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Cartan-Eilenberg Categories, Purity and Gorenstein Projetive Resolvents

Categorías de Cartan-Eileberg, Pureza y Resolventes Gorenstein Proyectivas

Dña. Sinem Odabaşi

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(Categorías de Cartan-Eilenberg, Pureza y Resolventes Gorenstein proyectivas)

Dª. Sinem Odabaşı

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AUTORIZA:

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La rosa
no buscaba la aurora:
casi eterna en su ramo,
buscaba otra cosa.

La rosa,
no buscaba ni ciencia ni sombra:
confín de carne y sueño,
buscaba otra cosa.

La rosa,
no buscaba la rosa.
Inmóvil por el cielo
buscaba otra cosa.

Federico García Lorca

CONTENTS

INTRODUCCIÓN

Desde finales del siglo XIX, los métodos homológicos basados en aspectos topológicos se han utilizado en contextos algebraicos, como en el caso la homología (o la cohomología) de grupos, de álgebras de Lie, etc. A mediados del siglo XX, Cartan y Eilenberg en [CE56] recapitularon todos los avances hasta la fecha en Álgebra Homológica Relativa e introdujeron nuevas e innovadoras ideas. Por ejemplo, los módulos proyectivos, los funtores exactos a izquierda, el funtor de torsión, etc. Sin embargo era de esperar que hubiese un marco más general que incluyera la cohomología de haces. Las categorías Abelianas, definidas en [Buc55], surgen de manera natural como el contexto adecuado en la búsqueda de éste objetivo.

La tesis de Verdier, [Ver96], bajo la supervisión de Grothendieck, trajo consigo otro enfoque en los funtores derivados, por medio de la categoría derivada. La categoría derivada $\mathbf{D}(\mathcal{A})$ de una categoría abeliana $\mathcal{A}$ es la categoría que se obtiene a partir de la categoría $\mathbf{C}(\mathcal{A})$ de complejos no acotados invirtiendo de manera formal los cuasi-isomorfismos. De este modo, si $\mathcal{F} : \mathcal{A} \to \mathcal{C}$ es un funtor aditivo entonces, bajo ciertas hipótesis razonables, existe un funtor $R\mathcal{F} : \mathbf{D}(\mathcal{A}) \to \mathbf{D}(\mathcal{C})$ con la propiedad de que si $A \in \mathcal{A}$ se considera como un complejo, entonces la cohomología del complejo $R\mathcal{F}(A)$ proporciona los funtores derivados a derecha usuales.

Los trabajos [Hoc45], [Hel58], [Buc59], [EM65] impulsaron el desarrollo del Álgebra Homológica Relativa. En esta teoría se estudian, no solo el funtor Ext, sino también sus subfuntores. De este modo, las resoluciones proyectivas e inyectivas se sustituyen por resoluciones por objetos proyectivos e inyectivos relativos y los epimorfismos y monomorfismos usuales por epimorfismos y monomorfismos propios.

Estas tres vertientes que acabamos de mencionar han dado lugar a nuevos descubrimientos en el campo del Álgebra Homológica y al desarrollo de nuevas áreas de estudio que continúan hasta el día de hoy. En la presente memoria nos centramos en tres tópicos dentro del campo del Álgebra Homológica (Relativa): las Categorías de Cartan-Eilenberg, la Pureza y las resolventes Gorenstein proyectivas.

Categorías de Cartan-Eilenberg: Como hemos mencionado anteriormente, en [CE56] Cartan y Eilenberg desarrollaron la forma de generar una familia de funtores $\{\mathcal{F}_n\}_{n\in\mathbb{Z}}$ a partir de un funtor aditivo $\mathcal{F}$ entre categorías de módulos. El primer método, basado en un procedimiento iterativo lento y elemental, conduce a la noción de funtores satélite. El otro, que se basa en resoluciones, da lugar a la creación de los funtores derivados. En caso de los funtores Hom y $\otimes$, las dos construcciones generan los mismos funtores. Con respecto a éste último, por ejemplo, una resolución proyectiva P_M de un módulo M es un complejo no-negativo de módulos proyectivos junto con un cuasi-isomorfismo $P_M \to M$. En este caso estamos considerando a M como un complejo cuya 0-componente es M y 0 en el resto. Debido a la propiedad que define a los módulos proyectivos, una resolución proyectiva reducida de un módulo será única salvo homotopía, lo cual permite obtener homologías isomorfas. Además, el funtor interno $\mathcal{H}om(P_M, -)$ en la categoría de complejos de módulos preserva cuasi-isomorfismos. Los complejos que verifican esta última propiedad reciben el nombre de complejos K-proyectivos. El módulo M y su resolución proyectiva reducida P_M son isomorfos en la categoría derivada $\mathbf{D}(R\text{-Mod})$. Los primeros trabajos en categorías derivadas estaban a menudo restringidos a complejos acotados inferiormente o directamente a complejos acotados, debido en gran medida a la necesidad de trabajar con resoluciones proyectivas. De hecho, para una categoría con suficientes proyectivos, un resultado clásico es que todo complejo acotado inferiormente, admite una resolución proyectiva por un complejo acotado inferiormente de módulos proyectivos, es decir, existe un cuasi-isomorfismo $P \to A$, siendo P un complejo de proyectivos acotado inferiormente. Por tanto es posible extender un funtor aditivo arbitrario $\mathcal{F} : \mathcal{A} \to \mathcal{C}$ a $\mathbf{D}_+(\mathcal{A}) \to \mathbf{D}_+(\mathcal{C})$.

En [Spa88], Spaltenstein probó que, en una categoría con límites directos exactos y con suficientes proyectivos, todo complejo no acotado es cuasi-isomorfo a un complejo K-proyectivo y que se podían usar los complejos K-proyectivos a la hora de calcular funtores derivados.

En [Qui67] Quillen dio otra aproximación al problema de derivar ciertos funtores tanto aditivos como no aditivos. En este trabajo Quillen introdujo la noción de estructura de modelos. En pocas palabras, en el caso de la categoría de complejos sobre una categoría abeliana $\mathcal{A}$, si podemos definir una estructura de modelos en $\mathbf{C}(\mathcal{A})$ en donde las equivalencias débiles sean los cuasi-isomorfismos, podemos asegurar la existencia de la categoría derivada y el funtor derivado queda determinado en términos de los reemplazamientos cofibrantes-fibrantes de la estructura de modelos. Sin embargo, en ocasiones sucede que los axiomas de estructura de modelos son excesivamente restrictivos y excluyen algunos casos interesantes.

En este contexto, en [GNPR10], los autores proponen una aproximación alternativa al álgebra homológica-homotópica teniendo presente el papel de los complejos K-proyectivos en las categorías derivadas. El punto de partida es una categoría $\mathcal{C}$ con un par de clases de morfismos (S, W) de modo que S está contenido en la saturación $\overline{W}$ de W. Los morfismos en S se denominan equivalencias fuertes y los de W equivalencias débiles. Un objeto (S, W)-cofibrante es un objeto $X \in \mathcal{C}$ tal que $\mathrm{Hom}_{\mathcal{C}[S^{-1}]}(X, w)$ es biyectiva para todo $w \in W$. Y una categoría de Cartan-Elenberg es una categoría que tiene equivalencias débiles y fuertes y con suficientes objectos cofibrantes. En otras palabras, su localización con respecto a las equivalencias débiles es equivalente a la subcategoría plena de la localización con respecto a las equivalencias fuertes formada por los objetos cofibrantes.

Vamos a ilustrar esta definición con un ejemplo sencillo. Consideremos la categoría de complejos de R-módulos, $\mathbf{C}(R\text{-Mod})$. En este caso, si tomamos como equivalencias fuertes a la clase de equivalencias homotópicas y como equivalencias débiles, la clase de cuasi-isomorfismos, la categoría de homotopía $\mathbf{K}(R\text{-Mod})$ y la categoría derivada

$\mathbf{D}(R\text{-Mod})$ son localizaciones de $\mathbf{C}(R\text{-Mod})$ con respecto a las equivalencias fuertes y débiles, respectivamente. Los objetos (S, W)-cofibrantes en este caso son exactamente los complejos K-proyectivos. Entonces $(\mathbf{C}(R\text{-Mod}), S, W)$ es una categoría de Cartan-Eilenberg a izquierda porque, para cada complejo A, existe un cuasi-isomorfismo $P \to A$, siendo P un complejo K-proyectivo.

A la vista del ejemplo anterior, la idea que subyace tras la definición de categoría de Cartan-Eilenberg es la de acercarse a la localización por equivalencias débiles por medio de la localización relativa de la subcategoría de objetos cofibrantes con respecto a las equivalencias fuertes. De este modo, si las equivalencias fuertes tienen buenas propiedades, la localización con respecto a las equivalencias débiles será más fácil de manejar. Esta es la abstracción formal de la idea que subyace a la aproximación de Cartan-Eilenberg y al funtor derivado. Es por esto por lo que los autores en [GNPR10] las denominan categorías de Cartan-Eilenberg. De hecho, toda estructura de modelos de Quillen proporciona una categoría de Cartan-Eilenberg pero el recíproco no es cierto en general, es decir, podemos encontrar categorías de Cartan-Eilenberg que no proceden de una categoría de modelos de Quillen (véase Proposition 4.3.6 y [GNPR10, Example 6.2.7]). Esta aproximación también permite extender la teoría clásica de derivación de funtores al caso no aditivo.

Pureza: Los morfismos más simples en la categoría de módulos son las secciones, es decir, el morfismo inducido por los sumandos directos de un módulo. La noción de submódulo puro generaliza, en cierto sentido, de forma satisfactoria la idea de sumando directo, al mismo tiempo que permite probar algunos resultados que no son necesariamente ciertos para sumandos directos. A modo de ejemplo, dado un submódulo de un módulo, puede que no sea posible extender dicho submódulo a un sumando directo que verifique ciertas restricciones en la cardinalidad, pero sin embargo sí es posible realizar dicha extensión de forma pura (consultar el Teorema 5.1.6).

Los submódulos puros se pueden definir de manera equivalente en términos del funtor Hom

y del funtor $\otimes$ y tambíen como límites directos de secciones. Constituyen la pieza fundamental de la versión de Álgebra Homológica Relativa conocida como Álgebra Homológica Pura. Existe otra definición equivalente de pureza en términos de sistemas de ecuaciones lineales. Hay varios trabajos en la literatura que se acercan a la pureza de este último modo (véase, a modo de ejemplo, [Pre09]), pero en la presente memoria no nos centraremos en este último punto de vista.

Una de las razones de la importancia de la pureza en el contexto del Álgebra Homológica Relativa es debido a su trascendencia en la Teoría de la Aproximación. Posiblemente el primer ejemplo que ilustra la idea de aproximación apareció en el Teorema syzygy de Hilbert [Hil90] con las resoluciones libres. Las resoluciones clásicas, dadas en [CE56], son las definidas por inyectivos y por proyectivos. En ausencia de objetos proyectivos e inyectivos en una categoría, la Teoría de la Aproximación busca calcular resoluciones por otras clases suficientemente buenas. La noción que necesitamos para poder encontrar tales resoluciones es la de clase de (pre)cubiertas o, su dual, el de clase de (pre)envolturas. De manera más precisa, una clase $\mathcal{F}$ de objetos en una categoría abeliana $\mathcal{C}$ se llama clase de precubiertas si para cada objeto $X \in \mathcal{C}$ existe una $\mathcal{F}$-precubierta $F \to X$, donde $F \in \mathcal{F}$. Esto quiere decir que cada morfismo $F' \to X$, con $F' \in \mathcal{F}$, admite una factorización $F' \to F \to X$. Para que la $\mathcal{F}$-precubierta $g : F \to X$ sea una $\mathcal{F}$-cubierta se requiere la condición adicional de que si un morfismo φ es tal que $g \circ \varphi = g$, entonces φ es un isomorfismo. La clase $\mathcal{F}$ será entonces una clase de cubiertas, si todo objeto admite una $\mathcal{F}$-cubierta. Si $\mathcal{F}$ es una clase de precubiertas, entonces todo objeto X admite una $\mathcal{F}$-resolución a izquierda única salvo homotopía. El concepto dual es el de clase de (pre)envolturas. Estos conceptos estan estrechamente ligados a los de pares de cotorsión, clases deconstructibles y categorías abelianas de modelos, [Hov02], [Sto13a], [Sal79], [Eno12].

La mayoría de las clases en las que estamos interesados son cerradas bajo subobjetos y cocientes puros, véase por ejemplo [Pin08], [BBE01], [EE05], [ET01], [RS98], [CPT10], [Gill04], [Gill06]. Para una relación más general entre (pre)envolturas, (pre)cubiertas y pureza, el lector puede consultar [Kra12] y [HJ08]. Como ejemplo, la categoría $\mathfrak{Qcoh}(X)$ de haces cuasi-coherentes sobre un esquema X no tiene, en general, suficientes proyectivos, pero sin embargo la clase $\mathcal{F}lat(X)$ de haces cuasi-coherentes planos es una clase de cubiertas, ver [EE05] para el argumento usando pureza. En [Gill07], Gillespie prueba que, bajo ciertas condiciones en el esquema X, existe una estructura de modelos monoidal en $\mathbf{C}(\mathfrak{Qcoh}(X))$, donde la $\mathcal{F}lat(X)$-resolución de un haz cuasi-coherente es un reemplazamiento cofibrante. Como consecuencia, el funtor Ext, que se calcula por medio de resoluciones inyectivas en $\mathfrak{Qcoh}(X)$ se puede calcular a partir de $\mathcal{F}lat(X)$-resoluciones.

La noción de pureza en R-Mod se puede extender de diversas formas a otras categorías. Aunque el punto en común de todas las posibles extensiones es que en cada caso es posible establecer un teorema del tipo de ”existencia de suficientes objetos puros” como se ha mencionado al comienzo de esta sección para R-Mod. Por ejemplo, la noción apropiada de pureza en $\mathfrak{Qcoh}(X)$ es la dada con respecto a los tallos, ya que es la que refleja la naturaleza geométrica, al igual que sucede con la definición de haz cuasi-coherente plano. Existe, sin embargo, otra definición categórica de pureza usando el funtor Hom y la definición categórica de objeto finitamente presentado. Crawley-Boevey en [Craw94] muestra que las categorías aditivas localmente finitamente presentadas constituyen el marco aditivo más general para definir una buena Teoría de Pureza. Recordemos que una sucesión en una categoría aditiva localmente finitamente presentada se dice que es pura cuando está proyectivamente generada por la clase de los objetos finitamente presentados. Dado un cardinal regular λ, las categorías localmente λ-presentables, que incluyen a las categorías de Grothendieck, proporcionan un marco general en el cual se pueden tratar aspectos relacionados con subobjetos λ-puros (ver [AR94, Section 2]). Para categorías trianguladas compactamente generadas, en [Bel00], los objetos compactos desempeñan el papel de los objetos finitamente presentados, de modo que los triángulos puros se definen por medio de objetos compactos.

Álgebra Homológica de Gorenstein: En una de las charlas del seminario Bourbaki acerca de la Dualidad, Grothendieck definió el concepto de anillo Gorenstein, para un anillo conmutativo R de tipo finito sobre un cuerpo. Un tal anillo será Gorenstein si es

Cohen-Macaulay y un cierto R-módulo es localmente libre de rango 1. En [Bas62], Bass caracteriza los anillos de dimensión autoinyectiva finita. Como se pone de manifiesto en la introducción de [Bas63], Serre observó que los dos conceptos coinciden en un contexto geométrico. La definición moderna de anillo Gorenstein y su condición equivalente aparece en [Bas63]: un anillo conmutativo Noetheriano R es Gorenstein si para cada ideal primo p, $\operatorname{id} R_p < \infty$. Esta es la definición que se usa habitualmente en Geometría Algebraica.

En [Aus67], Auslander introduce una nueva clase de módulos contenidos en la clase de módulos finitamente generados, a los que llamó módulos de G-dimensión 0. Entonces se definió la G-dimensión para módulos finitamente generados usando los módulos de G-dimensión 0. La teoría se desarrolló posteriormente en [AB69]. Uno de los aspectos más destacados de este nuevo invariante es que es más débil que la dimensión proyectiva pero aún así comparte con ella algunas propiedades. La teoría comenzó a interesar a los algebristas homológicos. Sin embargo, hasta este momento, estaba definida sólo para módulos finitamente generados y se comportaba bien para anillos específicos.

En [EJ95] Enochs y Jenda dieron una aproximación más general a la teoría. En este trabajo, ellos definen la noción de módulos "Gorenstein proyectivos" para un anillo arbitrario usando complejos totalmente acíclicos. Es una generalización de la noción de módulo finitamente generado con G-dimensión 0 en el sentido de Auslander. Es conocido que, para anillos Noetherianos, un módulo finitamente generado M es Gorenstein proyectivo si, y sólo si, tiene G-dimensión 0 (ver [Chr00, Theorem 4.1.4]). La ventaja de esta nueva aproximación también estriba en que permite definir nuevos conceptos tales como el de módulo Gorenstein inyectivo y módulo Gorenstein plano. Por tanto el artículo [EJ95] supone un nuevo enfoque en la teoría. Como resultado, tiene lugar la aparición de una nueva rama del Álgebra Homológica, que se conoce como Álgebra Homológica de Gorenstein. Diversos autores han trabajado intensamente en este campo y han estudiado su relación con la cohomología de Tate (por ejemplo; [AM02], [EJT93], [Chr00], [Iac05], [EJL04]). En términos generales, el principal problema en esta rama es conseguir las correspondientes versiones de Gorenstein de los resultados clásicos del Álgebra

Homológica para un anillo arbitrario. Pero ni siquiera se sabe la existencia de precubiertas Gorenstein proyectivas para un anillo arbitrario. En este sentido, existen intentos para debilitar la condición de un anillo de ser de Gorenstein ([Hol04], [Jør07]) para garantizar este resultado. Jorgensen en [Jør07] averigua la relación entre complejo dualizante y localización de Bousfield para complejos totalmente acíclicos. De este modo demuestra que la clase de los módulos Gorenstein proyectivos es una clase de precubiertas para anillos que admitan un complejo dualizante. El resultado extiende la clase de anillos conocidos hasta esa fecha para los cuales se conocía la existencia de precubiertas Gorenstein proyectivas.

A continuación vamos a resumir, por capítulos, los contenidos que se abordan en la presente memoria.

Los capítulos (I), (II) y (III) son las partes introductorias de la memoria. En ellos recapitulamos la terminología y los resultados básicos conocidos que usaremos a lo largo de la misma.

Capítulo IV

Este capítulo se centra en las Categorías de Cartan-Eilenberg a izquierda. De manera breve mencionaremos la localización de categorías y las categorías derivadas y recordaremos propiedades básicas de los complejos K-proyectivos, tanto en la categoría de homotopía como en la categoría derivada. Posteriormente nos centramos en categorías Cartan-Eilenberg a izquierda $(\mathcal{A}, S, W)$ y en sus propiedades más importantes. Veremos que los objetos (S, W)-cofibrantes y los objetos K-proyectivos presentan propiedades similares. Estudiamos, para una categoría $\mathcal{A}$, la estrecha relación que existe entre poseer una estructura de Cartan-Eilenberg a izquierda y el problema de la subcategoría co-ortogonal para $\mathcal{A}[S^{-1}]$, del mismo modo que las categorías abelianas de modelos quedan caracterizadas por la existencia de ciertos pares de cotorsión completos (los llamados pares de Hovey). De esta forma quedará claro que el problema de saber si una terna $(\mathcal{A}, S, W)$ constituye una categoría de

Cartan-Eilenberg a izquierda es equivalente a probar que la subcategoría $\overline{\mathcal{A}}_{cof}$, formada por los objetos (S, W)-cofibrantes, es una subcategoría coreflectiva de $\mathcal{A}[S^{-1}]$.

En [Pas12], el autor da un ejemplo y un contraejemplo de una categoría de Cartan-Eilenberg a izquierda en $\mathbf{C}_+(\mathcal{A})$ dependiendo de si $\mathcal{A}$ tiene o no suficientes proyectivos. En nuestro resultado principal de este capítulo, damos un método general de construcción de estructuras de Cartan-Eilenberg a izquierda que engloba en particular el ejemplo anterior. Pero incluso en el caso en que $\mathbf{C}_+(\mathcal{A})$ no es una categoría Cartan-Eilenberg a izquierda, como en el contraejemplo de [Pas12], nuestro método permite construir subcategorías de Cartan-Eilenberg a izquierda no triviales. La formulación precisa de nuestro resultado es la siguiente:

Teorema 4.4.5. Sea $(\mathcal{F}, \mathcal{B})$ un par de cotorsión en una categoría abeliana con suficientes $\mathcal{F}$-objetos. Supongamos que la clase $\mathcal{F}$ es una clase resolvente, $dg\,\tilde{\mathcal{B}} \cap \mathrm{Acic} = \tilde{\mathcal{B}}$ y que el par de cotorsión inducido $(dg\,\tilde{\mathcal{F}}, \tilde{\mathcal{B}})$ tiene suficientes proyectivos. Entonces $(\mathbf{C}_+(\mathcal{A}) \cap dg\,\tilde{\mathcal{B}}, S', W')$ es una categoría de Cartan-Eilenberg a izquierda.

Finalizaremos este capítulo mostrando la aplicabilidad del Teorema 4.4.5 en diferentes categorías, entre las cuales destacamos las categorías de haces y de módulos sobre un anillo Gorenstein.

Capitulo V

Este capítulo supone el comienzo de nuestra incursión en diferentes aspectos del Álgebra Homológica Pura. Podemos dividirlo en dos partes: en primer lugar estudiamos la teoría general de morfismos puros tanto geométricos como categóricos y una segunda parte sobre pureza en categorías de haces cuasi-coherentes. Empezamos recordando la noción de morfismos λ-puros en una categoría localmente λ-presentable, a los cuales denominaremos por monomorfismos puros categóricos. Posteriormente definimos los monomorfismos puros geométricos para categorías que tienen una estructura cerrada monoidal simétrica. De forma breve, son monomorfismos que se preservan bajo el producto tensorial con respecto a un objeto arbitrario. Estudiamos su relación con los monomorfismos puros categóricos. Para el caso en que la categoría ambiente es de Grothendieck cerrada monoidal simétrica, observamos que la noción de pureza categórica implica la pureza geométrica. Esta observación tendrá consecuencias destacables que se pondrán de manifiesto en los capítulos siguientes. Dedicaremos este capítulo a probar la existencia de (pre)envolturas puro inyectivas dependiendo de la pureza considerada en la categoría, lo cual nos proporciona la herramienta necesaria para proceder con el Álgebra Homológica Pura.

Teorema 5.2.6. Sea $\mathcal{C}$ una categoría de Grothendieck cerrada monoidal simétrica. Todo objeto admite una envoltura puro inyectiva geométrica.

En la tercera sección, nos centraremos en las categorías de haces de modulos $\mathcal{O}_X$-Mod y de haces cuasi-coherentes $\mathfrak{Qcoh}(X)$. En ellas surge, además, otra noción de pureza de manera natural. Se trata en ambos casos de categorías de Grothendieck que poseen una estructura cerrada monoidal simétrica. En el caso de $\mathcal{O}_X$-Mod probamos que la pureza geométrica se puede caracterizar en términos de los tallos (de ahí su nombre). Entonces la clase de los monomorfismos puros en $\mathcal{O}_X$-Mod entre objetos de $\mathfrak{Qcoh}(X)$ nos dará una nueva clase de sucesiones exactas cortas, que se pueden caracterizar en términos de las secciones sobre los abiertos afines. A esta clase de exactitud la llamaremos "exactitud pura en los tallos". De alguna forma ésta exactitud en la categoría $\mathfrak{Qcoh}(X)$ es más natural que la correspondiente a la pureza geométrica en $\mathfrak{Qcoh}(X)$, porque es, de hecho, el tipo de caracterización (en función de los tallos) que aparece al estudiar, por ejemplo, los haces cuasi-coherentes planos. Después de investigar la relación de la pureza en tallos con respecto a la pureza categórica, obtenemos el resultado que garantiza la existencia de resoluciones inyectivas puras en tallos en $\mathfrak{Qcoh}(X)$ sin suponer ninguna condición adicional en el esquema X:

Teorema 5.4.8. Sea X un esquema arbitrario. Todo haz cuasi-coherente sobre X tiene una envoltura puro inyectiva en tallos.

Capítulo VI.

Dedicamos este capítulo a estudiar algunas clases de haces cuasi-coherentes que se obtienen a partir de la pureza en tallos, así como sus propiedades homológicas. La idea que perseguimos es la de transferir algunos resultados conocidos en R-Mod a $\mathfrak{Qcoh}(X)$.

La clase $\mathcal{F}lat(X)$ de haces cuasi-coherentes planos sobre un esquema X ha sido amplio objeto de investigación en los últimos años, por tratarse de una elección natural para estudiar tanto la categoría de homotopía, como la categoría derivada de haces cuasi-coherentes ([EGPT12], [Gill07], [Hov01], [MS11], [Mur07], [HS13]). Como ya hemos mencionado antes, en [EE05] se prueba que la clase $\mathcal{F}lat(X)$ es una clase de cubiertas en $\mathfrak{Qcoh}(X)$ usando la pureza en tallos.

Dado un anillo asociativo R con identidad, un R-módulo a izquierda M se dice que es absolutamente puro si cada sistema de ecuaciones lineales finito con términos independientes en M, tiene una solución en M. Esto es equivalente a decir que M es un submódulo puro de todo R-módulo que lo contiene. En ciertos aspectos esta clase de módulos se comportan como los módulos inyectivos (véase [Mad67, Meg70, Pre09, Ste70] para un tratamiento general de módulos absolutamente puros y [Pin05] para un estudio más reciente). De hecho, los anillos Noetherianos se pueden caracterizar en términos de propiedades de los módulos absolutamente puros. Concretamente, R es Noetheriano si, y sólo si, la clase de los R-módulos absolutamente puros coincide con la clase de los R-módulos inyectivos ([Meg70]). En este capítulo exponemos las propiedades principales de los haces de módulos (localmente) absolutamente puros, tanto en la categoría $\mathfrak{Qcoh}(X)$ como en $\mathcal{O}_X$-Mod, para el caso en que X es un esquema localmente coherente. Por ejemplo, en la Proposición 6.2.7 mostramos que para comprobar la pureza absoluta local en $\mathfrak{Qcoh}(X)$ nos podemos restringir a un recubrimiento concreto por abiertos afines del esquema X. Además también vemos que los haces cuasi-coherentes localmente absolutamente puros son precisamente los $\mathcal{O}_X$-módulos absolutamente puros que son cuasi-coherentes, véase el Lema 6.2.8. Esto es análogo a la cuestión formulada en

[Hart77, II, §7, pg.135] para el caso de esquemas localmente Noetherianos (véase [Con00, Lemma 2.1.3]). Entonces caracterizamos los subesquemas cerrados localmente Noetherianos del espacio proyectivo $\mathbb{P}^n(A)$ (siendo A un anillo conmutativo y coherente) en términos de su clase de haces cuasi-coherentes absolutamente puros:

Proposición 6.2.11. Un subesquema cerrado $X \subseteq \mathbb{P}^n(A)$ es localmente Noetheriano si, y sólo si, cada haz cuasi-coherentes localmente absolutamente puro es localmente inyectivo.

Si X es un esquema Noetheriano, se sabe que la clase de los haces cuasi-coherentes localmente inyectivos es una clase de cubiertas en $\mathfrak{Qcoh}(X)$. Nosotros extendemos este resultado a la clase de los haces cuasi-coherentes localmente absolutamente puros sobre un esquema localmente coherente X.

Teorema 6.2.13. Sea X un esquema localmente coherente. Todo haz cuasi-coherente tiene una cubierta localmente absolutamente pura.

Como hemos señalado anteriormente, la categoría $\mathfrak{Qcoh}(X)$ es una categoría de Grothendieck, por tanto existen los productos. Sin embargo no es un problema fácil, en general, dar una descripción del objeto producto, debido en parte a que, a nivel de las secciones, el producto de módulos no se comporta bien con respecto a las localizaciones o, más generalmente, cuando lo tensorizamos por un módulo arbitrario, tomando como anillo base un anillo conmutativo (los productos directos no conmutan con el tensor en general). Incluso aunque se sabe que el producto tensor conmuta con productos directos al tensorizar por módulos finitamente presentados, no está claro que el objeto producto en $\mathfrak{Qcoh}(X)$ se pueda calcular a partir del producto de los módulos de secciones en cada abierto afín, si no imponemos condiciones adicionales en el haz estructural $\mathcal{O}_X$ asociado a X (por ejemplo que $\mathcal{O}_X(U)$ sea finitamente presentado como $\mathcal{O}_X(V)$-módulo, para cada par de abiertos afines $U \subseteq V$).

La ausencia de una descripción explícita del objeto producto trae consigo preguntas nuevas y relevantes relativas a la clase $\mathcal{F}lat(X)$ de haces cuasi-coherentes planos en X. Por ejemplo, Murfet en [Mur07, Remark B.7] plantea la cuestión de si la clase $\mathcal{F}lat(X)$ es cerrada bajo productos, cuando X es un esquema Noetheriano. Esta propiedad es crucial para probar que en $\mathbf{C}(A\text{-Mod})$, la categoría de complejos no acotados de A-módulos (siendo A un anillo conmutativo Noetheriano) el complejo $\mathcal{H}om_A(I, I')$ es un complejo de módulos planos, para complejos inyectivos $I, I' \in \mathbf{C}(A\text{-Mod})$. En este punto, queremos enfatizar que la noción usual de planitud en $\mathfrak{Qcoh}(X)$ no es categórica, como se demuestra en [ES12] (véase también [Rum10]). Recientemente Saorín y Šťovíček en [SS11, 4.2] han dado una solución positiva a esta cuestión para el caso de esquemas Dedekind. En su argumento ellos usan la caracterización de Crawley-Boevey de las subcategorías de preenvolturas de la categoría de objetos finitamente presentados en una categoría aditiva localmente finitamente presentada con productos (véase [Craw94, Theorem 4.2]). Entonces ellos prueban que si X es Dedekind, la categoría de haces coherentes localmente libres (los fibrados vectoriales) es una clase de preenvolturas dentro de la categoría de haces coherentes, obteniendo como consecuencia de ello que su clausura para límites directos, la clase $\mathcal{F}lat(X)$, es cerrada bajo productos.

Si X es un esquema afín, existe una equivalencia canónica entre las clases $\mathcal{F}lat(X)$ y $\mathcal{F}lat(R)$ (la clase de R-módulos planos) siendo $X = \mathrm{Spec}(R)$. Ahora, si X es de hecho Dedekind, es bien conocido que $\mathcal{F}lat(R)$ coincide con la clase de los R-módulos libres de torsión. De forma que, para un esquema arbitrario, tiene sentido definir la clase $\mathcal{F}$ de los haces cuasi-coherentes localmente libres de torsión como la clase de los objetos $\mathscr{F} \in \mathfrak{Qcoh}(X)$ tales que $\mathscr{F}(U)$ es un $\mathcal{O}_X(U)$-módulo libre de torsión, para cada abierto afín U. Esta clase contiene a $\mathcal{F}lat(X)$ en general y, de hecho, coincide con ella para esquemas Dedekind.

Teorema 6.3.7. Sea X un esquema de integridad. El producto $\mathscr{F}$ de una familia $\{\mathscr{F}_i\}_{i\in I}$ de haces cuasi-coherentes localmente libres en $\mathfrak{Qcoh}(X)$ es el subhaz cuasi-coherente maximal de

$\prod_{i\in I}\mathcal{F}_i$. De manera más precisa, es de la forma

$$\mathcal{F} = \sum_{\substack{\mathcal{M}\in\mathfrak{Qcoh}(X)\\ \mathcal{M}\subseteq\prod_{i\in I}\mathcal{F}_i}} \mathcal{M}.$$

Una consecuencia de este Teorema es que, para un esquema de integridad, la clase $\mathcal{F}$ induce una teoría de torsión hereditaria.

Hemos comentado anteriormente que en [EE05, Theorem 4.1] se prueba la existencia de cubiertas planas en $\mathfrak{Qcoh}(X)$. Nosotros probamos la existencia de cubiertas con respecto a la clase $\mathcal{F}$. Este resultado se conoce desde los años 60 para el caso de un esquema de integridad afín (véase [Eno63]).

Teorema 6.4.7. Todo haz cuasi-coherente sobre un esquema de integridad admite una cubierta localmente libre de torsión.

Capítulo VII

Este es el último capítulo que dedicamos a Álgebra Homológica Pura. En él seguimos considerando como marco general una categoría de Grothendieck cerrada monoidal simétrica. Nuestro objetivo final es la definición de la categoría derivada pura (geométrica), para especializarnos en el caso de un esquema.

Si $\mathcal{A}$ es una categoría localmente finitamente presentada, la pureza categórica define una estructura exacta pura en $\mathcal{A}$ y produce la categoría derivada pura $\mathbf{D}_{\mathrm{pur}}(\mathcal{A})$ estudiada, por ejemplo, por Christensen y Hovey en [CH02] y Krause [Kra12]. Recientemente en [Gil14] se prueba que esta categoría derivada pura se puede obtener como la categoría de homotopía de dos estructuras de modelos, usando los puro inyectivos y los puro proyectivos. Sin embargo, a menos que el esquema X sea afín, la pureza categórica definida anteriormente para categorías localmente finitamente presentadas no coincide con la pureza en tallos de $\mathfrak{Qcoh}(X)$. En el siguiente resultado, probamos que la categoría derivada relativa a la pureza geométrica aparece como la categoría de homotopía de una categoría de modelos.

Teorema 7.2.7. Sea $\mathcal{C}$ una categoría de Grothendieck cerrada monoidal simétrica y $\mathbf{C}(\mathcal{C})$ la categoría asociada de complejos. Entonces existe una estructura de modelos cofibrantemente generada en $\mathbf{C}(\mathcal{C})$ cuyos objetos triviales son los complejos acíclicos $\otimes$-puros, esto es, los complejos X para los que $X \otimes S$ es exacto para todo $S \in \mathcal{C}$. La estructura de modelos es exacta con respecto a la categoría exacta $\mathbf{C}(\mathcal{C})_{\otimes}$ de complejos tomando como clase propia de sucesiones exactas cortas las sucesiones puntualmente $\otimes$-puras de complejos. De hecho, la estructura de modelos es inyectiva en el sentido de que todos los complejos son cofibrantes y los complejos trivialmente fibrantes son los objetos inyectivos de $\mathbf{C}(\mathcal{C})_{\otimes}$, que son precisamente los complejos contractibles con componentes $\otimes$-puro inyectivas. Llamaremos a esta estructura de modelos la estructura de modelos $\otimes$-pura inyectiva en $\mathbf{C}(\mathcal{C})$ y su correspondiente categoría de homotopía es la categoría derivada $\otimes$-pura, denotada por $\mathbf{D}_{\otimes\text{-pur}}(\mathcal{C})$.

Como caso particular del teorema anterior, obtenemos, aplicando la Proposición 7.2.8, la siguiente aplicación a $\mathfrak{Qcoh}(X)$:

Corolario 7.2.9. Sea X un esquema cuasi-separado. Sea $\mathcal{E}$ la estructura exacta heredada por la pureza en tallos en $\mathfrak{Qcoh}(X)$ y consideremos la categoría de complejos no acotados $\mathbf{C}(\mathfrak{Qcoh}(X))$. Entonces, con respecto a la estructura exacta puntual inducida de $\mathcal{E}$, existe una estructura de modelos exacta e inyectiva en $\mathbf{C}(\mathfrak{Qcoh}(X))$. La categoría de homotopía correspondiente es la categoría derivada pura en tallos (o la categoría derivada pura geométrica) que denotaremos por $\mathbf{D}_{\text{stk-pure}}(\mathfrak{Qcoh}(X))$.

Dado que tenemos dos nociones diferentes de pureza en una categoría de Grothendieck cerrada monoidal simétrica, la pureza categórica y la geométrica, es natural preguntarse cuál es la relación entre ellas. En la Proposición 7.3.5 mostramos que hay un funtor canónico de la categoría derivada λ-pura, $\mathbf{D}_{\lambda\text{-pur}}(\mathcal{C})$, a la categoría derivada $\otimes$-pura, $\mathbf{D}_{\otimes\text{-pur}}(\mathcal{C})$, que admite un adjunto a derecha.

En [MS11] Murfet y Salarian definen lo que ellos llaman la categoría derivada pura de haces planos para un esquema Noetheriano y semi-separado, como el cociente de Verdier de la categoría de homotopía de los haces planos $\mathbf{K}(\mathcal{F}lat(X))$ con la subcategoría localizante $\mathbf{K}_{\mathrm{pac}}(\mathcal{F}lat(X))$ de los complejos acíclicos puros de haces. Los módulos planos están estrechamente relacionados con las categorías localmente finitamente presentadas, debido al Teorema de Representación de Crawley- Boevey [Craw94]. Este teorema establece que toda categoría localmente finitamente presentada $\mathcal{A}$ es equivalente a la subcategoría plena $\mathcal{F}lat(\mathcal{A})$ de $\mathrm{Mod}\text{-}A$, de A-módulos a derecha planos, donde A es el anillo funtor de $\mathcal{A}$, y la equivalencia da una correspondencia 1-1 entre sucesiones exactas puras en $\mathcal{A}$ y sucesiones exactas en $\mathcal{F}lat(A)$. Esta equivalencia se extiende también a nivel de estructuras de modelos y, en particular, a las categorías derivadas, de forma que obtenemos el siguiente:

Teorema 7.4.3. Sea $\mathcal{A}$ una categoría aditiva localmente finitamente presentada y sea $\mathcal{F}lat(A)$ su subcategoría equivalente de módulos planos en $\mathrm{Mod}\text{-}A$. Entonces $\mathbf{D}_{\mathrm{pur}}(\mathcal{A})$ es equivalente a $\mathbf{D}(\mathcal{F}lat(A))$, la categoría de homotopía de la estructura de modelos exacta e inyectiva en $\mathbf{C}(\mathcal{F}lat(A))$.

Capítulo VIII

Dedicamos este último capítulo a la versión de Álgebra Homológica Relativa conocida como Álgebra Homológica Gorenstein. Como hemos mencionado antes, el problema fundamental en este caso es traducir propiedades básicas del Álgebra Homológica al contexto relativo Gorenstein. En el capítulo nos centramos en los módulos Gorenstein proyectivos. El objetivo principal es estudiar tópicos relacionados con resoluciones a derecha por módulos Gorenstein proyectivos. En primer lugar, empezaremos considerando el problema de la existencia de preenvolturas Gorenstein proyectivas por módulos finitamente generados sobre un anillo local n-Gorenstein. De hecho, en [HJ11] ya se prueba que para un anillo Noetheriano local R, la clase de los modulos finitamente generados Gorenstein proyectivos es una clase de preenvolturas si, y sólo si, R es Gorenstein. Nosotros damos una demostración alternativa de este hecho:

Teorema 8.2.8. Sea R un anillo n-Gorenstein local. Todo módulo M tiene una preenvoltura Gorenstein proyectiva.

La razón que nos lleva a dar otra demostración de este resultado, es que nuestra demostración nos va a permitir calcular explícitamente las resoluciones a derecha Gorenstein proyectivas de módulos finitamente generados, obteniendo así el siguiente teorema:

Teorema 8.2.9. Sea R un anillo n-Gorenstein local. Si M es un R-módulo finitamente generado, entonces M tiene una resolución por la derecha Gorenstein proyectiva finita.

El siguiente objetivo es investigar ciertos funtores derivados por la izquierda relativos de Hom(-, -). Sobre un anillo conmutativo Noetheriano local, la existencia de una resolución a derecha proyectiva de un módulo finitamente generado está garantizada. Si, además, R es n-Gorenstein, sus cosyzygias son eventualmente Gorenstein proyectivas. Por tanto, para tales anillos, todo módulo finitamente generado tiene asociado un complejo totalmente acíclico de proyectivos. En otras palabras, una resolvente proyectiva completa $M \to \mathbf{P} \to \mathbf{T}$, siendo $\mathbf{T}$ un complejo de proyectivos. Denotamos por $\widehat{\mathrm{Ext}}_i(-,-)$ el funtor derivado a izquierda de Hom(-, -) el cual se obtiene al tomar un complejo proyectivo totalmente acíclico que aparece en alguna resolución a derecha proyectiva de la primera componente. Este funtor recibe el nombre de funtor derivado de Tate. En el siguiente resultado, obtenemos un resultado de balance de la homología de Tate, esto es, $\widehat{\mathrm{Ext}}_i(-,-)$ se puede calcular, o bien, tomando un complejo proyectivo totalmente acíclico que procede de la resolución a derecha proyectiva de la primera componente, o bien, a partir de una resolución a izquierda proyectiva de la segunda componente.

Teorema 8.3.2. Sea R un anillo n-Gorenstein local, y M, N R-módulos, donde M es finitamente generado. Si $M \to \mathbf{P} \to \mathbf{T}$ es una resolvente projectiva completa de M y $\mathbf{T}' \to \mathbf{P} \to N$ es una resolución proyectiva completa de N entonces $\widehat{\text{Ext}}_i(M,N) := H_i(\text{Hom}(\mathbf{T},N)) \cong H_i(\text{Hom}(M,\mathbf{T}'))$.

Otros funtores derivados a izquierda que centran nuestro interés son $\text{Ext}_i(-,-)$ y $\text{Gext}_i(-,-)$. Estos se obtienen usando una resolución a derecha proyectiva y una resolución a derecha Gorenstein proyectiva, respectivamente. En el siguiente resultado, obtenemos una sucesión exacta de tipo Avramov-Martsinkovsky (véase [AM02]) que conecta los tres funtores derivados a izquierda definidos, $\text{Ext}_i(-,-), \text{Gext}_i(-,-)$ y $\widehat{\text{Ext}}_i(-,-)$.

Teorema 8.3.3. Sea R un anillo n-Gorenstein local. Sea M un R-módulo finitamente generado y N un R-módulo. Entonces tenemos una sucesión exacta de funtores derivados a izquierda

$$0 \to \widehat{\text{Ext}}_{n-1}(M,N) \to \text{Ext}_{n-1}(M,N) \to \text{Gext}_{n-1}(M,N) \to \widehat{\text{Ext}}_{n-2}(M,N) \to \dots$$

$$\dots \to \text{Gext}_0(M,N) \to \widehat{\text{Ext}}_{-1}(M,N) \to 0.$$

INTRODUCTION

Starting from the end of the 19th century, topologically-based homological methods had been carried into algebraic systems, such as the homology and the cohomology of groups, Lie algebras... Cartan-Eilenberg in [CE56] summarized all developments done so far in Algebra as well as they brought some innovative ideas. For instance, projective modules, left exact functors, the torsion functor... However, it was foreseen that there would be a more general setting which includes the sheaf cohomology. Abelian categories, given by [Buc55], were involved in the search for that setting.

Verdier's thesis, [Ver96], under Grothendieck's supervisory, supplied another approach to the derived functor through the derived category. The derived category $\mathbf{D}(\mathcal{A})$ of an abelian category $\mathcal{A}$ is the category obtained from the category $\mathbf{C}(\mathcal{A})$ of complexes by formally inverting quasi-isomorphisms. So if $\mathcal{F} : \mathcal{A} \to \mathcal{C}$ is an additive functor then, under some reasonable conditions, there is a functor $R\mathcal{F} : \mathbf{D}(\mathcal{A}) \to \mathbf{D}(\mathcal{C})$ with the property that if $A \in \mathcal{A}$ is considered as a complex, then the cohomology of the complex $R\mathcal{F}(A)$ gives the ordinary right derived functors.

By the works [Hoc45], [Hel58], [Buc59], [EM65], Relative Homological Algebra took its way. In this theory, not only Ext functor but also its subfunctors are studied. So projective and injective resolutions are replaced by relative projective and injective objects and proper epimorphisms and monomorphisms.

These three steps that have just been mentioned led to new developments in Homological Algebra and new areas of study which continue to this day. This thesis is built on three topics in the realm of (Relative) Homological Algebra: Cartan-Eilenberg categories, purity and Gorenstein projective resolvents.

Cartan-Eilenberg categories: As pointed out above, Cartan-Eilenberg in [CE56] introduced how to generate a family of functors $\{\mathcal{F}_n\}_{n\in\mathbb{Z}}$ from an additive functor $\mathcal{F}$ between module categories. The slow and elementary iterative procedure leads to the notion of *satellite functors*. The other one, which is based on resolutions, gives rise to the *derived functors*. In case of Hom and $\otimes$, both of them coincide. For the latter one, for example, a deleted projective resolution P_M of a module M is a non-negative complex of projective modules with a quasi-isomorphism $P_M \to M$. Here M is thought as a complex whose 0th component is M and 0 anywhere else. Because of the property of projective modules, a deleted projective resolution of a module is unique up to homotopy, which permits to get isomorphic homology. Besides, the internal hom functor $\mathcal{H}om(P_M, -)$ on the category of complexes of modules preserves quasi-isomorphisms. Such complexes are called *K-projective complexes*. The module M and its deleted projective resolution P_M are isomorphic in $\mathbf{D}(R\text{-Mod})$. Early works on derived categories were often restricted to either bounded or bounded below complexes because of the need to work with projective resolutions. Indeed, for a category with enough projectives, it is a classical result that every bounded below complex A has a resolution by a bounded below complex of projectives, that is, a quasi-isomorphism $P \to A$ with a bounded below complex P of projectives. So it makes possible to extend any additive functor $\mathcal{F} : \mathcal{A} \to \mathcal{C}$ to $\mathbf{D}_+(\mathcal{A}) \to \mathbf{D}_+(\mathcal{C})$.

In [Spa88], Spaltenstein showed that for a category having exact direct limits and enough projectives, each unbounded complex is quasi-isomorphic to a K-projective complex and that one could use K-projective complexes to compute derived functors.

Another approach to the problem of extending certain (non)-additive functors to derived categories was given by [Qui67]. There, he introduced the model structure. Basically, in case of the category of complexes over an abelian category $\mathcal{A}$, if one finds a model structure on $\mathbf{C}(\mathcal{A})$ whose weak equivalences are quasi-isomorphisms, then its derived category exists and the derived functor is given in terms of fibrant-cofibrant replacements. But somehow axioms of the model structure are quite strong which excludes some interesting categories.

In [GNPR10], it was purposed another approach to homological-homotopical algebra by taking into account the role of K-projective complexes in derived categories. The initial data is a category $\mathcal{C}$ with a pair (S, W) of classes of morphisms such that S is included in the saturation, $\overline{W}$, of W. Morphisms in S and W are called strong and weak equivalences, respectively. An (S, W)-cofibrant object is an object $X \in \mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{C}[S^{-1}]}(X, w)$ is bijective for all $w \in W$. And a left Cartan-Eilenberg category is a category with weak and strong equivalences and enough cofibrant objects. In other words, its localization at weak equivalences is equivalent to the full subcategory of the localization at strong equivalences which consists of cofibrant objects.

Let us embody it by a basic example. Consider the category of complexes $\mathbf{C}(R\text{-Mod})$ over R-Mod. If we take the class of homotopy equivalences and quasi-isomorphisms as strong and weak equivalences, (S, W), respectively, then the homotopy category $\mathbf{K}(R\text{-Mod})$ and the derived category $\mathbf{D}(R\text{-Mod})$ are localizations of $\mathbf{C}(R\text{-Mod})$ at strong and weak equivalences, respectively. Besides, K-projective complexes are precisely (S, W)-cofibrant objects. Then $(\mathbf{C}(R\text{-Mod}), S, W)$ is a left Cartan-Eilenberg category because for each complex A, there is a quasi-isomorphism $P \to A$, with P a K-projective complex.

As seen in the previous paragraph, the idea of a Cartan-Eilenberg category is to approach to the localization at weak equivalences through the relative localization of the subcategory of cofibrant objects with respect to strong equivalences. So if strong equivalences are good enough to manage, then to handle the localization at weak equivalences would be easier. It is the more general formalism of the conventional idea which lays in the background of Cartan-Eilenberg's approach and the derived functor. This is why in [GNPR10] they call it like a Cartan-Eilenberg category. Besides, each Quillen model structure produces a Cartan-Eilenberg category but the converse need not be true, that is, there are Cartan-Eilenberg categories which don't come from any Quillen model categories, see Proposition 4.3.6 and [GNPR10, Example 6.2.7]. It enables also to extend the classical theory of deriving functors for even non-additive functors.

Purity: The simplest morphisms in a module category are sections, that is, direct summands. The concept of pure submodules provides a nice generalization of direct summands, a type of particularly well-behaved piece of a module. For example, maybe we may not extend any submodule to a direct summand with a cardinality restriction, but there is a kind of result which allows us to do it as a pure submodule, see Theorem 5.1.6.

Pure submodules are given equivalently in terms of Hom and $\otimes$ functors as well as they are precisely direct limit of sections. So it supplies the basic nontrivial example of Relative Homological Algebra, known as Pure Homological Algebra. There is also another equivalent definition by using linear system. There are several works from this perspective, see for example [Pre09], but we won't focus on this point of view.

The reason why purity takes an important place in Relative Homological Algebra is its efficient use in Approximation Theory. The first example possibly appeared in Hilbert's syzygy theorem in [Hil90] as free resolutions. The classical resolutions are injective and projective resolutions given in [CE56]. In the lack of projective and injective objects in a category, Approximation Theory looks for resolutions by possibly nice classes. Here, the concepts that we need in order to have suitable resolutions are (pre)covering and (pre)enveloping classes. These constitute the heart of this thesis. So, let us explain it a bit more. A class $\mathcal{F}$ of objects in an abelian category $\mathcal{C}$ is called precovering if for every object $X \in \mathcal{C}$ there is a morphism $F \to X$ with $F \in \mathcal{F}$ such that every morphism $F' \to X$, $F' \in \mathcal{F}$, has a factorization $F' \to F \to X$. In order to be covering, for each object $X \in \mathcal{C}$ there must be an $\mathcal{F}$-precovering morphism $g : F \to X$ such that in case $g \circ \varphi = g$, φ is an isomorphism. A precovering class $\mathcal{F}$ permits an object X to have a left $\mathcal{F}$-resolution which is unique up to homotopy. Its dual notion is a (pre)enveloping class. These are closely related with cotorsion pairs, deconstructible classes and abelian model categories, [Hov02], [Sto13a], [Sal79], [Eno12].

Most of the classes of our interest are closed under pure subobjects and pure quotients, for example see [Pin08], [BBE01], [EE05], [ET01], [RS98], [CPT10], [Gill04], [Gill06] and for more general relations between (pre)envelope-(pre)cover and purity see [Kra12] and [HJ08]. As an example, it is unlikely that the category of quasi-coherent sheaves $\mathfrak{Qcoh}(X)$ on a scheme X has enough projectives, but the class of flat quasi-coherent sheaves, $\mathcal{F}lat(X)$, is a covering class, see [EE05] by the purity argument. In [Gill07], Gillespie showed that under certain conditions on the scheme X, there is a monoidal model category structure on $\mathbf{C}(\mathfrak{Qcoh}(X))$ where a left $\mathcal{F}lat(X)$-resolution of a quasi-coherent sheaf is a cofibrant replacement. This results in that the Ext functor, which is calculated by injective resolutions in $\mathfrak{Qcoh}(X)$, can be computed through $\mathcal{F}lat(X)$-resolutions.

The purity in R-Mod has several extensions in different kind of categories. But the common point is that each one has a sort of enough pure-subobject result as mentioned at the beginning for R-Mod. For example, the appropriate purity in $\mathfrak{Qcoh}(X)$ is given on stalks as used in $\mathcal{F}lat(X)$. There is also a categorical concept of purity by using Hom functor and the categorical definition of finitely presentable object. It was shown by Crawley-Boevey in [Craw94] that locally finitely presentable additive categories were the most general additive setup to define a good purity theory. We recall that a sequence in a locally finitely presentable additive category is said to be pure whenever it is projectively generated by the class of finitely presentable objects. For some regular cardinal λ, locally λ-presentable categories, which include Grothendieck categories, provides a general setting in which well-behaved λ-pure subobjects are treated, see [AR94, Section 2]. For compactly generated triangulated categories, in [Bel00], compact objects are regarded as an analogous of finitely presentable objects, so pure triangles are defined through compact objects.

Gorenstein Homological Algebra: In a Bourbaki talk on Duality, Grothendieck defined a commutative ring R of finite type over a field to be Gorenstein if it is Cohen-Macaulay and a certain R-module is a locally free of rank 1. In [Bas62], Bass described rings of finite self injective dimension. As it is noted in the introduction of [Bas63], it was observed by Serre that both concepts coincide in a geometric context. And finally, the modern definition of Gorenstein rings and its equivalent condition was given in [Bas63]: a commutative Noetherian ring R is

Gorenstein if for each prime ideal p, $\mathrm{id}_{R_p} R_p < \infty$. This definition is mostly used in Algebraic Geomerty.

In [Aus67], Auslander introduced a new class of modules contained in the class of finitely generated modules, the so-called modules of G-dimension 0. And then the G-dimension was defined for finitely generated modules using modules having G-dimension 0. Later, the theory was developed in [AB69]. The nice part of this new invariant is that it is weaker than projective dimension and it satisfies some properties of projective dimension. The theory caught the attention of homological algebraists. But it was defined only for finitely generated modules and had good behaviour over some special rings.

A more general approach to the theory was given in [EJ95]. There, the authors defined the notion of 'Gorenstein projective' modules for any ring by using totally acyclic complexes. It is a generalization of finitely generated modules having G-dimension 0 in Auslander's sense. It is known that over a Noetherian ring, a finitely generated module M is Gorenstein projective if and only if it has G-dimension 0 (see [Chr00, Theorem 4.1.4]). The advantage of that generalization is also that it allows one to define new notions such as Gorenstein injective and Gorenstein flat modules. The work of [EJ95] gave a new perspective in the theory. As a result, a new branch in homological algebra which is known as Gorenstein homological algebra emerged. Several people have worked intensively in this area and have studied its relation with Tate cohomology (for example; [AM02], [EJT93], [Chr00], [Iac05], [EJL04]). Roughly speaking, the main problem in this branch is to get Gorenstein counterparts of Homological Algebra for a general ring. But even the existence of Gorenstein projective precovers over any ring isn't known yet. Several attempts were made to ease the condition of being Gorenstein ring, see ([Hol04], [Jør07]). In [Jør07], Jørgensen found out the relation between dualizing complex and Bousfield localization for totally acyclic complexes. So he proved that the class of Gorenstein projective modules is precovering over a ring which has a dualizing complex. The result extends the class of rings on which the existence of Gorenstein projective precovers is known.

Now, we shall summarize the contents of this memory.

Chapters (I), (II), (III) are the introductary parts of the memory. We recall some basic terminologies and results which will be used in the rest. Needless to say, it is not claimed originality in none of them.

Chapter IV

The main subject of this chapter is left Cartan-Eilenberg categories. We briefly mention the localization of categories and derived categories and we recall basic properties of K-projectives in the homotopy and the derived category. Later, we focus on left Cartan-Eilenberg categories $(\mathcal{A}, S, W)$ with their basic properties. We point out that (S, W)-cofibrat objects and K-projectives have similar properties. This concept of categories has a strong relation with the co-orthogonal subcategory problem for $\mathcal{A}[S^{-1}]$ while abelian model categories are characterized by two complete cotorsion pairs, the so-called Hovey pairs. To solve whether $(\mathcal{A}, S, W)$ is a left Cartan-Eilenberg category is the same as proving that the subcategory $\overline{\mathcal{A}}_{cof}$ consisting of (S, W)-cofibrant objects is a coreflective subcategory of $\mathcal{A}[S^{-1}]$.

In [Pas12], the author gives an example and a counterexample in $\mathbf{C}_+(\mathcal{A})$ to left Cartan-Eilenberg categories depending on whether $\mathcal{A}$ has enough projectives. In the main result of this chapter, we give a machinery through cotorsion pairs to produce left Cartan-Eilenberg categories, which includes that example. And even when $\mathbf{C}_+(\mathcal{A})$ is not a left Cartan-Eilenberg category like in the counterexample, this method will allow to produce some nontrivial left Cartan-Eilenberg subcategories. The precise formulation is the following:

Theorem 4.4.5. Let $(\mathcal{F}, \mathcal{B})$ be a cotorsion pair in an abelian category with enough $\mathcal{F}$ objects. Suppose that $\mathcal{F}$ is a resolving class, $dg\,\tilde{\mathcal{B}} \cap \mathrm{Acic} = \tilde{\mathcal{B}}$ and the induced cotorsion pair $(dg\,\tilde{\mathcal{F}}, \tilde{\mathcal{B}})$ has enough projectives. Then $(\mathbf{C}_+(\mathcal{A}) \cap dg\,\tilde{\mathcal{B}}, S', W')$ is a left Cartan-Eilenberg category.

We finish by illustrating the applicability of Theorem 4.4.5 in different settings, including categories of sheaves and modules over Gorenstein rings.

Chapter V

This chapter is the initiation of the series on topics in Pure Homological Algebra from various aspects. It can be divided into two parts: a general theory about geometric-categorical pure morphisms and purity in the category of quasi-coherent sheaves. We begin by recalling λ-pure morphisms in a locally λ-presentable category, which we call categorical pure. We define the geometrical pure monomorphism when a category has a closed symmetric monoidal structure. Shortly, it is a monomorphism which is preserved under the tensor product by any object. We look at its relation with the categorical pure monomorphisms. In case of a Grothendieck category with a closed symmetric monoidal structure, we observe that the categorical one implies the geometrical purity. This observation leads to several remarkable results that will be mentioned in the subsequent chapters. This chapter is basically dedicated to prove the existence of the pure injective (pre)envelopes depending on the purity and the category, which provides the necessary tool to proceed Pure Homological Algebra.

Theorem 5.2.6. Let $\mathcal{C}$ be a closed symmetric monoidal an Grothendieck category. Then every object can be (geometric) purely embedded in a geometric pure-injective object, that is, every object has a geometrical pure-injective preenvelope.

In the third section, we zoom in the categories $\mathcal{O}_X$-Mod and $\mathfrak{Qcoh}(X)$ where we face with another purity notion. Both categories are Grothendieck categories with closed symmetric monoidal structure. Once we focus in O_X-Mod, we see that the property of geometrical purity is exactly hidden in stalks. Moreover, the intersection of pure monomorphisms in $\mathcal{O}_X$-Mod with $\mathfrak{Qcoh}(X)$ provides a new class of short exact sequences, which are characterized on the sections over affine open subsets. We call that new notion of short exact sequences to be stalkwise pure-exact. Somehow stalkwise pure short exact sequences in $\mathfrak{Qcoh}(X)$ are more natural than the geometrical one because it serves as the complementary of flat quasi-coherent sheaves. After investigating the relation of stalkwise purity with the categorical pure, we get the result which guarantees the minimal stalkwise pure injective resolutions in $\mathfrak{Qcoh}(X)$ without assuming any condition on the scheme:

Theorem 5.4.8. Let X be any scheme. Each quasi-coherent sheaf in $\mathfrak{Qcoh}(X)$ has a stalkwise pure injective envelope which is a stalkwise pure monomorphism.

Chapter VI

This chapter is about some classes of quasi-coherent sheaves concerning to the stalkwise purity and their homological properties. The claim is to carry some known results in R-Mod into $\mathfrak{Qcoh}(X)$.

The class $\mathcal{F}lat(X)$ of flat quasi-coherent sheaves on a scheme X has been extensively used during the last years, as a natural choice for studying both the homotopy category and the derived category of quasi-coherent sheaves ([EGPT12], [Gill07], [Hov01], [MS11], [Mur07], [HS13]). As mentioned before, it was proved in [EE05] that $\mathcal{F}lat(X)$ is a covering class in $\mathfrak{Qcoh}(X)$ by using stalkwise purity.

Given an associative ring R with unit, a left R-module M is absolutely pure if every finite system of linear equations whose independent terms lie in M possesses a solution in M. This is equivalent to saying that M is a pure submodule of any R-module that contains it. In some aspects these behave like injective R-modules (see [Mad67, Meg70, Pre09, Ste70] for a general treatment of absolutely pure modules and [Pin05] for a revisited study). In fact, Noetherian rings can be characterized in terms of properties of absolutely pure modules. Namely, R is Noetherian if and only if the class of absolutely pure R-modules coincides with the class of injective R-modules ([Meg70]). We will exhibit the main properties of (locally) absolutely pure sheaves of modules, both in $\mathfrak{Qcoh}(X)$ and in $\mathcal{O}_X$-Mod, in case X is locally coherent scheme. For instance, we show in Proposition 6.2.7 that local absolutely purity in $\mathfrak{Qcoh}(X)$ can be checked on a particular affine covering of X. And we also see that locally absolutely pure quasi-coherent sheaves are precisely the absolutely pure $\mathcal{O}_X$-modules that are quasi-coherent, see Lemma 6.2.8. This is analogous to the question posted in [Hart77, II, §7, pg.135] for locally Noetherian schemes (cf. [Con00, Lemma 2.1.3]). Then we characterize locally Noetherian closed subschemes of the projective space $\mathbb{P}^n(A)$ (A a commutative and coherent ring) in terms of its class of absolutely pure quasi-coherent sheaves:

Proposition 6.2.11. A closed subscheme $X \subseteq \mathbb{P}^n(A)$ is locally Noetherian if and only if every locally absolutely pure quasi-coherent sheaf is locally injective.

If X is locally Noetherian scheme, it is known that the class of locally injective quasi–coherent sheaves is covering in $\mathfrak{Qcoh}(X)$. We extend this result to the class of locally absolutely pure quasi-coherent sheaves on a locally coherent scheme X.

Theorem 6.2.13. Let X be a locally coherent scheme. Then every quasi–coherent sheaf in $\mathfrak{Qcoh}(X)$ admits a locally absolutely pure cover.

As pointed out before, $\mathfrak{Qcoh}(X)$ is a Grothendieck category, so products always exist in it. However it is hard to know an explicit description of this object because, at the level of sections, the product of modules is not well-behaved in general with respect to localizations, or more generally, when tensoring by an arbitrary module with respect to a commutative ring (direct products do not commute with tensoring in general). But even though the tensor product does commute with products with respect to finitely presented modules, it is not clear whether the product object in $\mathfrak{Qcoh}(X)$ can be computed from the product module of sections at each affine open set if we do not impose extra assumptions on the sheaf of rings $\mathcal{O}_X$ attached to X (for instance if $\mathcal{O}_X(U)$ is finitely presented as $\mathcal{O}_X(V)$-module, for each affine open subsets

$U \subseteq V$).

The lack of an explicit description of the product object leads to new and relevant questions on the class $\mathcal{F}lat(X)$ of flat quasi-coherent sheaves on X. For instance, Murfet in [Mur07, Remark B.7] raises the question of whether $\mathcal{F}lat(X)$ is closed under products, for X a Noetherian scheme. This property is crucial to showing that in $\mathbf{C}(A\text{-Mod})$, the category of unbounded chain complexes of A-modules (A commutative Noetherian ring), the complex $\mathcal{H}om_A(I, I')$ is a complex of flat modules, for injectives $I, I' \in \mathbf{C}(A\text{-Mod})$. We point out that the usual notion of flatness in $\mathfrak{Qcoh}(X)$ is not categorical, as it is shown in [ES12], (see also [Rum10]). Recently Saorín and Šťovíček in [SS11, 4.2] have given a positive answer to this question for Dedekind schemes. In their argument they use Crawley-Boevey's characterization of preenveloping subcategories of the category of finitely presentable objects in a locally finitely presentable additive category with products (see [Craw94, Theorem 4.2]). So then they show that if X is Dedekind, the category of locally free coherent sheaves (the vector bundles) is preenveloping in the category of all coherent sheaves, obtaining as a byproduct that its closure under direct limits, the class $\mathcal{F}lat(X)$, is closed under products.

If X is affine, there is a canonical equivalence between $\mathcal{F}lat(X)$ and the class $\mathcal{F}lat(R)$ of flat R-modules, where $X = Spec(R)$. Now if X is also Dedekind it is well known that $\mathcal{F}lat(R)$ coincides with the class of torsion-free R-modules. So, for an arbitrary scheme, it makes sense to define the class $\mathcal{F}$ of locally torsion-free quasi–coherent sheaves as the class of $\mathscr{F} \in \mathfrak{Qcoh}(X)$ such that $\mathscr{F}(U)$ is a torsion-free $\mathcal{O}_X(U)$-module, for each affine open set U. This class contains $\mathcal{F}lat(X)$ in general, and indeed it coincides with it for Dedekind schemes.

Theorem 6.3.7. Let X be an integral scheme. The product $\mathscr{F}$ of a family $\{\mathscr{F}_i\}_{i\in I}$ of locally torsion-free quasi-coherent sheaves in $\mathfrak{Qcoh}(X)$ is the largest quasi-coherent subsheaf of $\prod_{i \in I} \mathcal{F}_i$. More concretely, it is of the form

$$\mathcal{F} = \sum_{\substack{\mathcal{M} \in \mathfrak{Qcoh}(X) \\ \mathcal{M} \subseteq \prod_{i \in I} \mathcal{F}_i}} \mathcal{M}.$$

One of the consequences of this theorem is that, for an integral scheme, the class $\mathcal{F}$ induces a hereditary torsion theory.

As mentioned before, flat covers in $\mathfrak{Qcoh}(X)$ exist by [EE05, Theorem 4.1]. We show the existence of covers with respect to the class $\mathcal{F}$. This was known from the sixties in case X is integral and affine due to [Eno63].

Theorem 6.4.7. Each quasi-coherent sheaf on an integral scheme has a locally torsion-free cover.

Chapter VII

This chapter is the final chapter on Pure Homological Algebra. We keep working on the general setting of a closed symmetric monoidal Grothendieck category. As the last aim, we deal with the (geometric) pure derived category, specially that of a scheme.

If $\mathcal{A}$ is a locally finitely presentable category, categorical purity defines a pure exact structure in $\mathcal{A}$ and yields the pure derived category $\mathbf{D}_{\text{pur}}(\mathcal{A})$ studied for example by Christensen and Hovey [CH02] and Krause [Kra12]. Recently in [Gil14] it has been shown that this pure derived category can be obtained as the homotopy category of two model category structures by using the pure projectives and the pure injectives. However, unless the scheme X is affine, the categorical purity defined above for locally finitely presentable categories does not coincide with the stalkwise purity on $\mathfrak{Qcoh}(X)$. In the next result, we prove that the derived category relative to $\otimes$-purity (= geometrical purity) comes up as a homotopy category of a model category.

Theorem 7.2.7. Let $\mathcal{C}$ be a closed symmetric monoidal Grothendieck category and $\mathbf{C}(\mathcal{C})$ the associated category of chain complexes. Then there is a cofibrantly generated model category structure on $\mathbf{C}(\mathcal{C})$ whose trivial objects are the $\otimes$-pure acyclic complexes; that is, complexes X for which $X \otimes S$ is exact for all $S \in \mathcal{C}$. The model structure is exact with respect to the exact category $\mathbf{C}(\mathcal{C})_{\otimes}$ of chain complexes along with the proper class of degreewise $\otimes$-pure exact sequences. In fact, the model structure is injective in the sense that every complex is cofibrant and the trivially fibrant complexes are the injective objects of $\mathbf{C}(\mathcal{C})_{\otimes}$, which are precisely the contractible complexes with $\otimes$-pure injective components. We call this model structure the $\otimes$-*pure injective model structure* on $\mathbf{C}(\mathcal{C})$ and its corresponding homotopy category is the $\otimes$-*pure derived category*, denoted $\mathbf{D}_{\otimes\text{-pur}}(\mathcal{C})$.

As a particular instance of the previous theorem, we get by applying Proposition 7.2.8, the following application to $\mathfrak{Qcoh}(X)$:

Corollary 7.2.9. Let X be a quasi-separated scheme. Let $\mathcal{E}$ be the exact structure coming from the stalkwise-purity in $\mathfrak{Qcoh}(X)$, and let us consider the category of unbounded complexes $\mathbf{C}(\mathfrak{Qcoh}(X))$. Then with respect to the induced degreewise exact structure from $\mathcal{E}$, there is an exact and injective model category structure on $\mathbf{C}(\mathfrak{Qcoh}(X))$. The corresponding homotopy category is the stalkwise-pure derived category (or geometric pure derived category), which we denote $\mathbf{D}_{\text{stk-pure}}(\mathfrak{Qcoh}(X))$.

Having two different notions of purity in a general closed symmetric monoidal Grothendieck category, the categorical purity and the geometrical purity, it is natural to ask what relationship there is between them. In Proposition we show that there is a canonical functor from the λ-pure derived category $\mathbf{D}_{\lambda\text{-pur}}(\mathcal{C})$ to the $\otimes$-pure derived category $\mathbf{D}_{\otimes\text{-pur}}(\mathcal{C})$, which admits a right adjoint.

In [MS11] Murfet and Salarian define what they call the pure derived category of flat sheaves for a semi-separated Noetherian scheme, as the Verdier quotient of the homotopy category of flat sheaves $\mathbf{K}(\mathcal{F}lat(X))$ with the localising subcategory $\mathbf{K}_{\text{pac}}(\mathcal{F}lat(X))$ of the pure acyclic complexes of sheaves. Flat modules are intimately related with locally finitely presentable categories due to Crawley-Boevey's Representation Theorem [Craw94]. This establishes that every additive locally finitely presentable additive category $\mathcal{A}$ is equivalent to the full subcategory $\mathcal{F}lat(A)$ of $\text{Mod-}A$ of unitary flat right A-modules, where A is the functor ring of $\mathcal{A}$ and the equivalence gives a 1-1 correspondence between pure exact sequences in $\mathcal{A}$ and exact sequences in $\mathcal{F}lat(A)$. This equivalence lifts to the level of model structures as well and in particular to the derived categories, so we get the following:

Theorem 7.4.3. Let $\mathcal{A}$ be a locally finitely presentable additive category and let $\mathcal{F}lat(A)$ be its equivalent full subcategory of flat modules in $\text{Mod-}A$. Then $\mathbf{D}_{\text{pur}}(\mathcal{A})$ is equivalent to $\mathbf{D}(\mathcal{F}lat(A))$, the homotopy category of the injective exact model category structure on $\mathbf{C}(\mathcal{F}lat(A))$.

Chapter VIII

Now, we change our route into Gorenstein Homological Algebra. As mentioned before, the main problem in the theory is to translate some basic properties of Homological Algebra into Gorenstein Homological Algebra. In the last chapter, the subject is Gorenstein projective modules. The main idea is to handle topics related to right resolutions by using Gorenstein projective modules. So firstly we begin by dealing with the existence of Gorenstein projective preenvelopes for finitely generated modules over a local n-Gorenstein ring. As a matter of fact, in [HJ11] it was already proved that over a local Noetherian ring R, the class of finitely generated Gorenstein projective modules is preenveloping if and only if R is Gorenstein.We give here an alternative proof of this fact:

Theorem 8.2.8. Let R be a local n-Gorenstein ring. Every finitely generated module M has a Gorenstein projective preenvelope.

The reason why we reprove it is that our proof enables us to compute right Gorenstein projective resolutions of finitely generated modules. So we get the next result.

Theorem 8.2.9. Let R be a local n-Gorenstein ring. If M is a finitely generated R-module then M has a finite right Gorenstein projective resolution.

The next claim is to investigate certain relative left derived functors of Hom(-, -). Over a commutative local Noetherian ring, the existence of a right projective resolution of a finitely generated module is guaranteed. If R is additionally n-Gorenstein, its coszygies are eventually Gorenstein projective. So over such a ring, any finitely generated module M gives us a totally acyclic complex of projectives, in other words, a complete projective resolvent $M \to \mathbf{P} \to \mathbf{T}$ with $\mathbf{T}$ a totally acyclic complex of projectives. We denote by $\widehat{\mathrm{Ext}}_i(-,-)$ the left derived functor of Hom(-, -) by taking a totally acyclic complex appearing in some right projective resolution of the first component. This is called the Tate derived functor. In the next result, we get a balance result on Tate homology, that is, $\widehat{\mathrm{Ext}}_i(-,-)$ can be computed by taking a totally acyclic complex arising from in the right projective resolution of the first component, or from a left projective resolution of the second component.

Theorem 8.3.2. Let R be a local n-Gorenstein ring and M, N be R-modules where M is finitely generated. If $M \to \mathbf{P} \to \mathbf{T}$ is a complete projective resolvent of M and $\mathbf{T}' \to \mathbf{P} \to N$ is a complete projective resolution of N then $\widehat{\mathrm{Ext}}_i(M,N) := H_i(\mathrm{Hom}(\mathbf{T}, N)) \cong H_i(\mathrm{Hom}(M, \mathbf{T}'))$.

Other left derived functors which are of our interest are $\mathrm{Ext}_i(-,-)$ and $\mathrm{Gext}_i(-,-)$. These are obtained by using a right projective resolution and a right Gorenstein projective resolution of the first component, respectively. In the next result, we get an Avramov-Martsinkovsky-type exact sequence connecting the left derived functors involving $\mathrm{Ext}_i(-,-), \mathrm{Gext}_i(-,-)$ and $\widehat{\mathrm{Ext}}_i(-,-)$ just as in [AM02].

Theorem 8.3.3. Let R be a local n-Gorenstein ring. Let M be a finitely generated R-module and N be an R-module. Then we have an exact sequence of left derived functors

$$0 \to \widehat{\text{Ext}}_{n-1}(M,N) \to \text{Ext}_{n-1}(M,N) \to \text{Gext}_{n-1}(M,N) \to \widehat{\text{Ext}}_{n-2}(M,N) \to \dots$$

$$\dots \to \text{Gext}_0(M,N) \to \widehat{\text{Ext}}_{-1}(M,N) \to 0.$$

CHAPTER ONE

CATEGORIES

This chapter is intended to be a notebook on categories by the author. It is introductory to category theory, so it contains very basic concepts and very well-known facts. Readers who are familiar to the language of categories may skip this chapter.

1.1 Categories

Let us begin by giving the definition of a category.

Definition 1.1.1. A *category* $\mathcal{C}$ consists of

C1) a class of '*objects*', $Ob(\mathcal{C})$,

C2) a set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of '*morphisms*', denoted by $f : X \to Y$, for each pair of objects $X, Y \in Ob(\mathcal{C})$,

C3) and a collection of mappings

$$\circ_{XYZ} : \mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$$

for each triple $X, Y, Z \in Ob(\mathcal{C})$. It is common to drop subscript expressions and to write $g \circ f$ for the image of $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$. These binary operations are called *compositions*.

The data must satisfy the following:

C4) For any $X \in Ob(\mathcal{C})$, there exists a kind of '*identity morphism*' $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$ with regard to $\mathrm{id}_X \circ f = f$ and $g \circ \mathrm{id}_X = g$ wherever these are defined.

C5) The composition '$\circ$' is associative, that is,

$$(f \circ g) \circ h = f \circ (g \circ h)$$

for any $h \in \text{Hom}_{\mathcal{C}}(X, Y)$, $g \in \text{Hom}_{\mathcal{C}}(Y, Z)$ and $f \in \text{Hom}_{\mathcal{C}}(Z, U)$.

For any morphism $f \in \text{Hom}_{\mathcal{C}}(X, Y)$, X and Y are called *domain* and *codomain* of f, respectively. We should point out that morphisms are not necessarily functions. It is just a set of some directed relations from its domain to codomain. So some authors prefer to call them 'arrows' instead of 'morphisms'. In fact, a morphism is intuitively referred to a structure-preserving mapping between two mathematical structures, because of etymological reasons of the word 'morph'. Many of main categories worked on have sets as objects and structure-preserving functions as morphisms. For instance,

Set : the category of all sets with functions,

Grp : the category of all groups with group homomorphisms,

Ab : the category of all abelian groups and homomorphisms,

R-**Mod** : the category of all left R-modules with R-module homomorphisms,

Ring : the category of all rings with ring homomorphisms,

Top : the category of all topological spaces with continuous maps.

But now in category theory, the use of the terminology 'morphism' is much more extended to that of functions.

A *subcategory* $\mathcal{C}'$ of a category $\mathcal{C}$ is a category whose class of objects, $Ob(\mathcal{C}')$, is a subclass of $Ob(\mathcal{C})$ with $\text{Hom}_{\mathcal{C}'}(X, Y) \subseteq \text{Hom}_{\mathcal{C}}(X, Y)$, for all $X, Y \in \mathcal{C}'$, and whose identity morphism id_X for each object $X \in \mathcal{C}'$ is the same as in $\mathcal{C}$.

For any categorical concept, there is a dual concept obtained by reversing the direction of morphisms in the definition. As a consequence, for every category $\mathcal{C}$, *the dual*, or *opposite*, category $\mathcal{C}^{op}$ of $\mathcal{C}$ has the same class of objects as $Ob(\mathcal{C})$ and $\mathrm{Hom}_{\mathcal{C}^{op}}(X,Y) := \mathrm{Hom}_{\mathcal{C}}(X,Y)$ for any pair of objects $X, Y \in \mathcal{C}$, that is, a morphism goes into $\mathcal{C}^{op}$ after switching its domain and codomain. It is easy to see $(\mathcal{C}^{op})^{op} = \mathcal{C}$.

Besides, any property or a categorical statement has a dual version. For example, if $\mathcal{P}_{\mathcal{C}}$ is a statement for a category $\mathcal{C}$, then $(\mathcal{P}^{op})_{\mathcal{C}}$ is the corresponding property of $\mathcal{C}^{op}$ phrased as a property of $\mathcal{C}$, that is, $(\mathcal{P}^{op})_{\mathcal{C}} := \mathcal{P}_{\mathcal{C}^{op}}$. Then we have:

Duality Principle: If a property $\mathcal{P}$ holds for all categories then $\mathcal{P}^{op}$ holds for all categories, as well.

For a category $\mathcal{C}$, the morphism class, $\mathrm{Mor}(\mathcal{C})$, is the disjoint union of morphism sets $\mathrm{Hom}_{\mathcal{C}}(X,Y)$, for all $X, Y \in \mathcal{C}$. For any pair of morphisms $f : X \to Y$ and $f' : X' \to Y'$, a morphism from f to f' is a pair (g,h), $g : X \to X'$ $h : Y \to Y'$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{g} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow{h} & Y' \end{array}$$

commutes. Let $(s,t) : f'' \to f$ and $(g,h) : f \to f'$ be morphisms in $\mathrm{Mor}(\mathcal{C})$. Then $(g,h) \circ (s,t) := (g \circ s, h \circ t)$ gives a composition in $\mathrm{Mor}(\mathcal{C})$. So $\mathrm{Mor}(\mathcal{C})$ is a category which is called *the arrow category for* $\mathcal{C}$. A category $\mathcal{C}$ is called *small* if $Ob(\mathrm{Mor}(\mathcal{C}))$ is a set. It is called λ-*small* for some regular cardinal λ if the cardinality of $Ob(\mathrm{Mor}(\mathcal{C}))$ is less than λ. In a special case, an $\aleph_0$-small category is called *finite*.

Definition 1.1.2. An equivalence relation $\sim$ on the class of morphisms of a category $\mathcal{C}$ is called a *congruence* on $\mathcal{C}$ if

(i) every equivalence class is contained in $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ for some $X, Y \in \mathcal{C}$,

(ii) $\sim$ is closed under composition, that is, if $f \sim f'$ and $g \sim g'$ then $g \circ f \sim g' \sim f'$ when compositions are defined.

Definition 1.1.3. Let $\mathcal{C}$ be a category with a congruence $\sim$. *The quotient category of $\mathcal{C}$ with respect to $\sim$, $\mathcal{C}/_\sim$,* is a category whose objects are the same as objects of $\mathcal{C}$ and whose morphisms are equivalence classes, $[f]$, that is, $\mathrm{Hom}_{\mathcal{C}/_\sim}(X, Y) := \mathrm{Hom}_{\mathcal{C}}(X, Y)/_\sim$ with composition law $[g] \circ [f] := [g \circ f]$ when defined.

The main examples of congruences and quotient categories are the homotopy relation on **Top** with homotopy category of topological spaces and its counterpart in algebra, the homotopy relation on the category of complexes $\mathbf{C}(\mathcal{C})$ with homotopy category $\mathbf{K}(\mathcal{C})$ of complexes, which will be detailed in Chapter 2.

Now, we give definitions of certain types of morphisms which play important roles in the theory.

Definition 1.1.4. Let $\mathcal{C}$ be a category and $f : X \to Y$ be a morphism in $\mathcal{C}$. It is said to be

section if there exists a morphism $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$.

retraction if there exists a morphism $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$.

isomorphism if there is a morphism $g : Y \to X$ in $\mathcal{C}$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$.

monomorphism if for any pair of morphisms $g_1, g_2 : A \rightrightarrows X$, the following holds:

$$f \circ g_1 = f \circ g_2 \;\Rightarrow\; g_1 = g_2,$$

epimorphism if for any pair of morphisms $h_1, h_2 : Y \rightrightarrows B$, the following holds:

$$h_1 \circ f = h_2 \circ f \;\Rightarrow\; h_1 = h_2.$$

bimorphism if f is both a monomorphism and an epimorphism.

The following is the list of some basic and known facts. Since retractions and epimorphisms are dual notions of sections and monomorphisms, respectively, we just state some of them for sections and monomorphisms. By duality principle, their dual come freely.

Remark 1.1.5. (i) Each class of morphisms in Definition 1.1.4 is closed under composition.

(ii) If f is an isomorphism, then a morphism g mentioned in the definition is unique. So we denote it $f^{-1} := g$. Furthermore, if f and h are isomorphisms, $(h \circ f)^{-1} = f^{-1} \circ h^{-1}$ when the composition is defined.

(iii) A morphism is an isomorphism if and only if it is both a section and a retraction.

(iv) Every section and isomorphism is a monomorphism.

(v) If $g \circ f$ is a section (monomorphism) then f is a section (monomorphism).

(vi) In $\mathbf{Set}, \mathbf{Grp}, \mathbf{Ab}$ and R-Mod, bimorphisms are precisely isomorphisms. But, for example, in $\mathbf{Top}$ it does not hold.

(vii) For many categories whose objects are sets and morphisms are structure-preserving functions, the notion of monomorphism is equivalent to the injectivity on the underlying set. For example, $\mathbf{Set}$, $\mathbf{Grp}$, $\mathbf{Ab}$, R-Mod, $\mathbf{Ring}$, $\mathbf{Top}$, ... In a category whose objects are sets and whose morphisms are some certain functions, it is clear that a morphism which is injective on the underlying set is a monomorphism. But the converse is not true in general. For instance, in the category of divisible abelian groups with group homomorphisms, the canonical morphism $\sigma : \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ of the additive groups of rational numbers to the quotient group by the group of integers is not injective but it is a monomorphism in that category.

(viii) Dually, in $\mathbf{Set}$, $\mathbf{Grp}$, $\mathbf{Ab}$, R-Mod, $\mathbf{Top}$, the notion of epimorphism is equal to the surjectivity on the underlying set. As mentioned in (vii), it doesn't always happen. For example, in the category of torsion-free abelian groups, a morphism $f : A \to B$ is an epimorphism if and only if the factor group $B/\operatorname{Im}(f)$ is a torsion group. Thus the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is an epimorphism but not surjective.

Let (X, f) and (X', f') be two pairs where $f : X \to Y$ and $f' : X' \to Y$ are monomorphisms in $\mathcal{C}$. $(X, f) \leq (X', f')$ if there is a morphism $h : X \to X'$ such that $f' \circ h = f$, The morphism h is necessarily a monomorphism. So it is an equivalence relation, i.e., $(X, f) \sim (X', f')$ if $(X, f) \leq (X', f')$ and $(X', f') \leq (X, f)$. In other words, (X, f) is equivalent to (X', f') if and only if there is an isomorphism $h : X \to X'$ such that $f' \circ h = f$. An equivalence class $[X, f]$ of a monomorphism f into Y is called a *subobject* of Y. Sometimes, by abuse of notation, we denote a subobject $[X, f]$ of Y as $X \subset Y$. At first sight, it seems that it is not well-defined to define subobjet of a subobject, because a subobject is a pair containing a morphism whose codomain is fixed. But by a small trick, we may get a subobject, that is, if $[T, t]$ is a subobject of X and $[X, f]$ is a subobject of Y, then $[T, f \circ t]$ is a subobject of Y. By light of this knowledge, we may write it as $T \subset X \subset Y$. A category $\mathcal{C}$ is called *well-powered* provided that each object in $\mathcal{C}$ has a representative class of subobjects that is a set.

The dual notion of subobject is a quotient object. Let (f, X) and (f', X') be two pairs where $f : Y \to X$ and $f' : Y \to X'$ are epimorphisms in $\mathcal{C}$. $(f', X') \leq (f, X)$ if there is a morphism $h : X \to X'$ such that $h \circ f = f'$. Again, we get an equivalence relation. (f, X) is said to be *equivalent* to (f', X') if $(f', X') \leq (f, X)$ and $(f, X) \leq (f', X')$. An equivalence class $[f, X]$ is called a *quotient object* of Y. A category $\mathcal{C}$ is called *co-(well-powered)* provided that each object in $\mathcal{C}$ has a representative class of quotient objects that is a set.

Definition 1.1.6. Let $\mathcal{C}$ be a category and X be an object. X is said to be

initial object if $\operatorname{Hom}_{\mathcal{C}}(X, M)$ consists of one element for every $M \in Ob(\mathcal{C})$,

terminal (final) object if $\operatorname{Hom}_{\mathcal{C}}(M, X)$ consists of one element for every $M \in Ob(\mathcal{C})$,

zero object if it is both an initial object and a final object.

It is clear that each of initial, final and zero objects of a category is unique up to isomorphisms, if it exists. If a category $\mathcal{C}$ has the zero object, we denote it by 0.

Proposition 1.1.7. *Let $\mathcal{C}$ be a category with the initial object X and the terminal object Y. Then the following are equivalent:*

(i) *$\mathcal{C}$ has the zero object.*

(ii) *X and Y are isomorphic.*

(iii) *$Hom_{\mathcal{C}}(Y, X) \neq \emptyset$.*

(iv) *$Hom_{\mathcal{C}}(A, B) \neq \emptyset$ for all $A, B \in \mathcal{C}$.*

Proof. The implications $(i \Rightarrow ii \Rightarrow iii)$ are easy. Suppose now (iii). So there is a morphism $f : Y \to X$. Since X is an initial object and Y is a terminal object, take $A \to X \to Y \to Z$, (iv) follows. Assume (iv). Take a morphism f from $\mathrm{Hom}_{\mathcal{C}}(Y, X)$. Then it is an isomorphism. Indeed, the composition of f with the unique morphism $h : X \to Y$ gives $f \circ h = \mathrm{id}_X$ and $h \circ f = \mathrm{id}_Y$ because Y is a terminal object and X is an initial object. □

If a zero object 0 exists, for any objects $X, Y \in \mathcal{C}$, the *zero map* $0 : X \longrightarrow Y$ is the unique morphism $A \to 0 \to B$.

Definition 1.1.8. A category $\mathcal{C}$ is called *preadditive* if for each pair of objects $X, Y \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is endowed with the structure of an abelian group in such a way that the composition of morphisms is biadditive with respect to these structures, that is,

$$g \circ (f_1 + f_2) = g \circ f_1 + g \circ f_2$$

whenever it is defined.

Note that if a category $\mathcal{C}$ is preadditive and has the zero object, a morphism f is a monomorphism in $\mathcal{C}$ if and only if it satisfies the following: $(f \circ g = 0 \;\Rightarrow\; g = 0)$. Its dual is valid for an epimorphism, as well.

1.2 Functors

If there are two mathematical objects, to compare them is always a natural stimulation. Here, the concept that serve as comparison of categories are functors.

Definition 1.2.1. A *functor* F from a category $\mathcal{C}$ to a category $\mathcal{D}$, $F : \mathcal{C} \to \mathcal{D}$, consists of:

- a mapping $Ob(\mathcal{C}) \to Ob(\mathcal{D})$, $X \rightsquigarrow F(X)$,
- a mapping $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ for all $X, Y \in \mathcal{C}$, $f \rightsquigarrow F(f)$

such that F preserves compositions and identity morphisms, that is, $F(g \circ f) = F(g) \circ F(f)$ when $g \circ f$ is defined in $\mathcal{C}$ and $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$. In case that $\mathcal{C}$ and $\mathcal{D}$ are preadditive, F is said to be *additive* if F preserves abelian structures of morphisms, i. e., $F(f + f') = F(f) + F(f')$ where $f, f' \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ for all $X, Y \in \mathcal{C}$.

Sometimes a functor as in the definition is called *covariant* . When a functor F satisfies all conditions above but changes directions of morphisms, it is called '*contravariant*', that is, it takes a morphism $f : X \to Y$ in $\mathcal{C}$ and send it to $F(f) : F(Y) \to F(X)$. In fact, a contravariant functor is nothing but just a covariant functor on the dual category of domain or codomain, i.e. if F is a contravariant functor, then F can be written as a covariant functor from $\mathcal{C}^{op}$ to $\mathcal{D}$ or from $\mathcal{C}$ to $\mathcal{D}^{op}$.

There are many examples of functors, but we just recall some of the basic ones that will be used pretty much in this memory.

Example 1.2.2. (Standard functors) Let $\mathcal{C}$ be any category.

(i) $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$, the identity functor on $\mathcal{C}$, keeps objects and morphisms the same.

(ii) If $\mathcal{C}'$ is a subcategory of $\mathcal{C}$, $\iota : \mathcal{C}' \hookrightarrow \mathcal{C}$ is the canonical inclusion functor that sends each morphism $f \in \mathcal{C}'$ to f in $\mathcal{C}$.

(iii) If $\mathcal{C}/_\sim$ is a quotient category of $\mathcal{C}$, then $Q : \mathcal{C} \to \mathcal{C}/_\sim$ is the canonical functor that assigns to each morphism f its equivalence class $[f]$.

(iv) There is a bifunctor

$$\mathrm{Hom}_{\mathcal{C}}(\text{-},\text{-}) : \mathcal{C}^{op} \times \mathcal{C} \to \mathbf{Set}$$

where $\mathrm{Hom}_{\mathcal{C}}(\text{-},\text{-})(A, B) := \mathrm{Hom}_{\mathcal{C}}(A, B)$ and for $f : A' \to A$ and $g : B \to B'$

$$\mathrm{Hom}_{\mathcal{C}}(f, g) : \ \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A', B')$$

is a function with $\mathrm{Hom}_{\mathcal{C}}(f, g)(h) := g \circ h \circ f$.

Definition 1.2.3. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Consider, for every pair of objects $X, Y \in \mathcal{C}$, the mapping

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y)).$$

The functor F is called

faithful if the mapping mentioned above is injective for all $X, Y \in \mathcal{C}$,

full if the mapping mentioned above is surjective for all $X, Y \in \mathcal{C}$,

fully faithful if it is both full and faithful.

Now we return to observing the set valued hom functors which have an essential role in Homological Algebra.

Definition 1.2.4. An object X is called

generator (cogenerator) if $\mathrm{Hom}_{\mathcal{C}}(X, \text{-})$ ($\mathrm{Hom}_{\mathcal{C}}(\text{-}, X)$) is a faithful functor,

projective (injective) if $\mathrm{Hom}_{\mathcal{C}}(X, \text{-})$ ($\mathrm{Hom}_{\mathcal{C}}(\text{-}, X)$) preserves epimorphisms. We denote Proj($\mathcal{C}$) or Proj for the class of projective objects in $\mathcal{C}$. The notation $\mathrm{Inj}(\mathcal{C})$ or Inj stands for the class of injective objects in $\mathcal{C}$.

Proposition 1.2.5. *An object X is projective in $\mathcal{C}$ if and only if for each epimorphism $f : A \to B$ and each morphism $g : X \to B$, there exists a morphism $h : X \to A$ such that $f \circ h = g$.*

Dually, an object X is injective if and only if for each monomorphism $f : A \to B$ and each morphism $g : A \to X$, there exists a morphism $h : B \to X$ such that $h \circ f = g$.

Proposition 1.2.6. *An object X is generator (cogenerator) if and only if whenever $f, g : A \rightrightarrows B$ are distinct morphisms, there exists a morphism $h : X \to A$ ($s : B \to X$) such that $f \circ h \neq g \circ h$ ($s \circ f \neq s \circ g$).*

Functors are machineries which link categories to other categories. Natural transformations do it for functors.

Definition 1.2.7. Let $F, G : \mathcal{C} \to \mathcal{D}$ be two functors. A *natural transformation* $\tau : F \to G$ is a family of morphisms $\{\tau_X\}_{X \in \mathcal{C}}$, $\tau_X : F(X) \to G(X)$, in $\mathcal{D}$ such that the following diagram is commutative for each morphism $f : X \to Y$ in $\mathcal{C}$

$$\begin{array}{ccc} F(X) & \xrightarrow{\tau_X} & G(X) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow{\tau_Y} & G(Y). \end{array}$$

An easy argument shows that if $\tau : F \to G$ and $\theta : G \to T$ are natural transformations then the componentwise composition $\theta \circ \tau = \{\theta_X \circ \tau_X\}_{X \in \mathcal{C}}$ is again a natural transformation. From this point, it is natural to consider the category of functors, Fun$(\mathcal{C}, \mathcal{D})$, from $\mathcal{C}$ to $\mathcal{D}$. Then for $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$ functors, the class $\mathrm{Nat}(F, G)$ of natural transformations would be the class of morphisms. For any functor F, the natural transformation $\mathrm{id}_F := \{\mathrm{id}_{F(X)}\}_{X \in \mathcal{C}}$ is the identity natural transformation for F. But the problem occurs in requiring to be $\mathrm{Nat}(F, G)$ a set. Under the condition of $\mathcal{C}$ a small category, that problem is solved. If $\mathcal{C}$ and $\mathcal{D}$ are preadditive and $\mathcal{C}$ is small, the full subcategory category Add$(\mathcal{C}, \mathcal{D})$ of all additive functors is preadditive.

Definition 1.2.8. Let $F, G : \mathcal{C} \to \mathcal{D}$ be two functors. A natural transformation $\tau : F \to G$ is said to be an *isomorphism* if there is a natural transformation $\sigma : G \to F$ such that $\sigma \circ \tau = \mathrm{id}_F$ and $\tau \circ \sigma = \mathrm{id}_G$. In that case, it is said that F is isomorphic to G and denoted by $F \cong G$.

Note that being a natural transformation τ an isomorphism is equivalent to being τ_X an isomorphism for each $X \in Ob(\mathcal{C})$.

Definition 1.2.9. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. It is called an

isomorphism if there is a functor $G : \mathcal{D} \to \mathcal{C}$ such that $F \circ G = \mathrm{id}_{\mathcal{D}}$ and $G \circ F = \mathrm{id}_{\mathcal{C}}$,

equivalence if there is a functor $G : \mathcal{D} \to \mathcal{C}$ such that $F \circ G \cong \mathrm{id}_{\mathcal{D}}$ and $G \circ F \cong \mathrm{id}_{\mathcal{C}}$.

Proposition 1.2.10. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then*

(i) it is an isomorphism if and only if it is fully faithful and bijective on objects,

(ii) it is an equivalence if and only if it is fully faithful and satisfies the following: for each object $D \in \mathcal{D}$ there is an object $X \in \mathcal{C}$ such that $F(X) \cong D$.

At first sight, our mathematical background makes us think that the right notion of ‘isomorphism’ between categories is that of functor isomorphism. But in Category Theory, we deal with objects and morphisms. Looking for isomorphic categories by isomorphism of functors is quite restrictive. Even though an object may not return to itself, it is still under control by equivalence functors. Besides, equivalence functors preserve and reflect all categorical properties. So in Category Theory, equivalence functors are sufficiently useful.

Now, let $\mathcal{C}$ be a small category. There is a canonical functor known as *Yoneda embedding*

$$Y : \mathcal{C} \to \text{Fun}(\mathcal{C}^{op}, \mathbf{Set}), \tag{1.2.1}$$

$Y(A) := \text{Hom}_{\mathcal{C}}(\text{-}, A)$. The reason why it is called embedding is the following.

Proposition 1.2.11. *(Yoneda Lemma)[Ste75, Proposition IV.7.3] Let $F : \mathcal{C}^{op} \to \mathbf{Set}$ be any functor. Then*

$$\text{Nat}(Y(A), F) \cong F(A)$$

for any $A \in \mathcal{C}$.

If $\mathcal{C}$ is preadditive, then Y will be an embedding in $\text{Add}(\mathcal{C}^{op}, \mathbf{Ab})$.

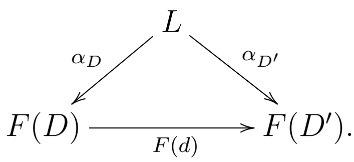
1.3 Limits and colimits

So far we have dealt with basic ingredients of category theory. Now it is time to put some flavor in it. In most of well-known categories, there are some special objects which make these categories more meaningful and help significantly to solve some problems within them, such as direct sums, direct limits, push-out diagrams,.. and their duals. Somehow we need some kind of concepts which are similar to them in more general categories. In this section, we introduce the general notions of limits and colimits of a functor, which involves these objects as particular cases.

Definition 1.3.1. For a given a functor $F : \mathcal{D} \to \mathcal{C}$, a *cone on F* consists of

- an object L in $\mathcal{C}$, and

- for every object $D \in \mathcal{D}$, a morphism $\alpha_D : L \to F(D)$ in $\mathcal{C}$, in such a way that for every morphism $d : D \to D'$ in $\mathcal{D}$, $\alpha_{D'} = F(d) \circ \alpha_D$, that is, the following diagram is commutative



Let $(L', \{\alpha'_D\}_{D\in\mathcal{D}})$ and $(L, \{\alpha_D\}_{D\in\mathcal{D}})$ be two cones on F. A *cone morphism* $f : L' \to L$ is a morphism on the underlying category $\mathcal{C}$ such that $\alpha_D \circ f = \alpha'_D$ for every $D \in \mathcal{D}$. So all cones on F constitute a category denoted Cone(F). For simplicity, sometimes we denote a cone just by the element L in order to prevent rudeness of notations because its morphisms come already with it.

Definition 1.3.2. If $F : \mathcal{D} \to \mathcal{C}$ is a functor, then a cone L on F is called *limit of* F provided that if $(L', \{\alpha'_D\}_{D\in\mathcal{D}})$ is a cone on F, then there is a unique cone morphism $f : L' \to L$.

In other words, a cone L on F is a limit of F if and only if it is a terminal object in Cone(F). Because of the essential uniqueness of terminal objects, if it exists, a limit of a functor is unique up to isomorphisms, as well. Therefore when a limit of a functor F exists, it is denoted by $\lim F$. Conversely, a terminal object of $\mathcal{C}$ can be written as a limit of a functor $F : I \to \mathcal{C}$, where I is the empty category. A category which admits all limits of functors whose domain is small is called *complete*.

Proposition 1.3.3. *[HS79, Theorem 29.3] Let F be a functor and let $(L, \{\alpha_D\}_{D\in\mathcal{D}})$ be a cone on F. Then the following are equivalent:*

(i) $(L, \{\alpha_D\}_{D\in\mathcal{D}})$ *is a limit of* F.

(ii) $(\mathit{Hom}(X, L), \{\mathit{Hom}(X, \alpha_D)\}_{D\in\mathcal{D}})$ *is a limit of* $\mathit{Hom}(X, -) \circ F$ *for each* $X \in \mathit{Ob}(\mathcal{C})$.

It means that a limit of a functor F is a sort of object $\lim F$ in such a way that every covariant $\mathrm{Hom}(X, -)$ has a distributive property over limit, that is,

$$\mathrm{Hom}(X, \lim F) \cong \lim \mathrm{Hom}(X, F(-))$$

for all $X \in \mathcal{C}$. As mentioned above, the notion of limit is the extract of certain objects satisfying a sort of universal properties, which are useful tools in the realm of Homological Algebra. We state here their definitions in terms of limits.

Definition 1.3.4. Let $\mathcal{C}$ be a category.

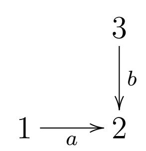
Equalizer : Let I be a category with two objects and morphisms

$$a, b : \ 1 \rightrightarrows 2$$

with identities. The limit of any functor $F : I \to \mathcal{C}$ is called the *equalizer of* $F(a)$ *and* $F(b)$ and denoted by $\mathrm{Eq}(F(a), F(b))$. A category $\mathcal{C}$ which admits all possible equalizers, that is, limits of all possible functors $F : I \to \mathcal{C}$ with I defined above, is said *to have equalizer*.

Product : Let I be a discrete category, that is, the only morphisms are identities. The limit of any functor $F : I \to \mathcal{C}$, if exists, is called *product of* $\{F(i)\}_{i \in I}$ and denoted by $\prod_I F(i)$. The canonical morphisms $\pi_i : \prod_I F(i) \to F(i)$ are called *projections*. If a category $\mathcal{C}$ admits limits of all functors $F : I \to \mathcal{C}$ where I is any discrete small category, then it is said *to have products*.

Pullback : Let I be a category with three objects and morphisms



with identities. The limit of any functor $F : I \to \mathcal{C}$ is called *pullback of* $F(a)$ *and* $F(b)$. A category which admits limits of any functor $F : I \to \mathcal{C}$ with I defined above is said *to have pullbacks*. As a special case, if $F(a)$ is a monomorphism, i.e. $F(a)$ is a subobject, the pullback of $F(a)$ and $F(b)$ is called *inverse image of* $F(1)$ *under* $F(b)$ and it is denoted by $F(b)^{-1}(F(1))$. If both $F(1)$ and $F(3)$ are subobjects of $F(2)$, it is called *intersection of* $F(1)$ *and* $F(3)$, denoted by $F(1) \cap F(3)$.

Inverse limit : Let I be a directed class, that is, a partially ordered class with the property that each pair of elements has an upper bound. The limit of any functor $F : I^{op} \to \mathcal{C}$ is called *inverse limit*.

Another notion which is a particular case of equalizers is that of kernel of a morphism.

Definition 1.3.5. Let $\mathcal{C}$ be a category with the zero object. Let $f : X \to Y$ be a morphism in $\mathcal{C}$. *A kernel of* f is a morphism $k : K \to X$ in $\mathcal{C}$ such that $f \circ k = 0$ and for every $k' : K' \to X$ with $f \circ k' = 0$, there exists a unique $g : K' \to K$ such that $k \circ g = k'$, in other words, it is an equalizer of the zero morphism, 0, and f, Eq$(0, f)$. If exists, it is denoted by Ker$f$. A category which admits kernels of all morphisms is said *to have kernels*.

Here is a list of some known facts for any category $\mathcal{C}$:

(i) For a pair of morphisms $f, g : X \rightrightarrows Y$ in $\mathcal{C}$, the canonical morphism Eq$(f) \to X$, if exists, is a monomorphism.

(ii) For a given family $\{C_i\}_{i \in I}$ of objects of $\mathcal{C}$, projections π_i, if $\prod_I C_i$ exists, are epimorphisms.

(iii) If $\mathcal{C}$ is a preadditive category with the zero object, Eq$(f, g) \cong$ Ker$(f - g)$ for each pair of morphisms $f, g : X \rightrightarrows Y$.

(iv) If $\mathcal{C}$ is a preadditive category with the zero object, a morphism $f : X \to Y$ is a monomorphism if and only if Ker$f$ exists and is equal to 0.

(v) Products are inverse limit of finite subproducts. So if $\mathcal{C}$ has inverse limits and finite products, then it has products. See [HS79, Proposition 22.5].

(vi) Limits preserve monomorphisms, i.e., if $F, G : I \rightrightarrows \mathcal{C}$ are functors with a natural transformation $\eta := (\eta_i)_{i \in I} : F \to G$ where each η_i is a monomorphism and if $\lim F$ and $\lim G$ exist, then the unique morphism $\lim F \to \lim G$ is a monomorphism.

(vii) If $\mathcal{C}$ has the zero object, then finite products and kernels, when they exist, can be obtained from certain type of pullback diagrams. The converse is true when a category has finite products and equalizers, that is, if $f : X_1 \to Y$ and $g : X_2 \to Y$ are morphisms in $\mathcal{C}$ then $\mathrm{Eq}(f \circ \pi_1, g \circ \pi_2)$ gives pullback of f, g. See [HS79, Theorem 21.3]

(viii) Every pullback of a monomorphism (retraction) is a monomorphism (retraction). See [HS79, Proposition 21.7].

We have chosen to introduce these special objects as a particular instance of limits. These special objects are closely related each other. But the converse is also possible, that is, under some conditions, limits can be written in terms of these special objects.

Proposition 1.3.6. *[Ste75, Proposition 8.2] Let $\mathcal{C}$ be a preadditive category with the zero object and let $F : I \to \mathcal{C}$ be a functor where I is a small category. If $\mathcal{C}$ has products and kernels then*

$$\lim F = \mathit{Ker}(\prod_{i \in I} F(i) \to \prod_{\lambda} F(t(\lambda)))$$

where the index of the second product runs over all morphisms, $\lambda : i \to j$, in I and $t(\lambda) := j$.

The dual of all notions given above: *cocones, colimits, coequalizers, cokernels, coproducts, pushouts, direct limits, cocompleteness*, respectively. So the duals of the given properties hold for them, as well. As a notation, if $X \subseteq Y$ a subobject in a category $\mathcal{C}$, it is common to denote Y/X for the cokernel of the inclusion $X \to Y$.

Note that the category of functors, $\mathbf{Func}(\mathcal{C}, \mathcal{D})$, with $\mathcal{C}$ a small category, inherits all the nice properties of $\mathcal{D}$. Limits and colimits in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ are obtained pointwise. The argument is valid for $\mathrm{Add}(\mathcal{C}, \mathcal{D})$ if $\mathcal{C}, \mathcal{D}$ are preadditive, see [Ste75, Section IV.7].

The next proposition shows a case where completeness and cocompleteness aren't far away from each other as one could think. In the second chapter, there will be another example, locally presentable categories, where cocompleteness implies completeness.

Proposition 1.3.7. *[HS79, Example 23.9.(11)] If $\mathcal{C}$ is a small category, then the following are equivalent:*

(i) $\mathcal{C}$ is complete.

(ii) $\mathcal{C}$ is cocomplete.

(iii) $\mathcal{C}$ is equivalent to a complete lattice.

1.4 Abelian categories

Abelian categories are the most coarse categories that ensure the existence of all required tools to do Homological Algebra. We should point out that the notions of kernels and cokernels in a category, as well as their existence, must be consistent in order to get nice factorization properties, in other words, to define exactness of sequences. It was firstly introduced by [Buc55] under the name exact category which is an abelian category without the requirement of the existence of direct sums. The importance of being preadditive light up here, see Theorem 1.4.7. Before giving directly the definition of Abelian category, we develop it according to what we need. All categories in this section are assumed to posses the zero object.

Definition 1.4.1. Let $f : X \to Y$ be a morphism in a category $\mathcal{C}$. Then it is said to be a

normal monomorphism if it is a kernel of a morphism in $\mathcal{C}$.

normal epimorphism if it is a cokernel of a morphism in $\mathcal{C}$.

Proposition 1.4.2. *[HS79, Corollary 39.4] Let $\mathcal{C}$ be a category having kernels and cokernels and f be a morphism in $\mathcal{C}$. Then*

(i) f is a normal monomorphism if and only if $f \cong Ker(Coker f)$.

(ii) f is a normal epimorphism if and only if $f \cong Coker(Ker f)$.

Note that being a normal monomorphism or epimorphism is preserved under equivalence, that is, for example, if $f : X \to Y$ and $f' : X' \to Y$ are equivalent monomorphisms where f is normal, so is f'.

Proposition 1.4.3. *[HS79, Proposition 39.8] Assume that $\mathcal{C}$ has kernels. If A and A' are subobjects of X, where A is normal, then $A \cap A'$ exists.*

Proof. Let $m : A \to X$ and $m' : A' \to X$ be subobject monomorphisms of X where $m = \mathrm{Ker}(f)$ for some $f : X \to Y$. It is easy to check that $\mathrm{Ker}(m \circ f)$ is the intersection of A and A'. □

Definition 1.4.4. A category $\mathcal{C}$ having kernels and cokernels is called

normal if each monomorphism is normal and every morphism is (epi, mono)-factorizable.

conormal if each epimorphism is normal and every morphism is (epi, mono)-factorizable.

exact if it is both normal and conormal.

We will see that the condition of (epi, mono)-factorization in exact category is automatic when a category is preadditive.

Corollary 1.4.5. *[HS79, Corollary 39.11] An exact category is well-powered if and only if it is co-(well-powered)*

Proof. It follows easily from Proposition 1.4.2. □

Lemma 1.4.6. *[HS79, Lemma 39.12] Suppose that $\mathcal{C}$ is a category having kernels and cokernels. If a morphism $f : X \to Y$ is of the form $f = h \circ g$ where g is a normal epimorphism, then Coker$(\text{Ker}f)$ is factorized over g.*

Proof. Since g is a normal epimorphism, by Proposition 1.4.2, $g = \text{Coker}(\text{Ker}g)$. The fact that $f \circ \text{Ker}(g) = 0$ implies the existence of a unique morphism $t : \text{Ker}g \to \text{Ker}f$ such that $\text{Ker}f \circ t = \text{Ker}g$. Then $\text{Coker}(\text{Ker}f) \circ \text{Ker}g = 0$, so there exists a unique morphism g' such that $\text{Coker}(\text{Ker}f) = g' \circ g$. □

Theorem 1.4.7. *[HS79, Theorem 39.13] Suppose that $\mathcal{C}$ has kernels and cokernels and each epimorphism is a normal epimorphism. Then the following are equivalent.*

(i) *$\mathcal{C}$ is conormal.*

(ii) *If f is a morphism in $\mathcal{C}$ with Ker$(f) = 0$, then f is a monomorphism.*

(iii) *If a morphism f is of the form $f = h \circ$ Coker$(\text{Ker}f)$, then h is a monomorphism.*

(iv) *Every morphism in $\mathcal{C}$ has (normal epi, mono)-factorization.*

Condition (ii) is always satisfied when a category is preadditive.

Proposition 1.4.8. *[HS79, Proposition 39.18] Suppose that $\mathcal{C}$ has kernels and cokernels and each epimorphism is a normal epimorphism. Then $\mathcal{C}$ is conormal.*

Proof. To prove (ii) in Theorem 1.4.7, take a morphism $f : X \to Y \in \mathcal{C}$ such that $\text{Ker}f = 0$. Suppose that there are morphisms $g, g' : A \rightrightarrows X$ in $\mathcal{C}$ with $f \circ g = f \circ g'$. So, we consider $m : X \to \text{Coeq}(g, g')$, coequalizer of g and g'. Thus there is a unique morphism $h : \text{Coeq}(g, g') \to X$ such that $f = h \circ m$. Note that $\text{id}_X : X \to \text{Coker}(\text{Ker}f) = X$. By Lemma 1.4.6, there is a morphism $m' : \text{Coeq}(g, g') \to X$ such that $m' \circ m = \text{id}_X$, which means that m is a section, so $g = g'$. □

By the duality principle, all results given above can be stated for normal categories, as well. So we have the following:

Proposition 1.4.9. *[HS79, Proposition 39.19] If $\mathcal{C}$ has equalizers and coequalizers such that each monomorphism and epimorphism are normal, then $\mathcal{C}$ is exact.*

A preadditive category with kernels and cokernels is normal (conormal) if and only if every monomorphism (epimorphism) is normal.

Theorem 1.4.10. *[HS79, Theorem 39.15] Let $\mathcal{C}$ be an exact category and let $f : X \to Y$ be a morphism in $\mathcal{C}$*

(i) The canonical morphism $\overline{f}$: Coker$(Kerf) \to$ Ker$(Cokerf)$ in the commutative diagram

$$\begin{array}{ccccccc} Kerf & \longrightarrow & X & \xrightarrow{f} & Y & \longrightarrow & Cokerf \\ & & \downarrow & & \uparrow & & \\ & & Coker(Kerf) & \xrightarrow{\overline{f}} & Ker(Cokerf) & & \end{array}$$

is an isomorphism and it is the unique (epi,mono)-factorization of f, up to isomorphism.

(ii) f is a monomorphism if and only if Ker$f = 0$ if and only if Coker$(Kerf) = X$.

(iii) f is an epimorphism if and only if Coker$f = 0$ if and only if Ker$(Cokerf) = Y$.

(iv) f is an isomorphism if and only if Ker$f$ = Coker$f = 0$.

There are two essential important concepts through which Homological Algebra in R-Mod is operated: the first one is compatible image, Im, and coimage, Coim, of morphisms, in order to define exact sequences, and the second one is connecting morphisms of homologies in order to get long exact sequences of Ext-sequences, from a short exact sequence. Now, an exact category enables us the first one: Let f be a morphism in an exact category $\mathcal{C}$, then

$$\operatorname{Im} f := \text{Ker}(\text{Coker}f),\ \text{Coim}f := \text{Coker}(\text{Ker}f).$$

By Theorem 1.4.10, $\operatorname{Im} f \cong \text{Coim}f$. In order to get the second, we need our category to have pullbacks and pushouts. Recall that in a preadditive category with the zero object, the existence of kernels (cokernels) and finite products (finite coproducts) is equivalent to the existence of pullbacks (pushouts). We should highlight another advantage of being preadditive: in a preadditive category $\mathcal{C}$, the coproduct and the product of any finite family of objects of $\mathcal{C}$ are the same (see [HS79, Proposition 40.8], see also [HS79, Theorem 40.13] for the relation between the preadditivity and admitting products).

Definition 1.4.11. A category $\mathcal{C}$ is called *additive* if it is preadditive and has finite products.

Definition 1.4.12. A category $\mathcal{C}$ is called *abelian* if it is both additive and exact, more concretely, it satisfies the following:

- A1) it is additive,
- A2) it admits the zero object,
- A2) it has kernels and cokernels,
- A3) each monomorphism is a normal monomorphism and each epimorphism is a normal epimorphism.

The following justifies our claim.

Theorem 1.4.13. *[HS79, Theorem 41.9] For any category $\mathcal{C}$, without requiring to admit the zero object, the following are equivalent:*

(i) $\mathcal{C}$ *is abelian.*

(ii) $\mathcal{C}$ *has pullbacks, pushouts, the zero object, and each monomorphism is a normal monomorphism and each epimorphism is a normal epimorphism.*

The following well-known facts in an abelian category can be found in [Fre64, Section 2.6].

Proposition 1.4.14. *Let* $\mathcal{C}$ *be an abelian category. Then*

(i) a morphism in $\mathcal{C}$ *is an isomorphism if and only if it is both a monomorphism and an epimorphism.*

(ii) The pullback of an epimorphism is an epimorphism.

(iii) The pushout of a monomorphism if a monomorphism.

(iv) Let $A \subseteq B \subseteq C$ *in* $\mathcal{C}$*. Then* $(C/A)/(B/A) \cong C/B$*.*

Proposition 1.4.15. *[Fre64, Proposition 3.35] Let* $\mathcal{C}$ *be an abelian category. If* $\mathcal{C}$ *has a generator then it is well-powered.*

Definition 1.4.16. Let $\mathcal{C}$ be an abelian category. A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ of morphisms in $\mathcal{C}$ is said to be *exact at* B if $\operatorname{Im} f \cong \operatorname{Ker} g$. *A short exact sequence* is an exact sequence of the form

$$0 \to A \to B \to C \to 0.$$

The next is known as 'Snake Lemma'.

Proposition 1.4.17. *[Hel58, Proposition 4.3] In an abelian category* $\mathcal{C}$*, a commutative diagram*

$$\begin{array}{ccccccc} & & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\ 0 & \longrightarrow & A' & \xrightarrow[f']{} & B' & \xrightarrow[g']{} & C' & & \end{array}$$

with exact rows induces an exact sequence

$$Ker a \to Ker b \to Ker c \to Coker a \to Coker b \to Coker c.$$

Definition 1.4.18. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between abelian categories. The functor F is said to be

right exact if $F(A) \to F(B) \to F(C) \to 0$ is exact for any exact sequence $A \to B \to C \to 0$,

left exact if $0 \to F(A) \to F(B) \to F(C)$ is exact for any exact sequence $0 \to A \to B \to C$,

exact if F is both right and left exact.

Note that a left or right exact functor between abelian categories is always additive, see [Fre64, Theorem 3.12 and 3.13]. A standard example Hom : $\mathcal{C}^{op} \times \mathcal{C} \to \mathbf{Ab}$ is left exact in each one of variables. Then an object P in an abelian category $\mathcal{C}$ is projective if and only if $\mathrm{Hom}_{\mathcal{C}}(P, \text{-})$ is exact. The dual statement is for an injective object.

Now we recall Yoneda extension classes. For an abelian category $\mathcal{C}$ and objects $A, C \in \mathcal{C}$, $\mathrm{Ext}^1(C, A)$ denotes the collection of equivalence classes of short exact sequences beginning with A and ending with C subject to: $\mathbb{E} \equiv \mathbb{E}'$ if and only if there is a morphism $\mathbb{E} \to \mathbb{E}'$, that is,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \Downarrow & & \downarrow & & \Downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & B' & \longrightarrow & C & \longrightarrow & 0. \end{array}$$

We sometimes write Ext for Ext^1. If $\mathbb{E} \in \mathrm{Ext}^1(C, A)$ and if $f : A \to A'$ and $g : C' \to C$ are morphisms in $\mathcal{C}$ then $\mathbb{E}g$ denotes the short exact sequence obtained by taking pullback of $\mathbb{E}$ over g. Analogously, $f\mathbb{E}$ stands for that of pushout of $\mathbb{E}$ over f. Then we have a functor

$$\mathrm{Ext}(g, f) : \mathrm{Ext}^1(C, A) \to \mathrm{Ext}^1(C', A')$$

which sends a short exact sequence $\mathbb{E}$ to $(f\mathbb{E})g \equiv f(\mathbb{E}g)$. If

$$0 \to A \to B \to C \to 0$$

is a short exact sequence then for an object $X \in \mathcal{C}$ the canonical sequences

$$0 \to \mathrm{Hom}(C,X) \to \mathrm{Hom}(B,X) \to \mathrm{Hom}(A,X) \\ \to \mathrm{Ext}^1(C,X) \to \mathrm{Ext}^1(B,X) \to \mathrm{Ext}^1(A,X)$$

$$0 \to \mathrm{Hom}(X,A) \to \mathrm{Hom}(X,B) \to \mathrm{Hom}(X,C) \\ \to \mathrm{Ext}^1(X,A) \to \mathrm{Ext}^1(X,B) \to \mathrm{Ext}^1(X,C)$$

are exact.

An *n-fold exact sequence starting at A and ending at C* is an exact sequence of the form

$$S: \ 0 \to A \to X_{n-1} \to \ldots \to X_0 \to C \to 0.$$

If T is any m-fold exact sequence starting at C, then *Yoneda composite* $S \circ T$ is the exact sequence composing S and T at C. The congruence relation on n-fold exact sequences is defined as the weakest reflexive, symmetric and transitive relation in such a way that it includes the congruence on short exact sequences and $(\mathbb{E}\beta) \circ \mathbb{E}' \equiv \mathbb{E} \circ (\beta\mathbb{E}')$ whenever the composites involved are all defined.

In order to define it, we first write an n-fold exact sequence S as the composite of n short exact sequences $\mathbb{E}_i$ in the form

$$S = \mathbb{E}_n \circ \mathbb{E}_{n-1} \circ \ldots \circ \mathbb{E}_1,$$

the $\mathbb{E}_i$ are unique up to isomorphism. $S \equiv S'$, where S' is an n-fold exact sequence with the same start and end, if S' can be obtained from S by a finite sequence of replacements of the following three types

- Replace any one factor $\mathbb{E}_i$ by a congruent short exact sequence;
- If two successive factors have the form $\mathbb{E}\beta \circ \mathbb{E}'$, replace them by $\mathbb{E} \circ \beta\mathbb{E}'$.
- If two successive factors have the form $\mathbb{E} \circ \beta\mathbb{E}'$, replace them by $\mathbb{E}\beta \circ \mathbb{E}'$.

Then $\mathrm{Ext}^n(C, A)$ stands for the collection of equivalence classes of n-fold exact sequences beginning with A and ending with C. We refer to [Mac95, Chapter III] for more detail on it.

1.5 Adjoint functors

In this section, we introduce the concept of adjoint functors which are strengthed concepts of equivalence functors. Recall that in case of equivalence functors, after applying functors, one expects to return to an object not equally but isomorphicly. The question raises up here: what would happen if we replace certain morphisms uniquely determined instead of isomorphisms? This is the starting point of adjointness which comes out in many diverse areas of mathematics. It provides us a wide class of functors and they are good enough to solve several problems.

Definition 1.5.1. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and D an object of $\mathcal{D}$.

A reflection of D **along** F is a pair (C_D, α_D) where

- C_D is an object of $\mathcal{C}$ and $\eta_D : D \to F(C_D)$ is a morphism of $\mathcal{D}$,
- If C is an object of $\mathcal{C}$ and $g : D \to F(C)$ is a morphism of $\mathcal{D}$, there exists a unique morphism $f : C_D \to C$ in $\mathcal{C}$ such that $F(f) \circ \eta_D = g$.

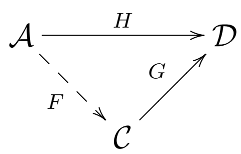
A coreflection of D along F is a pair (C_D, ξ_D) where

- C_D is an object of $\mathcal{C}$ and $\xi_D : F(C_D) \to D$ is a morphism of $\mathcal{D}$,
- If C is an object of $\mathcal{C}$ and $g : F(C) \to D$ is a morphism of $\mathcal{D}$, there exists a unique morphism $f : C \to C_D$ in $\mathcal{C}$ such that $\xi_D \circ F(f) = g$.

Completion of metric spaces, abelianization of groups and the order completion of partially-ordered sets are some of examples of reflections.

Proposition 1.5.2. *Let $G : \mathcal{C} \to \mathcal{D}$ and $H : \mathcal{A} \to \mathcal{D}$ be functors. Suppose that for each object $A \in \mathcal{A}$, the reflection of $H(A)$ along G exists and is given, $(C_{H(A)}, \eta_{H(A)})$. Then there exists a unique functor $F : \mathcal{A} \to \mathcal{C}$ such that*

- *for each $A \in \mathcal{A}$, $F(A) = C_A$,*
- *$\eta := \{\eta_A : H(A) \to G(C_A)\}_{A \in \mathcal{A}}$ is a natural transformation from H to $G \circ F$.*



As a consequence, in case $H = \mathrm{id}_D$, there exists a unique functor $F : \mathcal{D} \to \mathcal{C}$ with a natural transformation $\eta : \mathrm{id}_\mathcal{D} \to G \circ F$ and $F(D) = C_D$. Furthermore,

(i) F preserves colimits.

(ii) There is a unique functor natural transformation $\xi : F \circ G \to \mathrm{id}_\mathcal{C}$ such that

$$(G * \xi) \circ (\eta * G) = \mathrm{id}_G;$$

$$(\xi * F) \circ (F * \eta) = \mathrm{id}_F .$$

Proof. For the first, see [HS79, Lemma 26.9]. For (i) and (ii) see [HS79, Theorem 26.11]. □

The mission of the functor F is to take an object from $\mathcal{C}$ and attach it to the original object in $\mathcal{D}$ along G. It is clear from the definition that this assignment is unique up to isomorphism. In case of an equivalence G, F does it by an isomorphism. But the uniqueness is good enough for many situations. The dual statement for coreflections follows by Duality Principle.

Definition 1.5.3. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors. (F, G) is called an *adjoint pair* provided

- there exist natural transformations $\eta : \mathrm{id}_{\mathcal{C}} \to G \circ F$, called *unit of the adjunction*, and $\xi : F \circ G \to \mathrm{id}_{\mathcal{D}}$, *counit of adjunction*,
- $G(\xi_D) \circ \eta_{G(D)} = \mathrm{id}_{G(D)}$
  $\xi_{F(C)} \circ F(\eta_C) = \mathrm{id}_{F(C)}$.

In this case, F is called a *left adjoint of* G, G is said to be a *right adjoint of* F.

Theorem 1.5.4. *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors. The following are equivalent:*

(i) (F, G) *is an adjoint pair.*

(ii) There exists a natural transformation $\eta : \mathrm{id}_{\mathcal{C}} \to G \circ F$ *such that for every* $C \in \mathcal{C}$, $(F(C), \eta_C)$ *is a reflection of C along G.*

(iii) There exists a natural transformation $\xi : F \circ G \to \mathrm{id}_{\mathcal{D}}$ *such that for every* $D \in \mathcal{D}$, $(G(D), \xi_D)$ *is a coreflection of D along F.*

(iv) There is a natural isomorphism

$$Hom_{\mathcal{D}}(F(-), -) \to Hom_{\mathcal{C}}(-, G(-))$$

of set-valued bifunctors.

Proposition 1.5.5. *[Bor94, Proposition 3.2.2] If (F, G) is an adjoint pair, then F preserves colimits and G preserves limits.*

Theorem 1.5.6. *(Special adjoint functor theorem)[Bor94, Theorem 3.3.4] Consider a functor $G : \mathcal{D} \to \mathcal{C}$ and suppose the following conditions are satisfied:*

- *$\mathcal{D}$ is complete,*
- *G preserves small limits,*
- *$\mathcal{D}$ is well-powered,*
- *$\mathcal{D}$ has a cogenerating family.*

Then G has a left adjoint functor.

1.6 Grothendieck categories

There are two ways in R-Mod to prove the existence of injective envelopes. One is more peculiar to R-Mod, see [EJ00, Theorem 3.1.7]. The other one is more categorical and helps to set up categories with sufficient conditions in order to have enough injectives. In fact, the process is called *Quillen's small object argument*. It is based on the well-ordered direct limit of transfinite extensions of pushout diagrams. In order to prove that it is an embedding, direct limits need to be exact. And to control whether the output is injective, it is required to have a sort of result like Baer's criterion in R-Mod. Grothendieck proved in [Gro57] that an abelian category with exact direct limits and a generator has enough injectives.

Definition 1.6.1. Let $\mathcal{C}$ be an abelian category. It is called a *Grothendieck category* if it has a generator, direct limits and direct limits are exact.

Note that an abelian category has direct limits if and only if it has coproducts if and only if it has colimits. So Grothendieck categories are cocomplete. The importance of a generator is that it guarantees the category to be well-powered, see Proposition 1.4.15.

Proposition 1.6.2. *[Ste75, Proposition V.2.9] Let $\mathcal{C}$ be a Grothendieck category with a generator G. An object X is injective in $\mathcal{C}$ if and only if every diagram*

$$\begin{array}{ccc} G' & \hookrightarrow & G \\ \downarrow & \swarrow & \\ X & & \end{array}$$

can be completed to a commutative one.

Proposition 1.6.3. *Let $\mathcal{C}$ be a Grothendieck category. Then every object can be embedded in an injective object.*

Proof. Let X be an object. Then $E_1(X)$ is the pushout

$$\begin{array}{ccc} \bigoplus_{G' \leq G} G'^{(\mathrm{Hom}(G',X))} & \hookrightarrow & \bigoplus_{G' \leq G} G^{(\mathrm{Hom}(G',X))} \\ \downarrow & & \downarrow \\ X & \hookrightarrow & E_1(X). \end{array}$$

$E_\alpha(X)$ is defined inductively for every ordinal α. Namely, given $E_\alpha(X)$, then $E_{\alpha+1}(X) := E_1(E_\alpha(X))$. And if E_α is defined for every $\alpha < \beta$ where β is a limit ordinal, then $E_\beta := \varinjlim_{\alpha<\beta} E_\alpha$. If σ is an ordinal whose cofinality is greater then the cardinal of the set of subobjects of G, then $E_\sigma(X)$ is an injective since each subobject G' is σ-small. And the canonical morphism $X \to E_\sigma(X)$ is an inclusion because of exact direct limits. □

Definition 1.6.4. A subobject $A' \leq A$ is called *essential* if $B \cap A' \neq 0$ for every nonzero subobject $B \leq A$. More generally, a monomorphism $f : A' \to A$ is called *essential* if $\mathrm{Im}\, f$ is essential in A.

Definition 1.6.5. An *injective envelope* of an object A is an essential monomorphism $A \to$

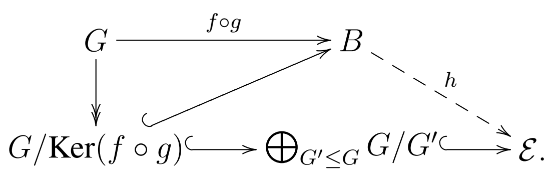
$E(A)$ where $E(A)$ is injective

All well-known properties on injective envelopes in R-Mod remain true in Grothendieck categories, as well. For instance, an injective envelope is uniquely determined, see [Ste75, Proposition 2.3]. An object is injective if and only if its essential extensions are isomorphisms, see [Ste75, Proposition 2.4]. In the light of these facts and Zorn's lemma, it is immediate the existence of injective envelopes in a Grothendieck category.

Proposition 1.6.6. *[Ste75, Proposition 2.5] Let $\mathcal{C}$ be a Grothendieck category. Then every object has an injective envelope.*

Proposition 1.6.7. *A Grothendieck category $\mathcal{C}$ admits an injective cogenerator $\mathcal{E}$.*

Proof. Let G be a generator for $\mathcal{C}$. Since $\mathcal{C}$ is well-powered $\bigoplus_{G' \leq G} G/G'$ is defined. $\mathcal{E}$ is an injective envelope of $\bigoplus_{G' \leq G} G/G'$. It is an injective cogenerator. Indeed, if $f : A \to B$ is a nonzero morphism, then there is a morphism $g : G \to A$ such that $f \circ g \neq 0$. Since $\mathcal{C}$ is an abelian category, $f \circ g$ has a factorization and it fits in the following commutative diagram



The morphism h exists because $\mathcal{E}$ is injective. Since $f \circ g$ is nonzero, then $h \circ f \circ g$ is nonzero, as well. $\square$

CHAPTER TWO
MORE TOPICS ON CATEGORIES

This chapter is devoted to some important classes of categories which are of our special interest. We recall some well-known facts.

The subject of the first section is locally presentable categories which were firstly introduced by [GU71]. Roughly, they are categories which have a small set of "compact" objects generating the category in terms of colimits. This property, which can be found in many categories of interest, makes the category manageable. Namely, they are precisely reflective subcategories of $\mathbf{Set}^{\mathcal{A}}$, set valued functors from some small-category $\mathcal{A}$, closed under λ-directed colimit ([AR94, 1.46]). Therefore, they are complete, cocomplete, well-powered and with a set of strong generators. They are also (co)well-powered. The theory enlarges the domain where some problems in homological algebra can be worked out, such as covering, enveloping ([Kra12]) and the orthogonal subcategory problem ([AR94, Section 1.C]).

One of the reason which makes the theory so important for us is that it enables a well-behaved concept of "pure morphisms" for more general categories, which are the λ-colimit completion of sections just as in R-Mod. In fact, pure short exact sequences in R-Mod are precisely direct limits of splitting short exact sequences. It will be detailed in Chapter (V). This section is based on the book [AR94], where locally presentable categories are discussed in detail.

In the second section, we speak of the category of complexes over an abelian category. They may arise from diverse areas of mathematics. But the reason why they occupy the heart of many theories is due to efficient uses in algebraic geometry-topology, combinatorial topology, etc. Complexes constitute, for example, a way to represent topological spaces through Algebra. For a given topological space there are various complexes associated to it, such as simplicial complexes, singular complexes, cellular complexes, etc. Besides, they constitute the essence of (Relative) Homological Algebra since they appear naturally as resolutions of objects. The category of complexes carries many of properties of its base category. We remind the reader some terminology and very basic properties on complexes. Some of the treatment on complexes can be found in [GR99].

Homotopy theory is an area in algebraic topology dealing with classifications of topological spaces up to homotopy equivalence, which is weaker than homeomorphisms. Roughly, a homotopy is about how to convert a space to another one continuously in time. It plays an important role as much as homeomorphisms because many of the homological tools are invariant under homotopy equivalence such as homology and homotopy groups: If two continuous maps of topological spaces are homotopic, their induced maps of singular homology groups are the same. Since topological spaces correspond to complexes in Algebra, there is a "homotopy" notion for chain maps of complexes. In fact, there are three different notions of homotopy: that of continuous maps of topological spaces, of simplical maps of simplical sets and that of morphisms of complexes. The implication goes in order. The connection between Homological Algebra and Algebraic Topology is treated very well in [GM03]. In addition, homotopy category is also a bridge between a category of complexes and its derived category. It is unlikely an abelian category. But Verdier in [Ver96] proved that it has a 'triangle structure' which allows to get a long exact sequence of homologies. In the third section, we recall some necessary ingredients for homotopy category.

The subject of the last section is to remember the equivalence between the category of quasi-coherent sheaves and some certain module representations of a quiver, which was pointed out in [EE05].

2.1 Presentable categories

We first start by giving the definition of λ-presentable objects in any category, which was introduced by [GU71]

Definition 2.1.1. Let λ be a regular cardinal. A set I with a preorder $\leq$ is called λ-*directed* if for any subset S with cardinality less than λ, there is an element $k \in I$ with $i \leq k$ for each $i \in S$.

A λ-*directed diagram* in a category $\mathcal{C}$ is a system of objects which is indexed over a λ-directed set, more concretely, a functor $F : I \to \mathcal{C}$ where I is a λ-directed set. Regarding to that, a colimit of a λ-directed diagram is called λ-*directed colimit in* $\mathcal{C}$.

Definition 2.1.2. Let X be an object in a category $\mathcal{C}$ and λ be a regular cardinal. It is said to be λ-*presentable* if $\mathrm{Hom}_{\mathcal{C}}(X, -)$ preserves λ-directed colimits. An object is called *presentable* if it is λ-presentable for some regular cardinal λ. In case $\lambda = \aleph_0$, it is called *finitely presentable*.

Proposition 2.1.3. *[AR94, Proposition 1.16] Let $F : \mathcal{D} \to \mathcal{C}$ be a functor where $\mathcal{D}$ is λ-small and for each $D \in \mathcal{D}$, $F(D)$ is λ-presentable in $\mathcal{C}$ for some regular cardinal λ. Then colim$F$, if exists, is λ-presentable in $\mathcal{C}$, as well.*

Definition 2.1.4. A category $\mathcal{C}$ is called *locally* λ-*presentable*, where λ is a regular cardinal, provided that

P1) it is cocomplete,

P2) it has a set of λ-presentable objects such that every object is a λ-directed colimit of objects from $\mathcal{A}$.

A category is called *locally presentable* if it is locally λ-presentable for some regular cardinal λ. In case $\lambda = \aleph_0$, it is said to be *locally finitely presentable*.

Let us denote by $\mathcal{C}_\lambda$ the class of all λ-presentable objects in a category $\mathcal{C}$. If $\mathcal{C}$ is locally λ-presentable, then there is a set in $\mathcal{C}_\lambda$ in such a way that every λ-presentable object is a direct summand of an object of it. This implies that $\mathcal{C}_\lambda$ is skeletally small, that is, it has a set of representatives, $\mathbf{Pres}_\lambda\mathcal{C}$. It is a small full subcategory of $\mathcal{C}$. The next proposition shows that the definition of locally presentable category given above by [AR94] is the same as the original one, that of [GU71].

Proposition 2.1.5. *[AR94, Theorem 1.20] A category is locally λ-presentable if and only if it is cocomplete and has a strong generator formed by λ-presentable objects.*

Let us consider the Yoneda functor for a locally λ-presentable category $\mathcal{C}$,

$$Y :\ \mathcal{C} \to \text{Fun}((\mathbf{Pres}_\lambda\mathcal{C})^{op}, \mathbf{Set}),$$

$Y(X) := \text{Hom}_{\mathcal{C}}(-, X)|_{\mathbf{Pres}_\lambda\mathcal{C}}$, which is fully faithful and preserves λ-colimits. Since $\mathcal{C}$ is cocomplete, it has a right adjoint, hence, it is equivalent to a reflective subcategory of $\text{Fun}((\mathbf{Pres}_\lambda\mathcal{C})^{op}, \mathbf{Set})$. As a matter of fact, locally presentable categories are precisely full, reflective subcategories of the category of set-valued functors over a small category. In addition to they are closed under λ-directed colimits, see [AR94, Theorem 1.46]. This implies that they are complete, as well.

Proposition 2.1.6. *[AR94, Corollary 1.60,] Let $\mathcal{C}$ be a locally λ-presentable category. Then the following holds:*

(i) The class of all monomorphisms are closed under λ-directed colimits, that is, the λ-directed colimit functor is exact.

(ii) λ-directed unions are λ-directed colimits.

As a consequence of the well-known Gabriel-Popescu Theorem, [GP64], locally presentable categories generate a wider domain involving Grothendieck categories.

Proposition 2.1.7. *Any Grothendieck category is locally presentable.*

Recall that in a Grothendieck category an object X is said to be λ-*generated* if $\mathrm{Hom}(X, -)$ preserves all λ-directed limits with all morphisms in the direct system being monomorphisms. Now let $\kappa \geq \lambda$ be regular cardinals. In a locally λ-presentable category, κ-presentable objects are precisely κ-small colimit of λ-presentable objects, see [AR94, Remark 1.30]. The following result gives a characterization of κ-presentable objects in a Grothendieck category in terms of κ-generated objects just as in R-Mod.

Proposition 2.1.8. *[Sto13a, Lemma A.2] Let $\mathcal{C}$ be a Grothendieck category which is λ-presentable for a regular cardinal λ. The following are equivalent for an object $X \in \mathcal{C}$ and a regular cardinal $\kappa \geq \lambda$:*

(i) X is κ-presentable.

(ii) X is κ-generated and whenever $0 \to K \to E \to X \to 0$ is a short exact sequence in $\mathcal{C}$ such that E is κ-generated, K is also κ-generated.

The other equivalent definition of a finitely presentable object in a module category R-Mod is to be a quotient of a morphism between finitely generated free modules. The following states it for sufficiently big infinite regular cardinals in Grothendieck categories, too.

Proposition 2.1.9. *[GU71, 7.6 and 9.3] Let $\mathcal{C}$ be a Grothendieck category such that $\mathcal{C}$ has a generating set $\mathcal{S}$ consisting of λ-presentable objects for some regular cardinal λ. Let X be an object of $\mathcal{C}$ and κ an infinite regular cardinal with $\kappa \geq \lambda$. Then*

(i) X is κ-generated if and only if there exists an exact sequence

$$\bigoplus_{i \in I} S_i \to X \to 0$$

with $|I| < \kappa$ and $S_i \in \mathcal{S}$ for all $i \in I$.

(ii) X is κ-presentable if and only if there exists an exact sequence

$$\bigoplus_{j\in J} S_j \to \bigoplus_{i\in I} S_i \to X \to 0$$

with $|J| < \kappa$, $|I| < \kappa$ and $S_i, S_j \in \mathcal{S}$ for all $i \in I$ and $j \in J$.

2.2 Category of complexes

Definition 2.2.1. Let $\mathcal{C}$ be an abelian category. A *chain complex* A is a chain of morphisms, which are called *differentials*, $\{d_n^A : A_n \to A_{n-1}\}_{n\in\mathbb{Z}}$ in $\mathcal{C}$ such that $d_{n+1}^A \circ d_n^A = 0$ for every $n \in \mathbb{Z}$. It is denoted as

$$\cdots \longrightarrow A_{n+1} \xrightarrow{d_{n+1}^A} A_n \xrightarrow{d_n^A} A_{n-1} \longrightarrow \cdots .$$

Definition 2.2.2. For any given complexes A and B, a *(chain) morphism* $f : A \to B$ is a $\mathbb{Z}$-indexed set of morphisms $\{f_n : A_n \to B_n\}_{n\in\mathbb{Z}}$ such that $d_n^B \circ f_n = f_{n-1} \circ d_n^A$, that is, the following diagram is commutative

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & A_{n+1} & \xrightarrow{d_{n+1}^A} & A_n & \xrightarrow{d_n^A} & A_{n-1} & \longrightarrow & \cdots . \\ & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\ \cdots & \longrightarrow & B_{n+1} & \xrightarrow[d_{n+1}^B]{} & B_n & \xrightarrow[d_n^B]{} & B_{n-1} & \longrightarrow & \cdots \end{array}$$

As a matter of fact, a category of complexes can be defined for any category, subject to $d_n \circ d_{n+1} = 0$. But here we are interested in the abelian case in order to deal with exact sequences easily.

To avoid any commotion of notations, we get rid of using superscripts on differential maps, we write d_n instead of d_n^A for any complex A. It is easy to observe that the class of all complexes over $\mathcal{C}$ constitutes a preadditive category, denoted by $\mathbf{C}(\mathcal{C})$, with chain morphisms and the canonical composition. The *nth cycle*, $Z_n(A)$, of a complex A is Kerd_n. Similarly, *the nth boundary*, $B_n(A)$, of A is $\operatorname{Im} d_{n+1}$. The *$n$th homology*, $H_n(A)$, of A is $Z_n(A)/B_n(A)$. A complex A is said to be *acyclic* (or *exact*) if $H_n(A) = 0$ for all $n \in \mathbb{Z}$, i.e.,

$$A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1}$$

is exact for each $n \in \mathbb{Z}$. We denote the class of acyclic complexes by Acic. A classical diagram chasing shows that $H_n : \mathbf{C}(\mathcal{C}) \to \mathcal{C}$ is an additive functor. There are two canonical ways to embed a category $\mathcal{C}$ in its category of complexes $\mathbf{C}(\mathcal{C})$ which are very useful for the theory. The first one is

$$S^n : \mathcal{C} \longrightarrow \mathbf{C}(\mathcal{C}),$$

$$X \rightsquigarrow \ldots \to 0 \to X \to 0 \to \ldots,$$

$$\begin{array}{ccccccccc} X & \rightsquigarrow & \cdots \longrightarrow & 0 & \longrightarrow & X & \longrightarrow & 0 & \longrightarrow \cdots \\ {\scriptstyle f}\downarrow & & & \downarrow & & \downarrow{\scriptstyle f} & & \downarrow & \\ Y & & \cdots \longrightarrow & 0 & \longrightarrow & Y & \longrightarrow & 0 & \longrightarrow \cdots \end{array}$$

where X and Y are located at nth place. For any complex $A \in \mathbf{C}(\mathcal{C})$, we have natural isomorphisms:

- $\mathrm{Hom}_{\mathbf{C}(\mathcal{C})}(S^n(X), A) \cong \mathrm{Hom}_{\mathcal{C}}(X, Z_n(A))$,
- $\mathrm{Hom}_{\mathbf{C}(\mathcal{C})}(A, S^n(X)) \cong \mathrm{Hom}_{\mathcal{C}}(\mathrm{Coker} d_{n+1}, X)$.

This means that (S^n, Z_n) and $(\mathrm{Coker} d_{n+1}, S^n)$ are adjoint pairs. So S^n preserves both any colimits and limits. The other one is

$$D^n : \mathcal{C} \longrightarrow \mathbf{C}(\mathcal{C})$$

$$X \rightsquigarrow \cdots \longrightarrow 0 \longrightarrow X \xrightarrow{\mathrm{id}} X \longrightarrow 0 \longrightarrow \cdots,$$

$$
\begin{array}{ccccccccccccc}
X & \rightsquigarrow & \cdots & \longrightarrow & 0 & \longrightarrow & X & \xrightarrow{\mathrm{id}} & X & \longrightarrow & 0 & \longrightarrow & \cdots \\
{\scriptstyle f}\downarrow & & & & \downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow & & \\
Y & & \cdots & \longrightarrow & 0 & \longrightarrow & Y & \xrightarrow{\mathrm{id}} & Y & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}
$$

where X and Y are placed at nth and $n-1$th place. Again, we have

- $\mathrm{Hom}_{\mathbf{C}(\mathcal{C})}(D^n(X), A) \cong \mathrm{Hom}_{\mathcal{C}}(X, A_n)$,
- $\mathrm{Hom}_{\mathbf{C}(\mathcal{C})}(A, D^{n+1}(X)) \cong \mathrm{Hom}_{\mathcal{C}}(A_n, X)$.

These imply that (D^n, p_n) and (p_n, D^{n+1}) are adjoint pairs where p_n is the canonical functor, $p_n(A) := A_n$. For any object $X \in \mathcal{C}$, there is a canonical exact sequence of complexes

$$0 \to S^{n-1}(X) \to D^n(X) \to S^n(X) \to 0.$$

As an application of these facts, if G is a generator for a category $\mathcal{C}$, then the family $\{D_n(G)\}_{n\in\mathbb{Z}}$ is a generating set for the category of complexes $\mathbf{C}(\mathcal{C})$ on $\mathcal{C}$.

Proposition 2.2.3. *Let $\mathcal{C}$ be an abelian category with a generator G and A be a complex in $\mathbf{C}(\mathcal{C})$. If the functor $Hom_{\mathbf{C}(\mathcal{C})}(-, A)$ leaves each short exact sequence, $n \in \mathbb{Z}$,*

$$0 \to S^{n-1}(G) \to D^n(G) \to S^n(G) \to 0$$

exact, then A is exact.

Proof. By assumption, we have an exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{C}}(G, Z_n(A)) \to \mathrm{Hom}_{\mathcal{C}}(G, A_n) \to \mathrm{Hom}_{\mathcal{C}}(G, Z_{n-1}(A)) \to 0$$

for each $n \in \mathbb{Z}$, it means

$$0 \to Z_n(A) \to A_n \to Z_{n-1}(A) \to 0$$

is exact since G is a generator of $\mathcal{C}$. $\square$

Consider a functor $F : I \to \mathcal{C}$. Then we have functors $S^0 \circ F : I \to \mathbf{C}(\mathcal{C})$ and $p_0 \circ S^0 \circ F : I \to \mathcal{C}$, where S^0 and p_0 preserve any limits and colimits and $p_0 \circ S^0 = \text{id}$. So colimit (limit) of F exists in $\mathcal{C}$ if and only if colimit (limit) of $S^0 \circ F$ exists in $\mathbf{C}(\mathcal{C})$. Conversely, let $F : I \to \mathbf{C}(\mathcal{C})$ be a functor. Then this generates a family of functors $\{F_n : I \to \mathcal{C}\}_{n \in \mathbb{Z}}$, $F_n = p_n \circ F$. There is a canonical natural transformation between F_n and F_{n-1} for each $n \in \mathbb{Z}$, i.e., $d_n : F_n(i) \to F_{n-1}(i)$. Firstly, we observe that if $\lim F$ exists, then $\lim(p_n \circ F) \cong p_n(\lim F)$ exists in $\mathcal{C}$ for each $n \in \mathcal{C}$. For the contrary, if $\lim F_n$ and $\lim F_{n-1}$ exist in $\mathcal{C}$, then there is a unique morphism $\lim F_n \to \lim F_{n-1}$ such that the diagram

$$\begin{array}{ccc} \lim F_n & \longrightarrow & \lim F_{n-1} \\ \downarrow & & \downarrow \\ F_n(i) & \xrightarrow[d_n]{} & F_{n-1}(i) \end{array}$$

is commutative for each $i \in I$. Because morphisms $\lim F_n \to \lim F_{n-1}$, $n \in \mathbb{Z}$, are determined uniquely and $F_{n-1}(i) \to F_n(i) \to F_{n+1}(i) = 0$ for each $i \in I$, $\lim F_{n+1} \to \lim F_n \to \lim F_{n-1}$ is the zero morphism. We denote $(\lim F_n)_n$ for the complex obtained by taking degreewise limit $\lim F_n$, $n \in \mathbb{Z}$, whenever each one exists. By universal property, we have the following:

Proposition 2.2.4. *Let $F : I \to \mathbf{C}(\mathcal{C})$ be a functor. Then $\lim F$ (colim$F$) of F exists in $\mathbf{C}(\mathcal{C})$ if and only if degreewise limits $\lim(p_n \circ F)$ (colimits colim$(p_n \circ F)$), $\forall n \in \mathbb{Z}$, exist in $\mathcal{C}$.*

As a consequence of Proposition 2.2.4, each of equalizers, kernels, pullbacks, limits and their duals in $\mathbf{C}(C)$ is obtained componentwise in $\mathcal{C}$. So $0 \to A \to B \to C \to 0$ is a short exact sequence in $\mathbf{C}(\mathcal{C})$ if and only if

$$0 \to A_n \to B_n \to C_n \to 0$$

is a short exact sequence in $\mathcal{C}$ for each $n \in \mathbb{Z}$.

Proposition 2.2.5. *Let $\mathcal{C}$ be a category. Then*

(i) $\mathcal{C}$ is abelian if and only if $\mathbf{C}(\mathcal{C})$ is abelian.

(ii) $\mathcal{C}$ is a Grothendieck category if and only if $\mathbf{C}(\mathcal{C})$ is a Grothendieck category.

If $\mathcal{C}$ is an abelian category, then two times application of Snake Lemma give a long exact sequence of homologies of complexes. Indeed, for a short exact sequence $0 \to A' \to A \to A'' \to 0$ of complexes, there is an exact complex

$$0 \to Z_n(A') \to Z_n(A) \to Z_n(A'') \to A'_{n-1}/B_n(A') \\ \to A_{n-1}/B_n(A) \to A''_{n-1}/B_n(A'') \to 0$$

for each $n \in \mathbb{Z}$. Again, applying Snake Lemma for the following commutative diagram

$$\begin{array}{ccccccc} & & A'_n/B_{n+1}(A') & \longrightarrow & A_n/B_{n+1}(A) & \longrightarrow & A''_n/B_{n+1}(A'') & \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow & \\ 0 & \longrightarrow & Z_{n-1}(A') & \longrightarrow & Z_{n-1}(A) & \longrightarrow & Z_{n-1}(A'') & \end{array}$$

we get the following long exact sequence of homologies

$$\ldots \to H_n(B) \to H_n(C) \to H_{n-1}(A) \to H_{n-1}(B) \to \ldots.$$

Proposition 2.2.6. *Let $\mathcal{C}$ be an abelian category. For any object $C \in \mathcal{C}$ and complex $A \in \mathbf{C}(\mathcal{C})$*

(i) $Ext_{\mathcal{C}}(C, A_n) \cong Ext_{\mathbf{C}(\mathcal{C})}(D^n(C), A)$.

(ii) $Ext_{\mathcal{C}}(A_n, C) \cong Ext_{\mathbf{C}(\mathcal{C})}(A, D^{n+1}(C))$.

(iii) $Ext_{\mathcal{C}}(C, Z_n(A)) \hookrightarrow Ext_{\mathbf{C}(\mathcal{C})}(S^n(C), A)$.

(iv) $Ext_{\mathcal{C}}(A_n/B_n(A), C) \hookrightarrow Ext_{\mathbf{C}(\mathcal{C})}(A, S^n(C))$.

Proof. See [Gill04, Lemma 3.1] for (i) and (ii). See [Gill08, Lemma 4.2]for (iii) and (iv). □

The following is easy to observe.

Proposition 2.2.7. *Let $\mathcal{C}$ be a category and A be a complex in $\mathbf{C}(\mathcal{C})$. Then we have:*

(i) A is a finitely presentable complex if and only if the set $\{i \in \mathbb{Z} : A_i \neq 0\}$ is finite (i. e. a bounded complex) and A_n is finitely presentable in $\mathcal{C}$, for each $n \in \mathbb{Z}$.

(ii) Let $\lambda > \aleph_0$ be a regular cardinal. Then A is a λ-presentable complex if and only if A_n is λ-presentable in $\mathcal{C}$ for each $n \in \mathbb{Z}$.

A complex A is said to be *bounded below (above)* if there exists $k \in \mathbb{Z}$ such that $A_n = 0$ for all $n < k$ $(n > k)$. In case it is bounded below and above, it is called *bounded*. We will denote by $\mathbf{C}_+(\mathcal{C}), \mathbf{C}_-(\mathcal{C}), \mathbf{C}_b(\mathcal{C})$ the subcategories consisting of bounded below, bounded above and bounded complexes, respectively. Now we introduce the truncation and suspension functors which are additive functors mapping a complex to a complex in $\mathbf{C}_+(\mathcal{C})$, $\mathbf{C}_-(\mathcal{C})$ and $\mathbf{C}(\mathcal{C})$. For any complex $A \in \mathbf{C}(\mathcal{C})$

- $\tau_{\geq n} : \mathbf{C}(\mathcal{C}) \to \mathbf{C}_+(\mathcal{C})$:

$$\tau_{\geq n}(A) \ : \ \ldots \to A_{n+2} \to A_{n+1} \to \mathrm{Ker}(d_n^A) \to 0 \to \ldots$$

- $\tau_{\leq n} : \mathbf{C}(\mathcal{C}) \to \mathbf{C}_-(\mathcal{C})$:

$$\tau_{\leq n}(A) : \ \ldots \to 0 \to 0 \to \mathrm{Coker}(d_{n+1}^A) \to A_{n-1} \to A_{n-2} \to \ldots.$$

- i-suspension functor for any $i \in \mathbb{Z}$, $[i] : \mathbf{C}(\mathcal{C}) \to \mathbf{C}(\mathcal{C})$:

  $A[i]$, is the complex given by $A[i]_n := A_{n-i}$ and $d_n^{A[i]} := (-1)^i d_{n-i}^A$ for any $n \in \mathbb{Z}$.

For any complex A and $m \in \mathbb{Z}$,

$$H_m(\tau_{\geq n}) = \begin{cases} H_m(A) & \text{if } m \geq n \\ 0 & \text{if } m < n \end{cases}$$

$$H_m(\tau_{\leq n}) = \begin{cases} H_m(A) & \text{if } m \leq n \\ 0 & \text{if } m > n \end{cases}$$

$$H_m(A[i]) = H_{m-i}(A).$$

There is an internal Hom-functor in $\mathbf{C}(\mathcal{C})$, $\mathcal{H}om_{\mathbf{C}(\mathcal{C})}(-,-) : \mathbf{C}(\mathcal{C})^{op} \times \mathbf{C}(\mathcal{C}) \to \mathbf{C}(\mathcal{C})$. For given complexes A and B, $\mathcal{H}om(A, B)$ is a complex whose nth module is

$$\mathcal{H}om(A,B)_n := \prod_{i \in \mathbb{Z}} \operatorname{Hom}(A_i, B_{i-n})$$

with differential $d_n : \mathcal{H}om(A,B)_n \to \mathcal{H}om(A,B)_{n-1}$, given by

$$d_n((f_i)_i) = (d^B_{i-n} \circ f_i - (-1)^n f_{i-1} \circ d^A_i)_i.$$

Then the nth cycle $Z_n(\mathcal{H}om(A,B))$ is the set of morphisms $\operatorname{Hom}_{\mathbf{C}(\mathcal{C})}(A, B[n])$. Suppose that $f := (f_i)_i$ is in the $B_n(\mathcal{H}om(A,B))$. Then there is a morphism $(g_i)_i \in \mathcal{H}om(A,B)_{n+1}$ such that $d_{i-n-1} \circ g_i - (-1)^{n+1} g_{i-1} \circ d_i = f_i$. We will see in the next section that it means $f \sim 0$, that is, a chain morphism $f : A \to B[n]$ and homotopic to zero. As a result, nth homology of $\mathcal{H}om(A,B)$, $H_n(\mathcal{H}om(A,B))$, is $\operatorname{Hom}_{\mathbf{C}(\mathcal{C})}(A, B[n])/_\sim$, that is, $\operatorname{Hom}_{\mathbf{K}(\mathcal{C})}(A, B[n])$.

2.3 Homotopy category of complexes

Definition 2.3.1. Let $\mathcal{C}$ be an additive category and $f, g : A \to B$ be two morphisms in $\mathbf{C}(\mathcal{C})$. It is said that f and g are *homotopic* if there are morphisms $s_i : A_i \to B_{i+1}$ for each $i \in \mathbb{Z}$, such that $f_i - g_i = d^B_{i+1} \circ s_i + s_{i-1} \circ d^A_i$. It is denoted by $f \sim g$.

The relation $\sim$ of being homotopic is an equivalence relation on the class of morphisms. It is well known that homotopy relation is closed under compositions, so it is a congruence on $\mathbf{C}(\mathcal{C})$. Then the quotient category (see Definition 1.1.3) of $\mathbf{C}(\mathcal{C})$ with respect to homotopy relation is called *the homotopy category of* $\mathcal{C}$ and denoted by $\mathbf{K}(\mathcal{C}) := \mathbf{C}(\mathcal{C})/_{\sim}$. It is also well-known that $\mathbf{K}(\mathcal{C})$ is an (pre)-additive category whenever $\mathcal{C}$ is (pre)-additive.

Definition 2.3.2. Let $f : A \to B$ be a morphism in $\mathbf{C}(\mathcal{C})$. Then it is called a

homotopy equivalence if there is a morphism $g : B \to A$ such that $g \circ f \sim \mathrm{id}_A$ and $f \circ g \sim \mathrm{id}_B$.

quasi-isomorphism if the corresponding homology morphism $H_n(f) : H_n(A) \to H_n(B)$ is an isomorphism for any $n \in \mathbb{Z}$.

It is clear from the definition that homotopy equivalences are precisely isomorphisms in $\mathbf{K}(\mathcal{C})$. The importance of homotopy equivalence comes from the fact that they are preserved under any additive functor from $\mathcal{C}$. Indeed, for any two homotopic morphisms $f \sim g$, $f_n - g_n = d_{n+1} \circ s_n + s_{n-1} \circ d_n$, and for any additive functor $F : \mathcal{C} \to \mathcal{D}$ we have $F(f_n) - F(g_n) = F(d_{n+1}) \circ F(s_n) + F(s_{n-1}) \circ F(d_n)$. As an application, homotopic morphisms give rise to the same morphisms on homologies, that is, if $f \sim g$ then $H_n(f) = H_n(g)$ for each $n \in \mathbb{Z}$. So any homotopy equivalence is a quasi-isomorphism, as well, and H_n is a functor from $\mathbf{K}(\mathcal{C})$. But the converse is not true, in general, a quasi-isomorphisms need not be a homotopy equivalence. For instance, the zero morphism $A \to 0$ from an exact complex A is always a quasi-isomorphism but it is a homotopy equivalence when it is contractible.

Definition 2.3.3. A complex A is said to be *contractible* if $\mathrm{id}_A \sim 0$.

The following is a standard consequence.

Proposition 2.3.4. *Let $\mathcal{C}$ be an abelian category and A be a complex in $\mathbf{C}(C)$. Then the following are equivalent:*

(i) A is contractible.

(ii) A is isomorphic to the zero complex 0 in $\mathbf{K}(\mathcal{C})$

(iii) A is an exact complex and

$$0 \to Z_n(A) \to A_n \to Z_{n-1}(A) \to 0$$

splits for all $n \in \mathbb{Z}$.

(iv) A is isomorphic to a direct sum of complexes of the form $D^n(X)$ where $X \in \mathcal{C}$ and $n \in \mathbb{Z}$.

The homotopy category $\mathbf{K}(\mathcal{C})$ is unlikely to be abelian even though $\mathcal{C}$ is an abelian category. The notions of monomorphisms and epimorphisms are not inherited by $\mathbf{C}(\mathcal{C})$, for example, a unit disc is homotopic to a point, that is, it is an isomorphism in the homotopy category but not even a monomorphism in the category of topological spaces or the inclusion of unit circle into unit disc is a homotopy equivalence, as well. There are few exact sequences in $\mathbf{K}(\mathcal{C})$, in fact, they are precisely split exact sequences in $\mathbf{K}(\mathcal{C})$, by the result of [Ver96].

In [Ver96], Verdier came up with a class of sequences in $\mathbf{K}(\mathcal{C})$, called *distinguished triangles*, which are nice enough to have a long exact sequences of homologies. Each morphism $[f] : A \to B$ in $\mathbf{K}(\mathcal{C})$ appears as a component in a distinguished triangle

$$C[-1] \longrightarrow A \xrightarrow{[f]} B \longrightarrow C$$

such that the left morphism and the right morphism act as kernel and cokernel of $[f]$, respectively. Categories with a class of sequences which satisfies some certain conditions as distinguished triangles do in $\mathbf{K}(\mathcal{C})$ are called *triangulated categories*, see for example [Nee01].

To get back to the issue at hand, the notion of mapping cone of a morphism in $\mathbf{C}(\mathcal{C})$ helps to construct distinguished triangles in $\mathbf{K}(\mathcal{C})$. The *mapping cone,* $c(f)$*, of* $f : A \to B$, a morphism in $\mathbf{C}(C)$, is a complex whose nth object is

$$c(f)_n := B_n \oplus A_{n-1}$$

with differential

$$d_n := \begin{pmatrix} d_n^B & 0 \\ f_{n-1} & -d_{n-1}^A \end{pmatrix} : \ B_n \oplus A_{n-1} \to B_{n-1} \oplus A_{n-2}.$$

There is also an exact sequence of complexes

$$0 \to B \to c(f) \to A[1] \to 0$$

which is a degreewise splitting exact sequence. Conversely, consider an exact sequence of complexes

$$0 \to B \to C \to A \to 0$$

which splits in each degree. Then for each $n \in \mathbb{Z}$, the short exact sequence $0 \to B_n \to C_n \to A_n \to 0$ is isomorphic to the canonical short exact sequence

$$0 \to B_n \to B_n \oplus A_n \to A_n \to 0.$$

Therefore C is of the form

$$\cdots \longrightarrow B_{n+1} \oplus A_{n+1} \xrightarrow{d_{n+1}} B_n \oplus A_n \xrightarrow{d_n} B_{n-1} \oplus A_{n-1} \xrightarrow{d_{n-1}} \cdots.$$

Commutativity of the diagram implies that the differential d_n of C must be of the form

$$\begin{pmatrix} d_n^B & 0 \\ \alpha_n & d_n^A \end{pmatrix}$$

where $\alpha_n : A_n \to B_{n-1}$, for each $n \in \mathbb{Z}$. Since $d_n \circ d_{n-1} = 0$, we get

$$d_n^B \circ \alpha_{n-1} + \alpha_n \circ d_{n-1}^A = 0$$

which leads to a morphism $\alpha := (\alpha_n) : A[-1] \to B$. So $c(\alpha) \cong C$. This means that an exact sequence $0 \to A \to C \to B \to 0$ which splits degreewise is isomorphic to a short exact sequence associated with a mapping cone

$$0 \to A \to c(\alpha) \to B \to 0. \tag{2.3.1}$$

We will see that α is unique up to homotopy.

The followings are well-known properties of the mapping cone.

Proposition 2.3.5. *Let $\mathcal{C}$ be an abelian category and $f : A \to B$ be a morphism in $\mathbf{C}(\mathcal{C})$. Then*

(i) $c(\mathrm{id}_A)$ *is contractible for each complex* $A \in \mathbf{C}(\mathcal{C})$.

(ii) f *in* $\mathbf{C}(\mathcal{C})$ *is a quasi-isomorphism if and only if* $c(f)$ *is exact.*

(iii) If $f \sim g$ *then* $c(f) \cong c(g)$.

(iv) $f \sim 0$ *if and only if*

$$0 \to B \to c(f) \to A[1] \to 0$$

is split exact in $\mathbf{C}(\mathcal{C})$.

(v)

$$0 \longrightarrow B \xrightarrow{\iota} c(f) \longrightarrow A[1] \longrightarrow 0$$

is split exact if and only if $[\iota]$ *is a section in* $\mathbf{K}(\mathcal{C})$*.*

Proof. (i) Take

$$s_n := \begin{pmatrix} 0 & \mathrm{id} \\ 0 & 0 \end{pmatrix} : A_n \oplus A_{n+1} \to A_{n-1} \oplus A_n$$

for each $n \in \mathbb{Z}$.

(ii) It follows from the associated exact sequence

$$0 \to B \to c(f) \to A[1] \to 0$$

of complexes and long exact sequence of homologies.

(iii) Suppose that $f \sim g : A \to B$ with homotopy morphisms $s_n : A_n \to B_{n+1}$ for each $n \in \mathbb{Z}$. Then

$$\begin{pmatrix} \mathrm{id} & 0 \\ s_{n-1} & \mathrm{id} \end{pmatrix} : B_n \oplus A_{n-1} \to B_n \oplus A_{n-1}$$

gives rise to an isomorphism of mapping cones $c(f)$ and $c(g)$.

(iv) Suppose that $f \sim 0$. By (iii), $0 \to B \to c(f) \to A[1] \to 0$ is isomorphic to $0 \to B \to c(0) \to A[1] \to 0$. Then the family of pointwise canonical projections $\{\pi_n B_n \oplus A_{n-1} \to B_n\}_{n\in\mathbb{Z}}$ is a chain morphism from $c(0)$ to B.

Conversely, suppose that it is split exact. Let $t : B[1] \to c(f)$ be a section. Then each t_n is of the form $(u_{n-1}, \mathrm{id}) : A_{n-1} \to B_n \oplus A_{n-1}$. Since t is a morphism of complexes, by the commutativity of each diagram,

$$d_n \circ u_{n-1} + f_{n-1} = -u_{n-2} \circ d_{n-1}.$$

Then $\{s_n : -u_n\}_{n\in\mathbb{Z}}$ is a family of homotopy maps between f and 0.

(v) Let $r := \{\begin{pmatrix} \alpha_n \\ t_{n-1} \end{pmatrix}\}_{n\in\mathbb{Z}}$ be a morphism from $c(f)$ to B, where $\alpha_n : B_n \to B_n$ and $t_{n-1} : A_{n-1} \to B_n$, in such a way that $[r] \circ [\iota] \sim \mathrm{id}_B$. Let $\{s_n : B_n \to B_{n+1}\}_{n\in\mathbb{Z}}$ be a homotopy between id_B and $r \circ \iota$. This implies that $\mathrm{id} - \alpha_{n-1} = s_{n-2} \circ d^B_{n-1} + d^B_n \circ s_{n-1}$ for each $n \in \mathbb{Z}$. Then

$$\begin{pmatrix} \mathrm{id} \\ t_{n-1} + s_{n-1} \circ f_{n-1} \end{pmatrix} : \quad B_n \oplus A_{n-1} \to B_n$$

gives rise to a chain map $c(f) \to B$ which is clearly a retraction of ι.

□

Proposition 2.3.6. *Let $\mathcal{C}$ be an abelian category. A complex P is projective (injective) in $\mathbf{C}(\mathcal{C})$ if and only if it is a contractible complex of projectives (injectives).*

More generally, a complex A has projective dimension $\leq n$ if and only if A is exact and each A_n and $Z_n(A)$ has projective dimension $\leq n$.

Proof. Sufficiency is clear by Proposition 2.3.4-(iv) and the fact that $D^n(X)$ is projective in $\mathbf{C}(\mathcal{C})$ if nd only if X is projective in $\mathcal{C}$. Now suppose that P is a projective complex. Then the short exact sequence

$$0 \to P[-1] \to c(\mathrm{id}) \to P \to 0$$

splits. By Proposition 2.3.5-(iv), $\mathrm{id} \sim 0$, that is, P is contractible. Besides, (p_n, D^{n+1}) is an adjoint pair where D^{n+1} is exact. So P_n is projective in $\mathcal{C}$ for each $n \in \mathbb{Z}$.

The other assertion is clear because exactness of complexes has the two-out-three property in a short exact sequence of complexes. □

Recall the internal hom, $\mathcal{H}om(-,-)$, of complexes from the previous section. Its nth homology is $\mathrm{Hom}_{\mathbf{K}(\mathcal{C})}(A, B[n])$. Then the following is immediate.

Corollary 2.3.7. *For $A, B \in \mathbf{C}(\mathcal{C})$, $\mathcal{H}om(A, B)$ is an exact complex if and only if $Hom_{\mathbf{K}(\mathcal{C})}(A, B[n]) = 0$ for all $n \in \mathbb{Z}$.*

For given complexes A, B, $\mathrm{Ext}_{dw}(A, B)$ is the subgroup of $\mathrm{Ext}(A, B)$ consisting of degreewise split short exact sequences $0 \to B \to X \to A \to 0$.

Proposition 2.3.8. *For given complexes $A, B \in \mathbf{C}(C)$, there is an isomorphism $Hom_{\mathbf{K}(\mathcal{C})}(A, B) \to Ext_{dw}(A[1], B)$.*

Proof. Let $\varphi : \mathrm{Hom}_{\mathbf{C}(\mathcal{C})}(A, B) \to \mathrm{Ext}_{dw}(A[1], B)$ be a morphism which sends a morphism $f : A \to B$ to the canonical exact sequence

$$0 \to B \to c(f) \to A[1] \to 0.$$

It is an epimorphism because of the argument discussed in 2.3.1. By Proposition 2.3.5-(iv), its kernel, Kerφ, is the set of morphisms $f : A \to B$ which are homotopic to zero. Then it induces an isomorphism

$$\overline{\varphi} : \mathrm{Hom}_{\mathbf{K}(\mathcal{C})}(A, B) \to \mathrm{Ext}_{dw}(A[1], B).$$

□

2.4 $\mathfrak{Qcoh}(X)$ as a category of representations

In this section we focus on the category of quasi-coherent sheaves and its equivalent category in terms of certain quiver representations. Firstly, we begin summarizing terminology. Most of them can be found in [GW10].

Let $(X, \mathcal{O}_X)$ be a scheme. We denote by $\mathcal{O}_X$-PreMod, $\mathcal{O}_X$-Mod and $\mathfrak{Qcoh}(X)$ the categories of $\mathcal{O}_X$-premodules, $\mathcal{O}_X$-modules and quasi-coherent sheaves over X, respectively. The notations $\mathfrak{PSh}(X)$ and $\mathfrak{Sh}(X)$ stand for the categories of presheaves and sheaves over X, respectively.

The category $\mathcal{O}_X$-PreMod is a locally finitely presentable category with enough projectives, see for example [PR04, Corollary 2.15]. The categories $\mathcal{O}_X$-Mod and $\mathfrak{Qcoh}(X)$ are known to be Grothendieck. For when $\mathcal{O}_X$-Mod is locally finitely presentable, see [PR04, Section 5].

For a presheaf $\mathcal{F}$ over X, remember that the *stalk of* $\mathcal{F}$ *at* $x \in X$ consists of the equivalence classes of the disjoint union of $\mathcal{F}(U)$, where U runs through all open neighborhoods of x,

$$\mathcal{F}_x = \big(\bigsqcup_{x\in U} \mathcal{F}(U)\big)/\sim$$

such that $a \sim b$ for $a \in \mathcal{F}(U)$ and $b \in \mathcal{F}(V)$ if and only if there exists an open neighborhood $W \subseteq U \cap V$ such that $a|_W = b|_W$. So, an element of $\mathcal{F}_x$ is represented by $s_x :=< U, s >$ for some open neighborhood U of x such that $s \in \mathcal{F}(U)$. Actually, s_x is the image of the section $s \in \mathcal{F}(U)$ in the stalk $\mathcal{F}_x$.

A scheme is called *quasi-compact* if it is quasi-compact as a topological space, that is, each of its open covers has a finite subcover. Therefore, $X = \mathrm{Spec}R$ is a quasi-compact topological space. It is not compact because $\mathrm{Spec}R$ is not Hausdorff in general.

Recall that for a morphism of schemes $f : X \to Y$, if $\mathcal{F}$ is a presheaf on X, then *the direct image* $f_*(\mathcal{F})$ is a presheaf defined on an open subset U of Y as $f_*(\mathcal{F})(U) := \mathcal{F}(f^{-1}(U))$. If $\mathcal{F}$ is either a sheaf or an $\mathcal{O}_X$-module then so is $f_*(\mathcal{F})$. But in general the direct image functor f_* doesn't preserve quasi-coherence. Let $x \in X$ and M be an $\mathcal{O}_{X,x}$-module. *The skyscraper* *sheaf* of M is the direct image sheaf $\iota_{*,x}(M)$ under the inclusion $\iota_x : \{x\} \hookrightarrow X$, precisely,

$$\iota_{*,x}(M)(U) = \begin{cases} M, & \text{if } x \in U \\ 0, & \text{otherwise} \end{cases}$$

If $\mathcal{G}$ is a presheaf on X, then the presheaf $f^+(\mathcal{G})$ is defined on an open subset U as

$$\varinjlim_{\substack{f(U)\subseteq V \\ V\subseteq Y}} \mathcal{G}(V)$$

where each V appeared in the index is open.

The *sheafification* functor $S : \mathfrak{Sh} \to \mathfrak{PSh}$, which is the left adjoint of the inclusion ι : $\mathfrak{PSh} \hookrightarrow \mathfrak{Sh}$, preserves the property of being $\mathcal{O}_X$-module and stalks due to its construction.

For a morphism $f : X \to Y$ of schemes, if $\mathcal{G}$ is a sheaf on Y, then $f^+(\mathcal{G})$ may not be a sheaf. So we consider its sheafification denoted by $f^{-1}\mathcal{G}$. It is called *the inverse image of* $\mathcal{G}$ *under* f. Then $(f^{-1}\mathcal{G})_x \cong \mathcal{G}_{f(x)}$. If X is an open subset of Y, then $f^{-1}\mathcal{G} \cong \mathcal{G} \mid_X$.

If $\mathcal{G} \in \mathcal{O}_X$-Mod, then $f^{-1}(\mathcal{G})$ is a $f^{-1}(\mathcal{O}_Y)$-module. But the map $\mathcal{O}_Y \to f_*\mathcal{O}_X$ corresponds to a map $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ by the adjoint property. So in order to convert it into an $\mathcal{O}_X$-module, it is tensorized by $\mathcal{O}_X$,

$$f^*(\mathcal{G}) := f^{-1}(\mathcal{G}) \otimes_{f^{-1}(\mathcal{O}_X)} \mathcal{O}_X.$$

Then we have a functor $f^* : \mathcal{O}_Y$-Mod $\to \mathcal{O}_X$-Mod whose stalk at $x \in X$ is $f^*(\mathcal{G})_x \cong \mathcal{G}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x}$. Note that the inverse image functor preserves tensor product.

Definition 2.4.1. Let X be a scheme. Then it is said to be

reduced if for every open subset $U \subseteq X$, $\mathcal{O}_X(U)$ has no nilpotent elements.

irreducible if for each decomposition $X = X_1 \cup X_2$ with X_1, X_2 closed subsets, one has $X_1 = X$ or $X_2 = X$.

integral if for any open subset $U \subseteq X$, the ring $\mathcal{O}_X(U)$ is an integral domain.

locally Noetherian If X can be covered by open affine subsets SpecA_i, where each A_i is a Noetherian ring.

Noetherian if it is both quasi-compact and locally Noetherian.

quasi-separated if the intersection of any two quasi-compact open subsets (or even open affines) is quasi-compact.

semi-separated if the intersection of two affine open subsets is again affine.

concentrated if it is both quasi-compact and quasi-separated.

Note that all types of schemes given in the definition have a local property, that is, for example, a scheme is locally Noetherian if and only if for every affine open subset Spec$A$ of X, A is a Noetherian ring. And a scheme is integral if and only if it is quasi-compact and irreducible.

Let $\mathcal{F}, \mathcal{G}$ be $\mathcal{O}_X$-modules. The *tensor product* $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is defined as the sheafification of the presheaf $U \to \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$, for each open subset $U \subseteq X$.

There is also an internal hom-functor in $\mathcal{O}_X$-Mod, $\mathcal{H}om$(-, -). The image $\mathcal{H}om(\mathcal{F}, \mathcal{G})(U)$ on an open subset $U \subseteq X$ is Hom$(\mathcal{F}|_U, \mathcal{G}|_U)$. For all $x \in X$, there is always a map

$$\mathcal{H}om(\mathcal{F}, \mathcal{G})_x \to \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x).$$

But in general, it is neither injective nor surjective.

For any $\mathcal{O}_X$-modules $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$, there are canonical isomorphisms

$$\text{Hom}(\mathcal{F}, \mathcal{H}om(\mathcal{G}, \mathcal{H})) \cong \text{Hom}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}, \mathcal{H}),$$

$$\mathcal{H}om(\mathcal{F}, \mathcal{H}om(\mathcal{G}, \mathcal{H})) \cong \mathcal{H}om(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}, \mathcal{H}),$$

$$\text{Hom}(\mathcal{O}_X, \mathcal{F}) \cong \mathcal{F}(X),$$

$$\mathcal{H}om(\mathcal{O}_X, \mathcal{F}) \cong \mathcal{F}.$$

That is, the category $\mathcal{O}_X$-Mod carries a closed symmetric monoidal structure, see Section 5.2.

The direct image functor doesn't commute neither with internal hom $\mathcal{H}om$ nor tensor product, that is, the canonical functors

$$f_*\mathcal{H}om(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om(f_*\mathcal{F}, f_*\mathcal{G}),$$

$$f_*(\mathcal{F}) \otimes f^*(\mathcal{G}) \to f_*(\mathcal{F} \otimes \mathcal{G})$$

aren't always monomorphisms nor epimorphisms. Besides, it doesn't preserve quasi-coherence unless X is Noetherian or f is quasi-compact and quasi-separated morphism.

Since $\mathfrak{Qcoh}(X)$ and $\mathcal{O}_X$-Mod are Grothendieck categories and the inclusion functor ι : $\mathcal{O}_X$-Mod $\hookrightarrow \mathfrak{Qcoh}(X)$ preserves limits and colimits, by the special adjoint functor theorem, Theorem 1.5.6, it has a right adjoint, $C : \mathfrak{Qcoh}(X) \to \mathcal{O}_X$-Mod known as *the coherator functor*. It redounds a closed symmetric monoidal structure on $\mathfrak{Qcoh}(X)$ because the tensor product of two quasi-coherent sheaves is quasi-coherent. The internal hom on $\mathfrak{Qcoh}(X)$ is defined as $C(\mathcal{H}om(\text{-}, \text{-}))$.

Recall that a quasi-coherent sheaf $\mathcal{F}$ is said to be *of type* κ, for κ an infinite cardinal, if each $\mathcal{F}(U)$ has fewer than κ generators over $\mathcal{O}_X(U)$ for each affine open subset $U \subseteq X$. Let κ be an infinite regular cardinal such that $\kappa > | \mathcal{O}_X(U) |$ for each affine open subset $U \subseteq X$ and

$\kappa >\mid H \mid$, where $H := \{\mathrm{res}_{UV} \mid$ for affine subsets $V \subseteq U \subseteq X\}$.

On the other hand, given $\mathcal{F} \in \mathfrak{Qcoh}(X)$, the *cardinality of* $\mathcal{F}$ is

$$\mid \mathcal{F} \mid := \sup\{\mid \mathcal{F}(U) \mid : \ U \in \mathcal{U}\},$$

(here $\mathcal{U}$ stands for the set of all affine open subsets of X). Note that if κ is as before, then $\mid \mathcal{F} \mid < \kappa$ if and only if $\mathcal{F}$ is of type κ.

Now we speak of the equivalent category of $\mathfrak{Qcoh}(X)$ given in [EE05, Section 2]. Note that by [Hart77, Proposition (2.5.2-(b,e))], an $\mathcal{O}_X$-module $\mathcal{F}$ for some scheme X is quasi-coherent if and only if it satisfies the following conditions on the affine open subsets:

(i) Let $V \subseteq U$ be two affine open subsets of the scheme X. Then we have an isomorphism of $\mathcal{O}_X(V)$-modules given by

$$\mathcal{O}_X(V) \otimes_{\mathcal{O}_X(U)} \mathcal{F}(U) \xrightarrow{\mathrm{id} \otimes f_{UV}} \mathcal{O}_X(V) \otimes_{\mathcal{O}_X(V)} \mathcal{F}(V) \cong \mathcal{F}(U)$$

where $f_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ is the restriction map of the $\mathcal{O}_X$-module $\mathcal{F}$.

(ii) Compatibility condition: If $W \subseteq V \subseteq U$ for affine open subsets W, V, U, then the composition

$$\mathcal{F}(U) \xrightarrow{f_{UV}} \mathcal{F}(V) \xrightarrow{f_{VW}} \mathcal{F}(W)$$

gives $\mathcal{F}(U) \xrightarrow{f_{UW}} \mathcal{F}(W)$.

Let $\mathcal{B}$ be a base of the scheme X containing affine open subsets such that $\mathcal{O}_X$ is $\mathcal{B}$-sheaf (see [EH00, Proposition I.12]). Now, define a quiver Q having the family $\mathcal{B}$ as the set of vertices, and an edge between two affine open subsets $U, V \in \mathcal{B}$ as the only one arrow $U \to V$ provided that $V \subsetneq U$. So the quiver comes equipped with the relation $U \to V \to W$ equals to $U \to W$ whenever defined. Fix this quiver. Take the representation $\mathbf{R}$ as $R(U) := \mathcal{O}_X(U)$ for each

$U \in \mathcal{B}$ and the restriction map $\rho_{UV} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$ for the edge $U \to V$.

An $\mathbf{R}$-*module* $\mathbf{M}$ is given by an $R(v)$-module $M(v)$, for each vertex $v \in V$, and an $R(v)$-linear morphism

$$\mathbf{M}(a) : M(v) \longrightarrow M(w)$$

for each edge $a : v \to w \in E$. We assume that $\mathbf{M}$ satisfies also the previous compatibility condition. Since $\mathbf{R}(a)$ is a ring homomorphism for an edge $a : v \to w$, the $R(w)$-module $M(w)$ can be thought as an $R(v)$-module.

An $\mathbf{R}$-module $\mathbf{M}$ is *quasi-coherent* if for each edge $a : v \to w$, the morphism

$$\mathrm{id}_{R(w)} \otimes_{R(v)} \mathbf{M}(a) : R(w) \otimes_{R(v)} M(v) \to R(w) \otimes_{R(w)} M(w)$$

is an $R(w)$-module isomorphism. The category of quasi-coherent $\mathbf{R}$-modules is denoted by $\mathbf{R}_{Qco}$-Mod.

Then the canonical functor

$$\Phi : \mathfrak{Qcoh}(X) \longmapsto \mathbf{R}_{Qco}\text{-Mod},$$

is an equivalence by [EH00, Proposition I.12] and [Hart77, Proposition (2.5.2-(b,e))]. This equivalence enables us to conceive the category of quasi-coherent sheaves better and to work with them easier. So we deal with $\mathbf{R}_{Qco}$-Mod rather than $\mathfrak{Qcoh}(X)$ when we have some concepts with local properties .

Let us make the equivalence concrete by an example.

Example 2.4.2. Let $X = \mathbb{P}^n_R$ be a projective scheme where $n \in \mathbb{N}$. Then again take a base containing affine open subsets $D_+(x_i)$ for all $i = 0, \dots n$, and all possible intersections. In this case, our base contains basic open subsets of this form

$$D_+(\prod_{i\in v} x_i),$$

where $v \subseteq \{0, 1, \ldots, n\}$.

So, the vertices of our quiver are all subsets of $\{0, 1, \ldots, n\}$ and we have only one edge $v \to w$ for each $v \subseteq w \subseteq \{0, 1, \ldots, n\}$ since $D_+(\prod_{i\in w} x_i) \subseteq D_+(\prod_{i\in v} x_i)$. Its representation has

$$\mathcal{O}_{\mathbb{R}^n_R}(D(\prod_{i\in v} x_i)) = R[x_0, \ldots, x_n]_{(\prod_{i\in v} x_i)}$$

on each vertex v, it is isomorphic to the polynomial ring on the ring R with the variables $\frac{x_j}{x_i}$ where $j = 0, \ldots, n$ and $i \in v$. We will denote this polynomial ring by $R[v]$. Then the representation $\mathbf{R}$ with respect to this quiver has vertex $R(v) = R[v]$ and edges $R[v] \hookrightarrow R[w]$ as long as $v \subseteq w$.

Finally, an $\mathbf{R}$-module $\mathbf{M}$ is quasi-coherent if and only if

$$S_{vw}^{-1} f_{vw} : S_{vw}^{-1} M(v) \longrightarrow S_{vw}^{-1} M(w) = M(w)$$

is an isomorphism as $R[w]$-modules for each $f_{vw} : M(v) \to M(w)$ where S_{vw} is the multiplicative group generated by the set $\{x_j/x_i |\, j \in w \setminus v,\ i \in v\} \cup \{1\}$ and $v \subset w$.

CHAPTER THREE

APPROXIMATIONS AND RELATIVE HOMOLOGICAL ALGEBRA

It has been always a natural question in any branch of mathematics to find properly a replacement of an object by an object of some special class. Depending on the context, 'properly' means that the replaced object respects certain operations and behaves as the original object. They are used to simplfy some arguments or to reduce problems to easier cases. As we will see later, for example, cofibrant and fibrant replacements in a model category are useful to control morphisms in the localization of a category, or a deleted projective resolution is a complex preserving quasi-isomorphisms, so the original object is represented by its deleted projective resolution in the derived category. In this chapter, we are interested in approximations in Homological Algebra. We continue getting to know must-have components of Homological Algebra.

The aim of the first section is to present the main concepts of Relative Homological Algebra, (pre)covers and (pre)envelopes. They are the concepts that allow us to approximate an object. All the well-known envelopes and covers in R-Mod such as injective envelopes by [EC53], projective covers by [Bas60] and torsion-free coverings by [Eno63] were defined and investigated separately. Enochs in [Eno81] brings them together with the notions of (pre)covers and (pre)envelopes. And in the same paper, he came up with the problem of the existence of flat covers, which is known as the Flat Cover Conjecture. These notions are known in Representation theory as *(minimal) right $\mathcal{F}$-approximations* and *(minimal) left $\mathcal{F}$-approximations* due to [AS80]. And preenvelopes are the same as *weak-reflections* in [AR94, Definition 4.5]. The good part of these concepts is that they are quite categorical concepts, hence, one can consider resolutions in any category and by any class of objects apart from the classical ones, injectives and projectives. We should point out the importance of the fact that these concepts are given in terms of Hom-functor: this gives that the attained resolutions are unique up to homotopy, see Comparison Theorem Proposition 3.1.7. So, all additive functors can be relatively derived.

In the second section, we introduce the principal tool, cotorsion pairs, of the general theory of approximations. We deal with approximations induced by a cotorsion pair. The notion was introduced by [Sal79] as an Ext-version of torsion theories for abelian groups. The problem, purposed by [Sal79], about when a cotorsion pair of abelian groups has enough injectives or projectives has been carried to any abelian categories. The theory looks for sufficient conditions on a cotorsion pair to have enough injectives or projectives. One of the results enlightening that problem was given by [ET01] with Salce Lemma: every cotorsion pair in R-Mod cogenerated by a set has enough projectives and injectives, i.e., it is complete. This result, as well as the purity notion, is the key point of the solution (see [BBE01]) to the Flat Cover Conjecture. It also displays the strong link of complete cotorsion pairs with approximations. Apart from the class of flat modules, the existence of projective precovers, injective envelopes, absolutely pure preenvelopes in R-Mod, etc,. can be deduced from this result, as well.

Model structures on a category are another kind of approximations, built by [Qui67]. An essential use in category theory is related to the existence of localization of categories, see Chapter (IV), because model structures make the localization category equivalent to a quotient category. We are interested in the connection between model categories and complete cotorsion pairs. Hovey in [Hov02] presented a one-to-one correspondence between abelian model categories and two compatible cotorsion pairs. Later Gillespie in [Gil11] extended it for a general setting, exact categories. In the last section, we stress this connection.

3.1 Covers and Envelopes

Let $\mathcal{C}$ be an abelian category.

Definition 3.1.1. Let $\mathcal{F}$ be a class of objects of $\mathcal{C}$.

- A morphism $\phi : F \to X$ of $\mathcal{C}$ is said to be an *$\mathcal{F}$-precover* of X if $F \in \mathcal{F}$ and if $\text{Hom}(F', F) \to \text{Hom}(F', X) \to 0$ is exact, for every $F' \in \mathcal{F}$. If any morphism $f : F \to F$ with $\phi \circ f = \phi$ is an isomorphism, then it is called an *$\mathcal{F}$-cover* of M. If every object has an *$\mathcal{F}$-(pre)cover*, then $\mathcal{F}$ is called a *(pre)covering class*.

- A morphism $\phi : X \to F$ of $\mathcal{C}$ is said to be an *$\mathcal{F}$-preenvelope* of X if $F \in \mathcal{F}$ and if $\text{Hom}(F, F') \to \text{Hom}(X, F') \to 0$ is exact, for every $F' \in \mathcal{F}$. If any morphism $f : F \to F$ such that $f \circ \phi = \phi$ is an isomorphism, then it is called an *$\mathcal{F}$-envelope* of M. If the class $\mathcal{F}$ is such that every object has an *$\mathcal{F}$-(pre)envelope*, then $\mathcal{F}$ is called a *(pre)enveloping class*.

In case $\mathcal{C} := R$-Mod and $\mathcal{F}$ is the class of either torsion free R-modules (over an integral domain R) or projective R-modules, the definition of $\mathcal{F}$-cover is not the same as the usual definitions of torsion-free covers as in [Eno63] and projective covers as in [Bas60] but they agree, see [Xu96, Theorem 1.2.12]. Dually, when $\mathcal{F}$ is the class of injective modules, the definition of $\mathcal{F}$-envelopes and injective envelopes agree, see [Xu96, Theorem 1.2.11].

The next well-known result is stated for modules in [Xu96], but it is known that its proof remains valid in any Grothendieck category.

Proposition 3.1.2. *[Bas06, Theorem 1.2] Let $\mathcal{C}$ be a Grothendieck category and $\mathcal{F}$ be a class of objects of $\mathcal{C}$ which is closed under direct limits. If an object $X \in \mathcal{C}$ has an $\mathcal{F}$-precover then X has an $\mathcal{F}$-cover, as well.*

$\mathcal{F}$-precovers or $\mathcal{F}$-preenvelopes are good enough to provide the machinery of relative homological algebra because $\mathcal{F}$-resolutions are unique up to homotopy. In addition, $\mathcal{F}$-covers and $\mathcal{F}$-envelopes give minimal ones among $\mathcal{F}$-precovers and $\mathcal{F}$-envelopes, respectively, in the sense that they are unique up to isomorphism and they are direct summand of any $\mathcal{F}$-precover and $\mathcal{F}$-envelopes. Note that $\mathcal{F}$-(pre)covers and $\mathcal{F}$-(pre)envelopes need not be epimorphisms and monomorphisms unless $\mathcal{F}$ contains a generating set and a cogenerating set for $\mathcal{C}$, respectively.

Definition 3.1.3. Let $\mathcal{F}$ be a class of objects of a category $\mathcal{C}$. *The right orthogonal class* $\mathcal{F}^{\perp}$ of $\mathcal{F}$ is

$$\mathcal{F}^{\perp} := \{X \in \mathcal{C} | \quad \text{Ext}(F, X) = 0, \forall F \in \mathcal{F}\}.$$

Dually, *the left orthogonal class* ${}^{\perp}\mathcal{F}$ of $\mathcal{F}$ is

$${}^{\perp}\mathcal{F} := \{X \in \mathcal{C} | \quad \text{Ext}(X, F) = 0, \forall F \in \mathcal{F}\}.$$

The left orthogonal class ${}^{\perp}\mathcal{F}$ of a class $\mathcal{F}$ is closed under coproducts, direct summands and extensions and contains all projective objects while the right orthogonal class $\mathcal{F}^{\perp}$ is closed under products, direct summands and extensions and contains all injective objects.

Definition 3.1.4. Let $\mathcal{C}$ be a category. Given a class of objects $\mathcal{F}$ of $\mathcal{C}$,

a special $\mathcal{F}$-precover of an object X is an epimorphism $\phi : F \to X$ such that $F \in \mathcal{F}$ and Ker$\phi \in \mathcal{F}^{\perp}$.

a special $\mathcal{F}$-preenvelope of an object X is a monomorphism $\phi : X \to F$ such that $F \in \mathcal{F}$ and Coker$\phi \in {}^{\perp}\mathcal{F}$.

From the long exact sequence of homology, we get that a special $\mathcal{F}$-precover is an $\mathcal{F}$-precover, as well. Dually for special $\mathcal{F}$-preenvelopes. The converse is not always true, but the following proposition, called Wakamatsu's lemma, says that from an $\mathcal{F}$-cover, it is possible to get a special $\mathcal{F}$-precover.

Proposition 3.1.5. *(Wakamatsu's lemma)[BR07, pg 28] Let $\mathcal{F}$ be a class which is closed under extensions. Then*

(i) If $\phi : F \to X$ is an $\mathcal{F}$-cover of X, then Ker$\phi \in \mathcal{F}^{\perp}$.

(ii) If $\phi : X \to F$ is an $\mathcal{F}$-envelope of X, Coker$\phi \in {}^{\perp}\mathcal{F}$.

For any class $\mathcal{F}$ of objects of a category $\mathcal{C}$, we say that a complex A is *Hom*$(\mathcal{F},-)$-*exact* when $\mathrm{Hom}(F,A)$ is an exact complex for every $F \in \mathcal{F}$. Dually, for $\mathrm{Hom}(-,\mathcal{F})$-exact complexes.

Definition 3.1.6. Let $\mathcal{C}$ be a category and $\mathcal{F}$ be a class of objects of $\mathcal{C}$. For a given object X of $\mathcal{C}$,

A left $\mathcal{F}$-resolution of X is a $\mathrm{Hom}(\mathcal{F},-)$-exact complex

$$\cdots \longrightarrow F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} X \longrightarrow 0\ ,$$

with $F_i \in \mathcal{F}$ for all $i \geq 0$, that is, each $F_0 \to X$ and $F_i \to \mathrm{Ker}d_{i-1}$, $i \geq 1$, are $\mathcal{F}$-precovers. In this case, $K_0 := X$, $K_i := \mathrm{Ker}d_{i-1}$, $i \geq 1$, are called the $0th$ and the ith $\mathcal{F}$-*syzygy* of X, respectively.

The deleted left $\mathcal{F}$-resolution of X, denoted by $\mathbf{F}_X$, is the complex

$$\cdots \longrightarrow F_1 \xrightarrow{d_1} F_0 \longrightarrow 0\ .$$

A right $\mathcal{F}$-resolution of X is a $\mathrm{Hom}(-,\mathcal{F})$-exact complex

$$0 \longrightarrow X \xrightarrow{d^0} F^0 \xrightarrow{d^1} F^1 \xrightarrow{d^2} \cdots\ ,$$

with $F^i \in \mathcal{F}$ for all $i \geq 0$, that is, $X \to F^0$ and $\mathrm{Coker}d^{i-1} \to F^i$, $i \geq 1$, are $\mathcal{F}$-preenvelopes. Furthermore, $C^0 := X$, $C^i := \mathrm{Coker}d^{i-1}$, $i \geq 1$, are called the $0th$ and the ith $\mathcal{F}$-*cosyzygy* of X, respectively.

The deleted right $\mathcal{F}$-resolution of X, denoted by $\mathbf{F}^X$, is the complex

$$0 \longrightarrow F^0 \xrightarrow{d^1} F^1 \xrightarrow{d^2} \cdots\ .$$

A complete $\mathcal{F}$-resolution of X is the complex

$$\ldots \to F_1 \to F_0 \to F^0 \to F^1 \to \ldots$$

obtained by composing a left and right $\mathcal{F}$-resolution of X over $F_0 \to X \to F^0$.

If $\mathcal{F}$ is a precovering class in a category $\mathcal{C}$ then every object has a left $\mathcal{F}$-resolution: Start with an $\mathcal{F}$-precover $d_0 : F_0 \to X$ of X and take its kernel, $K_1 := \text{Ker}d_0$, again an $\mathcal{F}$-precover $d_1 : F_1 \to K_1$ and so on. The following proposition is very well-known and its proof comes from classical arguments on commutative diagrams and universal properties of kernels and cokernels because of the fact that a left $\mathcal{F}$-resolution is a $\text{Hom}(\mathcal{F}, -)$-exact complex.

Proposition 3.1.7. *[Comparison Theorem] Let $\mathbf{F}_X \to X$ and $\mathbf{F}_{X'} \to X'$ be left $\mathcal{F}$-resolutions of X and X', respectively. Any morphism $\varphi : X \to X'$ induces a chain map $\phi : \mathbf{F}_X \to \mathbf{F}_{X'}$ which is unique up to homotopy. The dual one is for right resolutions.*

As a consequence of Comparison Theorem, any two left or right $\mathcal{F}$-resolutions of an object are homotopy equivalent.

3.2 Cotorsion pairs

All categories in this section will be abelian unless otherwise stated.

Definition 3.2.1. A *cotorsion pair*, or *cotorsion theory*, is a pair of classes $(\mathcal{F}, \mathcal{B})$ of objects of $\mathcal{C}$ such that $\mathcal{F}^{\perp} = \mathcal{B}$ and ${}^{\perp}\mathcal{B} = \mathcal{F}$.

Definition 3.2.2. A cotorsion pair $(\mathcal{F}, \mathcal{B})$ is said

to have enough injectives if for each object $C \in \mathcal{C}$, there exists a short exact sequence in $\mathcal{C}$

$$0 \to C \to B \to F \to 0$$

where $F \in \mathcal{F}$ and $B \in \mathcal{B}$.

to have enough projectives if for each object $C \in \mathcal{C}$, there exists a short exact sequence in $\mathcal{C}$

$$0 \to B \to F \to C \to 0$$

with $F \in \mathcal{F}$ and $B \in \mathcal{B}$.

complete if it has both enough injectives and projectives.

Definition 3.2.3. For a cotorsion pair $(\mathcal{F}, \mathcal{B})$ in a category $\mathcal{C}$, $\mathcal{C}$ is said to *have enough $\mathcal{F}$-objects* (*enough* $\mathcal{B}$-objects) if for every object $C \in \mathcal{C}$ there is an epimorphism (monomorphism) $F \to C \to 0$ $(0 \to C \to B)$ with $F \in \mathcal{F}$ $(B \in \mathcal{B})$.

The canonical cotorsion pairs in a category $\mathcal{C}$ are $(\mathcal{C}, \text{Inj})$ and $(\text{Proj}, \mathcal{C})$. But they are not always complete. A category $\mathcal{C}$ is said to *have enough projectives* if $(\text{Proj}, \mathcal{C})$ is complete, dually, to *have enough injectives* if $(\mathcal{C}, \text{Inj})$ is complete.

Proposition 3.2.4. *Let $\tau_1 := (\mathcal{F}_1, \mathcal{B}_1)$ and $\tau_2 := (\mathcal{F}_2, \mathcal{B}_2)$ be cotorsion pairs in $\mathcal{C}$ with $\mathcal{F}_1 \subseteq \mathcal{F}_2$. If τ_1 has enough projectives and τ_2 has enough injectives then τ_1 has enough injectives and τ_2 has enough projectives.*

Proof. Suppose that τ_1 has enough projectives and τ_2 has enough injectives. Then for any object $X \in \mathcal{C}$, there are short exact sequences $0 \to X \to B_2 \to F_2 \to 0$ and $0 \to B_1 \to F_1 \to F_2 \to 0$ where $B_2 \in \mathcal{B}_2$, $F_2 \in \mathcal{F}_2$, $F_1 \in \mathcal{F}_1$ and $B_1 \in \mathcal{F}_1$. We may embed them into a commutative diagram

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & B_1 & \Longrightarrow & B_1 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X & \longrightarrow & T & \longrightarrow & F_1 & \longrightarrow & 0 \\
 & & \Vert & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X & \longrightarrow & B_2 & \longrightarrow & F_2 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

with exact rows and columns. Since $\mathcal{B}_2 \subseteq \mathcal{B}_1, T \in \mathcal{B}_1$. So τ_1 has enough injectives.

Again by assumption, there are short exact sequences $0 \to B'_1 \to F'_1 \to X \to 0$ and $0 \to B'_1 \to B'_2 \to F'_2 \to 0$ with $B'_2 \in \mathcal{B}_2$, $F'_2 \in \mathcal{F}_2$, $F'_1 \in \mathcal{F}_1$ and $B'_1 \in \mathcal{F}_1$, which bring on the following commutative diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & B'_1 & \longrightarrow & F'_1 & \longrightarrow & X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \Vert & & \\
0 & \longrightarrow & B'_2 & \longrightarrow & T' & \longrightarrow & X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & F'_2 & \Longrightarrow & F'_2 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

with exact rows and columns. Then $T' \in \mathcal{F}_2$ and so τ_2 has enough projectives. $\square$

As a consequence of Proposition 2.3.4, we have:

Corollary 3.2.5. *Let $\mathcal{C}$ be a category and $(\mathcal{F}, \mathcal{B})$ a cotorsion pair in $\mathcal{C}$*

(i) If $\mathcal{C}$ has enough projectives and $(\mathcal{F}, \mathcal{B})$ has enough injectives then $(\mathcal{F}, \mathcal{B})$ is complete.

(ii) If $\mathcal{C}$ has enough injectives and $(\mathcal{F}, \mathcal{B})$ has enough projectives then $(\mathcal{F}, \mathcal{B})$ is complete.

(iii) If $\mathcal{C}$ has enough projectives and injectives, then $(\mathcal{F}, \mathcal{B})$ has enough injectives if and only if $(\mathcal{F}, \mathcal{B})$ has enough projectives.

The last assertion (iii) is called Salce lemma in case $\mathcal{C} := R$-Mod.

Definition 3.2.6. A class $\mathcal{F}$ of objects of an abelian category $\mathcal{C}$ is said to be

resolving provided that $\mathcal{F}$ is closed under extensions, Proj $\subseteq \mathcal{F}$ and if whenever

$$0 \to F' \to F \to F'' \to 0$$

is exact with $F, F'' \in \mathcal{F}$, F' belongs to $\mathcal{F}$, as well.

coresolving provided that $\mathcal{F}$ is closed under extensions, $\mathrm{Inj} \subseteq \mathcal{F}$ and if whenever

$$0 \to F' \to F \to F'' \to 0$$

is exact with $F, F' \in \mathcal{F}$, F'' belongs to $\mathcal{F}$, as well.

Definition 3.2.7. A cotorsion pair $(\mathcal{F}, \mathcal{B})$ is said to be *hereditary* if $\mathrm{Ext}^i(F, B) = 0$ for all $F \in \mathcal{F}$, $B \in \mathcal{B}$ and $i \geq 1$.

The following proposition ties up the notions of coresolving-resolving classes with hereditary cotorsion pairs.

Proposition 3.2.8. *[SS11, Lemma 4.25] Let $(\mathcal{F}, \mathcal{B})$ be a cotorsion pair in $\mathcal{C}$.*

(i) If it is hereditary, then $\mathcal{F}$ is resolving and $\mathcal{B}$ is coresolving.

(ii) If $\mathcal{C}$ has enough $\mathcal{F}$-objects and $\mathcal{F}$ is resolving then the cotorsion pair is hereditary.

(iii) If $\mathcal{C}$ has enough $\mathcal{B}$-objects and $\mathcal{B}$ is coresolving then the cotorsion pair is hereditary.

Proof. The part (i) is easy to observe. And (ii) and (iii) are dual to each other, so we just prove (ii). Assume that $\mathcal{C}$ has enough $\mathcal{F}$-objects and $\mathcal{F}$ is resolving. By induction, it is enough to prove $\mathrm{Ext}^2(F, B) = 0$ for all $F \in \mathcal{F}$ and $B \in \mathcal{B}$. Take an exact sequence

$$0 \to B \to X_1 \to X_2 \to F \to 0 \tag{3.2.1}$$

in $\mathcal{C}$. We split it into two short exact sequences.

$$0 \to K \to X_2 \to F \to 0,$$

$$0 \to B \to X_1 \to K \to 0.$$

By assumption, there is an epimorphism $F' \to X_2$ with $F' \in \mathcal{F}$. Then by usual pullback argument and equivalence relation on Yoneda extensions, the exact sequence 3.2.1 is equivalent to an exact sequence

$$0 \to B \to T \to F' \to F \to 0$$

with $\mathrm{Ker}(F' \to F) \in \mathcal{F}$ because $\mathcal{F}$ is a resolving class. Then the assertion follows. □

The next result is given in [AA02, Theorem 3.1] for the category R-Mod. But the argument is quite categorical. So we restate it here for abelian categories.

Proposition 3.2.9. *Let $(\mathcal{F}, \mathcal{B})$ be a hereditary cotorsion pair. Suppose that we have the* *following diagram with the exact row and exact columns*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \dashrightarrow & B_1 & \dashrightarrow & B & \dashrightarrow & B_2 & \dashrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \dashrightarrow & F_1 & \dashrightarrow & F & \dashrightarrow & F_2 & \dashrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

If the left and the right exact columns are special $\mathcal{F}$-precovers of X and Z, respectively, then the diagram can be completed to the commutative diagram with exact rows in such a way that the middle column is a special $\mathcal{F}$-precover of Y.

Now we discuss on some special cases of cotorsion pairs which guarantee the completeness.

Definition 3.2.10. Let $\mathcal{F}$ be any class of objects. The pair $(\mathcal{F}, \mathcal{F}^{\perp})$ (not necessarily a cotorsion pair!) is said to be

cogenerated by a set S if it satisfies the following:

$$X \in \mathcal{F}^{\perp} \Leftrightarrow \text{Ext}(F, X) = 0 \; \forall F \in S.$$

generated by a set S if it satisfies the following:

$$X \in \mathcal{F} \Leftrightarrow \text{Ext}(X, T) = 0 \; \forall T \in S.$$

Definition 3.2.11. Consider a direct system $(M_\alpha \mid \alpha \leq \lambda)$ of objects of $\mathcal{C}$ indexed on some ordinal λ. It is said to be *a continuous system of monomorphisms* if $M_0 = 0$, $M_\beta = \varinjlim_{\alpha<\beta} M_\alpha$ for each limit ordinal $\beta \leq \lambda$ and all the morphisms in the system are monomorphisms.

If $\mathcal{C}$ is a Grothendieck category, it is enough $M_\alpha \to M_{\alpha+1}$ to be a monomorphism for all $\alpha < \lambda$ in order to be a continuous direct system of monomorphisms, due to the exactness of the direct limit functor.

Definition 3.2.12. Let $\mathcal{S}$ be a class of objects which is closed under isomorphisms. An object M of $\mathcal{C}$ is said to be *$\mathcal{S}$-filtered* if there is a continuous system $(M_\alpha \mid \alpha \leq \lambda)$ of subobjects of M which satisfies $M = M_\lambda$ and $M_{\alpha+1}/M_\alpha$ is isomorphic to an object of $\mathcal{S}$ for each $\alpha < \lambda$. The chain, $(M_\alpha \mid \alpha \leq \lambda)$, is called *$\mathcal{S}$-filtration* of M.

The class of $\mathcal{S}$-filtered objects in $\mathcal{C}$ is denoted by Filt$(\mathcal{S})$. The relation $\mathcal{S} \subseteq$ Filt$(\mathcal{S})$ always holds. In case of being Filt$(\mathcal{S}) \subseteq \mathcal{S}$, the class $\mathcal{S}$ is said to be *closed under filtrations*. A class $\mathcal{F}$ of objects in $\mathcal{C}$ is called *deconstructible* if there is a set S such that $\mathcal{F} =$ Filt$(\mathcal{S})$.

Proposition 3.2.13. *[Sto13a, Lemma 1.6] Let $\mathcal{C}$ be a Grothendieck category. Every deconstructible class in $\mathcal{C}$ is closed under filtrations. In particular, it is closed under extensions and coproducts.*

Proof. Let $\mathcal{F} =$ Filt$(\mathcal{S})$ be a deconstructible class with a set $\mathcal{S}$ and let X be an object with $\mathcal{F}$-filtration, $(X_\alpha)_{\alpha \leq \sigma}$. The idea is to fill each gap between two consecutive subobjects with some $\mathcal{S}$-filtration and to shift ordinals. Consider the canonical short exact sequence $0 \to X_\alpha \to X_{\alpha+1} \to X_{\alpha+1}/X_\alpha \to 0$ for each $\alpha < \sigma$. By assumption, $X_\alpha/X_{\alpha+1}$ has an $\mathcal{S}$-filtration $(Y_{\gamma(\alpha+1)}/X_{\alpha+1})_{\gamma \leq \sigma_{\alpha+1}}$. Then $(Y_{\gamma\alpha})_{\gamma\alpha}$ with $\gamma \leq \sigma_\alpha$ for each $\alpha < \sigma$ gives the desired refined $\mathcal{S}$-filtration of X. $\square$

The dual concept of continuous system and filtrations are continuous inverse systems and inverse transfinite extensiones, see [EIJ07].

The original version of the following proposition, known as Eklof Lemma, was stated first for modules in [Ekl77]. But it is already known that it remains true for any abelian category.

Proposition 3.2.14. *(Eklof Lemma)[EO02, Proposition 3.1.1] Let $\mathcal{C}$ be an abelian category with direct limits and A, C be objects of $\mathcal{C}$. If A is the direct limit of a continuous chain of subobjects $\{A_\alpha\}_{\alpha<\lambda}$ for an ordinal number λ, such that Ext$(A_0, C) =$ Ext$(A_{\alpha+1}/A_\alpha, C) = 0$ for all $\alpha < \lambda$, then Ext$(A, C) = 0$.*

By Proposition 3.2.14, in such a category, we may deduce that if $(\mathcal{F}, \mathcal{B})$ is a cotorsion pair then $\mathcal{F}$ is closed under filtrations, $\mathcal{F} = \text{Filt}(\mathcal{F})$. And we already know that $\mathcal{F}$ is closed under direct summands and extensions. A natural question raises up here: Are these sufficient conditions for a class to be the left part of a cotorsion pair? Later, we will see that it is true for deconstructible classes having a generator of $\mathcal{C}$. Note that another consequence of Eklof Lemma is that if $\mathcal{F}$ is a deconstructible class, then the pair $(\mathcal{F}, \mathcal{F}^\perp)$ is cogenerated by a set.

Now, suppose that $\mathcal{C}$ is a Grothendieck category with a generator G. For an object $S \in \mathcal{C}$, there is a set $\mathcal{T}$ of short exact sequences

$$0 \to K \to T \to S \to 0$$

such that every short exact sequence

$$0 \to X \to Y \to S \to 0$$

can be obtained as a pushout diagram from a short exact sequence in $\mathcal{T}$. The argument is based on the generator G, but we refer to [Sto13, Proposition 5.3] for details. As a result, an object X belongs to $S^\perp$ if and only if $\text{Hom}(-, X)$ leaves exact each short exact sequence in $\mathcal{T}$. In this respect, cogenerated pairs $(\mathcal{F}, \mathcal{F}^\perp)$ in a Grothendieck category resemble the cotorsion pair $(\mathcal{C}, \text{Inj})$ because they admit a kind of 'Baer criterion'. Then Quillen's small object argument can be applied for them, as well.

Proposition 3.2.15. *Let $\mathcal{C}$ be a Grothendieck category and $\mathcal{F}$ be a class of objects of $\mathcal{C}$ closed under direct sums and extensions. If the pair $(\mathcal{F}, \mathcal{F}^\perp)$ is cogenerated by a set $\mathcal{S}$, then for any* *object $X \in \mathcal{C}$, there exists a short exact sequence*

$$0 \to X \to C \to F \to 0 \tag{3.2.2}$$

where $C \in \mathcal{F}^{\perp}$ and $F \in$ Filt$(\mathcal{S})$.

Proof. Let $S = \bigoplus_{S_i \in \mathcal{S}} S_i$. By assumption, $S \in \mathcal{F}$ and S is a generating object for $(\mathcal{F}, \mathcal{F}^{\perp})$. Now take an object $X \in \mathcal{C}$. Then $\mathbf{M}_1$ is the pushout

$$\begin{array}{ccc} \bigoplus_{\mathcal{T}} K^{(I_K)} & \hookrightarrow & \bigoplus_{\mathcal{T}} T^{(I_K)} \\ \downarrow & & \downarrow \\ X & \hookrightarrow & \mathbf{M}_1(X). \end{array}$$

where $I_K := \mathrm{Hom}(K, X)$ and the coproduct is indexed by all short exact sequences belonging to $\mathcal{T}$. By the same argument in Proposition 1.6.3, $\mathbf{M}_\alpha(X)$ is defined inductively. Note that $\mathbf{M}_1(X)/X \cong \bigoplus_{\mathcal{T}} S^{(I_K)}$. Due to the pushot property, $\mathbf{M}_{\alpha+1}/\mathbf{M}_\alpha$ is isomorphic to a coproduct of S. There is a regular ordinal λ for which each K is λ-presentable. Then for a regular cardinal $\beta > \lambda$, $\mathbf{M}_\beta(X)$ belongs to $\mathcal{F}^{\perp}$ and $\mathbf{M}_\beta(X)/X \cong (\varinjlim_{\alpha<\beta} \mathbf{M}_\alpha)/X \cong \varinjlim_{\alpha<\beta}(\mathbf{M}_\alpha/X)$. By construction, $\mathbf{M}_\beta(X)/X \in$ Filt$(\bigoplus_{\mathcal{T}} S)$. Then Proposition 3.2.13 implies that $\mathbf{M}_\lambda(X)/X \in$ Filt$(\mathcal{S})$. $\square$

Remark 3.2.16. The previous proposition is explicitly stated in [SS11, 2.13]. But it appears in [EEGO04, Theorem 2.5] with an extra hypothesis of that $\mathcal{F}$ is closed under well-ordered direct limits. Howewer it is just to say that F in 3.2.2 belongs to $\mathcal{F}$. A careful reader may notice that in the proof of [EEGO04, Theorem 2.5] what they find as a quotient of an $\mathcal{F}^{\perp}$-preenvelope of an object X is an object belonging to Filt$(\mathcal{S})$ without assuming that assumption on $\mathcal{F}$.

The following result is proved separately in [Eno12] for R-Mod and in [SS11] for Grothendieck categories. But we see that it is a direct consequence of Proposition 3.2.15. The proof is due to Prof. Sergio Estrada by a personal communication.

Proposition 3.2.17. *Any deconstructible class in a Grothendieck category is a precovering class.*

Proof. Let $\mathcal{F}$ be a deconstructible class and let X be an object in $\mathcal{C}$. Then by Proposition 3.2.15 and Proposition 3.2.13, the pair $(\mathcal{F}, \mathcal{F}^{\perp})$ has enough injectives. Now consider the subobject $\overline{X}$ of X which is the sum of all images of morphisms with domain in $\mathcal{F}$, that is,

$$\overline{X} := \sum_{\substack{\varphi \in \text{Hom}(F,X) \\ F \in \mathcal{F}}} \text{Im}(\varphi).$$

Since $\mathcal{C}$ is well-powered, this sum is well-defined. By Proposition 3.2.13, $\mathcal{F}$ is closed under coproducts, so there is a short exact sequence $0 \to K \to G \to \overline{X} \to$ with $G \in \mathcal{F}$. We consider the exact sequence $0 \to K \to C \to F \to 0$ with $F \in \mathcal{F}$ and $C \in \mathcal{F}^{\perp}$ obtained by Proposition 3.2.15. A standard pushout argument gives an $\mathcal{F}$-precover of $\overline{X}$, which is an $\mathcal{F}$-precover of X, as well because of the construction of $\overline{X}$.

□

In an abelian category $\mathcal{C}$, a cotorsion pair $(\mathcal{F}, \mathcal{B})$ is said to be *small* if $\mathcal{F}$ contains a generator and there is a set of monomorphism ι_S with cokernel $S \in \mathcal{F}$ such that $B \in \mathcal{B}$ if and only if B has an injective property with respect to each ι_S. For a Grothendieck category, in [EGPT12, Lemma 4.3], it was proved that a small cotorsion pair is just a cotorsion pair $(\mathcal{F}, \mathcal{B})$ cogenerated by a set where $\mathcal{F}$ has a generator.

Corollary 3.2.18. *[Sto13a, Proposition 2.9] Let $\mathcal{C}$ be a Grothendieck category and $\mathcal{F}$ be a class of objects of $\mathcal{C}$. Then*

(i) If $\mathcal{F}$ is a deconstructible class, $\mathcal{F} =$ Filt$(\mathcal{S})$, where $\mathcal{S}$ is a set, then the class of all direct summands of Filt(S) is a deconstructible class, as well.

(ii) If $(\mathcal{F}, \mathcal{B})$ is a cotorsion pair cogenerated by a set $\mathcal{S}$ such that $\mathcal{F}$ contains a generator G of $\mathcal{C}$, then $\mathcal{F}$ consists of direct summands of $(\mathcal{S} \cup G)$-filtered objects. In addition, it is complete.

Proof. (i) It follows from [Sto13a, Proposition 2.9].

(ii) It is a small cotorsion pair. Then it follows from [Gill07, Lemma 3.6]. As an alternative proof, note that Filt$(\mathcal{S} \cup G)$-precovers are epimorphisms because $G \in$ Filt$(\mathcal{S} \cup G)$. For $F \in \mathcal{F}$, there is a commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & X & \to & K & \to & F & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & B & \to & K'' & \to & F & \to & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & K' & = & K' & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array} \qquad (3.2.3)$$

where $B \in \mathcal{B}$, $K' \in$ Filt$(\mathcal{S} \cup G)$ and $K \to F$ is a Filt$(\mathcal{S} \cup G)$-precover by Proposition 3.2.15 and 3.2.17. By Proposition 3.2.13, $K'' \in$ Filt$(\mathcal{S} \cup G)$. Then F is a direct summand of K''. If we replace any object Y instead of F in the diagram 3.2.3, then the second exact row would be a special $\mathcal{F}$-precover of Y. Since the pair $(\mathcal{F}, \mathcal{B})$ has already special $\mathcal{B}$-preenvelopes, then it is complete.

□

We already know that a cotorsion pair whose left class is deconstructible is always a cotorsion pair cogenerated by a set. What Corollary 3.2.18 says is that the converse is true for a Grothendieck category when the left class of a cotorsion pair contains a generator for $\mathcal{C}$.

Corollary 3.2.19. *Let $\mathcal{C}$ be a Grothendieck category. If $\mathcal{F}$ is a deconstructible class which is closed under direct summands and contains a generator, then $(\mathcal{F}, \mathcal{F}^{\perp})$ is a complete cotorsion pair.*

Proof. Let $\mathcal{F} = \text{Filt}(\mathcal{S})$. Then $({}^{\perp}(\mathcal{F}^{\perp}), \mathcal{F}^{\perp})$ is a cotorsion pair cogenerated by $\mathcal{S}$ and ${}^{\perp}(\mathcal{F}^{\perp})$ contains a generator G for $\mathcal{C}$. Then by Corollary 3.2.18-(ii), ${}^{\perp}(\mathcal{F}^{\perp})$ consists of direct summands of Filt$(\mathcal{S} \cup G)$, that is, ${}^{\perp}(\mathcal{F}^{\perp}) \subseteq \mathcal{F}$. □

A typical example of the previous corollary is the class of flat R-modules, $\mathcal{F}lat(R)$. Flat modules are closed under pure-subobjects, pure-quotients and filtrations. Besides, the category R-Mod has enough pure-subobjects. In that manner, a filtration of a flat R-module can be constructed. Then the class $\mathcal{F}lat(R)$ constitutes a left class of a cotorsion pair. Modules in $\mathcal{F}lat(R)^{\perp}$ are called *cotorsion modules*.

Proposition 3.2.20. *[Bas06, Theorem 3.2] Let $\mathcal{C}$ be a Grothendieck category and $\mathcal{F}$ be a class of objects of $\mathcal{C}$ which is closed under coproducts and direct colimits. If there is a subset S of $\mathcal{F}$ such that every object of $\mathcal{F}$ can be written as a direct limit of objects from S, then $\mathcal{F}$ is covering.*

3.3 Model category and cotorsion pairs: Hovey correspondence

In this section, we focus on the connection between complete cotorsion pairs and model categories which was given by Hovey in [Hov02] in a general setting, that is, for proper classes in an abelian category. Later, it was carried into exact categories in a full generality by Gillespie in [Gil11]. We use this connection in order to define pure derived categories in Chapter (VII). But here we prefer to keep terminology and results needed as basic as possible in order to comprehend the underlying idea.

We shall begin with the formal definition of model categories. Recall that a morphism i :

$A \to B$ is said to *have the left lifting property with respect to* $p : X \to Y$, or p is said to *have the right lifting property with respect to* i, if for every commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

there exist a morphism $h : B \to X$ which makes the triangles commutative.

Definition 3.3.1. A *model category* is a complete and cocomplete category $\mathcal{A}$, together with three classes of maps (*weak equivalences, fibrations* and *cofibrations*) satisfying the following axioms:

M1) Two out of three axiom: If f and g are morphisms in $\mathcal{C}$ such that $g \circ f$ is defined and two of f, g and $g \circ f$ are weak equivalences, then so is the third.

M2) Retract axiom: If f and g are maps in $\mathcal{C}$ such that f is a retract of g in $\mathrm{Mor}(\mathcal{A})$ and g is one of a weak equivalence, a fibration or a cofibration, then so is f.

M4) Lifting axiom: Given a commutative diagram in $\mathcal{A}$

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y, \end{array}$$

i has the left lifting property with respect to p and that p has the right lifting property with respect to i if

(i) i is a cofibration and p is a fibration which is also a weak equivalence ($=$ *trivial fibration*), or

(ii) p is a fibration and i is a cofibration which is also a weak equivalence ($=$ *trivial cofibration*).

M5) Factorization axiom: Any morphism $f \in \mathcal{A}$ admits functorial factorizations

(i) $f = q \circ i$, where i is a cofibration and q is a trivial fibration, and

(ii) $f = p \circ j$, where p is a fibration and j is a trivial cofibration.

Definition 3.3.2. Let $\mathcal{A}$ be a model category and A be an object. Then it is called

(trivial) cofibrant if the unique morphism $0 \to A$ is a (trivial) cofibration,

(trivial) fibrant if the unique morphism $A \to 0$ is a (trivial) fibration.

Axioms of a model category allow to do cofibrant and fibrant replacements for any object. Indeed, for any object $X \in \mathcal{A}$, consider the zero morphism $0 \to X$. Then by (M5)-(i), it has a factorization as $0 \to Q(X) \to X$ where $0 \to Q(X)$ is a cofibration. So $Q(X)$ is a cofibrant object and $Q(X) \to X$ is said to be *cofibrant replacement of* X. Analogously, $X \to 0$ is factorized through a morphism $X \to R(X)$ where $R(X)$ is a fibrant object. Then $X \to R(X)$ is called a *fibrant replacement of* X.

There are two homotopy relations on the set of morphisms. One of them is *the left homotopy relation*. It is defined in terms of cylinder objects. The other one is *the right homotopy relation*, which is given through path objects. If given two objects X and Y are both cofibrant and fibrant, then the left and right homotopy relations on Hom-set coincide and form an equivalence relation on morphisms. It is denoted by $\sim$ and

$$\pi(X, Y) := \mathrm{Hom}_{\mathcal{C}}(X, Y)/_{\sim}.$$

We refer to [DS95] for more details. For a model category $\mathcal{A}$, *The homotopy category*, denoted by $Ho(\mathcal{A})$, is the localization of $\mathcal{A}$ with respect to weak equivalences, that is, $Ho(\mathcal{C}) := \mathcal{A}[\mathcal{W}^{-1}]$. In Section (IV.1), we will detail what the localization of a category is, but roughly it is a category where morphisms in W are invertible. The localization of a category with respect to any class $\mathcal{W}$ of morphisms can be always defined. However, in general it is not easy at all to understand its morphisms. The advantage of finding a model structure on $\mathcal{A}$ with $\mathcal{W}$

the class of weak equivalences is the following fact:

$$\mathrm{Hom}_{Ho(\mathcal{A})}(X, Y) \cong \pi(RQ(X), RQ(Y)).$$

It means that a big class of morphisms in the localization category is turned out being isomorphic to just a quotient of a morphism set.

Definition 3.3.3. An *abelian model category* is a complete and cocomplete abelian category $\mathcal{A}$ equipped with a model structure such that

(i) a map is a cofibration if and only if it is a monomorphism with cofibrant cokernel,

(ii) a map is fibration if and only if it is an epimorphism with fibrant kernel.

Definition 3.3.4. A subcategory of an abelian category $\mathcal{A}$ is called *thick* if it is closed under direct summands and whenever two out of three entries in a short exact sequence are in the class, so is the third.

For a model category $\mathcal{A}$,
$\mathcal{C}$: the class of all cofibrant objects,
$\mathcal{F}$: the class of all fibrant objects,
$\mathcal{W}$: the class of all trivial objects.

Proposition 3.3.5. *[Hov02, Proposition 4.1, Lemma 4.3] If $\mathcal{A}$ is an abelian model category, then $\mathcal{W}$ is a thick subcategory and the pairs $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$, $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are complete cotorsion pairs.*

The next proposition states the converse of Proposition 3.3.5 and completes the Hovey correspondence.

Proposition 3.3.6. *[Hov02, Section 5] Suppose $\mathcal{C}, \mathcal{F}$ and $\mathcal{W}$ are three classes of objects in a bicomplete abelian category $\mathcal{A}$ such that* *(i)* $\mathcal{W}$ *is thick,*

(ii) $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ *and* $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ *are complete cotorsion pairs.*

Then there exists a unique abelian model structure on $\mathcal{C}$ *such that* $\mathcal{C}$ *is the class of cofibrant objects,* $\mathcal{F}$ *is the class of fibrant objects and* $\mathcal{W}$ *is the class of trivial objects.*

(Trivial) cofibrations of the model structure mentioned in Proposition 3.3.6 are monomorphisms with cokernel in $\mathcal{C}$, ($\mathcal{C} \cap \mathcal{W}$) and (trivial) fibrations are epimorphisms with kernel in $\mathcal{F}$, ($\mathcal{F} \cap \mathcal{W}$). A weak equivalence is a morphism which has a factorization with a trivial cofibration followed by a trivial fibration. Then a cofibrant replacement of an object X is an epimorphism with kernel from $\mathcal{F} \cap \mathcal{W}$ and a fibrant replacement of X is a monomorphism with cokernel from $\mathcal{C} \cap \mathcal{W}$.

Regarding to the Hovey correspondence given in Proposition 3.3.5 and 3.3.6, two complete cotorsion pairs of the form $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ are called *Hovey pairs*. Then the result says that, for an abelian category, there is a bijection between abelian model structures on it and Hovey pairs.

Besides, in an abelian model category, there is a nice characterization of right and left homotopy relations on morphisms (see [Gil11, Proposition 4.3]). Let $f, g : X \rightrightarrows Y$ be two morphisms.

(i) f is right homotopic to g if and only if $g - f$ factors through an object from $\mathcal{C} \cap \mathcal{W}$.

(ii) f is left homotopic to g if and only if $g - f$ factors through an object from $\mathcal{F} \cap \mathcal{W}$.

(iii) If X and Y are both fibrant and cofibrant, f is homotopic to g if and only if $g - f$ factors through an object from $\mathcal{F} \cap \mathcal{C} \cap \mathcal{W}$.

The most known model structures in homological algebra are on the category of complexes. Let us expand on it. If an abelian category $\mathcal{A}$ has enough projectives and X is a bounded below complex over $\mathcal{A}$, then it is a classical result that there is a quasi-isomorphism $P \to X$ with a bounded below complex P of projectives. In [Qui67, Section I.1, Example-B], Quillen shows that there is a model structure on $\mathbf{C}_+(\mathcal{A})$ whose cofibrant objects are bounded below complexes of projectives and fibrant objects are all bounded below complexes.

A bounded below complex P of projectives has an essential property in $\mathbf{C}(\mathcal{A})$: the internal hom functor, $\mathcal{H}om(P, \text{-})$ preserves quasi-isomorphisms and epimorphisms in $\mathbf{C}(\mathcal{A})$. Therefore, a complex (not necessarily bounded below) which satisfies that property in $\mathbf{C}(\mathcal{A})$ is called *dg-projective*. In [AH86, 1.9], it was proved that for every complex X over R-Mod there is a quasi-isomorphism $P \to X$ with a dg-projective complex P. In fact, that quasi-isomorphism can be taken as an epimorphism. In [Hov99, Section 2.3], Hovey constructs a cofibrantly generated model structure in $\mathbf{C}(R\text{-Mod})$ whose cofibrant objects are dg-projective complexes. That abelian model structure is called *the projective model structure on* $\mathbf{C}(R\text{-}Mod)$ and its homotopy category is the derived category, $\mathbf{D}(R)$. The dual argument can be done for *dg-injectives* and we get *the injective model structure*. The Hovey pairs which correspond to the projective model structure on $\mathbf{C}(R\text{-Mod})$ are

$$(dg\,\text{Proj}, \mathrm{Acic}), \quad (\text{Proj}, \mathbf{C}(\mathcal{C}))$$

where $dg\,\text{Proj}$ is the class of dg-projective complexes. As a matter of fact, they are obtained by lifting the canonical cotorsion pair $(\text{Proj}, R\text{-Mod})$ to $\mathbf{C}(R\text{-Mod})$. In [Gill04], Gillespie formalized this phenomenon for more general cotorsion pairs. The advantage is that one could find a model structure on the category of complexes over an abelian category so that its homotopy category would be the derived category even though it doesn't have enough neither injectives nor projectives.

Definition 3.3.7. Let $(\mathcal{F}, \mathcal{B})$ be a cotorsion pair in an abelian category $\mathcal{A}$ and X be a complex in $\mathbf{C}(\mathcal{A})$. Then

(i) $X \in \tilde{\mathcal{F}}$ if X is exact and $Z_n(X) \in \mathcal{F}$ for all n.

(ii) $X \in \tilde{\mathcal{B}}$ if X is exact and $Z_n(X) \in \mathcal{B}$ for all n.

(iii) $X \in dg\,\tilde{\mathcal{F}}$ if $X_n \in \mathcal{F}$ for each n and $\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(X, B) = 0$ for all $B \in \tilde{\mathcal{B}}$.

(iv) $X \in dg\,\tilde{\mathcal{B}}$ if $X_n \in \mathcal{B}$ for each n and $\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(A, X) = 0$ for all $A \in \tilde{\mathcal{F}}$.

From now on, $(\mathcal{F}, \mathcal{B})$ will be a cotorsion pair in an abelian category. By [Gill04, Lemma 3.10], we already know $\tilde{\mathcal{F}} \subseteq dg\,\tilde{\mathcal{F}} \cap \mathrm{Acic}$ and $\tilde{\mathcal{B}} \subseteq dg\,\tilde{\mathcal{B}} \cap \mathrm{Acic}$.

We recall some related results from [Gill04] and [Gill07].

Proposition 3.3.8. *(i) If $\mathcal{A}$ has enough $\mathcal{F}$ objects then $(\tilde{\mathcal{F}}, dg\,\tilde{\mathcal{B}})$ is a cotorsion pair and $\mathbf{C}(\mathcal{A})$ has enough $\tilde{\mathcal{F}}$ objects.*

(ii) If $\mathcal{A}$ has enough $\mathcal{B}$ objects then $(dg\,\tilde{\mathcal{F}}, \tilde{\mathcal{B}})$ is a cotorsion pair and $\mathbf{C}(\mathcal{A})$ has enough $\tilde{\mathcal{B}}$ objects.

In case $(dg\,\tilde{\mathcal{F}}, \tilde{\mathcal{B}})$ and $(\tilde{\mathcal{F}}, dg\,\tilde{\mathcal{B}})$ are cotorsion pairs, they are called *the induced cotorsion pairs*.

Proposition 3.3.9. *(i) If $\tilde{\mathcal{F}}$ or $dg\,\tilde{\mathcal{F}}$ is resolving then $\mathcal{F}$ is resolving, as well.*

(ii) If $\tilde{\mathcal{B}}$ or $dg\,\tilde{\mathcal{B}}$ is coresolving then $\mathcal{B}$ is coresolving, as well.

Proposition 3.3.10. *If $\mathcal{F}$ is resolving then $dg\,\tilde{\mathcal{F}}$ is resolving. If $\mathcal{B}$ is coresolving then $dg\,\tilde{\mathcal{B}}$ is coresolving.*

The pairs $(dg\,\tilde{\mathcal{F}}, \tilde{\mathcal{B}})$ and $(\tilde{\mathcal{F}}, dg\,\tilde{\mathcal{B}})$ are called *compatible* if $\tilde{\mathcal{F}} = dg\,\tilde{\mathcal{F}} \cap \mathrm{Acic}$ and $\tilde{\mathcal{B}} = dg\,\tilde{\mathcal{B}} \cap \mathrm{Acic}$. The reason why we insist on them to be compatible is to get a model structure on $\mathbf{C}(\mathcal{A})$ whose weak equivalences would be precisely quasi-isomorphisms. As a consequence, its homotopy category would correspond to the derived category. For the canonical cotorsion pairs $(\mathrm{Proj}, \mathcal{A})$ and $(\mathcal{A}, \mathrm{Inj})$, the induced cotorsion pairs are always compatible.

Proposition 3.3.11. *(i) If $\tilde{\mathcal{F}} = dg\,\tilde{\mathcal{F}} \cap \mathrm{Acic}$ and $\mathcal{C}$ has enough $\mathcal{F}$ objects then $\mathcal{F}$ is resolving.*

(ii) If $\tilde{\mathcal{B}} = dg\,\tilde{\mathcal{B}} \cap \mathrm{Acic}$ and $\mathcal{C}$ has enough $\mathcal{B}$ objects then $\mathcal{B}$ is coresolving.

For an abelian category with enough projectives and enough injectives, the compatibility of the induced cotorsion pairs is equivalent to the cotorsion pair $(\mathcal{F}, \mathcal{B})$ to be hereditary, see [Gill04, Theorem 3.12]. This is also the case for a small cotorsion pair in a Grothendieck category, as well.

Proposition 3.3.12. *[Gill07, Corollary 3.9] Let $(\mathcal{F}, \mathcal{B})$ be a small cotorsion pair in a Grothendieck category $\mathcal{C}$. Then the following are equivalent:*

(i) $(dg\,\tilde{\mathcal{F}}, \tilde{\mathcal{B}})$ and $(\tilde{\mathcal{F}}, dg\,\tilde{\mathcal{B}})$ are compatible.

(ii) $\mathcal{F}$ is resolving, or equivalently, $\mathcal{B}$ is coresolving.

(iii) $(\mathcal{F}, \mathcal{B})$ is hereditary.

(iv) $\tilde{\mathcal{F}} = dg\,\tilde{\mathcal{F}} \cap \mathrm{Acic}$.

While the problem of compatibility of the induced cotorsion pairs is solved by adding some extra conditions on a category and on cotorsion pairs, the problem of completeness of the induced cotorsion pairs is not known yet. That is, for any abelian category the completness of a cotorsion pair $(\mathcal{F}, \mathcal{B})$ need not imply the completness of the induced pairs. But [YL14] proved that in a module category R-Mod, the induced cotorsion pairs of a complete and hereditary cotorsion pair are hereditary, complete and compatible. The proof is based on Proposition 3.2.9. Note that it implies that for every exact complex E there is a short exact sequence

$$0 \to B \to F \to E \to 0$$

with $B \in \tilde{\mathcal{B}}$ and $F \in \tilde{\mathcal{F}}$. Then by the same argument done in [YL14], we have the following:

Proposition 3.3.13. *If $(\mathcal{F}, \mathcal{B})$ is a hereditary complete cotorsion pair in a Grothendieck category $\mathcal{A}$, then $(\tilde{\mathcal{F}}, dg\,\tilde{\mathcal{B}})$ is a complete cotorsion pair.*

Proof. It is a cotorsion pair because $\mathcal{A}$ is a Grothendieck category. Completeness follows from standard pullback arguments and the fact that for every complex X there are short exact sequences

$$0 \to I \to E \to X \to 0$$

$$0 \to X \to I' \to E' \to 0$$

where E and E' are acyclic complexes and I and I' are dg-injectives. □

Another result is by [SS11] for Grothendieck categories.

Proposition 3.3.14. *[Sto13a, Theorem 4.2] Let $\mathcal{A}$ be a Grothendieck category and $\mathcal{F}$ be a decontructible class of objects in $\mathcal{A}$. Then the classes $\mathbf{C}(\mathcal{F})$ and $\tilde{\mathcal{F}}$ are deconstructible classes in $\mathbf{C}(\mathcal{A})$. In addition, if $\mathcal{F}$ has a generator for $\mathcal{A}$, then $dg\,\tilde{\mathcal{F}}$ is deconstructible, as well.*

The following fulfills the deficient part in Proposition 3.3.13 for deconstructible classes.

Proposition 3.3.15. *([SS11, Theorem 4.22]) Let $\mathcal{A}$ be a Grothendieck category and $\mathcal{F}$ be a deconstructible class such that*

(i) $\mathcal{F}$ is resolving and closed under direct summands, and

(ii) $\mathcal{F}$ has a generator of $\mathcal{A}$.

Then the induced pairs $(dg\,\tilde{\mathcal{F}}, \tilde{\mathcal{B}})$ and $(\tilde{\mathcal{F}}, dg\,\tilde{\mathcal{B}})$, $\mathcal{B} := \mathcal{F}^{\perp}$, are complete, compatible and hereditary cotorsion pairs in $\mathbf{C}(\mathcal{A})$.

There are other cotorsion pairs in $\mathbf{C}(\mathcal{A})$ by using a cotorsion pair $(\mathcal{F}, \mathcal{B})$ in $\mathcal{A}$ cogenerated by a set.

Proposition 3.3.16. *[Gill08, Section 3-4] Let* $(\mathcal{F}, \mathcal{B})$ *be a cotorsion pair in* $\mathcal{A}$.

(i) $({}^{\perp}\mathbf{C}(\mathcal{B}), \mathbf{C}(\mathcal{B}))$ *and* $(\mathbf{C}(\mathcal{F}), \mathbf{C}(\mathcal{F})^{\perp})$ *are cotorsion pairs .*

(ii) If $\mathcal{B}$ *contains a cogenerator of finite injective dimension, then* $(\mathbf{C}(\mathcal{F}) \cap \mathrm{Acic}, (\mathbf{C}(\mathcal{F}) \cap \mathrm{Acic})^{\perp})$ *is a cotorsion pair.*

(iii) If $\mathcal{F}$ *contains a generator of finite projective dimension, then* $({}^{\perp}(\mathbf{C}(\mathcal{B}) \cap \mathrm{Acic}), \mathbf{C}(\mathcal{B}) \cap \mathrm{Acic})$ *is a cotorsion pair*

(iv) $(\mathcal{F}, \mathcal{B})$ *is small and* $\mathcal{F}$ *contains a generator of finite projective dimension, then* $({}^{\perp}(\mathbf{C}(\mathcal{B}) \cap \mathrm{Acic}), \mathbf{C}(\mathcal{B}) \cap \mathrm{Acic})$ *and* $({}^{\perp}\mathbf{C}(\mathcal{B}), \mathbf{C}(\mathcal{B}))$ *are small, as well. Furthermore, that pair is a Hovey pair.*

Proposition 3.3.17. *[EJ11b, Theorem 7.4.2] Let* $(\mathcal{F}, \mathcal{B})$ *be a cotorsion pair in a Grothendieck category* $\mathcal{A}$ *which is cogenerated by set. Suppose that* $\mathcal{A}$ *has a projective generator. Then* $(\mathbf{C}(\mathcal{F}), \mathbf{C}(\mathcal{F})^{\perp})$ *and* $(\mathbf{C}(\mathcal{F}) \cap \mathrm{Acic}, (\mathbf{C}(\mathcal{F}) \cap \mathrm{Acic})^{\perp}$ *are small cotorsion pairs, as well. Furthermore, these constitute a Hovey pair, that is,* $\mathrm{Acic} \cap (\mathbf{C}(\mathcal{F}) \cap \mathrm{Acic})^{\perp}) = \mathbf{C}(\mathcal{F})^{\perp}$.

CHAPTER FOUR

CARTAN-EILENBERG CATEGORIES

The subject of this capitulo is Cartan-Eilenberg categories. They were introduced in [GNPR10]. There are to ways to derive an additive functor: The most common way based on [CE56] and developed by Grothendieck and his puppils is via the derived category and by using projective resolutions. The other one is by finding a model structure on the category and defining it on its homotopy category. The latter one is somehow restrictive. As for the first case, what is done to get the derived category $\mathbf{D}(\mathcal{C})$ is firstly to pass to the homotopy category $\mathbf{K}(\mathcal{C})$. Here, two deleted projective or injective resolutions of an object are isomorphic. And they are isomorphic to the object in derived category. A deleted projective resolution of an object plays an essential role in the theory because they preserve quasi-isomorphisms by the covariant internal $\mathcal{H}om$-functor. Complexes with that property are called K-projective (or q-projective). By these observations, left Cartan-Eilenberg categories provide two classes of morphisms (S, W), which resemble the relation between homotopy equivalences and quasi-isomorphisms, and (S, W)-cofibrant objects, which behave as K-projectives. In [GNPR10], they show that if a category $\mathcal{C}$ has enough projectives, then the canonical pair (S, W), where S and W are the class of homotopy equivalences and quasi-isomorphisms, respectively, on $\mathbf{C}_+(\mathcal{C})$ provides a left Cartan-Eilenberg category structure. In [Pas12], a counterexample on $\mathbf{C}_+(\mathcal{C})$ is showed when $\mathcal{C}$ hasn't enough projectives, that is, in such case, $\mathbf{C}_+(\mathcal{C})$ is not a left Cartan-Eilenberg category with the canonical pair (S, W). In this chapter, we prove that the first example comes from a cotorsion pair in $\mathcal{C}$ and we also supply a machinery that produces left Cartan-Eilenberg categories on $\mathbf{C}(\mathcal{C})$ or on a subcategory of $\mathbf{C}_+(\mathcal{C})$ for more general categories even though $\mathcal{C}$ have no enough projectives. The first and second sections are preliminary parts of this chapter. In the third section, we recall some facts from [GNPR10]. Original results can be found in the last section.

4.1 Localization of categories

This section is devoted to localization of categories. So we begin by giving the formal definition.

Definition 4.1.1. Let $\mathcal{C}$ be a category and W be a family of morphisms in $\mathcal{C}$. A category $\mathcal{C}[W^{-1}]$ with a functor $\gamma : \mathcal{C} \to \mathcal{C}[W^{-1}]$ is called a *localization of* $\mathcal{C}$ by W provided that

- for all $w \in W$, $\gamma(w)$ is an isomorphism,
- for any functor $F : \mathcal{C} \to \mathcal{D}$ such that $F(w)$ is an isomorphism for all $w \in W$, there exists a unique functor $F_W : \mathcal{C}[W^{-1}] \to \mathcal{D}$ such that the following diagram is commutative

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\gamma} & \mathcal{C}[W^{-1}] \\ {\scriptstyle F}\downarrow & \swarrow {\scriptstyle F_W} & \\ \mathcal{D} & & \end{array}$$

From the definition it is clear that if $\mathcal{C}[W^{-1}]$ exists then it is unique up to equivalence of categories. There is also an equivalence of categories $(\mathcal{C}[W^{-1}])^{op} \cong (\mathcal{C}^{op})[(W^{op})^{-1}]$. Here we should highlight the term 'category' in the definition. As we detail it presently, for any category and any class of morphisms there is always a potential candidate which satisfies nearly these conditions. In the progression of getting that 'potential candidate', it goes far from being Hom-set category. This potential candidate would be the solution to the problem if we didn't look for a category with a set of morphisms. Except for some special cases such as derived categories and reflective subcategories, the existence of such a localization of a category by any class of morphisms W is still an open problem.

As we have mentioned, we recall the construction of the 'potential candidate' $\mathcal{C}[W^{-1}]$. First we keep objects the same, $Ob(\mathcal{C}[W^{-1}]) := Ob(\mathcal{C})$.

Now, we put an artificial arrow $m_w : Y \to X$ in $\mathrm{Mor}(\mathcal{C})$ for each $w : X \to Y \in W$. We get a new class $\mathrm{Mor}(\mathcal{C}) \cup W'$, where W' is the class of added arrows for each $w \in W$.

In order to define compositions in $\mathrm{Mor}(\mathcal{C}) \cup W'$, we consider all possible finite paths in $\mathrm{Mor}(\mathcal{C}) \cup W'$. To make that definition of compositions well-defined, we put an equivalence relation on paths subject to *elementary operations*:

- $A \xrightarrow{f} B \xrightarrow{g} C \sim A \xrightarrow{g \circ f} C$,
- $X \xrightarrow{w} Y \xrightarrow{m_w} X \sim X \xrightarrow{\mathrm{id}} X$,
- $Y \xrightarrow{m_w} X \xrightarrow{w} Y \sim Y \xrightarrow{\mathrm{id}} Y$,
- $Y \xrightarrow{\mathrm{id}} Y \xrightarrow{m_w} X \sim Y \xrightarrow{m_w} X \xrightarrow{\mathrm{id}} X \sim Y \xrightarrow{m_w} X$,

that is, two paths with common beginning and common end are *equivalent* if one can be obtained from the other by finite elementary operations. So, we define $\mathrm{Hom}_{\mathcal{C}[W^{-1}]}(X, Y)$ as the equivalence classes of paths which begin with X and end with Y.

Now, $\gamma : \mathcal{C} \to \mathcal{C}[W^{-1}]$ is defined as the identity on objects and $\gamma(f)$ is the equivalence class of the path f, for a morphism $f : A \to B$ in $\mathcal{C}$. $\gamma(\mathrm{id}_X)$ is the identity morphism of X in $\mathcal{C}[W^{-1}]$. So for any $w : X \to Y$ in W, the equivalence class of m_w is the inverse of $\gamma(w)$ in $\mathcal{C}[W^{-1}]$, that is, $\gamma(w)$ is invertible. The pair $(\mathcal{C}[W^{-1}], \gamma)$ satisfies the universal property stated in Definition 4.1.1. Indeed, for a functor $F : \mathcal{C} \to \mathcal{D}$ which sends any morphism in W to an isomorphism in $\mathcal{D}$,

$$F_W(X) := F(X), \quad X \in Ob(\mathcal{C}),$$

$$F_W([f]) := F(f), \quad f \in \mathrm{Mor}(\mathcal{C}),$$

$$F_W([m_w]) := (F(w))^{-1}, \quad m_w \in W'.$$

$F_W : \mathcal{C}[W^{-1}] \to \mathcal{D}$ is well-defined and $F = F_W \circ \gamma$. The uniqueness of F_W is immediate because the inverse of $F(w)$, $w \in W$, is unique. If a class W of morphisms satisfies some nice properties, these morphisms have easier representations.

Definition 4.1.2. A class W of morphisms is said to be a *right multiplicative system* if it satisfies

RMS1) For all $X \in \mathcal{C}$, $\mathrm{id}_X \in W$,

RMS2) W is closed under compositions,

RMS3) given any morphisms, $f : X \to Y$ and $w : Z \to Y \in W$, it can be completed to a commutative diagram

$$\begin{array}{ccc} T & \xrightarrow{g} & Z \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle w} \\ X & \xrightarrow[f]{} & Y \end{array}$$

in $\mathcal{C}$, where $t \in W$,

RMS4) for given morphisms $f, g : X \to Y$ in $\mathcal{C}$, if there exists a morphism $t : Y \to Z \in W$ such that $t \circ f = t \circ g$ then there exists $w : V \to X \in W$ with $f \circ w = g \circ w$.

Its dual notion is called *left multiplicative system*. A class of morphisms is called a *multiplicative system* if it is both right and left multiplicative.

Proposition 4.1.3. *[GM03, Lemma III.2.8] Let W be a right multiplicative system in a category $\mathcal{C}$. Then morphisms in $\mathcal{C}[W^{-1}]$ can be described as equivalence classes of right roofs (w, f)*

$$\begin{array}{ccccc} & & T & & \\ & {\scriptstyle w}\swarrow & & \searrow{\scriptstyle f} & \\ X & & & & Y \end{array} \tag{4.1.1}$$

where $w \in W, f \in \mathrm{Mor}(\mathcal{C})$ with two right roofs $(w, f) \sim (w', f')$ if and only if there is another pair (r, h), $r \in W$, such that $w \circ r = w' \circ h$ and $f \circ r = f' \circ g$. The composition of two roofs (w, f) and (w', f') is the equivalence class of the roof $(w \circ t, f' \circ g)$ where (t, g) is obtained by (RMS3)

Let $(\mathcal{C}, W)$ be any category with any class W of morphisms and $\gamma : \mathcal{C} \to \mathcal{C}[W^{-1}]$ be the canonical functor. $\overline{W}$ denotes the class of all morphisms f in $\mathcal{C}$ such that $\gamma(f)$ is an isomorphism in $\mathcal{C}[W^{-1}]$. It is clear that $W \subseteq \overline{W}$.

Definition 4.1.4. A class of morphisms W is called saturated if $\overline{W} = W$.

We now return to classical cases of localizations of categories: the homotopy and the derived category. Remember from Section 2.3 that the homotopy category, $\mathbf{K}(\mathcal{C})$, is the category having the same objects of $\mathbf{C}(\mathcal{C})$, but Hom-set is the equivalence class of morphisms by 'homotopy' relation, $\mathrm{Hom}_{\mathbf{K}(\mathcal{C})}(A, B) = \mathrm{Hom}_{\mathbf{C}(\mathcal{C})}(A, B)/_{\sim}$. It is essentially the localization $\mathbf{C}(\mathcal{C})[S^{-1}]$ of $\mathbf{C}(\mathcal{C})$ by the class S of homotopy equivalences.

What we want is firstly to make two deleted projective or injective resolutions of an object identical and later to identify objects with their projective or injective resolutions. By Comparison Theorem, Proposition 3.1.7, we carry out the first claim by the homotopy category $\mathbf{K}(\mathcal{C})$. But an object is unlikely to be isomorphic to its deleted projective resolution in $\mathbf{K}(\mathcal{C})$. The following proposition says that the only objects which are isomorphic with their deleted projective resolutions are projectives.

Proposition 4.1.5. *Let $\mathbf{F}_X \to X$ be a left $\mathcal{F}$- resolution of X. It is an isomorphism in $\mathbf{K}(\mathcal{C})$ if and only if the resolution is contractible. The dual one is valid for deleted right $\mathcal{F}$-resolutions, as well.*

Proof. Consider the diagram

$$\begin{array}{lccccccccc}
\mathbf{F}_X : & \cdots & \longrightarrow & F_2 & \longrightarrow & F_1 & \xrightarrow{d_1} & F_0 & \longrightarrow & 0 \\
 & & & \downarrow & & \downarrow & & \downarrow{\scriptstyle d_0} & & \\
S^0(X) : & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & X & \longrightarrow & 0
\end{array}$$

If it is a homotopy equivalence then there is a morphism $k : S^0(M) \to \mathbf{F}_X$ such that $k \circ d_0 \sim \mathrm{id}_{\mathbf{F}_X}$ and $d_0 \circ k \sim \mathrm{id}_{S^0(X)}$. The latter means that $d_0 \circ k = \mathrm{id}_X$. Let $s_i : F_i \to F_{i+1}$, $i \geq 0$, be the homotopy between id and $k \circ d_0$. Then

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & F_2 & \longrightarrow & F_1 & \xrightarrow{d_1} & F_0 & \xrightarrow{d_0} & M & \longrightarrow & 0 \\
 & & \Downarrow & \overset{s_1}{\swarrow} & \Downarrow & \overset{s_0}{\swarrow} & \Downarrow & \overset{k}{\swarrow} & \Downarrow & & \\
\cdots & \longrightarrow & F_2 & \longrightarrow & F_1 & \xrightarrow{d_1} & F_0 & \xrightarrow{d_0} & M & \longrightarrow & 0
\end{array}$$

shows that the left $\mathcal{F}$-resolution of X is contractible. □

Derived categories come up by the necessity of solving the second claim. The formalism comes from Grothendieck and his school. *The derived category* of $\mathcal{C}$ is the localization, $\mathbf{C}(\mathcal{C})[W^{-1}]$, of $\mathbf{C}(\mathcal{C})$ by W, the class of quasi-isomorphisms. The class of quasi-isomorphisms, W, may not be neither a left nor right multiplicative system in $\mathbf{C}(\mathcal{C})$, but its image, $Q(W)$, in $\mathbf{K}(\mathcal{C})$ is a multiplicative system. Then morphisms in $\mathbf{D}(\mathcal{C}) = \mathbf{C}(\mathcal{C})[W^{-1}] \simeq \mathbf{K}(A)[Q(W)^{-1}]$ have nice representations as in 4.1.1.

Just as it happens in $\mathbf{K}(\mathcal{C})$, $\mathbf{D}(\mathcal{C})$ is rarely an abelian category, but it carries a triangulation, see [Ver96].

As for the existence of derived category, it is solved for almost all categories of interest. In [TLS00, Corollary 5.6], it was proved that $\mathbf{D}(\mathcal{C})$ has a set of morphisms if $\mathcal{C}$ is a Grothendieck category. But, by the correspondence between Hovey pairs and model structures on abelian categories mentioned in Section 3.3, the existence follows easily by the injective model structure on $\mathbf{C}(\mathcal{C})$.

A full additive subcategory $\mathbf{K}'$ of $\mathbf{K}(\mathcal{C})$ has a triangulation coming from $\mathbf{K}(\mathcal{C})$ if and only if

(i) $A \in \mathbf{K}'$ if and only if $A[1] \in \mathbf{K}'$, and

(ii) the cone of any map $u : A \to B$ in $\mathbf{K}'$ is homotopically equivalent to some complex in $\mathbf{K}'$.

In this case, $\mathbf{K}'$ is called a Δ-subcategory of $\mathbf{K}(\mathcal{C})$.

For a Δ-subcategory $\mathbf{K}'$ of $\mathbf{K}(\mathcal{C})$, we consider its derived category $\mathbf{D}'$ obtained by inverting quasi-isomorphisms in $\mathbf{K}'$. In order to make $\mathbf{D}'$ a subcategory of $\mathbf{D}(\mathcal{C})$, a sufficient condition is the following (or its dual argument):

For every quasi-isomorphism $X \to B$ in $\mathbf{K}(\mathcal{C})$ with $B \in \mathbf{K}'$, there exists a quasi-isomorphism $A \to X$ with $A \in \mathbf{K}'$.

If $\mathbf{D}'$ is a subcategory of $\mathbf{D}(\mathcal{C})$, then it is called a localizing subcategory of $\mathbf{D}(\mathcal{C})$. For more details see [Lip09], [Ver96].

The problem of how to derive a functor $F : \mathcal{C} \to \mathcal{C}'$ between any abelian categories $\mathcal{C}$ and $\mathcal{C}'$ is about extending F to a functor from $\mathbf{D}(\mathcal{C}) \to \mathbf{D}(C')$. If $\mathcal{C}$ has enough projectives, then deleted projective resolutions are not projective complexes but they have nice properties on quasi-isomorphisms.

Definition 4.1.6. A complex M is called *K-projective* if for any quasi-isomorphism $w : X \to Y$ and for any morphism $f : M \to Y$ in $\mathbf{K}(\mathcal{C})$ there is a unique morphism $g : M \to X$ in $\mathbf{K}(A)$ such that $w \circ g = f$ in $\mathbf{K}(\mathcal{C})$. Its dual is called *K-injective*.

Proposition 4.1.7. *[Spa88, Proposition 1.4] Let M be complex in $\mathbf{C}(\mathcal{C})$. The following are equivalent:*

(i) M is K-projective.

(ii) For every $X \in \mathbf{K}(\mathcal{C})$, the natural morphism $Hom_{\mathbf{K}(\mathcal{C})}(M, X) \to Hom_{\mathbf{D}(\mathcal{C})}(M, X)$ is a bijection.

(iii) For every quasi-isomorphism $w : X \to Y$, $\mathcal{H}om(M, w) : \mathcal{H}om(M, X) \to \mathcal{H}om(M, Y)$ is a quasi-isomorphism.

(iv) For every quasi-isomorphism $w : X \to Y$, Hom$_{\mathbf{K}(\mathcal{C})}(M, w)$: Hom$_{\mathbf{K}(\mathcal{C})}(M, X) \to$ Hom$_{\mathbf{K}(\mathcal{C})}(M, Y)$ is an isomorphism.

(v) For every exact sequence E, $\mathcal{H}om(M, E)$ is exact, that is, Hom$_{\mathbf{K}(A)}(M, E) = 0$.

A projective object in $\mathcal{C}$ can be seen as a K-projective complex in the category $\mathbf{C}(\mathcal{C})$.

Proposition 4.1.8. *[Spa88, Proposition 1.2] A complex of the form $S^n(C)$ is K-projective if and only if C is projective in $\mathcal{C}$.*

K-projective complexes are slightly different from dg-projective complexes mentioned in Section 3.3. A dg-projective complex is K-projective but the converse isn't true because an acyclic complex is K-projective if and only if it is contractible. So a complex is dg-projective if and only if it is a K-projective complex of projectives. In [Spa88, Corollary 3.5], Spaltenstein proved that in an abelian category with enough projectives and exact direct limits, every complex X has a left K-projective resolution, that is, a quasi-isomorphism $M \to X$ with a K-projective complex M.

4.2 Orthogonal subcategory problem

Definition 4.2.1. A full isomorphism-closed subcategory $\mathcal{C}'$ of $\mathcal{C}$ is said to be a *reflective (coreflective) subcategory* of $\mathcal{C}$ if the inclusion functor $\iota : \mathcal{C}' \to \mathcal{C}$ has a left (right) adjoint $r : \mathcal{C} \to \mathcal{C}'$. The functor r is called the *reflection (coreflection)*.

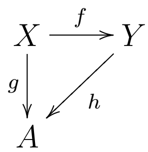
A limit of a functor in a reflective subcategory $\mathcal{C}'$ is a limit in $\mathcal{C}$ as well, because ι preserves limits. Conversely, if $\lim(\iota \circ F)$ or colim$(\iota \circ F)$ exists for a functor $F : I \to \mathcal{C}'$, then $\lim(\iota \circ F) \in \mathcal{C}'$ or r(colim$(\iota \circ F)$)=colimF. Then a reflective subcategory of a (finitely) complete or cocomplete category is (finitely) complete or cocomplete as well, (see [Bor94, Proposition 3.5.3 and 3.5.5]). Dually, it is true for coreflective subcategories.

Proposition 4.2.2. *[Mit65, Proposition V.5.3] Let $\mathcal{C}'$ be a coreflective subcategory of an abelian category $\mathcal{C}$ whose coreflector r preserves kernels. Then $\mathcal{C}'$ is also abelian.*

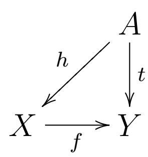
Proposition 4.2.3. *[AR94, Theorem 2.48] Let $\mathcal{C}$ be a locally presentable category. Then each full subcategory of $\mathcal{C}$ closed under limits and α-filtered colimits for some regular cardinal α is reflective in $\mathcal{C}$.*

Definition 4.2.4. Given a morphism $f : X \to Y$ and an object A of a category $\mathcal{C}$

f **is orthogonal to** A , $f \perp A$, when for every morphism $g : X \to A$, there exists a unique morphism $h : Y \to A$ such that $h \circ f = g$.



A **is orthogonal to** f , $A \perp f$, when for every morphism $t : A \to Y$, there exists a unique morphism $h : A \to X$ such that $f \circ h = t$.



Let $\mathcal{C}$ be a category and W a class of morphisms of $\mathcal{C}$. The *orthogonal subcategory of $\mathcal{C}$ determined by W* is the full subcategory $\mathcal{C}_W$ of $\mathcal{C}$, whose objects are those $A \in \mathcal{C}$ such that $w \perp A$ for every $w \in W$.

Proposition 4.2.5. *[Bor94, Proposition 5.4.4] Let $\iota : \mathcal{C}' \to \mathcal{C}$ be a reflective subcategory with reflection $r : \mathcal{C} \to \mathcal{C}'$. Let W be the class of all morphisms $f \in \mathcal{C}$ sent to an isomorphism by r. Given $C \in \mathcal{C}$, $\eta_C : C \to \iota(r(C))$ for the reflection of C. Then TFAE:*

(i) $X \in \mathcal{C}'$;

(ii) for all $f \in W$, $f \perp X$;

(iii) for all $C \in \mathcal{C}$, $\eta_C \perp X$.

By Proposition 4.2.5, if a subcategory is a reflective subcategory then it is an orthogonal subcategory determined by some class of morphisms W. The orthogonal subcategory problem looks for some conditions for the converse, that is, when an orthogonal subcategory of $\mathcal{C}$, determined by some class of morphisms W, is a reflective subcategory. This problem has an affirmative answer in a locally presentable category for a small orthogonal subcategory, i.e., that is determined by a set of morphisms.

Theorem 4.2.6. *[AR94, Theorem 1.39] Let* $\mathcal{C}$ *be a locally presentable category. Given a set* W *of morphisms of* $\mathcal{C}$*, the corresponding orthogonal subcategory* $\mathcal{C}_W$ *is reflective in* $\mathcal{C}$.

Proof. Since W is a set, then we may find a regular cardinal λ for which the domain and codomain of a morphism in W is λ-presentable. It is classical to observe that $\mathcal{C}_W$ is closed under limits and λ-directed colimits, see also [AR94, pg. 30-31]. Then it follows from Proposition 4.2.3. □

The link between reflective subcategories and localization of categories is the fact that every reflective subcategory can be seen as a localization of a category by some class of morphisms.

Proposition 4.2.7. *[Bor94, Proposition 5.3.1] Let* $\iota : \mathcal{C}' \to \mathcal{C}$ *be a reflective subcategory with reflection* r*. Let* W *be the class of morphisms* $f \in \mathcal{C}$ *such that* $r(f)$ *is an isomorphism. Then the localization of* $\mathcal{C}$ *by* W *exists and it is equivalent to* $r : \mathcal{C} \to \mathcal{C}'$*. Moreover,* W *is a left multiplicative system.*

4.3 Cartan-Eilenberg categories

Just as the relation among $\mathbf{K}(\mathcal{C})$, $\mathbf{D}(\mathcal{C})$ and K-projectives, in [GNPR10] a new concept, left Cartan-Eilenberg categories, was developed. For short, a category $(\mathcal{C}, S, W)$ with two classes of morphisms S, W is considered. And its localizations $\mathcal{C}[S^{-1}]$ and $\mathcal{C}[W^{-1}]$ are replaced instead of $\mathbf{K}(\mathcal{C})$ and $\mathbf{D}(\mathcal{C})$. Then (S, W)-cofibrant objects serves as K-projectives. In this section, we don't require Hom-set assumption in localizations of categories. So the existence of the localization of a category with respect to any class is guaranteed. We shall remind some terminology and results from [GNPR10].

Definition 4.3.1. A category $(\mathcal{C}, S, W)$ is a *category with strong and weak equivalences* if $S \subseteq \overline{W}$. Morphisms in S are called *strong equivalences* and those in W are called *weak equivalences*.

Since $S \subseteq \overline{W}$, the functor γ is factorized over the canonical functor $\delta : \mathcal{C} \to \mathcal{C}[S^{-1}]$. That is, there is a unique functor $\gamma' : \mathcal{C}[S^{-1}] \to \mathcal{C}[W^{-1}]$ such that $\gamma = \delta \circ \gamma'$.

Definition 4.3.2. Let $(\mathcal{C}, S, W)$ be a category with strong and weak equivalences. An object M is called (S, W)-*cofibrant*, or *cofibrant*, if for each morphism $w : X \to Y$ in W, the map

$$\mathrm{Hom}(M, w) : \mathrm{Hom}_{\mathcal{C}[S^{-1}]}(M, X) \to \mathrm{Hom}_{\mathcal{C}[S^{-1}]}(M, Y)$$

is bijective

Theorem 4.3.3. *Let $(\mathcal{C}, S, W)$ be a category with strong and weak equivalences and M be an object of $\mathcal{C}$. The following are equivalent.*

(i) M is cofibrant

(ii) For each $X \in \mathcal{C}$, $\gamma'_X : Hom_{\mathcal{C}[S^{-1}]}(M, X) \to Hom_{\mathcal{C}[W^{-1}]}(M, X)$ is bijective.

Definition 4.3.4. A category $(\mathcal{C}, S, W)$ with strong and weak equivalences is called *left Cartan-Eilenberg category* if for each object X of $\mathcal{C}$, there is a cofibrant object M with a morphism $\alpha : M \to X$ in $\mathcal{C}[S^{-1}]$ such that $\gamma'(\alpha)$ is an isomorphism in $\mathcal{C}[W^{-1}]$.

Let $\mathcal{C}_{cof}$ be the class of cofibrant objects and let $\overline{\mathcal{C}}_{cof}$ be the full subcategory of $\mathcal{C}[S^{-1}]$ consisting of cofibrant objects

Proposition 4.3.5. *A category $(\mathcal{C}, S, W)$ with strong and weak equivalences is a left Cartan-Eilenberg category if and only if*

$$\gamma' : \overline{\mathcal{C}}_{cof} \to \mathcal{C}[W^{-1}]$$

is an equivalence of categories.

The problem of being a left Cartan-Eilenberg category is closely related with the problem of being coreflective subcategory for $\overline{\mathcal{C}}_{cof}$ in $\mathcal{C}[S^{-1}]$.

If we are given a category $(\mathcal{C}, S, W)$ with strong and weak equivalences, we may consider the class $\delta(W)$ in $\mathcal{C}[S^{-1}]$. The subcategory $\overline{\mathcal{C}}_{cof}$ of cofibrant object is precisely determined by $\delta(W)$ in $\mathcal{C}[S^{-1}]$, that is, consists of objects M such that $M \perp \delta(w)$ in $\mathcal{C}[S^{-1}]$. Therefore the problem of whether a category $(\mathcal{C}, S, W)$ is left Cartan-Eilenberg is the same as that of whether $\overline{\mathcal{C}}_{cof}$ is a coreflective subcategory of $\mathcal{C}[S^{-1}]$.

In [GNPR10], it is proved that every model category contains a left Cartan-Eilenberg structure inside. For a model category $\mathcal{C}$, $\mathcal{C}_f$ and $\mathcal{C}_{cf}$ stands for the classes of Quillen fibrant and fibrant-cofibrant objects, respectively. For the equivalence relation $\sim_l$ transitively generated by the left homotopy, S_l will be the class of homotopy equivalences coming from $\sim_l$.

Proposition 4.3.6. *[GNPR10, Theorem 4.1.2] Let $\mathcal{C}$ be a model category and W_l be the class of weak equivalences in $\mathcal{C}_f$. Then $(\mathcal{C}_f, S_l, W_l)$ is a left Cartan-Eilenberg category with (S_f, W_f)-cofibrant objects are precisely objects in the class $\mathcal{C}_{cf}$*

4.4 Examples

In this section, we give new examples of left Cartan-Eilenberg categories and study its relation with cotorsion pairs. In view of the interlacing between Hovey pairs and abelian model category structures mentioned in Section 3.3, it seemed reasonable to expect some sort of connection between complete cotorsion pairs and left Cartan-Eilenberg categories. But it seems unlikely to get a cotorsion pair from a left Cartan Eilenberg category. Howewer we show in the next Proposition that we do get a left Cartan-Eilenberg category from a complete cotorsion pair on the category of complexes.

For a class $\overline{\mathcal{F}}$ of complexes, we denote by $\mathbf{K}(\overline{\mathcal{F}})$ for the homotopy category of $\overline{\mathcal{F}}$.

Proposition 4.4.1. *Let $\mathcal{C}$ be an abelian category and $(\overline{\mathcal{F}}, \overline{\mathcal{B}})$ be any complete cotorsion pair in $\mathbf{C}(\mathcal{C})$, where $\overline{\mathcal{F}}$ is closed under suspensions. Then there is a left Cartan-Eilenberg category structure on $\mathbf{C}(\mathcal{C})$ for which $\mathbf{K}(\overline{\mathcal{F}})$ is the class of cofibrant objects.*

Proof. For such a complete cotorsion pair $(\overline{\mathcal{F}}, \overline{\mathcal{B}})$, we know that the inclusion functor $\iota : \mathbf{K}(\overline{\mathcal{F}}) \hookrightarrow \mathbf{K}(\mathcal{C})$ has a right adjoint functor r by [EJ11b, Theorem 5.1.7]. So let S be the class of homotopy equivalences and W be the class of morphisms w such that $r([w])$ is an isomorphism. Then clearly, $S \subseteq W$ and the class $\mathbf{K}(\overline{\mathcal{F}})$ is the class of (S, W)-cofibrant objects using the fact that $\mathbf{K}(\overline{\mathcal{F}})$ is a coreflective subcategory. □

Example 4.4.2. Let $\mathcal{C}$ be a Grothendieck category and $(\mathcal{F}, \mathcal{B})$ be a cotorsion pair in $\mathcal{C}$ cogenerated by a set. Then by Proposition 3.3.16, there is a cotorsion pair $(\mathbf{C}(\mathcal{F}), \mathbf{C}(\mathcal{F})^{\perp})$ which is cogenerated by a set. If $\mathcal{C}$ has enough $\mathcal{F}$ objects, that cotorsion pair is complete. And also the class $\mathbf{C}(\mathcal{F})$ is closed under suspensions. By Proposition 4.4.1, the subcategory $\mathbf{K}(\mathbf{C}(\mathcal{F})) \hookrightarrow \mathbf{K}(\mathcal{C})$ is a coreflective subcategory. So $\mathbf{C}(\mathcal{C})$ is left Cartan-Eilenberg category with (S, W)-cofibrant objects are $\mathbf{K}(\mathcal{F}) := \mathbf{K}(\mathbf{C}(\mathcal{F}))$

If we have an abelian model category $\mathcal{C}$, then by Proposition 3.3.5 there is a Hovey pair,

$(\overline{\mathcal{F}}, \overline{\mathcal{B}} \cap \mathcal{W})$, $(\overline{\mathcal{F}} \cap \mathcal{W}, \overline{\mathcal{B}})$. Here, $\overline{\mathcal{B}}, \overline{\mathcal{F}}$ are Quillen fibrant, cofibrant objects, respectively. So, by Proposition 4.3.6, $(\overline{\mathcal{B}}, S, W)$ is a left Cartan-Eilenberg category with the class $\overline{\mathcal{B}} \cap \overline{\mathcal{F}}$ as the class of (S, W)-cofibrant objects, for some class of morphisms S, W.

Example 4.4.3. Let $\mathcal{C}$ be Grothendieck category and $(\mathcal{F}, \mathcal{B})$ be a small cotorsion pair in $\mathcal{C}$. Suppose that $\mathcal{F}$ contains a generator of finite projective dimension. Then by Proposition 3.3.16, there is a Hovey pair $({}^{\perp}\mathbf{C}(\mathcal{B}), \mathbf{C}(\mathcal{B}))$, $({}^{\perp}(\mathbf{C}(\mathcal{B}) \cap \mathrm{Acic}), \mathbf{C}(\mathcal{B}) \cap \mathrm{Acic})$. So by the above argument, $(\mathbf{C}(B), S, W)$ is a left Cartan-Eilenberg category with $\mathbf{C}(\mathcal{B}) \cap^{\perp} (\mathbf{C}(\mathcal{B}) \cap \mathrm{Acic})$ as (S, W)-cofibrant objects and W is the class of quasi-isomorphisms.

Example 4.4.4. Let $(\mathcal{F}, \mathcal{B})$ be a cotorsion pair in an abelian category with enough $\mathcal{F}$ and $\mathcal{B}$ objects. Then we have two cotorsion pairs $(dg\,\tilde{\mathcal{F}}, \tilde{\mathcal{B}})$ and $(\tilde{\mathcal{F}}, dg\,\tilde{\mathcal{B}})$. If the induced cotorsion pairs form a Hovey pair, that is, the induced cotorsion pairs are compatible, hereditary and complete, then by the above argument, the subcategory $(dg\,\tilde{\mathcal{B}}, S, W)$ is a left Cartan-Eilenberg category with $dg\,\tilde{\mathcal{F}} \cap dg\,\tilde{\mathcal{B}}$ as (S, W)-cofibrant objects, where W is the class of quasi-isomorphisms.

Consider the full subcategory $\mathcal{C} := \mathbf{C}_{+}(\mathcal{A}) \cap dg\,\tilde{\mathcal{B}}$ of $\mathbf{C}(\mathcal{A})$ for an abelian category $\mathcal{A}$. Then $\mathbf{K}_{\mathcal{C}}$ is the full subcategory of $\mathbf{K}_{+}(\mathcal{C})$ which is the quotient category of $\mathcal{C}$ by the induced homotopy relation. Let S' be the class of morphisms in $\mathcal{C}$ which are isomorphisms in $\mathbf{K}_{\mathcal{C}}$. Then it is precisely the class of homotopy equivalences which are in $\mathcal{C}$. Since the class S' is compatible with the homotopy congruence $\sim$ on $\mathcal{C}$, the localization $\mathcal{C}[S'^{-1}]$ of the category $\mathcal{C}$ is isomorphic to its homotopy category $\mathbf{K}_{\mathcal{C}}$.

It is easy to observe that $A \in \mathbf{K}_{\mathcal{C}}$ if and only if $A[1] \in \mathbf{K}_{\mathcal{C}}$. For any map $[u] : A \to B$ in $\mathbf{K}_{\mathcal{C}}$, there is an exact sequence

$$0 \to B \to c(u) \to A[1] \to 0$$

with $B, A[1] \in \mathbf{K}_{\mathcal{C}}$. Since $dg\,\tilde{\mathcal{B}}$ is closed under extensions, $c(u)$ is in $dg\,\tilde{\mathcal{B}}$ and due to being an exact sequence, it is also in $\mathbf{C}_{+}(\mathcal{C})$. So $\mathbf{K}_{\mathcal{C}}$ is a Δ-subcategory of $\mathbf{K}_{+}(\mathcal{C})$.

Let W' be the class of all quasi-isomorphisms in $\mathcal{C}$. Then $(\mathbf{C}_+(\mathcal{A}) \cap dg\,\tilde{\mathcal{B}}, S', W')$ is a category with strong and weak equivalences.

Theorem 4.4.5. *Let $(\mathcal{F}, \mathcal{B})$ be a cotorsion pair in an abelian category with enough $\mathcal{F}$ objects. Suppose that $\mathcal{F}$ is a resolving class, $dg\,\tilde{\mathcal{B}} \cap \mathrm{Acic} = \tilde{\mathcal{B}}$ and every complex in $\mathbf{C}(\mathcal{C})$ has a special $dg\,\tilde{\mathcal{F}}$-precover. Then $(\mathbf{C}_+(\mathcal{A}) \cap dg\,\tilde{\mathcal{B}}, S', W')$ is a left Cartan-Eilenberg category.*

Proof. The claim is to show that complexes of the form $X := \cdots \to X_{i+1} \to X_i \to 0 \to 0 \cdots$ in $\mathbf{C}_+(\mathcal{F}) \cap dg\,\tilde{\mathcal{B}}$ are (S', W')-cofibrant objects. For it, we need to show that they are K-projective in that category $\mathcal{C}$, that is, the natural morphism

$$\mathrm{Hom}(X, w) : \mathrm{Hom}_{\mathbf{K}_\mathcal{C}}(X, A) \to \mathrm{Hom}_{\mathbf{K}_\mathcal{C}}(X, B)$$

is an isomorphism for any quasi-isomorphism $w : A \to B$ in $\mathcal{C}$.

It is known that bounded below complexes which are in the class $dg\,\tilde{\mathcal{F}}$ are just of the form $X := \cdots X_{i+1} \to X_i \to 0 \to 0 \cdots$ with $X_j \in \mathcal{F}$ for each $j \in \mathbb{Z}$. In other words,

$$dg\,\tilde{\mathcal{F}} \cap \mathbf{C}_+(\mathcal{C}) = \mathbf{C}_+(\mathcal{F}).$$

Let $X \in \mathbf{C}_+(\mathcal{F}) \cap dg\,\tilde{\mathcal{B}} = dg\,\tilde{\mathcal{F}} \cap \mathbf{C}_+(\mathcal{C}) \cap dg\,\tilde{\mathcal{B}}$ and let $T \in \mathcal{C} \cap \mathrm{Acic}$. Since the pair $(\tilde{\mathcal{F}}, dg\,\tilde{\mathcal{B}})$ is compatible by assumption, that is, $\mathrm{Acic} \cap dg\,\tilde{\mathcal{B}} = \tilde{\mathcal{B}}$, $T \in \mathbf{C}_+(\mathcal{A}) \cap \tilde{\mathcal{B}}$. By definition of $dg\,\tilde{\mathcal{F}}$, $\mathrm{Hom}_{\mathbf{K}(\mathcal{C})}(X, T) = \mathrm{Hom}_{\mathbf{K}_\mathcal{C}}(X, T) = 0$.

Now for any quasi-isomorphism $w : A \to B$ in $\mathcal{C}$, consider the distinguished triangle associated to that morphism w

$$c(w)[-1] \longrightarrow A \xrightarrow{w} B \longrightarrow c(w)\,.$$

It gives us an exact sequence

$$\mathrm{Hom}_{\mathbf{K}_{\mathcal{C}}}(X, c(w)[-1]) \longrightarrow \mathrm{Hom}_{\mathbf{K}_{\mathcal{C}}}(X, A) \xrightarrow{\mathrm{Hom}(X,w)} \mathrm{Hom}_{\mathbf{K}_{\mathcal{C}}}(X, B) \longrightarrow \mathrm{Hom}_{\mathbf{K}_{\mathcal{C}}}(X, c(w)) .$$

Since w is a quasi-isomorphism, its cone $c(w)$ is exact. We already know that $c(w) \in \mathcal{C}$. So $\mathrm{Hom}_{\mathbf{K}_{\mathcal{C}}}(X, c(w)[-1]) = \mathrm{Hom}_{\mathbf{K}_{\mathcal{C}}}(X, c(w)) = 0$ and $\mathrm{Hom}(X, w)$ is an isomorphism. It means that such objects X are (S', W')-cofibrant objects in $\mathbf{C} + (\mathcal{A}) \cap dg\,\tilde{\mathcal{B}}$. By Theorem 4.3.3, we have the natural bijection

$$\mathrm{Hom}_{\mathbf{K}_{\mathcal{C}}}(X, B) = \mathrm{Hom}_{\mathcal{C}[S^{-1}]}(X, B) \to \mathrm{Hom}_{\mathcal{C}[W^{-1}]}(X, B) = \mathrm{Hom}_{\mathbf{D}_{\mathcal{C}}}(X, B).$$

Now we need to find a (S', W')-cofibrant replacement for each object A in $\mathcal{C}$, that is, a quasi-isomorphism $f : F \to A$ where $F \in \mathbf{C}_{+}(\mathcal{F}) \cap dg\,\tilde{\mathcal{B}}$. Let $A \in \mathcal{C}$. Since $(dg\,\tilde{\mathcal{F}}, \tilde{\mathcal{B}})$ has enough projectives, we have the following exact sequence

$$0 \longrightarrow K \xrightarrow{g} F \xrightarrow{f} A \longrightarrow 0$$

where $K \in \tilde{\mathcal{B}}$ and $F \in dg\,\tilde{\mathcal{F}}$. Since K is acyclic, the long exact sequence of homology implies that f is a quasi-isomorphism. We have found a quasi-isomorphic replacement of A by an object in $dg\,\tilde{\mathcal{F}}$. But it is needed to show that such a replacement can be done in our subcategory $\mathcal{C} = \mathbf{C}_{+}(\mathcal{A}) \cap dg\,\tilde{\mathcal{B}}$. Obviously, F is in $dg\,\tilde{\mathcal{B}}$ because of the fact that $dg\,\tilde{\mathcal{B}}$ is closed under extensions. Without lost of generality, suppose that $A_i = 0$ for all $i < 1$. Then the above short exact sequence is of the form

$$
\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & K_1 & \longrightarrow & K_0 & \longrightarrow & K_{-1} & \longrightarrow & K_{-2} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & F_1 & \xrightarrow{d_1^F} & F_0 & \xrightarrow{d_0^F} & F_{-1} & \xrightarrow{d_{-1}^F} & F_{-2} & \xrightarrow{d_{-2}^F} & \cdots \\
 & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_{-1}} & & \downarrow{\scriptstyle f_{-2}} & & \\
\cdots & \longrightarrow & A_1 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array}
$$

Consider the truncated complexes $\tau_{\leq -1}(F)$

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \text{Coker} d_0^F = \operatorname{Im} d_{-1}^F \hookrightarrow F_{-2} \longrightarrow F_{-3} \longrightarrow \cdots$$

and $\tau_{\geq -1}(F)$

$$\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow \text{Ker} d_{-1}^F = \operatorname{Im} d_0^F \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots .$$

The truncated complex $\tau_{\leq -1}(F)$ belongs to the class $\tilde{\mathcal{B}}$. Indeed, each F_i is isomorphic to the one K_i of K for $i \leq 0$. And it is isomorphic to the truncated complex $\tau_{\leq -1}(K)$ of K which is obviously still in $\tilde{\mathcal{B}}$.

$$
\begin{array}{cccccccccccccc}
\tau_{\geq -1}(F): & \cdots & \longrightarrow & F_1 & \longrightarrow & F_0 & \longrightarrow & \text{Ker} d_{-1}^F = \operatorname{Im} d_0^F & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots . \\
\downarrow{\scriptstyle \sigma'} & & & \downarrow{\scriptstyle \text{id}} & & \downarrow{\scriptstyle \text{id}} & & \hookdownarrow & & \downarrow & & \downarrow & & \\
F: & \cdots & \longrightarrow & F_1 & \xrightarrow{d_1^F} & F_0 & \xrightarrow{d_0^F} & F_{-1} & \xrightarrow{d_{-1}^F} & F_{-2} & \xrightarrow{d_{-2}^F} & F_{-3} & \xrightarrow{d_{-3}^F} & \cdots \\
\downarrow{\scriptstyle \sigma} & & & \downarrow{\scriptstyle \sigma_1} & & \downarrow{\scriptstyle \sigma_0} & & \downarrow{\scriptstyle \sigma_{-1} = d_{-1}^F} & & \downarrow{\scriptstyle \sigma_{-2} = \text{id}} & & \downarrow{\scriptstyle \sigma_{-3} = \text{id}} & & \\
\tau_{\leq -1}(F): & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \text{Coker} d_0^F = \operatorname{Im} d_{-1}^F & \hookrightarrow & F_{-2} & \longrightarrow & F_{-3} & \longrightarrow & \cdots
\end{array}
$$

As we noted before, $F \in dg\,\tilde{\mathcal{F}}$ and $\tau_{\leq -1}(F) \in \tilde{\mathcal{B}}$. Therefore, the natural map σ is homotopic to the zero map, that is, there is a set of morphisms $t = (t_i : F_i \to (\tau_{\leq -1}(F))_{i+1})_i$, with $\sigma = d^F \circ t + t \circ d^F$. For $i = -1$, $\sigma_{-1} = d^F_{-1} = t_{-2} \circ d^F_{-1}$. Then $t_{-2} \circ \iota = \mathrm{id}_{\mathrm{Im}\, d^F_{-1}}$, that is,

$$0 \longrightarrow \mathrm{Im}\, d^F_{-1} \hookrightarrow F_{-2} \xrightarrow{d^F_{-2}} \mathrm{Im}\, d^F_{-2} \longrightarrow 0$$

is a split exact sequence. But, since the class $\mathcal{F}$ is closed under direct summands, $\mathrm{Im}\, d^F_{-1} \in \mathcal{F}$. Considering the exact sequence

$$0 \longrightarrow \mathrm{Ker} d^F_{-1} \longrightarrow F_{-1} \xrightarrow{d_{-1}} \mathrm{Im}\, d^F_{-1} \longrightarrow 0\,.$$

From the fact that $\mathcal{F}$ is resolving, we follow that $\mathrm{Ker} d^F_{-1} \in \mathcal{F}$, that is, the truncated complex $\tau_{\geq -1}(F)$ is in $dg\,\tilde{\mathcal{F}}$. Now it remains to show that it is also in $dg\,\tilde{\mathcal{B}}$. But it easily follows from the truncation functor. In our case, $\tau_{\geq -1}$ preserves the exactness.

$$\begin{array}{ccccccccccccc}
 & & & & 0 & & 0 & & 0 & & 0 & & 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\tau_{\geq -1}(K): & & \cdots & \longrightarrow & K_1 & \longrightarrow & K_0 & \longrightarrow & \mathrm{Ker} d^F_{-1} = \mathrm{Im}\, d^F_0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \cdots \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\tau_{\geq -1}(F): & & \cdots & \longrightarrow & F_1 & \longrightarrow & F_0 & \longrightarrow & \mathrm{Ker} d^F_{-1} = \mathrm{Im}\, d^F_0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \cdots \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
A: & & \cdots & \longrightarrow & A_1 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \cdots \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & & & 0 & & 0 & & 0 & & 0 & & 0
\end{array}$$

where $A \in dg\,\tilde{\mathcal{B}}$ and $\tau_{\geq -1}(K) \in \tilde{\mathcal{B}}$, so $\tau_{\geq -1}(F) \in dg\,\tilde{\mathcal{B}}$. □

From Theorem 4.4.5, we may produce several non-trivial examples of left Cartan-Eilenberg categories. The first one was proved in [Pas12, Proposition 4.4.1].

Example 4.4.6. Let $\mathcal{A}$ be an abelian category with enough projectives. Then we have a complete and hereditary cotorsion pair $(\mathrm{Proj}, \mathcal{A})$. The induced cotorsion pairs from $(\mathrm{Proj}, \mathcal{A})$ in $\mathbf{C}(\mathcal{A})$ are $(\mathrm{Proj}\mathbf{C}(\mathcal{A}), \mathbf{C}(\mathcal{A}))$ and $(dg\,\mathrm{Proj}, \mathrm{Acic})$ where $dg\,\mathrm{Proj}$ denote the class of K-projective complexes, respectively. It is known that $(dg\,\mathrm{Proj}, \mathrm{Acic})$ has enough projectives. So by Theorem 4.4.5, $(\mathbf{C}_+(\mathcal{C}), S', W')$ is a left Cartan-Eilenberg category with K-projective complexes as (S', W')-cofibrant objects.

Example 4.4.7. Let R be a commutative Gorenstein ring, that is, a Noetherian ring with finite injective dimension, $\mathrm{id}_R\, R < \infty$. Let us consider the class of Gorenstein projective modules, GP, which will be detailed in Section 8.1. It is known that $(\mathrm{GP}, \overline{\mathcal{P}}(R))$ is a complete and hereditary cotorsion pair in R-Mod where $\overline{\mathcal{P}}(R)$ is the class of modules with finite projective dimension, see [Hov02, Theorem 8.3]. Then we have compatible, hereditary and complete induced cotorsion pairs $(dg\,\widetilde{\mathrm{GP}}, \widetilde{\overline{\mathcal{P}}}(R))$ and $(\widetilde{\mathrm{GP}}, dg\widetilde{\overline{\mathcal{P}}}(R))$ by [YL14]. So by Theorem 4.4.5, $(\mathbf{C}_+(\mathcal{C}) \cap dg\,\widetilde{\overline{\mathcal{P}}}(R), S', W')$ is a left Cartan-Eilenberg category.

Example 4.4.8. Let X be a semi-separated and quasi-compact scheme. Then the category of quasi-coherent sheaves $\mathfrak{Qcoh}(X)$ has a flat generator, see [TLS00, Proposition 1.1] and [Mur07, Proposition 16]. So the class $\mathcal{F}lat(X)$ of flat quasi-coherent sheaves contains a generator. And we already know that $\mathcal{F}lat(X) = \mathrm{Filt}(\mathcal{S})$, see [Gill07, Proposition 6.4], where $\mathcal{S}$ is the set of flat quasi-coherent sheaves which are $< \kappa$-generated for some regular cardinal κ. So we have a small cotorsion pair $(\mathcal{F}lat(X), \mathcal{F}lat(X)^{\perp})$. By Proposition 3.3.15, we have compatible, complete and hereditary induced cotorsion pairs $(dg\,\widetilde{\mathcal{F}lat(X)}, \widetilde{\mathcal{F}lat(X)^{\perp}})$ and $(\widetilde{\mathcal{F}lat(X)}, dg\,\widetilde{\mathcal{F}lat(X)}^{\perp})$. So, $\mathbf{C}_+(\mathfrak{Qcoh}(X)) \cap dg\,\widetilde{\mathcal{F}lat(X)}^{\perp}$ is a left Cartan-Eilenberg category.

In [Pas12], it is proved that $\mathbf{C}_+(\mathfrak{Qcoh}(\mathbb{P}^1_k))$, where $\mathbb{P}^1_k$ is the projective line, is not left Cartan-Eilenberg category. In the next example we show that under some conditions on the scheme, vector bundles allow to define a left Cartan-Eilenberg category by [EGPT12].

Example 4.4.9. Let X be a scheme having enough infinite-dimensional vector bundles (for example, a quasi-compact and quasi-separated scheme that admits an ample family of invertible sheaves, or a Noetherian, integral, separated, and locally factorial scheme). Let $\mathcal{F}$ be the class of all infinite dimensional vector bundles on X. Then by [EGPT12, Corollary 1.2], we have a small cotorsion pair $(\mathcal{F}, \mathcal{B})$ in $\mathfrak{Qcoh}(X)$. So $(\mathbf{C}_+(\mathfrak{Qcoh}(X)) \cap dg\,\tilde{\mathcal{B}})$ is a left Cartan-Eilenberg category. Note that, in general, $dg\,\tilde{\mathcal{B}} \neq \mathbf{C}(\mathfrak{Qcoh}(X))$, unless X is affine.

Remark 4.4.10. All the arguments that we have done work for the dual one, right Cartan-Eilenberg category. So their duals are true for right Cartan-Eilenberg categories, as well.

CHAPTER FIVE

PURITY

There are several ways to define a pure monomorphism in R-Mod. More categorical ones are given in terms of finitely presented modules, tensor product $\otimes$ or as the smallest proper class which is closed under direct limits and contains splitting short exact sequences. Tensor product is a monoidal structure in R-Mod. Then in a category which has no monoidal structure, it doesn't make sense to define purity through $\otimes$-functor. Since there is a categorical concept of finitely presentable object for any category, it is always possible to consider morphisms with a kind of projective property by finitely presentable objects, see [AR94, Section 2D]. But that notion fructifies when the category is locally finitely presentable, see [Craw94]. Analogously, in [AR94, Section 2D] λ-pure morphisms are defined for any regular cardinal λ and an important result claims that locally λ-presentable categories have enough λ-pure subobjects, see Theorem 5.1.6. In [AR94], Adámek and Rosický give also a characterization of λ-pure morphisms in a locally λ-presentable category, see Proposition 5.1.5. Concerning to these facts, the notion of λ-purity provides us an essential tool in Relative Homological Algebra for more general categories, see for example [Kra12]. We call that kind of pure morphisms as *categorical pure morphisms*. The first section is about basic terminology on categorical purity and we follow up [AR94] as a main source.

In the second section, we go one step further: what kind of relations there would be between purities if a category both has a monoidal structure $\otimes$ and is locally presentable. We follow the definition of [Fox76] in case a category has a monoidal structure. Pure monomorphisms are $-\otimes Y$-exact monomorphisms just as in R-Mod. We call that kind of pure monomorphisms as *geometrical* or *tensor pure monomorphisms*. We focus on Grothendieck categories with closed symmetric monoidal structures which appear abundantly: $\mathbf{Set}$, R-Mod, $\mathcal{O}_X$-Mod, $\mathfrak{Qcoh}(X)$... The key observation is that categorical pure morphisms are always geometrical pure, as well.

And our first result in this section is Theorem 5.2.6, which is about the existence of geometrical pure-injectives preenvelopes.

The third section is focused on the category of (quasi-coherent) $\mathcal{O}_X$-modules over any scheme. It is known that $\mathcal{O}_X$-Mod is a Grothendieck category with a symmetric closed monoidal structure. In this special case, the information of geometrical purity in $\mathcal{O}_X$-Mod is carried on stalks, see Proposition 5.3.1. A nicer characterization occurs when the domain and codomain of a morphism in $\mathcal{O}_X$ belong to $\mathfrak{Qcoh}(X)$. In this case, they are determined by sections of modules over affine open subsets, see Proposition 5.3.3. We call that kind of purity in $\mathfrak{Qcoh}(X)$ stalkwise-purity in $\mathfrak{Qcoh}(X)$. We should point out that $\mathfrak{Qcoh}(X)$ itself has a geometrical purity since $\mathfrak{Qcoh}(X)$ is also a closed symmetric monoidal category. We stress that stalkwise and geometrical purities are slightly different. Stalkwise pure monomorphisms are always geometrical pure in $\mathfrak{Qcoh}(X)$. But the converse doesn't hold in general because the stalkwise one requires to remain monic under tensor product with all $\mathcal{O}_X$-modules whereas the geometrical one only requires it for the $\mathcal{O}_X$-modules that are quasi-coherent. But this distinction disappears when X is quasi-separated, see Proposition 7.2.8. The stalkwise purity on $\mathfrak{Qcoh}(X)$ is more natural in Algebraic Geometry because it is in accord with flat quasi-coherent sheaves. For example, it plays an important role in the existence of covers by flat quasi-coherent sheaves, see [EE05]. In this section, we discuss the relation between $\aleph_0$-pure (categorical) and stalkwise pure monomorphisms in $\mathfrak{Qcoh}(X)$. We see that categorical pure morphisms are also stalkwise pure when the category has enough finitely presentable objects.

The aim of the fourth section is to prove the existence of stalkwise pure-injective envelopes in $\mathfrak{Qcoh}(X)$. By Theorem 5.2.6, we conclude that every $\mathcal{O}_X$-module is (geometric) purely embedded in a geometric pure-injective $\mathcal{O}_X$-Mod. In Theorem 5.4.6, we do it for $\mathfrak{Qcoh}(X)$ by using the coherator functor.

5.1 Purity in presentable categories

We begin by recalling the definition of an accessible category.

Definition 5.1.1. [AR94, Definition 2.1] Let λ be a regular cardinal. A category $\mathcal{C}$ is called λ-*accessible* if $\mathcal{C}$ has λ-directed colimits and $\mathcal{C}$ has a set of λ-presentable objects such that every object in $\mathcal{C}$ is a λ-directed colimit of objects from that set. A category is said to be *accessible* if it is λ-accessible for some regular cardinal λ.

It is clear from the definition that every locally λ-presentable category is λ-accessible. The converse is true when a category is cocomplete, or equivalently, complete.

Proposition 5.1.2. *[AR94, Corollary 2.47] A category $\mathcal{C}$ is locally λ-presentable if and only if it is cocomplete and λ-accessible if and only if it is complete and λ-accessible.*

Even though there are counterparts of most of results of locally presentable categories for accessible categories, one of the differences is that a λ-accessible category may not be μ-accessible for any regular cardinal $\mu \geq \lambda$, see [AR94, Theorem 2.11]. But it holds when 'any regular cardinal' is replaced by 'arbitrarily large regular cardinal', that is, for each accessible category, there is an arbitrarily large regular cardinal μ such that it is μ-accessible, see [AR94, Corollary 2.14].

Now, we introduce the main concept of this chapter, categorical pure morphisms.

Definition 5.1.3. [AR94, Definition 2.27] Let $f : A \to B$ be a morphism in $\mathcal{C}$. It is said to be λ-*pure* if for any commutative diagram

$$\begin{array}{ccc} A' & \xrightarrow{f'} & B' \\ {\scriptstyle u}\downarrow & \swarrow{\scriptstyle g} & \downarrow{\scriptstyle v} \\ A & \xrightarrow{f} & B \end{array}$$

where A', B' are λ-presentable, there is a morphism $g : B' \to A$ such that $u = g \circ f'$.

The following properties are easy to observe:

(i) $\aleph_0$-pure morphisms in R-Mod coincide with the usual pure monomorphisms in R-Mod.

(ii) Split monomorphisms are λ-pure for any λ.

(iii) The class of λ-pure morphisms are closed under compositions.

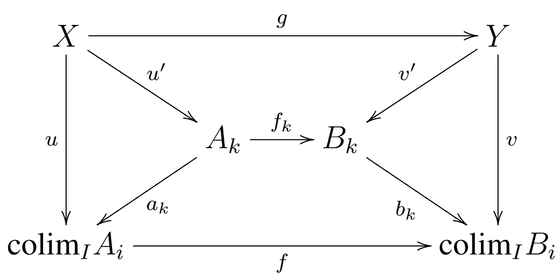
(iv) If $f \circ g$ is a λ-pure, then g is λ-pure.

(v) Every λ-pure morphism is λ'-pure for all $\lambda' \leq \lambda$.

Proposition 5.1.4. *[AR94, Proposition 2.29] Every λ-pure morphism in a λ-accessible category is a monomorphism.*

Proof. Let $f : A \to B$ be a λ-pure morphism. The important point here is to be able to write each object as a λ-colimit of λ-presentable objects. So it is enough to show that if $x_1, x_2 : X \rightrightarrows A$ are morphisms such that $f \circ x_1 = f \circ x_2$ with λ-presentable domain X then $x_1 = x_2$. In such case, there is a morphism $u : A' \to A$ with A' λ-presentable such that x_i has a factorization over u, that is, $x_i = u \circ x_i'$ for some $x_i' : X \to A'$, $i = 1, 2$. Again, $f \circ u$ has a factorization over some $\overline{v} : \overline{B} \to B$ with λ-presentable domain $\overline{B}$, $f \circ u = v \circ \overline{f}$. Since $\overline{v} \circ (\overline{f} \circ x_1') = \overline{v} \circ (\overline{f} \circ x_2')$, there are morphisms $v : B' \to B$ and $h : \overline{B} \to B'$ such that $\overline{v} = v \circ h$, $h \circ \overline{f} \circ x_1' = h \circ \overline{f} \circ x_2'$ and B' is λ-presentable. Since f is λ-pure, u has a factorization over $h \circ \overline{f}$, which implies $x_1 = x_2$. □

Let $g : X \to Y$ be a morphism in a category $\mathcal{C}$ where X, Y are λ-presentable objects for some regular cardinal λ. Then g is a λ-presentable object in $\mathrm{Mor}(\mathcal{C})$. Indeed, if there is a morphism $(u, v) : g \to \mathrm{colim} f_i$ for some λ-directed system $\{f_i : A_i \to B_i\}_{i \in I}$ in $\mathrm{Mor}(\mathcal{C})$, then there are morphisms $X \to A_i$ and $Y \to B_j$ over which u and v have a factorization, respectively. Since I is λ-directed, it can be found an object $k \in I$ for which u and v have factorization over the canonical morphisms $A_k \to \mathrm{colim}_I Ai$ and $B_k \to \mathrm{colim}_I B_i$, that is, there is a diagram



(5.1.1)

such that $u = a_k \circ u'$ and $v = b_k \circ v'$. Then we have $b_k \circ f_k \circ u' = b_k \circ v' \circ f'$. Due to being Y λ-presentable, again, there is an object $t \in I$ such that the diagram 5.1.1 is converted to a commutative one after replacing k instead of i, i.e., $g \to f$ has a factorization over $f_i \to f$.

The following proposition characterizes all λ-pure morphisms in a locally λ-presentable category. In such a category, the class of λ-pure morphisms is a free cocompletion of split monomorphisms with respect to λ-directed colimits. This characterization provides us an efficient tool to prove one of our main results, that will be explained later, in Chapter (VII).

Proposition 5.1.5. *[AR94, Proposition 2.30] Let $\mathcal{C}$ be any category. Then the following holds:*

(i) If $\mathcal{C}$ is a λ-accessible category, then the full subcategory of $\mathrm{Mor}(\mathcal{C})$ consisting of λ-pure morphisms is closed under λ-directed colimits and contains all sections.

(ii) If $\mathcal{C}$ is a locally λ-presentable category, then a morphism is a λ-pure morphism if and only if it is a λ-directed colimit of split monomorphisms.

Proof. (i) It follows from the above arguments.

(ii) Let f be a λ-pure morphism. Then we may write $f = \mathrm{colim}_I f_i$ as a λ-directed colimit of a system $(f_i : A_i \to B_i)_I$ with the canonical morphisms $(u_i, v_i) : f_i \to f$ where A_i and

B_i are λ-presentable for each $i \in I$. We consider pushouts of u_i and f_i,

$$\begin{array}{ccc} A_i & \xrightarrow{f_i} & B_i \\ {\scriptstyle u_i}\downarrow & & \downarrow{\scriptstyle u_i'} \\ A & \xrightarrow[f_i']{} & B_i' \end{array}$$

with the unique morphism $\alpha_i : B_i' \to B$ for each $i \in I$. It is obvious that $\{f_i'\}_{i\in I}$ is a λ-directed system with colim$f_i' = f$. Note that each f_i' is a split monomorphism because f is a λ-pure morphism and f_i' is in a pushout diagram, for each $i \in I$.

□

Another useful result related to purity which is of our interest is the following:

Theorem 5.1.6. *[AR94, Theorem 2.33] (Every λ-accessible category has enough λ-pure subobjects.) Let $\mathcal{C}$ be a λ-accessible category. There exist arbitrary large regular cardinals $\gamma \rhd \lambda$ such that every γ-presentable subobject A of B in $\mathcal{C}$ is contained in a λ-pure subobject $\overline{A}$ of B, where $\overline{A}$ is γ-presentable.*

Note that if $\mathcal{C}$ is an abelian category and locally λ-presentable category, then a monomorphism $f : A \to B$ is λ-pure monomorphism if and only if the short exact sequence

$$0 \to A \to B \to \text{Coker}f \to 0$$

is Hom$(X, \text{-})$-exact for every λ-presentable object $X \in \mathcal{C}$.

5.2 Purity in a closed symmetric monoidal Grothendieck category

In this section, we work on purity which comes from a closed symmetric monoidal structure. So we begin with a series of definitions regarding to monoidal structures on a category.

Definition 5.2.1. A *monoidal category* $(\mathcal{C}, \otimes, I, \alpha, l, r)$ consists of a category $\mathcal{C}$ and

M1) a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$,

M2) a functorial isomorphism

$$\alpha_{UVW} : \quad (U \otimes V) \otimes W \xrightarrow{\sim} U \otimes (V \otimes W)$$

(*associativity isomorphism*) of functors $\mathcal{C} \times \mathcal{C} \times \mathcal{C} \to \mathcal{C}$,

M3) a *unit object* $I \in \mathcal{C}$ and functorial isomorphisms

$$l_V : \quad I \otimes V \xrightarrow{\sim} V \ ,$$

$$r_V : \quad V \otimes I \xrightarrow{\sim} V$$

for $V \in \mathcal{C}$,

subject to two coherence axioms:

M4) Pentagon axiom: The following diagram

$$\begin{array}{ccc} & ((U \otimes V) \otimes W) \otimes Z & \\ \swarrow^{\alpha_{UVW} \otimes \mathrm{id}_Z} & & \searrow^{\alpha_{(U \otimes V)WZ}} \\ (U \otimes (V \otimes W)) \otimes Z & & (U \otimes V) \otimes (W \otimes Z) \\ \downarrow{\scriptstyle \alpha_{U(V \otimes W)Z}} & & \downarrow{\scriptstyle \alpha_{UV(W \otimes Z)}} \\ U \otimes ((V \otimes W) \otimes Z) & \xrightarrow[\mathrm{id}_U \otimes \alpha_{VWZ}]{} & U \otimes (V \otimes (W \otimes Z)) \end{array}$$

is commutative for all $U, V, W, Z \in \mathcal{C}$.

M5) Triangle axiom: For any $U, V \in \mathcal{C}$, the diagram

$$\begin{array}{ccc} (U \otimes I) \otimes V & \xrightarrow{\alpha} & U \otimes (I \otimes V) \\ & {\scriptstyle r \otimes \mathrm{id}} \searrow \quad \swarrow {\scriptstyle \mathrm{id} \otimes l} & \\ & U \otimes V & \end{array}$$

Definition 5.2.2. A *symmetry* c for a monoidal category $\mathcal{C}$ is a natural isomorphism c_{UV} : $U \otimes V \to V \otimes U$ in such a way that the following diagrams are commutative

$$\begin{array}{ccc} U \otimes V & \xrightarrow{c} & V \otimes U \\ & {\scriptstyle \mathrm{id}} \searrow & \downarrow {\scriptstyle c} \\ & & U \otimes V \end{array}$$

$$\begin{array}{ccccc} (U \otimes V) \otimes W & \xrightarrow{\alpha} & U \otimes (V \otimes W) & \xrightarrow{c} & (V \otimes W) \otimes U\ , \\ \downarrow {\scriptstyle c \otimes \mathrm{id}} & & & & \downarrow {\scriptstyle \alpha} \\ (V \otimes U) \otimes W & \xrightarrow[\alpha]{} & V \otimes (U \otimes W) & \xrightarrow[\mathrm{id} \otimes c]{} & V \otimes (W \otimes U) \end{array}$$

$$\begin{array}{ccc} I \otimes V & \xrightarrow{c} & V \otimes I\ . \\ & {\scriptstyle l} \searrow \quad \swarrow {\scriptstyle r} & \\ & V & \end{array}$$

A monoidal category $\mathcal{C}$ together with a symmetry is called a *symmetric monoidal category*.

Some interesting results emerge when $- \otimes V$ has a right adjoint.

Definition 5.2.3. A monoidal category $\mathcal{C}$ is said to be *closed* if each functor $- \otimes V : \mathcal{C} \to \mathcal{C}$ has a right adjoint $[V, -] : \mathcal{C} \to \mathcal{C}$,

$$\pi : \quad \mathrm{Hom}_{\mathcal{C}}(X \otimes V, Z) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X, [V.Z])\ .$$

$[X, Y]$ is called the *internal hom* of $X, Y \in \mathcal{C}$.

[Fox76] introduced the notion of pure morphisms with respect to a monoidal structure.

Definition 5.2.4. [Fox76] A monomorphism $f : X \to Y$ is called $\otimes$-pure if $f \otimes Z$ is a monomorphism for all $Z \in \mathcal{C}$.

In [Fox76], he considers locally λ-presentable categories having images of every morphism and in which $\otimes$ preserves λ-directed colimits. He gets a sort of enough pure subobjects result, see [Fox76, Theorem 3]. But the result is not enough for our necessities because a pure-completion of a γ-generated subobject of an object is not γ-generated anymore. In such a category, due to Proposition 5.1.5, we may observe that λ-pure morphisms are pure in the sense of $\otimes$. Therefore we get a better pure-completion of γ-presentable subobjects by Theorem 5.1.6.

From now on, we focus on a locally presentable abelian category $\mathcal{C}$ with a closed symmetric monoidal structure. Assume that $\mathcal{C}$ is locally λ-presentable. Therefore, there are two canonical classes $\mathcal{P}$ and $\mathcal{P}_{\otimes}$ of λ-pure and $\otimes$-pure short exact sequences, respectively. We generally call *categorical pure* and *geometrical pure* for the short exact sequences which belong to $\mathcal{P}$ and $\mathcal{P}_{\otimes}$, respectively. As a consequence of Proposition 5.1.5, $\mathcal{P} \subseteq \mathcal{P}_{\otimes}$. Besides, $\mathcal{C}$ has enough $\otimes$-pure subobjects, by Theorem 5.1.6.

One of the problems that we are interested in is whether there are enough injectives with respect to $\mathcal{P}$ and $\mathcal{P}_{\otimes}$. In case $\mathcal{P}$, it is only known when $\lambda = \aleph_0$. It was firstly proved in [GM93] for a locally finitely presentable Grothendieck category, but [Her03] showed the existence of categorical pure-injective envelopes for any locally finitely presentable category by embedding the category, through the Yoneda functor, into a locally finitely presentable Grothendieck category and by using the fact that it is equivalent to the subcategory of flat functors. In this case, the category of flat functors is closed under direct limits, so it is also closed under well-ordered direct limits. Then Proposition 3.2.15 is applicable.

Proposition 5.2.5. *[Her03] Let $\mathcal{C}$ be a locally finitely presentable category, then every object has a categorical pure-injective envelope.*

For more general regular cardinal λ, it isn't known the existence of injectives with respect to the class $\mathcal{P}$. But in the following theorem, we prove that a Grothendieck category with a closed symmetric monoidal structure $\otimes$ has enough injectives with respect to the proper class $\mathcal{P}_{\otimes}$.

Theorem 5.2.6. *Let $\mathcal{C}$ be a closed symmetric monoidal Grothendieck category. Then every object can be purely embedded in a pure-injective object, that is, every object has a geometrical pure-injective preenvelope.*

Proof. Let λ be a regular cardinal for which $\mathcal{C}$ is locally λ-presentable. For an injective cogenerator $\mathcal{E}$ of $\mathcal{C}$, it is easy to observe that each $[A, \mathcal{E}]$ is geometric pure-injective for all object $A \in \mathcal{C}$. Now the first claim is to show that an exact sequence $\mathbb{E}$ is geometric pure if and only if $\mathrm{Hom}(\mathbb{E}, [A, \mathcal{E}])$ is exact for every λ-presentable object A of $\mathcal{C}$. The necessity is clear since $[A, \mathcal{E}]$ is pure-injective. For the sufficiency, it is enough to show that $\mathbb{E} \otimes A$ is exact in $\mathcal{C}$ for any λ-presentable object A since $\otimes$ preserves any colimit. The assertion follows because $\mathrm{Hom}(\mathbb{E}, [A, \mathcal{E}]) \cong \mathrm{Hom}(\mathbb{E} \otimes A, \mathcal{E})$ is exact and $\mathcal{E}$ is a cogenerator.

Now let X be an object of $\mathcal{C}$. Consider the canonical morphism

$$\alpha : X \to \prod_{A \in \mathcal{A}} [A, \mathcal{E}]^{J_A},$$

where $\mathcal{A}$ is the set of representatives of λ-presentable objects and $J_A := \mathrm{Hom}(X, [A, \mathcal{E}])$. First, note that $\prod_{A \in \mathcal{A}} [A, \mathcal{E}]^{J_A}$ is a geometrical pure-injective. Now let $h : Y \to X$ be a morphism such that $\alpha \circ h = 0$. Then

$$\mathrm{Hom}(h, -) : \mathrm{Hom}_{\mathcal{C}}(X, [A, \mathcal{E}]) \to \mathrm{Hom}_{\mathcal{C}}(Y, [A, \mathcal{E}])$$

is zero for every $A \in \mathcal{A}$. By adjunction, it means that

$$\mathrm{Hom}(h \otimes \mathrm{id}_A, -) : \mathrm{Hom}_{\mathcal{C}}(X \otimes A, \mathcal{E}) \to \mathrm{Hom}_{\mathcal{C}}(Y \otimes A, \mathcal{E})$$

is the zero morphism. Since $\mathcal{E}$ is a cogenerator, we get $h \otimes \mathrm{id}_A = 0$ for all $A \in \mathcal{A}$. Then h must be zero, because $h \otimes \mathrm{id}_I$ is zero. This implies that α is a monomorphism. It is immediate that α is a $\otimes$-pure-monomorphism. □

5.3 Purity in $\mathfrak{Qcoh}(X)$

Let X be a scheme and $\mathcal{F}, \mathcal{G}$ be $\mathcal{O}_X$-modules. Recall from Section 2.4, the pair $(- \otimes -, \mathcal{H}om(-,-))$ makes $\mathcal{O}_X$-Mod a closed symmetric monoidal category. Then geometrical pure monomorphisms in $\mathcal{O}_X$-Mod have a special characterization.

Proposition 5.3.1. *Let $0 \to \mathcal{F} \xrightarrow{\tau} \mathcal{G}$ be an exact sequence in $\mathcal{O}_X$-Mod. The following conditions are equivalent:*

(i) the sequence is geometrical pure exact.

(ii) For each $x \in X$ the monomorphism $0 \to \mathcal{F}_x \xrightarrow{\tau} \mathcal{G}_x$ in $\mathcal{O}_{X,x}$-Mod, is pure.

Proof. $(i \Rightarrow ii)$ Let $M \in \mathcal{O}_{X,x}$-Mod. Then $i_{x,*}M$ (the skyscraper sheaf with respect to M) is an $\mathcal{O}_X$-module such that $(i_{x,*}M)_x = M$. Since $0 \to \mathcal{F} \xrightarrow{\tau} \mathcal{G}$ is pure,

$$0 \to i_{x,*}M \otimes \mathcal{F} \to i_{x,*}M \otimes \mathcal{G}$$

is exact, that is, for each $x \in X$,

$$0 \to (i_{x,*}M \otimes \mathcal{F})_x \to (i_{x,*}M \otimes \mathcal{G})_x$$

is exact in $\mathcal{O}_{X,x}$-Mod. But for each $\mathcal{A} \in \mathcal{O}_{X,x}$-Mod, $(i_{x,*}M \otimes \mathcal{A})_x \cong M \otimes \mathcal{A}_x$. Hence, from the previous, we follow that

$$0 \to M \otimes \mathcal{F}_x \to M \otimes \mathcal{G}_x$$

is exact in $\mathcal{O}_{X,x}$-Mod. So $0 \to \mathcal{F}_x \xrightarrow{\tau} \mathcal{G}_x$ is pure.

$(ii \Rightarrow i)$ Let $0 \to \mathcal{F} \xrightarrow{\tau} \mathcal{G}$ be an exact sequence in $\mathcal{O}_X$-Mod (so, for each $x \in X$, $0 \to \mathcal{F}_x \xrightarrow{\tau_x} \mathcal{G}_x$ is exact in $\mathcal{O}_{X,x}$-Mod). Given $\mathcal{M} \in \mathcal{O}_X$-Mod, the induced $\mathcal{M} \otimes \mathcal{F} \xrightarrow{id \otimes \tau} \mathcal{M} \otimes \mathcal{G}$ will be a monomorphism if, and only if, for each $x \in X$ the morphism of $\mathcal{O}_{X,x}$-modules

$$(\mathcal{M} \otimes \mathcal{F})_x \xrightarrow{(id \otimes \tau)_x} (\mathcal{M} \otimes \mathcal{G})_x$$

is such. But, for each $x \in X$, and $\mathcal{A} \in \mathcal{O}_X$-Mod, $(\mathcal{M} \otimes \mathcal{A})_x \cong \mathcal{M}_x \otimes \mathcal{A}_x$. So by (ii) it follows that $\mathcal{M} \otimes \mathcal{F} \xrightarrow{id \otimes \tau} \mathcal{M} \otimes \mathcal{G}$ is a monomorphism. Therefore $0 \to \mathcal{F} \xrightarrow{\tau} \mathcal{G}$ is geometrical pure. □

The following proposition shows that geometrical pure monomorphisms in $\mathcal{O}_X$-Mod whose domain and codomain belong to $\mathfrak{Qcoh}(X)$ carry the purity property on sections.

Proposition 5.3.2. *Let X be a scheme and $\mathcal{F}, \mathcal{G} \in \mathfrak{Qcoh}(X)$. The following conditions are equivalent:*

(i) $0 \to \mathcal{F} \xrightarrow{\tau} \mathcal{G}$ *is geometrical pure exact in $\mathcal{O}_X$-Mod.*

(ii) $0 \to \mathcal{F}(U) \xrightarrow{\tau_U} \mathcal{G}(U)$ *is pure in $\mathcal{O}_X(U)$-Mod, for each open affine $U \subseteq X$.*

Proof. $(i \Rightarrow ii)$ Let U be an affine open subset of X and $\imath : U \hookrightarrow X$ be the open immersion. And let $M \in \mathcal{O}_X(U)$-Mod. Then $\imath_*(\widetilde{M})$ is an $\mathcal{O}_X$-module. Therefore

$$0 \to \imath_*(\widetilde{M}) \otimes \mathcal{F} \to \imath_*(\widetilde{M}) \otimes \mathcal{G}$$

is exact. But then

$$0 \to (\imath_*(\widetilde{M}) \otimes \mathcal{F})(U) \to (\imath_*(\widetilde{M}) \otimes \mathcal{G})(U)$$

is exact in $\mathcal{O}_X(U)$-Mod, that is,

$$0 \to \imath_*(\widetilde{M})(U) \otimes \mathcal{F}(U) \to \imath_*(\widetilde{M})(U) \otimes \mathcal{G}(U)$$

is exact. Since, for each $\mathcal{O}_X(U)$-module A, $\imath_*(\widetilde{A})(U) = A$, we get that $0 \to M \otimes \mathcal{F}(U) \to M \otimes \mathcal{G}(U)$ is exact. Thus $0 \to \mathcal{F}(U) \to \mathcal{G}(U)$ is pure.

$(ii \Rightarrow i)$ This is immediate just by observing that, for each affine open set $U \subseteq X$, $(\mathcal{F} \otimes \mathcal{G})(U) \cong \mathcal{F}(U) \otimes \mathcal{G}(U)$, and that a morphism τ in $\mathcal{O}_X$-Mod is a monomorphism if, and only if, τ_U is a monomorphism in $\mathcal{O}_X(U)$-Mod. □

Proposition 5.3.3. *Let X be a scheme and $\mathcal{F}, \mathcal{G} \in \mathfrak{Qcoh}(X)$. The following statements are equivalent:*

(i) $0 \to \mathcal{F} \xrightarrow{\tau} \mathcal{G}$ *is geometrical pure exact in $\mathcal{O}_X$-Mod.*

(ii) There exists an open covering of X by affine open sets, $\mathcal{U} = \{U_i\}$, such that $0 \to \mathcal{F}(U_i) \xrightarrow{\tau_{U_i}} \mathcal{G}(U_i)$ *is pure in $\mathcal{O}_X(U_i)$-Mod.*

(iii) $0 \to \mathcal{F}_x \xrightarrow{\tau_x} \mathcal{G}_x$ *is pure in $\mathcal{O}_{X,x}$-Mod, for each $x \in X$.*

Proof. $(i \Rightarrow ii)$ It follows from Proposition 5.3.2.

$(ii \Rightarrow iii)$ Let $x \in X$. Then there exists $U_i \in \mathfrak{U}$ such that $x \in U_i = Spec(A_i)$, for some ring A_i. But then the claim follows by observing that $\mathcal{F}_x = (\widetilde{\mathcal{F}(U_i)})_x \cong \widetilde{\mathcal{F}(U_i)_x}$ and noticing that if $0 \to M \to N$ is pure exact in A_i-Mod, then $0 \to M_x \to N_x$ is pure exact in $(A_i)_x$-Mod.

$(iii \Rightarrow i)$ By Proposition 5.3.1, we know that τ is pure in $\mathcal{O}_X$-Mod. □

Remark 5.3.4. Note that $\mathfrak{Qcoh}(X)$ is a monoidal category with the tensor product induced from $\mathcal{O}_X$-Mod. Therefore there is also the notion of geometrical purity in $\mathfrak{Qcoh}(X)$ by using its own monoidal structure, that is, $0 \to \mathcal{F} \to \mathcal{G}$ is pure exact provided that it is $\mathcal{M} \otimes -$ exact, for each $\mathcal{M} \in \mathfrak{Qcoh}(X)$. Then a monomorphism in $\mathfrak{Qcoh}(X)$ as in Proposition 5.3.3 is always geometrical pure in $\mathfrak{Qcoh}(X)$. In Proposition 7.2.8, we see that in case X is quasi-separated the converse is true, as well. This is because the direct image functor $\imath_*(\widetilde{M})$

preserves quasi-coherence when X is quasi-separated, in the proof of Proposition 5.3.2. So we call *stalkwise pure-exact sequence* for that of Proposition 5.3.3 and *geometrical (tensor) pure-exact sequence* for that of induced by monoidal structure of $\mathfrak{Qcoh}(X)$, itself. The later one will be the subject of Chapter (VII).

Over an affine scheme X, the category of quasi-coherent sheaves on X is equivalent to the category $\mathcal{O}_X(X)$-Mod. So the following lemma can be easily obtained.

Lemma 5.3.5. *Let $\mathcal{F} \in \mathfrak{Qcoh}(X)$ and U be an affine open subset of X. Then $\mathcal{F} \mid_U$ is finitely presentable in $\mathfrak{Qcoh}(U)$ if and only if $\mathcal{F}(U)$ is finitely presented.*

Proposition 5.3.6. *[Mur2, proposition 75] Assume that X is semi-separated or concentrated. Let $\mathcal{F} \in \mathfrak{Qcoh}(X)$ and consider the following assertions.*

(i) $\mathcal{F}$ is a finitely presentable object in $\mathfrak{Qcoh}(X)$.

(ii) $\mathcal{F} \mid_U$ is finitely presentable in $\mathfrak{Qcoh}(U)$ for all affine open subsets $U \subseteq X$.

(iii) $\mathcal{F}_x$ is finitely presented for each $x \in X$.

Then the implications $i \Rightarrow ii \Rightarrow iii$ hold. If X is concentrated, then $i \Leftrightarrow ii$.

Proof. $(i \Rightarrow ii)$ We have to show that the canonical morphism

$$\psi : \varinjlim \mathrm{Hom}(\mathcal{F}|_U, \widetilde{B_i}) \to \mathrm{Hom}(\mathcal{F}|_U, \varinjlim \widetilde{B_i})$$

is an isomorphism for any direct system $\{\widetilde{B_i}, \varphi_{ij}\}_I$ of quasi-coherent $\mathcal{O}_X|_U$-modules. We have the following commutative diagram

$$\begin{array}{ccccc} \mathrm{Hom}(\mathcal{F}|_U, \widetilde{B_i}) & \longrightarrow & \varinjlim \mathrm{Hom}(\mathcal{F}|_U, \widetilde{B_i}) & \xrightarrow{\psi} & \mathrm{Hom}(\mathcal{F}|_U, \varinjlim \widetilde{B_i}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Hom}(\mathcal{F}, \imath_*(\widetilde{B_i})) & \longrightarrow & \varinjlim \mathrm{Hom}(\mathcal{F}, \imath_*(\widetilde{B_i})) & \xrightarrow{\psi'} & \mathrm{Hom}(\mathcal{F}, \varinjlim \imath_*(\widetilde{B_i})) \end{array} .$$

Columns are isomorphisms because of the adjoint pair $(\mathrm{res}_U, \imath_*)$. For the third column we also need to observe that, under the hypothesis on X, the direct image functor $\imath_*$ preserves direct limits. Since $\mathcal{F}$ is finitely presentable, the canonical morphism ψ' is an isomorphism. So ψ is an isomorphism.

$(ii \Rightarrow iii)$ $\mathcal{F}_x \cong M_p$ for some finitely presented R-module M and prime ideal p. Then (iii) follows because the localization of a finitely presented R-module is a finitely presented R_p-module. □

Definition 5.3.7. (cf. [Craw94, §3]) An exact sequence $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{T} \to 0$ in $\mathfrak{Qcoh}(X)$ is called *categorical pure* if the functor $\mathrm{Hom}(\mathcal{H}, -)$ leaves the sequence exact for every finitely presentable quasi-coherent $\mathcal{O}_X$-module $\mathcal{H}$.

We shall denote by $\mathbf{Pure_{fp}}$ the class of categorical pure short exact sequences in $\mathfrak{Qcoh}(X)$ and by $\mathbf{Pure}$ the class of stalkwise pure short exact sequences in $\mathfrak{Qcoh}(X)$, as in Proposition 5.3.3.

Proposition 5.3.8. *If $\mathfrak{Qcoh}(X)$ is a locally finitely presentable category then categorical pure short exact sequences are stalkwise pure exact, that is,* $\mathbf{Pure_{fp}} \subseteq \mathbf{Pure}$.

Proof. Let $\mathbb{E} \equiv 0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be an exact sequence in $\mathbf{Pure_{fp}}$. By assumption, $\mathcal{H} = \varinjlim \mathcal{H}_i$ where $\mathcal{H}_i$ is a finitely presentable object in $\mathfrak{Qcoh}(X)$ for each i. Now, for each i, the top row of the following pullback diagram,

$$\begin{array}{cccccccccc} \mathbb{E}^i = & 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{G}_i & \longrightarrow & \mathcal{H}_i & \longrightarrow & 0 \\ & & & \downarrow & & \downarrow & & \downarrow & & \\ & 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{H} & \longrightarrow & 0 \end{array}$$

is a categorical pure exact sequence ending with a finitely presentable object $\mathcal{H}_i$. Therefore, each $\mathbb{E}^i$ splits for every i. That is, $\mathbb{E} = \varinjlim \mathbb{E}^i$ where $\mathbb{E}^i$ is a splitting exact sequence for every

i. Now taking the stalk at $x \in X$, we get $\mathbb{E}_x = \varinjlim \mathbb{E}^i_x$. Then $\mathbb{E}^i_x$ is pure exact in $\mathcal{O}_{X,x}$-Mod for each $x \in X$, so is $\mathbb{E}_x$. Hence, by Proposition 5.3.3, $\mathbb{E}$ is a pure exact sequence in $\mathfrak{Qcoh}(X)$. □

5.4 Stalkwise pure injective envelopes in $\mathfrak{Qcoh}(X)$

Definition 5.4.1. A quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$ is said to be *fp-pure injective* (resp. *stalkwise pure injective*) if for every short exact sequence $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ in $\mathbf{Pure_{fp}}$ (resp. in $\mathbf{Pure}$) the sequence $0 \to \mathrm{Hom}(\mathcal{H}, \mathcal{M}) \to \mathrm{Hom}(\mathcal{G}, \mathcal{M}) \to \mathrm{Hom}(\mathcal{F}, \mathcal{M}) \to 0$ is exact. We shall denote by $\mathcal{P}inj_{fp}$ (resp. by $\mathcal{P}inj$) the class of all fp-pure injective quasi-coherent sheaves (resp. the class of all stalkwise pure injective quasi-coherent sheaves). In this section, when we say that an $\mathcal{O}_X$-module is pure injective, we mean that it is 'injective' with respect to all geometrical pure exact sequences in $\mathcal{O}_X$-Mod as in Proposition 5.3.1.

Remark 5.4.2.

- If X is concentrated then, by Proposition 5.3.8, $\mathcal{P}inj \subseteq \mathcal{P}inj_{fp}$.
- Clearly, every injective quasi-coherent $\mathcal{O}_X$-module is both fp-pure injective and stalkwise pure injective.

Theorem 5.4.3. *Let X be a concentrated scheme. Then every $\mathcal{M} \in \mathfrak{Qcoh}(X)$ admits an fp-pure injective envelope $\eta : \mathcal{M} \to \mathrm{PE}_{fp}(\mathcal{M})$. That is, $\mathcal{P}inj_{fp}$ is enveloping.*

Moreover the induced short exact sequence

$$0 \to \mathcal{M} \xrightarrow{\eta} \mathrm{PE}_{fp}(\mathcal{M}) \longrightarrow \frac{\mathrm{PE}_{fp}(\mathcal{M})}{\mathcal{M}} \to 0$$

is in $\mathbf{Pure_{fp}}$.

Proof. Since X is concentrated, $\mathfrak{Qcoh}(X)$ is a locally finitely presentable Grothendieck category. So the result follows from [Her03, theorem 6] (see also [GM93]). □

Now, we shall point out that $\mathcal{O}_X$-Mod is a Grothendieck category, hence it admits an injective cogenerator, $\mathcal{E}$. We consider the coherator functor C mentioned in Section 2.4. Then $C(\mathcal{E})$ is an injective cogenerator in $\mathfrak{Qcoh}(X)$ due to the fact that the inclusion functor $\iota : \mathfrak{Qcoh}(X) \hookrightarrow \mathcal{O}_X$-Mod is an exact functor with the right adjoint C.

We shall denote by $\mathcal{M}^\vee$ the *character $\mathcal{O}_X$-module* given by $\mathcal{M}^\vee = \mathcal{H}om(\mathcal{M}, \mathcal{E})$. There is a canonical map $ev : \mathcal{M} \to \mathcal{M}^{\vee\vee}$.

Proposition 5.4.4. *Given $\mathcal{M} \in \mathcal{O}_X$-Mod, the character $\mathcal{O}_X$-module $\mathcal{M}^\vee$ is pure injective in $\mathcal{O}_X$-Mod.*

Proof. Let $0 \to \mathcal{T} \to \mathcal{N} \to \mathcal{H} \to 0$ be a pure exact sequence in $\mathcal{O}_X$-Mod. Then

$$\mathrm{Hom}(\mathcal{N}, \mathcal{M}^\vee) \to \mathrm{Hom}(\mathcal{T}, \mathcal{M}^\vee) \to 0$$

is exact if and only if

$$\mathrm{Hom}(\mathcal{N} \otimes \mathcal{M}, \mathcal{E}) \to \mathrm{Hom}(\mathcal{T} \otimes \mathcal{M}, \mathcal{E}) \to 0$$

is exact. But the latter follows since $0 \to \mathcal{T} \otimes \mathcal{M} \to \mathcal{N} \otimes \mathcal{M}$ is exact and $\mathcal{E}$ is an injective cogenerator. □

Lemma 5.4.5. *Let $\mathcal{M}$ be a pure-injective $\mathcal{O}_X$-module. Then its coherator $C(\mathcal{M})$ is stalkwise pure injective in $\mathfrak{Qcoh}(X)$.*

Proof. Let $0 \to \mathcal{F} \to \mathcal{G}$ be a stalkwise pure exact sequence in $\mathfrak{Qcoh}(X)$. This means that it is pure exact in $\mathcal{O}_X$-Mod by Proposition 5.3.3. So we have an exact sequence

$$\mathrm{Hom}_{\mathcal{O}_X\text{-Mod}}(\mathcal{G}, \mathcal{M}) \to \mathrm{Hom}_{\mathcal{O}_X\text{-Mod}}(\mathcal{F}, \mathcal{M}) \to 0.$$

Since (ι, C) is an adjoint pair where $\iota : \mathfrak{Qcoh}(X) \hookrightarrow \mathcal{O}_X$-Mod, it implies that

$$\mathrm{Hom}_{\mathfrak{Qcoh}(X)}(\mathcal{G}, C(\mathcal{M})) \to \mathrm{Hom}_{\mathfrak{Qcoh}(X)}(\mathcal{F}, C(\mathcal{M})) \to 0$$

is exact. □

Corollary 5.4.6. *Every quasi-coherent sheaf $\mathcal{M}$ is a stalkwise pure subobject of a stalkwise pure injective quasi-coherent sheaf. In particular, the class of stalkwise pure injective quasi-coherent sheaves is preenveloping.*

Proof. As a consequence of Theorem 5.2.6, $\mathcal{M}$ can be embedded (geometric) purely into a pure injective $\mathcal{O}_X$-module $\mathcal{M}'$ in $\mathcal{O}_X$-Mod, $f : \mathcal{M} \hookrightarrow \mathcal{M}'$. So we apply the coherator functor on $\mathcal{M}'$, $C(\mathcal{M}')$. By Lemma 5.4.5, it is a stalkwise pure injective quasi-coherent sheaf. The adjoint pair (ι, C) allows to factorize f over $C(\mathcal{M}')$. Indeed, $\mathfrak{Qcoh}(X)$ is a coreflective subcategory of $\mathcal{O}_X$-Mod and $\mathcal{M}$ is quasi-coherent. So there is a unique morphism $\varphi : \mathcal{M} \to C(\mathcal{M}')$ over which f is factorized. Then φ is a stalkwise pure monomorphism in $\mathfrak{Qcoh}(X)$. □

In order to show that the class $\mathcal{P}inj$ in $\mathfrak{Qcoh}(X)$ is enveloping, we will apply [Xu96, Theorem 2.3.8] (this, in turn, uses [Xu96, Theorem 2.2.6]). The arguments in these proofs are categorical and can be easily extended to our setup in $\mathfrak{Qcoh}(X)$ by taking into account the following lemma:

Lemma 5.4.7. *For a given $\mathcal{M} \in \mathfrak{Qcoh}(X)$, the class of sequences in $\mathbf{Pure}$ of the form*

$$0 \to \mathcal{M} \to \mathcal{L} \to \mathcal{T} \to 0,$$

varying $\mathcal{L}, \mathcal{T} \in \mathfrak{Qcoh}(X)$ is closed under direct limits.

Proof. The argument is local and so it can be deduced from the corresponding result on module categories (see for example [Xu96, Proposition 2.3.7]). □

Combining Lemma 5.4.7 and Corollary 5.4.6 and applying the analogue to [Xu96, Theorem 2.3.8] for the category $\mathfrak{Qcoh}(X)$, we get

Theorem 5.4.8. *Every* $\mathcal{M} \in \mathfrak{Qcoh}(X)$ *admits a stalkwise pure injective envelope* $\eta : \mathcal{M} \to \mathrm{PE}(\mathcal{M})$. *That is,* $\mathcal{P}inj$ *is enveloping.*

Moreover the induced short exact sequence

$$0 \to \mathcal{M} \xrightarrow{\eta} \mathrm{PE}(\mathcal{M}) \longrightarrow \frac{\mathrm{PE}(\mathcal{M})}{\mathcal{M}} \to 0$$

is in $\mathbf{Pure}$.

CHAPTER SIX

RELATIVE HOMOLOGICAL ALGEBRA IN $\mathfrak{Qcoh}(X)$

This chapter is devoted to introducing some new classes in $\mathfrak{Qcoh}(X)$ related to purity and studying their homological properties.

Once we have a proper class $\mathcal{P}$, it is a standard argument to consider '$\mathcal{P}$-flat' objects, that is, objects such that every exact sequence ending with them belongs to $\mathcal{P}$. There is already a well-known notion of flat quasi-coherent sheaves, which are flat sheaves over the identity scheme morphism on X. Equivalently, these are determined by flatness on stalks. They play an important role in the theory of quasi-coherent sheaves because $\mathfrak{Qcoh}(X)$ has no enough projective objects generally. After all, in [EE05], the existence of flat covers in $\mathfrak{Qcoh}(X)$ was proved. Besides, Murfet and Salarian in [MS11] deal with the derived category of flat quasi-coherent sheaves over a Noetherian scheme because it is somehow an equipollent of the homotopy category of projectives. In the first section, we deal with $\mathcal{P}$-flats with respect to stalkwise and categorical purity. We study on the relations among these two notions of flatness and the usual one in $\mathfrak{Qcoh}(X)$.

In the second section we introduce and study (locally) absolutely pure sheaves in both $\mathcal{O}_X$-Mod and in $\mathfrak{Qcoh}(X)$. Remember from [Mad67] that an R-module A is *absolutely pure* if it is pure in every module containing it as a submodule. Absolutely pure modules are also studied with the terminology of FP-injectives ([Ste70]). It follows immediately from the definition that A is absolutely pure if, and only if, it is a pure submodule of some injective module. And therefore A is absolutely pure if and only if, $\mathrm{Ext}^1_R(M, A) = 0$ for each finitely presented R-module M. Since we have stalkwise pure exact sequences in categories of sheaves rather than categorical ones, we deal with stalkwise purity to define absolutely sheaves in $\mathcal{O}_X$-Mod and in $\mathfrak{Qcoh}(X)$. Firstly, in Proposition 6.2.7 we get that over a locally coherent scheme, absolutely pure quasi-coherent sheaves are determined by local information. It is very well-known that absolutely pure modules and injective modules coincide if and only if R is Noetherian. In

Proposition 6.2.11, we extend this result for a locally coherent subscheme $X \subseteq \mathbb{P}^n(R)$. Finally, in Theorem 6.2.13 we prove that the class of locally absolutely pure quasi-coherent sheaves is a covering class over a locally coherent scheme.

In the third section, we concentrate on the characterization of the product object of some special classes. The category of quasi-coherent sheaves over any scheme is known to admit products because it is complete. However their structure seems to be rather mysterious. For example, Murfet in his tesis [Mur07, Remark B.7] came up with the problem whether the class of flat quasi-coherent sheaves is closed under products over a Noetherian scheme in order to write down internal hom in the homotopy category of sheaves $\mathbf{K}(\mathrm{Inj}\, X)$ in terms of flat sheaves. In this section, we introduce the notion of locally torsion-free quasi-coherent sheaves over an integral scheme which is a generalization of flat quasi-coherent sheaves. By Theorem 6.3.7, we describe the structure of the product object of a family of locally torsion-free objects in $\mathfrak{Qcoh}(X)$, for an integral scheme X. Regarding to this fact, several applications relevant to products in the class of locally quasi-coherent sheaves over an integral scheme are provided. For example, the problem mentioned in [Mur07] was proved by [SS11] over a Dedekind scheme. It comes here as an application, see Corollary 6.3.11. And in Corollary 6.3.8, it is shown that the class of locally torsion-free quasi-coherent sheaves is the torsion-free class of a hereditary torsion-theory.

Finally, in the last section, we deal with homological properties of the class of torsion-free quasi-coherent sheaves. The fact that the class $\mathcal{F}$ of locally torsion-free quasi-coherent sheaves is the right part of a torsion theory in $\mathfrak{Qcoh}(X)$ results in that each $\mathcal{M} \in \mathfrak{Qcoh}(X)$ admits an $\mathcal{F}$-*reflection* and thus $\mathcal{F}$ is a *reflective* class in $\mathfrak{Qcoh}(X)$. So, in particular, we deduce that $\mathcal{F}$ is enveloping. This section is devoted to proving that the class $\mathcal{F}$ over an integral scheme is also covering.

6.1 Flat quasi-coherent sheaves

We recall that a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is *flat* if $\mathcal{F}\otimes -$ is exact in $\mathcal{O}_X$-Mod. Equivalently, $\mathcal{F}(U)$ is flat as an $\mathcal{O}_X(U)$-module for each affine open subset $U \subseteq X$, or $\mathcal{F}_x$ is flat as an $\mathcal{O}_{X,x}$-module for each $x \in X$. We will denote by $\mathcal{F}lat(X)$ the class of all flat quasi-coherent sheaves.

Definition 6.1.1. A quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is called *tensor flat* (resp. *fp-flat*) if every short exact sequence in $\mathfrak{Qcoh}(X)$ ending in $\mathcal{F}$ is stalkwise pure exact (resp. is categorical pure). We shall denote by $\mathcal{F}lat_{\otimes}$ (resp. by $\mathcal{F}lat_{fp}$) the class of all tensor flat quasi-coherent sheaves (resp. the class of all fp-flat quasi-coherent sheaves).

Proposition 6.1.2. *Let $\mathcal{F} \in \mathfrak{Qcoh}(X)$. If $\mathcal{F}$ is flat, then it is also tensor flat. In case X is semi-separated, the converse also holds.*

Proof. Let $0 \to \mathcal{T} \to \mathcal{G} \to \mathcal{F} \to 0$ be an exact sequence in $\mathfrak{Qcoh}(X)$. Given an affine open $U \subseteq X$, $0 \to \mathcal{T}(U) \to \mathcal{G}(U) \to \mathcal{F}(U) \to 0$ is also exact. Since $\mathcal{F}(U)$ is a flat $\mathcal{O}_X(U)$-module, we deduce from Proposition 5.3.2 that $\mathcal{F}$ is tensor flat.

If X is a semi-separated scheme then the direct image functor $\imath_*$ for the inclusion map $\imath : U \hookrightarrow X$, where U is affine, is exact. Let $\mathcal{F} \in \mathfrak{Qcoh}(X)$ be tensor flat. We need to show that $\mathcal{F}(U)$ is a flat $\mathcal{O}_X(U)$-module, for each affine open subset $U \subseteq X$. Let

$$0 \longrightarrow A \longrightarrow B \longrightarrow \mathcal{F}(U) \longrightarrow 0$$

be an exact sequence of $\mathcal{O}(U)$-modules. By the previous observation, we have an exact sequence

$$0 \longrightarrow \imath_*(\widetilde{A}) \longrightarrow \imath_*(\widetilde{B}) \longrightarrow \imath_*(\widetilde{\mathcal{F}|_U}) \longrightarrow 0\ .$$

If we take the pullback of the morphism $\imath_*(\widetilde{B}) \to \imath_*(\widetilde{\mathcal{F}|_U})$ and the canonical morphism $\mathcal{F} \to$

$\imath_*(\widetilde{\mathcal{F}|_U})$, we get the commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \imath_*(\widetilde{A}) & \longrightarrow & \mathcal{H} & \longrightarrow & \mathcal{F} & \longrightarrow & 0\,. \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \imath_*(\widetilde{A}) & \longrightarrow & \imath_*(\widetilde{B}) & \longrightarrow & \imath_*(\widetilde{\mathcal{F}|_U}) & \longrightarrow & 0 \end{array}$$

Since $\imath_*(\widetilde{A})$ and $\mathcal{F}$ are quasi-coherent, $\mathcal{H}$ is quasi-coherent. By assumption, the first row is stalkwise pure exact, so by Proposition 5.3.2 each image under affine open subset is pure exact. From this diagram, it can be deduced that $\mathcal{H}(U) \cong \imath_*(\widetilde{B})(U) = B$. So the short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow \mathcal{F}(U) \longrightarrow 0$ is pure, and then $\mathcal{F}(U)$ is a flat $\mathcal{O}_X(U)$-module. □

Corollary 6.1.3. *Assume that $\mathfrak{Qcoh}(X)$ is locally finitely presentable (for instance if X is concentrated). Then $\mathcal{F}lat_{fp} \subseteq \mathcal{F}lat_{\otimes}$. If X is semi-separated then $\mathcal{F}lat_{fp} \subseteq \mathcal{F}lat_{\otimes} = \mathcal{F}lat$.*

Proof. This follows from Proposition 5.3.8 and the previous Proposition. □

Remark 6.1.4. The inclusions in Corollary 6.1.3 are strict. Namely in [ES12, Corollary 4.6] it is shown that $\mathcal{F}lat_{fp} = 0$ in case $X = \mathbf{P}^n(R)$. In general there is a large class of projective schemes X such that $\mathcal{F}lat_{fp} = 0$ in $\mathfrak{Qcoh}(X)$ (see [ES12, Theorem 4.4]).

6.2 Absolutely pure sheaves

An R-module A is *absolutely pure* if it is pure in every module containing it as a submodule. A is absolutely pure if and only if it is a pure submodule of some injective module if and only if $\mathrm{Ext}^1_R(M, A) = 0$ for each finitely presented R-module M.

In this section we will study (locally) absolutely pure sheaves in both $\mathcal{O}_X$-Mod and in $\mathfrak{Qcoh}(X)$ in terms of stalkwise pure exact sequences.

Definition 6.2.1. Let $(X, \mathcal{O}_X)$ be a scheme.

(i) Let $\mathcal{F}$ be in $\mathcal{O}_X$-Mod. $\mathcal{F}$ is *absolutely pure* in $\mathcal{O}_X$-Mod if every exact sequence $0 \to \mathcal{F} \to \mathcal{G}$ in $\mathcal{O}_X$-Mod is geometrical pure exact in $\mathcal{O}_X$-Mod.

(ii) Let $\mathcal{F}$ be a quasi-coherent sheaf on X. $\mathcal{F}$ is called *absolutely pure* in $\mathfrak{Qcoh}(X)$ if every exact sequence $0 \to \mathcal{F} \to \mathcal{G}$ in $\mathfrak{Qcoh}(X)$ is stalkwise pure exact.

(iii) Let $\mathcal{F}$ be a quasi-coherent sheaf on X. $\mathcal{F}$ is called *locally absolutely pure* if $\mathcal{F}(U)$ is absolutely pure over $\mathcal{O}_X(U)$ for every affine open $U \subseteq X$.

Lemma 6.2.2. *All these notions of locally absolutely purity of quasi-coherent sheaves and absolutely purity in $\mathcal{O}_X$-Mod and in $\mathfrak{Qcoh}(X)$ are closed under taking pure subobjects.*

Proof. It follows from the fact that if $f \circ g$ is a pure monomorphism, for each notion of purity, with monomorphisms f and g, then g is a pure monomorphism. □

Lemma 6.2.3. *Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. The following are equivalent:*

(i) $\mathcal{F}$ is absolutely pure in $\mathcal{O}_X$-Mod.

(ii) $\mathcal{F} |_{U_i}$ is absolutely pure in $\mathcal{O}_X |_{U_i}$ -Mod for a cover $\{U_i\}$ of X.

Proof. $(i \Rightarrow ii)$ Let $U \subseteq X$ be open. Then the extension of $\mathcal{F} |_U$ by zero outside U, $j_!(\mathcal{F} |_U)$, is contained in $\mathcal{F}$. Since the stalk of $j_!(\mathcal{F} |_U)$ is $\mathcal{F}_x$ if $x \in U$ and 0 otherwise, $j_!(\mathcal{F} |_U)$ is a pure subsheaf of $\mathcal{F}$ in $\mathcal{O}_X$-Mod. So $j_!(\mathcal{F} |_U)$ is absolutely pure in $\mathcal{O}_X$-Mod, too.

Now let $\mathcal{G}$ be any $\mathcal{O}_X |_U$-module with an exact sequence $0 \to \mathcal{F} |_U \to \mathcal{G}$. Then

$$0 \to j_!(\mathcal{F} |_U) \to j_!(\mathcal{G})$$

is still exact in $\mathcal{O}_X$-Mod. So it is pure in $\mathcal{O}_X$-Mod. But this means that

$$0 \to (j_!(\mathcal{F} |_U))_x \to (j_!(\mathcal{G}))_x$$

is pure for all $x \in X$. For $x \in U$, that exact sequence is equal to the exact sequence

$$0 \to (\mathcal{F} \mid_U)_x \to (\mathcal{G})_x \quad \text{and} \quad j_!(\mathcal{F} \mid_U) \mid_U = \mathcal{F} \mid_U$$

and $(j_!(\mathcal{G})) \mid_U = \mathcal{G}$. That proves the desired implication.

$(ii \Rightarrow i)$ Let $0 \to \mathcal{F} \to \mathcal{G}$ be an exact sequence in $\mathcal{O}_X$-Mod. In order to show that it is pure exact, we need to show that the morphism induced on the stalk is pure exact, for every $x \in X$. But the restriction functor to open subsets is left exact and $(\mathcal{F} \mid_U)_x = \mathcal{F}_x$. So the claim follows. □

Lemma 6.2.4. *Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. If $\mathcal{F}_x$ is absolutely pure for all $x \in X$ then $\mathcal{F}$ is absolutely pure in $\mathcal{O}_X$-Mod.*

Proof. Let $0 \to \mathcal{F} \to \mathcal{G}$ be an exact sequence in $\mathcal{O}_X$-Mod. To be pure in $\mathcal{O}_X$-Mod is equivalent to be pure at the induced morphism on the stalk for every $x \in X$. So that proves our implication. □

Let $X = Spec(R)$ be an affine scheme. The next proposition shows that in order to check that a quasi–coherent $\mathcal{O}_X$-module $\widetilde{A}$ is absolutely pure, it suffices that its restrictions $\widetilde{A}|_{D(s_i)}$, $i = 1, \ldots, n$, be absolutely pure, where $\cup_{i=1}^{n} D(s_i) = X$, and $s_1, \ldots, s_n \in R$.

Proposition 6.2.5. *Let R be a ring and $s_1, s_2, \ldots, s_n$ a finite number of elements of R which generate the unit ideal. Let A be an R-module. If A_{s_i} is absolutely pure over R_{s_i} for every $i = 1, \ldots, n$ then A is absolutely pure over R.*

Proof. Given $A \subseteq B$, we want to prove the canonical morphism $M \otimes A \to M \otimes B$ is injective for every module M. Let $K = \text{Ker}(M \otimes A \to M \otimes B)$. Then by our hypothesis we get $K_{s_i} = 0$ for each $i = 1, \ldots, n$. So if $x \in K$, then ${s_i}^{h_i} x = 0$ for some $h_i \geq 0$. But the set $\{s_1, s_2, \ldots, s_n\}$ generates R. So we have $s_1 t_1 + \ldots + s_n t_n = 1$ for some $t_1, \ldots, t_n \in R$. And also $(s_1 t_1 + \ldots + s_n t_n)^h x = 0$ if $h > h_1 + \ldots + h_n - 1$, i.e., $x = 1.x = 1^h x = 0$. □

Let $X = Spec(R)$ be an affine scheme. Now we will see that in order to check that a quasi–coherent $\mathcal{O}_X$-module $\widetilde{A}$ is absolutely pure, it suffices to check that, for each $P \in X$, each stalk $\widetilde{M}_P$ is an absolutely pure $\mathcal{O}_{X,P}$-module.

Proposition 6.2.6. *If A_P is absolutely pure over R_P for every prime ideal P then A is absolutely pure over R.*

Proof. Let M be a finitely presented R-module. We want to prove that
$\mathrm{Ext}^1_R(M, A) = 0$. Since M is finitely presented,

$$(\mathrm{Ext}^1_R(M, A))_P \cong \mathrm{Ext}^1_{R_P}(M_P, A_P) = 0.$$

Since this is true for each prime ideals P, $\mathrm{Ext}^1_R(M, A) = 0$. So A is absolutely pure. □

Both Propositions 6.2.5 and 6.2.6 do not assume any condition on the ring R. Their converses are not true in general. However they are if R is coherent, see [Pin05, Theorem 3.21]. So it makes sense to define a notion of locally absolutely pure quasi-coherent sheaves over a locally coherent scheme. A scheme $(X, \mathcal{O}_X)$ is *locally coherent* provided that $\mathcal{O}_X(U)$ is a coherent ring, for each affine open subset $U \subseteq X$. Since coherence descends along faithfully flat morphisms of rings (see [Har66, Corollary 2.1]), it follows that X is locally coherent if, and only if, $\mathcal{O}_X(U_i)$ is coherent for each $i \in I$ of some affine open covering $\{U_I\}_{i\in I}$ of X. So over a locally coherent scheme, the next proposition states that in order to prove whether a quasi-coherent sheaf is locally absolutely pure, it is sufficient to look at some cover by affine subsets of X. And these show that locally absolutely purity is a stalkwise property.

Proposition 6.2.7. *Let $(X, \mathcal{O}_X)$ be a locally coherent scheme. Then the following conditions are equivalent for a quasi-coherent sheaf $\mathcal{F}$:*

(i) $\mathcal{F}(U)$ is absolutely pure for every affine U.

(ii) $\mathcal{F}(U_i)$ is absolutely pure for all $i \in I$ for some cover $\{U_i\}_{i\in I}$ of affine open subsets.

(iii) $\mathcal{F}_x$ *is absolutely pure for all* $x \in X$.

Proof. We just need to prove the implications $(ii \Rightarrow iii)$ and $(iii \Rightarrow i)$. By [Pin05, Theorem 3.21] the localization of an absolutely pure module over a coherent ring is again absolutely pure, so the first implication follows. For the second, Let $\mathcal{F}(U) \cong M$ for an $\mathcal{O}_X(U)$-module M. By assumption, $\mathcal{F}(U)_P \cong M_P$ is absolutely pure for all prime ideal P of $\mathcal{O}_X(U)$. Hence, $\mathcal{F}(U) = M$ is also absolutely pure by Proposition 6.2.6. □

The next lemma shows that the locally absolutely pure objects in $\mathfrak{Qcoh}(X)$ on a locally coherent scheme X are exactly the absolutely pure $\mathcal{O}_X$-modules which are quasi-coherent.

Lemma 6.2.8. *Let* X *be a locally coherent scheme and* $\mathcal{F}$ *be a quasi-coherent sheaf. Then* $\mathcal{F}$ *is locally absolutely pure if and only if* $\mathcal{F}$ *is absolutely pure in* $\mathcal{O}_X$*-Mod.*

Proof. It follows by Lemma 6.2.3 and Proposition 6.2.7. □

At this point, we may consider the relation between absolutely pure quasi-coherent sheaves and locally absolutely pure quasi-coherent sheaves.

Lemma 6.2.9. *Let* X *be a locally coherent scheme. Every locally absolutely pure quasi-coherent sheaf is absolutely pure in* $\mathfrak{Qcoh}(X)$.

Proof. This follows from Proposition 6.2.7 and Proposition 5.3.3. □

The converse of Lemma 6.2.9 is not clear in general. But it is true if $X = Spec(R)$ is affine and R is coherent, or X is locally Noetherian. The first case is clear since $\mathfrak{Qcoh}(X) \cong \mathcal{O}_X(X)$-Mod. For the second, let $\mathcal{F}$ be an absolutely pure in $\mathfrak{Qcoh}(X)$ and $E(\mathcal{F})$ be its injective envelope in $\mathfrak{Qcoh}(X)$. Then $0 \to \mathcal{F} \to E(\mathcal{F})$ is pure exact. So, for each affine open subset $U \subseteq X$, $0 \to \mathcal{F}(U) \to E(\mathcal{F})(U)$ is pure exact in $\mathcal{O}_X(U)$-Mod. But $E(\mathcal{F})(U)$ is an injective

$\mathcal{O}_X(U)$-module and $\mathcal{F}(U)$ is a pure submodule of it. Hence $\mathcal{F}(U)$ is absolutely pure, for each affine $U \subseteq X$. So, $\mathcal{F}$ is a locally absolutely pure quasi-coherent sheaf.

Proposition 6.2.10. *Let X be a locally coherent scheme. If the class of injectives sheaves in $\mathcal{O}_X$-Mod is equal to the class of absolutely pure sheaves in $\mathcal{O}_X$-Mod, then X is a locally Noetherian scheme.*

Proof. Suppose that these classes are equal. Let M be an absolutely pure $\mathcal{O}_X(U)$-module where U is an affine open subset. Then the sheaf $j_!(\widetilde{M})$ obtained by extending $\widetilde{M}$ by zero outside U is an absolutely pure $\mathcal{O}_X$-module by Lemma 6.2.4. By assumption, it is injective in $\mathcal{O}_X$-Mod. So, its restriction $(j_!(\widetilde{M})) \mid_U = \widetilde{M}$ is injective in $\mathcal{O}_X \mid_U$ -Mod. Since $\widetilde{M}$ is quasi-coherent, it is injective in $\mathfrak{Qcoh}(U)$ which implies that M is injective $\mathcal{O}_X(U)$-module. So $\mathcal{O}_X(U)$ is a Noetherian ring and hence X is a locally Noetherian scheme. □

Recall that a closed subscheme X of $\mathbb{P}^n(R)$ is given by a quasi-coherent sheaf of ideals, i.e. we have an ideal $I_v \subseteq R[v]$ for each v with $R[w] \otimes_{R[v]} I_v \cong I_w$ when $v \subseteq w$. This means $I_v \to I_w$ is the localization by the same multiplicative set as above. But then $R[v]/I_v \to R[w]/I_w$ is also a localization. So, by abusing the notation, we shall also denote by $\mathbf{R}$ the structural sheaf of rings attained to X.

Proposition 6.2.11. *A closed subscheme $X \subseteq \mathbb{P}^n(R)$ which is locally coherent (for instance if R is coherent) is locally Noetherian if and only if locally absolutely pure quasi-coherent sheaves are locally injective.*

Proof. "If" part is clear. Indeed, if a scheme is locally Noetherian, then all classes of locally absolutely pure, absolutely pure, locally injective and injective quasi-coherent sheaves are equal, by [Hart66, II, Proposition 7.17, Theorem 7.18].

For the "only if" part, suppose that the class of locally injective and locally absolutely pure quasi-coherent sheaves are equal. As given in Example 2.4.2, we deal with a cover

$\{D_+(\prod_{i\in v} x_i)\}_{v\subseteq\{1,\ldots,n\}}$ of basic affine open subsets of X since locally absolutely purity is independent of choice of the base by Proposition 6.2.7. Let M be an absolutely pure $R[v]$-module for some $v \subseteq \{1,\ldots,n\}$. By taking its direct image $\iota_*(\widetilde{M})$, we get a locally absolutely pure quasi-coherent sheaf on X. Indeed, $\iota_*(\widetilde{M})(D_+(\prod_{i\in w} x_i)) = S_{vw}^{-1}M(v)$ for $v \subseteq w$ is absolutely pure $R[w]$-module by [Pin05, Theorem 3.21] and $\iota_*(\widetilde{M})(D_+(\prod_{i\in w} x_i)) = \widetilde{M}(D_+(\prod_{i\in w} x_i) \cap D_+(\prod_{i\in v} x_i))$ as $R[w]$-module for $v \nsubseteq w$. But

$$\widetilde{M}(D_+(\prod_{i\in w} x_i) \cap D_+(\prod_{i\in v} x_i)) = S_{v(v\cup w)}^{-1}M(v)$$

is absolutely pure as $R[(v\cup w)]$-module and since $R[(v\cup w)] = S_{v(v\cup w)}^{-1}R[w]$, it is also absolutely pure as $R[w]$-module, by [Pin05, Theorem 3.20]. By assumption $\iota_*(\widetilde{M})$ is locally injective, that is, $(\iota_*(\widetilde{M}))(D_+(\prod_{i\in v} x_i)) = M$ is injective. So, $R[v]$ is Noetherian, by [Meg70, Theorem 3]. This implies that X is locally Noetherian. □

Note that the class of locally absolutely pure quasi-coherent sheaves over a locally coherent scheme is closed under direct limits and coproducts since absolutely pure modules over coherent rings are closed under direct limits, [Pin05, Proposition 2.4].

The next proposition, which will be used several times in the rest of chapter, is valid for any scheme.

Proposition 6.2.12. *[EE05, Proposition 3.3] Let X be any scheme. Let $Q = (V, E)$ and $\mathbf{R}$ be its quiver and ring representation mentioned in Section 2.4. Let $\mathcal{F}$ be a quasi-coherent sheaf. Let κ be an infinite cardinal such that $|R(v)| \leq \kappa$ and such that $\max|E|, |V| \leq \kappa$. Let $M_v \subseteq \mathcal{F}(v)$ be subsets with $|M_U| \leq \kappa$ for all $v \in V$. Then there is a stalkwise pure quasi-coherent subsheaf $\mathcal{F}' \subseteq \mathcal{F}$ with $M_v \subseteq \mathcal{F}'(v)$ for all $v \in V$ and $|\mathcal{F}| \leq \kappa$.*

Theorem 6.2.13. *Let X be a locally coherent scheme. The class of locally absolutely pure quasi-coherent sheaves is a covering class.*

Proof. First note that over a coherent ring, a quotient of an absolutely pure module by a pure submodule is again absolutely pure [Pin05, Proposition 4.2]. So, using that, we can say that a quotient of a locally absolutely pure quasi-coherent sheaf by a stalkwise pure quasi-coherent subsheaf is again locally absolutely pure.

Let λ be the cardinality (see Section 2.4) of the scheme X. By Proposition 6.2.12, there is an infinite cardinal κ such that for every quasi-coherent sheaf can be written as a sum of quasi-coherent subsheaves of type κ. In fact, every subsheaf with type κ of a quasi-coherent sheaf $\mathcal{F}$ can be embedded in a quasi-coherent subsheaf of type κ which is stalkwise pure in $\mathcal{F}$. Let $\mathcal{S}$ be the set of locally absolutely pure quasi-coherent sheaves of type κ. By combining this with the fact that the class of locally absolutely pure quasi-coherent sheaves is closed under taking quotient by a stalkwise pure quasi-coherent sheaf, it follows that each locally absolutely pure quasi-coherent sheaf admits an $\mathcal{S}$-filtration. So, every locally absolutely pure sheaf is filtered by the ones of type κ.

On the other hand, since absolutely pure modules are closed under extensions and direct limits over a coherent ring, every quasi-coherent sheaf on a locally coherent scheme possessing an $\mathcal{S}$-filtration is also locally absolutely pure quasi-coherent. So, the class of locally absolutely pure quasi-coherent sheaves is equal to the class Filt$(\mathcal{S})$ of all $\mathcal{S}$-filtered quasi-coherent sheaves. So, that class is precovering by Proposition 3.2.17. Being closed under direct limits also implies that the class of locally absolutely pure quasi-coherent sheaves is covering. $\square$

6.3 Locally torsion-free quasi-coherent sheaves

Definition 6.3.1. A torsion theory for an abelian category $\mathcal{C}$ is a pair $(\mathcal{T}, \mathcal{F})$ of classes of objects $\mathcal{C}$ such that

(i) $\mathrm{Hom}(T, F) = 0$ for all $T \in \mathcal{T}, F \in \mathcal{F}$.

(ii) If $\mathrm{Hom}(C, F) = 0$ for all $F \in \mathcal{F}$, then $C \in \mathcal{T}$.

(iii) If $\mathrm{Hom}(T, C) = 0$ for all $T \in \mathcal{T}$, then $C \in \mathcal{F}$.

In that case, $\mathcal{T}$ is called a *torsion class* while $\mathcal{F}$ is called a *torsion-free* class.

In fact, being a torsion class in an abelian category is equivalent to being closed under quotient objects, coproducts and extensions. And its dual form is valid for a torsion-free class.

A torsion theory $(\mathcal{T}, \mathcal{F})$ in an abelian category $\mathcal{C}$ is called *hereditary* if the torsion class is closed under subobjects, or equivalently, in case $\mathcal{C}$ is a Grothendieck category, the torsion-free class is closed under injective envelopes. And it is called of *finite type* when its torsion-free class is closed under direct limits. The prototypical example of a hereditary torsion theory of finite type comes from the category of modules over an integral domain where $\mathcal{T}$ is the class of all torsion modules and $\mathcal{F}$ is the class of all torsion-free modules.

Note that if a scheme X is integral then all its restriction maps $\mathcal{O}_X(V) \to \mathcal{O}_X(U)$ between any open subsets $U \subseteq V$ are monomorphisms, see [GW10, Proposition 3.29]. From now on all schemes are assumed to be integral.

We start this section by proving that locally torsion quasi–coherent sheaves are easily shown to induce a torsion theory in $\mathfrak{Qcoh}(X)$:

Proposition 6.3.2. *Let $\mathcal{T}$ be the class of quasi-coherent sheaves over X whose modules of sections over all affine open sets are torsion. Then $\mathcal{T}$ is a torsion class of a hereditary torsion theory in $\mathfrak{Qcoh}(X)$.*

Proof. Since $\mathcal{T}$ is closed under extensions, quotients, coproducts and subobjects, it is the torsion part of a hereditary torsion theory $(\mathcal{T}, \mathcal{F}_{\mathcal{T}})$ where $\mathcal{F}_{\mathcal{T}}$ consists of the $\mathcal{M} \in \mathfrak{Qcoh}(X)$ having just the zero morphism from each element of $\mathcal{T}$. $\square$

Now let $\mathcal{F}$ be the class in $\mathfrak{Qcoh}(X)$ of *locally torsion-free quasi-coherent sheaves*, that is, $\mathscr{F} \in \mathcal{F}$ whenever $\mathscr{F}(U)$ is torsion-free $\mathcal{O}_X(U)$-module, for each open set U in X. First of all, we claim that being locally torsion-free is a Zariski-local notion in $\mathfrak{Qcoh}(X)$.

Lemma 6.3.3. *Let R be an integral domain and M be an R-module. Then the following are equivalent:*

(i) *M is torsion-free.*

(ii) *M_P is torsion-free as R_P-module for each prime ideal P.*

(iii) *M_m is torsion-free as R_m-module for each maximal ideal m.*

Proof. The implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ are easy. For $(iii) \Rightarrow (i)$, assume that there is a nonzero torsion element $x \in M$. Then the ideal $\mathrm{Ann}_R(x) = \{r \in R \mid \quad rx = 0\}$ is neither zero nor R since x is not zero. Consider a maximal ideal m containing $\mathrm{Ann}_R(x)$. Then $\frac{x}{1}$ is not zero in M_m. But $\frac{r}{1}.\frac{x}{1} = 0$ for any nonzero $r \in \mathrm{Ann}_R(x)$. By the assumption, $\frac{r}{1} = 0$ in R_m, that is, $tr = 0$ for some $t \in R \backslash m$. But since $r \neq 0$ and R is an integral domain, $t = 0$ which yields a contradiction. □

Proposition 6.3.4. *Let $\mathscr{F}$ be a quasi-coherent sheaf over X. Then the following are equivalent:*

(i) *$\mathscr{F} \in \mathcal{F}$.*

(ii) *There is a cover $\mathcal{U}$ of X containing affine open subsets such that $\mathscr{F}(U)$ is torsion-free for all $U \in \mathcal{U}$.*

(iii) *$\mathscr{F}_x$ is torsion-free for each $x \in X$.*

Proof. $(i \Rightarrow ii)$ is clear.

$(ii \Rightarrow iii)$. Let $x \in X$. Since $\mathcal{U}$ is a cover of X with affine open sets, there is an affine open set $U \in \mathcal{U}$ containing x. Since $\mathcal{F}$ is quasi-coherent, $\mathcal{F}_x = M_P$, where $\mathcal{F}(U) = M$ is torsion-free and P is the prime ideal of $\mathcal{O}_X(U)$ corresponding to $x \in U$. This proves (iii).

$(iii \Rightarrow i)$. Let U' be any affine open subset of X and P be any prime ideal of $\mathcal{O}_X(U')$. By assumption, $\mathcal{F}_x = (\mathcal{F} \mid_{U'})_x = \mathcal{F}(U')_P$ is torsion-free, where $x \in U'$ corresponding to the prime ideal P. Hence Lemma 6.3.3 implies that $\mathcal{F}(U')$ is torsion-free. Note that the set of affine open subsets of X constitutes a base for the scheme X. Let V be any open subset of X and f be a nonzero section of $\mathcal{F}(V)$ such that $r.f = 0$ for some $r \in \mathcal{O}_X(V)$. Since f is nonzero, by sheaf axioms, there is a covering $\{U_i\}_I$ of affine open sets such that $\mathrm{res}_{VU_i}(f) \neq 0$ for some $i \in I$, where $\mathrm{res}_{VU_i} : \mathcal{F}(V) \to \mathcal{F}(U_i)$ is the restriction map of $\mathcal{F}$ for $U_i \subseteq V$. But $\mathrm{res}_{VU_i}(r.f) = r|_{U_i}.\mathrm{res}_{VU_i}(f) = 0$. Here $|_{U_i}$ is the restriction map $\mathcal{O}_X(V) \to \mathcal{O}_X(U_i)$. Since $\mathcal{F}(U_i)$ is torsion-free, $r|_{U_i} = 0$. But X is an integral scheme, so all its restriction maps between any open subsets are monic. This implies that $r = 0$. Hence we conclude that $\mathcal{F}(V)$ is torsion-free for any open subset $V \subseteq X$. □

We focus now on studying the quasi–coherent sheaves in the class $\mathcal{F}$. It is not difficult to see that $\mathcal{F}$ is closed under subobjects and extensions. But proving that $\mathcal{F}$ is closed under products requires more work and will be the main goal of this section. We will show in Corollary 6.3.8 that the class $\mathcal{F}$ induces a torsion theory $(\mathcal{T}_{\mathcal{F}}, \mathcal{F})$ on $\mathfrak{Qcoh}(X)$. We start with the following:

Lemma 6.3.5. *If $\mathcal{F}$ is a quasi-coherent sheaf over X which is in the class $\mathcal{F}$, then its restriction maps between affine open subsets are monomorphisms.*

Proof. Let $V \subseteq U$ be affine open subsets and $\mathrm{res}_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ be its restriction map between these affine open subsets. Suppose $0 \neq x \in \mathcal{F}(U)$ such that $\mathrm{res}_{UV}(x) = 0$. Remember that $\mathcal{O}_X(V) \otimes_{\mathcal{O}_X(U)} \mathcal{F}(U)$ and $\mathcal{F}(V)$ are isomorphic by the map $\mathrm{id}_{\mathcal{O}_X(V)} \otimes_{\mathcal{O}_X(U)} \mathrm{res}_{UV}$. Then $(\mathrm{id}_{\mathcal{O}_X(V)} \otimes_{\mathcal{O}_X(U)} \mathrm{res}_{UV})(1 \otimes x) = 0$. This implies that $1 \otimes x = 0$ in $\mathcal{O}_X(V) \otimes_{\mathcal{O}_X(U)} \mathcal{F}(U)$. Since $\mathcal{O}_X(V)$ is flat as $\mathcal{O}_X(U)$-module, we infer from [Ste75, Proposition 8.8, I] that there exists a matrix $A_{n\times 1}$ with coefficients from $\mathcal{O}_X(U)$ and a vector $S_{1\times n}$ with coefficients from $\mathcal{O}_X(V)$ such that $A.x = 0$ and $1 = S.A$. But since x is torsion-free and nonzero, $A = 0$. This contradicts to $1 = S.A$. $\square$

By Lemma 6.3.5 and the sheaf axiom, we can deduce that for every non-empty affine open subset $U \subseteq X$ and for every affine open covering $U = \bigcup_i U_i$, we have $\mathcal{F}(U) = \bigcap_i \mathcal{F}(U_i)$.

We will analyze the interaction between the product in $\mathfrak{Qcoh}(X)$ and the product in $\mathfrak{Sh}(X)$ for objects which lie in $\mathcal{F}$.

Proposition 6.3.6. *Let $\{\mathcal{F}_i\}_I$ be a family of locally torsion-free quasi-coherent sheaves over X. Its product $\mathcal{F}$ in $\mathfrak{Qcoh}(X)$ is a subsheaf of $\prod_I \mathcal{F}_i$.*

Proof. By the universal property of the product object, there is a unique morphism $\alpha : \mathcal{F} \to \prod \mathcal{F}_i$. Then we need to show that Kerα is the zero sheaf. Note that $(\text{Ker}\alpha)(U) = \text{Ker}\alpha_U$ for every open subset U. Firstly, we will prove that Ker$\alpha \in \mathfrak{Qcoh}(X)$. For each affine open subsets $V \subseteq U$, let us consider the following diagram

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\text{Ker}\alpha_U & \longrightarrow & \text{Ker}\alpha_V \\
\downarrow & & \downarrow \\
\mathcal{F}(U) & \xrightarrow{\text{res}_{UV}} & \mathcal{F}(V) \\
{\scriptstyle \alpha_U}\downarrow & & \downarrow{\scriptstyle \alpha_V} \\
\prod \mathcal{F}_i(U) & \underset{(\text{res}^i_{UV})_I}{\hookrightarrow} & \prod \mathcal{F}_i(V)
\end{array}$$

where res^i_{UV} is the restriction map of $\mathcal{F}_i$ between affine open subsets $V \subseteq U$ for all i. The last morphism of the diagram, $(\text{res}^i_{UV})_I : \prod \mathcal{F}_i(U) \to \prod \mathcal{F}_i(V)$, is monic because by Lemma 6.3.5 each morphism res^i_{UV} is monic for all $i \in I$ and any direct product of monomorphisms is a monomorphism. Columns are exact, as well. If we tensorize it by $\mathcal{O}_X(V)$, we get the following diagram

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ \mathcal{O}_X(V)\otimes_{\mathcal{O}_X(U)} \mathrm{Ker}\alpha_U & \xrightarrow{f} & \mathrm{Ker}\alpha_V \\ \downarrow & & \downarrow \\ \mathcal{O}_X(V)\otimes_{\mathcal{O}_X(U)} \mathcal{F}(U) & \xrightarrow{g} & \mathcal{F}(V) \\ \downarrow{\scriptstyle \mathrm{id}\otimes\alpha_U} & & \downarrow{\scriptstyle \alpha_V} \\ \mathcal{O}_X(V)\otimes_{\mathcal{O}_X(U)} \prod \mathcal{F}_i(U) & \xhookrightarrow[\mathrm{id}\otimes(\mathrm{res}^i_{UV})_I]{} & \prod \mathcal{F}_i(V). \end{array}$$

Since $\mathcal{O}_X(V)$ is flat as $\mathcal{O}_X(U)$-module, columns remain exact and $\mathrm{id}\otimes(\mathrm{res}^i_{UV})_I$ is a monomorphism. The morphism $g := \mathrm{id}_{\mathcal{O}_X(V)}\otimes_{\mathcal{O}_X(U)} \mathrm{res}_{UV}$ is an isomorphism by the quasi-coherence of $\mathcal{F}$. Then f is an isomorphism. This implies that $\mathrm{Ker}\alpha$ is zero. Indeed, the morphism $\mathrm{Ker}\alpha \hookrightarrow \mathcal{F} \to \mathcal{F}_i$ is the zero morphism for each $i \in I$. The universal property of the product $\mathcal{F}$ in $\mathfrak{Qcoh}(X)$ implies that there is a unique morphism $\mathrm{Ker}\alpha \to \mathcal{F}$. So $\mathrm{Ker}\alpha = 0$. $\square$

Theorem 6.3.7. *The product $\mathcal{F}$ of a family $\{\mathcal{F}_i\}_{i\in I}$ of torsion-free quasi-coherent sheaves in $\mathfrak{Qcoh}(X)$ is the largest quasi-coherent subsheaf of $\prod_{i\in I}\mathcal{F}_i$. More concretely, it is of the form*

$$\mathcal{F} = \sum_{\substack{\mathcal{M}\in\mathfrak{Qcoh}(X)\\ \mathcal{M}\subseteq\prod_{i\in I}\mathcal{F}_i}} \mathcal{M}.$$

Proof. By Proposition 6.3.6, we know that $\mathcal{F}$ is a quasi-coherent subsheaf of $\prod_{i\in I}\mathcal{F}_i$. Now, let $\mathcal{F}'$ be a quasi-coherent subsheaf in $\prod_{i\in I}\mathcal{F}_i$. Consider the morphism

$$\mathcal{F}' \hookrightarrow \prod_{i\in I}\mathcal{F}_i \xrightarrow{\pi_i} \mathcal{F}_i$$

for each $i \in I$. By the universal property of products, there is a unique morphism $f : \mathcal{F}' \to \mathcal{F}$ such that $(\pi_i \mid_{\mathcal{F}'}) = (\pi_i \mid_{\mathcal{F}}) \circ f$. But for an open subset U of X, the projection map $\pi_i(U) : \prod_{i\in I}\mathcal{F}_i(U) \to \mathcal{F}_i(U)$ is the canonical one. So we can deduce that the morphism f that we have obtained is an inclusion. This proves that $\mathcal{F}$ is the largest quasi-coherent subsheaf of $\prod_{i\in I}\mathcal{F}_i$. It implies that

$$\mathcal{F} = \sum_{\substack{\mathcal{M}\in\mathfrak{Qcoh}(X)\\ \mathcal{M}\subseteq\prod_{i\in I}\mathcal{F}_i}} \mathcal{M}.$$

□

Corollary 6.3.8. *The class $\mathcal{F}$ in $\mathfrak{Qcoh}(X)$ is closed under arbitrary products. In particular it induces a torsion theory of finite type in $\mathfrak{Qcoh}(X)$.*

Proof. $\mathcal{F}$ is closed under direct limits and under arbitrary products in view of Proposition 6.3.6. Since it is also closed under subobjects and extensions it is the right part of a torsion theory of finite type $(\mathcal{T}_{\mathcal{F}}, \mathcal{F})$ in $\mathfrak{Qcoh}(X)$. □

Now let us consider the torsion theory $(\mathcal{T}, \mathcal{F}_{\mathcal{T}})$ provided by Proposition 6.3.2. Then it is easy to see that $\mathcal{F} \subseteq \mathcal{F}_{\mathcal{T}}$. But the reverse inclusion is not known yet. If we restrict ourselves to the case of a quasi-separated scheme X, we can say something more. In order to show the equality over a quasi-separated scheme X, we will use the other torsion theory induced by Corollary 6.3.8. Recall that the torsion theory obtained from the class $\mathcal{F}$ in $\mathfrak{Qcoh}(X)$ is the pair $(\mathcal{T}_{\mathcal{F}}, \mathcal{F})$. We prove the equality $\mathcal{T}_{\mathcal{F}} = \mathcal{T}$ in the following.

Corollary 6.3.9. *Let X be a quasi-separated scheme. Then the pair $(\mathcal{T}, \mathcal{F})$ constitutes a torsion theory in $\mathfrak{Qcoh}(X)$.*

Proof. The claim is to show that $\mathcal{T} = \mathcal{T}_{\mathcal{F}}$ where $(\mathcal{T}_{\mathcal{F}}, \mathcal{F})$ is the torsion theory obtained from Corollary 6.3.8. As a first step, we will show that the direct image functor on $U \subseteq X$, where U is affine, preserves locally torsion-freeness. Let M be a torsion-free $\mathcal{O}_X(U)$-module for an affine subset U. Then the quasi-coherent sheaf $\widetilde{M}$ associated to M is locally torsion-free quasi-coherent $\mathcal{O}_X|_U$-module by Lemma 6.3.3 and Proposition 6.3.4.

Now, if we consider the direct image functor for the inclusion $\iota : U \hookrightarrow X$, then $\iota_*(\widetilde{M})$ is again a locally torsion-free quasi-coherent $\mathcal{O}_X$-module. Recall that if a scheme is quasi-separated, then a direct image functor ι_* for any inclusion $\iota : U \hookrightarrow X$ with affine open subset U preserves quasi-coherence. For any open subset $V \subseteq X$, $\iota_*(\widetilde{M})(V) = \widetilde{M}(V \cap U)$ as $\mathcal{O}_X(V)$-module where $\mathcal{O}_X(V) \hookrightarrow \mathcal{O}_X(V \cap U)$. But $\widetilde{M}(V \cap U)$ is torsion-free as $\mathcal{O}_X(V \cap U)$-module since $V \cap U \subseteq U$. Then it is also torsion-free as $\mathcal{O}_X(V)$-module. Quasi-coherence of $\iota_*(\widetilde{M})$ follows from the fact that X is quasi-separated.

Now let $\mathcal{T} \in \mathcal{T}_{\mathcal{F}}$. We need to show that the module of sections of $\mathcal{T}$ over each affine open set is torsion. Let U be an affine open set and $\mathcal{T}(U) \to M$ be any morphism of $\mathcal{O}_X(U)$-modules where M is torsion-free. Then we have a morphism $\mathcal{T}|_U \to \widetilde{M}$ in $\mathfrak{Qcoh}(U)$. Applying the direct image functor ι_*, we get a morphism $\iota_*(\mathcal{T}|_U) \to \iota_*(\widetilde{M})$ in $\mathfrak{Qcoh}(X)$, where $\iota_*(\widetilde{M})$ is locally-torsion free. But there is a canonical morphism $\mathcal{T} \to \iota_*(\mathcal{T}|_U)$ which is the identity for all open subsets $U' \subseteq U$. So the composition $\mathcal{T} \to \iota_*(\mathcal{T}|_U) \to \iota_*(\widetilde{M})$ is zero by assumption. It gives us that the morphism $\mathcal{T}(U) \to M$ is zero. So $\mathcal{T}$ is a locally torsion quasi-coherent sheaf. □

Corollary 6.3.10. *The product $\mathcal{F}$ of a family $\{\mathcal{F}_i\}_{i\in I}$ of flat quasi-coherent sheaves in $\mathfrak{Qcoh}(X)$ is the largest quasi-coherent subsheaf of $\prod_{i\in I}\mathcal{F}_i$. More concretely, it is of the form*

$$\mathcal{F} = \sum_{\substack{\mathcal{M}\in\mathfrak{Qcoh}(X)\\ \mathcal{M}\subseteq\prod_{i\in I}\mathcal{F}_i}} \mathcal{M}.$$

Proof. This follows by noticing that every flat quasi–coherent sheaf is in fact locally torsion-free. □

Now we get another proof of [SS11, Proposition 4.16].

Corollary 6.3.11. *Let X be a Dedekind scheme. The class $\mathcal{F}lat(X)$ of flat quasi–coherent sheaves is closed under taking products in $\mathfrak{Qcoh}(X)$.*

Proof. Let $\{\mathcal{F}_i\}_{i\in I}$ be a family of flat quasi–coherent sheaves. By Corollary 6.3.10, the product object $\mathcal{F}$ of the family $\{\mathcal{F}_i\}_{i\in I}$ in $\mathfrak{Qcoh}(X)$ is a subsheaf of $\prod_I \mathcal{F}_i$ whose modules of sections over open subsets are torsion-free. Hence $\mathcal{F} \in \mathcal{F}$. But for a Dedekind scheme the classes $\mathcal{F}$ and $\mathcal{F}lat(X)$ coincide, so we are done. □

6.4 Torsion-free covers in $\mathfrak{Qcoh}(X)$

In the previous Corollary 6.3.8 we showed that the class $\mathcal{F}$ of locally torsion-free quasi–coherent sheaves is the right part of a torsion theory in $\mathfrak{Qcoh}(X)$. One immediate consequence of this is that each $\mathcal{M} \in \mathfrak{Qcoh}(X)$ admits an $\mathcal{F}$*-reflection* and thus $\mathcal{F}$ is a *reflective* class in $\mathfrak{Qcoh}(X)$. So, in particular, we deduce that $\mathcal{F}$ is enveloping.

This section is devoted to proving that the class $\mathcal{F}$ is also covering, that is, each $\mathcal{M} \in \mathfrak{Qcoh}(X)$ admits an $\mathcal{F}$-cover.

Let κ be an infinite regular cardinal such that $\kappa >\mid \mathcal{O}_X(U) \mid$ for each affine open subset $U \subseteq X$ and $\kappa >\mid H \mid$, where $H := \{\text{res}_{UV} \mid \text{ for affine subsets } V \subseteq U \subseteq X\}$. See Section 2.4 for the cardinality of a quasi-coherent sheaf and κ-type quasi-coherent sheaves.

Lemma 6.4.1. *Let $\mathcal{S}$ be the set of isomorphism classes of quasi-coherent sheaves in $\mathcal{F}$ of type κ. Then $\mathcal{F} = Filt(\mathcal{S})$.*

Proof. Let $\mathcal{F} \in \mathcal{F}$ and $x \in \mathcal{F}(U)$ for some affine open subset U. By Proposition 6.2.12, there is a quasi-coherent pure subsheaf $\mathcal{G} \subseteq \mathcal{F}$ of type κ containing x. Here, the purity is considered in the sense of tensor product. It easy to check that it has the global and local properties on affine open subsets, i.e., $\mathcal{G}(U)$ is a pure submodule of $\mathcal{F}(U)$ for each affine open subset U. This implies that $r\mathcal{G}(U) = r\mathcal{F}(U) \cap \mathcal{G}(U)$ for all $r \in \mathcal{O}(U)$, and so $\mathcal{F}(U)/\mathcal{G}(U)$ is torsion-free for all affine open subsets U. Therefore $\mathcal{F}/\mathcal{G} \in \mathcal{F}$.

By transfinite induction, we will construct an $\mathcal{S}$-filtration for each object in $\mathcal{F}$. For $\mathcal{F} \in \mathcal{F}$, consider $\mu = \mid \mathcal{F} \mid$ and $\mathcal{F}_0 = 0$, $\mathcal{F}_1 := \mathcal{G}$ obtained as above. For $\alpha < \mu$, if $x + \mathcal{F}_\alpha(U) \in (\mathcal{F}/\mathcal{F}_\alpha)(U) = \mathcal{F}(U)/\mathcal{F}_\alpha(U)$, there is a pure quasi-coherent subsheaf $\mathcal{F}_{\alpha+1}/\mathcal{F}_\alpha$ of type κ containing $x + \mathcal{F}_\alpha$. For a limit ordinal $\beta \leq \mu$, $\mathcal{F}_\beta := \varinjlim_{\alpha<\beta} \mathcal{F}_\alpha$. Then, $(\mathcal{F}_\alpha \mid \alpha \leq \mu)$ is an S-filtration for $\mathcal{F}$.

From that construction, we get that $\mathcal{F} \subseteq \text{Filt}(\mathcal{S})$. Now, let $(\mathcal{M}_\alpha \mid \alpha \leq \lambda)$ be an $\mathcal{S}$-filtration of a quasi-coherent sheaf $\mathcal{M}$. We have that $\mathcal{M}_1 = \mathcal{M}_1/\mathcal{M}_0 = \mathcal{M}_1/0$ is in $\mathcal{F}$. And now if we suppose that $\mathcal{M}_\alpha \in \mathcal{F}$, for $\alpha < \lambda$, we have a short exact sequence

$$0 \longrightarrow \mathcal{M}_\alpha \longrightarrow \mathcal{M}_{\alpha+1} \longrightarrow \mathcal{M}_{\alpha+1}/\mathcal{M}_\alpha \longrightarrow 0,$$

where $\mathcal{M}_\alpha, \mathcal{M}_{\alpha+1}/\mathcal{M}_\alpha$ are in $\mathcal{F}$. Therefore, $\mathcal{M}_{\alpha+1}$ is also in $\mathcal{F}$. Since $\mathcal{F}$ is closed under direct limits, $\mathcal{M}_\alpha$ is locally torsion-free whenever α is a limit ordinal. This implies that $\mathcal{M}_\lambda = \mathcal{M}$ is locally torsion-free. Hence $\mathcal{F} = \text{Filt}(\mathcal{S})$. □

We will adapt the arguments of [Eno12] to the category $\mathfrak{Qcoh}(X)$ to infer in Theorem 6.4.7 that $\mathcal{F}$ is covering. Since the set of affine open subsets of X is a base of the scheme and uniquely determines quasi–coherent sheaves over it, we will often use the images of a quasi–coherent sheaf on affine open subsets.

Given a filtration $(\mathcal{F}_\alpha)_{\alpha\leq\kappa}$ and $(\mathcal{F}'_\alpha)_{\alpha\leq\kappa}$ of $\mathcal{F} \subseteq \mathcal{F}'$, respectively, we say that the filtration $(\mathcal{F}'_\alpha)_{\alpha\leq\kappa}$ of $\mathcal{F}'$ is compatible with the one $(\mathcal{F}_\alpha)_{\alpha\leq\kappa}$ of $\mathcal{F}$ if $(\mathcal{F}_\alpha)_{\alpha\leq\kappa}$ is obtained by restricting $(\mathcal{F}'_\alpha)_{\alpha\leq\kappa}$ to $\mathcal{F}$, that is, $\mathcal{F}'_\alpha \cap \mathcal{F} = \mathcal{F}_\alpha$ for all $\alpha \leq \kappa$.

Lemma 6.4.2. *Assume that $(\mathcal{F}_\alpha)_{\alpha\leq\kappa}$ is a filtration of $\mathcal{F}$ in $\mathfrak{Qcoh}(X)$. If $\mid \mathcal{F}'/\mathcal{F} \mid < \kappa$ where $\mathcal{F} \subseteq \mathcal{F}' \in \mathfrak{Qcoh}(X)$, then there is a filtration $(\mathcal{F}'_\alpha)_{\alpha\leq\kappa}$ which is compatible with the one of $\mathcal{F}$ and except for possibly one $\beta < \kappa$, $\mathcal{F}_{\alpha+1}/\mathcal{F}_\alpha$ is isomorphic to $\mathcal{F}'_{\alpha+1}/\mathcal{F}'_\alpha$ and $\mathcal{F}_{\beta+1}/\mathcal{F}_\beta$ is a direct summand of $\mathcal{F}_{\beta+1}/\mathcal{F}_\beta$ with the complement $\mathcal{F}'/\mathcal{F}$.*

Proof. For each affine open $U \subseteq X$, S_U denotes the $\mathcal{O}_X(U)$ submodule of $\mathcal{F}'(U)$ which is generated by representatives of $\mathcal{F}'(U)/\mathcal{F}(U)$. By assumption, $| S_U |< \kappa$. We can complete these subsets to a quasi-coherent subsheaf of $\mathcal{F}'$, say $\mathcal{S}$, containing these submodules $S_U \subseteq \mathcal{S}(U)$ and with the cardinality $< \kappa$.

We know that there exists $\beta_U < \kappa$ for each affine $U \subseteq X$ such that $\mathcal{S}(U) \cap \mathcal{F}_\beta = \mathcal{S}(U) \cap \mathcal{F}$ since κ is a regular cardinal and is the length of the filtration. Consider $\beta := \cup_U \beta_U$. Now define the new filtration as $\mathcal{F}'_\alpha = \mathcal{F}_\alpha$ for $\alpha \leq \beta$, and $\mathcal{F}'_\alpha = \mathcal{F}_\alpha + \mathcal{S}$ for $\alpha > \beta$. Since these are quasi-coherent, $(\mathcal{F}'_{\alpha+1}/\mathcal{F}'_\alpha)(U) = \mathcal{F}'_{\alpha+1}(U)/\mathcal{F}'_\alpha(U)$ and $(\mathcal{F}_\alpha + \mathcal{S})(U) = \mathcal{F}(U) + \mathcal{S}(U)$ for affine open subsets and by using the fact $\mathcal{S}(U) \cap \mathcal{F}_\alpha(U) = \mathcal{S}(U) \cap \mathcal{F}(U)$ for each $\alpha \geq \beta$, the claims mentioned in the lemma follow. □

The next corollary says that in $\mathfrak{Qcoh}(X)$ it is possible to convert a filtration of any length and whose quotient between consecutive factors is bounded by κ into a filtration with κ-length. Recall that for a given class $\mathcal{C}$, $\mathbf{Sum}(\mathcal{C})$ is the class of direct sums of objects which are isomorphic to some in $\mathcal{C}$.

Corollary 6.4.3. *Let $\mathcal{C}$ be a class of quasi-coherent sheaves with cardinality $< \kappa$. If a quasi-coherent sheaf $\mathcal{F}$ has a $\mathcal{C}$-filtration, then it has a $\mathbf{Sum}(\mathcal{C})$-filtration of length κ.*

Proof. It easily follows by making transfinite induction on the length of the given filtration and by using Lemma 6.4.2. □

Let $\mathcal{F}' \in \mathbf{Sum}(\mathcal{C})$ with a given direct sum decomposition $\mathcal{F}' = \oplus_{i \in I} \mathcal{N}_i$ such that each $\mathcal{N}_i$ is isomorphic to some object in the class $\mathcal{C}$. As defined in [Eno12] for modules, we call a quasi–coherent subsheaf $\mathcal{F} \subseteq \mathcal{F}'$ to be a nice subsheaf relative to this direct sum decomposition if $\mathcal{F} = \oplus_{j \in J} \mathcal{N}_j$ for some subset $J \subseteq I$. And $\mathcal{F}$ is a nice subsheaf of $\mathcal{F}' \in$ Filt($\mathbf{Sum}(\mathcal{C})$) if, when we give $\mathcal{F}$ the induced filtration $(\mathcal{F}_\alpha)_{\alpha \leq \sigma}$, the image of the canonical map $\mathcal{F}_{\alpha+1}/\mathcal{F}_\alpha \to \mathcal{F}'_{\alpha+1}/\mathcal{F}'_\alpha$

is a nice subsheaf of $\mathcal{F}'_{\alpha+1}/\mathcal{F}'_{\alpha}$ relative to the given direct sum decomposition of $\mathcal{F}'_{\alpha+1}/\mathcal{F}'_{\alpha}$ for each $\alpha < \sigma$.

Lemma 6.4.4. *Let $\mathcal{F}$ be a quasi–coherent sheaf and $\mathcal{M} \in \mathbf{Sum}(\mathcal{F})$. Assume that we have a morphism $f : \mathcal{M} \to \mathcal{N}$ in $\mathfrak{Qcoh}(X)$. Then there exists a nice quasi-coherent subsheaf $\mathcal{T}$ contained in Ker(f) relative to a direct sum decomposition of $\mathcal{M} \in \mathbf{Sum}(\mathcal{F})$ such that $\mid \mathcal{M}/\mathcal{T} \mid \leq \mid \mathcal{F} \mid^{\mid Hom(\mathcal{F},\mathcal{N})\mid}$.*

Proof. Note that $\mathcal{M} = \oplus_{i\in I}\mathcal{F}_i$, where $\mathcal{F}_i = \mathcal{F}$ for each $i \in I$. Then the morphism f is of the form $(f_i)_{i\in I}$, where $f_i : \mathcal{F}_i \to \mathcal{N}$ for each $i \in I$. Now, we define an equivalence relation on I for the fixed morphism $f = (f_i)_{i\in I}$ as follows: $i \sim j$ if and only if $f_i = f_j$. If J is a subset of I which is representatives of equivalence classes, we define the subpresheaf $\mathcal{F}'_i$ of $\mathcal{M}$ for each $i \in I\backslash J$ with $j \in J$ and $i \sim j$ such that $\mathcal{F}'_i(U)$ includes sections, on each open subset $U \subseteq X$, having $x \in \mathcal{F}(U)$ in the i'th component, $-x$ in the j'th one and 0 in the others. This subpresheaf is isomorphic to $\mathcal{F}_i(U)$. So it is a quasi-coherent sheaf which is isomorphic to $\mathcal{F}_i$. And also $f_U(\mathcal{F}'_i) = 0$ for each $i \in I\backslash J$ and for each affine open subset $U \subseteq X$. Then $\mathcal{T} := \oplus_{i\in I\backslash J}\mathcal{F}'_i \subseteq \text{Ker} f$. The exact sequence

$$0 \longrightarrow (\oplus_{i\in I\backslash J}\mathcal{F}'_i)(U) \hookrightarrow \mathcal{M}(U) = (\oplus_{i\in I}\mathcal{F}_i)(U) \xrightarrow{h_U} (\oplus_{i\in J}\mathcal{F}_i) \longrightarrow 0\,,$$

where the map $h_U((x_i)_{i\in I}) = (y_j)_{j\in J}$, $y_j = \sum_{\substack{i\in I\\ i\sim j}} x_i$, is splitting since the map t which is defined on each affine subset $U \subseteq X$ as $t_U : \mathcal{M}(U) \to (\oplus_{i\in I\backslash J}\mathcal{F}'_i)(U)$, $t_U(x_i)_{i\in I} = (y_i)_{i\in I}$ such that $y_i = x_i$ for each $i \in I$ and $y_j = \sum_{\substack{i\in I\backslash J\\ i\sim j}} x_i$ for each $j \in J$, is compatible with restriction maps and gives us the identity map when composed with the inclusion map from $(\oplus_{i\in I\backslash J}\mathcal{F}'_i)(U)$. Therefore, $\mathcal{M} \cong (\oplus_{i\in J}\mathcal{F}_i) \oplus (\oplus_{i\in I\backslash J}\mathcal{F}'_i)$. So, the quasi–coherent subsheaf $\mathcal{T} = \oplus_{i\in I\backslash J}\mathcal{F}'_i$ is nice in $\mathcal{M}$ relative to some direct sum decomposition in $\mathbf{Sum}(\mathcal{F})$. Finally, $\mid J \mid \leq \mid \text{Hom}(\mathcal{F},\mathcal{N}) \mid$ implies that $\mid \oplus_{i\in J}\mathcal{F}_i \mid = \mid \mathcal{M}/\mathcal{T} \mid \leq \mid \mathcal{F} \mid^{\mid \text{Hom}(\mathcal{F},\mathcal{N})\mid}$. □

Lemma 6.4.5. *If $\mathcal{M} \in \mathbf{Sum}(\mathcal{F})$ and $\varepsilon := 0 \longrightarrow \mathcal{N} \longrightarrow \mathcal{T} \xrightarrow{f} \mathcal{M} \longrightarrow 0$ is an exact sequence in $\mathfrak{Qcoh}(X)$, then it is isomorphic to an exact sequence $0 \to \mathcal{N} \to \mathcal{T}' \oplus \mathcal{V} \to$*

$\mathcal{M}' \oplus \mathcal{V} \to 0$ *where* $\mathcal{V}$ *is a nice quasi–coherent subsheaf of* $\mathcal{M}$ *and* $| \mathcal{M}/\mathcal{V} | \leq | \mathcal{F} |^{|Ext(\mathcal{F},\mathcal{N})|}$

Proof. Suppose $\mathcal{M} = \oplus_{i \in I} \mathcal{F}_i$ where $\mathcal{F}_i = \mathcal{F}$ for all $i \in I$. Then for each $i \in I$, we can consider the quasi–coherent sheaf associated to the one defined on affine open subsets $U \subseteq X$ as $f^{-1}(\mathcal{F}_i)(U) = f_U^{-1}(\mathcal{F}_i(U))$ with the map $f_U : f_U^{-1}(\mathcal{F}_i(U)) \to \mathcal{F}_i$. Note that this assignment gives rise to a well-defined quasi–coherent sheaf, as it satisfies both the cocycle condition and the quasi–coherent condition. Indeed, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{N} & \longrightarrow & f^{-1}(\mathcal{F}_i) & \longrightarrow & \mathcal{F}_i & \longrightarrow & 0\,. \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{N} & \longrightarrow & \mathcal{T} & \longrightarrow & \oplus_{i \in I} \mathcal{F}_i & \longrightarrow & 0 \end{array}$$

Now, as we did before, we define an equivalence relation on I as $i \sim j$ if and only if there exists a commutative diagram morphism

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{N} & \longrightarrow & f^{-1}(\mathcal{F}_i) & \longrightarrow & \mathcal{F}_i = \mathcal{F} & \longrightarrow & 0\,. \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle h_{ij}} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & \mathcal{N} & \longrightarrow & f^{-1}(\mathcal{F}_j) & \longrightarrow & \mathcal{F}_j = \mathcal{F} & \longrightarrow & 0 \end{array}$$

Consider the set J of representatives of equivalence classes. Then, for each $i \in I \backslash J$, we define a quasi–coherent subsheaf $\mathcal{V}_i$ of $\mathcal{M}$ as follows: for each affine open subsets $U \subseteq X$, $\mathcal{V}_i(U)$ consists of elements from $\mathcal{M}(U) = \oplus_I \mathcal{F}_i(U)$ having c_U in i'th and $-c_U$ in j'th component, where $c_U \in \mathcal{F}(U)$ and 0 for others, and where $j \in J$ with $j \sim i$. It is easy to see that $\mathcal{V}_i$ is isomorphic to $\mathcal{F}_i$. In fact, for a fixed $i_0 \in I \backslash J$, we define a map $t_U : (\oplus_I \mathcal{F}_i)(U) = \oplus_I \mathcal{F}_i(U) \to \mathcal{V}_{i_0}(U)$ such that $t_U((c_U^i)_{i \in I}) = (c_U'^i)_{i \in I}$ where $c_U'^{i_0} = c_U^{i_0}$, $c_U'^j = -c_U'^{i_0}$, where $j \in J$ and $j \sim i_0$, and $c_U'^i = 0$. Then, $t_U \circ i_U = \mathrm{id}$ for each affine set U. This map is compatible with the restriction maps on every affine open subsets $V \subseteq U \subseteq X$. So it can be extended to $\oplus_I \mathcal{F}_i$. Finally, $\mathcal{V}_{i_0}$ is a direct summand of $\oplus_I \mathcal{C}_i$, for each $i_0 \in I$.

Now, we define a quasi–coherent sheaf $f^{-1}(\mathcal{V}_i)$ for each $i \in I \backslash J$ with $j \sim i$ and $j \in J$ such that

$$(f^{-1}(\mathcal{V}_i))(U) :=$$

$$f_U^{-1}(\mathcal{V}_i) = \{a_u + b_u \mid \ a_u \in f_U^{-1}(\mathcal{F}_i(U)), b_u \in f_U^{-1}(\mathcal{F}_j(U)) \text{ and } -\pi_i f_U(a_u) = \pi_j f_U(b_u)\}.$$

For each affine U and $i \in I \backslash J$, we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{N}(U) & \xrightarrow{\iota_u} & (f^{-1}(\mathcal{V}_i))(U) & \xrightarrow{f} & \mathcal{V}_i(U) & \longrightarrow & 0\,. \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{N}(U) & \longrightarrow & \mathcal{T}(U) & \longrightarrow & \oplus_{i\in I}\mathcal{F}_i(U) & \longrightarrow & 0 \end{array}$$

Now, for each affine set U, we can define the morphism given by $\sigma_U : (f^{-1}(\mathcal{V}_i))(U) \to \mathcal{N}(U)$, $\sigma_U(a_U + b_U) := (h_{ij})_U(a_U) + b_U$. Then σ_U is compatible with restriction maps, and we have $\sigma_U \circ \iota_U = \mathrm{id}_{\mathcal{N}(U)}$, that is, the first row splits for each affine set U. Hence $\mathcal{V}_i$ has an isomorphic image which is a direct summand in $f^{-1}(V_i)$. Since $\mathcal{V}_i$ is also a direct summand in $\mathcal{M}$, we can deduce that $\mathcal{V}_i$ has an isomorphic image in $\mathcal{T}$, which is a direct summand. Combining all of them and considering the quasi–coherent subsheaf $\mathcal{V} := \oplus_{i\in I\backslash J}\mathcal{V}_i$ of $\mathcal{M}$, we identify it with its isomorphic image in $\mathcal{T}$. So, the original exact sequence ε is reduced to the desired one. And also $\mathcal{M}/\mathcal{V} \simeq \oplus_{j\in J}\mathcal{F}_j$, so the claim on the cardinality follows.

□

Now if we focus on the class $\mathcal{F}$ of locally torsion-free quasi-coherent sheaves, we can combine all previous results to infer the following:

Lemma 6.4.6. *Let λ be a cardinal. There is a cardinal μ such that for each morphism $f : \mathcal{M} \to \mathcal{N}$ where $\mathcal{M} \in$ Filt($\mathcal{S}$) and $\mid \mathcal{N} \mid \leq \lambda$ there is a quasi–coherent subsheaf $\mathcal{T}$ of $\mathcal{M}$ contained in Ker$f$ such that $\mathcal{F}/\mathcal{T} \in$ Filt($\mathcal{S}$) and $\mid \mathcal{F}/\mathcal{T} \mid \leq \mu$.*

Proof. Using Corollary 6.4.3, Lemma 6.4.4 and 6.4.5, we can apply a transfinite induction on κ mentioned in Lemma 6.4.1 to find a cardinal as done in [Eno12, Theorem 5.1]. □

Theorem 6.4.7. *Each quasi-coherent sheaf over X has an $\mathcal{F}$-cover.*

Proof. Let $\mathcal{N}$ be a quasi-coherent sheaf and λ be the cardinality of $\mathcal{N}$. By Lemma 6.4.1 and Lemma 6.4.6, there is a cardinal μ such that, for each morphism $f : \mathcal{F} \to \mathcal{N}$ where $\mathcal{F} \in \mathcal{F}$, there is a quasi-coherent submodule $\mathcal{T}$ of $\mathcal{F}$ contained in Ker$f$ with $\mathcal{F}/\mathcal{T} \in \mathcal{F}$ of type μ. Then

$$\mathcal{M} := \bigoplus_{\substack{\mathcal{T} \in S' \\ h \in \mathrm{Hom}(\mathcal{T},\mathcal{N})}} \mathcal{T}_h,$$

with the canonical morphism $\sigma : \mathcal{M} \to \mathcal{N}$, is a precover of $\mathcal{N}$ (where $\mathcal{S}'$ is the isomorphism class of locally torsion-free quasi-coherent sheaves of type μ). Finally, since $\mathcal{F}$ is closed under direct limits, by [Xu96, Theorem 2.2.12] (whose proof is valid for any Grothendieck category, so in particular for $\mathfrak{Qcoh}(X)$) $\mathcal{M}$ has an $\mathcal{F}$-cover. $\square$

Remark. Theorem 6.4.7 may be also derived from Lemma 6.4.1, Proposition 3.2.17 and Proposition 3.1.2. It could be also deduced from Proposition 3.2.20 by using the same argument as the proof of Lemma 6.4.1 and the fact that the class $\mathcal{F}$ is closed under directed colimits.

CHAPTER SEVEN

PURE DERIVED CATEGORY OF MONOIDAL CATEGORIES

The goal of this chapter is to define the pure derived category of the category of quasi-coherent sheaves $\mathfrak{Qcoh}(X)$. Regarding to this, in the first section, we recall some basic terminology and notions on exact categories. This section enjoys the survey article of [Büh10] on exact categories. An exact category, shortly, is a category where we are allowed to do Homological Algebra by means of Yoneda extensions. It includes some short exact sequences which makes the category resembles an abelian category. And the theory on abelian categories can be built for exact categories, as well. Besides, we recall the notion of Grothendieck type, introduced by [SS11], which allows to show the existence of enough injectives relative to an exact structure, just as in Grothendieck categories. In order to do so, we put all the necessary conditions on the exact structure, such as, possessing a generator, closure under well ordered direct limits and all objects are small.

Concerning to locally λ-presentable categories, introduced in Chapter (II), there is a proper class $\mathcal{P}$ projectively generated by λ-presentable objects. So the category has enough $\mathcal{P}$-projectives. In case $\lambda := \aleph_0$, by the result of [Her03], it has also enough $\mathcal{P}$-injectives. In [Gil14], the pure derived category $\mathbf{D}_{pure}(\mathcal{C})$ of a locally finitely presentable category $\mathcal{C}$ comes from an injective and projective model structure on $\mathbf{C}(\mathcal{C})$ with the canonical proper class defined degreewise. Since the most natural purity on $\mathfrak{Qcoh}(X)$ is that of stalkwise defined, see Section 5.4, the aim of this chapter is to show that the stalkwise pure derived category of $\mathfrak{Qcoh}(X)$ is also the homotopy category of an injective model structure on $\mathbf{C}(\mathcal{C})$. In order to proceed by using the argument of [CH02], we take advantage of two concepts: monoidal categories and exact structures of Grothendieck type. By using the result on λ-purity, we prove in Lemma 7.2.2 that $\mathfrak{Qcoh}(X)$ with the geometrical ($\otimes$) pure exact sequences is of Grothendieck type. That enables to use Lemma 7.1.15, [Sto13, Lemma 7.10], to define $\mathbf{D}_{\otimes\text{-pur}}(\mathfrak{Qcoh}(X))$. And finally, in case X is quasi-separated, $\otimes$-purity and stalkwise purity coincide, so we are done.

One may wonder why we don't prove it directly for the stalkwise purity. In Section ($V.4$), we have proved the existence of stalkwise pure-injective envelopes. But we still don't know whether this exact structure is efficient in the sense of Definition 7.1.10 because of condition (Ef4). The explicit question is whether $\mathfrak{Qcoh}(X)$ with the stalkwise pure exact structure has a generator.

In the third section, we tie up the two pure-derived categories of a closed symmetric monoidal Grothendieck category $\mathcal{C}$. The λ-pure derived category, $\mathbf{D}_{\lambda\text{-pur}}(\mathcal{C})$, comes from a projective model structure while $\otimes$-pure derived category comes from an injective model structure. So it seems that they are jointed to each other. We prove in Proposition 7.3.5 that there is a Quillen adjunction between them.

In the last section, we restrict ourselves to a locally finitely presentable category. We recall the result of [Craw94], exhibiting the equivalence between a locally finitely presentable category with a categorical pure-exact structure and a category of Set-valued and direct limit preserving functors from a small category, which are called *flat functors*. In fact, they correspond to flat modules over a ring with enough idempotents. All together, we conclude in Theorem 7.4.3 that to consider the categorical pure derived category of such a category is the same as the derived category of flat modules over a ring with enough idempotents. Note that Murfet and Salarian in [MS11] define the stalkwise pure-derived category of flat sheaves over a semi-separated Noetherian scheme as the analogous to the homotopy category of projectives. As a matter of fact, it is precisely the derived category of flat sheaves. Theorem 7.4.4 shows that, for *any* scheme X, we can realize the derived category of flat sheaves as the homotopy category of a model structure on $\mathbf{C}(\mathcal{F}lat(X))$ which is injective with respect to the exact structure.

7.1 Exact Categories

Let $\mathcal{C}$ be an additive category.

Definition 7.1.1. An *exact category* is an additive category $\mathcal{C}$ equipped with a family $\mathcal{E}$ of kernel-cokernel sequences

$$C' \overset{i}{\hookrightarrow} C \overset{p}{\twoheadrightarrow} C'' ,$$

called *short exact sequences of* $\mathcal{C}$ (or *conflations*), with i and p *admissible monic* (or *inflation*) and *epic* (or *deflation*), respectively, subject to:

E0) $\mathcal{E}$ is closed under isomorphisms.

E1) The identity morphism id_C, $C \in \mathcal{C}$, is both an admissible monic and epic.

E2) Admissible monics are closed under compositions.

E3) Admissible epics are closed under compositions.

E4) The pushout of an admissible monic along any morphism exists and yields an admissible monic.

E5) The pullback of an admissible epic along any morphism exists and yields an admissible epic.

The following consequences are immediate from the definition.

Proposition 7.1.2. *[Büh10, Lemma 2.7,Prop 2.9] Let $(\mathcal{C}, \mathcal{E})$ be an additive exact category. Then*

(i) A morphism is an isomorphism if and only if it is both an admissible monic and an admissible epic.

(ii) Every split exact sequence belongs to any exact structure.

(iii) The direct sum of two short exact sequences is short exact.

One of the differences between proper classes of exact sequences and exact structures is that in case $j \circ i$ is an admissible monic, the morphism i is not always implied to be an admissible monomorphism, and dually for admissible epimorphisms. The following proposition, which was one of the axioms of Quillen's original definition, see [Qui74, pg 99], is a consequence of the exact category axioms proved by [Yon60, Corollary p. 525]. Then we may conclude that an exact structure on category is a proper class of exact sequences if $\mathcal{C}$ is abelian.

Proposition 7.1.3. *(Obscure Axiom)[Büh10, Proposition 2.16] Let $i : A \to B$ be a morphism in $\mathcal{C}$ with a cokernel. If there exists a morphism $j : B \to C$ in $\mathcal{C}$ such that the composite $j \circ i$ is an admissible monic, then i is an admissible monic. Dually, for admissible epics.*

The advantage of an exact structure is to enable us to consider relative Yoneda-Ext classes, and therefore to do Relative Homological Algebra. They are like an abelian structure inside a category. Any additive category admits an abelian structure even though it is not abelian, by taking just the split short exact sequences. Most of diagramatic facts in an abelian category continue to be available in an exact category, as well, such as Five Lemma, Noether Isomorphism, 3×3 Lemma... see [Büh10, Section 3.] for a detailed treatment.

If one has an abelian category $\mathcal{C}$ with a full subcategory $\mathcal{C}'$ closed under extensions, then $\mathcal{C}'$ has a canonical exact category structure that contains all short exact sequences with components from $\mathcal{C}'$. The converse is also true, that is, every exact category is a full subcategory of an abelian category.

The relative $\mathcal{E}$-Yoneda extension $\mathrm{Ext}_{\mathcal{E}}(X, Y)$ consists of all equivalence class of short exact sequences

$$Y \overset{i}{\hookrightarrow} C \overset{p}{\twoheadrightarrow} X$$

which belong to $\mathcal{E}$. So all terminologies and concepts related in an abelian category can be transformed into the relevant ones in an exact category by replacing admissible monic or epic instead of usual monomorphisms and epimorphisms, such as cotorsion pairs, projectives and injectives, injective and projective resolutions.

In order to get the exact category version of Hovey pairs, that is, model category structures compatible with the exact structure, the class of admissible monics and admissible epics must be closed under retractions, because in this case cofibrations are admissible monics with certain cokernels and fibrations are admissible epics with certain kernels. Gillespie in [Gil11] proved that this requirement on exact categories is equivalent to being the underlying category weakly idempotent complete.

Lemma 7.1.4. *[Büh10, Lemma 7.1-Corollary 7.5] For an exact category $\mathcal{C}$, the following are equivalent:*

(i) Every section has a cokernel.

(ii) Every retraction has a kernel.

(iii) Every section is an admissible monic.

(iv) Every retraction is an admissible epic.

Definition 7.1.5. An exact category category is called *weakly idempotent complete* if it satisfies one of the equivalent conditions in Lemma 7.1.4.

To be weakly idempotent complete doesn't come freely for any exact category. For example, by Eilenberg Swindle, for any projective module P there is a set I such that $P \oplus R^{(I)} \cong R^{(I)}$. So the projection of $R^{(I)}$ into $R^{(I)}$ is a retraction but it doesn't admit kernel in the category of free modules unless P is free.

Weakly idempotent complete exact categories are the natural set up where the hypothesis in Obscure Axiom is always satisfied, see [Büh10, Proposition 7.6].

The following proposition is due to [Gil11, Proposition 2.4]

Proposition 7.1.6. *Let $(\mathcal{C}, \mathcal{E})$ be an exact category. Then the following are equivalent:*

(i) $(\mathcal{C}, \mathcal{E})$ *is weakly idempotent complete.*

(ii) The class of admissible monics is closed under retractions.

(iii) The class of admissible epics is closed under retractions.

Since the main object of this chapter is to construct a pure derived category on a closed symmetric monoidal Grothendieck category, we need to speak of complexes over an exact category. As we already know from Chapter (II), for any additive category $\mathcal{C}$, the category of complexes $\mathbf{C}(\mathcal{C})$ and the homotopy category $\mathbf{K}(\mathcal{C})$ are always definable and they are additive. Furthermore, there is a canonical exact category structure on $\mathbf{C}(\mathcal{C})$ which is carried from an exact category $\mathcal{C}$, defined as degreewise. Unless explicitly stated otherwise, $\mathbf{C}(\mathcal{C})$ will always carry this exact structure.

In an abelian category, there are two ways of describing derived categories: as a localization with respect to quasi-isomorphisms or as a Verdier quotient $\mathbf{K}(\mathcal{C})/\mathbf{C}_{ac}(\mathcal{C})$ where $\mathbf{C}_{ac}(\mathcal{C})$ denotes the class of all exact complexes. The later one forms the definition of derived category for more general exact categories, $\mathbf{D}(\mathcal{C}) := \mathbf{K}(\mathcal{C})/\mathbf{C}_{ac}(\mathcal{C})$. So firstly, let us recall what an acyclic complex is in an exact category.

Definition 7.1.7. A chain complex A over an exact category $\mathcal{C}$ is said to be *acyclic*, or *exact*, if each differential d_n factors as $A_n \twoheadrightarrow Z_{n-1}(A) \rightarrowtail A_{n-1}$ such that $Z_n(A) \rightarrowtail A_n \twoheadrightarrow Z_{n-1}(A)$ is a short exact sequence.

Let $\mathbf{K}_{ac}(\mathcal{C})$ be the full subcategory $\mathbf{K}(\mathcal{C})$ consisting of acyclic complexes. Through a result from [Nee90, 1.1], $\mathbf{K}_{ac}(\mathcal{C})$ is closed under mapping cones, therefore, it is a full triangulated subcategory of $\mathbf{K}(\mathcal{C})$.

As for quasi-isomorphisms, in the abelian case, to be a morphism f quasi-isomorphism is equivalent to the fact that its mapping cone, $c(f)$, is acyclic. Regarding to this, a chain morphism $f : A \to B$ over an exact category $\mathcal{C}$ is called a *quasi-isomorphism* if its cone $c(f)$ is homotopy equivalent to an acyclic complex. We should highlight here that $c(f)$ need not be in $\mathbf{K}_{ac}(\mathcal{C})$, since $\mathbf{K}_{ac}(\mathcal{C})$ is closed under isomorphisms as long as the category $\mathcal{C}$ is idempotent complete, see [Büh10, Proposition 10.9].

Now what we need is to ensure that an exact category has enough injective objects, as it happens in a Grothendieck category. Recall that the proof of the existence of injective envelope in a Grothendieck category is based on transfinite colimits of pushouts. The mission of having a generator is to enable to control the subobjects of any object, therefore, every object will be 'small' for a sufficiently large enough cardinal. And also it permits to find a kind of Baer criterion for categories. By this motivation, [SS11] come up with enough conditions on exact categories through which these ideas can be proceeded.

Definition 7.1.8. [Sto13, Definition 3.2] Let α be an ordinal number, and let $(X_\beta, f_{\beta\beta'})_{\beta<\beta'<\alpha}$ be a direct system indexed by α in $\mathcal{C}$. Such a system is called a *α-sequence* if for each limit ordinal $\beta < \alpha$, the object X_β together with the morphisms $f_{\mu\beta}$, $\mu < \beta$, is a colimit of the direct subsystem $(X_\mu, f_{\mu,\mu'})_{\mu<\mu'<\beta}$.

Definition 7.1.9. [Sto13, Definition 3.3] If $\mathcal{C}$ is an exact category, κ is a cardinal number and $\mathcal{D}$ is a class of morphisms of $\mathcal{C}$, then an object $X \in \mathcal{C}$ is called *κ-small relative to $\mathcal{D}$* if for every infinite regular cardinal $\alpha \geq \kappa$ and every α-sequence $(E_\beta, f_{\beta\beta'})_{\beta<\beta'<\alpha}$ in $\mathcal{C}$ such that $f_{\beta,\beta+1} \in \mathcal{D}$ for all $\beta + 1 < \alpha$, the canonical map of sets

$$\varinjlim_{\beta<\alpha} \mathrm{Hom}_{\mathcal{C}}(X, E_\beta) \to \mathrm{Hom}(X, \varinjlim_{\beta<\alpha} E_\beta)$$

is an isomorphism. The object X is *small relative to $\mathcal{D}$* if it is κ-small relative to $\mathcal{D}$ for some cardinal κ.

Definition 7.1.10. [Sto13, Definition 3.4] An exact category $\mathcal{C}$ is called *efficient* if

Ef1) $\mathcal{C}$ is weakly idempotent complete.

Ef2) Arbitrary transfinite compositions of inflations exist and are themselves inflations.

Ef3) Every object of $\mathcal{C}$ is small relative to the class of all inflations.

Ef4) $\mathcal{C}$ admits a generator. That is, there is an object $G \in \mathcal{C}$ such that every $X \in \mathcal{C}$ admits a deflation $G^{(I)} \to X \to 0$.

Definition 7.1.11. [Sto13, Definition 3.11] An exact category $\mathcal{C}$ is said to be of *Grothendieck type* if it is efficient and it is deconstructible in itself, i.e, there is a set of objects $S \subset \mathcal{C}$ such that $\mathcal{C} = \text{Filt}(S)$.

Proposition 7.1.12. *[Sto13, Theorem 3.16] Let $\mathcal{C}$ be a Grothendieck category and $\mathcal{C}' \subseteq \mathcal{C}$ be a deconstructible class which is closed under retracts. Then $\mathcal{C}'$ together with the collection of all short exact sequences in $\mathcal{C}$ whose terms belong to $\mathcal{C}$ is an exact category of Grothendieck type.*

Proposition 7.1.13. *[Sto13, Corollary 5.9] Let $(\mathcal{C}, \mathcal{E})$ be an exact category of Grothendieck type and* Inj *the class of injective objects with respect to $\mathcal{E}$. Then $(\mathcal{C}, \text{Inj})$ is a functorially complete cotorsion pair in $\mathcal{C}$.*

Lemma 7.1.14. *[Sto13, Lemma 7.9] Let $\mathcal{C}'$ be a full subcategory of a Grothendieck category which is deconstructible and closed under retracts, considered with the induced exact structure. Then $\mathbf{C}_{ac}(\mathcal{C}')$ is deconstructible in $\mathbf{C}(\mathcal{C}')$.*

The next lemma is the exact category version of Proposition 3.3.13.

Lemma 7.1.15. *[Sto13, Lemma 7.10] Let $\mathcal{C}$ be an exact category of Grothendieck type such that $\mathbf{C}_{ac}(\mathcal{C})$ is deconstructible in $\mathbf{C}(\mathcal{C})$. If $(\mathcal{F}, \mathcal{B})$ is a complete hereditary cotorsion pair in $\mathcal{C}$, then $(\tilde{\mathcal{F}}, \tilde{\mathcal{F}}^{\perp})$ is a complete (and hereditary) cotorsion pair in $\mathbf{C}(\mathcal{C})$.*

Theorem 7.1.16. *[Sto13, Theorem 7.11] Let $\mathcal{C}$ be an exact category of Grothendieck type such that $\mathbf{C}_{ac}(\mathcal{C})$ is deconstructible in $\mathbf{C}(\mathcal{C})$. Then there is a hereditary model structure on $\mathbf{C}(\mathcal{C})$ such that*

(i) every object of $\mathbf{C}(\mathcal{C})$ is cofibrant.

(ii) The trivial objects are precisely the acyclic complexes.

(iii) The class of fibrat objects is $\mathbf{C}_{ac}(\mathcal{C})^{\perp}$.

In particular we have Ho$\mathbf{C}(\mathcal{C}) = \mathbf{D}(\mathcal{C})$*, the unbounded derived category of* $\mathcal{C}$*.*

7.2 The pure-injective model structure

Let $\mathcal{C}$ be a closed symmetric monoidal Grothendieck category. In Section 5.2 we noted that $\mathcal{C}$ is locally λ-presentable for some regular cardinal λ, and that there are two generally different notions of purity. Let $\mathcal{P}$ denote the proper class of λ-pure short exact sequences in $\mathcal{C}$ and $\mathcal{P}_{\otimes}$ denote the proper class of geometrical pure short exact sequences in $\mathcal{C}$. Recall that we have the containment $\mathcal{P} \subseteq \mathcal{P}_{\otimes}$. Our main interest in this section will be the $\otimes$-pure exact structure. So throughout the rest of this section, when we say pure exact we will always mean geometrical pure exact, unless explicitly stated otherwise. We will denote by $\mathbf{C}(\mathcal{C})_{\otimes}$ the exact structure consisting of $\mathbf{C}(\mathcal{C})$ along with the componentwise pure exact sequences.

Proposition 7.2.1. *Let* $\{P_i;\ \psi_{ij} : P_i \to P_j\}_I$ *be a* λ*-directed system in* $\mathcal{C}$*. Then the canonical morphism* $\bigoplus_I P_i \to \mathrm{colim} P_i \to 0$ *is* λ*-pure epic. So it is also geometrical pure epic.*

Proof. For any $\rho : i \to j$, $s(\rho) = i$ and $t(\rho) = j$. For each $i \in I$, let us denote by $\iota_i : P_i \to \bigoplus_I P_i$ and $\pi_i : \bigoplus P_i \to P_i$ the canonical injection and projection maps respectively. Consider $l_\rho := \iota_{t(\rho)} \circ \psi_{ij} - \iota_{s(\rho)} : P_{s(\rho)} \to \bigoplus_I P_i$, which is monic for all morphism ρ in I, (actually it splits). So, we have the induced morphism $(l_\rho)_\rho : \bigoplus_\rho P_{s(\rho)} \to \bigoplus_I P_i$. We know that $\mathrm{colim} P_i =$ Coker$(l_\rho)_\rho$. That is, there is an exact sequence

$$0 \longrightarrow \textstyle\sum_\rho \mathrm{Im}\, l_\rho \longrightarrow \bigoplus_I P_i \xrightarrow{g} \mathrm{colim} P_i \longrightarrow 0\ .$$

Then $\alpha_i := g \circ \iota_i$ is the family of morphisms $\alpha_i : P_i \to \mathrm{colim} P_i$ with $\alpha_j \circ \psi_{ij} = \alpha_i$ for each $\rho : i \to j$. Let $f : H \to \mathrm{colim} P_i$ be a morphism where H is λ-presentable. Then f factors through α_i for some i, that is, there is a morphism $f' : H \to P_i$ such that $\alpha_i \circ f' = f$. But $\alpha_i = g \circ \iota_i$, $g \circ \iota_i \circ f' = f$. That is, that exact sequence is $\mathrm{Hom}(H, -)$-exact for each λ-presentable object H, which means that it is λ-pure exact. □

Lemma 7.2.2. *$\mathcal{C}$ with the pure exact structure is an exact category of Grothendieck type.*

Proof. It is routine to check that $\mathcal{C}$ along with the pure exact sequences form an exact category. So we check Definitions 7.1.10 and 7.1.11. First, we must prove that it is efficient. (Ef1) is clear since $\mathcal{C}$ is abelian and (Ef2) is also clear since the tensor product preserves any colimit. Any object $X \in \mathcal{C}$ is κ-presentable for some cardinal κ, so (Ef3) easily follows. Now (Ef4) follows from Proposition 7.2.1. In detail since $\mathcal{C}$ is locally λ-presentable we have a set of λ-presentable objects for which each $X \in \mathcal{C}$ is the colimit of a λ-directed system $\{P_i;\ \psi_{ij} : P_i \to P_j\}_I$ with each P_i in that set. Then by Proposition 7.2.1 the canonical morphism $\bigoplus_I P_i \to \mathrm{colim} P_i \to 0$ is a pure epimorphism as required. So we conclude that $\mathcal{C}$ with the pure exact structure is efficient. Finally by Theorem 5.1.6 and the fact that if $A \leq A' \leq B$ is such that $A \leq B$ and $A'/A \leq B/A$ are pure-monic in $\mathcal{C}$ then $A' \leq B$ is also pure-monic, we infer that there is a regular cardinal γ such that $\mathcal{C} = \mathrm{Filt}(\mathcal{C}^{\leq\gamma})$. Here $\mathcal{C}^{\leq\gamma}$ is the class of all γ-presentable objects in $\mathcal{C}$ and the filtration is built in $\mathcal{C}$ with the pure-exact structure. □

A complex C in $\mathbf{C}(\mathcal{C})$ is called *$\otimes$-acyclic* if it is acyclic in $\mathbf{C}(\mathcal{C})_{\otimes}$, the exact category of chain complexes with the pointwise pure exact structure. This means each sequence $0 \to Z_nC \to C_n \to Z_{n-1}C \to 0$ is pure, or equivalently, $C \otimes S$ is exact for all $S \in \mathcal{C}$. We shall denote by $\mathbf{C}_{\otimes\text{-}ac}(\mathcal{C})$ the class of all $\otimes$-acyclic complexes. Our aim is to construct the relative derived category of $\mathcal{C}$ with respect to the $\otimes$-pure proper class, that is, whose trivial objects are the $\otimes$-acyclic complexes. To achieve this aim, we will use the proper class version of Hovey's correspondence between cotorsion pairs and model category structures mentioned in Proposition 3.3.5 and Proposition 3.3.6. We note that when the underlying category is abelian, an exact structure on the category is the same thing as a proper class [Gil14, Appendix B]. So the language of abelian model structures from [Hov02] and the language of exact model structures from [Gil11] and [Sto13] are the same thing when the underlying category is abelian.

Let $\mathcal{P}inj$ denote the class of objects in $\mathcal{C}$ having the injective property with respect to the proper class $\mathcal{P}_{\otimes}$, the geometrical pure short exact sequences in $\mathcal{C}$. We will call an object in $\mathcal{P}inj$ *pure-injective*. From Theorem 5.2.6, we already know that $(\mathcal{C}, \mathcal{P}inj)$ is a complete cotorsion pair in $\mathcal{C}$ with the pure exact structure. But by Proposition 7.1.13 and Lemma 7.2.2, we reprove that $(\mathcal{C}, \mathcal{P}inj)$ is a hereditary complete cotorsion pair. In particular, every object in $\mathcal{C}$ can be purely embedded in a pure-injective object.

We now define the following classes in $\mathbf{C}(\mathcal{C})$, which will turn out to be the fibrant and trivially fibrant objects in our model structure for the $\otimes$-pure derived category:

$$dg\,\mathcal{P}inj = \{L \in \mathbf{C}(\mathcal{C}) : \ L_n \in \mathcal{P}inj \text{ and each map } E \to L \text{ is homotopic to } 0, \forall E \in \mathbf{C}_{\otimes\text{-}ac}(\mathcal{C})\}$$

and

$$\widetilde{\mathcal{P}inj} = \{T \in \mathbf{C}_{\otimes\text{-}ac}(\mathcal{C}) : \ Z_n T \in \mathcal{P}inj\}.$$

One can check that $\widetilde{\mathcal{P}inj}$ is the class of injective objects in the exact category $\mathbf{C}(\mathcal{C})_{\otimes}$ of chain complexes with the pointwise pure-exact structure. Moreover, they are precisely the contractible complexes with pure-injective components. We want to apply relative version of Proposition 3.3.6 to the pairs $(\mathbf{C}_{\otimes\text{-}ac}(\mathcal{C}), dg\,\mathcal{P}inj)$ and $(\mathbf{C}(\mathcal{C}), \widetilde{\mathcal{P}inj})$. So we have to show that these two pairs are complete cotorsion pairs in $\mathbf{C}(\mathcal{C})_{\otimes}$.

First, we will prove that $\mathbf{C}_{\otimes\text{-}ac}(\mathcal{C})$ is deconstructible. We will start with the following lemma.

Lemma 7.2.3. *Let X be $\otimes$-acyclic and X' be a subcomplex of X. Assume that*

(i) X' is acyclic.

(ii) $Z_n X' \subseteq Z_n X$ is $\otimes$-pure, for each $n \in \mathbb{Z}$.

Then X' is $\otimes$-acyclic and $X'_n \subseteq X_n$ is $\otimes$-pure, for each $n \in \mathbb{Z}$.

Proof. By hypothesis, we have the commutative diagram below with the top row exact, the bottom row pure exact, and the outer vertical arrows pure monomorphisms.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z_nX' & \longrightarrow & X'_n & \longrightarrow & Z_{n-1}X' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Z_nX & \longrightarrow & X_n & \longrightarrow & Z_{n-1}X & \longrightarrow & 0 \end{array}$$

Since the composite $Z_nX' \subseteq Z_nX \subseteq X_n$ is pure, we get that the composite $Z_nX' \subseteq X'_n \subseteq X_n$ is also pure. It follows immediately that $Z_nX' \subseteq X'_n$ is pure. So the top row is pure exact and now the snake lemma can be used to show that the middle vertical arrow is also a pure monomorphism. □

Proposition 7.2.4. *There is a regular cardinal γ such that every $\otimes$-acyclic complex X has a $\mathbf{C}_{\otimes\text{-}ac}(\mathcal{C})^{\leq\gamma}$-filtration. That is, $\mathbf{C}_{\otimes\text{-}ac}(\mathcal{C}) = Filt(\mathbf{C}_{\otimes\text{-}ac}(\mathcal{C})^{\leq\gamma})$, where $\mathbf{C}_{\otimes\text{-}ac}(\mathcal{C})^{\leq\gamma}$ is the class of γ-presentable $\otimes$-acyclic complexes, and the monomorphisms in the filtration are with respect to the degreewise pure exact structure.*

Proof. The class $\mathbf{C}_{\otimes\text{-}ac}(\mathcal{C})$ is closed under direct limits, so it suffices to show that there is a regular cardinal γ satisfying that: given $A \subseteq X \neq 0$ where $X \in \mathbf{C}_{\otimes\text{-}ac}(\mathcal{C})$ and A is γ-presentable, there exists a γ-presentable $X' \neq 0$ such that $A \subseteq X' \subseteq X$, and $X' \in \mathbf{C}_{\otimes\text{-}ac}(\mathcal{C})$, and $X'_n \subseteq X_n$ is $\otimes$-pure for each $n \in \mathbb{Z}$. Once we show this, a standard argument utilizing properties of the $\otimes$-purity will allow for the construction of the desired filtration of X.

Since $\mathcal{C}$ is Grothendieck, it is locally λ-presentable and so $\mathbf{C}(\mathcal{C})$ is also locally λ-presentable. Let $0 \neq X \in \mathbf{C}_{\otimes\text{-}ac}(\mathcal{C})$. By Theorem 5.1.6, there is a regular cardinal γ such that every γ-presentable subcomplex $A \subseteq X$ can be embedded in a γ-presentable subcomplex $X' \subseteq X$ which is a λ-pure embedding. According to Lemma 7.2.3 we just need to check that X' is acyclic and that $Z_nX' \subseteq Z_nX$ is $\otimes$-pure for all $n \in \mathbb{Z}$. Now for any λ-presentable $L \in \mathcal{C}$

we have that $S^n(L)$ is a λ-presentable complex. Therefore, applying $\text{Hom}_{\mathbf{C}(\mathcal{C})}(S^n(L), -)$ to $0 \to X' \to X \to X/X' \to 0$, yields a short exact sequence

$$0 \to \text{Hom}_{\mathbf{C}(\mathcal{C})}(S^n(L), X') \to \text{Hom}_{\mathbf{C}(\mathcal{C})}(S^n(L), X) \to \text{Hom}_{\mathbf{C}(\mathcal{C})}(S^n(L), X/X') \to 0.$$

But the canonical isomorphism $\text{Hom}_{\mathcal{C}}(L, Z_nY) \cong \text{Hom}_{\mathbf{C}(\mathcal{C})}(S^n(L), Y)$ gives us a short exact sequence $0 \to \text{Hom}_{\mathcal{C}}(L, Z_nX') \to \text{Hom}_{\mathcal{C}}(L, Z_nX) \to \text{Hom}_{\mathcal{C}}(L, Z_n(X/X')) \to 0$. Since $\mathcal{C}$ is locally λ-presentable, there it has a generating set consisting of λ-presentable objects and so it follows that $0 \to Z_nX' \to Z_nX \to Z_n(X/X') \to 0$ is a short exact sequence. In fact we have just shown that this is a λ-pure exact sequence in $\mathcal{C}$. So it is also $\otimes$-pure exact. It now only remains to show that X' is itself exact. For this, we apply the snake lemma to

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z_nX' & \longrightarrow & Z_nX & \longrightarrow & Z_n(X/X') & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & X'_n & \longrightarrow & X_n & \longrightarrow & (X/X')_n & \longrightarrow & 0 \end{array}$$

to conclude we have a short exact sequence $0 \to B_{n-1}X' \to B_{n-1}X \to B_{n-1}(X/X') \to 0$ for all n. We then turn around and apply the snake lemma to

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B_nX' & \longrightarrow & B_nX & \longrightarrow & B_n(X/X') & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Z_nX' & \longrightarrow & Z_nX & \longrightarrow & Z_n(X/X') & \longrightarrow & 0 \end{array}$$

and use that $B_nX = Z_nX$ to conclude that $B_nX = Z_nX$ (and $B_n(X/X') = Z_n(X/X')$). □

Corollary 7.2.5. *The pair $(\mathbf{C}_{\otimes\text{-}ac}(\mathcal{C}), dg\,\mathcal{P}inj)$ is a complete (and hereditary) cotorsion pair in $\mathbf{C}(\mathcal{C})_\otimes$.*

Proof. As noted after Proposition 7.1.13, we have that $(\mathcal{C}, \mathcal{P}inj)$ is a complete hereditary cotorsion pair in $\mathcal{C}$ with the pure exact structure. So by Lemma 7.1.15 and Proposition 7.2.4, we infer that $(\mathbf{C}_{\otimes\text{-}ac}(\mathcal{C}), \mathbf{C}_{\otimes\text{-}ac}(\mathcal{C})^\perp)$ is a complete (and hereditary) cotorsion pair in $\mathbf{C}(\mathcal{C})_\otimes$. So it only remains to show that $\mathbf{C}_{\otimes\text{-}ac}(\mathcal{C})^\perp$ coincides with the class $dg\,\mathcal{P}inj$. By definition of

$dg\,\mathcal{P}inj$, it is clear $dg\,\mathcal{P}inj \subseteq \mathbf{C}_{\otimes\text{-}ac}(\mathcal{C})^{\perp}$. Now let $X \in \mathbf{C}_{\otimes\text{-}ac}(\mathcal{C})^{\perp}$. It is enough to show that each X_i is pure-injective. Let $0 \to X_i \to A \to B \to 0$ be pure-exact sequence in $\mathcal{C}$. Then we get a short exact sequence of complexes $0 \to X \to \overline{A} \to D^i(B) \to 0$ in $\mathbf{C}(\mathcal{C})$ by taking the pushout of $X_i \to X_{i-1}$ and $X_i \to A$ and where $\overline{A}_i = A$. Since pure-monomorphisms are closed by forming pushouts the sequence is degreewise pure-exact. By assumption the sequence splits and so, in particular, it splits on each degree. Hence X_i is pure-injective. $\square$

Proposition 7.2.6. *The pair $(\mathbf{C}(\mathcal{C}), \widetilde{\mathcal{P}inj})$ is a complete (and hereditary) cotorsion pair in $\mathbf{C}(\mathcal{C})_{\otimes}$. Moreover, $\widetilde{\mathcal{P}inj} = dg\,\mathcal{P}inj \cap \mathbf{C}_{\otimes\text{-}ac}(\mathcal{C})$.*

Proof. It can be easily observed that $\mathbf{C}(\mathcal{C})_{\otimes}$, the exact category of chain complexes with the degreewise pure-exact structure, is of Grothendieck type. Indeed, $\mathbf{C}(\mathcal{C})$ is a Grothendieck category and any λ-pure subobject gives us a degreewise λ-pure monomorphism. So it is a degreewise $\otimes$-pure monomorphism as well. Note that, colimits in $\mathbf{C}(\mathcal{C})$ are computed pointwise. So we again can apply Proposition 5.1.6 to argue that $\mathbf{C}(\mathcal{C})_{\otimes}$ is deconstructible in itself. Then by Proposition 7.1.13 we get that $(\mathbf{C}(\mathcal{C}), \mathrm{Inj})$ is a complete cotorsion pair in $\mathbf{C}(\mathcal{C})_{\otimes}$. But here $\mathrm{Inj} = \widetilde{\mathcal{P}inj}$.

Since $\widetilde{\mathcal{P}inj}$ consists of contractible complexes of pure-injectives, $\widetilde{\mathcal{P}inj} \subseteq dg\,\mathcal{P}inj \cap \mathbf{C}_{\otimes\text{-}ac}(\mathcal{C})$. For the converse, let $X \in dg\,\mathcal{P}inj \cap \mathbf{C}_{\otimes\text{-}ac}(\mathcal{C})$. By assumption, the identity map $X \to X$ is homotopic to zero, so X is a contractible complex of pure-injectives. So $X \in \widetilde{\mathcal{P}inj}$. $\square$

Theorem 7.2.7. *Let $\mathcal{C}$ be a closed symmetric monoidal Grothendieck category and $\mathbf{C}(\mathcal{C})$ the associated category of chain complexes. Then there is a cofibrantly generated model category structure on $\mathbf{C}(\mathcal{C})$ whose trivial objects are the $\otimes$-pure acyclic complexes; that is, complexes X for which $X \otimes S$ is exact for all $S \in \mathcal{C}$. The model structure is exact with respect to the exact category $\mathbf{C}(\mathcal{C})_{\otimes}$ of chain complexes along with the proper class of degreewise $\otimes$-pure exact sequences. In fact, the model structure is injective in the sense that every complex is cofibrant and the trivially fibrant complexes are the injective objects of $\mathbf{C}(\mathcal{C})_{\otimes}$, which are precisely the contractible complexes with $\otimes$-pure injective components. We call this model structure the*

$\otimes$-pure injective model structure on $\mathbf{C}(\mathcal{C})$ and its corresponding homotopy category is the $\otimes$-pure derived category, denoted $\mathbf{D}_{\otimes\text{-pur}}(\mathcal{C})$.

Firstly, we recall what kind of relation there are between the two notion of purity in $\mathfrak{Qcoh}(X)$, that of monoidal structure and that of stalkwise defined as in Section 5.4. The definition of $\otimes$-pure used here is slightly different than the one used in Section 5.4. But they agree when X is quasi-separated. See [EEO14, Remark 3.5].

Proposition 7.2.8. *Let X be a quasi-separated scheme, and $f : \mathcal{F} \to \mathcal{G}$ a monomorphism in $\mathfrak{Qcoh}(X)$. The following assertions are equivalent:*

(i) f is $\otimes$-pure.

(ii) There exists an open affine covering $\mathcal{U} = \{U_i\}_{i\in I}$ of X such that f_{U_i} is pure in $\mathcal{O}_X(U_i)$-Mod.

(iii) f is stalkwise pure.

Proof. $(ii \Rightarrow iii)$ and $(iii \Rightarrow i)$ have been already proved in Proposition 5.3.3. So we just prove $(i \Rightarrow ii)$. Let U be an affine open subset of $\mathcal{U}$ and $\iota : U \hookrightarrow X$ be the open immersion. And let $M \in \mathcal{O}_X(U)$-Mod. Since X is quasi-separated, then $\iota_*(\widetilde{M})$ is a quasi-coherent $\mathcal{O}_X$-module. Therefore

$$0 \to \iota_*(\widetilde{M}) \otimes \mathcal{F} \to \iota_*(\widetilde{M}) \otimes \mathcal{G}$$

is exact. But then

$$0 \to (\iota_*(\widetilde{M}) \otimes \mathcal{F})(U) \to (\iota_*(\widetilde{M}) \otimes \mathcal{G})(U)$$

is exact in $\mathcal{O}_X(U)$-Mod, that is

$$0 \to \iota_*(\widetilde{M})(U) \otimes \mathcal{F}(U) \to \iota_*(\widetilde{M})(U) \otimes \mathcal{G}(U)$$

is exact. Since, for each $\mathcal{O}_X(U)$-module A, $\iota_*(\widetilde{A})(U) = A$, we get that $0 \to M \otimes \mathcal{F}(U) \to M \otimes \mathcal{G}(U)$ is exact. Thus $0 \to \mathcal{F}(U) \to \mathcal{G}(U)$ is pure. $\square$

Corollary 7.2.9. *Let X be a quasi-separated scheme. Let $\mathcal{E}$ be the exact structure coming from the stalkwise-purity in $\mathfrak{Qcoh}(X)$, and let us consider the category of unbounded complexes $\mathbf{C}(\mathfrak{Qcoh}(X))$. Then with respect to the induced degreewise exact structure from $\mathcal{E}$, there is an exact and injective model category structure on $\mathbf{C}(\mathfrak{Qcoh}(X))$. The corresponding homotopy category is the* stalkwise-pure derived category *(or* geometric pure derived category*), which we denote $\mathbf{D}_{\text{stk-pure}}(\mathfrak{Qcoh}(X))$.*

7.3 Relationship between the two pure derived categories

Suppose that $\mathcal{C}$ is a closed symmetric monoidal Grothendieck category. In this section we get an adjunction between the two derived categories obtained from the proper class $\mathcal{P}$ of the λ-pure short exact sequences and the proper class $\mathcal{P}_\otimes$ of the geometrical pure short exact sequences. By [Gil14], we have the λ-pure derived category $\mathbf{D}_{\lambda\text{-pur}}(\mathcal{C})$ and the λ-pure projective model structure on $\mathbf{C}(\mathcal{C})_{\mathcal{P}}$ whose trivial objects are the λ-pure exact complexes. This model structure corresponds to Hovey pairs in $\mathbf{C}(\mathcal{C})_{\mathcal{P}}$ that we denote by $(dg\,\lambda\text{-Pproj}, \mathbf{C}_{\lambda\text{-}ac}(\mathcal{C}))$ and $(\widetilde{\text{Proj}}, \mathbf{C}(\mathcal{C}))$. In particular, $\mathbf{C}_{\lambda\text{-}ac}(\mathcal{C})$ denotes the class of λ-pure exact complexes.

From the previous section, we have the Hovey pairs $(\mathbf{C}_{\otimes\text{-}ac}(\mathcal{C}), dg\,\mathcal{P}inj)$ and $(\mathbf{C}(\mathcal{C}), \widetilde{\mathcal{P}inj})$ on $\mathbf{C}(\mathcal{C})_\otimes$. So the derived category $\mathbf{D}_{\otimes\text{-pur}}(\mathcal{C})$ has an injective model structure whose trivial objects are the $\otimes$-acyclic complexes, while $\mathbf{D}_{\lambda\text{-pur}}(\mathcal{C})$ has a projective model structure whose trivial objects are the λ-pure acyclic complexes.

Definition 7.3.1. Suppose $\mathcal{C}$ and $\mathcal{D}$ are model categories.

(i) We call a functor $F : \mathcal{C} \to \mathcal{D}$ a *left Quillen functor* if F is a left adjoint and preserves cofibrations and trivial cofibrations.

(ii) We call a functor $U : \mathcal{D} \to \mathcal{C}$ a *right Quillen functor* if U is a right adjoint and preserves fibrations and trivial fibrations.

(iii) Suppose (F, U, φ) is an adjunction from $\mathcal{C}$ to $\mathcal{D}$. That is, F is a functor $\mathcal{C} \to \mathcal{D}$, U is a functor $\mathcal{D} \to \mathcal{C}$, and φ is a natural isomorphism $\mathrm{Hom}(FA, B) \to \mathrm{Hom}(A, UB)$ expressing U as a right adjoint of F. We call (F, U, φ) a *Quillen adjunction* if F is a left Quillen functor.

Lemma 7.3.2. *[Hov99, Lemma 1.3.4] Suppose $(F, U, \varphi) : \mathcal{C} \to \mathcal{D}$ is an adjunction, and $\mathcal{C}$ and $\mathcal{D}$ are model categories. Then (F, U, φ) is a Quillen adjunction if and only if U is a right Quillen functor.*

Definition 7.3.3. Suppose $\mathcal{C}$ and $\mathcal{D}$ are model categories.

(i) If $F : \mathcal{C} \to \mathcal{D}$ is a left Quillen functor, define the total left derived functor $LF : \mathrm{Ho}\mathcal{C} \to \mathrm{Ho}\mathcal{D}$ to be the composite

$$\mathrm{Ho}\mathcal{C} \xrightarrow{\mathrm{Ho}Q} \mathrm{Ho}\mathcal{C}_c \xrightarrow{\mathrm{Ho}F} \mathrm{Ho}\mathcal{D} .$$

Given a natural transformation $\tau : F \to F'$ of left Quillen functors, define the total derived natural transformation L_τ to be $\mathrm{Ho}\tau \circ \mathrm{Ho}Q$, so that $(L\tau)_X = \tau_{QX}$.

(ii) If $U : \mathcal{D} \to \mathcal{C}$ is a right Quillen functor, define the total right derived functor RU : $\mathrm{Ho}\mathcal{D} \to \mathrm{Ho}\mathcal{C}$ of U to be the composite

$$\mathrm{Ho}\mathcal{D} \xrightarrow{\mathrm{Ho}R} \mathrm{Ho}\mathcal{D}_f \xrightarrow{\mathrm{Ho}U} \mathrm{Ho}\mathcal{C} .$$

Given a natural transformation $\tau : U \to U'$ of right Quillen functors, define the total derived natural transformation $R\tau$ to be $\mathrm{Ho}\tau \circ \mathrm{Ho}R$, so that $R\tau_X = \tau_{RX}X$.

Lemma 7.3.4. *[Hov99, Lemma 1.3.10] Suppose $\mathcal{C}$ and $\mathcal{D}$ are model categories and (F, U, φ) : $\mathcal{C} \to \mathcal{D}$ is a Quillen adjunction. Then LF and RU are part of an adjunction $L(F, U, \varphi) = (LF, RU, R\varphi)$, which we call the derived adjunction.*

Proposition 7.3.5. $\mathrm{id} : \mathbf{C}(\mathcal{C})_{\mathcal{P}} \to \mathbf{C}(C)_{\otimes}$ *is a left Quillen functor. So there is a Quillen adjunction between* $\mathbf{D}_{\lambda\text{-pur}}(\mathcal{C})$ *and* $\mathbf{D}_{\otimes\text{-pur}}(\mathcal{C})$

Proof. Clearly id is a left adjoint functor of $\mathrm{id} : \mathbf{C}(\mathcal{C})_{\otimes} \to \mathbf{C}(C)_{\mathcal{P}}$. Also, id preserves cofibrations and trivial cofibrations. Indeed, a cofibration f in $\mathbf{C}(\mathcal{C})_{\mathcal{P}}$ is a degreewise λ-pure monomorphism with cokernel in $dg\,\lambda\text{-Pproj}$. Such an f is a cofibration in $\mathbf{C}(\mathcal{C})_{\otimes}$ as well since here the cofibrations are the degreewise $\otimes$-pure monomorphisms. Also, any trivial cofibration f in $\mathbf{C}(\mathcal{C})_{\mathcal{P}}$ is a degreewise λ-pure monomorphism with cokernel in $\widetilde{\text{Proj}}$, in particular, contractible. So it is a trivial cofibration in $\mathbf{C}(\mathcal{C})_{\otimes}$.

By Lemma 7.3.2, $\mathrm{id} : \mathbf{C}(\mathcal{C})_{\otimes} \to \mathbf{C}(C)_{\mathcal{P}}$ is a right Quillen functor.

From Lemma 7.3.4, the total left derived functor $L(\mathrm{id}) : \mathbf{D}_{\lambda\text{-pur}}(\mathcal{C}) \to \mathbf{D}_{\otimes\text{-pur}}(\mathcal{C})$ and the total right derived functor $R(\mathrm{id}) : \mathbf{D}_{\otimes\text{-pur}}(\mathcal{C}) \to \mathbf{D}_{\lambda\text{-pur}}(\mathcal{C})$ gives us an adjunction $(L(\mathrm{id}), R(\mathrm{id}))$. By definition, $L(\mathrm{id})(X)$ is its cofibrant replacement in $\mathbf{C}(\mathcal{C})_{\mathcal{P}}$, that is, $L(\mathrm{id})(X) \in dg\,\lambda\text{-Pproj}$. Dually, $R(\mathrm{id})(X)$ is its fibrant replacement in $\mathbf{C}(\mathcal{C})_{\otimes}$, so $R(\mathrm{id})(X) \in dg\,\mathcal{P}inj$. ∎

7.4 The pure derived category of flat sheaves via model structures

Let $\mathcal{A}$ be a locally finitely presentable additive category. We start by recalling the following representation theorem due to Crawley-Boevey (see also [Pre09, Chapter 16] for a nice exposition).

Theorem 7.4.1 (Crawley-Boevey)**.** *Every finitely accessible additive category $\mathcal{A}$ is equivalent to the full subcategory $\mathcal{F}lat(A)$ of the category* Mod-A *of unitary right A-modules consisting of flat right A-modules where A is the functor ring of $\mathcal{A}$ (that is, a ring with enough idempotents). This equivalence gives a 1-1 correspondence between pure exact sequences in $\mathcal{A}$ and exact* *sequences in $\mathcal{F}lat(A)$.*

In other words, $\mathcal{A}$ with its pure exact structure is equivalent to $\mathcal{F}lat(A)$ with its canonical exact structure inherited from $\mathrm{Mod}\text{-}A$. In particular, the equivalence takes injective objects in $\mathcal{A}$ (pure-injectives) to injective objects in $\mathcal{F}lat(A)$ (cotorsion flat modules). Similarly it preserves projectives, taking pure-projectives in $\mathcal{A}$ (retracts of direct sums of finitely presentable objects) to projective modules in $\mathcal{F}lat(A)$. Also each exact category is of Grothendieck type with the class of acyclic complexes being deconstructible. This leads to injective model structures on the associated chain complex categories with their inherited degreewise exact structures. On the other hand, each of the exact categories $\mathcal{A}$ and $\mathcal{F}lat(A)$ possesses a set of projective generators leading to projective model structures. Concentrating on the exact category $\mathbf{C}(\mathcal{F}lat(A))$, we have the following fact from [Gil13, Corollary 7.4 and 7.5].

Lemma 7.4.2. *There is an injective model structure on $\mathbf{C}(\mathcal{F}lat(A))$ in which every object is cofibrant and the fibrant objects are dg-cotorsion complexes which are flat on each degree. The trivial objects are the acyclic complexes in $\mathbf{C}(\mathcal{F}lat(A))$. This class coincides with $\widetilde{\mathcal{F}}$, the class of exact complexes with flat cycles. On the other hand, there is a projective model structure on $\mathbf{C}(\mathcal{F}lat(A))$ having the same class of trivial objects. Here every object is fibrant and the cofibrant objects are the complexes consisting of a projective module in each degree.*

On the other hand we learned from [Gil14] that the (usual, i.e. categorical) pure derived category of a locally finitely presentable category $\mathcal{A}$ can be obtained as the homotopy category of both an injective and projective model category structure on the exact category $\mathbf{C}(\mathcal{A})_{dw-pur}$. This denotes the exact category of chain complexes with the degreewise pure exact structure. So in view of the previous comments we have the following alternative way of defining the pure derived category of $\mathcal{A}$.

Theorem 7.4.3. *Let $\mathcal{A}$ be a locally finitely presentable additive category and $\mathcal{F}lat(A)$ its equivalent full subcategory of flat modules in $\mathrm{Mod}\text{-}A$. The (categorical) pure derived* *category of $\mathcal{A}$, $\mathbf{D}_{\text{pur}}(\mathcal{A})$, is equivalent to the derived category of the exact category $\mathcal{F}lat(A)$, $\mathbf{D}(\mathcal{F}lat(A))$.*

Proof. Using the equivalence of Crawley-Boevey discussed above, the acyclic complexes in the exact category $\mathbf{C}(\mathcal{A})_{dw-pur}$, which are the pure acyclic complexes, correspond to the class $\widetilde{\mathcal{F}}$ of acyclic complexes in $\mathbf{C}(\mathcal{F}lat(A))$. The injective model structure on $\mathbf{C}(\mathcal{A})_{dw-pur}$ is completely determined by the injective cotorsion pair (Pure acyclic complexes, DG-pure-injectives). This corresponds to the injective cotorsion pair ($\widetilde{\mathcal{F}}$, dg-cotorsion complexes of flats) in Lemma 7.4.2. There is a similar correspondence for the projective model structures. We note that by [Craw94, Lemma 1] the exact structures are each weakly idempotent complete and so by [Gil13, Lemma 3.1] a map is a weak equivalence in either model structure if and only if it factors as an admissible monomorphism (inflation) with trivial cokernel followed by an admissible epimorphism (deflation) with trivial kernel. From this we see that weak equivalences in $\mathbf{C}(\mathcal{A})_{dw-pur}$ correspond to weak equivalences in $\mathbf{C}(\mathcal{F}lat(A))$. So the homotopy categories $\mathbf{D}_{\text{pur}}(\mathcal{A})$ and $\mathbf{D}(\mathcal{F}lat(A))$ must be equivalent. □

Note that the two injective cotorsion pairs in the above proof may each be thought of as the "DG-injective" cotorsion pairs, but with respect to their exact structure. Similarly the projective cotorsion pairs may be thought of as the "DG-projective" cotorsion pairs with respect to these exact structures.

So it seems clear that in order to gain a better understanding of the pure derived category, one should focus on the derived category of flat modules. In [MS11] Murfet and Salarian define the *pure derived category of flat sheaves* for a semi-separated noetherian scheme. But a close inspection of their definition reveals that they are considering the *derived category of flat sheaves* in the above sense. Theorem 7.4.4 shows that, for *any* scheme X, we can realize the derived category of flat sheaves as the homotopy category of a model structure on $\mathbf{C}(\mathcal{F}lat(X))$ which is injective with respect to the exact structure.

Theorem 7.4.4. *Let X be a scheme, and $\mathcal{F}lat(X)$ the category of quasi-coherent flat sheaves. There is an injective exact model structure on $\mathbf{C}(\mathcal{F}lat(X))$. So every object is cofibrant and the fibrant objects are dg-cotorsion complexes which are flat on each degree. The trivial objects are those in $\mathbf{C}_{ac}(\mathcal{F}lat(X)) = \widetilde{\mathcal{F}}$, the class of acyclic complexes with flat cycles. The corresponding homotopy category is the derived category of flat sheaves, $\mathbf{D}(\mathcal{F}lat(X))$.*

Proof. The category of quasi-coherent sheaves is Grothendieck and the class $\mathcal{F}lat(X)$ of flat quasi-coherent sheaves is deconstructible. So by Proposition 7.1.12 we get that $\mathcal{F}lat(X)$ inherits the structure of an exact category of Grothendieck type. Moreover, by Lemma 7.1.14 we get that $\mathbf{C}_{ac}(\mathcal{F}lat(X)) = \widetilde{\mathcal{F}}$ is deconstructible in the exact category $\mathbf{C}(\mathcal{F}lat(X))$. Then we conclude, using Theorem 7.1.16, that $(\widetilde{\mathcal{F}}, \widetilde{\mathcal{F}}^{\perp})$ is an injective model structure in $\mathbf{C}(\mathcal{F}lat(X))$. It is left to argue that $\widetilde{\mathcal{F}}^{\perp} = \mathcal{Y} \cap \mathbf{C}_{ac}(\mathcal{F}lat(X))$ where $\mathcal{Y}$ is the class of dg-cotorsion complexes in $\mathbf{C}(\mathfrak{Qcoh}(X))$. We are calling a complex $Y \in \mathbf{C}(\mathfrak{Qcoh}(X))$ *dg-cotorsion* if each Y_n is cotorsion and every chain map $F \to Y$ is null homotopic whenever F is in $\widetilde{\mathcal{F}}$. So then $\mathcal{Y} \cap \mathbf{C}_{ac}(\mathcal{F}lat(X))$ is the class of complexes Y with each Y_n cotorsion flat and with every chain map $F \to Y$ being null homotopic whenever F is in $\widetilde{\mathcal{F}}$. But now using that the injective objects in $\mathcal{F}lat(X)$ are the cotorsion flats, we can argue as in Corollary 7.2.5 that this coincides with $\widetilde{\mathcal{F}}^{\perp}$ in $\mathbf{C}(\mathcal{F}lat(X))$. $\square$

CHAPTER EIGHT
GORENSTEIN PROJECTIVE RESOLVENTS

Gorenstein projective and injective modules constitute the heart of Gorenstein Homological Algebra, whose current definitions were introduced in [EJ95]. The subject of this chapter is Gorenstein projective modules. The first section is the preliminary part of the chapter.

In the second section, we prove the existence of Gorenstein projective preenvelopes for finitely generated modules over a local n-Gorenstein ring. So we deal with right Gorenstein projective resolutions, see Theorem 8.2.8 and Theorem8.2.9.

In the third section, we deal with certain left derived functors of $\mathrm{Hom}(-,-)$ by using (complete) projective and Gorenstein projective resolvents. We prove that each one has a balance property, see Proposition 8.3.2. And all of them are tied up in a long exact complex, see Theorem 8.3.3.

8.1 Gorenstein projective modules

In this section, we recall some basic definitions and facts that will be used later. Throughout the paper, we assume that R is a commutative, associative ring with unity.

R-Mod, R-mod denote the category of R-modules and finitely generated R-modules, respectively. $\mathrm{Proj}R, \mathrm{proj}R, \mathrm{FP}R$ and $\mathrm{Abs}R$ denote full subcategories of R-Mod consisting of projective, finitely generated projective, finitely presented and absolutely pure modules, respectively.

Definition 8.1.1. Let R be Noetherian. It is called Gorenstein if $\mathrm{id}\, R < \infty$.

In the literature, some authors call such a ring to be Iwanaga-Gorenstein since it is different from the one which is used in Algebraic Geometry. In Algebraic Geometry, a ring is called Gorenstein if for each prime ideal p, $\operatorname{id}_{R_p} R_p < \infty$, see [Bas63]. But in case $\mathrm{Krulldim}(R) < \infty$, this is equivalent to being Iwanaga-Gorenstein ring. In this thesis, we deal with Iwanaga-Gorenstein rings. So when we say 'Gorenstein ring', it is always referred to the Iwanaga version of Gorenstein, see [Iwa80]. In case $\operatorname{id} R \leq n$ for some $n \in \mathbb{N}$, we sometimes call the ring *n-Gorenstein*.

If we consider the class Proj$R$, we already know that Proj$R$ is a precovering class in R-Mod. So every R-module has a projective resolution.

If R is local and Noetherian, then it is semiperfect. So, for any finitely generated module M, its dual will have a projective cover $R^n \to M^*$. This allows to map M to a projective module, $M \to M^{**} \hookrightarrow (R^n)^* \cong R^n$, which will be its projective envelope. This means that over such a ring, the class proj$R$ is covering and enveloping for the subcategory R-mod. So every finitely generated module has a minimal right and left proj-resolution.

Let

$$\ldots \to P_1(M) \to P_0(M) \to M \to 0$$

be a projective resolution of M. Now, suppose that M has a right Proj-resolution (projective resolvent)

$$0 \to M \to P^0(M) \to P^1(M) \to \ldots.$$

By patching these left and right Proj-resolutions of M, we get a complex

$$\cdots \longrightarrow P_1(M) \longrightarrow P_0(M) \longrightarrow P^0(M) \longrightarrow P^1(M) \longrightarrow \cdots.$$
$$P_0(M) \dashrightarrow M \dashrightarrow P^0(M)$$

Renumerating we get:

$$\mathbf{T}: \quad \cdots \longrightarrow P_1(M) \longrightarrow P_0(M) \longrightarrow P_{-1}(M) \longrightarrow P_{-2}(M) \longrightarrow \cdots.$$

This is a complete Proj-resolution of M. For any $P \in$ Proj, Hom$(\mathbf{T}, P)$ may not be exact at some points $i > 0$. So we call an exact sequence $\mathbf{T}$ of projective modules to be *totally acyclic* if it is Hom$(-,$ Proj$)$-exact. The importance of totally acyclic complexes comes from the connection between G- $\dim$ and Gorenstein projective modules. [AB69] defined the Gorenstein dimension, G-dim, for finitely generated modules. [EJ95] extended it for non-finite modules using totally acyclic complexes.

A *complete projective resolution* of M is a diagram $\mathbf{T}_M \xrightarrow{u} \mathbf{P}_M \xrightarrow{v} M$ where $\mathbf{P}_M \xrightarrow{v} M$ is a left projective resolution of M, $\mathbf{T}_M$ is a totally acyclic complex and u_i is bijective for $i \gg 0$. It is known that this resolution for any module M is unique up to homotopy, [Vel06]. Dually, a *complete projective resolvent* of M is a diagram $M \xrightarrow{v} \mathbf{P}^M \xrightarrow{u} \mathbf{T}^M$ where $M \xrightarrow{v} \mathbf{P}^M$ is a right projective resolution of M, $\mathbf{T}^M$ is a totally acyclic complex and u_i is bijective for $i \gg 0$.

Definition 8.1.2. An R-module M is called Gorenstein projective if there is a totally acyclic complex $\mathbf{T}$ such that $M =$ Ker$(P_0 \to P_{-1})$.

If R is coherent and M is finitely generated, M is Gorenstein projective if and only if there is an exact sequence $\mathbf{T}$ of finitely generated projective modules such that $M =$ Ker$(P_0 \to P_{-1})$ and it remains exact under the contravariant functor Hom$(-, R)$.

Proposition 8.1.3. *[Chr00, Proposition 4.1.3] Let R be Noetherian and M be a finitely generated R-module. M is Gorenstein projective if and only if G-dim$(M) = 0$*

An R-module M is called *reduced* if it doesn't have a nonzero projective direct summand. Suppose that M has a projective envelope. Then a projective preenvelope $f : M \to P$ is a projective envelope if and only if P has no nonzero projective direct summand containing the image of f.

8.2 Gorenstein Projective Resolvents

In this section, we prove that over a commutative local n-Gorenstein ring every finitely generated module has a finite right Gorenstein projective resolution. In this section, we assume that R is a commutative and n-Gorenstein ring for some $n \geq 0$.

Proposition 8.2.1. *[EJ11b, Proposition 10.2.12] Let M be an R-module and let*

$$\ldots \to P_i(M) \to \ldots \to P_1(M) \to P_0(M) \to M \to 0$$

be a projective resolution of M. If C_i = Ker$(P_i(M) \to P_{i-1}(M))$ then $C_i \hookrightarrow P_i(M)$ is a projective preenvelope for all $i \geq n$.

Corollary 8.2.2. *Let M and N be R-modules with projective resolutions*

$$\ldots \to P_i(M) \to \ldots \to P_1(M) \to P_0(M) \to M \to 0$$

$$\ldots \to P_i(N) \to \ldots \to P_1(N) \to P_0(N) \to N \to 0.$$

Let $C_i :=$ Ker$(P_i(M) \to P_{i-1}(M))$ and $C_i' :=$ Ker$(P_i(N) \to P_{i-1}(N))$ for each i. If $f : C_i \to C_i'$ is any homomorphism for some $i \geq n$, then there are morphisms $f_j : P_j(M) \to P_j(N)$ for all $j \geq n$ which make commutative the diagrams obtained by the above resolutions.

Proof. By Proposition 8.2.1, $C_i \hookrightarrow P_i(M)$ is a projective preenvelope for all $i \geq n$ while $P_{i+1}(N) \to C_i'$ is a projective precover for all i. Hence if we have a morphism $f : C_i \to C_i'$ for some $i \geq n$ then it gives us morphisms $f_i : P_i(M) \to P_i(N)$ and $f_{i+1} : P_{i+1}(M) \to P_{i+1}(N)$ which commute with differential maps and f. Since these resolutions are exact, we have morphisms $\overline{f}_i : C_{i-1} \to C'_{i-1}$ and $\overline{f}_{i+1} : C_{i+1} \to C'_{i+1}$. The same argument can be repeated from the morphisms $\overline{f}_i$ and $\overline{f}_{i+1}$ if $i-1 \geq n$. So we get morphisms $f_j : P_j(M) \to P_j(N)$ for all $j \geq n$. $\square$

Corollary 8.2.3. *Let R be local. Let M be a finitely generated R-module. If the minimal projective resolution*

$$\ldots \to P_i(M) \to \ldots \to P_1(M) \to P_0(M) \to M \to 0$$

is eventually periodic, then it becomes periodic after n steps.

Proof. Eventually periodic means that there are positive integers m and p such that $P_{t+1}(M) \to P_t(M) \to P_{t-1}(M) \cong P_{t+1+p}(M) \to P_{t+p}(M) \to P_{t-1+p}(M)$ for $t > m$. Suppose that $m > n$ (otherwise it is clear). Consider the minimal resolution of C_{p-1}

$$\ldots \to P_{m+p}(M) \to \ldots \to P_{p+1}(M) \to P_p(M) \to C_{p-1} \to 0.$$

Then by Corollary 8.2.2 with $N = C_{p-1}$ and by the isomorphism $C_{m+1} \to C_{m+1+p}$, we get isomorphisms $P_{i+2}(M) \to P_{i+1}(M) \to P_i(M) \simeq P_{i+2+p}(M) \to P_{i+1+p}(M) \to P_{i+p}(M)$ for all $i \geq n$. $\square$

Proposition 8.2.4. *[EJ11b, Proposition 10.2.15] Let M be a finitely generated R-module with a projective resolvent*

$$0 \to M \to P^0(M) \to P^1(M) \to P^2(M) \to \ldots.$$

Let $D^{i-1} :=$ Ker$(P^i(M) \to P^{i+1}(M))$. Then $P^i \to D^i$ is a projective precover for all $i \geq n-2$. If it is minimal, then D^i is reduced for all $i \geq n-1$.

Corollary 8.2.5. *Let M and N be finitely generated R-modules having projective resolvents*

$$0 \to M \to P^0(M) \to P^1(M) \to P^2(M) \to \ldots$$

$$0 \to N \to P^0(N) \to P^1(N) \to P^2(N) \to .$$

Let $D^{i-1} = Ker(P^i(M) \to P^{i+1}(M))$ *and* $D'^{i-1} := Ker(P^i(N) \to P^{i+1}(N))$*. If we have* $f : D^i \to D'^i$ *for some* $i \geq n-2$*, then it can be completed to morphisms* $f^j : P^j(M) \to P^j(N)$ *for all* $j \geq n-2$*.*

Proof. Using the previous proposition and the same process as in 8.2.2. □

Proposition 8.2.6. *[EJ11b, Theorem 10.2.14] Let* M *be an* R*-module and let*

$$\ldots \to P_i(M) \to \ldots \to P_1(M) \to P_0(M) \to M \to 0$$

be a projective resolution. Then $C_i := Ker(P_i(M) \to P_{i-1}(M))$ *is Gorenstein projective for all* $i \geq n-1$*.*

Proposition 8.2.7. *[EJ11b, Theorem 10.2.16] Let* M *be a finitely generated* R*-module having a projective resolvent*

$$0 \to M \to P^0(M) \to \ldots \to P^1(M) \to P^2(M) \to \ldots.$$

Let $D^i := Ker(P^{i+1}(M) \to P^{i+2}(M))$*, then* D^i *is Gorenstein projective for all* $i \geq n-2$*.*

The next result is proved in [Jør07] and [EJ11b, Corollary 11.8.3]) by different ways. Here, we reprove it since the process used in the proof will be needed later to get a finite Gorenstein projective resolvent in Theorem 8.2.9 for finitely generated modules.

Theorem 8.2.8. *Let* R *be local. Every finitely generated module* M *has a finitely generated Gorenstein projective preenvelope.*

Proof. Let

$$\mathbf{P}: \quad 0 \to M \to P^0 \to P^1 \to \ldots$$

be a projective resolvent of M. Consider its right partial resolution

$$\overline{\mathbf{P}}: \quad 0 \to M \to P^0 \to P^1 \to \ldots \to P^{n-2} \to D \to 0$$

where $D := \text{Ker}(P^{n-1} \to P^n)$. By Proposition 8.2.7, D is a Gorenstein projective module. Take a partial projective resolution of D,

$$\overline{\mathbf{L}}: \quad 0 \to K \to D^0 \to \ldots \to D^{n-2} \to D \to 0.$$

This can be completed to a morphism $\overline{\mathbf{u}} : \overline{\mathbf{P}} \to \overline{\mathbf{L}}$ with $u : M \to K$. It is a Gorenstein projective preenvelope in the stable homotopy category $St(R)$, that is, if G is any Gorenstein projective with partial right projective resolution

$$\mathbf{P}_G: \quad 0 \to G \to P^0(G) \to \ldots \to P^{n-2}(G) \to T \to 0$$

and if there is a morphism $g : M \to G$, then this map can be completed to a morphism $\overline{\mathbf{P}} \to \mathbf{P}_G$ with $s : D \to T$. By using that morphism s, we can get a morphism $\overline{\mathbf{L}} \to \mathbf{P}_G$ with $t : K \to G$. These morphisms $\overline{\mathbf{P}} \to \mathbf{P}_G$ and $\overline{\mathbf{P}} \to \overline{\mathbf{L}} \to \mathbf{P}_G$ are not the same but by some standard calculations it can be seen that their difference $g - t \circ f$ is factorized through $M \to P^0$.

So the claim is to prove that $\alpha := (d, u) : M \to P^0 \oplus K$ is a Gorenstein projective preenvelope, where d is the map $M \to P^0$. $P^0 \oplus K$ is Gorenstein projective since K is Gorenstein projective and P^0 is projective. Now, if $g : M \to G$ is a morphism with a Gorenstein projective module G, by the above argument, there is a map $t : K \to G$ such that $g - t \circ u$ is factorized over d, i.e., there exists $h : P^0 \to G$ with $h \circ d = (g - t \circ u)$. Then $(h, t) : P^0 \oplus K \to G$ is the desired map. □

Theorem 8.2.9. *Let R be local. If M is a finitely generated R-module then M has a finite right Gorenstein projective resolution.*

Proof. Let

$$\overline{\mathbf{P}}: \quad 0 \to M \to P^0 \to P^1 \to \ldots \to P^{n-2} \to D \to 0$$

$$\overline{\mathbf{L}}: \quad 0 \to K \to D^0 \to \ldots \to D^{n-2} \to D \to 0$$

be complexes with the morphism $\overline{\mathbf{u}} : \overline{\mathbf{P}} \to \overline{\mathbf{L}}$ as in the proof of Theorem 8.2.8. Then this gives an exact sequence of complexes

$$0 \to \overline{\mathbf{L}} \to c(\overline{\mathbf{u}}) \to \overline{\mathbf{P}}[1] \to 0$$

where $c(\overline{\mathbf{u}})$, the mapping cone of $\overline{\mathbf{u}}$, is given by:

$$c(\overline{\mathbf{u}}) = 0 \to M \to K \oplus P^0 \to \ldots \to D^{n-2} \oplus D \to D \to 0.$$

Since that short exact sequence is a degreewise split exact sequence of complexes with both $\overline{\mathbf{P}}$ and $\overline{\mathbf{L}}$ being $\mathrm{Hom}(-, \mathrm{Proj})$-exact complexes, $c(\overline{\mathbf{u}})$ is $\mathrm{Hom}(-, \mathrm{Proj})$-exact as well.

There is also an exact sequence

$$0 \to \overline{D} \to c(\overline{\mathbf{u}}) \to c(\widetilde{\mathbf{u}}) \to 0$$

with $\overline{D} = 0 \to D \xrightarrow{\mathrm{id}} D \to 0$ with D in the (n-1)th and (n-2)th place, and zeros everywhere else. Again it is a degreewise short exact sequence of complexes and $\overline{D}$ and $c(\overline{\mathbf{u}})$ are $\mathrm{Hom}(-, \mathrm{Proj})$-exact complexes, so $c(\widetilde{\mathbf{u}})$ is also $\mathrm{Hom}(-, \mathrm{Proj})$-exact,

$$c(\widetilde{\mathbf{u}}) = 0 \to M \to K \oplus P^0 \to D^0 \oplus P^1 \to \ldots \to D^{n-3} \oplus P^{n-2} \to D^{n-2} \to 0.$$

We need to show that $c(\widetilde{\mathbf{u}})$ is a right Gorenstein projective resolution of M. By the proof of Theorem 8.2.8, we already know that $M \xrightarrow{\alpha} K \oplus P^0$ is a Gorenstein projective preenvelope, where $\alpha := (d, u)$. So it is enough to show that $\mathrm{Coker}(\alpha) \to D^0 \oplus P^1$ is a Gorenstein projective preenvelope. Note that the complex

$$0 \to \text{Coker}(\alpha) \to D^0 \oplus P^1 \to \ldots \to D^{n-2} \to 0$$

is also Hom$(-, \text{Proj})$-exact where Coker(α) is finitely generated. So it is a right projective resolution of Coker(α) with finitely generated projectives. By the proof of Theorem 8.2.8 (with D replaced by the zero module in this case) Coker$(\alpha) \to D^0 \oplus P^1$ is a Gorenstein projective preenvelope of Coker(α).

Continuing in this fashion we obtain that the complex $c(\widetilde{\mathbf{u}})$ is Hom$(-, \mathcal{GP})$-exact. Thus $c(\widetilde{\mathbf{u}})$ is a right Gorenstein projective resolution of M. □

8.3 Tate Derived Functors

In this section, we consider three derived functors of Hom by using right projective resolutions. We discuss balance results and show that there exists a long exact sequence connecting them.

Definition 8.3.1. Let $\mathcal{C}, \mathcal{D}$ and $\mathcal{E}$ be abelian categories and $T : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ be an additive functor contravariant in the first variable and covariant in the second. Let $\mathcal{F}$ and $\mathcal{G}$ be classes of objects of $\mathcal{C}$ and $\mathcal{D}$, respectively. Then T is said to be left balanced by $\mathcal{F} \times \mathcal{G}$ if for each object $M \in \mathcal{C}$, there is a $T(-, \mathcal{G})$-exact complex

$$0 \to M \to F^0 \to F^1 \to \ldots$$

with each $F^i \in \mathcal{F}$, and if for every object $N \in \mathcal{D}$, there is a $T(\mathcal{F}, -)$-exact complex

$$\ldots \to G_1 \to G_0 \to N \to 0$$

with $G_i \in \mathcal{G}$.

It is clear from the definition that if $\mathcal{F}$ is a preenveloping and precovering class in R-Mod then Hom$(-,-)$ is left balanced by $\mathcal{F} \times \mathcal{F}$ on R-Mod $\times$ R-Mod. For example, If R is a local Noetherian ring, Hom$(-,-)$ is left balanced by proj$R \times$ Proj$R$ on R-mod $\times$ R-Mod. So for any $M \in R$-mod and $N \in R$-Mod, homologies at n, $H_n(\text{Hom}(\mathbf{P}^M, N)) \cong H_n(\text{Hom}(M, \mathbf{P}_N))$, where $\mathbf{P}^M$ is a deleted right projective resolution of M and $\mathbf{P}_N$ is a deleted left projective resolution of N. We denote it as Ext$_n(M,N)$, left derived functor of Hom$(-,-)$.

Another example of our interest is the class $\mathcal{GP}$ of Gorenstein projective modules. If R is a local n-Gorenstein ring, Hom$(-,-)$ is left balanced by $\mathcal{GP}_{fg} \times \mathcal{GP}$ on R-mod $\times$ R-Mod, where $\mathcal{GP}_{fg}$ is the class of finitely generated Gorenstein projective modules. So we denote by Gext$_n(-,-)$ the left derived functor by means of Gorenstein projective modules.

Let M be an R-module having complete projective resolvent $M \to \mathbf{P}^M \to \mathbf{T}^M$. Since that representation is unique up to homotopy, we may consider the derived functor $\widehat{\text{Ext}}_i^a(M,N) := H_i(\text{Hom}(\mathbf{T}^M, N))$. For an R-module N having complete projective resolution $\mathbf{T}'_N \to \mathbf{P}_N \to N$ we can define another derived functor $\widehat{\text{Ext}}_i^b(M,N) := H_i(\text{Hom}(M, \mathbf{T}'_N))$. Now we show that the two procedures yield the same derived functors.

Proposition 8.3.2. *Let R be local and n-Gorenstein. The derived functors $\widehat{Ext}_i^a(-,-)$ and $\widehat{Ext}_i^b(-,-)$ are equal on R-mod $\times$ R-Mod.*

Proof. Let

$$0 \to M \to P^0(M) \to \ldots \to P^{n-1}(M) \to P^n(M) \to \ldots$$

be a right projective resolution of finitely generated R-module M and

$$\ldots \to P_2(N) \to P_1(N) \to P_0(N) \to N \to 0$$

be a left projective resolution of N. By Proposition 8.2.7, $D = \text{Ker}(P^n(M) \to P^{n+1}(M))$ is Gorenstein projective. So there is a totally acyclic complex $\mathbf{T}^M$ such that $(\mathbf{T}^N)^i = P^i(M)$ for all $i \geq n$ and $D = \text{Ker}((\mathbf{T}^M)^n \to (\mathbf{T}^M)^{n+1})$. By Proposition 8.2.6, $C = \text{Ker}(P_{n-1}(N) \to P_{n-2}(N))$ is Gorenstein projective. Therefore there is a totally acyclic complex $\mathbf{T}'_N$ of C such that $(\mathbf{T}'_N)_i = P_i(N)$ for all $i \geq n$ and $C = \text{Ker}((\mathbf{T}'_N)_{n-1} \to (\mathbf{T}'_N)_{n-2})$. Shortly, we will use $\mathbf{T}$ and $\mathbf{T}'$ instead.

Consider the short exact sequence

$$0 \to C_0 \to P_0(N) \to N \to 0.$$

If we apply the functor $\text{Hom}(\mathbf{T}, -)$, we get a short exact sequence of complexes

$$0 \to \text{Hom}(\mathbf{T}, C_0) \to \text{Hom}(\mathbf{T}, P_0(N)) \to \text{Hom}(\mathbf{T}, N) \to 0.$$

Since $\mathbf{T}$ is a totally acyclic complex, we have an isomorphism $H_i(\text{Hom}(\mathbf{T}, N)) \cong H_{i-1}(\text{Hom}(\mathbf{T}, C_0))$ for any i. Applying the same process for $0 \to C_1 \to P_1(N) \to C_0 \to 0$, we get $H_i(\text{Hom}(\mathbf{T}, N)) \cong H_{i-2}(\text{Hom}(\mathbf{T}, C_1))$. Finally, we obtain $H_i(\text{Hom}(\mathbf{T}, N)) \cong H_{i-n}(\text{Hom}(\mathbf{T}, C))$ where $C_{n-1} = C = \text{Ker}(P_{n-1}(N) \to P_{n-2}(N))$.

By [EEI12, Theorem 3.3], $H_{i-n}(\text{Hom}(\mathbf{T}, C)) \cong H_{i-n}(\text{Hom}(D, \mathbf{T}'))$. Recall that

$$0 \to M \to P^0(M) \to P^1(M) \to \dots P^{n-1}(M) \to D \to 0$$

is a partial right projective resolution of M. Let $F^i := \text{Coker}(P^{i-1}(M) \to P^i(M))$ where $P^{-1} := M$. If we consider the exact sequence $F^{n-2} \to P^{n-1}(M) \to D \to 0$ whose first map is a projective preenvelope map, then the functor $\text{Hom}(-, \mathbf{T}')$ converts it to a short exact sequence of complexes

$$0 \to \text{Hom}(D, \mathbf{T}') \to \text{Hom}(P^{n-1}(M), \mathbf{T}') \to \text{Hom}(F^{n-2}, \mathbf{T}') \to 0.$$

Since $\mathbf{T}'$ is a totally acyclic complex of projectives, we obtain an isomorphism

$H_{i-n}(\mathrm{Hom}(D, \mathbf{T}')) \cong H_{i-n+1}(\mathrm{Hom}(F^{n-2}, \mathbf{T}'))$. Again by applying the same argument, we get $H_{i-n}(\mathrm{Hom}(D, \mathbf{T}')) \cong H_i(\mathrm{Hom}(M, \mathbf{T}'))$, that is, $H_i(\mathrm{Hom}(\mathbf{T}, N)) \cong H_{i-n}(\mathrm{Hom}(\mathbf{T}, C)) \cong H_{i-n}(\mathrm{Hom}(D, \mathbf{T}')) \cong H_i(\mathrm{Hom}(M, \mathbf{T}'))$. $\square$

Over a commutative local n-Gorenstein ring, we denote by $\widehat{\mathrm{Ext}}_i(-,-)$ the derived functors $\widehat{\mathrm{Ext}}_i^a(-,-) = \widehat{\mathrm{Ext}}_i^b(-,-)$ on R-mod $\times$ R-Mod.

Theorem 8.3.3. *Let R be local and n-Gorenstein. Let M be a finitely generated R-module and N be an R-module. Then we have an exact sequence of left derived functors*

$$0 \to \widehat{Ext}_{n-1}(M, N) \to Ext_{n-1}(M, N) \to Gext_{n-1}(M, N) \to \widehat{Ext}_{n-2}(M, N) \to \dots$$

$$\dots \to Gext_0(M, N) \to \widehat{Ext}_{-1}(M, N) \to 0.$$

Proof. Take a right projective resolution of M

$$\mathbf{P} = 0 \to M \to P^0 \to P^1 \to \dots.$$

Since $D := \mathrm{Ker}(P^{n-1} \to P^n)$ is Gorenstein projective, let us consider a partial totally acyclic complex of D

$$\mathbf{L} = 0 \to K \to D^0 \to D^1 \to \dots \to D^{n-2} \to P^{n-1} \to P^n \to \dots.$$

By the above we have a map of complexes $\overline{\mathbf{u}} : \mathbf{P} \to \mathbf{L}$ with u_j as in the proof of Theorem 8.2.9 for $0 \leq j \leq n-2$ and with $u_j = \mathrm{id}_{P^j}$ for all $j \geq n-1$.

Note that the right Gorenstein projective resolution $c(\widetilde{u})$ of M obtained in the proof of

Theorem 8.2.9, is the mapping cone of the morphism

$$\begin{array}{lccccccccccc}
\widetilde{\mathbf{P}}: & 0 & \longrightarrow & M & \xrightarrow{d} & P^0 & \xrightarrow{d} & \cdots & \longrightarrow & P^{n-2} & \longrightarrow & 0\,. \\
\downarrow{\scriptstyle \widetilde{\mathbf{u}}} & & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle u_o} & & & & \downarrow{\scriptstyle u_{n-2}} & & \\
\widetilde{\mathbf{L}}: & 0 & \longrightarrow & K & \xrightarrow{d'} & D^0 & \xrightarrow{d'} & \cdots & \longrightarrow & D^{n-2} & \longrightarrow & 0
\end{array}$$

Now consider the tail complex of $\mathbf{P}$:

$$\mathbf{T}: \qquad \cdots \longrightarrow 0 \longrightarrow P^{n-1} \xrightarrow{-d} P^n \xrightarrow{-d} \cdots$$

with morphisms

$$\begin{array}{lccccccccccccccc}
\widetilde{\mathbf{P}}: & \cdots & \longrightarrow & 0 & \longrightarrow & M & \xrightarrow{d} & P^0 & \xrightarrow{d} & \cdots & \longrightarrow & P^{n-2} & \longrightarrow & 0 & \longrightarrow & \cdots\,. \\
\downarrow{\scriptstyle \widetilde{\mathbf{u}}} & & & & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle u_o} & & & & \downarrow{\scriptstyle u_{n-2}} & & \downarrow & & \\
\widetilde{\mathbf{L}}: & \cdots & \longrightarrow & 0 & \longrightarrow & K & \xrightarrow{d'} & D^0 & \xrightarrow{d'} & \cdots & \longrightarrow & D^{n-2} & \longrightarrow & 0 & \longrightarrow & \cdots \\
\downarrow{\scriptstyle -d'} & & & & & \downarrow & & \downarrow & & & & \downarrow{\scriptstyle -d'} & & \downarrow & & \\
\mathbf{T}: & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & P^{n-1} & \xrightarrow{-d} & P^n & \xrightarrow{-d} & \cdots
\end{array}$$

Note that $-d' \circ \widetilde{\mathbf{u}} = -d$ and $c(-d') = \mathbf{L}[1]$ and $c(d) = \mathbf{P}[1]$. So we have a distinguished triangle

$$c(\widetilde{\mathbf{u}}) \to c(-d' \circ \widetilde{\mathbf{u}}) \to c(d') \to c(\widetilde{\mathbf{u}})[1]$$

which is isomorphic to

$$c(\widetilde{\mathbf{u}}) \to \mathbf{P}[1] \to \mathbf{L}[1] \to c(\widetilde{\mathbf{u}})[1].$$

Applying the Hom$(-, N)$ functor to this distinguished triangle, we get the exact sequence

$$0 \to \widehat{\text{Ext}}_{n-1}(M,N) \to \text{Ext}_{n-1}(M,N) \to \text{Gext}_{n-1}(M,N) \to \widehat{\text{Ext}}_{n-2}(M,N) \to \dots$$

$$\dots \to \text{Gext}_0(M,N) \to \widehat{\text{Ext}}_{-1}(M,N) \to 0.$$

□

REFERENCES

[AR94] Adámek, J. & Rosický, J. (1994). Locally presentable and accessible categories. London Mathematical Society Lecture Note Series, 189. Cambridge University Press, Cambridge.

[AA02] Akinci, K. D. & Alizade, R. (2002). Special precovers in cotorsion theories. (English summary) Proc. Edinb. Math. Soc. (2) 45, no. 2, 411-420.

[TLS00] Alonso Tarrío, L.; Jeremías López, A. & Souto Salorio, M. J. (2000). Localization in categories of complexes and unbounded resolutions. (English summary) Canad. J. Math. 52, no. 2, 225-247.

[Aus67] Auslander, M. (1967). Anneaux de Gorenstein, et torsion en algèbre commutative, Secrétariat mathématique, Paris, Séminaire d'Algèbre commutative dirigé par Pierre Samuel, 1966/67. Texte rédigé, d'après des exposés de Maurice Auslander, par Marquerite Mangeney, Christian Peskine et Lucien Szpiro. École Normale Supérieure de Jeunes Filles.

[AB69] Auslander, M. & Bridger, M. (1969). Stable module theory. Memoirs of the American Mathematical Society, No. 94, American Mathematical Society, Providence, R.I.

[AS80] Auslander, M. & Smalø, S. O. (1980). Preprojective modules over Artin algebras. J. Algebra 66, no. 1, 61-122.

[AH86] Avramov, L.L. & Halperin, S. (1986). Through the looking glass: A dictionary between rational homotopy theory and local algebra, Lecture Notes in Mathematics 1183 (Springer, Berlin) l-27.

[AM02] Avramov, L. L. & Martsinkovsky, A. (2002). Absolute, relative, and Tate cohomology of modules of finite Gorenstein dimension. Proc. London Math. Soc. (3) 85, no. 2, 393-440.

[Bas60] Bass, H. (1960). Finitistic dimension and a homological generalization of semi-primary rings. Trans. Amer. Math. Soc. 95, 466-488.

[Bas62] Bass, H. (1962). Injective dimension in Noetherian rings. Trans. Amer. Math. Soc. 102, 18–29.

[Bas63] Bass, H. (1963). On the ubiquity of Gorenstein rings. Math. Z., 82, 8–28.

[Bek00] Beke, T. (2000). Sheafifiable homotopy model categories. Math. Proc. Camb. Phil. Soc. 129, no. 3, 447–475.

[Bel00] Beligiannis, A. (2000). Relative Homological Algebra and purity in triangulated categories. Journal of Algebra 227, 268-361.

[BR07] Beligiannis, A. & Reiten, I. (2007). Homological and homotopical aspects of torsion theories. (English summary) Mem. Amer. Math. Soc. 188, no. 883.

[BBE01] Bican, L.; El Bashir, R. & Enochs, E. (2001). All modules have flat covers. (English summary) Bull. London Math. Soc. 33, no. 4, 385-390.

[Bor94] Borceux, F. (1994). Handbook of categorical algebra. 1. Basic category theory. Encyclopedia of Mathematics and its Applications, 50. Cambridge University Press, Cambridge.

[Buc55] Buchsbaum, D. (1955). Exact categories and Duality. Trans. Amer. Math. Soc. 80, 1-34.

[Buc59] Buchsbaum, D. (1959). A note on homology in categories. Ann. of Math. (2), 69, 66-74.

[Büh10] Bühler, T. (2010). Exact categories. (English summary) Expo. Math. 28, no. 1, 169.

[CE56] Cartan, H. & Eilenberg, S. (1956). Homological algebra. Princeton University Press, Princeton, N. J.

[Chr00] Christensen, L. W. (2000). Gorenstein Dimensions. Lecture Notes in Mathematics, 1747. Springer-Verlag, Berlin.

[CH02] Christensen, J. D. & Hovey, M. (2002). Quillen model structures for relative homological algebra, Math. Proc. Camb. Phil. Soc. 133(2) , 261-293.

[Coh59] Cohn, P. M. (1959). On the free product of associative rings. Math. Z. 71, 380–398.

[Con00] Conrad, B. (2000). Grothendieck duality and base change. Lecture Notes in Mathematics, Vol. 1750, Springer-Verlag.

[Craw94] Crawley-Boevey, W. (1994). Locally finitely presented additive categories. Comm. Algebra 22, 1641-1674.

[CPT10] Crivei, S.; Prest, M. & Torrecillas, B. (2010). Covers in finitely accessible categories. Proc. Amer. Math. Soc. 138, 1213-1221.

[DS95] Dwyer, W. G. & Spaliński, J. (1995). Homotopy theories and model categories. Handbook of algebraic topology, 73-126, North-Holland, Amsterdam.

[EC53] Eckmann, B. & Schopf, A. (1953). Über injektive Moduln. Arch. Math. (Basel) 4, 75-78.

[EM65] Eilenberg, S. & Moore, J. C. (1965). Foundations of relative homological algebra. Mem. Amer. Math. Soc. No. 55, 39 pp.

[EH00] Eisenbud, D., & Harris, J. (2000). The geometry of schemes. New York: Springer-Verlag.

[Ekl77] Eklof, P. C. (1977). Homological algebra and set theory. Trans. Amer. Math. Soc. 227, 207-225.

[ET01] Eklof, P. C. & Trlifaj, J. (2001). How to make Ext vanish. (English summary) Bull. London Math. Soc. 33, no. 1, 41-51.

[Bas06] El Bashir, R. (2006). Covers and directed colimits. Algebr. Represent. Theory 9, no. 5, 423-430.

[Eno63] Enochs, E. E. (1963). Torsion free covering modules. Proc. Amer. Math. Soc. 14 884-889.

[Eno71] Enochs, E. E. (1971). Torsion free covering modules. II. Arch. Math. (Basel) 22, 37-52.

[Eno81] Enochs, E. E. (1981). Injective and flat covers, envelopes and resolvents. Israel J. Math. 39, no. 3, 189-209.

[Eno12] Enochs, E. E. (2012). Shortening filtrations. Sci. China Math. 55, no. 4, 687-693.

[EE05] Enochs, E. & Estrada, S. (2005). Relative homological algebra in the category of quasi-coherent sheaves. Adv. Math. 194, no. 2, 284-295.

[EEGO04] Enochs, E.; Estrada, S.; García Rozas, J. R. & Oyonarte, L. (2004). Flat covers in the category of quasi-coherent sheaves over the projective line. Comm. Algebra 32, no. 4, 1497-1508.

[EEI12] Enochs, E.; Estrada, S. & Iacob, A. (2012). Balance with unbounded complexes. Bull. London Math. Soc., 44, no. 3, 439-442.

[EEIO14] Enochs, E., Estrada, S. ; Iacob, A. & Odabaşı, S. (2014). Gorenstein projective resolvents. Submitted.

[EEO14] Enochs, E.E.; Estrada, E. & Odabaşi, S. (2014). Pure injective and absolutely pure sheaves. P. Edinburgh Math. Soc. Accepted.

[EIJ07] Enochs, E. E.; Iacob, A. & Jenda, O. M. G. (2007). Closure under transfinite extensions. (English summary) Illinois J. Math. 51, no. 2, 561-569.

[EJ95] Enochs, E. E. & Jenda, O. M. G. (1995). Gorenstein injective and projective modules. Math. Z. 220, no. 4, 611-633.

[EJ00] Enochs, E. E. & Jenda, O. M. G. (2000). Relative homological algebra. de Gruyter Expositions in Mathematics, 30. Walter de Gruyter & Co., Berlin.

[EJ11b] Enochs, E. E. & Jenda, O. M. G. (2011). Relative homological algebra. Volume 2. de Gruyter Expositions in Mathematics, 30. Walter de Gruyter GmbH & Co. KG, Berlin.

[EJL04] Enochs, E. E.; Jenda, O. M. G. & Lopez-Ramos, J. A. (2004). The existence of Gorenstein flat covers. Math. Scand. 94, no. 1, 46-62.

[EJT93] Enochs, E. E.; Jenda, O. M. G. & Torrecillas, B. (1993). Gorenstein flat modules, Nanjing Daxue Xuebao Shuxue Bannian Kan 10, 1-9.

[EO01] Enochs, E. E. & Oyonarte, L. (2001). Flat covers and cotorsion envelopes of sheaves. Proc. Amer. Math. Soc. 130, no. 5, 1285-1292.

[EO02] Enochs, E. & Oyonarte, L. (2002). Covers, envelopes and cotorsion theories, Nova Science Publishers, Inc., New York.

[EGO14] Estrada, S.; Gillespie, J & Odabaşı, S. (2014). Pure exact structures and the pure derived category of a scheme. Submitted.

[EGPT12] Estrada, S.; Guil Asensio, P.; Prest, M. & Trlifaj, J. (2012). Model category structures arising from Drinfeld vector bundles. (English summary) Adv. Math. 231, no. 3-4, 1417-1438.

[ES12] Estrada, S. & Saorín, M. Locally finitely presented categories with no flat objects. Forum Math. DOI:10.1515/forum-2012-0054.

[Fox76] Fox, T. F. (1976). Purity in locally presentable monoidal categories. J. Pure Appl. Algebra 8, no. 3, 261-265.

[Fre64] Freyd, P. (1964). Abelian categories. An introduction to the theory of functors. Harper's Series in Modern Mathematics Harper & Row, Publishers, New York.

[GU71] Gabriel, P. & Ulmer, F. (1971). Lokal präsentierbare Kategorien. (German) Lecture Notes in Mathematics, Vol. 221. Springer-Verlag, Berlin-New York.

[GP64] Popescu, N. & Gabriel, P. (1964) Caractérisation des catégories abéliennes avec générateurs et limites inductives exactes. (French) C. R. Acad. Sci. Paris 258, 4188-4190.

[GR99] García Rozas, J. R. (1999). Covers and envelopes in the category of complexes of modules. Chapman & Hall/CRC Research Notes in Mathematics, 407. Chapman & Hall/CRC, Boca Raton, FL.

[GM93] García, J. L. & Martínez Hernández, J. (1993). Purity through Gabriel's functor rings. Third Week on Algebra and Algebraic Geometry (SAGA III) (Puerto de la Cruz, 1992). Bull. Soc. Math. Belg. Sér. A 45, no. 1-2, 137-152.

[GM03] Gelfand, S. I. & Manin, Y. I. (2003). Methods of homological algebra. Second edition. Springer Monographs in Mathematics. Springer-Verlag, Berlin.

[Gill04] Gillespie, J. (2004). The flat model structure on Ch(R). Trans. Amer. Math. Soc., 356(8), 3369-3390.

[Gill06] Gillespie, J. (2006). The flat model structure on complexes of sheaves. Trans. Amer. Math. Soc., 358(7), 2855-2874.

[Gill07] Gillespie, J. (2007). Kaplansky classes and derived categories. Math. Z., 257, 811-843.

[Gill08] Gillespie, J. (2008). Cotorsion pairs and degreewise homological model structures. Homology, Homotopy Appl. 10, no. 1, 283-304.

[Gil11] Gillespie, J. (2011). Model structures on exact categories. (English summary) J. Pure Appl. Algebra 215 , no. 12, 2892-2902.

[Gil13] Gillespie, J. Exact model structures and recollements, arXiv:1310.7530.

[Gil14] Gillespie, J. The derived category with respect to a generator, arXiv: 1406.2514.

[GW10] Görtz, U. & Wedhorn, T. (2010). Algebraic geometry I. Schemes with examples and exercises. Advanced Lectures in Mathematics. Vieweg + Teubner, Wiesbaden.

[Gro57] Grothendieck, A. (1957). Sur quelques points d'algèbre homologique. (French) Tôhoku Math. J. (2) 9, 119-221.

[GNPR10] Guillén, F.; Navarro, V.; Pascual, P. & Roig, A. (2010). A Cartan-Eilenberg approach to homotopical algebra. J. Pure Appl. Algebra 214, no. 2, 140-164.

[Har66] Harris, M. E. (1966). Some results on coherent rings. Proc. Amer. Math. Soc. 17, 474–479.

[Hart66] Hartshorne, R. (1966). Residues and duality. Lecture Notes in Mathematics 20, Springer-Verlag, Berlin-New York.

[Hart77] Hartshorne, R. (1977). Algebraic geometry. New York: Springer-Verlag.

[Hel58] Heller, A. (1958). Homological algebra in abelian categories. Ann. of Math. (2) 68 484-525.

[HS79] Herrlich, H. & Strecker, G. E. (1979). Category theory. An introduction. Second edition. Sigma Series in Pure Mathematics, 1. Heldermann Verlag, Berlin.

[Her03] Herzog, I. (2003). Pure-injective envelopes. J. Algebra Appl. 2, no. 4, 397-402.

[Hil90] Hilbert, D. (1890). Über die Theorie der Algebraischen Formen. Math. Annalen 36, 473-534.

[Hoc45] Hochschild, G. (1945). On the cohomology groups of an associative algebra. Annals of Math. 46, 58-67.

[Hol04] Holm, H. (2004). Gorenstein homological dimensions. J. Pure Appl. Algebra, 189, no. 1-3, 167-193.

[HJ08] Holm, H. & Jørgensen, P. (2008). Covers, precovers and purity. Illinois J. Math. 52, no. 2, 691-703

[HJ11] Holm, H. & Jørgensen, P. (2011). Rings without a Gorenstein analogue of the Govorov-Lazard theorem. Q. J. Math. 62, no. 4, 977-988.

[HS13] Hosseini, E., & Salarian, S. (2013). A cotorsion theory in the homotopy category of flat quasi-coherent sheaves. Proc. Amer. Math. Soc. 141, no. 3, 753-762.

[Hov01] Hovey, M. (2001). Model category structures on chain complexes of sheaves, *Trans. Amer. Math. Soc. 353*, 2441-2457.

[Hov99] Hovey, M. (1999). Model categories. Mathematical Surveys and Monographs, 63. American Mathematical Society, Providence, RI.

[Hov02] Hovey, M. (2002). Cotorsion pairs, model category structures, and representation theory. Math. Z. 241, no. 3, 553-592.

[Iac05] Iacob, A. (2005). Balance in generalized Tate cohomology. Comm. Algebra, 33, no. 6, 2009-2024.

[Iwa80] Iwanaga, Y. (1980). On rings with finite self-injective dimension II. Tsukuba J. Math., Vol. 4, 107-113.

[Jør07] Jørgensen, P. (2007). Existence of Gorenstein projective resolutions and Tate cohomology. J. Eur. Math. Soc. 9, no. 1, 59–76.

[Kra12] Krause, H. (2012). Approximations and adjoints in homotopy categories, Math. Ann. 353 (3), 765–781.

[Lip09] Lipman, J. (2009) Notes on derived functors and Grothendieck duality. (English summary) Foundations of Grothendieck duality for diagrams of schemes, 1259, Lecture Notes in Math., 1960, Springer, Berlin.

[Mac95] Mac Lane, S. (1995). Homology. Reprint of the 1975 edition. Classics in Mathematics. Springer-Verlag, Berlin.

[Mad67] Maddox, B. (1967). Absolutely pure modules. Proc. Amer. Math. Soc. 18, 155–158.

[Maş00] Maşiek, V. (2000). Gorenstein dimension and torsion of modules over commutative Noetherian rings. Special issue in honor of Robin Hartshorne. Comm. Algebra, 28, no. 12, 5783-5811.

[Meg70] Megibben, C. (1970). Absolutely pure modules. Proc. Amer. Math. Soc. 26, 561–566.

[Mit65] Mitchell, B. (1965). Theory of categories. Pure and Applied Mathematics, Vol. XVII Academic Press, New York-London.

[Mur1] Murfet, D. An adjunction for modules over projective schemes. Available at `http://www.therisingsea.org`

[Mur2] Murfet, D. Modules over a scheme. Available at `http://www.therisingsea.org`

[Mur07] Murfet, D. (2007). The Mock homotopy category of projectives and Grothendieck duality, PhD thesis, Australian National University, available at `http://www.therisingsea.org/thesis.pdf`.

[MS11] Murfet, D & Salarian, S. (2011). Totally acyclic complexes over noetherian schemes. Adv. Math. 226, 1096–1133.

[Nee90] Neeman, A. (1990). The derived category of an exact category. J. Algebra 135, no. 2, 388-394.

[Nee01] Neeman, A. (2001). Triangulated categories. Annals of Mathematics Studies, 148. Princeton University Press, Princeton, NJ.

[Nee08] Neeman, A. (2008). The homotopy category of flat modules, and Grothendieck duality. (English summary) Invent. Math. 174, no. 2, 255-308.

[Oda14] Odabaşı, S. (2014). Locally torsion-free quasi-coherent sheaves. J. Pure Appl. Algebra 218, no. 9, 1760-1770.

[Oda14a] Odabaşı, S. (2014). Cotorsion pairs and Cartan-Eilenberg categories. Submitted.

[Pas12] Pascual, P. (2012). Some remarks on Cartan-Eilenberg categories. Collect. Math. 63, no. 2, 203-216.

[Pin05] Pinzon, K. R. (2005). Absolutely pure modules. University of Kentucky Doctoral Dissertations. Available at `http://uknowledge.uky.edu`

[Pin08] Pinzon, K. R. (2008). Absolutely pure covers. Comm. Algebra, 36, no. 6, 2186–2194.

[Pre09] Prest, M. (2009). Purity, spectra and localisation. Encyclopedia of Mathematics and its Applications, 121, Cambridge University Press, Cambridge.

[PR04] Prest, M. & Ralph, A. Locally finitely presented categories of sheaves of modules. Available at
`www.maths.manchester.ac.uk/~mprest/shvfp101.ps`

[Prü23] Prüfer, H. (1923). Untersuchungen über die Zerlegbarkeit der abzählbaren primären abelschen Gruppen. Math. Z. 17, 35–61.

[Qui67] Quillen, D. G. (1967). Homotopical algebra. Lecture Notes in Mathematics, No. 43, Springer-Verlag, Berlin-New York.

[Qui74] Quillen, D. (1974). Higher K-theory for categories with exact sequences. New developments in topology (Proc. Sympos. Algebraic Topology, Oxford, 1972), pp. 95-103. London Math. Soc. Lecture Note Ser., No. 11, Cambridge Univ. Press, London.

[RS98] Rada, J. & Saorín, M. (1998). Rings characterized by (pre)envelopes and (pre)covers of their modules, Comm. Algebra 26, 899-912.

[SRV00] Rada, J.; Saorín, M. & del Valle, A. (2000). Reflective subcategories. (English summary) Glasg. Math. J. 42, no. 1, 97-113.

[Rot79] Rotman, J. J. (1979). An introduction to Homological Algebra. Academic Press.

[Rum10] Rump, W. (2010). Flat covers in abelian and in non-abelian categories, *Advances in Math., 225*, Issue 3, 1589-1615.

[Sal79] Salce, L. (1979). Cotorsion theories for abelian groups. Symposia Mathematica, Vol. XXIII (Conf. Abelian Groups and their Relationship to the Theory of Modules, INDAM, Rome, 1977), pp. 1132, Academic Press, London-New York.

[SS11] Saorín, M.; Šťovíček, J. (2011). On exact categories and applications to triangulated adjoints and model structures. (English summary) Adv. Math. 228, no. 2, 968-1007.

[Spa88] Spaltenstein, N. (1988). Resolutions of unbounded complexes. Compositio Math. 65, no. 2, 121-154.

[Ste67] Stenström, B. (1967). Pure submodules. Ark. Mat. 7, 159–171.

[Ste68] B. Stenström, B. (1968). Purity in functor categories. J. Algebra. 8, 352–361.

[Ste70] Stenström, B. (1970). Coherent rings and FP-injective modules. J. London Math. Soc. 2, 323–329.

[Ste75] Stenström, B. (1975). Rings of Quotients. GMW **217**, Springer-Verlag, New York-Heidelberg.

[Sto13] Šťovíček, J. (2013). Exact model categories, approximation theory, and cohomology of quasi-coherent sheaves. Advances in representation theory of algebras, 297-367, EMS Ser. Congr. Rep., Eur. Math. Soc., Zrich.

[Sto13a] Šťovíček, J. (2013). Deconstructibility and the Hill lemma in Grothendieck categories. (English summary) Forum Math. 25, no. 1, 193-219

[Vel06] Veliche, G. (2006). Gorenstein projective dimension for complexes. Trans. Amer. Math. Soc. 358, 1257–1283.

[Ver96] Verdier, J. L. (1996). Des catgories drives des catgories abliennes. (French. French summary) [On derived categories of abelian categories] With a preface by Luc Illusie. Edited and with a note by Georges Maltsiniotis. Astrisque No. 239, (1997).

[Xu96] Xu, J. (1996). Flat covers of modules. Lecture Notes in Mathematics, Vol. 1634, Springer-Verlag, .

[YL14] Yang, G. & Liang, L. (2014). All modules have Gorenstein flat precovers. Comm. Algebra 42, no. 7, 3078-3085.

[Yon60] Yoneda, N. (1960). On Ext and exact sequences. J. Fac. Sci. Univ. Tokyo Sect. I 8, 507-576.

INDEX