right), \\ \mathcal{R}_2 &= \mathcal{P}\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}, (1,1)\right), \\ \mathcal{R}_3 &= \mathcal{P}\left(\begin{bmatrix} 0 & 0 & -1 \end{bmatrix}, -1\right). \end{aligned} \tag{4.38}$$

The FORE compensator is a full reset compensator. Therefore, the existence and uniqueness of the solution of the saturated reset control system over the interval $[0, \infty)$ for all initial condition $\mathbf{x}_0 \in \mathbb{R}^3$, is guaranteed by Propositions 2.2 and 4.2. In addition, the assumptions made throughout the chapter hold for this system, and thus, {(9,10,-5), (-9,-10,5), $\mathbf{0}$} are the unique equilibrium points of the saturated reset control system by Proposition 4.6.

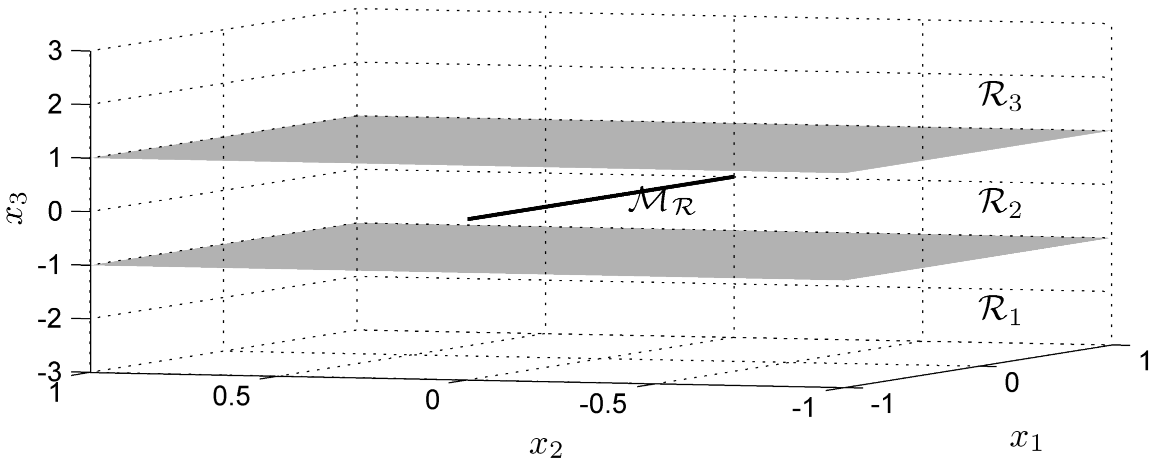



Figure 4.10 Regions of saturation, $\mathcal{R}_j$, $j \in \{1, 2, 3\}$.

On the other hand, in order to obtain the directed graphs, it is proposed the following hyperplanes to divide each region of saturation:

$$\begin{array}{ccc} \overbrace{\mathcal{H}([\ 0\ \ 0.1\ \ 0\ ], -1)}^{\mathcal{R}_1} & \overbrace{\mathcal{H}([\ 0\ \ 0.1\ \ 1\ ], 0)}^{\mathcal{R}_2} & \overbrace{\mathcal{H}([\ 0\ \ 0.1\ \ 0\ ], 1)}^{\mathcal{R}_3} \\ \mathcal{H}([\ 1\ \ 0.9\ \ 0\ ], 0) & \mathcal{H}([\ 1\ \ 0.9\ \ 0\ ], 0) & \mathcal{H}([\ 1\ \ 0.9\ \ 0\ ], 0) \\ \mathcal{H}([\ 0\ \ -1\ \ -2\ ], 0) & \mathcal{H}([\ 0\ \ -1\ \ -2\ ], 0) & \mathcal{H}([\ 0\ \ -1\ \ -2\ ], 0) \end{array} \tag{4.39}$$

Note that the above hyperplanes are designed, such that there is not an equilibrium point inside any polytope. Now, the algorithm in Section 4.4 is applied. In this case, the initial region of convergence $\mathcal{R}_C^0$ has been obtained by the Corollary 4.1. Fig. 4.11 shows in blue the set of CRCs polytopes obtained by the procedure. In addition, it can be checked by using Proposition 4.6 that Condition 1 holds, and thus, all the CRCs polytopes are regions of convergence. The region of convergence obtained by the method proposed in [187] (Corollary 4.1) is also shown in Fig. 4.11. Again it can be concluded that the region of convergence obtained by the proposed procedure is significantly larger. It should be mentioned that the resulting region of convergence is unbounded in some directions.

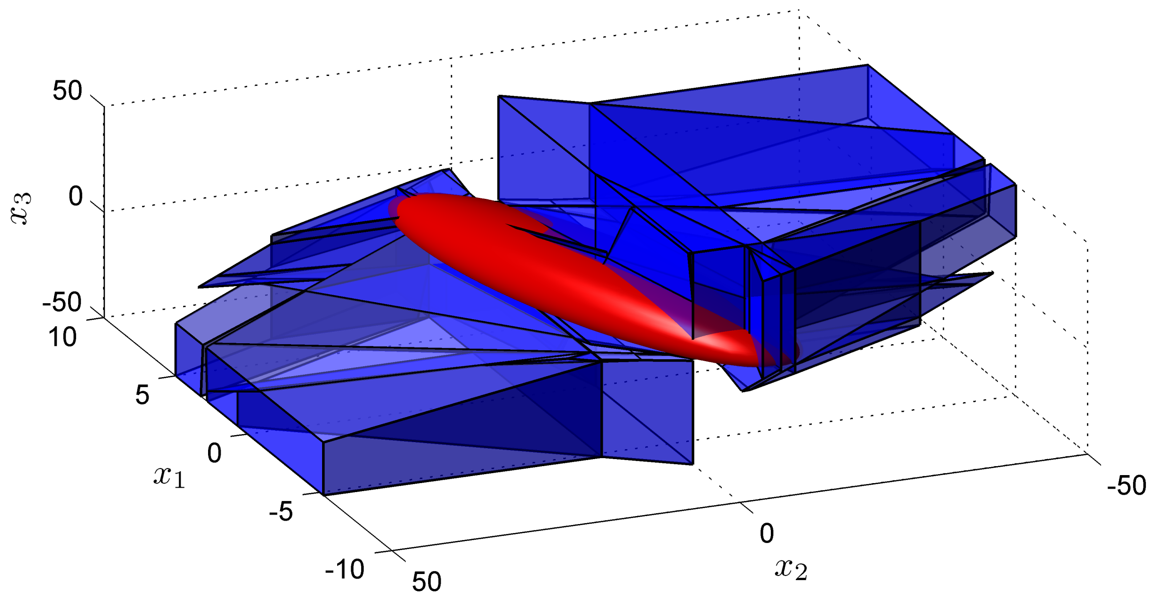



Figure 4.11 Obtained state-space partition: Region of convergence: proposed method (blue and red) and Corollary 4.1 (red).

# Chapter 5

# Design and Applications

The PI+CI compensator with its design improvements lead into a large number of design parameters compare with its counterpart PI. In this chapter, we focus our attention on the development of a systematic method for PI+CI tuning. The analysis of processes modeled by first and second order systems and integrator plus dead time systems results into a set of simple tuning rules. Finally, the benefits of the PI+CI compensator and the proposed tuning rules are shown by two real applications of process control: an in-line pH control process and a liquid level control process.

## 5.1 PI+CI Tuning Rules for First and Second Order Plants

In this section, the simplest model processes are considered, first and second order plants. Even in this simple cases the fundamental limitations of the LTI compensator leave still room to improve its performance, as it is shown in [46] for a simple Clegg integrator and an integrator plant. In particular, the PI+CI compensator has shown to overcome the performance of its counterpart PI in such a simple cases. For instance, the proper value of the reset ratio, obtained in [25], yields into a flat step setpoint response after the first reset instant for first order plants. However, the reset benefits vanish for greater order plants, especially in processes without pure integral term, where an undesirable undershoot appears [168]. For this reason, we consider the PI+CI compensator with two improvements: a variable reset ratio and a variable band resetting law. In comparison to the approach in [268], here it is provided a systematic method for tuning the reset ratio, instead of tuning rules based on heuristic methods. In addition, the PI+CI compensator is provided with a switching reset ratio, since the tuning of the variable reset ratio will be different from setpoint tracking and disturbance rejection. Throughout the section, the switching law will be omitted, assuming that the switching is performed exactly at the disturbance step. This assumption is made since the setpoint tracking and disturbance rejection will be studied separately. Finally, in spite of the fact that the process has not time-delay, a variable band resetting law is also considered.

The PI+CI compensator with these improvements may be represented in a state-space form as follows:

$$PI+CI: \begin{cases} \left.\begin{array}{l} \dot{\mathbf{x}}_r(t) = A_r\mathbf{x}_r(t) + B_r e(t), \\ \dot{p}_r(t) = 0, \end{array}\right\} & (e(t), \dot{e}(t)) \notin \mathcal{R}, \\ \left.\begin{array}{l} \mathbf{x}_r(t^+) = A_\rho \mathbf{x}_r(t), \\ p_r(t^+) = \mathcal{P}_{\sigma(t)}(\mathbf{x}_r(t), e(t)), \end{array}\right\} & (e(t), \dot{e}(t)) \in \mathcal{R}, \\ v(t) = C_r(p_r(t))\mathbf{x}_r(t) + D_r e(t), & \end{cases} \tag{5.1}$$

where the matrices $A_r$, $B_r$, $C_r$, $D_r$, and $A_\rho$ are given by

$$A_r = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_r = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C_r(p_r(t)) = k_i \begin{bmatrix} 1 - p_r(t) & p_r(t) \end{bmatrix}, \quad D_r = k_p, \tag{5.2}$$

and $A_\rho = diag(1, 0)$. For the variable band resetting law, the resetting set $\mathcal{R}$ is simply defined by

$$\mathcal{R} = \{(e(t), \dot{e}(t)) \in \mathbb{R}^{2n} : \theta\dot{e}(t) + e(t) = 0\} \tag{5.3}$$

Note that the resetting set for the zero crossing resetting law is directly obtained by $\theta = 0$. Combination of constant/variable reset ratio and zero crossing/variable reset band resetting laws gives a variety of PI+CI compensators, with different design parameters. In the more basic structure, that is constant reset ratio and zero crossing resetting law, the only design parameter is the constant reset ratio (besides the $\text{PI}_{base}$ parameters). That is, both functions $\mathcal{P}_1$ and $\mathcal{P}_2$ (reference and disturbance respectively) are constants. On the other hand, for variable reset ratio and variable band resetting law, the functions $\mathcal{P}_1$ and $\mathcal{P}_2$ (nonconstant) and the parameter $\theta$ have to be designed.

### 5.1.1 Reset Control System Analysis

Consider the autonomous reset control system (2.37) presented in Chapter 2, where now the reset compensator is the PI+CI (5.1). Note that the matrices of the PI+CI are time-variable[1], so that the state-space realization of the reset control system is

$$\begin{cases} \left.\begin{array}{l} \dot{\mathbf{x}}_r(t) = A(t)\mathbf{x}(t), \\ \dot{p}_r(t) = 0, \end{array}\right\} & (t, \mathbf{x}(t)) \notin \mathcal{S}, \\ \left.\begin{array}{l} \mathbf{x}(t^+) = A_R\mathbf{x}(t), \\ p_r(t^+) = \mathcal{P}_{\sigma(t)}(\mathbf{x}(t)), \end{array}\right\} & (t, \mathbf{x}(t)) \in \mathcal{S}, \\ y(t) = C\mathbf{x}(t) \end{cases} \tag{5.4}$$

with the initial condition of the reset ratio, $p_r(0) = p_{r0}$. In addition, the matrices of the system are $A_R = diag\left(I_{n_p}, A_\rho, I_{n_1+n_2}\right)$ and

$$A(t) = \begin{bmatrix} A_p - B_pD_rC_p & B_pC_r(p_r(t)) & B_pD_rC_1 & B_pC_2 \\ -B_rC_p & A_r & B_rC_1 & 0 \\ 0 & 0 & A_1 & 0 \\ 0 & 0 & 0 & A_2 \end{bmatrix}, \tag{5.5}$$

$$C = \begin{bmatrix} C_p & 0 & 0 & 0 \end{bmatrix}. \tag{5.6}$$

The resetting set $\mathcal{S}$ is given by

$$\mathcal{S} = \{(t, \mathbf{x}) \in \mathbb{R}_+ \times \mathbb{R}^n : C_M(t)\mathbf{x} = 0\}. \tag{5.7}$$

In the more general case of variable band resetting law, $C_M(t)$ is given by $C_M(t) = [-C_p\ 0\ C_1\ 0]\,(\theta A(t) + I)$. In the particular case of zero crossing resetting law ($\theta = 0$), the result is $C_M = \begin{bmatrix} -C_p & 0 & C_1 & 0 \end{bmatrix}$. Note that in general, the CI state $x_{ci}$ is set to zero at the reset instants $t_k$, $k = 1, 2, \ldots$, given by $(t_k, \mathbf{x}(t_k)) \in \mathcal{S}$, that in general will depend on the initial condition of the exosystems.

The analysis of the reset control system is based on the fact that the response of a reset control system to a given reference or a disturbance, can be exactly determined as a sequence of LTI systems responses. Note that the solution of (5.4) in the interval $(t_k, t_{k+1}]$, $k = 0, 1, \ldots$ (it is assumed $t_0 = 0$) is simply given by the LTI base system solution

$$\mathbf{x}(t) = e^{A(t_k)(t-t_k)}\mathbf{x}(t_k^+) \tag{5.8}$$

[1]The variable reset ratio and the variable band resetting law imply that the reset actions depend somehow on the time. For this reason the notation $\mathcal{S}$ is applied, instead of $\mathcal{M}$ (used for state-dependent IDS).

and at the reset instants $\mathbf{x}(t_k^+) = A_R\mathbf{x}(t_k)$. Thus, the closed-loop error is $e(t) = C_E\mathbf{x}(t) = \begin{bmatrix} -C_p & 0 & C_1 & 0 \end{bmatrix}\mathbf{x}(t)$. Now, for a given $k \in \{0, 1, \ldots\}$, define $e_k(t)$, as $e_k(t) = e(t + t_k) = C_E e^{A(t_k)t}\mathbf{x}(t_k^+)$, which is well defined for any $t \geq 0$, and then its Laplace transform is

$$E_k(s) = C_E(sI - A(t_k))^{-1}\mathbf{x}(t_k^+). \tag{5.9}$$

As a result, the error signal $e$ is determined by the sequence $(e_k)_{k=0,1,2,\ldots}$, where $e_k$ is the inverse Laplace transform of $E_k$, and $\mathbf{x}(t_0^+) = \mathbf{x}(0^+) = (\mathbf{x}_{p0}, x_{i0}, 0, \mathbf{w}_{10}, \mathbf{w}_{20})$; it is simply given by

$$e(t) = \begin{cases} e_0(t), & t \in [0, t_1], \\ e_1(t - t_1), & t \in (t_1, t_2], \\ e_2(t - t_2), & t \in (t_2, t_3], \\ \ldots \end{cases} \tag{5.10}$$

Following, it will be analyzed how the PI+CI parameters enter in the error dynamics as given by (5.9) and (5.10). The cases of tracking step reference signals and rejecting step disturbances will be considered independently.

### Step Reference and Disturbance Responses

Firstly, we consider the case of tracking a given step reference $r$ with height $w_{10}$, generated by the exosystem $\Sigma_1$. In this case, the exosystem matrices are simply $A_1 = 0$ and $C_1 = 1$, and the exosystem initial condition is the step height $w_{10}$. On the other hand, it is assumed that $d = 0$. Consider by simplicity a $n_p$-order plant P with transfer function

$$P(s) = \frac{b_0}{s^{n_p} + a_{n_p-1}s^{n_p-1} + \cdots a_1 s + a_0}. \tag{5.11}$$

Using (5.9), it can be obtained that the error $E_{r,k}(s)$ (the subscript $r$ has been added to denote that it is a tracking error) is given by (see Appendix E)

$$E_{r,k}(s) = \frac{(w_{10} - x_{n_p}(t_k))s^{n_p} + \cdots + (w_{10}a_1 - x_1(t_k))s + w_{10}a_0 - b_0 k_i(1 - p_r(t_k^+))x_i(t_k)}{s^{n_p+1} + a_{n_p-1}s^{n_p} + \cdots + a_1 s^2 + (a_0 + b_0 k_p)s + b_0 k_i} \tag{5.12}$$

for $k = 0, 1, 2, \ldots$, where $\mathbf{x}_p = (x_1, x_2, \ldots, x_{n_p})$ is the plant state and $x_i$ is the integrator state (the non-resetting term of the PI+CI compensator). In general, the error signal dynamic between reset instants will be defined by the zeros and poles of $E_{r,k}(s)$. Regarding the tuning of the PI+CI compensator, it can be concluded that the poles only depend on the $\text{PI}_{base}$ compensator, with parameters $k_p$ and $k_i$, while the reset action only affects the zeros, in particular it directly affects the independent term of the numerator polynomial through the parameter $p_r$.

Consider now the case of a step disturbance of height $w_{20}$ (the exosystem matrices are $A_2 = 0$ and $C_2 = 1$) and zero reference ($r = 0$), then the error response is now given by (see Appendix E)

$$E_{d,k}(s) = -\frac{x_{n_p}(t_k)s^{n_p} + \cdots + x_1(t_k)s + w_{20}b_0 + b_0k_i(1 - p_r(t_k^+))x_i(t_k)}{s^{n_p+1} + a_{n_p-1}s^{n_p} + \cdots + a_1s^2 + (a_0 + b_0k_p)s + b_0k_i} \tag{5.13}$$

for $k = 0, 1, 2, \ldots$. Again, the analysis of the error dynamic between reset instants is rather simple. In $E_{d,k}(s)$, the PI$_{base}$ compensator parameters affect its poles, while the reset action can be only used to modify the zeros location through the parameter $p_r$, that directly affects the independent term of the numerator polynomial.

On the other hand, for the PI+CI design, the plant state and and the compensator state will be assumed to be at rest, that is, the initial condition is $\mathbf{x}(0) = (\mathbf{0}, \mathbf{0}, w_{10}, w_{20})$.

### 5.1.2 A Design Method for First Order Plants

Let tackle first the simplest case of first order plant given by

$$P(s) = \frac{b_0}{s + a_0} \tag{5.14}$$

and the zero crossing resetting law. In this case, $E_{r,k}(s)$ and $E_{d,k}(s)$ result in particularly simple expressions. By using (5.12) and (5.13), they are directly obtained

$$E_{r,k}(s) = \frac{a_0w_{10} - b_0k_i(1 - p_r(t_k^+))x_i(t_k)}{s^2 + (a_0 + b_0k_p)\,s + b_0k_i} \tag{5.15}$$

and

$$E_{d,k}(s) = -\frac{b_0w_{20} + b_0k_i(1 - p_r(t_k^+))x_i(t_k)}{s^2 + (a_0 + b_0k_p)\,s + b_0k_i}. \tag{5.16}$$

#### PI+CI Design for Flat Response

*Reference tracking.* Firstly, consider the case of step tracking. By simple inspection of (5.15), it is clear that a proper election of $p_r(t_k^+)$, $k = 1, 2, \ldots$, can remove the overshoot of an arbitrarily fast step response at the first reset instant, producing a flat response and reaching the steady-state at the first reset instant, simply by doing $E_{r,k}(s) = 0$, $k = 1, 2, \ldots$. Since the reset compensator is initially at rest, that is

$x_i(0) = 0$, then the value $p_r(0)$ is irrelevant (it will be generally taken as $p_r(0) = p_{r_0} = 0$) and the first signal error $E_{r,0}(s)$ is exactly the error produced by the base compensator $\text{PI}_{base}$ in the interval $[0, t_1]$. Clearly, a flat step response is impossible to obtain with a LTI compensator, and thus it is an example of the reset compensation overcoming the fundamental limitation of LTI compensators. The design rule is to set the reset ratio $p_r$ by the following function

$$\mathcal{P}_1(\mathbf{x}(t)) = 1 - \frac{a_0 w_{10}}{b_0 k_i x_i(t)}. \tag{5.17}$$

For a step height $w_{10} > 0$, note that $x_i(t_1)$ is the integral of the error up to the instant $t_1$; as a result, $x_i(t_1) > 0$ and thus $p_r(t_k^+) < 1$. Otherwise, if $w_{10} < 0$ then $x_i(t_1) < 0$ and again $p_r(t_k^+) < 1$. On the other hand, the equation (5.17) produces a flat response after the first reset instant, $t_1$. For $w_{10} > 0$, the control signal has to be reduced at $t_1$ in order to obtain a smaller overshoot. However, if $p_r(t_k^+) < 0$ then $1 - p_r(t_k^+) > 1$ and the control signal is increased. Therefore, the tuning rule (5.17) gives always $p_r(t_k^+) > 0$. Otherwise, if $w_{10} < 0$ the same reasoning can be used.

In general, the reset ratio will depend on the plant DC-gain $k = \frac{b_0}{a_0}$, and the step height $w_{10}$, and also the integral state $x_i$ at the reset instant $t_k$. Note that since the error is zero at the first reset instant $t_1$, then $x_i(t_k) = x_i(t_1)$ for $k = 2, 3, \ldots$, that is the integrator state has a constant value after the first reset instant, and thus the value $p_r(t_k^+) = p_r(t_1^+)$ is also constant for any $k = 1, 2, \ldots$. By direct substitution of (5.17) in (5.15), it is obtained that $E_{r,k}(s) = 0$ for $k = 1, 2, \ldots$. On the other hand, although $x_i(t_1)$ can be approximately computed by solving the base system equation, it is not possible to obtain an explicit solution. From a control point of view, it is more convenient to simply obtain the value $x_i(t_1)$ on-line from the reset compensator and then computing the value $p_r(t_1^+)$ to be used afterwards.

*Disturbance Rejection.* For a flat response after the first reset instant, in relation to a step disturbance response, it is also possible to find proper values of $p_r(t_k^+)$, by simply making $E_{d,k}(s) = 0$ in (5.16). The result is that $\mathcal{P}_2$ is simply given by

$$\mathcal{P}_2(\mathbf{x}(t)) = 1 + \frac{w_{20}}{k_i x_i(t)}. \tag{5.18}$$

By solving the base system equations, it is possible to obtain $x_i(t_1)$ in a closed-form. The result is

$$x_i(t_1) = -\frac{w_{20}}{k_i}(1 + e^{\frac{\alpha}{\beta}\pi}), \tag{5.19}$$

where $\alpha \pm i\beta$ are the poles of the transfer function (5.16). Thus, by direct substitution in (5.18), it yields into a constant reset ratio defined by

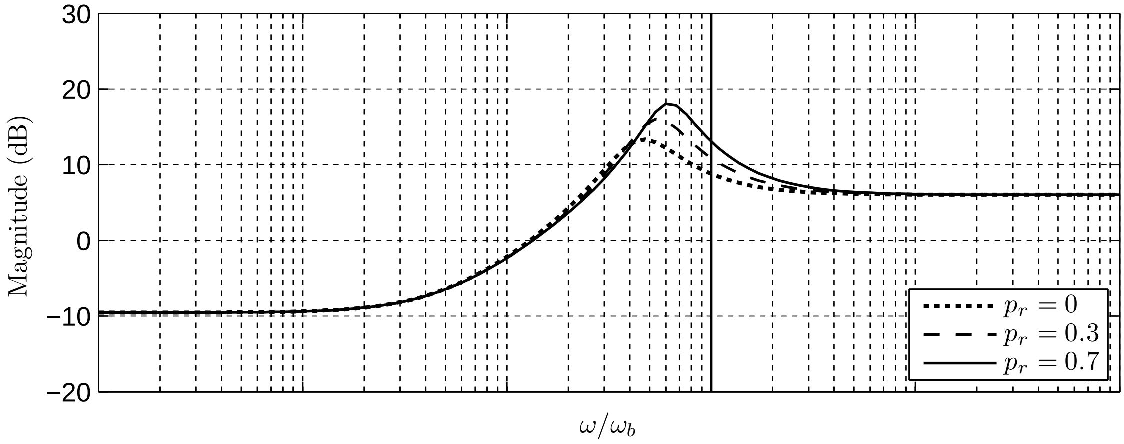
$$\mathcal{P}_2(\mathbf{x}(t)) = 1 - \frac{1}{1 + e^{\frac{\alpha\pi}{\beta}}}. \tag{5.20}$$

A heuristic design method of a PI+CI compensator for a first order plant, with the constant reset ratio $p_r$ given by (5.20) both for tracking and disturbance rejection, has been already proposed in [36]. Note that when (5.20) is used for step tracking, a flat response is obtained but only after the second reset instant.

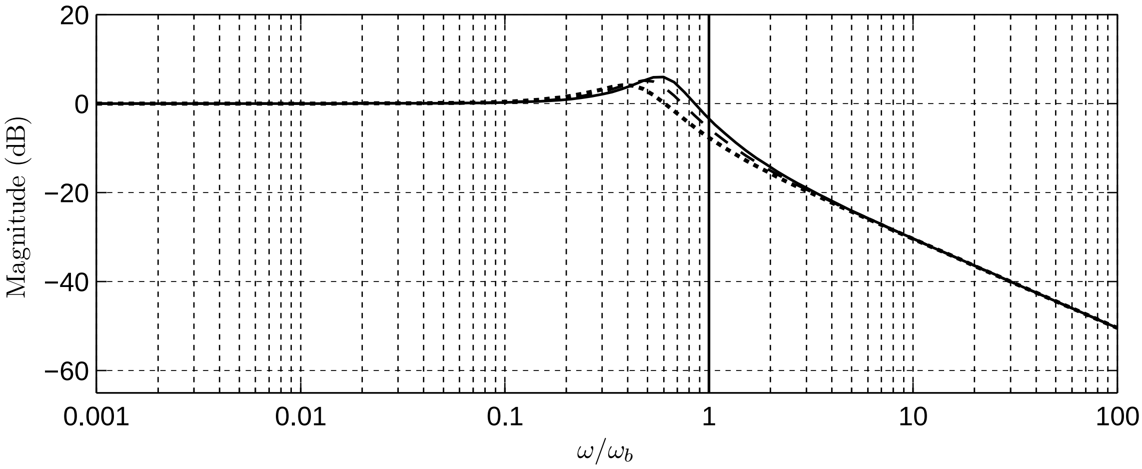
*Robustness.* For reference tracking, the reset ratio as given by (5.17) is only affected by the plant DC-gain $k$ for robustness analysis. Note that the rest of the parameters are known in advance, and the value $x_i(t_k)$ may be obtained on-line. The result is that if $k$ can be well modeled and has low variations, then the design is very robust. And, in addition, a flat response is obtained independently of the plant time constant $1/a_0$. On the other hand, regarding robustness against disturbance rejection, the design based on (5.20) is very robust against uncertainty, since the function $\mathcal{P}_2$ only depend on the location of the closed-loop poles of the base control system. In particular, $p_r$ only depends on the closed-loop poles damping ratio $\xi$, since $\frac{\alpha}{\beta} = \frac{\xi}{\sqrt{1-\xi^2}}$. Thus, a good design of the base PI compensator with low variation of the damping ratio against uncertainty will give a very robust reset control system.

*Sensor noise effect (cost of feedback).* Consider a first order plant with $a_0 = 0.5$ and $b_0 = 1.5$, and a PI+CI compensator with zero crossing resetting law and the parameters $k_p = 2$ and $\tau_i = 0.15$. Fig. 5.1a shows the frequency response magnitude vs. frequency for different values of the reset ratio. In addition, Fig. 5.1b represents the different frequency responses from reference to closed-loop outputs, and gives a reference of the operational bandwidth $\omega_B$. It is clearly shown that PI+CI compensation produces a similar cost of feedback for noise frequencies well beyond $\omega_B$.

*Integrating plants.* In the case of a pure integrator, the parameter $a_0$ of (5.14) is 0, then from equation (5.15) we get that a flat step response for reference tracking is obtained with $p_r(t_k) = 1$. On the other hand, the equation (5.20) is also valid for disturbance rejection since it is not directly affected by $a_0$; note that in this case the reset ratio will be less than 1.




(a) Frequency response from noise to control input.

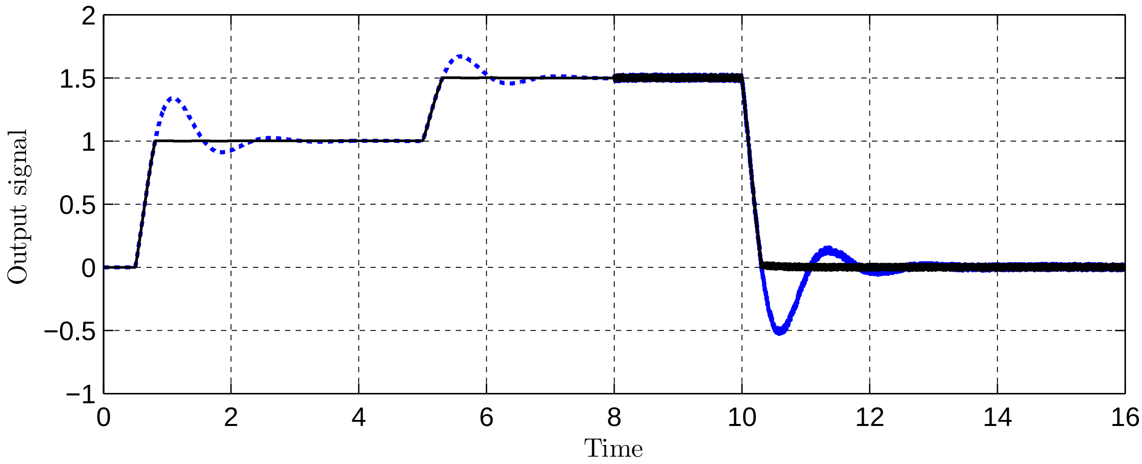



(b) Frequency response from reference to closed-loop output.

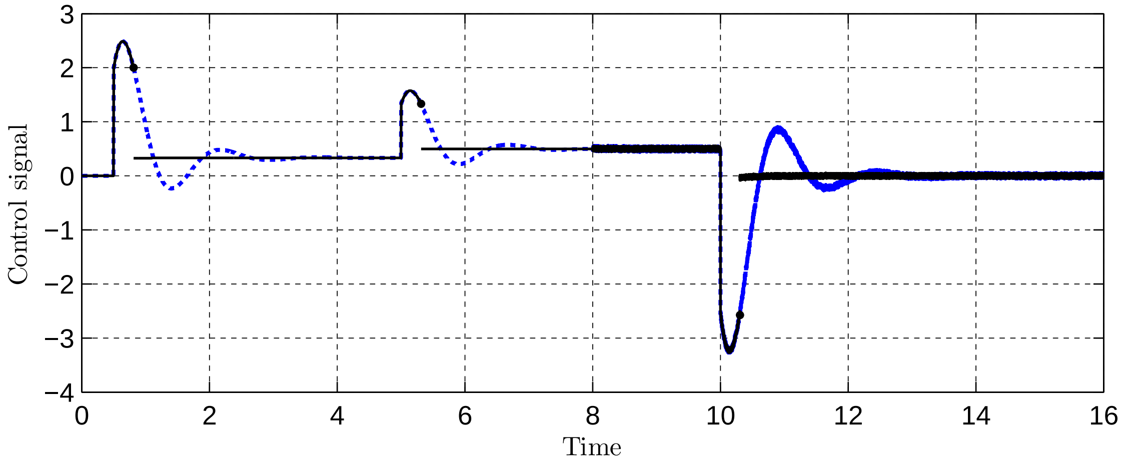
Figure 5.1 Frequency responses of the reset control system composed by a first order plant ($a_0 = 0.5$, $b_0 = 1.5$) and a PI+CI compensator ($k_p = 2$, $\tau_i = 0.15$).

**Example 5.1.** Consider a first order plant P with $a_0 = -0.5$ and $b_0 = 1.5$ and the PI+CI compensator given by (5.1). Design specifications are to track step references and reject step disturbances, with a fast response and without overshoot. In fact, a fast flat response (impossible to achieve by LTI compensation) is desired, and the above tuning rules for PI+CI compensation will be used. A base PI compensator with parameters $k_p = 2$ and $\tau_i = 0.15$ is chosen to obtain a fast response. Then, the PI+CI compensator is designed by using $\mathcal{P}_1$ and $\mathcal{P}_2$ given by (5.17) and (5.20), respectively. Fig. 5.2a shows step responses for the base PI and the PI+CI compensators. Note that the responses of the PI and the PI+CI are identical after a step change and until the first reset instant, and that the overshoot is eliminated at the first reset time. The control signal of the PI+CI remains constant from the first reset instant, as it is shown in Fig. 5.2b. In addition, sensor noise (bandwidth lower limit over 50 rad/s) is added in the simulation. As it has been discussed above, both PI and PI+CI exhibit similar characteristics in terms of cost of feedback. On the other hand, step disturbances are considered in the simulation shown in Fig. 5.3. Again, overshooting is completely eliminated at the first reset instant, making clear the superior performance of the PI+CI against the PI compensator.

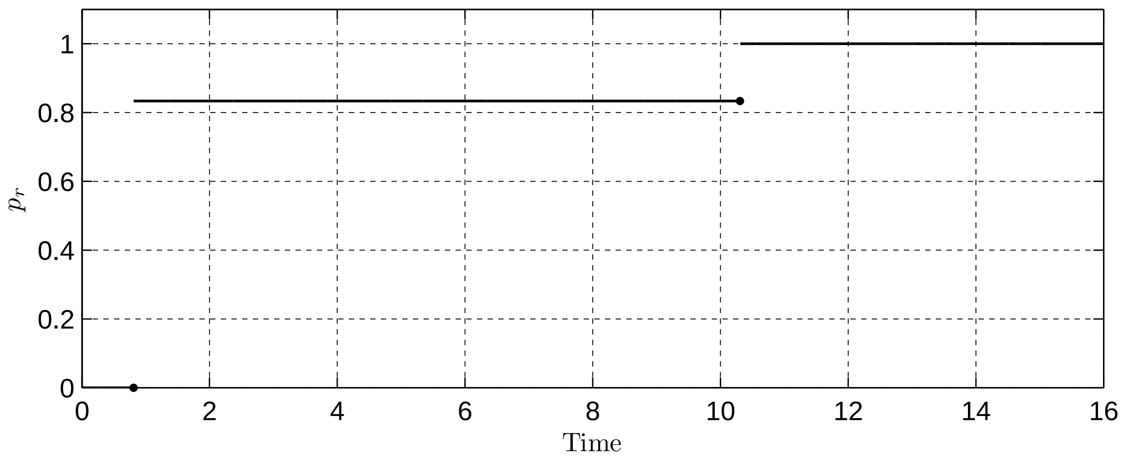
Finally, robustness against parametric uncertainty is checked by simulation, the results are given in Fig. 5.4. In this case, a simulation of the reset control system is given for a step reference, jointly with a step disturbance, for different values of the DC-gain $k = 2.4, 3, 3.6$. Note that the reset compensator has been tuned for the nominal value $k = 3$, and thus variations of $\pm 20\%$ over the nominal value has been considered. The results show that the PI+CI is robust enough against gain variations; note that PI+CI is not sensible to time constant variations in the sense that a flat response is always obtain after the first reset instant, being the only limit the allowed value of the control input.




(a) Output signal.

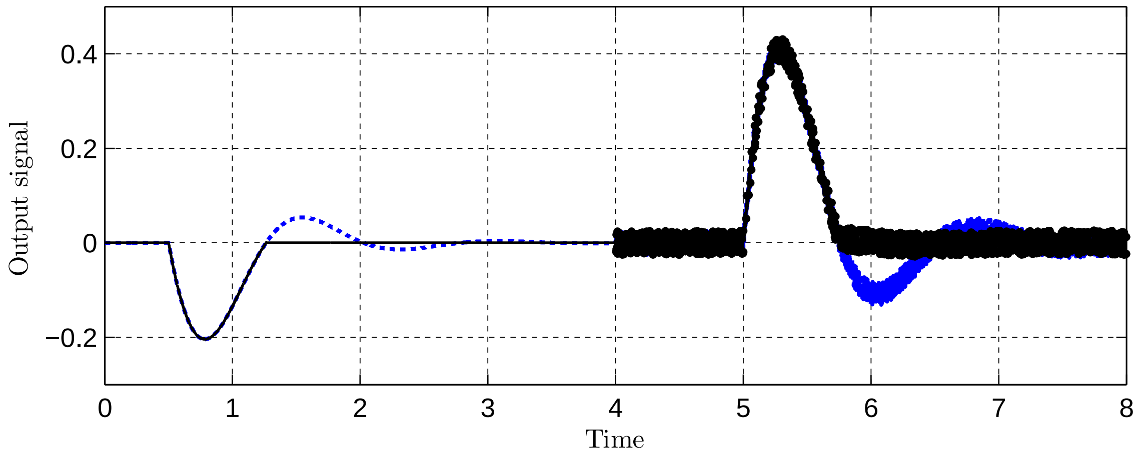



(b) Control signal.

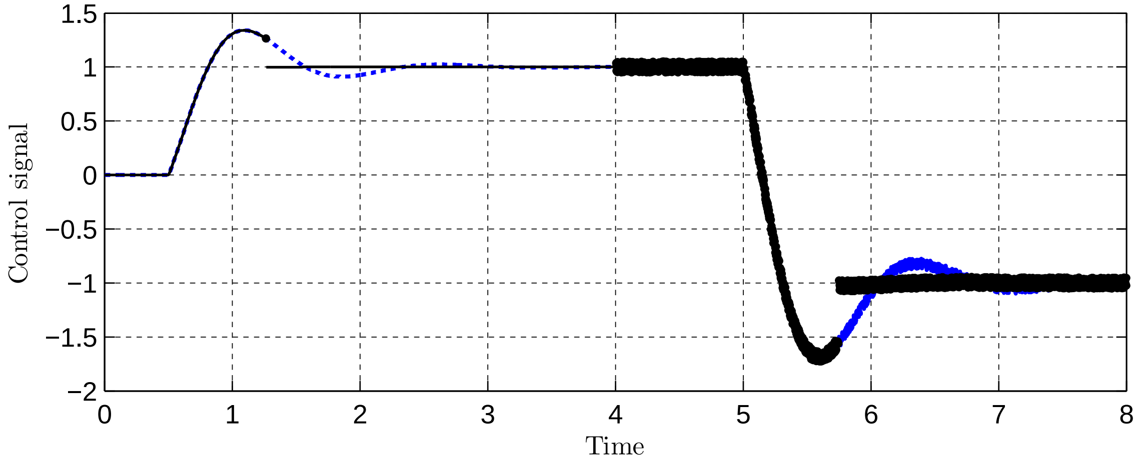



(c) Reset ratio.

Figure 5.2 Output signals, control signals, and reset ratio of the reset control system and its base PI from the Example 5.1 for a sequence of step setpoints. $PI_{base}$ (dotted-blue) and PI+CI (solid-black).

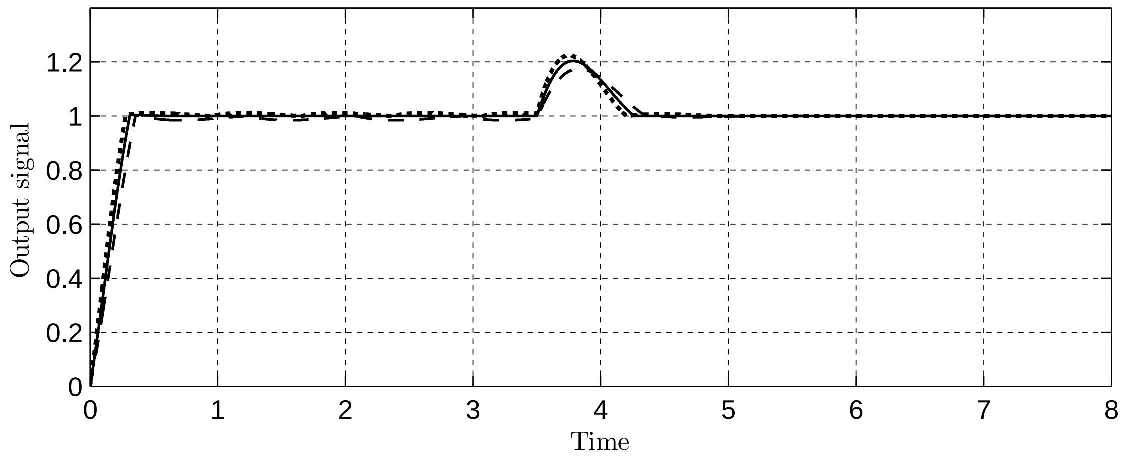



(a) Output signal.

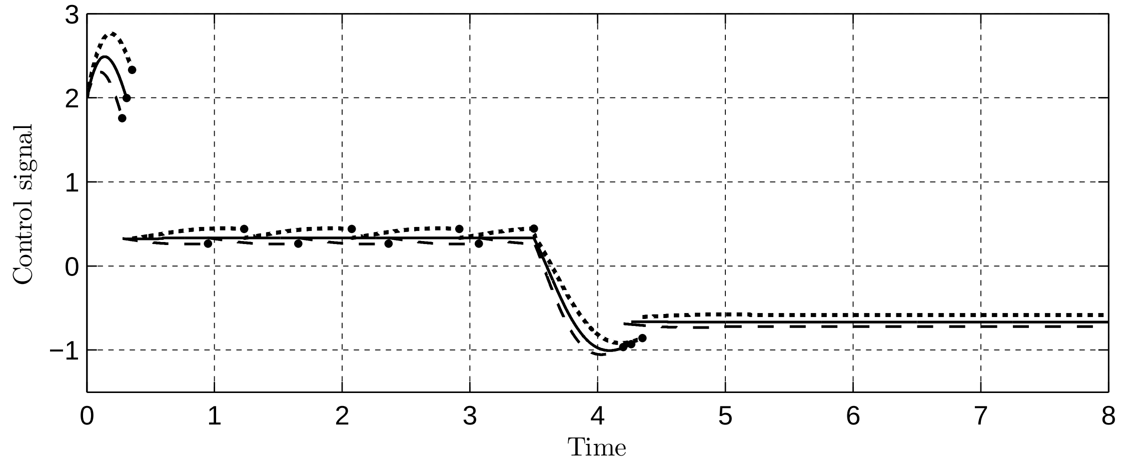



(b) Control signal.

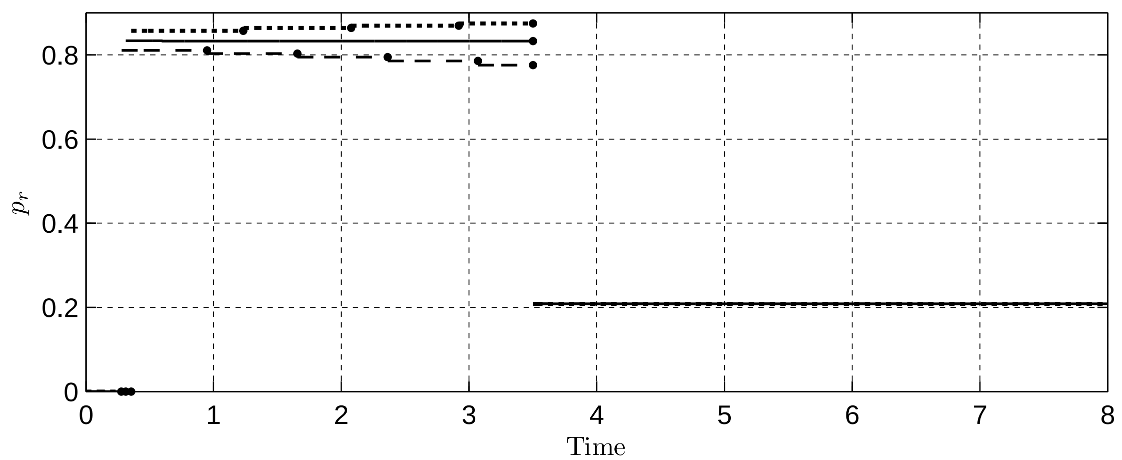
Figure 5.3 Output signals and control signals of the reset control system and its base PI from the Example 5.1 for a sequence of step disturbances. $PI_{base}$ (dotted-blue) and PI+CI (solid-black).




(a) Output signal.




(b) Control signal.




(c) Reset ratio.

Figure 5.4 Output signals, control signals and reset ratios of the reset control system from the Example 5.1 with a 20% of uncertainty in the plant gain. $k = 2.4$ (dashed), $k = 3$ (solid), and $k = 3.6$ (dotted).

### 5.1.3 A Design Method for Second Order Plants

In this section, it is considered a slightly more complicated plant, that is a second order plant given by

$$P(s) = \frac{b_0}{s^2 + a_1 s + a_0}. \tag{5.21}$$

The step error response, $E_{r,k}(s)$, and the step disturbance error response, $E_{d,k}(s)$, can be computed by equations (5.12) and (5.13). As a result, it is obtained

$$E_{r,k}(s) = \frac{(w_{10} - x_2(t_k))\, s^2 + (a_1 w_{10} - x_1(t_k))\, s + a_0 w_{10} - b_0 k_i (1 - p_r(t_k^+)) x_i(t_k)}{s^3 + a_1 s^2 + (a_0 + b_0 k_p)\, s + b_0 k_i}, \tag{5.22}$$

$$E_{d,k}(s) = -\frac{x_2(t_k) s^2 + x_1(t_k) s + b_0 w_{20} + b_0 k_i (1 - p_r(t_k^+)) x_i(t_k)}{s^3 + a_1 s^2 + (a_0 + b_0 k_p)\, s + b_0 k_i}. \tag{5.23}$$

Note that in contrast to (5.15) and (5.16), now it is not possible to make zero $E_{r,k}(s)$ and $E_{d,k}(s)$ for some designed value of $p_r(t_k^+)$, since the numerator of both transfer functions cannot be cancelled. In the following, a design method for the PI+CI will be developed, based on the minimization of the $H_2$-norm of the functions $E_{r,k}$ and $E_{d,k}$ (or equivalently the $\mathcal{L}_2$-norm of the time-domain error signals $e_{r,k}$ and $e_{d,k}$), for $k = 1, 2, \ldots$. Therefore, the value of $e_0$ is determined by the base PI compensator, which is assumed to be well designed, and thus, the design method is focused on how to choose the variable reset ratio and/or the variable band resetting law.

The minimization criterion consists in minimizing the *energy* of the different error signals $e_k$, that composed the closed-loop error $e(t)$ (see Fig. 5.5). Note that the $H_2$-norm has been used because the integral of the error in the interval $(t_k, t_{k+1}]$ cannot be computed analytically; and even in the case it would be possible, the minimization problem would not have solution. In order to illustrate this fact, consider a zero crossing resetting law then we get

$$E_{r,k}(s) = \frac{(a_1 w_{10} - x_1(t_k))\, s + a_0 w_{10} - b_0 k_i (1 - p_r(t_k^+)) x_i(t_k)}{s^3 + a_1 s^2 + (a_0 + b_0 k_p)\, s + b_0 k_i}. \tag{5.24}$$

The minimum $\mathcal{L}_2$-norm for the above transfer function in the interval $[0, \Delta]$ (with $\Delta$ the instant of the next zero crossing of the impulsive response of the transfer function) is obtained for

$$\frac{a_0 w_{10} - b_0 k_i (1 - p_r(t_k^+)) x_i(t_k)}{a_1 w_{10} - x_1(t_k)} \to -\infty \tag{5.25}$$

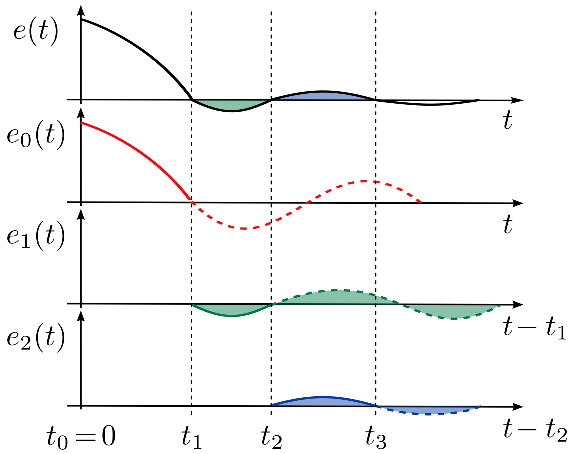



Figure 5.5 Illustration of the error signals sequence with zero crossing resetting law.

and then, the minimization problem is not bounded.

In the following, both reference tracking and disturbance rejection will be investigated separately.

**Tuning Rule for Zero Crossing Resetting Law**

Firstly, the PI+CI compensator with variable reset ratio and zero crossing resetting law will be investigated. In this case, the step error response and the step disturbance response are given by equations (5.22) and (5.23), respectively, where the terms $(w_{10} - x_2(t_k))$ and $x_2(t_k)$ are equal to zero.

*Reference tracking.* Clearly, the reset ratio $p_r(t_k^+)$ produces different responses depending on where the zero of $E_{r,k}(s)$ is located. Let us start with a simple structure for the reset ratio

$$\mathcal{P}_1(\mathbf{x}(t)) = 1 - \frac{a_0 w_{10} - \delta \dot{e}(t_k)}{b_0 k_i x_i(t_k)}, \tag{5.26}$$

where $\delta$ is a design parameter. Then, after some computation, $E_{r,k}(s)$ can be rewritten as

$$E_{r,k}(s) = \frac{\dot{e}(t_k)(s+\delta)}{s^3 + a_1 s^2 + (a_0 + b_0 k_p)\, s + b_0 k_i}. \tag{5.27}$$

Note that the term $\dot{e}(t_k)$ is constant over $(t_k, t_{k+1}]$, and thus, it is not difficult to minimize the $H_2$-norm of $E_{r,k}(s)$, for $k = 1, 2, \ldots$. Note also that, although the expression of the reset ratio is similar to the case of first order plants, in this case the values of $x_i(t_k)$ are in general different to $x_i(t_1)$ and thus the reset ratio is not necessarily constant after the first reset instant.

In this case, the optimum value $\delta = 0$ is obtained (see Appendix E). As a result, the design rule for a PI+CI compensator with zero crossing resetting law is

$$\mathcal{P}_1(\mathbf{x}(t)) = 1 - \frac{a_0 w_{10}}{b_0 k_i x_i(t)}. \tag{5.28}$$

*Disturbance rejection.* A similar approach can be applied to improve the performance of step disturbance rejection. For instance, if the reset ratio is set as

$$p_r(t_k) = 1 + \frac{b_0 w_{20} + \delta \dot{e}(t_k)}{b_0 k_i x_i(t_k)} \tag{5.29}$$

for $k = 1, 2, \ldots$, then the error is again given by

$$E_{d,k}(s) = \frac{\dot{e}(t_k)(s+\delta)}{s^3 + a_1 s^2 + (a_0 + b_0 k_p)\, s + b_0 k_i} \tag{5.30}$$

and the minimum of the $H_2$-norm would be again obtained for $\delta = 0$. However, since the disturbance step height $w_{20}$ cannot be assumed to be known in control practice, in order to compute the reset ratio by (5.29) (otherwise a simple feedforward compensator could be simply used to remove the step disturbance), an extra mechanism has to be used.

A good strategy is to use a disturbance observer [238, 265]; for instance, if the *unknown input observer* [159] is used, then the estimation of the disturbance, $\hat{d}(t)$ is simply given by

$$\begin{cases} \dot{\hat{\mathbf{x}}}_o(t) = A_o \hat{\mathbf{x}}_o(t) + B_o v(t) + L(y(t) - C_o \hat{\mathbf{x}}_o(t)), \\ \hat{d}(t) = \begin{bmatrix} C_2 & 0 \end{bmatrix} \hat{\mathbf{x}}_o(t) \end{cases} \tag{5.31}$$

with

$$A_o = \begin{bmatrix} A_2 & 0 \\ B_p C_2 & A_p \end{bmatrix}, \quad B_o = \begin{bmatrix} 0 \\ B_p \end{bmatrix}, \quad C_o = [\, 0 \;\; C_p \,],$$
$$L = \begin{bmatrix} L_w \\ L_p \end{bmatrix}, \quad \hat{\mathbf{x}}_o = \begin{bmatrix} \hat{w}_2 \\ \hat{\mathbf{x}}_p \end{bmatrix}. \tag{5.32}$$

The observer gain matrices $L_w$ and $L_p$ should be chosen such that $d(t) - \hat{d}(t) \approx 0$ for $t \approx t_1$, and thus a good estimation of the perturbation is available at the first reset instant. By using the estimation of the disturbance the reset ratio is set by the function

$$\mathcal{P}_2(\mathbf{x}(t), \hat{d}(t)) = 1 + \frac{\hat{d}(t)}{k_i x_i(t)}. \tag{5.33}$$

### Tuning Rule for Variable Band Resetting Law

In the following, the additional use of a variable band resetting law will be investigated. Thus, there is a new design parameter $\theta \geq 0$, that will be mainly used to produce a smother response, reducing the adverse undershoot effect of the zero crossing reset actions. Obviously, for $\theta = 0$ the zero crossing resetting law is recovered.

*Reference tracking.* Again, the starting point is the computation of $E_{r,k}(s)$. For step references of height $w_{10}$, the result is (see Appendix E)

$$E_{r,k}(s) = \frac{(w_{10} - x_{p_2}(t_k))s(s + a_1 - \frac{1}{\theta}) + a_0 w_{10} - b_0 k_i (1 - p_r(t_k^+))x_i(t_k)}{s^3 + a_1 s^2 + (a_0 + b_0 k_p)\, s + b_0 k_i}. \tag{5.34}$$

In this case, there are two design parameters, $\theta$ and $\mathcal{P}_1(\mathbf{x}(t))$, which may be used to modify the step response after reset instants. A first simple approach is to set a minimization problem with only the parameter $\theta$. In this case, the function $\mathcal{P}_1(\mathbf{x}(t))$ is again simply given by the (5.28). Note that $(w_{10} - x_{p_2}(t_k)) = e(t_k) = -\theta\dot{e}(t_k)$, and thus, the transfer function $E_{r,k}(s)$ can be rewritten as follows:

$$E_{r,k}(s) = \frac{-\dot{e}(t_k)(\theta s^2 + (a_1\theta - 1)s)}{s^3 + a_1 s^2 + (a_0 + b_0 k_p)\, s + b_0 k_i}. \tag{5.35}$$

Minimising the $H_2$-norm of $E_{r,k}(s)$ (see Appendix E), the following value for the parameter $\theta$ is obtained

$$\theta = \frac{a_1}{a_1^2 + a_0 + b_0 k_p}. \tag{5.36}$$

Another tuning rule consists in setting the minimization problem with two parameters, coming from the variable reset ratio and the variable band resetting law. Firstly, a new parameter $\delta$ is introduced, such as

$$\mathcal{P}_1(\mathbf{x}(t)) = 1 - \frac{a_0 w_{10} - \delta e(t)}{b_0 k_i x_i(t)}, \tag{5.37}$$

and thus, the transfer function $E_{r,k}(s)$ is given by

$$E_{r,k}(s) = \frac{-\dot{e}(t_k)(\theta s^2 + (a_1\theta - 1)s + \delta\theta)}{s^3 + a_1 s^2 + (a_0 + b_0 k_p)\, s + b_0 k_i}. \tag{5.38}$$

The minimization problem yields into the following tuning rule (see Appendix E)

$$\delta = \frac{b_0 k_i}{a_1}, \tag{5.39}$$

$$\theta = \frac{a_1^2}{a_1^3 + a_1(a_0 + b_0 k_p) - b_0 k_i}. \tag{5.40}$$

Regarding the variation of the new design parameters $\theta$ and $\delta$ respect to $k_p$ and $k_i$, it can be easily deduced from equations (5.36), (5.39) and (5.40) that an increment of $k_p$ implies that $\theta$ decreases and if $k_i$ increases then both $\delta$ and $\theta$ increases.

*Disturbance rejection.* The closed-loop error for a step disturbance is given in this case by

$$E_{d,k}(s) = -\frac{-x_{p2}(t_k)s(s + a_1 - \frac{1}{\theta}) + b_0 w_{20} + b_0 k_i (1 - p_r(t_k))x_i(t_k)}{s^3 + a_1 s^2 + (a_0 + b_0 k_p)\, s + b_0 k_i}. \tag{5.41}$$

If the disturbance observer (5.31) is used with a reset ratio set by the function

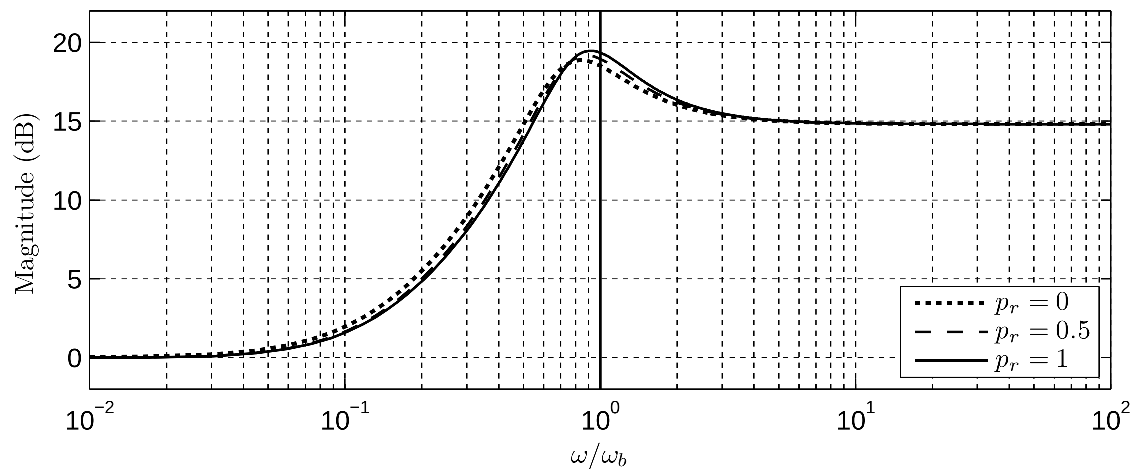
$$\mathcal{P}_2(\mathbf{x}(t), \hat{d}(t)) = 1 + \frac{b_0 \hat{d}(t) + \delta e(t)}{b_0 k_i x_i(t)}. \tag{5.42}$$

The minimization problem may be solved following similar arguments to those of Appendix E. Assuming that $\hat{d}(t_k) = w_{20}$, the values of $\delta$ and $\theta$ are the same as in the reference tracking case, that is (5.39) and (5.40).

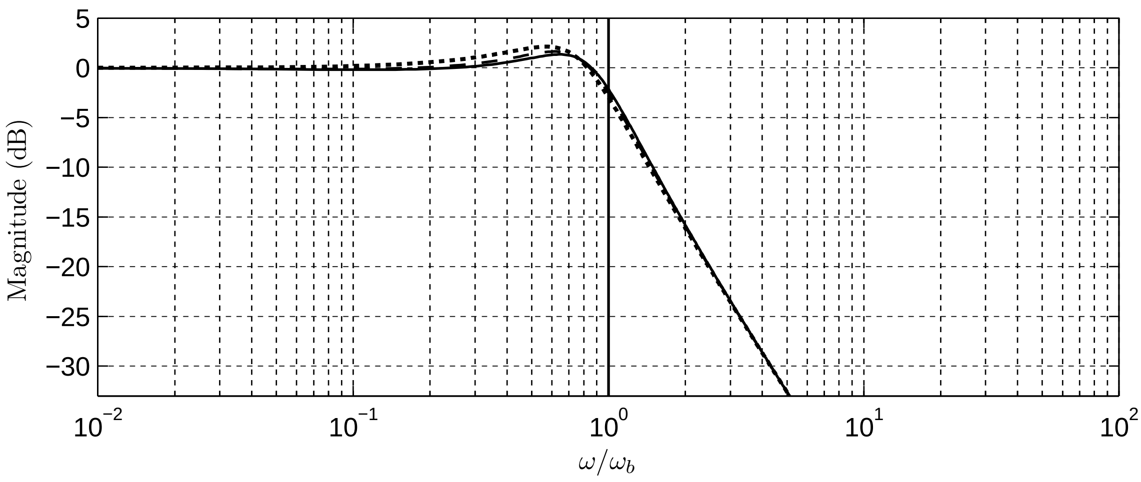
*Robustness.* For reference tracking, the reset ratio can be given by (5.28) or (5.37). In the first case, it is only affected by the plant DC-gain $k$. In the second case, the reset ratio and the other parameters depend on all the parameters of the plant, $a_1$, $a_0$, and $b_0$. Consider the different effects of the uncertainty of all the plant parameters over the parameters $\delta$ and $\theta$. If $\delta$ is decreased from its nominal value, then the response will be similar to the case $\delta = 0$. Otherwise, if $\delta$ increases then the reset ratio is also increased and the overshoot will be reduced. However, if $\delta$ is too large then the response would be overdamped. A simple way to avoid that behavior is by saturating the reset ratio, in order to keep it between 0 and 1. On the other hand, if the parameter $\theta$ is decreased, then the response will be close to the zero crossing case. Otherwise, if $\theta$ increases then the overshoot will be reduced and the response would be smoother. However, for large values of $\theta$ the reset action has almost no effect at the first reset instant, and since the other reset instants are obtained for $\dot{e}(t_k) \approx 0$ (note that $\dot{e}(t_k) = -e(t_k)/\theta \Rightarrow \lim_{\theta\to\infty} \dot{e}(t_k) = 0$), the response would be deteriorated.

*Sensor noise effect (cost of feedback).* Consider a second order plant ($a_1 = 6$, $a_0 = 5$, $b_0 = 5$) and a PI+CI compensator ($k_p = 5.5$, $\tau_i = 0.8$) with variable band resetting law ($\theta = 0.1$). As in the first order case, the reset control system sensibility respect to noise can be analyzed by the describing function of the PI+CI; in this case, due to the variable band resetting law, the describing function is obtained from (2.57). Fig. 5.6a shows the frequency response magnitude vs frequency for different values of the reset ratio, and Fig. 5.6b represents the different frequency responses from reference to closed-loop output. It can be seen again that the cost of feedback of the PI+CI with variable band resetting law is similar to the PI$_{base}$ ($p_r = 0$) for noise frequencies beyond $\omega_B$.

*Integrating plants.* This is a particular case of (5.21) where $a_0 = 0$. Thus, the PI+CI tuning rule can be directly applied by making $a_0 = 0$ in equations (5.28) and (5.37) for step tracking. Again, the suitable reset ratio for zero crossing resetting law is 1, that is the compensator is a full reset compensator. In spite of the fact that a full reset action is used, the asymptotic property of the compensator is kept for a step reference, since the plant has an integrator. On the other hand, the tuning rules for disturbance rejection (5.33) and (5.42) (zero crossing and variable band resetting law) also apply, since the parameter $a_0$ does not appear in the equations. Again, in general it is expected the reset ratio to be less than one for zero steady-state error.




(a) Frequency response from noise to control input.




(b) Frequency response from reference to closed-loop output.

Figure 5.6 Frequency responses of the reset control system composed by a second order plant ($a_1 = 6$, $a_0 = 5$, $b_0 = 5$) and a PI+CI compensator ($k_p = 5.5$, $\tau_i = 0.8$) with variable band resetting law ($\theta = 0.1$).

| Method | $k_p$ | $\tau_i$ | $\delta$ | $\theta$ | Overshoot % | Undershoot % | IAE$_r$ |
|---|---|---|---|---|---|---|---|
| SIMC | 5.5 | 0.8 | - | - | 23.00 | 3.31 | 0.57 |
| PI+CI$_{ZC}$ | 5.5 | 0.8 | 0 | 0 | 18.23 | 5.71 | 0.53 |
| PI+CI$_{VB}$ | 5.5 | 0.8 | 5.73 | 0.096 | 14.10 | 3.52 | 0.49 |

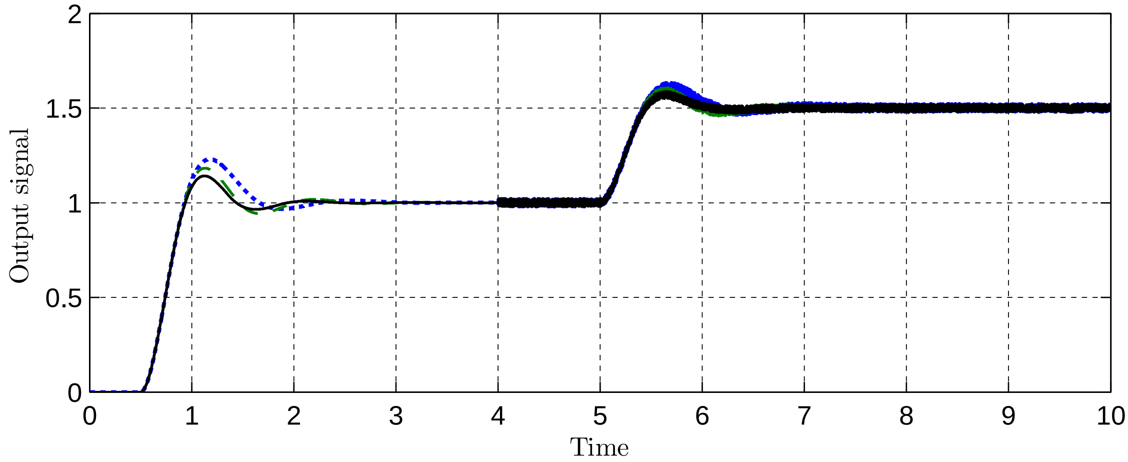
Table 5.1 Compensators setting and performance indices for the Example 5.2.

**Example 5.2.** In this example the above PI+CI tuning methods are applied to a second order plant with low damping

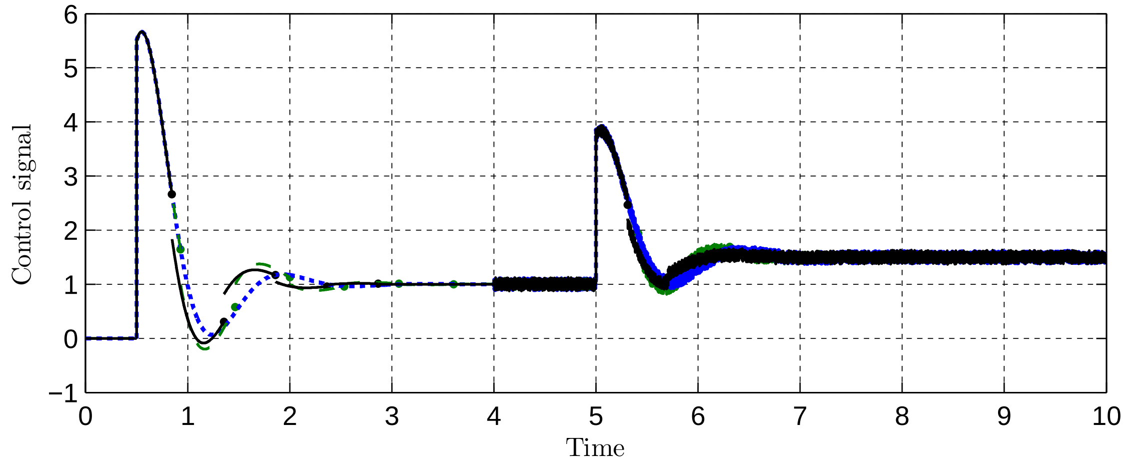
$$P(s) = \frac{5}{s^2 + 6s + 5}. \tag{5.43}$$

The PI$_{base}$ setting is obtained by the SIMC tuning rule ([254]). The setting of the PI$_{base}$ and the PI+CI compensator with zero crossing resetting law (PI+CI$_{ZC}$) and variable band resetting law (PI+CI$_{VB}$) is shown in Table 5.1. Fig. 5.7 shows the step responses, the control signal and the reset ratio for the three compensators. Note that the overshoot is significantly reduced by both reset PI compensators, specially for the case of variable band resetting law. However, the zero crossing resetting law increases the undershoot and the resulting response is more oscillating. This undesirable effect is practically avoided by using the variable band. In addition, sensor noise is added in the simulation to show that the proposed control strategy is not sensitive to the sensor noise. Note that the three compensators exhibit a similar behavior in presence of noise, as it has been previously discussed by the describing function analysis. On the other hand, in order to reduce the effect of noise over the variable band resetting law, a first order low pass filter with time constant $0.3\theta \approx 0.03$ has been used. In addition, robustness against parametric uncertainty is checked by simulation for the PI+CI$_{VB}$ case, and the results are shown in Fig. 5.8. The simulation consists of a step responce of the PI+CI compensator with variable band for different values of the DC-gain of the plant $k = 0.9, 1, 1.1$. The PI+CI$_{VB}$ compensator has been tuned for the nominal value $k = 1$.

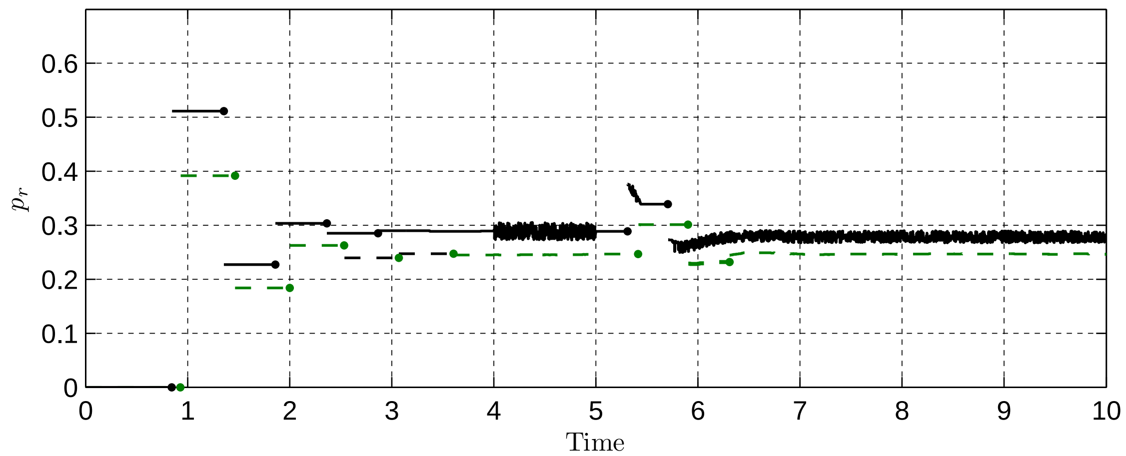
Finally, in order to get a better comparison between the different compensators, performance indices such as the overshoot, the undershoot, and the IAE are obtained in Table 5.1. From these indices, it can be concluded that PI+CI$_{VB}$ gets the best performance indices, significantly improving PI compensation without increasing the cost of feedback.




(a) Output signal.

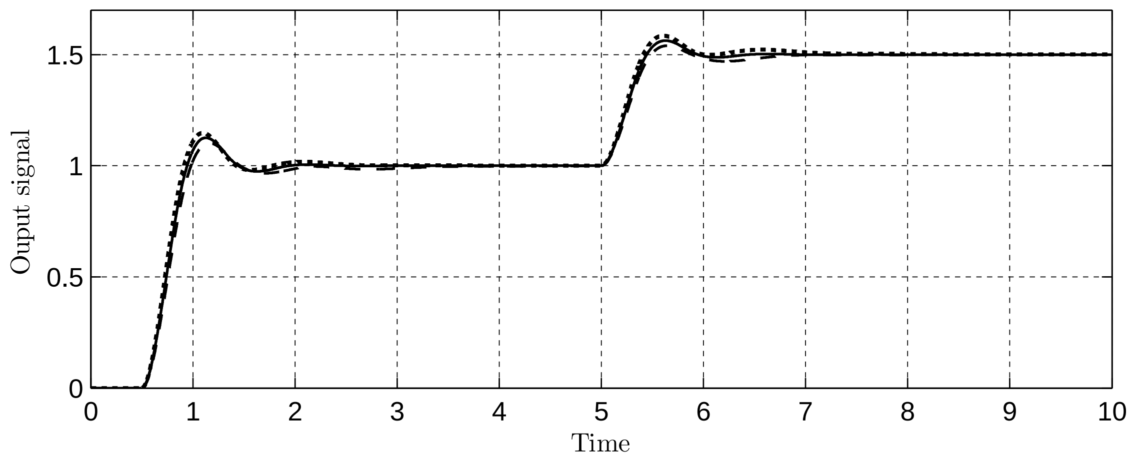



(b) Control signal.

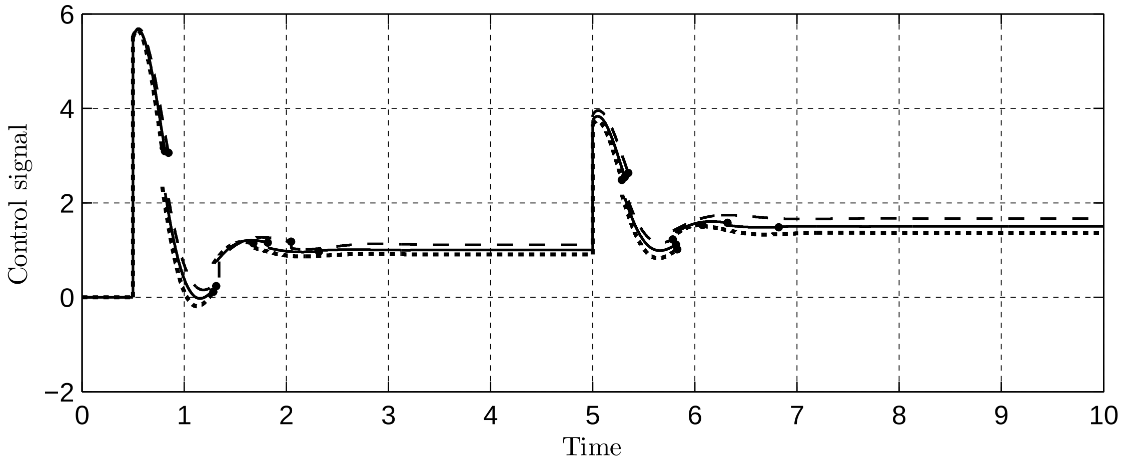



(c) Reset ratio.

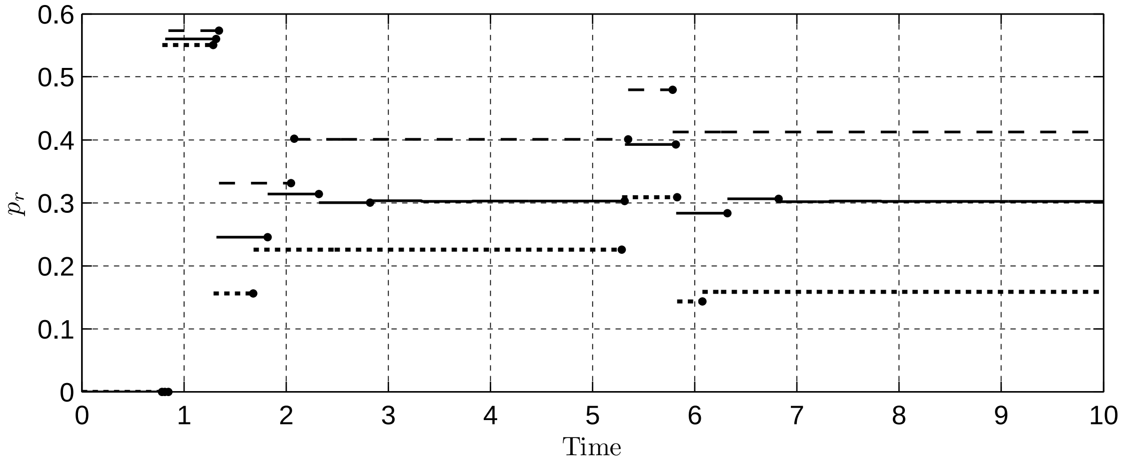
Figure 5.7 Output signals, control signals, and reset ratio of the reset control system and its base PI from the Example 5.2 for a sequence of step setpoints. PI (dotted-blue), PI+CI with zero crossing resetting law (dashed-green) and PI+CI with variable band resetting law (solid-black).




(a) Output signal.




(b) Control signal.




(c) Reset ratio.

Figure 5.8 Output signals, control signals and reset ratios of the reset control system from the Example 5.2 with a 10% of uncertainty in the plant gain. $k = 0.9$ (dashed), $k = 1$ (solid), and $k = 1.1$ (dotted).

| Method | $k_p$ | $\tau_i$ | $\delta$ | $\theta$ | Undershoot % | IAE$_d$ |
|---|---|---|---|---|---|---|
| SIMC | 5.5 | 0.3 | - | - | 23.00 | 3.31 |
| PI+CI$_{ZC}$ | 5.5 | 0.3 | 0 | 0 | 18.23 | 5.71 |
| PI+CI$_{VB}$ | 5.5 | 0.3 | 5.73 | 0.1 | 14.10 | 3.52 |

Table 5.2 Compensators setting and performance indices for the Example 5.3.

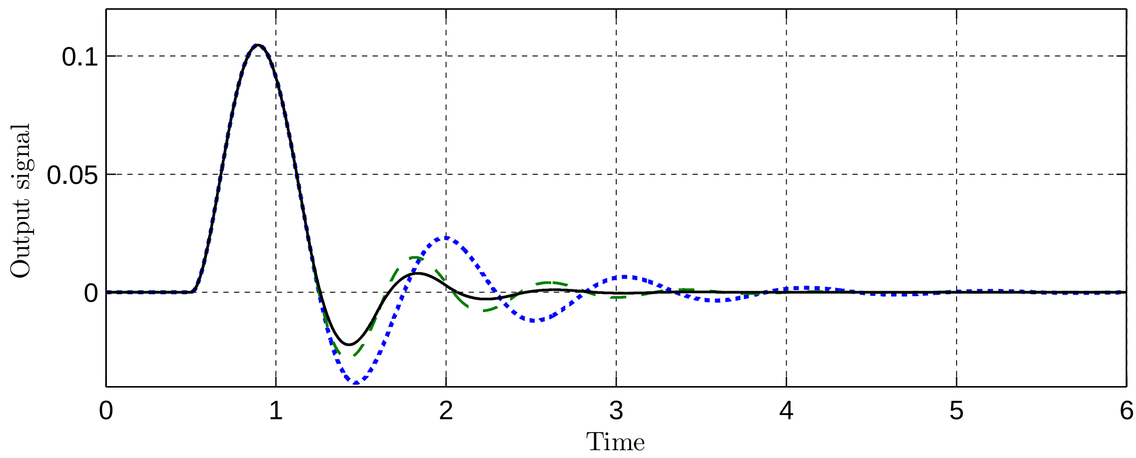
**Example 5.3.** A step disturbance rejection problem is considered for the plant of the previous example. The PI+CI compensator is now tuned by using the reset ratio (5.42); note that the tuning rules for the parameters $\theta$ and $\delta$ are identical to the case of reference tracking. In this case, the PI$_{base}$ compensator has been tuned with $k_p = 8$ and $\tau_i = 0.3$, producing a fast oscillating response. The setting of the PI and the PI+CI and the performance indices are shown in Table 5.2. In Fig. 5.9, the step disturbance response for the PI, PI+CI$_{ZC}$ and PI+CI$_{VB}$. The control signal and the reset ratio are also shown. It is again clear that the compensator PI+CI$_{VB}$ obtains the best disturbance rejection, with the smallest settling time.

### 5.1.4 Well-posed Reset Instants and Asymptotic Tracking

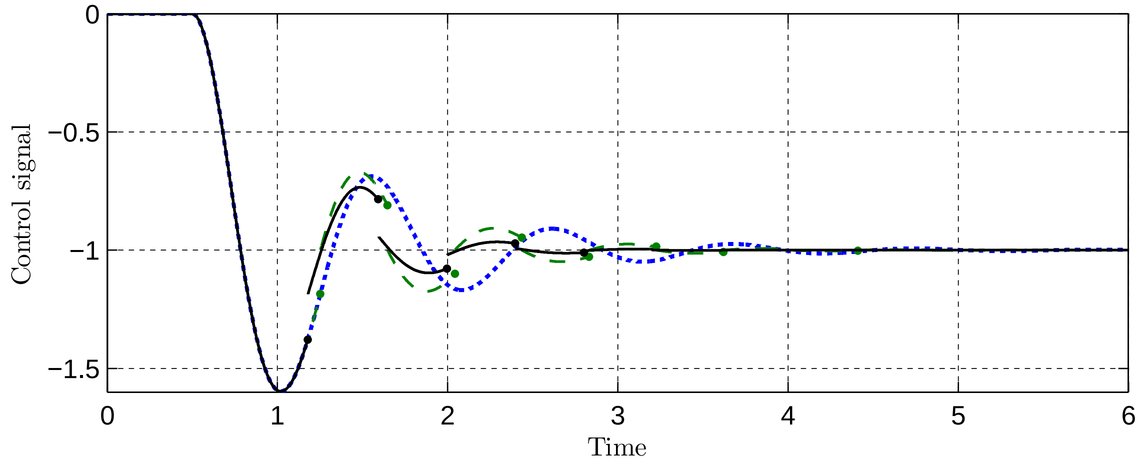
In this section, we analyze two important issues regarding the design of the reset control system: the reset control system has well-posed reset instants, so that no time regularization is necessary, and the reset control system asymptotically tracks a step reference signal $r$, that is, $\lim_{t\to\infty} e(t) = \infty$. Similarly, it is also of interest to guarantee that the reset control system with the proposed design asymptotically rejects a step disturbance ($e(t) \to 0$ as time goes infinity).

For the sake of simplicity, we analyze only the case of step reference, but the same approach can be applied for step disturbances. Note that the representation of the error signal by a sequence of impulses responses of LTI system (equation (5.12) and (5.13)) is applicable whenever the state $x_{ci}$ is zero.

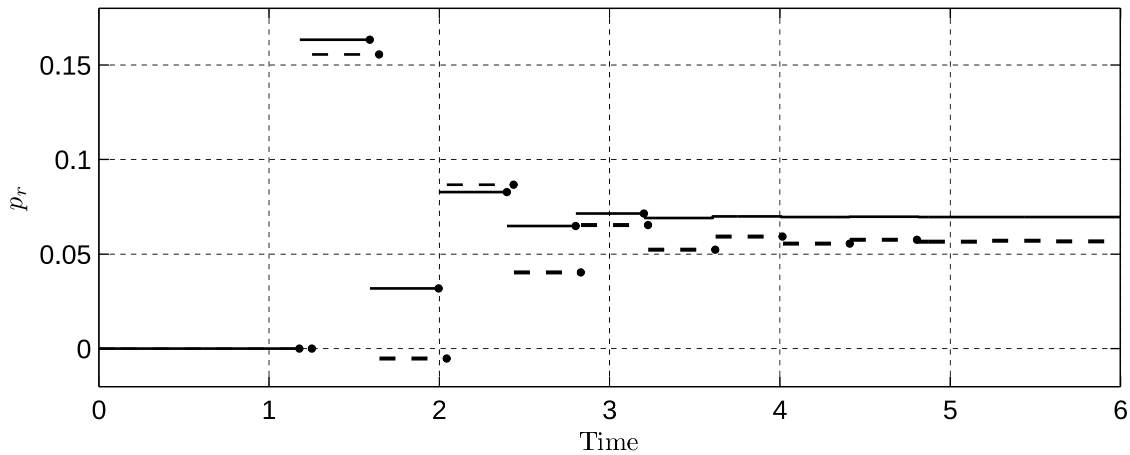
First, we focus on the case of second order plant. In this case, the sequence $E_{r,k}$ is given by (5.27) and (5.38) for zero crossing resetting law and variable band resetting law, respectively. Note that the transfer functions $E_{r,k}$ can be represented in the state-space, such that their impulse responses coincide with the initial condition response on the state-space. In addition, the numerator of the transfer function changes for every reset




(a) Output signal.




(b) Control signal.




(c) Reset ratio.

Figure 5.9 Output signals, control signals, and reset ratio of the reset control system and its base PI from the Example 5.2 for a step disturbance. PI (dotted-blue), PI+CI with zero crossing resetting law (dashed-green) and PI+CI with variable band resetting law (solid-black).

instant. Therefore, we can define an IDS whose output coincides with the error signal of the reset control system for any initial condition with $x_{ci}(0) = 0$. The counterpart IDS is given by

$$\begin{cases} \dot{\mathbf{x}}(t) = A\mathbf{x}(t), & C_M\mathbf{x}(t) \neq 0, \\ \mathbf{x}(t^+) = A_R\mathbf{x}(t), & C_M\mathbf{x}(t) = 0, \\ e(t) = C\mathbf{x}(t), \end{cases} \tag{5.44}$$

where the initial condition and the matrices $A$ and $C$ are given by

$$A = \begin{bmatrix} 0 & 0 & -b_0k_i \\ 1 & 0 & -(a_0 + b_0k_p) \\ 0 & 1 & -a_1 \end{bmatrix}, \; C = [\; 0 \;\; 0 \;\; 1 \;] \;, \; \mathbf{x}_0 = \begin{bmatrix} a_0w_{10} - b_0k_i(1 - p_r(0))x_i(0) \\ a_1w_{10} - x_1(0) \\ w_{10} - x_2(0) \end{bmatrix}. \tag{5.45}$$

Matrices $A_R$ and $C_M$ depend on the resetting law: for zero crossing the matrices are given by

$$A_R = \begin{bmatrix} 0 & \delta & -\delta a_1 \\ 0 & 1 & -a_1 \\ 0 & 0 & 0 \end{bmatrix}, \quad C_M = C. \tag{5.46}$$

If a variable band resetting law is used, then the matrices are

$$A_R = \begin{bmatrix} 0 & \delta\theta & -\delta\theta a_1 \\ 0 & (\theta a_1 - 1) & -a_1(\theta a_1 - 1) \\ 0 & \theta & -a_1\theta \end{bmatrix}, \quad C_M = C(\theta A + I). \tag{5.47}$$

Suppose that for an initial condition $\mathbf{x}_0 \notin \mathcal{M}$ there is not reset actions, then the asymptotic tracking is subjected to the base system. Therefore, if the parameter $k_i > 0$, then the reset control system tracks asymptotically a step reference (assuming a stable base system). On the contrary, now let $t_1$ be the first reset instant (if $\mathbf{x}_0 \in \mathcal{M}$ then $t_1 = 0$) and take $\mathbf{x}(t_1^+)$ as the new initial condition in the system (5.44). Then, the reset control system tracks asymptotically a step reference if the trivial solution of the system (5.44) is asymptotically stable. By simple inspection of the transfer functions $E_{r,k}(s)$ as given by (5.27) and (5.38), it is directly deduced that all the error signals are identical up to a constant (the error derivative at the reset instant), that is $e_{r,k}(t) = -\dot{e}_{r,k-1}(t_k - t_{k-1})e^*(t)$ for some function $e^*(t)$. Therefore, considering the more general case of variable band resetting law the reset instant $t_{k+1}$ is defined by $t_{k+1} = t_k + \Delta$, where $\Delta$ satisfies

$$\theta\dot{e}_{r,k}(\Delta) + e_{r,k}(\Delta) = -\dot{e}_{r,k-1}(t_k - t_{k-1})(\theta\dot{e}^*(\Delta) + \dot{e}^*(\Delta)) = 0. \tag{5.48}$$

Hence, $\Delta$ does not depend on the reset instant and then it is concluded that the reset instants are periodic with (fundamental) period $\Delta > 0$. The zero crossing resetting law is a particular case in which $\theta = 0$. Since the reset instants are periodic, it is clear that the system has well-posed reset instants. Regarding the asymptotic stability, a simple sufficient condition for the asymptotic stability of the IDS (5.44) is that the matrix $A_R e^{A\Delta}$ be Schur-stable (see, e.g., [61]), that is, all its eigenvalues belong to the interior of the unit circle.

Finally, the case of first order plant is much simpler. Following the same approach, the matrices of the system (5.44) are now given by $C_M = C$, $A_R = \text{diag}(0, 0)$, and

$$A = \begin{bmatrix} 0 & -b_0 k_i \\ 1 & -(a_0 + b_0 k_p) \end{bmatrix}, \; C = [\; 0 \;\; 1 \;] \;, \; \mathbf{x}_0 = \begin{bmatrix} a_0 w_{10} - b_0 k_i (1 - p_r(0)) x_i(0) \\ 0 \end{bmatrix}. \tag{5.49}$$

Note that all the elements of the matrix $A_R$ are 0, since $p_r$ is set to make the error zero at the first reset instant. Therefore, there is only one reset instant and the stability is directly deduced from the stability of the base system.

## 5.2 PI+CI Tuning Rule for Integrating Plus Dead Time Processes

Thy dynamic of many industrial processes can be properly modeled by an integrator plus dead time system (IPDT) (see, e.g., [275]); some examples are liquid level of tanks, batch distillation columns, oil water-gas separators in oil industry. In general, an IPDT system P is modeled by the transfer function

$$P(s) = \frac{k e^{-hs}}{s} \tag{5.50}$$

with two parameters $k$ and $h$, where $k$ is the system (velocity) gain, and $h$ is the time-delay. In general, all stable processes with very long time constants may also be approximated by integrating models in the working frequency range. For instance, a lag dominant first order plus dead time system (FOPDT) with gain $\hat{k}$, time constant $\tau$, and time-delay $\hat{h}$, may be well-approximated by (5.50) with gain $k = \hat{k}/\tau$ and time-delay $h = \hat{h}$, when $\tau >> \hat{h}$. Control of IPDT processes may be challenging due to the existence of a pole at the origin, which yields into an unstable open-loop. In addition, it is well-known that the control of these processes implies a big trade-off between servo and regulatory control, or in other terms, between setpoint response and output disturbance rejection ([4, 106, 115]).

Regarding the PI+CI compensator, first, note that the feedback interconnection is affected by a time-delay, and thus, its stability cannot be directly guaranteed. Therefore, a strategy of stabilization by time-dependent conditions will be applied. In addition, the PI+CI is provided with a variable band resetting law to overcome the harmful effect of the time-delay. Finally, it is also considered a switching reset ratio so that we take advantage of the reset actions for both servo and regulatory problems. As a result, the PI+CI with these improvements may be represented by

$$\begin{cases} \left.\begin{array}{l} \dot{\mathbf{x}}_r(t) = A_r\mathbf{x}_r(t) + B_r(p_r(t))e(t), \\ \dot{p}_r(t) = 0, \\ \dot{\Delta}(t) = 1, \end{array}\right\} & (e(t),\dot{e}(t)) \notin \mathcal{R} \vee \Delta(t) < \Delta_m, \\ \left.\begin{array}{l} \mathbf{x}_r(t^+) = A_\rho\mathbf{x}_r(t), \\ p_r(t^+) = \mathcal{P}_{\sigma(t)}(\mathbf{x}_r(t), e(t)), \\ \Delta(t^+) = 0, \end{array}\right\} & ((e(t),\dot{e}(t)) \in \mathcal{R} \wedge \Delta(t) \geq \Delta_m) \vee \Delta(t) = \Delta_M, \\ v(t) = C_r\mathbf{x}_r(t) + D_r e(t), & \end{cases} \tag{5.51}$$

where $\Delta(0) = 0$, $\mathbf{x}_r = (x_i, x_{ci})^\top \in \mathbb{R}^2$ is the compensator state: $x_i$ is the integrator state, and $x_{ci}$ corresponds to the Clegg integrator state. Matrices $A_r$, $A_\rho$, and $D_r$ are given in (5.2), and $B_r$ and $C_r$ are as follows:

$$B_r(p_r(t)) = \begin{bmatrix} 1 - p_r(t) \\ p_r(t) \end{bmatrix}, \qquad C_r = k_i \begin{bmatrix} 1 & 1 \end{bmatrix}. \tag{5.52}$$

The design parameter of the variable band resetting law is set equal to the time-delay, so that there is one less tuning parameter. In this way, the resetting set $\mathcal{R}$ is given by

$$\mathcal{R} = \{(e(t),\dot{e}(t),\Delta(t)) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}_+ : (h\dot{e}(t) + e(t) = 0 \wedge \Delta(t) \geq \Delta_m) \vee \Delta(t) = \Delta_M\} \tag{5.53}$$

On the other hand, it has been shown that a P+CI compensator ($p_r = 1$) improves step tracking compared to a partial reset PI+CI compensator ($p_r < 1$) in integrating processes without losing the fundamental asymptotic property (zero state-state error for step set point). Nevertheless, when the process is disturbed at the input by a step signal un undesirable non-zero error is obtained in the steady-state. Therefore, it is proposed to set the reset ratio function $\mathcal{P}_{\sigma(t)}$ by $\mathcal{P}_1 = 1$ and $\mathcal{P}_2 = p_r^*$, where $p_r^*$ is the reset ratio to be used for disturbance rejection. As a result, the PI+CI has three tuning parameters $k_p$, $k_i$, and $p_r^*$.

### 5.2.1 A Base PI Tuning Rule

The control of IPDT processes has been widely studied (see, e.g., [275] and the references therein) and, as a result, a multitude of PI tuning methods have been proposed in the literature ([212]). In general, the tuning rules for both PI and PID compensators may be classified as follows (classification suggested in [274]):

- *Process reaction curve tuning rules:* These tuning rules are based on process model data determined from a measured open-loop step response. The most well-known tuning of this type was devise by Ziegler and Nichols in 1942 ([308]). A sample of tuning rules based on this approach is [80, 95, 277].

- *Optimisation-based tuning rules:* Tuning rules can be also obtained by minimising a suitable performance index, such as the Integral Absolute Error (IAE), the Integral Time Absolute Error (ITAE) and the Integral Square Error (ISE). The performance criteria are typically specified for optimum regulator ([135, 233]), optimum servo action ([5, 241]) or simultaneously both servo and regulation ([7, 225]).

- *Direct synthesis tuning rules:* In these tuning rules the compensator parameters are retrieved by imposing a desired closed-loop response. They may be specified using time domain metric ([76, 254]), such as achieving the desired poles of the closed-loop response, or a frequency domain metric ([7, 195]), such as specified gain margin and/or phase margin.

- *Tuning rules for robustness:* These tuning rules are design to provide the closed-loop system with a robust stability and/or a robust performance criterion ([69, 77]).

- *Ultimate cycle tuning rules:* The tuning rules in this category are based on a nonparametric model given by the ultimate gain and ultimate frequency. The first tuning rule was proposed by Ziegler and Nichols in [308]. Other tuning rules for IPDT processes are [49, 192].

In particular, Simple/Skogestad Internal Model Control (SIMC) method [254] has became popular since 2003, mainly due to its simplicity and good trade-off between robustness and performance [115].

### The SIMC Tuning Rule

It is based on the Internal Model Control (IMC) approach [243], and the closely related direct synthesis method [255]. In general, the goal is to obtain a setpoint response given by a first order transfer function with a time constant $\tau_c$ and the unavoidable time-delay $h$. The PI tuning rule for IPDT processes is

$$k_p = \frac{1}{k(\tau_c + h)}, \qquad \tau_i = 4(\tau_c + h). \tag{5.54}$$

Small values of $\tau_c$ favour the speed and disturbance rejection over the stability and robustness. On the other hand, $\tau_c$ is recommended in [254] to take the value of the time-delay ($\tau_c = h$), since it gives good robustness margins (a sensitivity peak of 1.7) and almost optimal performance in terms of the balanced servo/regulatory IAE (optimality constrained to a sensitivity peak of 1.7). The resulting tuning rule with the recommended value is simply given by

$$k_p = \frac{0.5}{kh}, \qquad \tau_i = 8h. \tag{5.55}$$

### A Modification of the SIMC Tuning Rule

In SIMC, $\tau_i$ is set to improve the robustness by avoiding slow frequency oscillations, giving as a result an overdamped (or critically damped) response. Thus, this rule is not appropriate for the base compensator of a reset control system, since the absence of oscillations prevents the reset actions to occur[2]. In this work, a simple modification of the SIMC tuning rule to give an oscillatory response is proposed. The oscillations are well characterized by the closed-loop characteristic polynomial, that can be obtained from the numerator of the transfer function $1 + PI_{base}(s)P(s)$ and using the approximation $e^{-sh} \approx 1 - hs$; it is given by

$$\tau_i(\frac{1}{kk_p} - h)s^2 + (\tau_i - h)s + 1. \tag{5.56}$$

Note that the characteristic polynomial has the standard form $\tau_0^2 s^2 + 2\tau_0 \xi s + 1$, and oscillations occurs when $\xi < 1$. For PI parameters such as $k_p \ll \frac{1}{kh}$ and $\tau_i \gg h$ it is

[2]The step setpoint response of the base system with the SIMC tuning rule has overshoot, and thus, at least one reset action is performed resulting in a reduction of the overshoot. However, the step disturbance response does have no oscillation and no reset actions are performed.

obtained

$$\tau_0 = \sqrt{\tau_i(\frac{1}{kk_p} - h)} \approx \sqrt{\frac{\tau_i}{kk_p}}, \tag{5.57}$$

$$\xi = \frac{\tau_i - h}{2\tau_0} \approx \frac{1}{2}\sqrt{kk_p\tau_i}, \tag{5.58}$$

from where a simple rule is

$$k_p = \frac{2\xi}{k\tau_0}, \qquad \tau_i = 2\xi\tau_0. \tag{5.59}$$

Finally, if $\tau_0 = \alpha\xi h$ is used for some scalar $\alpha$, it is simpler to characterize the above approximations. The result is that the modified rule to obtain an oscillatory response is

$$k_p = \frac{2}{k\alpha h}, \qquad \tau_i = 2\alpha\xi^2 h, \tag{5.60}$$

where $\xi \leq 1$ and $\alpha \gg 2$. Note that the SIMC tuning rule (5.55) (with $\tau_c = h$) is recovered for $\alpha = 4$ and $\xi = 1$, and that for the response to be oscillatory necessarily $\xi < 1$. In this work, a value $\alpha = 4$ will be also used and $\xi$ will be adopted as a new tuning parameter for the reset compensator, and thus, the modified SIMC tuning rule for the compensator $\text{PI}_{base}$ is

$$k_p = \frac{0.5}{kh}, \qquad \tau_i = 8\xi^2 h. \tag{5.61}$$

Although $\xi$ is not the exact value of the damping factor (some approximations have been performed), it is a measure of the system damping in the line followed by the SIMC tuning rule. Note that the same type of approximation is performed, and for $\alpha = 4$ and $\xi = 1$ the SIMC tuning rule is recovered. As $\xi$ becomes smaller a more oscillating response is produced and this is essential for reset compensation. In addition, for small values of $\xi$, we may obtain an unstable response. Although, it has been shown that a reset compensator can stabilize an unstable base system, from a practical point of view, it is adequate to consider that the parameters $k_p$ and $\tau_i$ are set such that the base system is stable. For a closed-loop system formed by a PI and an IPDT process, the region of stabilizing parameters is bounded in the $(\hat{k}_p, \hat{k}_i)$-plane by the parametric curve (see for example [244]) given by

$$\hat{k}_p = k_p kh = \omega sin(\omega), \qquad \hat{k}_i = \frac{k_p kh^2}{\tau_i} = \omega^2 cos(\omega) \tag{5.62}$$

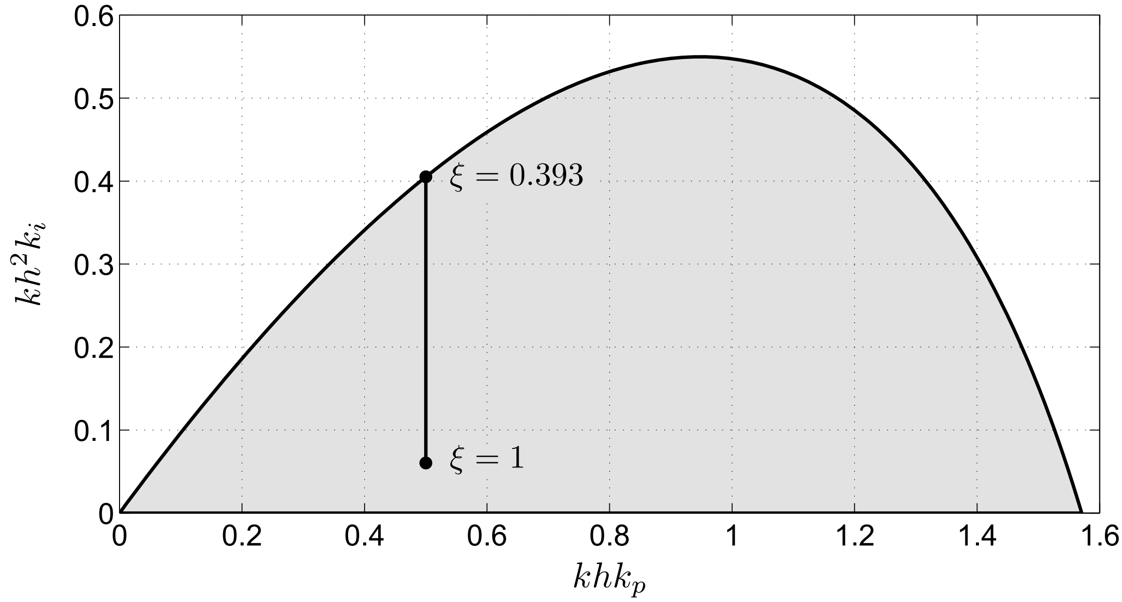



Figure 5.10 $\mathrm{PI}_{base}$ stabilizing region in the $(\hat{k}_p, \hat{k}_i)$-plane (shadow area), and values of the parameters for the modified SIMC rule (vertical line); the SIMC rule corresponds to $\xi = 1$.

for $\omega \in [0, \frac{\pi}{2}]$ and the line $\hat{k}_i = 0$. Hence, the proposed modified SIMC rule (5.61) gives an stabilizing $\mathrm{PI}_{base}$ compensator if the following conditions are satisfied

$$\omega sin(\omega) = 0.5, \qquad 8\xi^2\omega^2 cos(\omega) \geq 0.5, \tag{5.63}$$

and finally, it results that $\xi \geq 0.393$. Hence, the design parameter $\xi$ is assumed to be in the interval $[0.4, 1]$ for the base closed-loop system to be stable (see Fig. 5.10).

### 5.2.2 A PI+CI Tuning Rule

The reset compensator PI+CI will be tuned by using the equations (5.61) for its base system, which still leaves two parameters to be tuned: $\xi$ and $p_r^\star$. A parameter $\xi < 1$ assured an oscillatory response which is needed for the reset actions to be effective, and the parameter $p_r^*$ will be used to improve the disturbance rejection. In the following, the approach of Section 5.1 will be adapted to provide a tuning rule for the $p_r^*$.

Let $e(t)$, $t \geq 0$, be the error signal of the closed-loop system due to a step disturbance of height $w_{20}$, then it is defined the sequence of instants $(T_k)$, $k = 0, 1, \ldots$ such that $h = T_0 < T_1 < T_2 \ldots$ and for all $k \geq 0$ it is satisfied $e(T_k) = 0$. Following the approach of Section 5.1, this signal may be determined by a sequence $(e_k)_{k=0,1,2,...,}$ (the initial conditions of the compensator and the plant are assumed to be zero) as follows:

$$e(t) = \begin{cases} 0, & t \in [0, T_0], \\ e_0(t - T_0), & t \in [T_0, T_1], \\ e_1(t - T_1), & t \in (T_1, T_2], \\ e_2(t - T_2), & t \in (T_2, T_3], \\ \dots \end{cases} \tag{5.64}$$

Note that in this case, $e_k$ defines the error signal between two instants in which the error is zero, instead of reset instants (the zero crossings instants and the reset instants are different due to the variable band resetting law). Consider the closed-loop state $\mathbf{x} = (x_p, x_i, x_{ci}, w_2)^\top$, then it is obtained that every $e_k$ is given by

$$e_k(t) = e_{1,k}(t) + e_{2,k}(t) = Ce^{A(t-T_k)}\mathbf{x}(T_k) + \int_0^h Ce^{A\tau} A_d \mathbf{x}(t - \tau)d\tau, \tag{5.65}$$

where

$$A = \begin{bmatrix} 0 & kk_i(1 - p_r^*) & kk_i p_r^* & k \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{5.66}$$

$$A_d = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{5.67}$$

$$C = \begin{bmatrix} -1 & 0 & 0 & 0 \end{bmatrix}, \tag{5.68}$$

$$\mathbf{x}(T_k) = \begin{bmatrix} 0 & x_i(T_k) & x_{ci}(T_k) & w_{20} \end{bmatrix}^\top. \tag{5.69}$$

By a simple inspection of $A_d$, it is clear that $e_{2,k}$ is not directly affected by the states $x_i$ and $x_{ci}$. Therefore, we cannot directly affect this term by setting the $p_r^*$. For this reason, we focus on $e_{1,k}$, which can be obtained as the impulse response of the following transfer function (same procedure as in Appendix E)

$$E_{1,k}(s) = -\frac{kw_{20} + kk_i(1 - p_r^*)x_i(T_k)}{s^2} - \frac{kk_i p_r^* x_{ci}(T_k)}{s^2}. \tag{5.70}$$

Note that $E_{1,k}(s)$ can be made zero by using a variable reset ratio, but it should be change every reset instant by using the value of the compensator states at the future instant $T_k$, which is not possible. Linear approximations such as $x_i(T_k) = \frac{hx_i(t_k)}{2}$ may be used as a simple solution. Nevertheless, the change of the reset ratio affects the matrix $A$, and thus, $e_{2,k}$ is also modified, which may yield into a deterioration of the performance. A simpler tuning rule for a constant reset ratio can be developed if we focus our attention on the first instants $t_1$ and $T_1$. Note that the error signal of the reset control system and the base system are identical in the interval $[0, t_1 + h]$ and almost identical in $[t_1 + h, T_1]$ (if the prediction of the variable band resetting law is exact the interval $[t_1 + h, T_1]$ is zero). Normally, the variable band resetting law yields to $t_1 + h \approx T_1$. In addition, $x_i(T_1) \gg x_{ci}(T_1)$, since $x_{ci}$ was set to zero approximately $h$ units time before $T_1$. Therefore, it is clear that simply taking the following reset ratio

$$p_r^* = 1 - \frac{w_{20}}{k_i x_i(T_1)}, \tag{5.71}$$

the error signal $e_{1,1}(t)$ will be smaller, and thus, the performance of the compensator will be improved. As it was mentioned above, $x_i(T_1)$ can be approximated by the step disturbance response of the base system. Considering the plant (5.50) and a PI compensator given by

$$PI(s) = \frac{k_p(s + \frac{1}{\tau_i})}{s}. \tag{5.72}$$

It is clear that the transfer function $T_d(s)$ from the disturbance to the error signal can be approximated by using $e^{-hs} \approx 1 - hs$, resulting in the following expression

$$T_d(s) = \frac{P(s)}{1 + P(s)PI(s)} \approx \frac{kse^{-hs}}{(1 - kk_p h)s^2 + kk_p(1 - \frac{h}{\tau_i})s + \frac{kk_p}{\tau_i}}. \tag{5.73}$$

Solving for the above equation, the value $x_i(T_1)$ is simply given by

$$x_i(T_1) = -\frac{w_{20}}{k_i}(1 + e^{\frac{\alpha}{\beta}\pi}), \tag{5.74}$$

where $\alpha \pm i\beta$ are the poles of the transfer function (5.73). The values of $k_p$ and $\tau_i$ given in (5.61) yield to

$$\frac{\alpha}{\beta} = -\frac{\sqrt{2}(8\xi^2 - 1)}{8\xi}. \tag{5.75}$$

Thus, by direct substitution in (5.71), it is obtained the following tuning rule for the reset ratio

$$p_r^* = 1 - \frac{1}{1+e^{-f\pi}}, \quad f = \frac{\sqrt{2}(8\xi^2-1)}{8\xi}, \tag{5.76}$$

where $\xi$ is the tuning parameter used in Section 5.2.1. Note that if $\xi$ is large (approximately greater than 0.8), the response may have small oscillations or even no oscillations, and then the reset actions would have no effect.

### 5.2.3 A Case Study

In this section, the benefits of the proposed reset compensator will be illustrated by simulation. Consider an IPDT system given by (5.50) with $k = 0.2$ and $\theta = 7.4$ (this system has been used by the distillation column model given in [69, 77]). In order to compare PI+CI and PI compensators, it will be considered the following tuning rules. Five tuning rules for PI compensation:

- SIMC (with $\tau_c = \theta$) [254].
- AMIGO [139].
- The method in [195], that will be referred to as MaShe.
- An optimal tuning that minimized $\text{IAE}_c = (\text{IAE}_r + \text{IAE}_d)/2$, referred to as Opt. $\text{IAE}_c$ ($\text{IAE}_r$ and $\text{IAE}_d$ are the IAE with respect to the unit step setpoint response and the unit step disturbance response, respectively).
- An optimal tuning that minimized $\text{IAE}_c$ subject to an percentage overshoot less than 46% (setpoint response), referred to as Const. Opt. $\text{IAE}_c$ (for comparison purposes, the maximum percentage overshoot of 46% obtained by a PI+CI compensator with $\xi \in [0.4, 1]$ has been used).

And three PI+CI compensators (time-triggered, with $\rho = 150$) with:

- $\xi = 0.4$.
- $\xi = 0.7$.
- $\xi = 0.55$ (minimizes $\text{IAE}_c$).

On the other hand, the variable band resetting law has been implemented without filter, since an ideal noise-free case has been considered in the simulation. The obtained compensators parameters are shown in Table 5.3.

| Method | $k_p$ | $\tau_i$ | $p_r^*$ | Overshoot % | IAE$_r$ | IAE$_d$ |
|---|---|---|---|---|---|---|
| SIMC | 0.337 | 59.20 | - | 27.74 | 29.02 | 175.9 |
| AMIGO | 0.236 | 51.80 | - | 28.94 | 35.54 | 222.9 |
| MaShe | 0.608 | 37.00 | - | 74.26 | 27.08 | 60.88 |
| Opt. IAE$_c$ | 0.533 | 30.76 | - | 70.26 | 26.34 | 57.64 |
| Const. Opt. IAE$_c$ | 0.398 | 37.19 | - | 45.85 | 26.34 | 93.48 |
| PI+CI - $\xi = 0.4$ | 0.337 | 9.47 | 0.40 | 39.14 | 22.51 | 87.60 |
| PI$_{base}$ - $\xi = 0.4$ | 0.337 | 9.47 | - | 124.0 | >300 | >1000 |
| PI+CI - $\xi = 0.7$ | 0.337 | 29.00 | 0.09 | 16.74 | 17.65 | 94.8 |
| PI$_{base}$ - $\xi = 0.7$ | 0.337 | 29.00 | - | 50.05 | 30.19 | 97.60 |
| PI+CI - $\xi = 0.55$ (Opt.) | 0.337 | 17.90 | 0.19 | 23.84 | 18.93 | 82.64 |
| PI$_{base}$ - $\xi = 0.55$ | 0.337 | 17.90 | - | 74.50 | 42.35 | 108.6 |

Table 5.3 Compensators setting and performance indices for an IPDT process with $k = 0.2$ and $h = 7.4$.

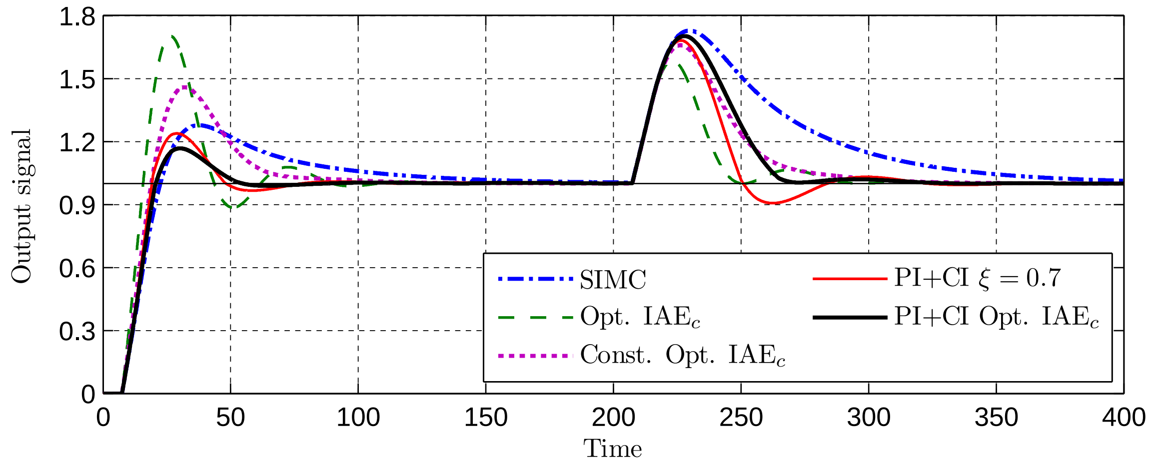
**Performance Analysis**

A simulation of the closed-loop system with the different compensators has been performed. The simulation consists of the response to a unit step setpoint at $t = 0$, and an input disturbance of amplitude 0.25 at $t = 200s$. Fig. 5.11 shows the responses for several compensators (not all of them are included for the sake of clarity). Moreover, Fig. 5.12 shows the response of the PI+CI compensators and its base PI compensators (PI$_{base}$). Several performance indices has been computed: percentage overshoot (setpoint response), IAE$_r$, and IAE$_d$. All the results are shown in Table 5.3. Several conclusions are obtained:

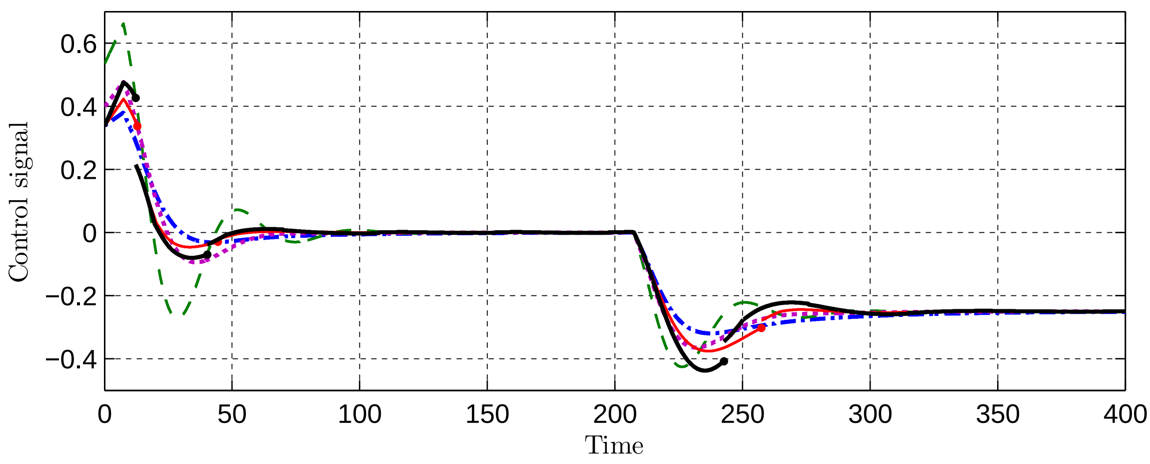
- The setpoint response of the PI+CI compensator improves when $\xi$ increases in the interval $[0.4, 0.7]$. However, note that the setpoint response gets slower as $\xi$ increases, and thus, the IAE$_r$ may increase for larger values of $\xi$ ($> 0.7$). The disturbance rejection is better for values around 0.55, and it gets worse either when $\xi$ increases or decreases around that value.

- In comparison with its base compensator PI$_{base}$, reset compensator PI+CI supposes a significant improvement of the performance (both setpoint and disturbance) in all the considered cases[3]. This is reasonable, since the modified SIMC

[3] The reader is referred to [198] for a detailed study of the PI+CI overcoming the base PI when they are applied to an IPDT process.

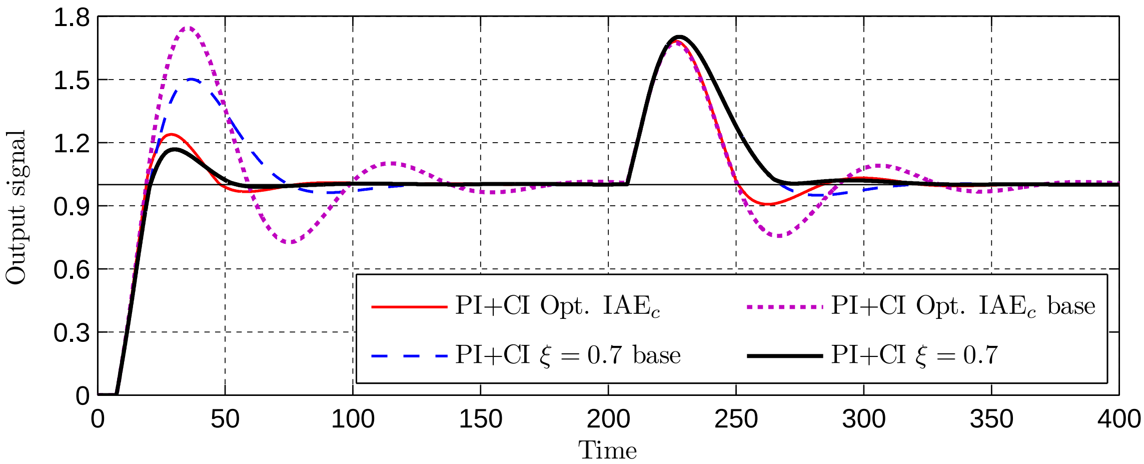



(a) Output signal.

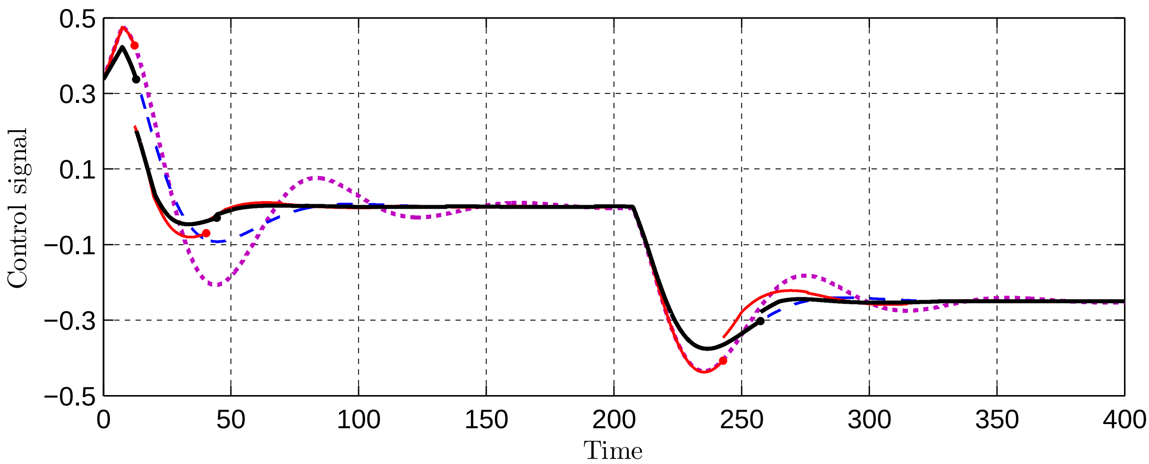



(b) Control signal.

Figure 5.11 Output signals and control signals for several PI and PI+CI compensators applied to an IPDT process with $k = 0.2$ and $h = 7.4$.




(a) Output signal.




(b) Control signal.

Figure 5.12 Output signals and control signals for several PI+CI compensators and its base compensators applied to an IPDT process with $k = 0.2$ and $h = 7.4$.

| | Overshoot % | $\text{IAE}_r$ | $\text{IAE}_d$ |
|---|---|---|---|
| SIMC | 41.81 | 33.37 | 176.1 |
| AMIGO | 33.41 | 41.72 | 237.8 |
| MaShe | 110.80 | 65.83 | 145.2 |
| Opt. $\text{IAE}_c$ | 103.38 | 50.40 | 101.2 |
| Const. Opt. $\text{IAE}_c$ | 67.17 | 30.42 | 94.08 |
| PI+CI - $\xi = 0.4$ | 66.00 | 36.83 | 242.4 |
| PI+CI - $\xi = 0.7$ | 34.93 | 21.96 | 102.8 |
| PI+CI - $\xi = 0.55$ (Opt.) | 44.79 | 25.05 | 109.7 |

Table 5.4 Worst-case performance indices for several PI and PI+CI compensators applied to an IPDT process ($k = 0.2$ and $h = 7.4$) with a 15% of uncertainty in the gain and the time-delay.

rule produces a detuned oscillating response with respect to the SIMC tuning rule.

- In comparison to the rest of the PI settings, PI+CI compensator clearly shows better performance for setpoint tracking in all cases. However, disturbance rejection is not as better as the optimal PI compensator (Opt. $\text{IAE}_c$). However, note that in this case PI compensation results in a percentage overshoot of 70%; if the PI compensator has to satisfy an percentage overshoot specification of 46% (which is the worst-case of the PI+CI compensator) then PI+CI compensator overcomes the performance of PI compensator in all cases.

**Robustness analysis**

In order to illustrate the performance robustness of the proposed tuning rule against parametric uncertainty, it is considered a perturbation in $k$ and $h$ of ±15% with respect to their nominal values. Maximum and minimum values of gain and time-delay are defined as follows:

$$\begin{aligned} k_M &= 1.15k = 0.23, \quad k_m = 0.85k = 0.17, \\ h_M &= 1.15h = 8.51, \quad h_m = 0.85h = 6.29. \end{aligned} \tag{5.77}$$

The several performance indices (% overshoot, $\text{IAE}_r$, and $\text{IAE}_d$) are now obtained for the four combinations of maximum and minimum values of gain and time-delay. The results are collected in a bar graph (Fig. 5.13); in addition, Table 5.4 shows the

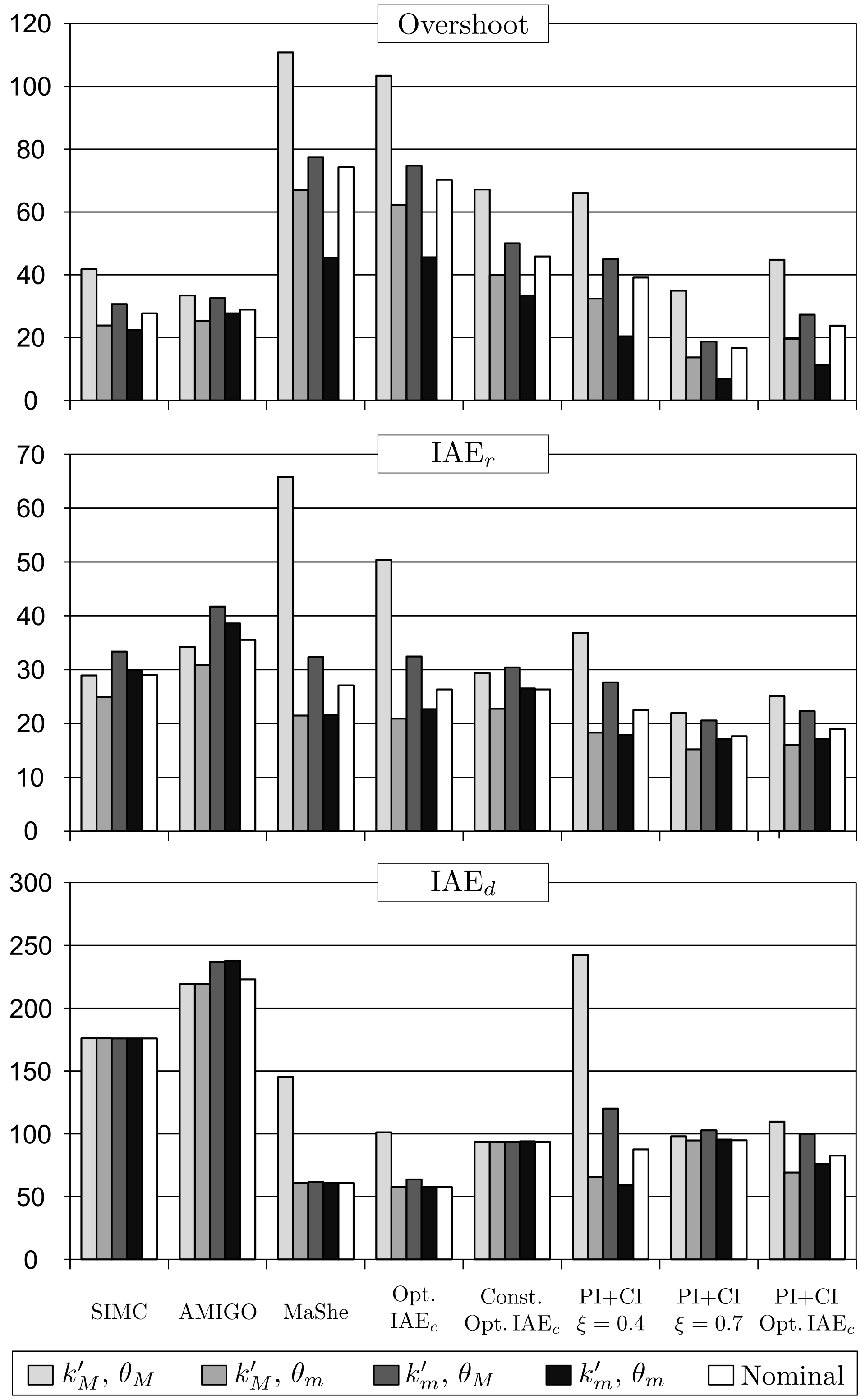



Figure 5.13 Performance indices for several PI and PI+CI compensators applied to an IPDT process ($k = 0.2$ and $h = 7.4$) with a 15% of uncertainty in the gain and the time-delay.

worst-case performance indices values for the different compensators. Note that, in general, the reset compensator PI+CI with $\xi = 0.7$ gives significantly best worst-case performance indices. The worst-case $\text{IAE}_d$ is slightly better for other compensators like the Const. Opt. $\text{IAE}_c$ PI compensator, but at the cost of doubling the worst-case overshoot percentage in this case. It is also true that PI-SIMC and PI-AMIGO are the most robust in the sense that they give the slightest variations of performance indices in comparison with its nominal values; but note that in general these worst case indices are significantly worse (except in worst-case % overshoot for PI-AMIGO that is slightly better) than the PI+CI with $\xi = 0.7$. In addition, regarding PI+CI compensation, the parameter $\xi$ does not always give a direct balance between performance and robustness. Note that Table 5.3 and 5.4 shows that this balance only occurs for disturbance rejection: for $\xi = 0.55$ it is obtained the best $\text{IAE}_d$ value, while the best worst case $\text{IAE}_d$ value is obtained for $\xi = 0.7$. However, the best values of nominal and worst-case indices % overshoot and $\text{IAE}_r$ are always obtained for $\xi = 0.7$.

## 5.3 Summary of Tuning Rules

Tuning rules have been developed that essentially consist in using a previously tuned PI compensator and resetting some percentage of the integral term. In general, this partial reset of the integral term results in an improvement on the closed-loop transient response, reducing the overshoot percentage and overall the settling time corresponding to the design without reset, and without degrading the speed of response as well as robustness and cost of feedback. It has been studied three different plants; a first order plant, a second order plant, and integrating plus dead time plant, and several design improvements of the PI+CI compensator have been proposed in order to overcome the PI compensation. As a result, simple analytical tuning rules are provided for the three processes under study. A summary of the tuning rulers may be found in Table 5.5. For first and second order plants, it is not provided tuning rules for the base PI compensator, which is assumed to be well tuned for the reset compensation (oscillatory response). The tuning rules is focused on the setting of the variable reset ratio for both step setpoint tracking and step disturbance rejection. In addition, we take advantages of the benefits of the variable band resetting law on reducing the undershoot. On the other hand, a simple modified SIMC tuning rule with a free design parameter is developed for IPDT processes, in order to ensure an oscillatory response. In addition, a fixed reset ratio is considered, whose value for disturbance rejection is given as a function of the free design parameter.

| Process | $k_p$ | $\tau_i$ | $\theta$ | $\mathcal{P}_1$ (setpoint) | $\mathcal{P}_2$ (disturbance) |
|---|---|---|---|---|---|
| First order | - | - | 0 | $1-\dfrac{a_0 w_{10}}{b_0 k_i x_i(t)}$ | $1-\dfrac{1}{1+e^{-\frac{\alpha}{\beta}\pi}}$ |
| Second order (zero crossing) | - | - | 0 | $1-\dfrac{a_0 w_{10}}{b_0 k_i x_i(t)}$ | $1-\dfrac{\hat{d}(t)}{b_0 k_i x_i(t)}$ |
| Second order (variable band) | - | - | $\dfrac{a_1^2}{\lambda}$ | $1-\dfrac{a_0 w_{10}-\delta e(t)}{b_0 k_i x_i(t)}$ | $1-\dfrac{\hat{d}(t)-\delta e(t)}{b_0 k_i x_i(t)}$ |
| IPDT | $\dfrac{0.5}{kh}$ | $8\xi^2 h$ | h | 1 | $1-\dfrac{1}{1+e^{-f\pi}}$ |

$$\lambda = a_1^3 + a_1(a_0+b_0k_p) - b_0k_i, \quad \delta = \frac{b_0k_i}{a_1}, \quad f = \frac{\sqrt{2}(8\xi^2-1)}{8\xi}$$

Table 5.5 Tuning rules for first/second order processes and IPDT processes.

## 5.4 Industrial Processes Applications

This section is devoted to the application of the reset compensator PI+CI with the tuning rules developed in the previous sections, testing the control strategies by means of experiments in real plants. Firstly, the tuning rule for second order processes of Section 5.1 is applied to a pH control process. The second application consists in controlling the level of water in a cylindrical tank by a PI+CI compensator tuned with the tuning rule given in Section 5.2.

### 5.4.1 Reset Control of an In-line pH Process

As described in the Arrhenius theory, an acid is a substance that ionizes in water to give hydrogen ions ($H^+$) whereas a base is a substance that ionizes in water to give hydroxyl ions ($OH^-$). The concentration of hydrogen and hydroxide ions determines whether a mixed solution is acidic or alkaline. As described in [78], the concentration of $H^+$ and $OH^-$ in an aqueous solution can vary over an extremely wide range. Thus it is very convenient to measure the acidity of the solution by using the logarithm of the concentration of hydrogen ions, rather than the concentration itself. This concept of pH scaling for measuring the acidity of a substance was introduced by Soren Sørensen in 1909:

$$pH = -\log_{10}[H^+]. \tag{5.78}$$

The range of values for the pH in aqueous solutions is $[0, 14]$. Solutions with a pH less than 7 are said to be acidic and solutions with a pH greater than 7 are basic or alkaline. A neutral solution has exactly a pH of 7 (pure water).

The presence of a logarithm in the definition of the pH implies, as it is well-known, that the neutralization process is highly nonlinear (S-shaped curve). Thus, there are regions where the process is almost non-sensitive to variations on the input, while there are other regions where a very small change of the input leads into a significant change in the output. As a result, a pH process plant is one of the most challenging industrial processes to model and control. The control of pH arises in a wide range of industries including wastewater treatment, biotechnology, pharmaceuticals and chemical processing. This makes the pH control an interesting and active research subject. Multitude control strategies has been applied to pH control problem, for instance: neural networks [87, 147], fuzzy control [10], switching control [52], internal model control [90], on-line identification [278], $H_\infty$ robust control [105]. The pH process to be controlled in this section is part of a bigger industrial plant with a particular configuration, that to the author's knowledge, makes the previous control strategies non applicable (at least directly). The vast majority of the previous works use one or more Continuous Stirred Tank Reactors (CSTRs), where an strong stirring performs an almost instantaneous mixture of the reagents. An alternative to using CSTRs is to control pH in-line, as the influent flows towards its outlet, by injecting neutralising reagent directly into the pipeline.

The process to be studied in this section is an in-line pH process. Potable water is considered as product, which is slightly alkaline due to the water purification treatment. For the pH control experiments, we treat the water with a discontinuous injection of nitric acid ($HNO_3$). The discontinuous injection provides robustness against pressure changes in the product line. However, it leads into a longer time to the mixture be homogenous. For this reason, the previous work [62] (pH control in the same industrial plant) considers a configuration of the process in which the sensor is placed far front the acid injection point. This configuration is less sensitive to the sensor noise, but

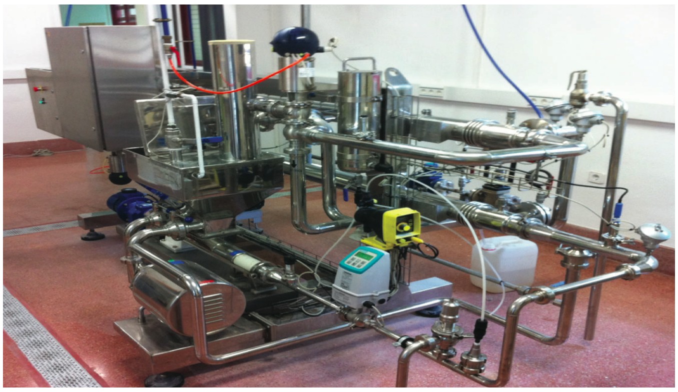


Figure 5.14 Food industry pilot plant for pH control experiments.

it introduces a large time-delay. On the contrary, the work [230] uses an alternative configuration with the sensor closer to the injection point, which leads into a small time-delay and an increment of the noise sensitivity.

### In-line pH Control Setup

The pH process is part of the canned food industry pilot plant shown in Fig. 5.14. The process consists of a tank with a capacity of about 50 liters where the product is stored and a screw pump that creates a continuous flow of product. Then there is a dosing pump that introduces a small amounts of a strong acid in the stream. This pump can only make a natural number of strokes per minute (spm), between 1 and 100, giving a flow rate of 1.6 liters per hour (lph) at 100 spm. After the injection of the acid, the product goes through a static mixer, which is 350mm long. Finally, the pH is measured by an Orbisint CPS11 pH sensor.

For the purposes of this section, we consider the setup of [230], in which the system is modeled as a first order system with dominant time constant. In this setup, the product is potable water from the supply network and no thermal treatment is done. The strong acid is nitric acid with a concentration of 10%. Let $A^-$ be the unknown alkaline component of the potable water, then the neutralization reaction is described by

$$A^- + HNO_3 \rightarrow HA + NO_3^- . \tag{5.79}$$

The operation point is set to pH=7.5, with a steady product flow rate around 295 lph. The identification of the plant is accomplished by a set of steps as input, then the outputs are analyzed in the Identification MATLAB Toolbox, and a FOPDT model is

| Method | $k_p$ | $\tau_i$ | $\delta$ | $\theta$ | Overshoot % | Undershoot % | IAE$_r$ |
|---|---|---|---|---|---|---|---|
| SIMC | 45.83 | 40.17 | - | - | 22.5 | 7.5 | 20.32 |
| PI+CI | 45.83 | 40.17 | $6.23e^{-10}$ | 10 | 12.5 | 10 | 17.14 |

Table 5.6 Compensators parameters and performance indices for the in-line pH process.

obtained

$$P(s) = \frac{K}{(\tau s + 1)} e^{-hs}, \tag{5.80}$$

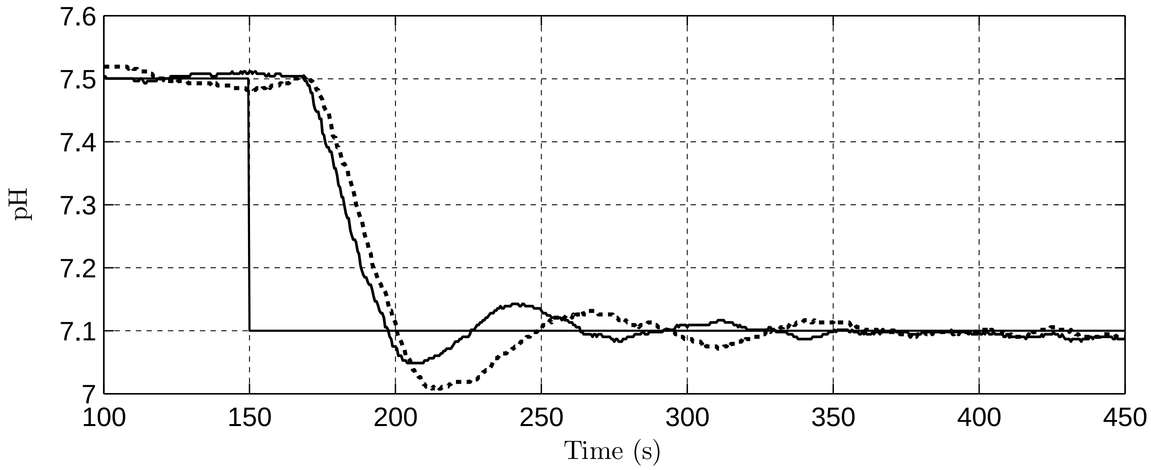
where $K = -0.03$, $\tau = 40.17$ and $h = 15$.

In order to applied the tuning rule proposed in Section 5.1, a second order model without time-delay is obtained for the pH process. Since the time constant of the process is greater than its time-delay, a possible model is simply obtained by

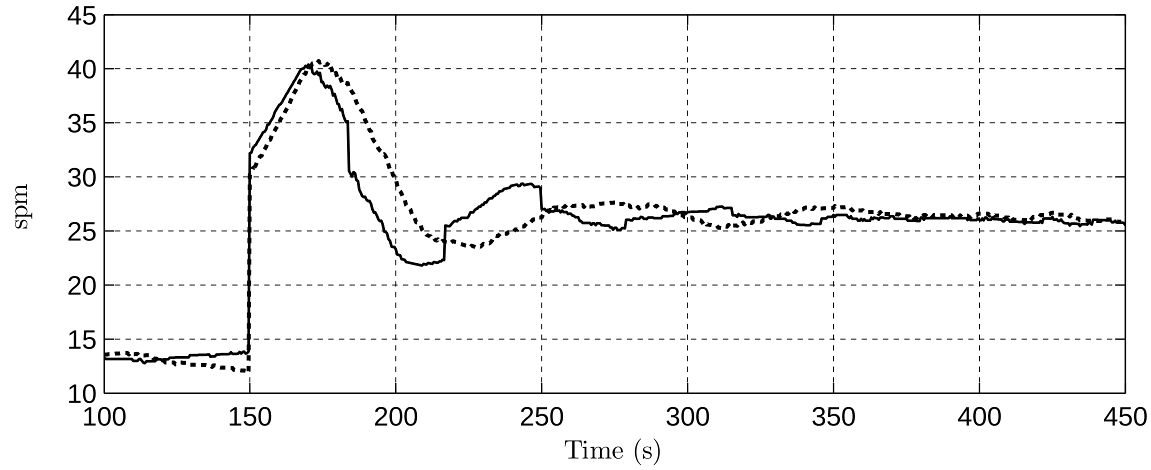
$$\tilde{P}(s) = \frac{K}{((\tau - h)s + 1)(2hs + 1)}. \tag{5.81}$$

This approximation is used in [254] to apply a PI tuning rule for FOPDT processes to second order processes. The proposed PI+CI tuning rule is compared with a well-tuned PI compensator. In this case, the SIMC tuning rule is used to set the parameters $k_p$ and $\tau_i$ of the PI. In addition, the PI with these parameters is considered as a base PI for the PI+CI compensator. The rest of the parameters of the PI+CI are set by applying the tuning rule in Table 5.5 to the model $\tilde{P}$. The compensators parameters are shown in Table 5.6. On the other hand, a moving average filter has been used to reduce the effect of the noise on the derivative of the variable band resetting law. In addition, the parameter $\theta$ has been incremented to 10 (from 9.29), in order to reduce the extra phase introduced by the filter.

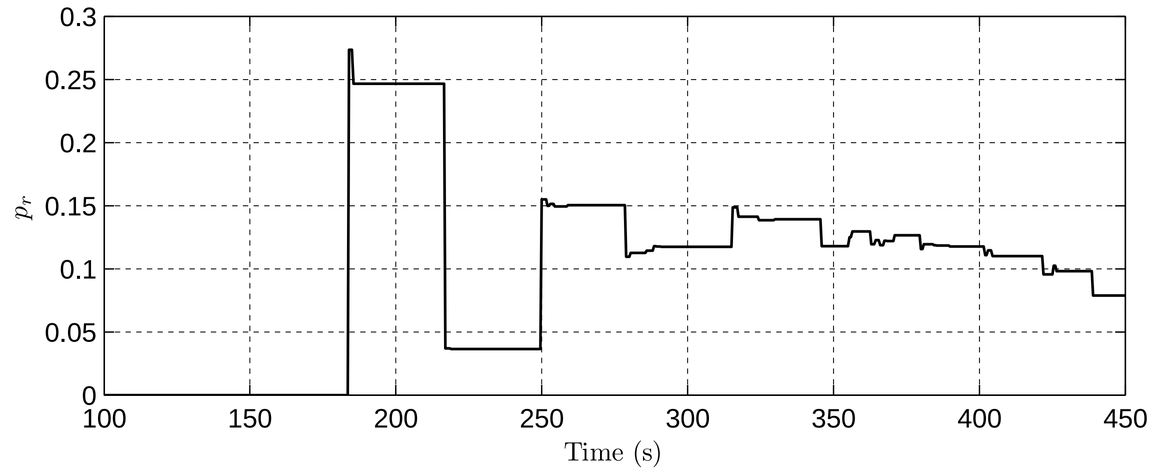
The control system is designed to improve the step tracking of the pH process. Fig. 5.15 shows the step responses of the PI and the PI+CI compensators, as well as the control signal and the reset ratio. In addition, the overshoot, undershoot and IAE$_r$ performance indices are shown in Table 5.6. In general, it can be seen that the PI+CI gives better performance than the PI compensator. However, despite a variable reset ratio is used, the undershoot of the response increases respect to the PI compensator. Note that the tuning rule was design for a second order process, but the process under study is not a real second order process.




(a) Output signal.




(b) Control signal.




(c) Control signal.

Figure 5.15 Output signals (pH value), control signals (wpm), and reset ratio for a PI and a PI+CI applied to the in-line pH process. PI (dotted) and PI+CI (solid).

Figure 5.16 Liquid level process.

### 5.4.2 Reset Control of Liquid Level in a Tank

The liquid level control system is applied to many industries: in food processing, pharmaceutical industries, water purification systems, industrial chemical processing and spray coating, boilers, etc. So the liquid level control is very important to industry process control ([70, 213, 309]). Typical actuators used in liquid level control systems include pumps, motorized valves, on-off valves, etc. In addition, level sensors such as displacement float, capacitance probe, pressure sensor ([41]), etc., provide liquid level measurement for the purpose of feedback control.

#### Liquid Level Tank Control Setup

The liquid level experiments are performed in a cylindrical tank (see Fig. 5.16), where the liquid is potable water. The height and the diameter of the tank are 1.15 and 0.057 meters, respectively. The tank has been designed to reduce the saturation phenomenon during the experiments for a fair comparison of the applied compensators. A small manual valve is placed at the bottom of the tank, which allows to set the output flow rate. The water flows out the tank only due to the gravity force, and thus, the outflow rate is given by $Q_o(t) = \alpha\sqrt{2gh(t)}$, where $g$ is the gravitational acceleration constant, $\alpha$ is the gross-sectional area of the output orifice (bottom manual valve) and $h(t)$ is the level of the liquid. An industrial helicoidal impeller pump is used to pump the water from an storage tank up to the controlled tank. The pump provides a continuous flow rate, which is nearly proportional to the control signal given to the variable frequency drive. An steady-state identification of the relation between the control signal (%) and the flow rate yield to the following expression for the input flow rate $Q_i(t) = (u_p(t) + 0.48)/587.52$, where $u_p(t) \in [3, 100](\%)$. The level of water is measured by a low price solid-state sensor with resistive output.

A model of the process can be derived easily by considering the balance equation based on an instantaneous rate-of-change:

$$\dot{h}(t) = \frac{1}{A}(Q_i(t) - Q_o(t)), \tag{5.82}$$

where $h(t)$ is the liquid level, $Q_i$ is the controlled input flow rate, $Q_o$ is the uncontrolled output flow rate from the tank, and $A$ is the cross-sectional area of the tank. If the difference $Q_i(t) - Q_o(t)$ is considered as the process input variable and the level $h$ is the output variable, the process exhibits clearly an integral dynamic. In addition, a time-delay $h$ may occur because of the dynamics of the sensor and the actuator. In addition, the process is controlled to be in the operation point 1 meter or closed to it (small reference steps). For control purpose, we consider that the output flow rate is known and constant, and thus, it is used to compute the signal of the pump. Note that variations of the output flow rate will be considered as input disturbance, being the compensator responsible for rejecting then.

The parametric identification of the process is simply done through an step response experiment. Specifically, the identification was done by setting the input flow rate equal to the output flow rate when the level of the liquid is 1 meter. In this case, the diameter of the output orifice is approximately 0.0033 meters, thus the output flow rate is 0.038 lps (liters per second), which corresponds to $u_p = 21.9\%$. For the identification, we used a step signal in $u_p$ from 21.9% to 50%. Finally, the identification of the process at this operation point leads to the process model

$$P(s) = \frac{0.37e^{-0.55s}}{s}. \tag{5.83}$$

Several liquid level control experiments will be performed with a PI compensator and a reset PI+CI compensator. The PI+CI is provided with a switching reset ratio with triggered-by-counter switching law ($N_s = 5$), an stabilization by time-dependent conditions, where $\Delta_m$ and $\Delta_M$ are obtained by Proposition 3.10 (note that the proposition has to be applied for both values of the reset ratio, $p_r = 0.99$ and $p_r = p_r^*$) and finally, a variable band resetting law. In addition, a first order filter is applied to the derivative of the variable band resetting law. By the analysis in Section 2.4.3, the time

| Method | $k_p$ | $\tau_i$ | $p_r^*$ | $\Delta_m$ | $\Delta_M$ | Max. Overshoot % | $\text{IAE}_r$ | $\text{IAE}_d$ |
|---|---|---|---|---|---|---|---|---|
| SIMC | 2.43 | 4.42 | - | - | - | 52.57 | 0.38 | 0.24 |
| PI+CI - $\xi = 0.6$ | 2.43 | 1.59 | 0.15 | 0.1 | 2.81 | 24.86 | 0.23 | 0.14 |

Table 5.7 Compensators parameters and performance indices for the liquid level process.

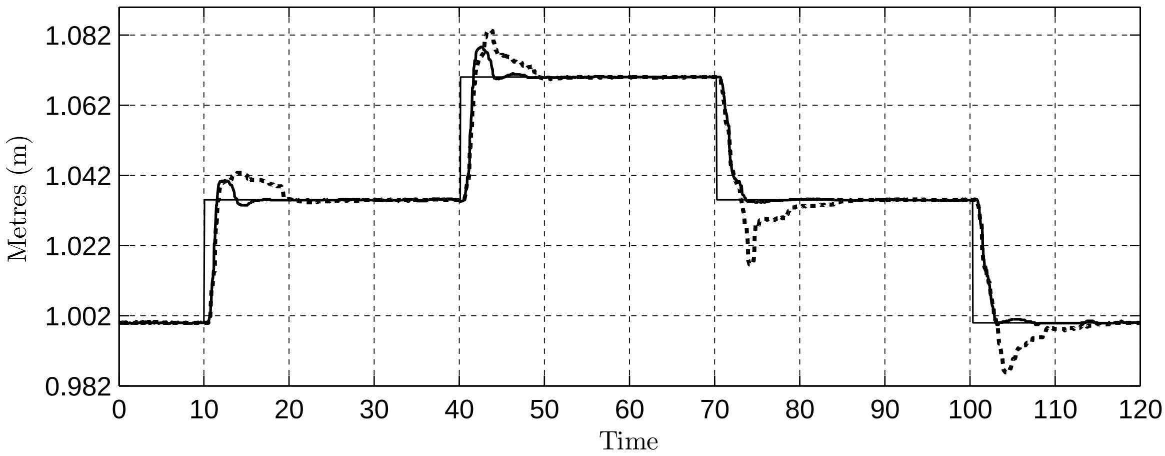
constant of the filter is set to $0.1 < 0.4h = 0.22$. The PI+CI performance is compared with a PI compensator tuned by the SIMC method. Table 5.7 shows the setting for both compensators.

**Experiments Results**

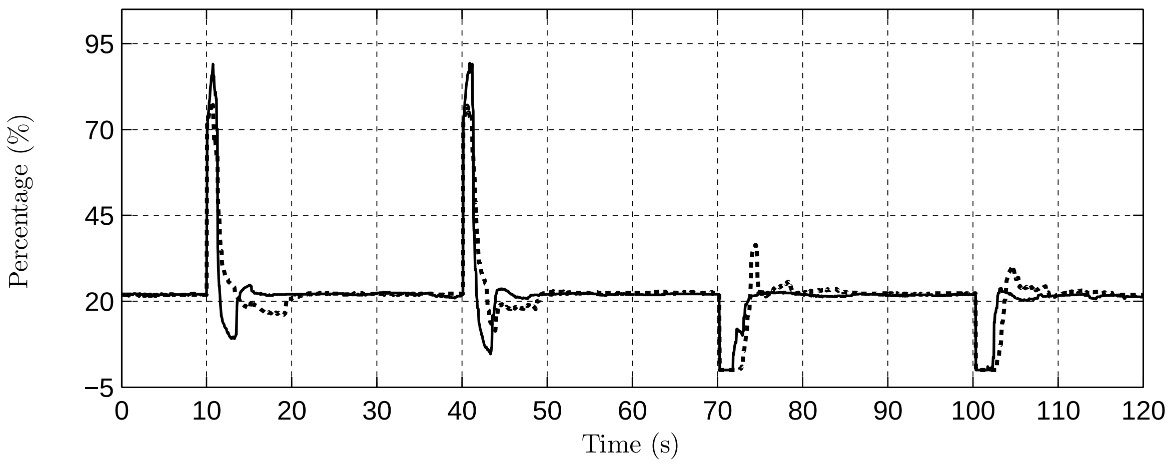
*Step reference tracking.* Fig. 5.17 shows the responses and the control signal for the PI and PI+CI compensators, and for several step references; the PI compensator produces a step response with an overshoot of 20% in the first step and 38% in the second. The overshoot in the down steps increases due to the saturation of the control signal, reaching the 52%. In general, the response of the PI compensator is smooth and produces a longer settling time, as it can be expected for the SIMC tuning rule. However, the responses of the PI+CI compensator are faster and thanks to the reset actions the overshoot and the settling time are smaller than in the PI case. In the down steps, the response of the PI+CI is also better, producing an almost flat response, which suggests good properties of the reset compensator for saturated processes.

*Step disturbance rejection.* In Fig. 5.18, it can be seen the water level and the control signal in presence of disturbances. In this case, in order to ensure a fair comparison of the compensators and to allow the repetition of the experiments, the disturbance is produced as a change in the estimated output flow rate, that is, the disturbance is introduced by software. This is similar to a real change in the manual output valve. It has been checked that the PI+CI also has a better disturbance rejection; it has a faster response, and in addition, the reset actions avoid a large overshoot.

For a quantitative comparison, measures like overshoot and IAE have been calculated using the experimental data, the results are given in Table 5.7. The best performance is achieved by the PI+CI compensator in both setpoint tracking and disturbance rejection, with an improvement of at least a 40% in the IAE index.

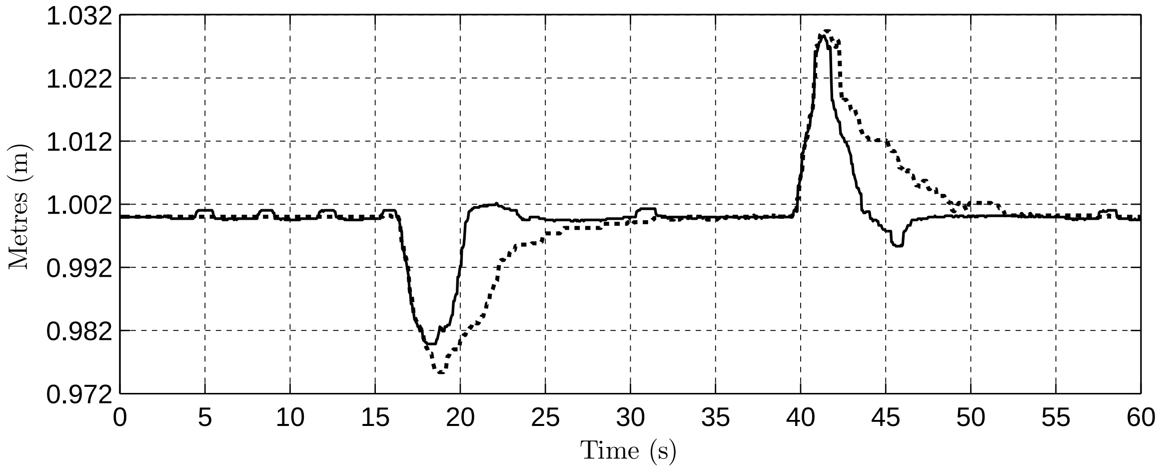



(a) Output signal.

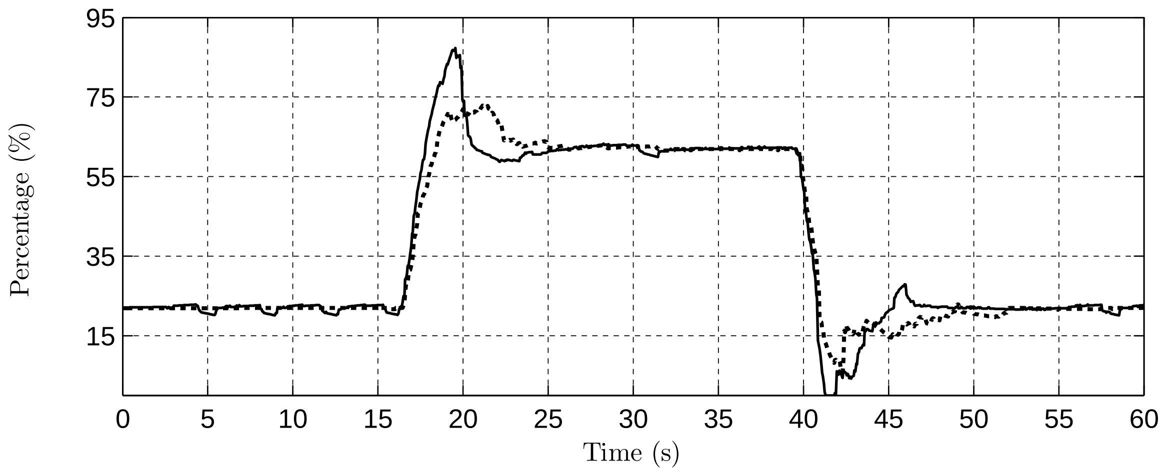



(b) Control signal.

Figure 5.17 Output signals (water level) and control signals (percentage of the pump) for a PI and a PI+CI applied to the liquid level process with a sequence of step setpoints. PI (dotted) and PI+CI (solid).




(a) Output signal.




(b) Control signal.

Figure 5.18 Output signals (water level) and control signals (percentage of the pump) for a PI and a PI+CI applied to the liquid level process with a sequence of step disturbances. PI (dotted) and PI+CI (solid).

# Chapter 6

# Conclusions and Future Works

Finally, we summarize the main results and contributions of the thesis. In addition, we give several open problems in the line of the thesis, and potential lines of research.

## 6.1 Conclusions

Due to the variety of the topics addressed, the exposition of the contributions will be performed by discussing each chapter separately.

**Chapter 1 and 2. Introduction and Reset Control Systems.**

In the first chapter, we performed an analysis of the bibliography in the research field of reset control systems. A complete (under the criterion of the author) chronological list of the publications on reset control, showing that the interest of the control community has been increased in the last decade. The number of publications per year in the last years has been triple.

On the other hand, (time-delay) reset control systems had been described under the framework of impulsive (delay) dynamical systems, which provided a solid theoretical basis for the analysis of the systems. In addition, the description of the reset control systems as IDDS allows the generalization and application of the particular results to a wider class of systems.

Finally, several modifications of the PI+CI compensator have been analyzed, and new design improvements, such as switching reset ratio and time-dependent stabilization, have been proposed.

**Chapter 3. Stability Analysis of IDDSs.**

In chapter 3, we addressed the stability analysis of impulsive delay dynamical systems, considering time-delay reset control systems as the main application. The results can be separated in two lines: analysis of state-dependent IDDS and stabilization by time-dependent conditions. In the first line, a general Lyapunov-Krasovskii-based criterion was developed. Local asymptotic stability conditions for a particular class of IDDS with nonlinear and time-varying base system was provided by the application of the general results. In addition, several stability criteria in the form of LMIs were obtained for checking the stability of state-dependent IDDS with a LTI base system. The conditions have been shown to be less conservative than the results in the literature for time-delay reset control systems. The examples showed that the global asymptotic stability is guaranteed for larger values of the time-delay.

The main idea in the second line is to relax the conditions imposed on the LK functional, allowing bounded increments of the functional after the reset actions and due to the reset actions. This is accomplished by imposing restriction on the reset intervals. IDDS with stable and unstable base systems are analyzed separately. In the first case, we imposed a lower bound on the reset interval, allowing the functional to increase during a short amount of time. The examples showed that the new criterion guarantees the stability of time-delay reset control systems for larger values of the time-delay. In the second case, a lower and upper bound of the reset intervals are applied, such that increments of the LK function may occur during the whole reset interval. The main contribution is thus a method for the stabilization of IDDS with unstable base systems, which is indeed applicable to time-delay reset control systems, unlike the other results in the literature. As a particular case of study, it is considered a reset control system comprises a LTI plant and a PI+CI compensator. By a new realization of the PI+CI, we showed that the stability of a partial solution of the reset control system guaranteed the stability of the complete solution.

Finally, it should be emphasize that most of the results for time-delay reset control systems do not deal with the effect of the discontinuous solutions into the LK functional and its derivative. However, this thesis shows that it is necessary to impose extra conditions on the functional in order to guarantees the correctness of the results.

**Chapter 4. Reset Control Systems with Saturation.**

In Chapter 4, we have taken into account the presence of a limited actuator. This nonlinearities have been shown to have a great impact on the stability of the closed-loop system, limiting the stability of the equilibrium point to a local region (the region of attraction). We have presented a procedure to obtain an estimation of the region of attraction for a reset control system. The method is based on a polytopic partition of the state-space and a description of the system by a directed graph, representing the trajectories of the system by links between the nodes of the graph. Several theoretical results are provided in order to guarantee the convergence of the trajectories to the origin. We provided an algorithm that starts with a preliminary approximation of the region of attraction (for instance obtained by the previous results) and as a result, it produces a larger estimation.

**Chapter 5. Design and Applications.**

In Chapter 5, we addressed the tuning of the PI+CI compensator. We developed a systematic method for tuning the reset ratio and the variable band resetting law. First, it has been shown that the step response between two consecutive reset instants can be analyzed by the impulse response of a LTI system in a given set, giving as result an error sequence. Therefore, simple tuning rules have been developed for first and second order plants. For first order plants, the tuning method provides a flat step response that reaches the steady-state at the first reset instant; for second order plants, several tuning methods have been developed, based on the minimization of the $H_2$-norm of each function of the error sequence, and although they do not give a flat response, they produce significant improvements over PI compensation without increasing the cost of feedback. Second, the method has been extended to tune the PI+CI compensator to control IPDT processes. The proposed PI+CI compensator has a variable band resetting law and a switching reset ratio (with two modes: setpoint tracking and load disturbance rejection). It is provided a simple tuning rule for all the parameters of the PI+CI, leaving the parameter $\xi$ (related with the damping rate of the base system) as a design parameter. A case study has shown that, in general, PI+CI compensation gives better performance indices and also better worst- case performance indices (with respect to parametric uncertainty) than PI compensators tuned with several methods including SIMC, AMIGO, etc. Finally, the proposed tuning rules have been tested in real experiments: an in-line pH process and a liquid level process. The experiments showed the superior performance of the PI+CI over a well-tuned PI compensator.

## 6.2 Future Works

In this section, we provides several open questions and potential line of research.

The new stability criteria developed in this thesis open a multitude of potential of future works. Let list some of them. First, it would be interesting to extend the results to the input/output stability. It has been proved that reset control systems overcomes the performance of the base system in terms of the $H_2$-norm and the $\mathcal{L}_2$ gain. Since the results in chapter 3, remove the necessity of a stable base system, it appears a potential tools for proving the superior performance (for instance $H_2$-norm and the $\mathcal{L}_2$ gain) of reset compensation over its linear counterpart for processes with time-delay. At the same time, the computation of this performance measurements could be applied to the multiobjective reset control synthesis via LMI optimization.

Regarding the reset control systems with saturation, note that the after-reset set is assumed to be inside the region of nonsaturation in order to guarantee the well-posed of the reset instants. However, this is a strong assumption that is not usually satisfied when the compensator comprises a proportional term. A first attempt could be to limit the definition of the reset control system to a region of the state-space in which the assumption is satisfied, but the estimation of the region of attraction will be necessarily limited to this region. Therefore, we think that the first efforts should pursue the formalization of basic properties such as existence and uniqueness of the solution, as well as continuous dependence on the initial conditions, that have been shown to be a key point in the developed procedure. Another limitation of the method is that do not take into account the trajectories that reach the initial approximation of the region of attraction after several reset actions. Finally, a more ambitious future work could be the application of the proposed stability results to the region of attraction estimation, following the line of saturated systems without reset actions.

On the other hand, a general systematic method for analysis of the response of reset control systems, based on a set of LTI systems, has been proposed. Nevertheless, it has been only applied to the tuning of low order systems. From the application point of view, it is interesting to consider higher order systems and also systems with time-delay. For the last case, a first step could be the approximation of the time-delay by rational functions (for instance Padé approximation). In the line of Section 5.2, we may consider the analysis of the responses of the time-delay reset control system by the response of a set of LTI systems to more general inputs, instead of the simple impulsive response, and then, contemplate for instance the optimisation of the $\mathcal{L}_2$ gain.

Finally, a multitude of design improvements have been proposed for the PI+CI compensator, which result useful in real applications. However, the modifications lead into a substantial increment of the complexity of the reset control system (time-varying, switching signals, etc.), becoming the analysis of the system under the IDS framework more difficult. Therefore, it should be considered, whenever it is necessary, the study of the reset control system under more general frameworks, for instance the HI framework.

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# Glossary

## Symbols

| | |
|---|---|
| $\mathbb{N}$ | Set of natural numbers. |
| $\mathbb{Z}_-$ | Set of nonpositive integer numbers. |
| $\mathbb{R}$ | Set of real numbers. |
| $\mathbb{R}_+$ | Set of nonnegative real numbers (including 0). |
| $\mathbb{R}^n$ | Real coordinate space of $n$ dimensions. |
| $\mathbb{R}^{n\times m}$ | Set of real matrices with $n$ rows and $m$ columns. |
| $\mathbb{S}^n$ | Set of symmetric real matrices of dimension $n \times n$. |
| $\mathbb{S}^n_+$ | Set of positive definite real matrices of dimension $n \times n$. |
| $I_n$ | $n$-dimensional identity matrix ($I$ when the dimension can be deduced). |
| $0_n$ | $n$-dimensional zero matrix (0 when the dimension can be deduced). |
| $\mathbf{0}$ | The zero vector. |
| $\mathbb{I}$ | Interval $[0, T) \subset \mathbb{R}_+$ for some $T$ finite or infinite. The relation of the interval with some data is denoted by a subindex. |
| $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$ | Set of continuous functions from $\mathbb{R}_+$ to $\mathbb{R}$. |
| $\mathcal{PC}(\Omega, \mathbb{R}^n)$ | Set of piecewise continuous functions from $\Omega \subseteq \mathbb{R}_+$ to $\mathbb{R}^n$. |
| $\mathcal{PAC}(\Omega, \mathbb{R}^n)$ | Set of piecewise absolutely continuous functions from $\Omega \subseteq \mathbb{R}_+$ to $\mathbb{R}^n$. |
| $\mathcal{K}$ | if $\alpha \in \mathcal{K}$ then $\alpha$ is a strictly increasing continuous function with $\alpha(0) = 0$. |
| $\vee$ | Logical disjunction. |
| $\wedge$ | Logical conjunction. |
| $f(t^+)$ | $f(t^+) = \lim_{\epsilon\to 0, \epsilon>0} f(t+\epsilon)$ for a function $f \in \mathcal{PC}(\Omega, \mathbb{R}^n)$, $t \in \Omega$ |
| $\limsup$ | Supremum limit. |
| $\mathrm{proj}_{[1,m]}(\mathcal{Z})$ | Orthogonal projection of $\mathcal{Z} \subset \mathbb{R}^n$ on the subspace $\mathbb{R}^m$ with $m \leq n$. |
| $\mathcal{R}(A)$ | Column space of matrix $A \in \mathbb{R}^{n\times m}$. |
| $\mathcal{N}(A)$ | Null space of matrix $A \in \mathbb{R}^{n\times m}$. |
| $\mathrm{span}\,\mathcal{V}$ | Subspace spanned by the set of vectors $\mathcal{V} = \{\mathbf{v}_i\}$, $i \in \{1, 2, \ldots\}$. |
| $\lVert A \rVert$ | $\mathcal{L}_2$-norm of the real matrix $A$. |

| | |
|---|---|
| $\lambda_m(A)$ | Minimum real eigenvalue of the real matrix $A$. |
| $\lambda_M(A)$ | Maximum real eigenvalue of the real matrix $A$. |
| $A^\top$ | Transpose of the real matrix $A$. |
| $He(A)$ | $He(A) = A + A^\top$ for a matrix $A$. |
| $\text{diag}(A, B)$ | Block matrix composed by the matrices $A$ and $B$. |
| $\text{int}(\Omega)$ | Interior of the set $\Omega$ (the relative interior in Chapter 4). |
| $\partial\Omega$ | Boundary of the set $\Omega$ (the relative boundary in Chapter 4). |
| $(\mathbf{x}, \mathbf{y})$ | The vector $[\mathbf{x}^\top \mathbf{y}^\top]^\top$. |
| $\mathcal{R}(s)$ | The set of rational functions. |
| $\lVert f \rVert_{\mathcal{L}_2}$ | $\mathcal{L}_2$-norm of the function $f \in \mathcal{PAC}(\Omega, \mathbb{R}^n)$. |
| $\mathbf{x} \prec \mathbf{y}$ | The set of inequalities $\mathbf{x}_i < \mathbf{y}_i$ with $\mathbf{x}_i$ and $\mathbf{y}_i$ the elements of the vectors. |
| $\mathbf{x} \preceq \mathbf{y}$ | The set of inequalities $\mathbf{x}_i \leq \mathbf{y}_i$ with $\mathbf{x}_i$ and $\mathbf{y}_i$ the elements of the vectors. |
| $\mathbb{T}$ | Set of reset instants. The relation of the set with some data is denoted by a subindex. |
| $\text{dom}\, f$ | The domain of the function $f$. |
| $A > 0$ | The matrix $A \in \mathbb{S}^n$ is positive definite ($<$ negative definite). |
| $A \geq 0$ | The matrix $A \in \mathbb{S}^n$ is positive semidefinite ($\leq$ negative semidefinite). |

## Acronyms

| | |
|---|---|
| CI | Clegg Integrator |
| FOPDT | First Order Plus Dead Time |
| FORE | First Order Reset Element |
| HI | Hybrid Inclusions |
| IAE | Integral Absolute Error |
| IDDS | Impulsive Delay Dynamical System |
| IDS | Impulsive Dynamical System |
| IPDT | Integral Plus Dead Time |
| ISE | Integral Square Error |
| ITAE | Integral Time Absolute Error |
| LK | Lyapunov-Krasovskii |
| LMI | Linear Matrix Inequality |
| LTI | Linear Time-Invariant |
| PI | Proportional-Integral |
| PID | Proportional-Integral-Derivative |
| PI+CI | Proportional-Integral plus Clegg Integrator |
| QFT | Quantitative Feedback Theory |

# Appendix A

# Linear Matrix Inequalities

**Definition A.1 (Linear Matrix Inequality (LMI)).** A *Linear Matrix Inequality* is an inequality of the form

$$F(\mathbf{x}) := F_0 + \sum_{i=1}^{m} F_i \mathbf{x}_i > 0, \tag{A.1}$$

where $\mathbf{x} \in \mathbb{R}^m$ and $L_i \in \mathbb{S}^n$, $i = 1, \ldots, m$.

The inequality means that $F(x)$ is a positive definite matrix, that is,

$$\mathbf{y}^\top F(\mathbf{x})\mathbf{y} > 0, \forall \mathbf{y} \in \mathbb{R}^n \setminus \{\mathbf{0}\}. \tag{A.2}$$

The symmetric matrices $F_i$, $i = 0, 1, \ldots, m$ are given and $\mathbf{x}$ is the variable. Thus, $F(\mathbf{x})$ is an affine function of the elements $\mathbf{x}$.

The inequality (A.1) is a *strict* LMI. If $F(\mathbf{x})$ is only required to be positive semidefinite, then it is referred to as *nonstrict* LMI.

**Definition A.2 (Feasible LMI).** A LMI $F(x) > 0$ is *feasible* if the set $\{\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\} : F(\mathbf{x}) > 0\}$ is nonempty.

Any feasible nonstrict LMI can be reduced to an equivalent strict LMI that is feasible by eliminating implicit equality constraints and then reducing the resulting LMI by removing any constant nullspace ([266]).

The notation (A.1) of an LMI is not common in the field of automatic control and system theory. In this field, an LMI is usually found as follows (example of Lyapunov stability condition of a LTI system):

$$A^\top P + PA < 0, \qquad P > 0, \tag{A.3}$$

where the matrix $P > 0$ is the variable and $A \in \mathbb{R}^{n \times n}$ is given data. It is not possible to give a general formulation of LMIs where matrices are variable since there is a large variety of differences forms. Nevertheless, any LMI in *matrix variable* form can be written into the form (A.1) (the converse is not necessarily true) ([53]).

Interior Points (IP) algorithms are mainly used to solve numerically LMIs ([53]) since 1994, when Nesterov, Nemirovskii and Alizadeh (see [205]) extend IP methods for Semidefinite Programming (SDP) which is the class of problems where LMIs belong. At this time, there are many solvers for SDP, for instance SeDuMi [257], SPDT3 [263], etc. However, all this solvers have been developed for the mathematical framework of SDP and the LMI notation in the field of automatic control and control theory is based on a matrix representation. In order to simplify the used of the solvers, softwares called *parsers* have been developed as interface between these notations, for instance Yalmip [186].

# Appendix B

# Quadratic Integral Inequalities

In this appendix, it is provided as set of quadratic integral inequalities that play an important role in the stability problem of time-delay systems. The reader is referred to [117, 281] for further details.

**Lemma B.1** (**Basic inequality**). *Assume that* $a(\alpha) \in \mathbb{R}^n$ *and* $b(\alpha) \in \mathbb{R}^n$*, then for any positive definite matrix* $R \in \mathbb{S}^n$*, the following holds*

$$-2\int_{\Omega} a(\alpha)^\top b(\alpha) d\alpha \leq \int_{\Omega} a^\top(\alpha) R a(\alpha) + b^\top(\alpha) R^{-1} b(\alpha) d\alpha. \tag{B.1}$$

**Lemma B.2** (**Park's inequality** [226]). *Assume that* $a(\alpha) \in \mathbb{R}^n$ *and* $b(\alpha) \in \mathbb{R}^n$ *are given for* $\alpha \in \Omega$*. Then, for any positive definite matrix* $X \in \mathbb{S}^n$ *and any matrix* $M \in \mathbb{R}^{n\times n}$*, the following holds*

$$-2\int_{\Omega} a(\alpha)^\top b(\alpha) d\alpha \leq \int_{\Omega} \begin{bmatrix} a(\alpha) \\ b(\alpha) \end{bmatrix}^\top \begin{bmatrix} X & XM \\ \star & (M^\top X + I)X^{-1}(XM + I) \end{bmatrix} \begin{bmatrix} a(\alpha) \\ b(\alpha) \end{bmatrix}. \tag{B.2}$$

**Lemma B.3** (**Moon's inequality** [200]). *Assume that* $a(\alpha) \in \mathbb{R}^{n_x}$ *and* $b(\alpha) \in \mathbb{R}^{n_y}$ *are given for* $\alpha \in \Omega$*, and* $N \in \mathbb{R}^{n_x \times n_y}$*. Then, for any matrices* $X \in \mathbb{R}^{n_x \times n_x}$*,* $Y \in \mathbb{R}^{n_x \times n_y}$*, and* $Z \in \mathbb{R}^{n_y \times n_y}$*, the following holds*

$$-2\int_{\Omega} a(\alpha)^\top N b(\alpha) d\alpha \leq \int_{\Omega} \begin{bmatrix} a(\alpha) \\ b(\alpha) \end{bmatrix}^\top \begin{bmatrix} X & Y - N \\ \star & Z \end{bmatrix} \begin{bmatrix} a(\alpha) \\ b(\alpha) \end{bmatrix}, \tag{B.3}$$

*where*

$$\begin{bmatrix} X & Y \\ \star & Z \end{bmatrix} \geq 0. \tag{B.4}$$

**Lemma B.4 (Jensen's inequality).** *Let $\phi$ be a convex function and $f(x)$ is integrable over $[a(t), b(t)]$, $a(t) < b(t)$ for some parameter $t \in U$. Then the following inequality holds*

$$\phi\left(\int_{a(t)}^{b(t)} f(x)dx\right) \leq |b(t) - a(t)| \int_{a(t)}^{b(t)} \phi(f(x))dx \tag{B.5}$$

The Jensen's inequality is widely used in approaches based on Lyapunov-Krasovskii functionals as a efficient bounding technique. In those cases, the inequality is defined as follows:

$$\left(\int_a^b f(x)dx\right)^\top P \left(\int_a^b f(x)dx\right) \leq |b - a| \int_a^b f(x)^\top P f(x)dx \tag{B.6}$$

with $P > 0$. The convex function is $\phi(z) = z^\top P z$, since $P > 0$.

# Appendix C

# Some Theoretical Results on Calculus

## C.1 Fundamental theorem of calculus

**Proposition C.1 (Second fundamental theorem of calculus).** *Consider a function $f$ with $\dot{f}$ continuous on an open interval $I$, then for each $a$, $b \in I$, we have*

$$f(b) = f(a) + \int_a^b \dot{f}(\alpha)d\alpha. \tag{C.1}$$

**Corollary C.1.** *Consider a left-continuous function $f \in \mathcal{PAC}([a,b],\mathbb{R}^n)$ with $\dot{f} \in \mathcal{PC}$ the upper right-hand derivative of $f$. The function $f$ is assumed to have $M \geq 0$ jump discontinuities at $(\theta_j)_{j=1}^M$ with $a \leq \theta_1 < \theta_2 < \cdots < \theta_M < b$. Then, it is obtained*

$$f(b) = f(a) + \int_a^b \dot{f}(\alpha)d\alpha + \sum_{j=1}^{M}(f(\theta_j^+) - f(\theta_j). \tag{C.2}$$

*Proof.* The proof follows directly from the application of the second fundamental theorem of calculus to the intervals $[a,\theta_1]$, $[\theta_M, b]$, and $[\theta_j, \theta_{j+1}]$, $j = 1,\ldots,M-1$. □

## C.2 Integration by parts

Consider two continuous functions $f$ and $g$ with $\dot{f}$ and $\dot{g}$ continuous on an open interval $I$, then for each $a$, $b \in I$, the following equality hold:

$$\int_a^b f(\alpha)\dot{g}(\alpha)d\alpha = f(b)g(b) - f(a)g(a) - \int_a^b \dot{f}(\alpha)g(\alpha)d\alpha. \tag{C.3}$$

The above equation is derived from the derivative product rule and the application of the second fundamental theorem of calculus. Consider now that the function $g$ is left-continuous and $g \in \mathcal{PAC}([a,b],\mathbb{R}^n)$ with $\dot{g} \in \mathcal{PC}$ the upper right-hand derivative of $g$. In addition, the function $g$ is assumed to have $M \geq 0$ jump discontinuities at $(\theta_j)_{j=1}^{M}$ with $a \leq \theta_1 < \theta_2 < \cdots < \theta_M < b$. In this case, an equation for the integration by parts can be obtained by the application of the equation (C.3) to the intervals $[a,\theta_1]$, $[\theta_M,b]$, and $[\theta_j,\theta_{j+1}]$, $j = 1,\ldots,M-1$.

$$\int_a^b f(\alpha)\dot{g}(\alpha)d\alpha = f(b)g(b) - f(a)g(a) - \int_a^b \dot{f}g(\alpha)d\alpha f(b) - \sum_{j=1}^{M}(f(\theta_j)g(\theta_j^+) - f(\theta_j)g(\theta_j)). \tag{C.4}$$

# Appendix D

# Legendre Polynomials

## D.1 Definition and Properties

**Definition D.1.** A Legendre polynomials over the interval $[-h, 0]$ is defined by

$$L_k(\alpha) = (-1)^k \sum_{l=0}^{k} p_l^k \left( \frac{\alpha + h}{h} \right)^l \tag{D.1}$$

with $p_l^k = (-1)^l \binom{k}{l} \binom{k+l}{l}$ for all $k \in \mathbb{N}$.

The set of Legendre polynomials $\{L_k, k \in \mathbb{N}\}$ forms a basis for the space of piecewise continuous functions defined on the interval, so any such function can be written as a linear combination of Legendre polynomials (see, e.g., *Fouries-Legendre series* in [164]).

The Legendre polynomials described in the above definition satisfy the following properties:

1. *Orthogonality:*

$$\int_{-h}^{0} L_k(\alpha) L_l(\alpha) d\alpha = \begin{cases} 0, & k \neq l, \\ \frac{h}{2k+1}, & k = l, \end{cases} \quad \forall (k, l) \in \mathbb{N}^2. \tag{D.2}$$

2. *Boundary conditions:*

$$L_k(0) = 1, L_k(-h) = (-1)^k \quad \forall k \in \mathbb{N}. \tag{D.3}$$

3. *Differentiation:*

$$\dot{L}_k(\alpha) = \begin{cases} 0, & k = 0, \\ \sum_{i=0}^{k-1} \frac{2i+1}{h}(1-(-1)^{k+i})L_i(\alpha), & k \geq 1, \end{cases} \quad \forall k \in \mathbb{N}. \tag{D.4}$$

## D.2 Bessel-Legendre Inequalities

In [252], several inequalities are developed based on the Legendre polynomials. The proofs can be found in [252].

**Lemma D.1.** *Let $\mathbf{x} \in \mathcal{C}$, $R \in \mathbb{S}_+^n$, and $h > 0$, then the inequality*

$$\int_{-h}^{0} \mathbf{x}^\top(\alpha) R \mathbf{x}(\alpha) d\alpha \geq \frac{1}{h} \sum_{k=0}^{N} (2k+1) \Omega_k^\top R \Omega_k \tag{D.5}$$

*holds, for all $N \in \mathbb{N}$, where $\Omega_k = \int_{-h}^{0} L_k(\alpha)\mathbf{x}(\alpha)d\alpha$, $k = 0, \ldots, N$.*

**Corollary D.1.** *Let $\mathbf{x}$ be such that $\dot{x} \in \mathcal{C}$, $R \in \mathbb{S}_+^n$, and $h > 0$, then the integral inequality*

$$\int_{-h}^{0} \dot{\mathbf{x}}^\top(\alpha) R \dot{\mathbf{x}}(\alpha) d\alpha \geq \frac{1}{h} \xi_N^\top \left[ \sum_{k=0}^{N} (2k+1) \Gamma_N^\top(k) R \Gamma_N(k) \right] \xi_N \tag{D.6}$$

*holds, for all $N \in \mathbb{N}$, where*

$$\xi_N = \begin{cases} \begin{bmatrix} \mathbf{x}^\top(0) & \mathbf{x}^\top(-h) \end{bmatrix}^\top, & N = 0 \\ \begin{bmatrix} \mathbf{x}^\top(0) & \mathbf{x}^\top(-h) & \frac{1}{h}\Omega_0^\top & \ldots & \frac{1}{h}\Omega_{N-1}^\top \end{bmatrix}^\top, & N > 0 \end{cases} \tag{D.7}$$

$$\Gamma_N(k) = \begin{cases} [I \quad -I], & N = 0 \\ \begin{bmatrix} I & (-1)^{k+1} I & \gamma_{Nk}^0 I & \ldots & \gamma_{Nk}^{N-1} I \end{bmatrix}, & N > 0 \end{cases} \tag{D.8}$$

$$\gamma_{Nk}^i = \begin{cases} -(2i+1)(1-(-1)^{k+i}), & i \leq k \\ 0, & i > k \end{cases} \tag{D.9}$$

*and $\Omega_k = \int_{-h}^{0} L_k(\alpha)\mathbf{x}(\alpha)d\alpha$.*

# Appendix E

# Technical details of Chapter 5

This appendix contains the technical details for the development of the design procedure exposed in Chapter 5. First, it is exposed the computation of the set of transfer functions that describe the error signal. Second, it is solved the minimization problem for second order plants and zero crossing resetting law and variable band resetting law.

## E.1 Closed-loop Error Sequence

Closed-loop response of the reset control system is given by the error sequence (5.9)-(5.10). The numerator polynomial of $E_{r,k}(s)$, $k = 0, 1, 2, \ldots$, can be computed from the determinant of the matrix

$$\begin{bmatrix} A(t_k) - sI & \mathbf{x}(t_k^+) \\ C_E & 0 \end{bmatrix}, \tag{E.1}$$

while the denominator polynomial is simply the determinant of $A(t_k) - sI$. For the step tracking case ($d = 0$), the system equations can be simplified by eliminating the state $\mathbf{w}_2$ corresponding to the disturbance. Thus, the closed-loop state at the reset instants is $\mathbf{x}(t_k^+) = (x_1(t_k), \cdots, x_{n_p}(t_k), x_i(t_k), 0, w_{10})$ (by simplicity $t_k$ will be removed in the following). In addition, using an observer form $(A_p, B_p, C_p)$ for the realization of the plant transfer function (5.11), and equations (5.2), (2.36) and (5.5), the determinant of

(E.1) is

$$\begin{vmatrix} -s & 0 & \cdots & 0 & -a_0 - b_0 k_p & b_0 k_i (1-p_r) & b_0 k_i p_r & b_0 k_p & x_1 \\ 1 & -s & \cdots & 0 & -a_1 & 0 & 0 & 0 & x_2 \\ 0 & 1 & \cdots & 0 & -a_2 & 0 & 0 & 0 & x_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n_p-1} - s & 0 & 0 & 0 & x_{n_p} \\ 0 & 0 & \cdots & 0 & -1 & -s & 0 & 1 & x_i \\ 0 & 0 & \cdots & 0 & -1 & 0 & -s & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & -s & w_{10} \\ 0 & 0 & \cdots & 0 & -1 & 0 & 0 & 1 & 0 \end{vmatrix}. \tag{E.2}$$

Therefore, by using the following identity

$$\begin{vmatrix} -s & 0 & \cdots & 0 & \alpha \\ 1 & -s & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{vmatrix} = (-1)^{n_p+1}\alpha, \tag{E.3}$$

it is not difficult to see that the determinant (E.2) is given by

$$-s^2(s \begin{vmatrix} -s & 0 & \cdots & 0 & x_1 \\ 1 & -s & \cdots & 0 & x_2 \\ 0 & 1 & \cdots & 0 & x_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & x_{n_p} \end{vmatrix} + w_{10} \begin{vmatrix} -s & 0 & \cdots & 0 & -a_0 - b_0 k_p \\ 1 & -s & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n_p-1} - s \end{vmatrix} \tag{E.4}$$

$$+(-1)^{n_p-1} b_0 (k_p w_0 + k_i (1 - p_r(t_k) x_i(t_k)) \tag{E.5}$$

and, in addition, by using the identity

$$\begin{vmatrix} -s & 0 & \cdots & 0 & \beta_{n_p} \\ 1 & -s & \cdots & 0 & \beta_{n_p-1} \\ 0 & 1 & \cdots & 0 & \beta_{n_p-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \beta_1 \end{vmatrix} = (-1)^{n_p+1}\beta_{n_p} - s \begin{vmatrix} -s & 0 & \cdots & 0 & \beta_{n_p-1} \\ 1 & -s & \cdots & 0 & \beta_{n_p-2} \\ 0 & 1 & \cdots & 0 & \beta_{n_p-3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \beta_1 \end{vmatrix}, \tag{E.6}$$

the determinant results in the polynomial

$$s^2((w_{10} - x_{n_p}(t_k))s^{n_p} + \cdots + (w_{10}a_1 - x_1(t_k))s + w_{10}a_0 - b_0k_i(1 - p_r(t_k))x_i(t_k)). \quad \text{(E.7)}$$

Therefore, an analogous approach can be applied to obtain the polynomial denominator of $E_{r,k}(s)$, as the determinant of $A(t_k) - sI$. The result is that the term $s^2$ is cancelled in (E.7), and finally the expression (5.12) is obtained.

On the other hand, for the disturbance rejection case ($r = 0$), the error $E_{d,k}(s)$ may be obtained using a similar approach (details are omitted by brevity). Note that the polynomial denominator is identical to the reference tracking case.

## E.2 $H_2$-norm Minimization

In general, for a function $E \in \mathbb{R}(s)$ the $H_2$-norm is defined as

$$\|E\|_2 = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} |E(j\omega)|^2 d\omega\right)^{\frac{1}{2}} \quad \text{(E.8)}$$

and is related with the $\mathcal{L}_2$-norm of its inverse Laplace transform $e$, that is $\|e\|_{\mathcal{L}_2} = \left(\int_{-\infty}^{\infty} |e(t)|^2 dt\right)^{\frac{1}{2}} = \|E\|_2$.

For the error signal given by (5.9), $E_k(s) = C_E(sI - A(t_k))^{-1}\mathbf{x}(t_k^+)$ (where $t_k$ is a constant regarding the next reasoning), it is can be obtained that its $H_2$-norm is given by [166]:

$$\|E_k\|_2^2 = \mathbf{x}(t_k^+)^\top L_o \mathbf{x}(t_k^+), \quad \text{(E.9)}$$

where $L_o > 0$ is the observability Gramian matrix which can be obtained from the following Lyapunov equation

$$A(t_k)^\top L_o + L_o A(t_k) + C_E^\top C_E = 0. \quad \text{(E.10)}$$

In particular, for an error signal given by

$$E_k(s) = C_E(sI - A(t_k))^{-1}\mathbf{x}(t_k^+) = \frac{\beta_2 s^2 + \beta_1 s + \beta_0}{s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0} \quad \text{(E.11)}$$

for some constants $\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2$, it is obtained that the Gramian matrix $L_o \in \mathbb{R}^{3\times 3}$ has the form

$$L_o = \begin{pmatrix} l_1 & 0 & l_2 \\ 0 & l_3 & 0 \\ l_2 & 0 & l_4 \end{pmatrix} \tag{E.12}$$

for some constants $l_i$, $i = 1, \cdots, 4$, and that the error $H_2$-norm is simply

$$\|E_k\|_2^2 = \beta_0^2 l_1 + \beta_1^2 l_3 + \beta_2^2 l_4 + 2\beta_0\beta_2 l_2. \tag{E.13}$$

In the following, equation (E.13) will be used to solve the minimization problem of section 5.1.3; only the reference tracking case will be derived, the disturbance rejection case follows by using similar arguments and is omitted for brevity.

*Zero crossing resetting law.* In this case, the error is

$$E_{r,k}(s) = \frac{\dot{e}(t_k)(s+\delta)}{s^3 + a_1 s^2 + (a_0 + b_0 k_p)\, s + b_0 k_i} \tag{E.14}$$

and it depends on the parameter $\delta$ (note that $\dot{e}(t_k)$ is constant); by using (E.14), its $H_2$-norm is simply

$$\|E_{r,k}\|_2^2 = \dot{e}^2(t_k)(\delta^2 l_1 + l_3), \tag{E.15}$$

where $l_1, l_3 > 0$. Thus, it is clear that the minimum of the $H_2$-norm is achieved for $\delta = 0$.

*Variable band resetting law.* The resetting law is given by $\theta\dot{e}(t_k) + e(t_k) = 0$. Removing the dependence on $t_k$ by simplicity, the error is given by

$$E_{r,k}(s) = \frac{(w_{10} - x_2)\, s^2 + (a_1 w_{10} - x_1)\, s + a_0 w_{10} - b_0 k_i (1-p_r) x_i}{s^3 + a_1 s^2 + (a_0 + b_0 k_p)\, s + b_0 k_i}, \tag{E.16}$$

where

$$\theta\dot{e} + e = 0 \rightarrow \frac{1}{\theta} = -\frac{\dot{e}}{e} = -\frac{a_1 x_2 - x_1}{w_{10} - x_2}. \tag{E.17}$$

Now, adding the null term $(a_1 x_2 - a_1 x_2)s$ to the numerator of (E.16), $E_{r,k}(s)$ can be rewritten as follows:

$$E_{r,k}(s) = \frac{(w_{10} - x_2)\, s \left(s + a_1 - \frac{1}{\theta}\right) + a_0 w_{10} - b_0 k_i (1-p_r) x_i}{s^3 + a_1 s^2 + (a_0 + b_0 k_p)\, s + b_0 k_i}, \tag{E.18}$$

and finally, for the reset ratio (5.37), the result is

$$E_{r,k}(s) = \frac{-\dot{e}(t_k)\left(\theta s^2 + (\theta a_1 - 1)\, s + \delta\theta\right)}{s^3 + a_1 s^2 + (a_0 + b_0 k_p)\, s + b_0 k_i}, \tag{E.19}$$

and thus, the $H_2$-norm given by

$$\|E_{r,k}\|_2^2 = \dot{e}^2(t_k)(\Delta^2 l_1 + (\theta a_1 - 1)^2 l_3 + \theta^2 l_4 + 2\Delta\theta l_2), \tag{E.20}$$

where $\Delta = \delta\theta$, depends on both parameters $\theta$ and $\delta$. In order to solve the minimization problem, the term $\dot{e}(t_k)$ is considered constant.

The simple case $\delta = 0$ corresponds to $\|E_{r,k}\|_2^2 = \dot{e}^2(t_k)((\theta a_1 - 1)^2 l_3 + \theta^2 l_4)$, and the minimum is obtained at

$$\theta = \frac{a_1 l_3}{a_1^2 l_3 + l_4}. \tag{E.21}$$

A simple computation of the observability Gramian matrix leads into the following value of $\theta$ as function of the process parameters, and $k_p$

$$\theta = \frac{a_1}{a_1^2 + a_0 + b_0 k_p}. \tag{E.22}$$

The minimization problem with respect the two parameters can also be solved, since the right-hand of (E.20) is a bivariate quadratic function (in the variables $\theta$ and $\Delta$) which cannot be negative since it represents a norm. In addition, $l_1 > 0$, and thus, the function has a minimum at

$$\delta = -\frac{l_2}{l_1}, \tag{E.23}$$

$$\theta = \frac{a_1 l_1 l_3}{a_1^2 l_1 l_3 + l_1 l_4 - l_2^2}. \tag{E.24}$$

Replacing the values of the observability Gramian matrix into the above equations, the following equations are obtained

$$\delta = \frac{b_0 k_i}{a_1}, \tag{E.25}$$

$$\theta = \frac{a_1^2}{a_1^3 + a_1(a_0 + b_0 k_p) - b_0 k_i}. \tag{E.26}$$

# Appendix F

# Describing Function

## F.1 Introduction

Describing function analysis is a widely known technique to study frequency response of nonlinear systems. It is an extension of linear frequency response analysis. In linear systems, transfer functions depend only on the frequency of the input signal. In nonlinear systems, when a specific class of input signal such as a sinusoidal is applied to a nonlinear element, the output signal is not a sinusoidal. You can describe approximately the transfer characteristics of the nonlinearity by a function that depends not only on frequency, but also on input amplitude. This function is referred to as a describing function. For instance, describing function analysis has a wide area of applications from frequency response analysis to prediction of limit cycles. For a detailed analysis of the describing function the reader is referred to [107].

Regarding reset control systems, the describing function analysis has been used in several works (see [38, 234] and the references therein), where an analysis of limit cycles, well-posedness and stability has been accomplished. The describing function of the CI compensator with zero crossing resetting law is given by

$$CI(\omega) = \frac{1.62}{j\omega} e^{-j38.1^\circ}. \tag{F.1}$$

If the fixed band with parameter $\delta$ is applied as a resetting law, then the describing function is as follows:

$$CI_\delta(E,\omega) = \frac{1}{j\omega}\left(1 + \frac{j4\sqrt{1-\left(\frac{\delta}{E}\right)^2}}{\pi} e^{j\sin^{-1}\left(\frac{\delta}{E}\right)}\right). \tag{F.2}$$

Finally, the describing function of the CI with a more sophisticated resetting law, that is the variable band resetting law, is given by

$$CI_\delta(\omega) = \frac{1}{j\omega}\left(1 + \frac{j2}{\omega}\left(1 + e^{j2tan^{-1}(wh)}\right)\right). \tag{F.3}$$

## F.2 CI with Filtered Variable Band Resetting Law

Let the function $e_f(t)$ be the error signal filtered by the following first order law pass filter

$$F(s) = \frac{1}{\tau_f s + 1}. \tag{F.4}$$

The response of the filter to a sinusoidal input $e(t) = E\sin(\omega t)$ is given by

$$e_f(t) = \frac{E\tau_f}{(\tau_f\omega)^2+1}\left(e^{-\frac{1}{\tau_f}t}\omega + \frac{1}{\tau_f}\sin(\omega t) - \omega\cos(\omega t)\right) \tag{F.5}$$

with the following time-derivative

$$\dot{e}_f(t) = \frac{E}{(\tau_f\omega)^2+1}\left(-\omega e^{-\frac{1}{\tau_f}t} + \tau_f\omega^2\sin(\omega t) + \omega\cos(\omega t)\right). \tag{F.6}$$

Consider now the variable band resetting law given by $\theta\dot{e}_f(t) + e(t) = 0$, and let parameterize the time constant of the filter as follows $\tau_f = \gamma\theta$, with $\gamma \in [0,1]$. For the error signal $e(t) = E\sin(\omega t)$ and its derivative (F.6), the resetting law is written as follows:

$$\theta\dot{e}_f(t) + e_f(t) = \frac{E}{(\gamma\theta\omega)^2+1}\left(-\theta\omega e^{-\frac{1}{\gamma\theta}t} + (1+\gamma(\theta\omega)^2 + (\gamma\theta\omega)^2)\sin(\omega t) + \theta\omega\cos(\omega t)\right). \tag{F.7}$$

Equating to zero and considering the steady-state, we obtain the following equation

$$(1+\gamma(\theta\omega)^2 + (\gamma\theta\omega)^2)\sin(\omega t) + h\omega\cos(\omega t) = 0 \tag{F.8}$$

that yields to

$$\tan(\omega t) = -\frac{h\omega}{1+\gamma(1+\gamma)(\theta\omega)^2}, \tag{F.9}$$

$$t = \frac{1}{\omega}\left(n\pi - \tan^{-1}\left(\frac{h\omega}{1+\gamma(1+\gamma)(\theta\omega)^2}\right)\right) \tag{F.10}$$

for some integer $n > 0$. Now considering that the parameter $\delta$ of the describing function of the CI with fixed band is variable with $\delta = E\sin(\omega t)$ and $t$ given by (F.10). Substuting $\delta$ into (F.2), after some computation, it is obtained the following describing function

$$CI_{\theta,\gamma}(\omega) = \tfrac{1}{j\omega}\left(1+\tfrac{2j}{\pi}\left(1+e^{2j\tan^{-1}\left(\frac{\omega\theta}{1+\gamma(1+\gamma)(\theta\omega)^2}\right)}\right)\right). \tag{F.11}$$