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FINITE DIMENSIONAL MESH ALGEBRAS AND THEIR HOMOLOGICAL PROPERTIES

ÁLGEBRAS DE MALLA FINITO DIMENSIONALES Y SUS PROPIEDADES HOMOLÓGICAS

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INFORMA: Que la Tesis Doctoral titulada "FINITE DIMENSIONAL MESH ALGEBRAS AND THEIR HOMOLOGICAL PROPERTIES" ha sido realizada por Dña. Estefanía Andreu Juan bajo la inmediata dirección y supervisión de D. Manuel Saorín Castaño, y que el Departamento de Matemáticas ha dado su conformidad para que sea presentada ante la Comisión de Doctorado.

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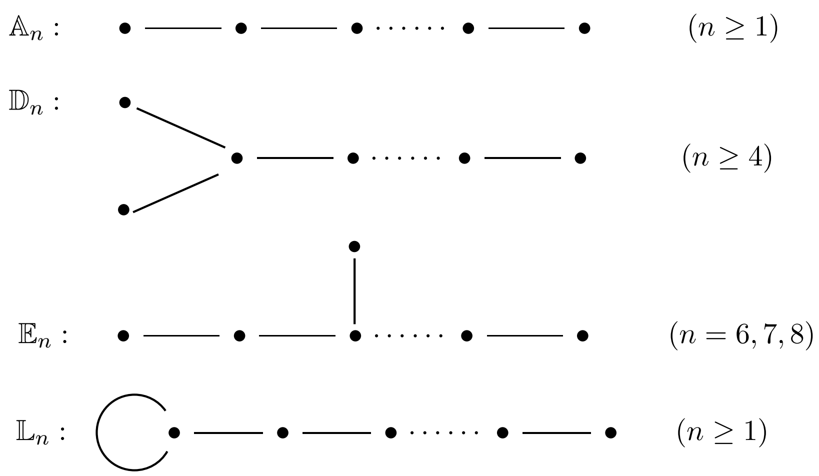
Introducción

La noción de álgebra preproyectiva apareció por primera vez a finales de los años 70 de la mano de Gelfand y Ponomarev en su trabajo [42] sobre teoría de representación de quivers finitos sin ciclos orientados. Esencialmente, dado un diagrama finito no orientado Δ, el álgebra preproyectiva asociada, usualmente denotada por $P(\Delta)$, se define a partir del álgebra de caminos obtenida al reemplazar cada uno de los ejes de Δ por un par de flechas opuestas e identificando, posteriormente, todos y cada uno de los ciclos de longitud 2 en un mismo vértice. Derivado de su estudio surgieron las primeras aplicaciones en problemas de clasificación de álgebras de tipo finito ([22],[23]) y se establecieron conexiones con otros tipos de álgebras como las envolventes universales o las de conglomerado ([40], [41]). Además, al margen de dicha rama, las álgebras preproyectivas están presentes en muchas y distintas disciplinas de las Matemáticas. Entre otras, cabe destacar que desempeñan un papel especial en la aproximación de haces perversos a grupos cuánticos de Lusztig ([60], [61]) y que han servido como herramienta para abordar problemas de geometría diferencial [53] o estudiar deformaciones no conmutativas de singularidades de Klein [21].

Un hecho sobradamente conocido a día de hoy es que, cuando el cuerpo base K es algebraicamente cerrado, el álgebra $P(\Delta)$ es finito dimensional únicamente en el caso en que Δ es uno de los diagramas de Dynkin generalizados $\mathbb{A}_n$, $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$ ó $\mathbb{L}_n$:



Históricamente, en el ámbito de las álgebras finito dimensionales, el operador sigicia, Ω_Λ, que asocia a cada Λ-módulo M el núcleo de su cubierta proyectiva $P_\Lambda(M) \longrightarrow M$, ha supuesto una herramienta muy útil tanto para establecer relaciones entre los distintos módulos como para obtener información sobre la estructura de la propia álgebra. Por ejemplo, el hecho de que en un álgebra Λ todos sus módulos simples sean Ω_Λ-periódicos implica que ésta es autoinyectiva, es decir, que las clases de módulos proyectivos e inyectivos coinciden. Por módulo Ω_Λ- periódico entendemos cualquier Λ-módulo M para el que existe un entero $r > 0$ tal que $\Omega^r_\Lambda(M)$ es isomorfo a M. En tal caso, llamamos periodo de M al menor de los enteros positivos satisfaciendo la propiedad anterior. Dicho operador pasa de ser útil a resultar fundamental en el estudio de un tipo concreto de álgebras finito dimensionales contenidas en la clase de las autoinyectivas y conocidas como periódicas. Se dice que un álgebra Λ que es periódica cuando lo es como módulo sobre su álgebra envolvente $\Lambda^e := \Lambda \otimes \Lambda^{op}$, o equivalentemente, vista como Λ-bimódulo. Una propiedad común que poseen las álgebras preproyectivas de dimensión finita es que, salvo en el caso de $\Delta = \mathbb{A}_1$,$P(\Delta)$ es Ω- periódica de periodo a lo sumo 6.

En toda disciplina matemática resulta natural y habitual tratar de generalizar conceptos de manera que se conserven las propiedades más relevantes. Con ese objetivo, K. Erdmann y A. Skowroński introdujeron en [29] una nueva clase de álgebras asociadas a Diagramas de Dynkin que contiene a las preproyecitvas finito dimensionales y que han suscitado un gran interés en los últimos tiempos en el marco general de las álgebras de dimensión finita. Las llamaron álgebras de malla m-fold y constituyen precisamente la clase de álgebras autoinyectivas Λ para las que Ω^3_Λ permuta las clases de isomorfía de los módulos simples. Tomando como referencia el citado trabajo, si Δ es uno de los de quivers de Dynkin $\mathbb{A}_r$, $\mathbb{D}_r$ ó $\mathbb{E}_n$ $(n = 6, 7, 8)$, un álgebra de malla m-fold de tipo Δ es un cociente B/G del álgebra de malla $B = B(\Delta)$ de un quiver de translación $\mathbb{Z}\Delta$ por un grupo G de automorfismos débilmente admisibles de $\mathbb{Z}\Delta$. Posteriormente, gracias a un resultado de A. Dugas ([25] Teorema 3.1), se supo que las álgebras de malla m-fold son ciertamente las álgebras de malla de quivers de translación que resultan ser finito dimensionales. También es conocido que, además de a las álgebras preproyecivas finito dimensionales, esta clase contiene a las álgebras estables de Auslander de todas las álgebras autoinyectivas estándar de representación finita ([25]) y a las álgebras de Auslander-Reiten estables de varias singularidades de hipersuperficie. Además, por [14] [Sección 6] se sabe que estas álgebras son periódicas.

A finales de los años 90, en su trabajo *Triangulated categories and geometry*, M. Kontsevich definió el concepto de dimensión de Calabi-Yau en el contexto de las catergorías trianguladas $Hom-$finitas, es decir, sobre K-categorías trianguladas para las que el espacio vectorial de morfismos entre dos objetos cualesquiera tiene dimensión finita. Bajo estas hipótesis, decimos que la K-categoría $\mathcal{T}$ con funtor de suspensión $\sum : \mathcal{T} \longrightarrow \mathcal{T}$ es de Calabi-Yau si existe un cierto número natural n de manera que $\sum^n$ es un funtor de Serre, o lo que es lo mismo, si existe un número natural n tal que $DHom_\mathcal{T}(X, -)$ y $Hom_\mathcal{T}(-, \sum^n X)$ son naturalmente isomorfos como funtores cohomológicos $\mathcal{T}^{op} \longrightarrow K - \text{mod}$. De ser así, al menor número natural m satisfaciendo que $\sum^m$ es un funtor de Serre se le llama dimensión de Calabi-Yau de $\mathcal{T}$ y escribimos CY-dim$(\mathcal{T}) = m$. Las categorías trianguladas de Calabi-Yau están presentes en muchos campos de la Matemática y la Física Teórica. En La Teoría de la Representación de álgebras, la noción desempeña un papel muy significativo en el estudio de las álgebras y las categorías de conglomerado ([55]).

Cuando Λ es un álgebra autoinyectiva finito dimensional y ${}_{\Lambda}\underline{\text{mod}}$ es su categoría de módulos estable, entonces el funtor sicigia $\Omega_\Lambda : {}_{\Lambda}\underline{\text{mod}} \longrightarrow {}_{\Lambda}\underline{\text{mod}}$ es una equivalencia de categorías y ${}_{\Lambda}\underline{\text{mod}}$ tiene estructura de categoría triangulada considerando Ω_Λ^{-1} como funtor de suspensión. La condición Calabi-Yau en esta categoría ha sido profundamente estudiada (ver p.e. [12], [25], [28], [35], [50], [51], ...) e igualmente se ha relacionado con el concepto de álgebra de Calabi-Yau Frobenius, tal y como fue definido por C. Eu and T. Schedler en [35]: se dice que el álgebra Λ es de Calabi-Yau Frobenius cuando $\Omega_{\Lambda^e}^{-r-1}(\Lambda)$ es isomorfa a $D(\Lambda) = Hom_K(\Lambda, K)$ como Λ-bimódulos, para algún entero $r \geq 0$. Nótese que si el álgebra Λ es Calabi-Yau Frobenius, entonces se tiene que ${}_{\Lambda}\underline{\text{mod}}$ es de Calabi-Yau. Además, habida cuenta de que $\Omega_{\Lambda^e}^{-r-1} \otimes_\Lambda ?$ y Ω_Λ^{-r-1} son naturalmente isomorfos como funtores ${}_{\Lambda}\underline{\text{mod}} \longrightarrow {}_{\Lambda}\underline{\text{mod}}$, la dimensión de Calabi-Yau de la categoría ${}_{\Lambda}\underline{\text{mod}}$ es menor o igual que el menor de los enteros $r \geq 0$ tal que $\Omega_{\Lambda^e}^{-r-1}(\Lambda) \cong D(\Lambda)$ como Λ-bimódulos, número que llamaremos en lo que sigue dimensión de Calabi-Yau Frobenius de Λ y denotaremos por CYF-dim(Λ). Sin embargo, en general, no se sabía si éstos dos números coinciden.

Posterior a las investigaciones realizadas en [12] y [28], las álgebras de Auslander estables Λ de un álgebra autoinyectiva de tipo de representación finito cuya categoría de módulos estable ${}_{\Lambda}\underline{\text{mod}}$ es Calabi-Yau quedaron completamente determinadas en [25] y [51] en función de su tipo. Inspirado por el trabajo de C. Riedtmann en [66], H. Asashiba define en [6] el tipo del álgebra de Auslander estable de un álgebra autoinyectiva de tipo de representación finito, que es invariante salvo equivalencia derivada, como la terna (Δ, f, t) donde Δ es el diagrama de Dynkin asociado, f es la frecuencia y t es el orden de torsión. En el primero de los trabajos mencionados ([25]), A. Dugas identifica tales álgebras cuando t=1 ó 3 y en muchos de, pero no todos, los casos con $t = 2$. Los casos restantes cuando $t = 2$ han sido determinados muy recientemente por S.O.Ivanov-Y.V.Volkow en [51]. Por tanto, teniendo en cuenta de que las álgebras de malla finito dimensionales son autoinyectivas, de manera natural surgen entonces una serie de cuestiones:

Cuestión 1: ¿Cuáles son las álgebras de malla finito dimensionales cuya categoría de módulos estable es de Calabi-Yau? ¿Cuáles son Calabi-Yau Frobenius? Y finalmente, ¿ cuál es la relación entre ambas dimensiones?

En términos de bimódulos, un álgebra básica finito dimensional Λ es autoinyectiva justamente cuando existe un isomorfismo de Λ-bimódulos entre $D(\Lambda)$ y el bimódulo torcido ${}_1\Lambda_\eta$, para algún automorfismo η de Λ. Dicho automorfismo está unívocamente determinado, salvo automorfismo interior, y recibe el nombre de automorfismo de Nakayama de Λ. Según esto, el problema de decidir cuándo Λ es Calabi-Yau Frobenius forma parte de un problema más general que consiste en determinar bajo qué condiciones $\Omega_{\Lambda^e}^r(\Lambda)$ es isomorfa a un bimódulo torcido ${}_1\Lambda_\varphi$, para algún automorfismo φ de Λ, que quedará entonces determinado salvo automorfismo interior. Por un resultado debido a Green-Snachall-Solberg ([44]), esta última condición sobre un álgebra finito dimensional fuerza a la misma a ser autoinyectiva. Es más, observemos que cuando φ es la identidad, o un automorfismo interior, obtenemos precisamente la definición de álgebra periódica. Determinar las álgebras autoinyectivas que son periódicas es una cuestión, a día de hoy, ampliamente abierta. No obstante, hay un número considerable de trabajos en la literatura al respecto en donde varias álgebras periódicas, entre ellas las álgebras de malla finito dimensionales, han sido identificadas (véase p.e. [14], [24], [29]). Sin embargo, aun incluso conociendo que un álgebra es periódica, el cálculo explícito de su periodo resulta, habitualmente, una tarea dura y complicada. En el contexto de las álgebras de malla finito dimensionales se tienen resultados sólo en un muy pocos casos. Más concretamente, de los artículos [11], [32] y [67] sabemos que el periodo es 6 para todas las álgebras preprojectivas de diagramas de Dynking generalizados siempre que $\mathrm{Char}(K) \neq 2$. Asimismo, en [25], se obtiene el periodo del álgebra de Auslander estable de un álgebra autoinyectiva de tipo de representación finito de tipo (Δ, f, t) igual a $(\mathbb{D}_4, f, 3)$, $(\mathbb{D}_n, f, 2)$ con $n > 4$ y $f > 1$ impar, ó $(\mathbb{E}_6, f, 2)$. De nuevo parece natural plantearse la siguiente cuestión:

Cuestión 2: ¿Cuál es exactamente el periodo de un álgebra de malla finito dimensional?

Otro de los problemas que ha llamado especialmente la atención el ámbito de las álgebras autoinyectivas finito dimensionales consiste en caracterizar aquellas que son simétricas o débilmente simétricas. Decimos que un álgebra Λ es simétrica cuando es isomorfa a $D(\Lambda)$ como Λ-bimódulo. Ésto último equivale a decir que el funtor de Nakayama $DHom_\Lambda(-, \Lambda) \cong D(\Lambda) \otimes_\Lambda - : \Lambda - \mathrm{Mod} \to \Lambda - \mathrm{Mod}$ es naturalmente isomorfo al funtor identidad. Si debilitamos la condición sobre dicho funtor a que únicamente conserve las clases de isomorfía de los módulos simples obtenemos precisamente la definición de álgebra débilmente simétrica. Dirigimos nuestra mirada entonces a responder las siguientes preguntas:

Cuestión 3: ¿Qué álgebras de malla finito dimensionales son débilmente simétricas? ¿Cuáles de ellas son a su vez simétricas?

Desde que fuera introducida por G. Hochschild en 1945 en su trabajo [47], la teoría de (co)homología que recibe su propio nombre ha sido extensamente estudiada teniendo una gran influencia, entre otros, en el campo de las álgebras finito dimensionales. Si Λ es un álgebra finito dimensional, para cada $i \geq 0$, llamamos i-ésimo grupo de cohomología de Hochschild al K-espacio vectorial $HH^i(\Lambda) := \mathrm{Ext}^i_{\Lambda^e}(\Lambda, \Lambda)$. Ocurre entonces que, junto con el producto de Yoneda, el K-espacio vectorial $\oplus_{i \geq 0} HH^i(\Lambda)$ admite estructura de K-álgebra graduada-conmutativa comúnmente conocida como anillo de cohomología de Hochschild de Λ, usualmente denotado por $HH^*(\Lambda)$. Los grados más bajos de este anillo tienen interpretaciones estrechamente relacionadas con estructuras clásicas del álgebra y también de la geometría algebraica. De hecho, $HH^0(\Lambda)$ y $HH^1(\Lambda)$ coinciden con el centro y el espacio de derivaciones exteriores del álgebra, respectivamente, mientras que $HH^2(\Lambda)$ controla su teoría de deformación: si éste es cero, entonces el álgebra Λ resulta ser rígida. En lo que respecta a su estructura multiplicativa, el anillo de cohomología $HH^*(\Lambda)$ de un álgebra autoinyectiva y finito dimensional resulta tener un gran interés en conexión con el estudio de variedades de módulos y con cuestiones sobre su relación con el álgebra de Yoneda de Λ. Ésta se define como el álgebra graduada $E(\Lambda) = \mathrm{Ext}^*_\Lambda(\Lambda/J, \Lambda/J)$ donde $J = J(\Lambda)$ denota el radical de Jacobson de Λ. De hecho, inspirados por la teoría de representación modular de grupos finitos donde, entre otros, Carlson ([18], [19]) y Benson [10] desarrollaron la teoría de variedades de módulos, Snashall y Solberg ([68], véase también [27]) iniciaron el estudio de variedades de módulos sobre álgebras finito dimensionales arbitrarias, reemplazando el anillo de grupo de cohomología $HH^*(G, K)$ por el anillo de cohomología de Hochschild $HH^*(\Lambda)$ del álgebra en cuestión. Para la nueva teoría, generalmente se requiere que Λ sea autoinyectiva y que satisfaga algunas condiciones de generación finita, las cuales siempre se da cuando Λ es periódica. Sin embargo, poco más sabe acerca del anillo de cohomología de Hochschild $HH^*(\Lambda)$ de un álgebra finito dimensional Λ salvo que, siempre y cuando Λ sea periódica, existe un isomorfismo de álgebras $HH^*(\Lambda)/\mathcal{N} \cong K[x]$ donde $\mathcal{N}$ denota el ideal generado por los elementos nilpotentes y x es un elemento homogéneo de $HH^*(\Lambda)$ cuyo grado coincide precisamente con el periodo de Λ. Esta propiedad aparece por primera vez enunciada explícitamente en ([44], Proposición 1.1). Sin embargo se trata de una aplicación directa con $A = \Lambda \otimes \Lambda^{op}$ y $M = \Lambda$, cuando Λ es periódica, de un resultado más general debido a Carlson [17]. Dicho resultado nos dice que cuando $A = KG$ es un álgebra de grupo siendo G finito, que es el prototipo de álgebras autoinyectivas finito dimensionales, y M es un A-módulo periódico de periodo q, entonces $\mathrm{Ext}^*_A(M, M)/\mathcal{N} \cong K[x]$ donde $x \in \mathrm{Ext}^q_A(M.M)$ y N es el ideal de $\mathrm{Ext}^*_A(M, M)$ generado por los elementos nilpotentes. De hecho, el resultado es igualmente válido para cualquier álgebra autoinyectiva finito dimensional.

Las cuestiones anteriores sugieren que encontrar patrones sobre el comportamiento de las componentes homogéneas de $HH^*(\Lambda)$ con respecto al producto de Yoneda, en casos particulares donde la estructura multiplicativa de $HH^*(\Lambda)$ sea computable, puede ayudar a dar algunas claves acerca de cómo abordarlas. En lo que respecta a las álgebras preprojectivas finito dimensionales, la estructura del anillo de cohomología de Hochschild es conocida en el caso de tipo $\mathbb{A}_n$ sobre un cuerpo de caracterítica arbitraria ([30], [31]) y en los casos $\mathbb{D}_n$ y $\mathbb{E}_6$ cuando el cuerpo tiene característica cero. Basándonos en el concepto de tipo extendido (Δ, m, t) de un álgebra de malla finito dimensional, definido en el Capítulo 2 de esta memoria, resulta que las álgebras preproyectivas anteriores son, ciertamente, las álgebras de malla de tipo extendido $(\Delta, 1, 1)$ donde $\Delta = \mathbb{A}_n$, $\mathbb{D}_n$ ó $\mathbb{E}_6$. Un paso hacia adelante en esta dirección sería considerar el caso en que Λ es un álgebra de malla de tipo $(\Delta, 1, t)$ con $t > 1$, las cuáles se corresponden con las conocidas en tiempos modernos como álgebras preproyectivas generalizadas. En esta tesis abordaremos el caso $\Delta = \mathbb{A}_n$.

Cuestión 4: ¿Cómo son los grupos de cohomología de Hochschild $HH^i(\Lambda)$ de un álgebra preproyectiva genereralizada Λ de tipo $\mathbb{A}_n$? ¿Cuál es la estructura multiplicativa del anillo de cohomología asociado $HH^*(\Lambda)$?

Esta monografía se articula en 6 capítulos relacionados entre sí y que proporcionan la respuesta a cada una de las preguntas anteriormente formuladas.

Capítulo 1

Este primer capítulo, que bien podría considerarse un capítulo preliminar, constituye la base fundamental para desarrollar los contenidos propios del estudio en lo que se refiere a la primera parte de la tesis. Situados en el contexto general de las álgebras con suficientes idempotentes, los conceptos principales que se manejan son los de álgebra graduada pseudo-Frobenius y sus correspondientes forma y automorfismo de Nakayama, todos ellos introducidos en la Sección 1.3 de este trabajo.

Diremos que un álgebra graduada con suficientes idempotentes, débilmente básica y localmente finito dimensional es pseudo-Frobenius si los objetos proyectivos finitamente generados y los inyectivos finitamente cogenerados coinciden en su categoría de módulos graduados. Entre otros resultados, cabe destacar la caracterización que presentamos de dichas álgebras (Teorema 1.3.2). Las nociones pertinentes pueden encontrarse en dicha sección.

TEOREMA. Sea $A = \oplus_{h\in H} A_h$ un álgebra graduada débilmente básica con suficientes idempotentes. Consideremos las siguientes afirmaciones:

1. Las categorías de A-módulos graduados por la izquierda y por la derecha, $A - Gr$ y $Gr - A$, son Frobenius.
2. $D({}_AA)$ y $D(A_A)$ son A-módulos graduados proyectivos.
3. A es pseudo-Frobenius.
4. Existe una forma graduada de Nakayama $(-,-) : B \times B \longrightarrow K$.

Entonces se verifica la siguiente cadena de implicaciones:

$$1) \Longrightarrow 2) \Longrightarrow 3) \Longleftrightarrow 4).$$

Cuando A es localmente acotada graduada, se tiene además que 4) $\Longrightarrow$ 2). Finalmente, si A es localmente graduada Noetheriana, las cuatro afirmaciones son equivalentes.

Cuando el álgebra satisface la condición 1 del teorema anterior se dice que es Quasi-Frobenius. Como se puede intuir, las álgebras pseudo-Frobenius son, en el marco de las álgebras graduadas con suficientes idempotentes, o equivalentemente, de las K–categorías graduadas, el concepto análogo al que representan las álgebras autoinyectivas finito dimensionales en el contexto de las álgebras asociativas unitarias. Sería entonces natural preguntarse acerca de la existencia de un automorfismo del álgebra A que jugase un papel similar al del automorfismo de Nakayama para álgebras finito dimensionales autoinyectivas. En el siguiente resultado garantizamos dicha existencia (Corolario 1.3.6):

COROLARIO. Sea $A = \oplus_{h\in H} A_h$ un álgebra graduada pseudo-Frobenius y sea $(e_i)_{i\in I}$ una familia distinguida débilmente básica de idempotentes ortogonales. Si A es localmente acotada(graduada), entonces se satisfacen las siguientes condiciones:

1. Existe un automorfismo (no graduado) de álgebras $\eta : A \longrightarrow A$, que permuta los idempotentes e_i y conserva los elementos homogéneos tal que ${}_1A_\eta$ es isomorfa a $D(A)$ como A-bimódulos no graduados.

2. Si la aplicación grado $\mathbf{h} : I \longrightarrow H$ asociada a la forma de Nakayama $(-,-) : A \times A \longrightarrow K$ toma un valor constante h, entonces η se puede elegir para que sea graduado y tal que $D(A)$ sea isomorfo a ${}_1A_\eta[h]$ como A-bimódulos graduados.

El automorfismo η recibe el nombre de automorfismo de Nakayama de A.

A pesar de que no aportamos ideas genuinas en el proceso de pasar del contexto de las álgebras asociativas unitarias a las graduadas con suficientes idempotentes conviene señalar que, hasta donde sabemos, el concepto de álgebra pseudo-Forbenius y sus asociados, como la forma de Nakayama y el automorfismo de Nakayama, no han sido desarrollados anteriormente y resultan claves para el resto de este trabajo.

En la Sección 1.4 revisamos la teoría de cubrimientos desde el punto de vista de las álgebras graduadas con suficientes idempotentes con especial énfasis en el caso particular en el que el funtor cubrimiento es del tipo $F : A \longrightarrow A/G$ donde A es un álgebra graduada con suficientes idempotentes y G es un grupo de automorfismos de A de grado 0 que permuta los idempotentes e_i. En la segunda parte de dicha sección estudiamos bajo qué condiciones podemos garantizar la conservación de la condición pseudo-Frobenius via el funtor de cubrimiento. Como se enuncia a continuación (Proposición 1.4.3), ésto siempre ocurre cuando la forma de Nakayama asociada al álgebra A en cuestión es lo que llamamos G-invariante, es decir, cuando se satisface que $(a^g, b^g) = (a, b)$ para cualesquiera $a, b \in A$ y $g \in G$.

PROPOSICIÓN. Sea $A = \oplus_{h \in H} A_h$ un álgebra graduada localmente acotada y básica (débilmente escindida), con $(e_i)_{i \in I}$ como familia distinguida de idempotentes ortogonales homogéneos, y sea G un grupo que actúa sobre A como automorfismos graduados que permutan los e_i y que actúa libremente sobre los objetos. Supongamos que A es pseudo-Frobenius graduada y que admite una forma de Nakayama G-invariante $(-,-) : A \times A \longrightarrow K$. Entonces $\Lambda = A/G$ es un álgebra graduada localmente acotada (débilmente escindida) y pseudo-Frobenius cuya forma graduada de Nakayama viene inducida por $(-,-)$.

De hecho, aprovechamos para finalizar la sección mostrando que, bajo las hipótesis de la proposición anterior, el automorfismo de Nakayama η de A induce a su vez el autormofismo de Nakayama $\bar{\eta}$ de A/G (Corolario 1.4.5).

Capítulo 2

El segundo capítulo de esta memoria está dedicado a profundizar y obtener información clave sobre algunos aspectos del álgebra de malla de un diagrama de Dynkin como es su automorfismo de Nakayama. Si bien tal álgebra no es el objeto principal de estudio de esta tesis, nuestro interés en la misma reside en que el hecho de que toda álgebra de malla finito dimensional proviene en cierto sentido de un álgebra de este tipo.

Al inicio de este capítulo, concretamente en la Sección 2.2, recordamos la definición del quiver de traslación $\mathbb{Z}\Delta$ de un diagrama de Dynkin Δ. A continuación, en la Subsección 2.3.1, introducimos la noción de álgebra de malla $B = B(\Delta)$. Dicha álgebra se define como el álgebra cociente $K\mathbb{Z}\Delta/I$, donde I es el ideal homogéneo generado por unas relaciones conocidas como relaciones de malla. Asimismo, presentamos una breve lista de propiedades esenciales y bien conocidas de B (Proposición 2.3.1). Además, cabe destacar el hecho de que el álgebra B es un álgebra graduada con suficientes idempotentes que resulta ser pseudo-Frobenius.

Seguidamente, en la Subsección 2.3.2, mostramos la lista de las álgebras de malla finito dimensionales e introducimos la definición de tipo extendido para tales álgebras que juega un papel fundamental en este trabajo. De hecho, todos los resultados principales del Capítulo 3 de esta memoria sobre álgebras de malla finito dimensionales están dados en términos de su tipo extendido. Dicha noción está basada en el hecho de que cada álgebra de malla finito dimensional Λ es isomorfa a un álgebra de órbitas B/G donde B es el álgebra de malla asociada a un cierto diagrama de Dynkin Δ y G es un grupo de automorfismos débilmente admisibles de $\mathbb{Z}\Delta$ vistos como automorfismos de B. Es más, es bien conocido el hecho de que Λ admite como cubierta de Galois a su correspondiente de malla algebra B. Con la idea de simplificar algunos de los cálculos , finalizamos la Sección 2.3 modificando ligeramente las relaciones de malla orginales. Básicamente la idea consiste en convertir cada una de las relaciones de malla, que inicialmente es una suma, en una diferencia de caminos.

El resultado principal de este capítulo, que aparece en la Sección 2.4, es el Teorema 2.4.2 donde definimos explícitamente, para cualquier elección de (Δ, G), un automorfismo graduado de Nakayama η de B verificando la propiedad de ser G-invariante. Aquí G-invariante significa que η conmuta con los elementos de G. Como consecuencia de este resultado se deriva una fórmula precisa para un automorfismo graduado de Nakayama de cualquier álgera de malla finito dimensional.

He aquí el Teorema mencionado:

TEOREMA. Sea Δ un quiver de Dynkin y sea $G =< \varphi >$ un grupo de automorfismos débilmente admisibles de $\mathbb{Z}\Delta$. Si η es el automorfismo graduado de B que actúa como la permutación de Nakayama ν sobre los vértices y como se indica en la siguiente lista sobre las flechas, entonces η es un automorfismo de Nakayama de B tal que $\eta \circ g = g \circ \eta$, para todo $g \in G$.

1. Cuando $\Delta = \mathbb{A}_n$ y φ es arbitrario, $\eta(\alpha) = \nu(\alpha)$ para todo $\alpha \in (\mathbb{Z}\Delta)_1$
2. Cuando $\Delta = \mathbb{D}_{n+1}$:
   (a) Si $n + 1 \geq 4$ y $\varphi = \tau^m$ entonces:
      i. $\eta(\alpha) = -\nu(\alpha)$, siempre y cuando $\alpha : (k, i) \longrightarrow (k, i + 1)$ sea una flecha hacia arriba con $i \in \{2, ..., n - 1\}$.
      ii. $\eta(\alpha) = \nu(\alpha)$, siempre y cuando $\alpha : (k, i) \longrightarrow (k+1, i-1)$ sea una flecha hacia abajo con $i \in \{3, ..., n\}$.

   iii. $\eta(\varepsilon_i) = (-1)^i \nu(\varepsilon_i)$, para la flecha $\varepsilon_i : (k,2) \longrightarrow (k,i)$ $(i = 0,1)$,
   iv. $\eta(\varepsilon'_i) = (-1)^{i+1} \nu(\varepsilon'_i)$, para la flecha $\varepsilon'_i : (k,i) \longrightarrow (k+1,2)$ $(i = 0,1)$.

(b) Si $n+1 > 4$ y $\varphi = \rho\tau^m$ entonces:
   i. $\eta(\alpha) = -\nu(\alpha)$, siempre y cuando α sea una flecha hacia arriba como las anteriores o $\alpha : (k,i) \longrightarrow (k+1,i-1)$ sea una flecha hacia abajo como las anteriores tal que $k \equiv -1 \pmod m$.
   ii. $\eta(\alpha) = \nu(\alpha)$, siempre y cuando $\alpha : (k,i) \longrightarrow (k+1,i-1)$ sea una flecha hacia abajo tal que $k \not\equiv -1 \pmod m$
   iii. Para las flechas restantes, si q y r son el cociente y el resto que resultan al dividir k entre m, entonces
   $\eta(\varepsilon_i) = (-1)^{q+i} \nu(\varepsilon_i)$ $(i = 0,1)$.
   $\eta(\varepsilon'_i) = (-1)^{q+i+1} \nu(\varepsilon'_i)$, cuando $r \neq m-1$, y $\eta(\varepsilon'_i) = (-1)^{q+i} \nu(\varepsilon'_i)$ en otro caso.

(c) Si $n+1 = 4$ y $\varphi = \rho\tau^m$ (véase el convenio 2.3.7), entonces:
   i. $\eta(\varepsilon_i) = \nu(\epsilon_i)$, siempre y cuando $\epsilon_i : (k,2) \to (k,i)$ $(i = 0,1,3)$
   ii. $\eta(\varepsilon'_i) = -\nu(\varepsilon'_i)$, siempre y cuando $\epsilon'_i : (k,i) \to (k+1,2)$ $(i = 0,1,3)$.

3. Cuando $\Delta = \mathbb{E}_6$:

(a) Si $\varphi = \tau^m$ entonces:
   i. $\eta(\alpha) = \nu(\alpha)$ y $\eta(\alpha') = -\nu(\alpha')$, donde $\alpha : (k,1) \to (k,2)$ y $\alpha' : (k,2) \to (k+1,1)$.
   ii. $\eta(\beta) = \nu(\beta)$ y $\eta(\beta') = -\nu(\beta')$, donde $\beta : (k,2) \to (k,3)$ y $\beta' : (k,3) \to (k+1,2)$.
   iii. $\eta(\gamma) = \nu(\gamma)$ y $\eta(\gamma') = -\nu(\gamma')$, donde $\gamma : (k,3) \to (k,4)$ y $\gamma' : (k,4) \to (k+1,3)$.
   iv. $\eta(\delta) = -\nu(\delta)$ y $\eta(\delta') = \nu(\delta')$, donde $\delta : (k,4) \to (k,5)$ y $\delta' : (k,5) \to (k+1,4)$.
   v. $\eta(\varepsilon) = -\nu(\varepsilon)$ y $\eta(\varepsilon') = \nu(\varepsilon')$, donde $\varepsilon : (k,3) \to (k,0)$ y $\varepsilon' : (k,0) \to (k+1,3)$.

(b) Si $\varphi = \rho\tau^m$, (k,i) es el origen de la flecha considerada, q y r son el cociente y el resto que resultan al dividir k entre m, entonces:
   i. $\eta(\alpha) = \nu(\alpha)$.
   ii. $\eta(\alpha') = -\nu(\alpha')$.
   iii. $\eta(\beta) = (-1)^q \nu(\beta)$
   iv. $\eta(\beta') = (-1)^{q+1} \nu(\beta')$
   v. $\eta(\gamma) = (-1)^q \nu(\gamma)$
   vi. $\eta(\gamma') = \nu(\gamma')$, donde ó q es impar y $r \neq m-1$ ó q es par y $r = m-1$, y $\eta(\gamma') = -\nu(\gamma')$ en otro caso.
   vii. $\eta(\delta) = -\nu(\delta)$
   viii. $\eta(\delta') = \nu(\delta')$.

ix. $\eta(\varepsilon) = -\nu(\varepsilon)$
x. $\eta(\varepsilon^{'}) = -\nu(\varepsilon^{'})$, cuando $r = m-1$, and $\eta(\varepsilon^{'}) = \nu(\varepsilon^{'})$ en otro caso.

4. Cuando $\Delta = \mathbb{E}_7$, $\varphi = \tau^m$, y entonces:
   i $\eta(a)$ está definida como en 3.(a) para cualquier flecha a contenida en la copia de $\mathbb{E}_6$.
   ii $\eta(\zeta) = \nu(\zeta)$ y $\eta(\zeta^{'}) = -\nu(\zeta^{'})$, donde $\zeta : (k,5) \to (k,6)$ y $\zeta^{'} : (k,6) \to (k+1,5)$.
5. Cuando $\Delta = \mathbb{E}_8$, $\varphi = \tau^m$, y entonces:
   i $\eta(a)$ está definida como en 4 para cualquier flecha a contenida en la copia de $\mathbb{E}_7$.
   ii $\eta(\theta) = \nu(\theta)$ y $\eta(\theta^{'}) = -\nu(\theta^{'})$, donde $\theta : (k,6) \to (k,7)$ y $\theta^{'} : (k,7) \to (k+1,6)$.

Capítulo 3:

Los resultados de este capítulo versan sobre las preguntas 1, 2 y 3. Comenzamos el capítulo presentando dos resultados claves. El primero de ellos (Lema 3.2.1) sirve para determinar cuándo dos automorfismos graduados G-invariantes de un álgebra de malla B inducen, salvo conjugación, el mismo automorfismo del álgebra de malla finito dimensional $\Lambda = B/G$. El segundo (Proposición 3.2.2) identifica el subrupo H de los enteros s tales que el automorfismo y la permutación de Nakayama de Λ, $\bar{\eta}$ y $\bar{\nu}$ respectivamente, coinciden, salvo automorfismo interior, en su s-ésima potencia. Es decir, H está formado por los $s \in \mathbb{Z}$ tales que $\bar{\eta}^s\bar{\nu}^{-s}$ es un automorfismo interior de Λ. El subgrupo H resulta crucial en todas y cada una de las preguntas que se abordan en este capítulo.

Con los dos resultados previos como herramientas principales, pasamos a la Sección 3.3 dedicada exclusivamente a responder la pregunta 3. El único teorema de dicha sección identifica completamente las álgebras de malla finito dimensionales que son débilmente simétricas o simétricas:

TEOREMA. Sea Λ un álgebra de malla m-fold de tipo extendido (Δ, m, t), siendo c_Δ el número de Coxeter de Λ. Si Λ es débilmente simétrica, entonces $t = 1$ ó $t = 2$ y, cuando $\text{char}(K) = 2$ ó $\Delta = \mathbb{A}_r$, tal álgebra es además simétrica. Es más, se verifican las siguientes afirmaciones:

1. Cuando $t = 1$, Λ es débilmente simétrica si, y sólo se, Δ es $\mathbb{D}_{2r}$, $\mathbb{E}_7$ ó $\mathbb{E}_8$ y m es divisor de $\frac{c_\Delta}{2} - 1$. Cuando $\text{char}(K) \neq 2$, tal álgebra es simétrica si, y sólo si, m es par.
2. Cuando $t = 2$ y $\Delta \neq \mathbb{A}_{2n}$, Λ es débilmente simétrica si, y sólo si, m divide a $\frac{c_\Delta}{2} - 1$ y, además, el cociente de la división es impar, en el caso $\Delta = \mathbb{A}_{2n-1}$, y par, en el caso $\Delta = \mathbb{D}_{2r}$. Cuando $\text{char}(K) \neq 2$, tal álgebra es simétrica si, y sólo si, $\Delta = \mathbb{A}_{2n-1}$ ó m es impar.
3. Cuando $(\Delta, m, t) = (\mathbb{A}_{2n}, m, 2)$, i.e. $\Lambda = \mathbb{L}_n^{(m)}$, el álgebra es (débilmente) simétrica si, y sólo si, $2m-1$ divide a $2n-1$.

La Sección 3.4 está destinada a la determinación del periodo y la dimensión de Calabi-Yau de un álgebra de malla m-fold. En primer lugar, Subsección 3.4.1, calculamos la parte inicial de una resolución proyectiva minimal y G-invariante de B como B-bimódulo graduado (Proposición). En particular, probamos un hecho ciertamente relevante para nuestros propósitos y es precisamente que $\Omega^3_{B^e}(B)$ es siempre isomorfa a ${}_\mu B_1$ para un cierto automorfismo graduado μ de B que está en el centralizador de G y cuya fórmula describimos explícitamente (Proposición 3.4.3). Como consecuencia de la G-invarianza de los resultados anteriores obtenemos que el automorfismo inducido $\bar{\mu}$ de $\Lambda = B/G$ satisface la propiedad de que $\Omega^3_{\Lambda^e}(\Lambda) \cong {}_{\bar{\mu}}\Lambda_1$. Esta propiedad resulta ser fundamental tanto en el desarrollo de los contenidos como en la obtención de los resultados.

Seguidamente, introducimos el concepto de automorfismo establemente interior que se trata, en general, de una condición más débil que la de automorfismo interior. Concretamente, decimos que un automorfismo σ de Λ es establemente interior si el funtor ${}_\sigma(-) \cong_\sigma \Lambda_1 \otimes_\Lambda - : \Lambda - \underline{\text{mod}} \longrightarrow \Lambda - \underline{\text{mod}}$ es naturalmente isomorfo al funtor identidad. Sin embargo, en lo que se refiere a las álgebras de malla m-fold, ambos conceptos coinciden en una gran número de casos. Por ejemplo, ésto siempre sucede cuando Λ es un álgebra de longitud de Loewy mayor o igual que 4 (véase el Lema 3.4.6).

Las respuestas a las cuestiones 1 y 2 planteadas en este capítulo aparecen en las Subsecciones 3.4.3 y 3.4.4. En la primera de ellas se calcula de forma explícita el periodo de cualquier álgebra de malla m-fold Λ, es decir, el menor de los enteros positivos r tales que $\Omega^3_{\Lambda^e}(\Lambda)$ es isormofa a Λ como Λ-bimódulo. Distinguimos primeramente el caso $\Delta = \mathbb{A}_2$ (Proposición 3.4.8), donde el álgebra tiene longitud de Loewy 2:

> PROPOSICIÓN. Sea Λ un álgebra autoinyectiva y conexa con longitud de Loewy 2. Entonces se verifican las siguientes afirmaciones:
>
> 1. Si $\text{char}(K) = 2$ ó $\Lambda = \mathbb{A}_2^{(m)}$, i.e. $|Q_0|$ es par, entonces el periodo de Λ es $|Q_0|$.
> 2. Si $\text{char}(K) \neq 2$ y $\Lambda = \mathbb{L}_1^{(m)}$, i.e. $|Q_0|$ es impar, entonces el periodo de Λ es $2|Q_0|$.

Los restantes casos se recogen en el Teorema 3.4.12, cuyo enunciado para característica $\neq 2$ dice:

> TEOREMA. Sea Λ un álgebra de malla m-fold de tipo extendido (Δ, m, t), donde $\Delta \neq \mathbb{A}_1, \mathbb{A}_2$, denotemos por $\pi = \pi(\Lambda)$ al periodo de Λ y, para cada entero positivo k, denotemos por $O_2(k)$ al mayor número natural r tal que 2^r divide a k. Cuando $\text{char}(K) \neq 2$, el periodo de Λ viene dado como sigue:
>
> 1. Si $t = 1$ entonces:
>   (a) Cuando Δ es $\mathbb{A}_r$, $\mathbb{D}_{2r-1}$ ó $\mathbb{E}_6$, el periodo es $\pi = \frac{6m}{\text{mcd}(m, c_\Delta)}$.
>   (b) Cuando Δ es $\mathbb{D}_{2r}$, $\mathbb{E}_7$ ó $\mathbb{E}_8$, el periodo es $\pi = \frac{3m}{\text{mcd}(m, \frac{c_\Delta}{2})}$, cuando m es par, y $\pi = \frac{6m}{\text{mcd}(m, \frac{c_\Delta}{2})}$, cuando m es impar.

2. Si $t = 2$ entonces:
   (a) Cuando Δ es $\mathbb{A}_{2n-1}$, $\mathbb{D}_{2r-1}$ ó $\mathbb{E}_6$, el periodo es $\frac{6m}{\text{mcd}(2m, m+\frac{c_\Delta}{2})}$, cuando $O_2(m) \neq O_2(\frac{c_\Delta}{2})$, y $\pi = \frac{12m}{\text{mcd}(2m, m+\frac{c_\Delta}{2})}$ en otro caso.
   (b) Cuando $\Delta = \mathbb{D}_{2r}$, el periodo es $\frac{6m}{\text{mcd}(2m, \frac{c_\Delta}{2})} = \frac{6m}{\text{mcd}(2m, 2r-1)}$.
   (c) Cuando $\Delta = \mathbb{A}_{2n}$, i.e. $\Lambda = \mathbb{L}_n^{(m)}$, el periodo es $\pi = \frac{6(2m-1)}{\text{mcd}(2m-1, 2n+1)}$
3. Si $t = 3$ entonces $\pi = 3m$, cuando m es par, y $6m$, cuando m es impar.

En la parte final del capítulo, donde se estudian las dimensiones de Calabi-Yau, resaltamos dos resultados. Por una parte, combinando las siguientes proposiciones, se identifica la relación precisa entre la dimensión estable de Calabi-Yau y la dimensión de Calabi-Yau Frobenius de un álgebra de malla m-fold mostrando que ambas dimensiones pueden diferir cuando $\Delta = \mathbb{A}_2$ pero siempre coinciden cuando $\Delta \neq \mathbb{A}_r$, para $r = 1, 2$.

PROPOSICIÓN. Sea Λ un álgebra autoinyectiva con longitud de Loewy 2. Entonces Λ es siempre establemente Calabi-Yau y se verifican las siguientes igualdades:

1. Si $\text{char}(K) = 2$ ó $\Lambda = \mathbb{A}_2^{(m)}$, i.e. $|Q_0|$ es par, entonces $CY - \dim(\Lambda) = CYF - \dim(\Lambda) = 0$.
2. Si $\text{char}(K) \neq 2$ y $\Lambda = \mathbb{L}_1^{(m)}$, i.e., $|Q_0|$ es impar, entonces $CY - \dim(\Lambda) = 0$ y $CYF - \dim(\Lambda) = 2m - 1 = |Q_0|$.

PROPOSICIÓN. Sea Λ un álgebra de malla m-fold de tipo de Dynkin Δ distinto de $\mathbb{A}_r$, para $r = 1, 2, 3$. Entonces Λ es establemente Calabi-Yau si, y sólo si, es Calabi-Yau Frobenius. En tal caso, se verifica la igualdad $CY - \dim(\Lambda) = CYF - \dim(\Lambda)$.

Para finalizar el capítulo y lo que sería la primera parte de esta tesis proporcionamos, para las álgebras de malla m-fold, un criterio para determinar cuándo son establemente Calabi-Yau, junto con una identificación en tal caso de la dimensión estable de Calabi-Yau. El caso en el que K tiene característica 2 se trata en el Corolario 3.4.18. Cuando $\text{char}(K) \neq 2$, el resultado dice:

TEOREMA. Supongamos que $\text{char}(K) \neq 2$ y que Λ es un álgebra de malla m-fold de tipo extendido (Δ, m, t), donde $\Delta \neq \mathbb{A}_1, \mathbb{A}_2$. Adoptamos la convención de que si a, b, k son tres enteros fijados, entonces $au \equiv b \pmod{k}$ significa que u es el menor entero positivo satisfaciendo la congruencia. El álgebra es Calabi-Yau Frobenius si, y sólo si, es establemente Calabi-Yau. Es más, tenemos que $CYF - \dim(\Lambda) = CY - \dim(\Lambda)$ y se verifican las siguientes afirmaciones:

1. Si $t = 1$ entonces
   (a) Cuando Δ es $\mathbb{A}_r$, $\mathbb{D}_{2r-1}$ ó $\mathbb{E}_6$, el álgebra es establemente Calabi-Yau si, y sólo si, $\text{mcd}(m, c_\Delta) = 1$. Entonces $CY - \dim(\Lambda) = 6u + 2$, donde $c_\Delta u \equiv -1 \pmod{m}$.

(b) Cuando Δ es $\mathbb{D}_{2r}$, $\mathbb{E}_7$ ó $\mathbb{E}_8$, el álgebra es establemente Calabi-Yau si, y sólo si, $\mathrm{mcd}(m, \frac{c_\Delta}{2}) = 1$. Entonces:

i. $CY - \dim(\Lambda) = 3u + 2$, donde $\frac{c_\Delta}{2}u \equiv -1 \pmod m$, siempre y cuando m sea par;

ii. $CY - \dim(\Lambda) = 6u + 2$, donde $c_\Delta u \equiv -1 \pmod m$, siempre y cuando m sea impar;

2. Si $t = 2$ entonces

(a) Cuando Δ es $\mathbb{A}_{2n-1}$, $\mathbb{D}_{2r-1}$ or $\mathbb{E}_6$, el álgebra es establemente Calabi-Yau si, y sólo si, $\mathrm{mcd}(2m, m + \frac{c_\Delta}{2}) = 1$. Entonces $CY - \dim(\Lambda) = 3u + 2$, donde $(m + \frac{c_\Delta}{2})u \equiv -1 \pmod{2m}$.

(b) Cuando $\Delta = \mathbb{D}_{2r}$, el álgebra es establemente Calabi-Yau si, y sólo si, $\mathrm{mcd}(m, 2r - 1) = 1$ y m es impar. Entonces $CY - \dim(\Lambda) = 3u + 2$, donde $(2r - 1)u \equiv -1 \pmod{2m}$.

(c) Cuando $\Delta = \mathbb{A}_{2n}$, el álgebra es establemente Calabi-Yau si, y sólo si, $\mathrm{mcd}(2m - 1, 2n + 1) = 1$. Entonces $CY - \dim(\Lambda) = 6u - 1$, donde $(m + n)(2u - 1) \equiv -1 \pmod{2m - 1}$

3. Si $t = 3$ entonces el álgebra no es establemente Calabi-Yau.

Los resultados de los Capítulos 1, 2 y 3 aparecen en [5].

Capítulo 4:

Los tres últimos capítulos de esta memoria abordan la cuestión 4 mencionada más arriba, es decir, el estudio del anillo de cohomología de Hochschild de las álgebras de malla m-fold de tipo extendido $(\mathbb{A}_{2n}, 1, 2)$ y $(\mathbb{A}_{2n-1}, 1, 2)$ conocidas como álgebras preproyectivas generalizadas $\mathbb{L}_n$ y $\mathbb{B}_n$, respectivamente. Comienza con un cuarto capítulo introductorio donde se presentan brevemente los conceptos y resultados necesarios para abordar tal problema. Así, en una primera sección se recuerda una noción importante como es el producto de Yoneda de extensiones y se define el concepto fundamental de anillo de cohomología de Hochschild de un álgebra Λ que se denotará por $HH^*(\Lambda)$. En un principio, Λ será un álgebra considerada sobre un anillo conmutativo R, que además, se supone proyectiva como R-módulo. Más adelante, se asumirá que R es un cuerpo y, para enfatizar este hecho, escribiremos $R = K$ en tal caso. En la Sección 4.3 se introduce el anillo de cohomología de Hochschild estable de un R-álgebra Λ, que sea además Gorenstein proyectiva como Λ-bimódulo, denotado por $\underline{HH}^*(\Lambda)$. Cuando el álgebra es simétrica, cosa que siempre ocurre cuando $\Lambda = \mathbb{L}_n$ ó $\Lambda = \mathbb{B}_n$ con n par, obtenemos que $HH_*(\Lambda) \cong D(HH^*(\Lambda))$ como $HH^*(\Lambda)$-módulos. Además, en la Sección 4.4 se prueba que, a parte de la graduación homológica canónica, en la que $HH^n(\Lambda)$ es la componente homogénea de grado n, el anillo $HH^*(\Lambda)$ hereda la graduación inducida por la longitud de caminos del álgebra de caminos del quiver en cuestión, que se llamará en lo que sigue "graduación por longitud", dotando a $HH^*(\Lambda)$ de una estructura de R-álgebra bigraduada ($= \mathbb{Z} \times \mathbb{Z}$ graduada). La Sección 4.5 está dedicada a las álgebras de Frobenius introducidas por Eu-Schedler en [35]. Merece la pena mencionar que cuando Λ es un álgebra de Frobenius, entonces el anillo de cohomología estable $\underline{HH}^*(\Lambda)$ es graduado-conmutativo (Proposición 4.5.2), además de ser una localización de su versión clásica (Proposición 4.5.6). Finalmente se incluye la Sección 4.6, dedicada especícfamente a las álgebras autoinyectivas. Muchos de los resultados de este apartado son sencillas aplicaciones de los resultados obtenidos en el capítulo 1 de esta memoria sobre álgebras pseudo-Frobenius.

Capítulo 5:

Los resultados de este capítulo, que trata sobre el estudio del anillo de cohomología de Hochschild del álgebra preproyectiva generalizada $\mathbb{L}_n$ sobre un anillo conmutativo R, aparecen publicados en [3] y [4] para los casos de característica distinta a 2 e igual a 2, respectivamente. Concretamente, describimos su estructura como álgebra bigraduada con el producto de Yoneda mediante una presentación explícita dada por generadores homogéneos y relaciones. Resulta importante tener en cuenta dos aspectos en lo que se refiere al caso $\mathbb{L}_n$. Por una parte, nuestra estrategia para probar el resultado principal consiste en abordar primero el caso en que R es un cuerpo y posteriormente deducir de éste el resultado más general en el que R es un anillo conmutativo en el que 2 es invertible. Por otra parte, conviene señalar que los resultados de este capítulo fueron los primeros que se obtuvieron y, por tanto, son anteriores al momento en que nos percatamos del cambio de relaciones que facilitaba ciertos cálculos y que se presenta en la Subsección 2.3.3 de esta memoria. Así, a lo largo de este capítulo se consideraran las relaciones de malla originales, que vienen dadas como sumas de caminos en lugar de diferencias.

Como suele ser habitual, se tratará separadamente el caso en el que el cuerpo tiene característica igual a 2 (véase la Sección 5.5). La diferencia más notable entre ambos casos es que, cuando la característica es distinta de dos, el álgebra $\mathbb{L}_n$ tiene periodo 6 mientras que en el caso contrario, el periodo es exactamente 3. Como se menciona en la primera parte de la introducción, dicha periodicidad se traslada a su vez a los grupos de cohomología. En la Sección 5.2 proporcionamos los elementos necesarios para nuestros cálculos: la sección se inicia con la definición del álgebra $\mathbb{L}_n$ mediante su quiver y relaciones y, a continuación, se muestra una base dualizable de la misma, que será la utilizada, cuya existencia está garantizada por el hecho de ser simétrica. Inmediatamente, se proporciona una resolución proyectiva minimal del álgebra como bimódulo sobre sí misma que induce a su vez el complejo de cocadena que se utiliza para calcular la cohomología (Proposición 3.4.2).

Con la información obtenida en la sección anterior se procede al cálculo de las dimensiones de los espacios de cohomología y homología de Hochschild, así como las de los espacios de homología cíclica en característica cero. En particular, tomando $\Lambda := \mathbb{L}_n$, se da una base canónica para cada $HH^i(\Lambda)$ formada por elementos homogéneos con respecto a la graduación por longitud (Proposición 5.3.10). La técnica que se sigue consiste en identificar previamente la estructura de cada $HH^i(\Lambda)$ como módulo sobre el centro $Z(\Lambda) = HH^0(\Lambda)$.

El resultado principal de este capítulo, que aparece en la Sección 5.4 y trata la estructura multiplicativa de $HH^*(\Lambda)$ es el Teorema 5.4.1, a partir del cuál obtenemos una presentación del anillo de cohomología de Hochschild estable de Λ, $\underline{HH}^*(\Lambda)$, dada por generadores y relaciones.

TEOREMA. Sea $\Lambda = \mathbb{L}_n$ el álgebra preprojectiva generalizada $\mathbb{L}_n$ sobre un anillo conmutativo R en el que 2 es invertible. Se verifican las siguientes afirmaciones para el anillo de cohomología de Hochschild $HH^*(\Lambda)$:

1. $HH^*(\Lambda)$ es la R-álgebra bigraduada-conmutativa dada por
   a) Generadores: $x_0, x_1, \ldots, x_n, y, z_1, \ldots, z_n, t_1, t_2, \ldots t_{n-1}, \gamma, h$
   b) Relaciones:
      i) $x_i\xi = 0$ para cada $i = 1, \ldots, n$ y cada generador ξ.
      ii) $x_0^n = y^2 = x_0 z_j = x_0 t_i = y t_i = t_i t_k = 0$, for $j = 1, \ldots, n \quad i, k = 1, \ldots n-1)$
      iii) $z_j z_k = (-1)^{k-j+1}(2j-1)(n-k+1)x_0^{n-1}\gamma$, para $1 \leq j \leq k \leq n$.
      iv) $z_j\gamma = (-1)^j(n-j+1)x_0^{n-1}h$, para $j = 1, ..., n$
      v) $\gamma^2 = z_1 h$
      vi) $y z_j = (2n+1)\sum_{1\leq k\leq j-1}(-1)^{j-k}(j-k)t_k + (-1)^{j-1}(2j-1)y z_1$, para $j = 2, ..., n$
      vii) $z_k t_j = \delta_{jk} x_0^{n-1} y\gamma$, para $k = 1, \ldots, n \quad j = 1 \ldots, n-1$
      viii) $t_j\gamma = \delta_{1j} x_0^{n-1} y h$, para $j = 1, \ldots, n-1$.
2. La graduación homológica en $HH^*(\Lambda)$ viene determinada por las igualdades $\deg(x_i) = 0$, $\deg(y) = 1$, $\deg(z_j) = 2$, $\deg(t_k) = 3$, $\deg(\gamma) = 4$ y $\deg(h) = 6$.
3. La graduación por longitud en $HH^*(\Lambda)$ viene determinada por las igualdades $\mathrm{ldeg}(x_0) = 2$, $\mathrm{ldeg}(x_i) = 2n-1$, para $i \neq 0$, $\mathrm{ldeg}(y) = 0$, $\mathrm{ldeg}(z_j) = -2$, $\mathrm{ldeg}(t_k) = -2$, $\mathrm{ldeg}(\gamma) = -2n-2$ y $\mathrm{ldeg}(h) = -4n-2$.
4. La multiplicación por h induce un isomorfismo $HH^i(\Lambda) \xrightarrow{\cong} HH^{i+6}(\Lambda)$, para cada $i > 0$.
5. Cada $HH^i(\Lambda)$ es un R-módulo libre, siendo la siguiente una lista de las correspondientes bases (see Proposition 5.3.10):
   (a) Para $HH^0(\Lambda)$: $\{x_0, x_0^2, \ldots, x_0^{n-1}, x_1, \ldots, x_n\}$.
   (b) Para $HH^1(\Lambda)$: $\{y, x_0 y, x_0^2 y, \ldots x_0^{n-1} y\}$.
   (c) Para $HH^2(\Lambda)$: $\{z_1, \ldots, z_n\}$.
   (d) Para $HH^3(\Lambda)$: $\{t_1, \ldots, t_{n-1}, y z_1\}$.
   (e) Para $HH^4(\Lambda)$: $\{x_0^{n-1}\gamma, \ldots, x_0\gamma, \gamma\}$.
   (f) Para $HH^5(\Lambda)$: $\{x_0^{n-1} y\gamma, \ldots, x_0 y\gamma, y\gamma\}$.
   (g) Para $HH^6(\Lambda)$: $\{h, x_0 h, \ldots, x_0^{n-1} h\}$.

   En particular $\dim(HH^0(\Lambda)) = 2n$ y $\dim(HH^i(\Lambda)) = n$, para todo $i > 0$, donde $\dim(-)$ denota el rango como R-módulo libre.

Finalizamos el capítulo con la Sección 5.5 donde se presenta el resultado análogo al anterior cuando el cuerpo K tiene caracteriística 2 (Teorema 5.5.1).

Capítulo 6:

En el sexto y último capítulo de esta memoria se completa la respuesta a la cuestión 4 mediante el estudio del anillo de cohomología de Hochschild del álgebra preproyectiva generalizada $\mathbb{B}_n$ sobre un cuerpo de característica distinta de 2. En tal caso, gracias al Teorema 3.4.12 se sabe que $\mathbb{B}_n$ tiene periodo 6 y, por consiguiente, los espacios de cohomología también tienen periodo 6. La estrategia utilizada en este último capítulo es considerablemente distinta a la del anterior. A diferencia del caso $\mathbb{L}_n$, donde el álgebra se presentaba directamente mediante su quiver y relaciones sin atender a su relación con el quiver de traslación $\mathbb{Z}\mathbb{A}_{2n}$, $\mathbb{B}_n$ sí será considerada como el álgebra de órbitas $\mathbb{Z}\mathbb{A}_{2n-1}/\langle\rho\tau\rangle$ siendo τ y ρ la traslación de Auslander-Reiten y el automorfismo dado por la reflexión natural de $\mathbb{Z}\mathbb{A}_{2n-1}$, respectivamente, o equivalentemente, considerada como el álgebra de malla finito dimensional de tipo extendido $(\mathbb{A}_{2n-1}, 1, 2)$ (véase la Subsección 6.5.1). La ventaja en este caso es que se pueden utilizar los resultados obtenidos en los tres primeros capítulos de esta memoria para conseguir información esencial para nuestros propósitos. Así, en la Subsección 6.5.2 y usando el Teorema 6.5.3, se calcula la matriz de Cartan de $\mathbb{B}_n$. En la Subsección 6.5.4, y a partir de la resolución proyectiva del álgebra de malla $B = B(\mathbb{A}_{2n-1})$ como B-bimódulo, se describe la de $\mathbb{B}_n$ que induce, como se presenta en la Subsección 6.5.5, el complejo de cocadena que induce a su vez la cohomología de Hochschild. Se continúa con la Sección 6.6 en la que, considerando $\Lambda = \mathbb{B}_n$, se identifica la estructura de cada espacio de cohomología $HH^i(\Lambda)$ como $Z(\Lambda)$-bimódulo y, como herramienta necesaria, el ideal $I = \mathcal{P}(\Lambda, \Lambda)$ de $Z(\Lambda) = End_{\Lambda^e}(\Lambda)$ formado por todos los endomorfismos de Λ como bimódulo que se factorizan a través de un bimódulo proyectivo. Dicha descripción depende de hecho de la paridad de n, lo cuál se debe principalmente a que el automorfismo de Nakayama es la identidad cuando n es par y τ cuando n es impar. Es decir, Λ es simétrica cuando n es par pero ni siquiera es débilmente simétrica cuando n es impar. Para finalizar el capítulo y por tanto esta memoria, a lo largo de la Sección 6.7 presentamos los dos resultados principales (para n impar y n par) que describen mediante generadores y relaciones la estructura como álgebra bigraduada del anillo de cohomología de Hochschild $HH^*(\Lambda)$ sobre un cuerpo de característica $\neq 2$.

El Teorema 6.7.1 trata el caso n impar:

TEOREMA. Sea n impar y sea Λ el álgebra preproyectiva generalizada $\mathbb{B}_n$ sobre un cuerpo de característica $\neq 2$ y véase a $HH^*(\Lambda)$ como álgebra bigraduada, donde $\mathrm{bideg}(r) = (\mathrm{hdeg}(r), \mathrm{ldeg}(r))$, para cualquier elemento homogéneo $r \in HH^*(\Lambda)$. Considérese los siguientes elementos de $HH^*(\Lambda)$:

a) $x = \sum_{3\leq i\leq 2n-3} c_i \in HH^0(\Lambda) = Z(\Lambda)$, donde c_i es el ciclo de longitud 4 en i;

b) $y \in HH^1(\Lambda)$ representado por $\tilde{y} : \oplus_{a\in Q_1}\Lambda e_{i(a)} \otimes e_{t(a)}\Lambda \longrightarrow \Lambda$, donde $\tilde{y}(e_{i(a)} \otimes e_{t(a)}) = a$, para todo $a \in Q_1$;

c) $h \in HH^6(\Lambda)$ representado por la aplicación multiplicación $\oplus_{i\in Q_0}\Lambda e_i \otimes e_i\Lambda \longrightarrow \Lambda$ y

d) En caso que $\mathrm{char}(K)$ divida a n, el elemento $v \in HH^5(\Lambda)$ representado por $\tilde{v} : \oplus_{i\in Q_0}\Lambda e_{\tau(i)} \otimes e_i\Lambda \longrightarrow \Lambda$, donde $\tilde{v}(e_{\tau(i)} \otimes e_i) = \delta_{in}e_n$, para todo $i \in Q_0$, siendo δ_{in} el símbolo de Kronecker.

Entonces se tiene que $\mathrm{bideg}(x) = (0, 4)$, $\mathrm{bideg}(y) = (1, 0)$, $\mathrm{bideg}(h) = (6, -4n)$ y $\mathrm{bideg}(v) = (5, -2n - 2)$ y se verifican las siguientes afirmaciones:

1. Si $\mathrm{char}(K)$ no divide a n, entonces $HH^*(\Lambda)$ es el álgebra conmutativa con generadores x, y, h, sujeta a las relaciones:

$$x^{\frac{n+1}{2}} = 0, \qquad x^{\frac{n-1}{2}}y = 0, \qquad x^{\frac{n-1}{2}}h = 0 \qquad \text{y} \qquad y^2 = 0.$$

2. Si $\mathrm{char}(K)$ divide a n, entonces $HH^*(\Lambda)$ es el álgebra bigraduada conmutativa con generadores x, y, v, h, sujeta a las relaciones

$$x^{\frac{n+1}{2}} = 0, \qquad x^{\frac{n-1}{2}}y = 0, \qquad xv = 0, \qquad y^2 = 0 \qquad yv = 0 \qquad \text{y} \qquad v^2 = 0.$$

Para n par el resultado es el siguiente (Teorema 6.7.10)

TEOREMA. Sea n par, sea Λ el álgebra preproyectiva generalizada B_n sobre un cuerpo de característica $\neq 2$ y véase a $HH^*(\Lambda)$ como álgebra bigraduada, donde $\mathrm{bideg}(r) = (\mathrm{hdeg}(r), \mathrm{ldeg}(r))$, para cualquier elemento homogéneo $r \in HH^*(\Lambda)$. Considérese los siguientes elementos de $HH^*(\Lambda)$:

a) $x, x_1, ..., x_{2n-1}$ en $HH^0(\Lambda) = Z(\Lambda)$ dados como se sigue:
   (a) $x = \sum_{3\leq i\leq 2n-3} c_i$, donde c_i es el ciclo no nulo de longitud 4 en i;
   (b) $\{x_1, ..., x_{n-2}\}$ donde $x_{2k} = \omega_{2k}$ y $x_{2k-1} = \sum_{r=1}^{k} \omega_{2r-1}$, para todo $0 < k < \frac{n}{2}$;
   (c) $x_{n-1} = \sum_{1\leq k\leq \frac{n}{2}} (\omega_{2k-1} - \omega_{2n-2k+1})$, $x_n = \omega_n$ y $x_i = \omega_i + \omega_{2n-i}$, para todo $n < i \leq 2n - 1$.

b) $y \in HH^1(\Lambda)$ representado por $\tilde{y} : \oplus_{a\in Q_1}\Lambda e_{i(a)} \otimes e_{t(a)}\Lambda \longrightarrow \Lambda$, donde $\tilde{y}(e_{i(a)} \otimes e_{t(a)}) = a$;

c) $z \in HH^2(\Lambda)$ representado por $\tilde{z} : \oplus_{i\in Q_0}\Lambda e_{\tau(i)} \otimes e_i\Lambda \longrightarrow \Lambda$, donde $\tilde{z}(e_{\tau(i)} \otimes e_i) = \delta_{in}e_n$;

d) $t \in HH^3(\Lambda)$ representado por $\tilde{t} : \oplus_{i\in Q_0}\Lambda e_{\tau(i)} \otimes e_i\Lambda \longrightarrow \Lambda$, donde $\tilde{t}(e_{\tau(i)} \otimes e_i) = \delta_{in}w_n$;

e) $u \in HH^4(\Lambda)$ representado por $\tilde{u} = \oplus_{a\in Q_1}\Lambda e_{\tau(i(a))} \otimes e_{t(a)}\Lambda \longrightarrow \Lambda$, donde $\tilde{u}(e_{\tau(i(a))} \otimes e_{t(a)}) = \frac{1}{2}(\delta_{a,\alpha_n}\alpha_n - \delta_{a,\beta_n}\beta_{n-1})$;

f) $v_1, ..., v_{n-2} \in HH^5(\Lambda)$, donde cada uno de los v_j está representado por $\tilde{v}_j = \oplus_{i\in Q_0}\Lambda e_i \otimes e_i\Lambda \longrightarrow \Lambda$, donde:
   (a) $\tilde{v}_j(e_i \otimes e_i) = (\delta_{ij} - \delta_{i,2n-j})e_i$, siempre y cuando j sea par;
   (b) $\tilde{v}_j(e_i \otimes e_i) = (\delta_{ij} - \delta_{i,j+2} - \delta_{i,2n-j} + \delta_{i,2n-j-2})e_i$, siempre y cuando j sea impar.

g) $h \in HH^6(\Lambda)$ representado por la aplicación multiplicación $\oplus_{i\in Q_0}\Lambda e_i \otimes e_i\Lambda \longrightarrow \Lambda$.

Los grados de longitud de estos elementos son $\text{ldeg}(x) = 4$, $\text{ldeg}(x_i) = 2n - 2$, $\text{ldeg}(y) = 0$, $\text{ldeg}(z) = \text{ldeg}(t) = -2$, $\text{ldeg}(u) = -2n$, $\text{ldeg}(v_j) = -2n - 2$ y $\text{ldeg}(h) = -4n$.

Además, como álgebra, $HH^*(\Lambda)$ está generada por estos elementos, sujeta a las relaciones graduado-conmutativas con respecto a la graduación homológica junto con las siguientes relaciones:

1. $x^{\frac{n}{2}} = xz = xt = xv_i = 0$
2. $x_i\xi = 0$, para cada generador ξ, excepto en el caso $i \leq n - 2$ y $\xi = v_i$ ó $\xi = h$
3. $x_iv_i = x^{\frac{n}{2}-1}yu$, para todo $i \leq n - 2$.
4. $y^2 = yt = 0$
5. $yz = -nt$
6. $yv_{2k} = 4nx_{2k-1}h$ y $yv_{2k-1} = -4nx_{2k}h$, para todo $1 \leq k < \frac{n}{2}$
7. $z^2 = -nx^{\frac{n}{2}-1}u$
8. $zt = x^{\frac{n}{2}-1}yu$
9. $zv_j = 0$, para todo $j = 1, ..., n - 2$
10. $zu = 0$
11. $t^2 = tu = tv_j = 0$, para todo $j = 1, ..., n - 2$
12. $u^2 = 0$
13. $uv_j = 0$, para todo $j = 1, ..., n - 2$
14. $v_{2k}v_j = 4n\delta_{2k-1,j}x^{\frac{n}{2}}uh$, para todo $1 \leq k < \frac{n}{2}$ y $j = 1, 2, ..., n - 2$.

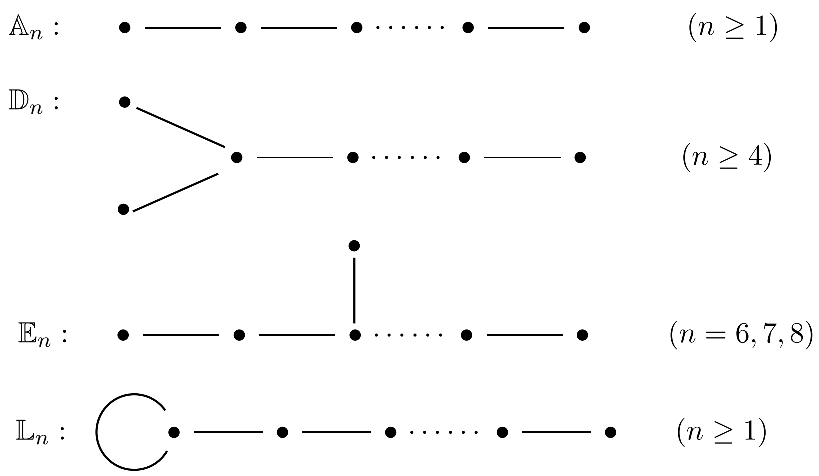
Los capítulos 5 y 6 de la memoria muestran que se pueden observar diferencias en el comportamiento del anillo de cohomología de Hochschild $HH^*(\Lambda)$ para $\Lambda = \mathbb{L}_n$ y $\mathbb{B}_n$, y, más aún, entre los casos n impar y n par de $\mathbb{B}_n$. Por ejemplo, para $\mathbb{L}_n$, $HH^*(\Lambda)$ es siempre conmutativa, como también lo es para $\mathbb{B}_n$, cuando n es impar. Sin embargo, para $\mathbb{B}_n$ con n par, y salvo que $\text{Char}(K)$ divida a n, la estructura conmutiativa de $HH^*(\Lambda)$ se destruye por completo.

Al final de la memoria hemos incluido una lista de las referencias bilbiográficas que hemos manejado.

Introduction

The notion of preprojective algebra first appeared in the late 70s in the work of Gelfand and Ponomarev [42] on the representation theory of finite quivers without oriented cycles. Essentially, given a non-oriented graph Δ, the associated preprojective algebra, usually denoted by $P(\Delta)$, is obtained from Δ by replacing each edge by a pair of two opposite arrows and identifying, afterwards, all cycles of length 2 starting at the same vertex. They found their first applications in classification problems of algebras of finite type ([22], [23]) and have been linked to universal enveloping algebras and cluster algebras ([40], [41]). They also occur in very diverse parts of mathematics. For instance, they play a special role in Lusztig's perverse sheaf approach to quantum groups ([60], [61]) and have been used to tackle differential geometry problems [53] or to study non-commutative deformations of Kleinian singularities [21].

When $R = K$ is an algebraically closed field, it is well known that $P(\Delta)$ is finite dimensional if and only if Δ is a disjoint union of generalized Dynkin graphs, $\mathbb{A}_n$, $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$ or $\mathbb{L}_n$:



Historically, in the context of finite dimensional algebras, the syzygy operator, Ω_Λ, which assigns to every Λ-module M the kernel of its projective cover $P_\Lambda(M) \longrightarrow M$, has become a very useful tool to connect modules as well as to obtain information about the structure of the algebra itself. For instance, if all the simple modules are Ω_Λ-periodic, then Λ is a self-injective algebra, that is, the class of projective and injective modules coincide. By an Ω_Λ-periodic module we mean a module M satisfying that there exists an integer $r > 0$ such that $\Omega_\Lambda^r(M)$ is isomorphic to M. In such case, the smallest natural number satisfying the previous property is called the period of M. This operator is not only useful, but also fundamental, when studying a particular kind of finite dimensional algebras contained in the class of self-injective algebras and known as periodic algebras. An algebra Λ is called periodic when it is periodic as a module over its enveloping algebra $\Lambda^e = \Lambda \otimes \Lambda^{op}$, or equivalently, as a Λ-bimodule. An important common feature of the preprojective algebras of generalized Dynkin type is that, except for $\Delta = \mathbb{A}_1$, $P(\Delta)$ is $(\Omega-)$periodic of period at most 6.

In every mathematical discipline, it is natural and also usual to try to generalize concepts in such a way that the most relevant properties are preserved. With this idea in mind, K. Erdmann and S. Skowroński introduced in [29] a new class of algebras associated to Dynkin diagrams which contains the preprojective algebras and which has deserved a lot of attention in recent times in the general context of finite dimensional algebras. They were called m-fold mesh algebras and are precisely the self-injective algebras Λ for which Ω_Λ^3 permutes the isomorphism classes of simple modules. Following the aforementioned work, if Δ is one of the Dynkin quivers $\mathbb{A}_n$, $\mathbb{D}_r$ or $\mathbb{E}_n$ $(n = 6, 7, 8)$, an m-fold mesh algebra of type Δ is a quotient B/G of the mesh algebra $B = B(\Delta)$ of a stable translation quiver $\mathbb{Z}\Delta$ by a weakly admissible group of automorphisms G of $\mathbb{Z}\Delta$. Later, by a result of Dugas ([25] Theorem 3.1), it was known that the m-fold mesh algebras are precisely the mesh algebras of stable translation quivers which are finite dimensional. This class of algebras properly contains the stable Auslander algebras of all standard representation-finite self-injective algebras (see [25]) and also the Auslander-Reiten algebras of several hypersurface singularities (see [29][Section 8]). Moreover, by [14][Section 6], all the algebras in the class are periodic.

In the late 90s, in his work entitled *Triangulated categories and geometry*, M. Kontsevich defined the notion of Calabi-Yau dimension for Hom finite triangulated K-categories, that is, for triangulated K-categories for which the K-vector space of morphisms between two any objects is finite dimensional. Under these hypotheses, the K-category $\mathcal{T}$, with suspension functor $\sum : \mathcal{T} \longrightarrow \mathcal{T}$, is called Calabi-Yau, when there is a natural number n such that $\sum^n$ is a Serre functor, or equivalently, when there exists a natural number n such that $DHom_{\mathcal{T}}(X, -))$ and $Hom_{\mathcal{T}}(-, \sum^n X)$ are naturally isomorphic as cohomological functors $\mathcal{T}^{op} \longrightarrow K-mod$. In such case, the smallest natural number m such that $\sum^m$ is a Serre functor is called the Calabi-Yau dimension of $\mathcal{T}$ and we write CY-dim$(\mathcal{T}) = m$. Calabi-Yau triangulated categories appear in many fields of Mathematics and Theoretical Physics. In Representation Theory of algebras, the concept plays an important role in the study of cluster algebras and cluster categories (see [55]).

When Λ is a self-injective finite dimensional algebra and ${}_\Lambda\underline{mod}$ is its stable module category, then the syzygy functor $\Omega_\Lambda : {}_\Lambda\underline{mod} \longrightarrow {}_\Lambda\underline{mod}$ is an equivalence of categories and ${}_\Lambda\underline{mod}$ has a structure of triangulated category with the inverse of the syzygy functor, Ω_Λ^{-1}, as suspension functor. The Calabi-Yau condition on this category has been deeply studied (see, e.g., [28], [12], [35], [25], [50], [51],...) and it has been related with that of Frobenius Calabi-Yau algebra, as defined by Eu and Schedler ([35]): the algebra Λ is called Calabi-Yau Frobenius when $\Omega_{\Lambda^e}^{-r-1}(\Lambda)$ is isomorphic to $D(\Lambda) = \mathrm{Hom}_K(\Lambda, K)$ as Λ-bimodules, for some integer $r \geq 0$. Notice that if the algebra Λ is Calabi-Yau Frobenius, then $\Lambda - \underline{mod}$ is Calabi-Yau. Also, taking into account that $\Omega_{\Lambda^e}^{-r-1} \otimes_{\Lambda} ?$ and Ω_{Λ}^{-r-1} are naturally isomorphic functors ${}_{\Lambda}\underline{\text{mod}} \longrightarrow {}_{\Lambda}\underline{\text{mod}}$, the Calabi-Yau dimension of ${}_{\Lambda}\underline{\text{mod}}$ is less or equal than the smallest integer $r \geq 0$ such that $\Omega_{\Lambda^e}^{-r-1}(\Lambda)$ is isomorphic to $D(\Lambda)$ as Λ-bimodules, a number which will be called from now on the Calabi-Yau Frobenius dimension of Λ and will be denoted by CYF-dim(Λ). However, in general, it is not known whether these two numbers are equal.

After earlier work in [12] and [28], the determination of the stable Auslander algebras Λ of a representation-finite self-injective algebra such that ${}_{\Lambda}\underline{mod}$ is Calabi-Yau is done in [25] and [51] in terms of its type. Inspired by the work of Riedtmann ([66]), H. Asashiba defined in [6] the type of an stable Auslander algebra of a self-injective algebra of finite representation type, which is invariant under derived equivalence, as the triple (Δ, f, t) where Δ is the associated Dynkin diagram, f is the frequency and t is the torsion order. In the first of these two papers [25], A. Dugas identifies such algebras when t is 1 or 3, and also in many cases with $t = 2$. The remaining cases for $t = 2$ have been recently settled by Ivanov-Volkow ([51]). Hence, bearing in mind that the m-fold mesh algebras are self-injective, some natural questions arise:

Question 1: Which are the m-fold mesh algebras whose stable module category is Calabi-Yau? Which are Calabi-Yau Frobenius? And finally, which is the relation between both dimensions?

In terms of bimodules, a basic finite dimensional algebra Λ is self-injective precisely when there is an isomorphism of Λ-bimodules between $D(\Lambda)$ and the twisted bimodule ${}_1\Lambda_{\eta}$, for some automorphism η of Λ. This automorphism is uniquely determined up to inner automorphism and is called the Nakayama automorphism of Λ. According to this, the problem of deciding when Λ is Calabi-Yau Frobenius is part of a more general problem which consists of determining under which conditions $\Omega_{\Lambda^e}^{r}(\Lambda)$ is isomorphic to a twisted bimodule ${}_1\Lambda_{\varphi}$, for some automorphism φ of Λ, which is then determined up to inner automorphism. By a result of Green-Snashall-Solberg ([44]), this condition on a finite dimensional algebra forces it to be self-injective. Moreover, observe that when φ is the identity, or an inner automorphism, we precisely obtain the definition of periodic algebra. The problem of determining the self-injective algebras which are periodic is, nowadays, widely open. However, there is a lot of work in the literature where several classes of periodic algebras, including the m-fold mesh algebras, have been identified (see, e.g., [14], [29], [24]). Nevertheless, even when an algebra Λ is known to be periodic, it is usually difficult and hard to calculate explicitly its period. In the context of the m-fold mesh algebras, the explicit calculation of their period has been done only in very few cases. Concretely, from the papers [67], [32] and [11] we know that the period is 6 for all preprojective algebras of generalized Dynkin type, whenever Char$(K) \neq 2$. In addition, in [25], the period is calculated when Λ is the stable Auslander algebra of a standard representation-finite self-injective algebra of type (Δ, f, t) equal to $(\mathbb{D}_4, f, 3)$, $(\mathbb{D}_n, f, 2)$, with $n > 4$ and $f > 1$ odd, or $(\mathbb{E}_6, f, 2)$. Again, a natural question arises:

Question 2: Which is the period of an m-fold mesh algebra?

Another interesting problem that has attracted special attention in the context of self-injective finite dimensional algebras is that of characterizing those which are symmetric or weakly symmetric. We say that an algebra Λ is symmetric when it is isomorphic to $D(\Lambda)$ as Λ-bimodule. This is equivalent to saying that the Nakayama functor $DHom_\Lambda(-,\Lambda) \cong D(\Lambda) \otimes_\Lambda - : \Lambda - \text{Mod} \to \Lambda - \text{Mod}$ is naturally isomorphic to the identity functor. If we weaken the condition imposed to the functor in order to preserve the isomorphism classes of simple modules we obtain precisely the definition of weakly symmetric algebra. Therefore, one could ask:

Question 3: Which of the m-fold mesh algebras are weakly symmetric? Which of them are in turn symmetric?

Since it was introduced by G. Hochschild in 1945 in his work [47], the (co)homology theory which is known by his own name has been study in depth having a marked influence, among others, in the field of the finite dimensional algebras. Given a finite dimensional algebra Λ, for each $i \geq 0$, the K-vector space $HH^i(\Lambda) := \text{Ext}^i_{\Lambda^e}(\Lambda,\Lambda)$ is called the i-th Hochschild cohomology group. It turns out then that, with the Yoneda product, the K-vector space $\oplus_{i\geq 0} HH^i(\Lambda)$ admits a structure of graded-commutative K-algebra commonly known as the Hochschild cohomology ring of Λ and usually denoted by $HH^*(\Lambda)$. The lowest degrees of this ring have very concrete interpretations closely related to classical algebraic and geometric structures. For instance, $HH^0(\Lambda)$ and $HH^1(\Lambda)$ coincide with the center and the space of outer derivations of the algebra, respectively, while $HH^2(\Lambda)$ controls its deformation theory: if it is zero, then the algebra is rigid. Concerning the multiplicative structure, the Hochschild cohomology ring $HH^*(\Lambda)$ of a self-injective finite dimensional algebra is of great interest in connection with the study of varieties of modules and with questions about its relationship with the Yoneda algebra of Λ. This is the graded algebra $E(\Lambda) = \text{Ext}^*_\Lambda(\Lambda/J,\Lambda/J)$, where $J = J(\Lambda)$ denotes the Jacobson radical of Λ. Indeed, with inspiration from modular representation theory of finite groups, where the theory of varieties of modules had been developed by Carlson ([18], [19]), Benson ([10]) and others, Snashall and Solberg ([68], see also [27]) started the study of varieties of modules over arbitrary finite dimensional algebras, replacing the group cohomology ring $HH^*(G,K)$ by the Hochschild cohomology ring $HH^*(\Lambda)$ of the considered algebra Λ. For the new theory, one generally requires Λ to be self-injective and $HH^*(\Lambda)$ to satisfy some finite generation conditions, which are always satisfied when Λ is periodic. However, little else is known about the Hochschild cohomology ring $HH^*(\Lambda)$ of a finite dimensional algebra Λ but the fact that, whenever Λ is periodic, there exists an isomorphism of algebras $HH^*(\Lambda)/\mathcal{N} \cong K[x]$ where $\mathcal{N}$ is the ideal generated by the nilpotent elements and x is an homogeneous element of $HH^*(\Lambda)$ whose degree coincides with the period of Λ. This property first appeared explicitly in ([44], Proposición 1.1). Nevertheless, it is just a direct application with $A = \Lambda \otimes \Lambda^{op}$ and $M = \Lambda$, when Λ is periodic, of a more general result by Carlson [17]. Such result states that when $A = KG$ is a group algebra of a finite group G, which is the prototypical example of self-injective finite dimensional algebras, and M is a periodic A-module of period q, then $\text{Ext}^*_A(M,M)/\mathcal{N} \cong K[x]$ where $x \in \text{Ext}^q_A(M.M)$ and $\mathcal{N}$ is the ideal of $\text{Ext}^*_A(M,M)$ generated by the nilpotent elements. Indeed, the result is equally valid for any self-injective finite dimensional algebra.

The former questions suggest that finding patterns of behavior of the homogeneous elements of $HH^*(\Lambda)$ with respect to the Yoneda product, in particular cases where the multiplicative structure of $HH^*(\Lambda)$ is computable, can help to give some hints on how to deal with them. Regarding the finite dimensional preprojective algebras, the structure of the Hochschild cohomology ring, when $R = K$ is a field, is known for type $\mathbb{A}_n$ in arbitrary characteristic ([30], [31]) and, in the case of a field of characteristic zero, for types $\mathbb{D}_n$ and $\mathbb{E}$ [33]. Following the notion of extended type (Δ, m, t) of an m-fold mesh algebra, introduced in Chapter 2, it turns out that the preprojective algebras mentioned before are in fact the m-fold mesh algebras of extended type $(\Delta, 1, 1)$ where $\Delta = \mathbb{A}_n, \mathbb{D}_n$ or $\mathbb{E}_6$. A step forward in that direction would be to consider the case when Λ is an m-fold mesh algebra of extended type $(\Delta, 1, t)$ with $t > 1$, which correspond to the algebras known nowadays as generalized preprojective algebras. In this thesis we tackle the case $\Delta = \mathbb{A}_n$.

Question 4: How are the Hochschild cohomology groups $HH^i(\Lambda)$ of a generalized preprojective algebra Λ of type $\mathbb{A}_n$? Which is the multiplicative structure of the associated Hochschild cohomology ring $HH^*(\Lambda)$?

This thesis is organized in 6 chapters which are related between them and provide an answer to the questions formulated above.

Chapter 1

This first chapter, which may be considered as a preliminary chapter, is the fundamental basis for developing the contents of the first part of this work. Placed in the general context of algebras with enough idempotents, the main concepts we shall work with are those of pseudo-Frobenius algebra and their associated Nakayama form and automorphism, all of them introduced in Section 1.3.

We will say that a graded algebra with enough idempotents, weakly basic and locally finite dimensional is pseudo-Frobenius if the finitely generated projective objects and the finitely cogenerated injective objects coincide in its category of graded modules. Among other results, it is worth mentioning the characterization presented of such algebras (Theorem 1.3.2). The appropriated notions can be found in that section.

THEOREM. Let $A = \oplus_{h\in H} A_h$ be a weakly basic graded algebra with enough idempotents. Consider the following assertions:

1. $A - Gr$ and $Gr - A$ are Frobenius categories
2. $D({}_AA)$ and $D(A_A)$ are projective graded A-modules
3. A is graded pseudo-Frobenius
4. There exists a graded Nakayama form $(-,-) : B \times B \longrightarrow K$.

Then the following chain of implications holds:

$$1) \Longrightarrow 2) \Longrightarrow 3) \Longleftrightarrow 4).$$

When A is graded locally bounded, also 4) $\Longrightarrow$ 2) holds. Finally, if A is graded locally Noetherian, then the four assertions are equivalent.

When the algebra satisfies the condition 1 of the previous theorem, it is called Quasi-Frobenius. As one can suspect, pseudo-Frobenius graded algebras with enough idempotents are the analogue in the context of graded algebras with enough idempotents, or equivalently graded K-categories, of what finite dimensional self-injective algebras are in the context of associative unital algebras. Therefore, we may ask about the existence of an automorphism of the algebra A playing a similar role to that of the Nakayama automorphism for self-injective finite dimensional algebras. The next result guarantees its existence (Corollary 1.3.6):

COROLLARY. Let $A = \oplus_{h\in H} A_h$ be a graded pseudo-Frobenius algebra and let $(e_i)_{i\in I}$ be a weakly basic distinguished family of orthogonal idempotents. If A is graded locally bounded, then following assertions hold:

1. There is an automorphism of (ungraded) algebras $\eta : A \longrightarrow A$, which permutes the idempotents e_i and maps homogeneous elements onto homogeneous elements, such that ${}_1A_\eta$ is isomorphic to $D(A)$ as an ungraded A-bimodule.
2. If the degree map $\mathbf{h} : I \longrightarrow H$ associated to the Nakayama form $(-,-) : A \times A \longrightarrow K$ takes constant value h, then η can be chosen to be graded and such that $D(A)$ is isomorphic to ${}_1A_\eta[h]$ as graded A-bimodules.

The automorphism η is called the Nakayama automorphism of A.

Although there are no genuine new ideas in the process of passing from unital ungraded algebras to graded algebras with enough idempotents, as far as we know, the concept of pseudo-Frobenius algebras and its associated ones, like Nakayama form and Nakayama automorphism, had not been developed before in such a generality and they are crucial for the rest of the work.

In Section 1.4, we revisit covering theory from the point of view of graded algebras with enough idempotents emphasizing the particular case where the covering functor is of type $F : A \longrightarrow A/G$ where A is a graded algebra with enough idempotents and G is a group of automorphisms of A of degree 0 which permutes the idempotents e_i. In the second part of this section we study under which conditions we can guarantee the preservation of the pseudo-Frobenius condition via the usual covering functor. As stated below (Proposition 1.4.3), this always occurs when the associated Nakayama form of A is what we call G-invariant, that is, when it satisfies that $(a^g, b^g) = (a, b)$ for any $a, b \in A$ and $g \in G$.

PROPOSITION. Let $A = \oplus_{h\in H} A_h$ be a (split weakly) basic graded locally bounded algebra, with $(e_i)_{i\in I}$ as distinguished family of orthogonal homogeneous idempotents, and let G be a group which acts on A as graded automorphisms which permute the e_i and which acts freely on objects. Suppose that

> A is graded pseudo-Frobenius admitting a G-invariant graded Nakayama form $(-,-) : A \times A \longrightarrow K$. Then $\Lambda = A/G$ is a (split weakly) basic graded locally bounded pseudo-Frobenius algebra whose graded Nakayama form is induced from $(-,-)$.

Moreover, we end the section by showing that, under the hypothesis of the previous proposition, the Nakayama automorphism η of A induces in turn the Nakayama automorphism $\bar{\eta}$ de A/G (Corollary 1.4.5).

Chapter 2

The second chapter of this work is devoted to study in depth and obtain some useful information about the mesh algebra of a Dynkin diagram as its Nakayama automorphism. Despite of the fact that this algebra is not the main object of study of this thesis, our interest lies in the fact that every m-fold mesh algebra arises in some sense from such an algebra.

In the beginning of this chapter, concretely in Section 2.2, we recall the definition of the stable translation quiver $\mathbb{Z}\Delta$ of a Dynkin diagram Δ. Next, in Subsection 2.3.1, we introduce the notion of the mesh algebra $B = B(\Delta)$. This algebra is defined as the quotient algebra $K\mathbb{Z}\Delta/I$, where I is the homogeneous ideal generated by the so-called mesh relations. In addition, we present a short list of well known essential properties of B (Proposition 2.3.1). It is worth mentioning that the algebra B is a graded algebra with enough idempotents which turns out to be pseudo-Frobenius.

Then, in Subsection 2.3.2, we provide the complete list of the m-fold mesh algebras and we introduce the definition of extended type for such algebras which plays a crucial role throughout this work. Indeed, all our main results on m-fold mesh algebras in Chapter 3 are given in terms of their extended type. This notion is based in the fact that every m-fold mesh algebra Λ is isomorphic to an orbit algebra B/G where B is the mesh algebra associated to a Dynkin diagram Δ and G is a weakly admissible group of automorphisms of $\mathbb{Z}\Delta$ viewed as automorphisms of B. Moreover, it is a well known fact that Λ admits as a Galois cover its corresponding mesh algebra B. With the idea of simplifying calculations, we end Section 2.3 by performing a change of relations which, roughly speaking, transforms sums of paths of length 2 into differences.

The main result of this chapter, which appears in Section 2.4, is Theorem 2.4.2 where we explicitly define, for any choice of (Δ, G), a graded Nakayama automorphism η of B satisfying the property of being G-invariant. Here G-invariant means that η commutes with the elements in G. As a consequence, we derive a precise formula for the graded Nakayama automorphism of any m-fold mesh algebra.

The aforemention Theorem is the following:

> THEOREM. Let Δ be a Dynkin quiver with the labeling of vertices and the orientation of the arrows of Subsection 2.3.1, and let $G =< \varphi >$ be a weakly admissible automorphism of $\mathbb{Z}\Delta$. If η is the graded automorphism of B which

acts as the Nakayama permutation on the vertices and acts on the arrows as indicated in the following list, then η is a Nakayama automorphism of B such that $\eta \circ g = g \circ \eta$, for all $g \in G$.

1. When $\Delta = \mathbb{A}_n$ and φ is arbitrary, $\eta(\alpha) = \nu(\alpha)$ for all $\alpha \in (\mathbb{Z}\Delta)_1$
2. When $\Delta = \mathbb{D}_{n+1}$:
   (a) If $n+1 \geq 4$ and $\varphi = \tau^m$ then:
      i. $\eta(\alpha) = -\nu(\alpha)$, whenever $\alpha : (k,i) \longrightarrow (k,i+1)$ is an upward arrow with $i \in \{2,...,n-1\}$.
      ii. $\eta(\alpha) = \nu(\alpha)$, whenever $\alpha : (k,i) \longrightarrow (k+1,i-1)$ is downward arrow with $i \in \{3,...,n\}$.
      iii. $\eta(\varepsilon_i) = (-1)^i \nu(\varepsilon_i)$, for the arrow $\varepsilon_i : (k,2) \longrightarrow (k,i)$ $(i = 0,1)$,
      iv. $\eta(\varepsilon_i') = (-1)^{i+1} \nu(\varepsilon_i')$, for the arrow $\varepsilon_i' : (k,i) \longrightarrow (k+1,2)$ $(i = 0,1)$.
   (b) If $n+1 > 4$ and $\varphi = \rho\tau^m$ then:
      i. $\eta(\alpha) = -\nu(\alpha)$, whenever α is an upward arrow as above or $\alpha : (k,i) \longrightarrow (k+1,i-1)$ is downward arrow as above such that $k \equiv -1 \pmod m$.
      ii. $\eta(\alpha) = \nu(\alpha)$, whenever $\alpha : (k,i) \longrightarrow (k+1,i-1)$ is downward arrow such that $k \not\equiv -1 \pmod m$
      iii. For the remaining arrows, if q and r are the quotient and rest of dividing k by m, then
      $\eta(\varepsilon_i) = (-1)^{q+i} \nu(\varepsilon_i)$ $(i = 0,1)$.
      $\eta(\varepsilon_i^{'}) = (-1)^{q+i+1} \nu(\varepsilon_i^{'})$, when $r \neq m-1$, and $\eta(\varepsilon_i^{'}) = (-1)^{q+i} \nu(\varepsilon_i^{'})$ otherwise
   (c) If $n+1 = 4$ and $\varphi = \rho\tau^m$ (see the convention 2.3.7), then:
      i. $\eta(\varepsilon_i) = \nu(\epsilon_i)$, whenever $\epsilon_i : (k,2) \to (k,i)$ $(i = 0,1,3)$
      ii. $\eta(\varepsilon_i') = -\nu(\varepsilon_i')$, whenever $\epsilon_i' : (k,i) \to (k+1,2)$ $(i = 0,1,3)$.
3. When $\Delta = \mathbb{E}_6$:
   (a) If $\varphi = \tau^m$ then:
      i. $\eta(\alpha) = \nu(\alpha)$ and $\eta(\alpha^{'}) = -\nu(\alpha^{'})$, where $\alpha : (k,1) \to (k,2)$ and $\alpha^{'} : (k,2) \to (k+1,1)$.
      ii. $\eta(\beta) = \nu(\beta)$ and $\eta(\beta^{'}) = -\nu(\beta^{'})$, where $\beta : (k,2) \to (k,3)$ and $\beta^{'} : (k,3) \to (k+1,2)$.
      iii. $\eta(\gamma) = \nu(\gamma)$ and $\eta(\gamma^{'}) = -\nu(\gamma^{'})$, where $\gamma : (k,3) \to (k,4)$ and $\gamma^{'} : (k,4) \to (k+1,3)$.
      iv. $\eta(\delta) = -\nu(\delta)$ and $\eta(\delta^{'}) = \nu(\delta^{'})$, where $\delta : (k,4) \to (k,5)$ and $\delta^{'} : (k,5) \to (k+1,4)$.
      v. $\eta(\varepsilon) = -\nu(\varepsilon)$ and $\eta(\varepsilon^{'}) = \nu(\varepsilon^{'})$, where $\varepsilon : (k,3) \to (k,0)$ and $\varepsilon^{'} : (k,0) \to (k+1,3)$.
   (b) If $\varphi = \rho\tau^m$, (k,i) is the origin of the given arrow, q and r are the quotient and rest of dividing k by m, then:

i. $\eta(\alpha) = \nu(\alpha)$.
ii. $\eta(\alpha^{'}) = -\nu(\alpha^{'})$.
iii. $\eta(\beta) = (-1)^q\nu(\beta)$
iv. $\eta(\beta^{'}) = (-1)^{q+1}\nu(\beta^{'})$
v. $\eta(\gamma) = (-1)^q\nu(\gamma)$
vi. $\eta(\gamma^{'}) = \nu(\gamma^{'})$, when either q is odd and $r \neq m-1$ or q is even and $r = m-1$, and $\eta(\gamma^{'}) = -\nu(\gamma^{'})$ otherwise.
vii. $\eta(\delta) = -\nu(\delta)$
viii. $\eta(\delta^{'}) = \nu(\delta^{'})$.
ix. $\eta(\varepsilon) = -\nu(\varepsilon)$
x. $\eta(\varepsilon^{'}) = -\nu(\varepsilon^{'})$, when $r = m-1$, and $\eta(\varepsilon^{'}) = \nu(\varepsilon^{'})$ otherwise.

4. When $\Delta = \mathbb{E}_7$, $\varphi = \tau^m$, and then:

   i $\eta(a)$ is given as in 3.(a) for any arrow a contained in the copy of $\mathbb{E}_6$.
   ii $\eta(\zeta) = \nu(\zeta)$ and $\eta(\zeta^{'}) = -\nu(\zeta^{'})$, where $\zeta : (k,5) \to (k,6)$ and $\zeta^{'} : (k,6) \to (k+1,5)$.

5. When $\Delta = \mathbb{E}_8$, $\varphi = \tau^m$, and then:

   i $\eta(a)$ is given as in 4 for any arrow a contained in the copy of $\mathbb{E}_7$.
   ii $\eta(\theta) = \nu(\theta)$ and $\eta(\theta^{'}) = -\nu(\theta^{'})$, where $\theta : (k,6) \to (k,7)$ and $\theta^{'} : (k,7) \to (k+1,6)$.

Chapter 3:

The results in this chapter deal with questions 1, 2 and 3. We start the chapter by proving two key results. The first of them (Lemma 3.2.1) determines in particular when two G-invariant automorphisms of the mesh algebra B induce, up to conjugation, the same automorphism of the m-fold mesh algebra $\Lambda = B/G$. The second one (Proposition 3.2.2) identifies the subgroup H consisting of the integers s such that the Nakayama automorphism and the Nakayama permutation of Λ, $\bar{\eta}$ and $\bar{\nu}$ respectively, coincide, up to inner automorphism, in their s-th power. That is, H consists of the $s \in \mathbb{Z}$ such that $\bar{\eta}^s\bar{\nu}^{-s}$ is a inner automorphism of Λ. The subgroup H is crucial for our purposes in this chapter.

With all our tools in place, Section 3.3 is devoted to answer question 3. The only theorem of this section identifies all symmetric and weakly symmetric m-fold mesh algebras:

THEOREM. Let Λ be an m-fold mesh algebra of extended type (Δ, m, t) and c_Δ be the Coxeter number of Δ. If Λ is weakly symmetric then $t = 1$ or $t = 2$ and, when $\text{char}(K) = 2$ or $\Delta = \mathbb{A}_r$, such an algebra is also symmetric. Moreover, the following assertions hold:

1. When $t = 1$, Λ is weakly symmetric if, and only if, Δ is $\mathbb{D}_{2r}$, $\mathbb{E}_7$ or $\mathbb{E}_8$ and m is a divisor of $\frac{c_\Delta}{2} - 1$. When $\text{char}(K) \neq 2$, such an algebra is symmetric if, and only if, m is even.

2. When $t = 2$ and $\Delta \neq \mathbb{A}_{2n}$, Λ is weakly symmetric if, and only if, m divides $\frac{c_\Delta}{2} - 1$ and, moreover, the quotient of the division is odd, in case $\Delta = \mathbb{A}_{2n-1}$, and even, in case $\Delta = \mathbb{D}_{2r}$. When $\text{char}(K) \neq 2$, such an algebra is symmetric if, and only if, $\Delta = \mathbb{A}_{2n-1}$ or m is odd.
3. When $(\Delta, m, t) = (\mathbb{A}_{2n}, m, 2)$, i.e. $\Lambda = \mathbb{L}_n^{(m)}$, the algebra is (weakly) symmetric if, and only if, $2m - 1$ divides $2n - 1$.

In Section 3.4 we determine the period and the Calabi-Yau dimension of all m-fold mesh algebras. First, in Subsection 3.4.1 we calculate explicitly the initial part of a 'G-invariant' minimal projective resolution of B as a graded B-bimodule. We prove in particular that $\Omega^3_{B^e}(B)$ is always isomorphic to ${}_\mu B_1$, for a graded automorphism μ of B which is in the centralizer of G and which is explicitly calculated (Proposition 3.4.3). As a consequence of the G-invariance of our previous results, we get that the induced automorphism $\bar{\mu}$ of $\Lambda = B/G$ has the property that $\Omega^3_{\Lambda^e}(\Lambda) \cong {}_{\bar{\mu}}\Lambda_1$ and this is fundamental in the rest of the work.

Next, we introduce the concept of stably inner automorphism which is, in general, a weaker condition than the condition of inner automorphism. Concretely, we shall say that an automorphism σ of Λ is stably inner if the functor ${}_\sigma(-) \cong_\sigma \Lambda_1 \otimes_\Lambda - : \Lambda - \underline{\text{mod}} \longrightarrow \Lambda - \underline{\text{mod}}$ is naturally isomorphic to the identity functor. However, regarding the m-fold mesh algebras, both concepts coincide in many cases. For instance, when the algebra Λ has Loewy length greater or equal than 4 (see Lemma 3.4.6).

We answer the questions 1 and 2 in Subsections 3.4.3 and 3.4.4. In the first one we compute explicitly the period of every m-fold mesh algebra, that is, the smallest positive integer r such that $\Omega^r_{\Lambda^e}(\Lambda)$ is isomorphic to Λ as a Λ-bimodule. We first distinguish the case $\Delta = \mathbb{A}_2$ (Proposition 3.4.8), where the algebra has Lowey length 2:

PROPOSITION. Let Λ be a connected self-injective algebra of Loewy length 2. The following assertions hold:

1. If $\text{char}(K) = 2$ or $\Lambda = \mathbb{A}_2^{(m)}$, i.e. $|Q_0|$ is even, then the period of Λ is $|Q_0|$.
2. If $\text{char}(K) \neq 2$ and $\Lambda = \mathbb{L}_1^{(m)}$, i.e. $|Q_0|$ is odd, then the period of Λ is $2|Q_0|$.

We deal with the remaining cases in Theorem 3.4.12, whose statement in characteristic $\neq 2$ is the following:

THEOREM. Let Λ be an m-fold mesh algebra of extended type (Δ, m, t), where $\Delta \neq \mathbb{A}_1, \mathbb{A}_2$, let $\pi = \pi(\Lambda)$ denote the period of Λ and, for each positive integer k, denote by $O_2(k)$ the biggest of the natural numbers r such that 2^r divides k. When $\text{char}(K) \neq 2$, the period of Λ is given as follows:

1. If $t = 1$ then:
   (a) When Δ is $\mathbb{A}_r$, $\mathbb{D}_{2r-1}$ or $\mathbb{E}_6$, the period is $\pi = \frac{6m}{\gcd(m, c_\Delta)}$.

(b) When Δ is $\mathbb{D}_{2r}$, $\mathbb{E}_7$ or $\mathbb{E}_8$, the period is $\pi = \frac{3m}{\gcd(m, \frac{c_\Delta}{2})}$, when m is even, and $\pi = \frac{6m}{\gcd(m, \frac{c_\Delta}{2})}$, when m is odd.

2. If $t = 2$ then:

   (a) When Δ is $\mathbb{A}_{2n-1}$, $\mathbb{D}_{2r-1}$ or $\mathbb{E}_6$, the period is $\frac{6m}{\gcd(2m, m+\frac{c_\Delta}{2})}$, when $O_2(m) \neq O_2(\frac{c_\Delta}{2})$, and $\pi = \frac{12m}{\gcd(2m, m+\frac{c_\Delta}{2})}$ otherwise.

   (b) When $\Delta = \mathbb{D}_{2r}$, the period is $\frac{6m}{\gcd(2m, \frac{c_\Delta}{2})} = \frac{6m}{\gcd(2m, 2r-1)}$.

   (c) When $\Delta = \mathbb{A}_{2n}$, i.e. $\Lambda = \mathbb{L}_n^{(m)}$, the period is $\pi = \frac{6(2m-1)}{\gcd(2m-1, 2n+1)}$

3. If $t = 3$ then $\pi = 3m$, when m is even, and $6m$, when m is odd.

At the end of the chapter, where we study Calabi-Yau dimensions, we emphasize two results. On one hand, combining the following propositions, we find the precise relation between the stable Calabi-Yau and the Calabi-Yau Frobenius dimensions of an m-fold mesh algebra showing that both dimensions may differ when $\Delta = \mathbb{A}_2$ but are always equal when $\Delta \neq \mathbb{A}_r$, for $r = 1, 2$.

PROPOSITION. Let Λ be a connected self-injective algebra of Loewy length 2. Then Λ is always a stably Calabi-Yau algebra and the following equalities hold:

1. If $\mathrm{char}(K) = 2$ or $\Lambda = \mathbb{A}_2^{(m)}$, i.e. $|Q_0|$ is even, then $CY - \dim(\Lambda) = CYF - \dim(\Lambda) = 0$.
2. If $\mathrm{char}(K) \neq 2$ and $\Lambda = \mathbb{L}_1^{(m)}$, i.e., $|Q_0|$ odd, then $CY - \dim(\Lambda) = 0$ and $CYF - \dim(\Lambda) = 2m - 1 = |Q_0|$.

PROPOSITION. Let Λ be an m-fold mesh algebra of Dynkin type Δ different from $\mathbb{A}_r$, for $r = 1, 2, 3$. Then Λ is stably Calabi-Yau if, and only if, it is Calabi-Yau Frobenius. In such case the equality $CY - \dim(\Lambda) = CYF - \dim(\Lambda)$ holds.

To end this chapter and also the first part of this thesis, we provide for the class of the m-fold mesh algebras, a criterion to determine when they are stably Calabi-Yau together with an identification in such case of the stable Calabi-Yau dimension. Corollary 3.4.18 deals with the case when K has characteristic 2. When $\mathrm{char}(K) \neq 2$, we have the following:

THEOREM. Let us assume that $\mathrm{char}(K) \neq 2$ and let Λ be the m-fold mesh algebra of extended type (Δ, m, t), where $\Delta \neq \mathbb{A}_1, \mathbb{A}_2$. We adopt the convention that if a, b, k are fixed integers, then $au \equiv b \pmod{k}$ means that u is the smallest positive integer satisfying the congruence. The algebra is Calabi-Yau Frobenius if, and only if, it is stably Calabi-Yau. Moreover, we have $CYF - \dim(\Lambda) = CY - \dim(\Lambda)$ and the following assertions hold:

1. If $t = 1$ then

(a) When Δ is $\mathbb{A}_r$, $\mathbb{D}_{2r-1}$ or $\mathbb{E}_6$, the algebra is stably Calabi-Yau if, and only if, $\gcd(m, c_\Delta) = 1$. Then $CY - \dim(\Lambda) = 6u + 2$, where $c_\Delta u \equiv -1 \pmod{m}$.

(b) When Δ is $\mathbb{D}_{2r}$, $\mathbb{E}_7$ or $\mathbb{E}_8$, the algebra is stably Calabi-Yau if, and only if, $\gcd(m, \frac{c_\Delta}{2}) = 1$. Then:

   i. $CY - \dim(\Lambda) = 3u + 2$, where $\frac{c_\Delta}{2} u \equiv -1 \pmod{m}$, whenever m is even;

   ii. $CY - \dim(\Lambda) = 6u + 2$, where $c_\Delta u \equiv -1 \pmod{m}$, whenever m is odd;

2. If $t = 2$ then

   (a) When Δ is $\mathbb{A}_{2n-1}$, $\mathbb{D}_{2r-1}$ or $\mathbb{E}_6$, the algebra is stably Calabi-Yau if, and only if, $\gcd(2m, m + \frac{c_\Delta}{2}) = 1$. Then $CY - \dim(\Lambda) = 3u + 2$, where $(m + \frac{c_\Delta}{2})u \equiv -1 \pmod{2m}$.

   (b) When $\Delta = \mathbb{D}_{2r}$, the algebra is stably Calabi-Yau if, and only if, $\gcd(m, 2r-1) = 1$ and m is odd. Then $CY - \dim(\Lambda) = 3u + 2$, where $(2r - 1)u \equiv -1 \pmod{2m}$.

   (c) When $\Delta = \mathbb{A}_{2n}$, the algebra is stably Calabi-Yau if, and only if, $\gcd(2m - 1, 2n + 1) = 1$. Then $CY - \dim(\Lambda) = 6u - 1$, where $(m + n)(2u - 1) \equiv -1 \pmod{2m - 1}$

3. If $t = 3$ then the algebra is not stably Calabi-Yau.

The results of Chapters 1, 2 and 3 appear in [5].

Chapter 4:

The three last chapters of this dissertation deal with the question 4 mentioned above, that is, the study of the Hochschild cohomology ring of the m-fold mesh algebras of extended type $(\mathbb{A}_{2n}, 1, 2)$ and $(\mathbb{A}_{2n-1}, 1, 2)$, known as the generalized preprojective algebras $\mathbb{L}_n$ and $\mathbb{B}_n$, respectively. We start with the introductory Chapter 4, where we briefly present the notions and results needed to approach the problem. Hence, in the first section we recall the notion of the Yoneda product of extensions and we define the fundamental concept of the Hochschild cohomology ring of an algebra Λ which will be denoted by $HH^*(\Lambda)$. Initially, Λ will be an algebra over a commutative ring R that, in addition, it is supposed to be projective as an R-module. Later in this work, we will assume that R is a field and, in order to emphasize this fact, we will write $R = K$ in such case. In Section 4.3 we introduce the stable Hochschild cohomology ring of an R-algebra Λ, which is also supposed to be Gorenstein projective as Λ-bimodule, denoted by $\underline{HH}^*(\Lambda)$. When the algebra is symmetric, for instance, when $\Lambda = \mathbb{L}_n$ or $\Lambda = \mathbb{B}_n$ with n even, we obtain that $HH_*(\Lambda) \cong D(HH^*(\Lambda))$ as $HH^*(\Lambda)$-modules. Also, in Section 4.4 we show that, apart from the homological grading in which $HH^n(\Lambda)$ is the homogeneous component of degree n, the ring $HH^*(\Lambda)$ inherits the grading by path length of Λ, making $HH^*(\Lambda)$ into a bigraded ($= \mathbb{Z} \times \mathbb{Z}$ graded) R-algebra . In Section 4.5 we review the notion and essential properties of Frobenius algebras, as introduced by Eu-Schedler in [35]. It is worth mentioning that when Λ is a Frobenius algebra, then the stable Hochschild cohomology ring $\underline{HH}^*(\Lambda)$ is graded-commutative (Proposition 4.5.2) and also a localization of its classical version (Proposition 4.5.6). Finally, we include Section 4.6, which is devoted to self-injective algebras. Most of the results given in this section are easy applications of the results obtained in Chapter 1 on pseudo-Frobenius algebras.

Chapter 5:

The results in this chapter, concerning the study of the Hochschild cohomology ring of the generalized preprojective algebra $\mathbb{L}_n$ over a commutative ring R, appear in [3] and [4] for the cases when the characteristic is different and equal to 2, respectively. Specifically, we describe its structure as a bigraded algebra under the Yoneda product by giving an explicit presentation by homogeneous generators and relations. It is important to keep in mind two considerations regarding the case $\mathbb{L}_n$. On one hand, our approach to the proof of the main theorem is to first prove it when R is a field, and then to deduce from this the general statement when R is a commutative ring on which 2 is invertible. On the other hand, we shall point out that the results of this chapter were obtained before we realized about the change of relations that simplifies some calculations and which is presented in Subsection 2.3.3 of this thesis. Thus, throughout this chapter we will consider the original mesh relations, given as sums of paths instead of differences.

As usual, we separate the case when the field has characteristic 2 from the rest (see Section 5.5). The most remarkable difference is that, when the characteristic is different from 2, the algebra $\mathbb{L}_n$ has period 6 while, on the contrary, the period is exactly 3 when Char $\neq$ 2. As we mentioned in the first part of the introduction, this periodicity is translated to the cohomology groups. In Section 5.2 we provide the elements needed for our calculations: we begin the section by introducing the definition of the algebra $\mathbb{L}_n$ given by its quiver and relations and then, since it is symmetric, we show the dualizable basis which will be used. Next, we give the minimal projective resolution of the algebra as a graded bimodule that induces the cochain complex calculating the cohomology (Proposition 3.4.2).

With all our tools in place, we proceed to calculate the dimensions of the Hochschild (co)homology spaces, as well as the cyclic homology spaces in zero characteristic. In particular, putting $\Lambda := \mathbb{L}_n$, we give a canonical basis of each $HH^i(\Lambda)$ consisting of homogeneous elements with respect to the length degree (Proposition 5.3.10). This is done by identifying previously the structure of each $HH^i(\Lambda)$ as a module over $Z(\Lambda) = HH^0(\Lambda)$.

The main result of this chapter, given in Section 5.4 and which shows the multiplicative structure of de $HH^*(\Lambda)$, is Theorem 5.4.1, from which we obtain a presentation of the stable Hochschild cohomology ring of Λ, $\underline{HH}^*(\Lambda)$, given by generators and relations.

THEOREM. Let Λ be the generalized preprojective algebra $\mathbb{L}_n$ over a commutative ring R on which 2 is invertible. The following assertions hold for the Hochschild cohomology ring $HH^*(\Lambda)$:

1. $HH^*(\Lambda)$ is the commutative bigraded R-algebra given by

a) Generators: $x_0, x_1, \ldots, x_n, y, z_1, \ldots, z_n, t_1, t_2, \ldots t_{n-1}, \gamma, h$

b) Relations:

i) $x_i\xi = 0$ for each $i = 1, \ldots, n$ and each generator ξ.

ii) $x_0^n = y^2 = x_0z_j = x_0t_i = yt_i = t_it_k = 0$, $(j = 1, \ldots, n \quad i, k = 1, \ldots n-1)$

iii) $z_jz_k = (-1)^{k-j+1}(2j-1)(n-k+1)x_0^{n-1}\gamma$, for $1 \leq j \leq k \leq n$.

iv) $z_j\gamma = (-1)^j(n-j+1)x_0^{n-1}h$, for $j = 1, ..., n$

v) $\gamma^2 = z_1h$

vi) $yz_j = (2n+1)\sum_{1\leq k\leq j-1}(-1)^{j-k}(j-k)t_k + (-1)^{j-1}(2j-1)yz_1$, for $j = 2, ..., n$

vii) $z_kt_j = \delta_{jk}x_0^{n-1}y\gamma$, for $k = 1, \ldots, n \quad j = 1\ldots, n-1$

viii) $t_j\gamma = \delta_{1j}x_0^{n-1}yh$, for $j = 1, \ldots, n-1$.

2. The homological grading on $HH^*(\Lambda)$ is determined by the equalities $\deg(x_i) = 0$, $\deg(y) = 1$, $\deg(z_j) = 2$, $\deg(t_k) = 3$, $\deg(\gamma) = 4$ and $\deg(h) = 6$.

3. The length grading on $HH^*(\Lambda)$ is determined by the equalities $\text{ldeg}(x_0) = 2$, $\text{ldeg}(x_i) = 2n-1$, for $i \neq 0$, $\text{ldeg}(y) = 0$, $\text{ldeg}(z_j) = -2$, $\text{ldeg}(t_k) = -2$, $\text{ldeg}(\gamma) = -2n-2$ and $\text{ldeg}(h) = -4n-2$.

4. Multiplication by h gives an isomorphism $HH^i(\Lambda) \xrightarrow{\cong} HH^{i+6}(\Lambda)$, for each $i > 0$.

5. All $HH^i(\Lambda)$ are free R-modules, and the following are bases for them (see Proposition 5.3.10):

(a) For $HH^0(\Lambda)$: $\{x_0, x_0^2, \ldots, x_0^{n-1}, x_1, \ldots, x_n\}$.

(b) For $HH^1(\Lambda)$: $\{y, x_0y, x_0^2y, \ldots x_0^{n-1}y\}$.

(c) For $HH^2(\Lambda)$: $\{z_1, \ldots, z_n\}$.

(d) For $HH^3(\Lambda)$: $\{t_1, \ldots, t_{n-1}, yz_1\}$.

(e) For $HH^4(\Lambda)$: $\{x_0^{n-1}\gamma, \ldots, x_0\gamma, \gamma\}$.

(f) For $HH^5(\Lambda)$: $\{x_0^{n-1}y\gamma, \ldots, x_0y\gamma, y\gamma\}$.

(g) For $HH^6(\Lambda)$: $\{h, x_0h, \ldots, x_0^{n-1}h\}$.

In particular $\dim(HH^0(\Lambda)) = 2n$ and $\dim(HH^i(\Lambda)) = n$, for all $i > 0$, where $\dim(-)$ denotes the rank as a free R-module.

We end the chapter with Section 5.5 where we present the analogous result to the previous theorem when the field K has characteristic 2 (Theorem 5.5.1).

Chapter 6:

In this final chapter we complete the answer to question 4 by the study of the Hochschild cohomology ring of the generalized preprojective algebra $\mathbb{B}_n$ over a field of characteristic different from 2. In such case, due to Theorem 3.4.12 we know that $\mathbb{B}_n$ has period 6 and, consequently, the cohomology spaces have also period 6. The strategy used in this last chapter is rather different from the previous one. Unlike the case $\mathbb{L}_n$, where the algebra was given by its quiver and relations with no mention to its relationship with the stable translation quiver $\mathbb{Z}\mathbb{A}_{2n}$, $\mathbb{B}_n$ will be considered as the orbit algebra $\mathbb{Z}\mathbb{A}_{2n-1}/\langle\rho\tau\rangle$ where τ and ρ are the Auslander-Reiten translation and the automorphism given by the natural reflection of $\mathbb{Z}\mathbb{A}_{2n-1}$, respectively, or equivalently, considered as the m-fold mesh algebra of extended type $(\mathbb{A}_{2n-1}, 1, 2)$ (see Subsection 6.5.1). The advantage in this case is that we can use the results obtained in the first three chapters in order to get relevant information for our purposes. In Section 6.5.2 and using Theorem 6.5.3, we compute the Cartan matrix of $\mathbb{B}_n$. In Subsection 6.5.4, and from the projective resolution of the mesh algebra $B = B(\mathbb{A}_{2n-1})$ as B-bimodule, we describe that of $\mathbb{B}_n$ which induces, as shown in Subsection 6.5.5, the cochain complex that induces in turn the Hochschild cohomology. We continue with Section 6.6 where, putting $\Lambda = \mathbb{B}_n$, we identify the structure of each cohomology space $HH^i(\Lambda)$ as $Z(\Lambda)$-bimodule and, as a necessary tool, the ideal $I = \mathcal{P}(\Lambda, \Lambda)$ of $Z(\Lambda) = End_{\Lambda^e}(\Lambda)$ consisting of all endomorphisms of Λ as a bimodule that factor through a projective bimodule. This description depends on the parity of n, which is basically due to the fact that the Nakayama automorphism is the identity when n is even and τ when n is odd. That is, Λ is symmetric when n is even but it is not even weakly symmetric when n is odd. To end the chapter, and therefore this thesis, throughout Section 6.7 we present the two major results (for n odd and n even) which describe by means of generators and relations the structure as a bigraded algebra of the Hochschild cohomology ring $HH^*(\Lambda)$ over a field of characteristic $\neq 2$.

Theorem 6.7.1 deals with the case n odd:

THEOREM. Let n be odd and let Λ be the generalized preprojective algebra $\mathbb{B}_n$ over a field of characteristic $\neq 2$ and let view $HH^*(\Lambda)$ as a bigraded algebra (see Section 2). Consider the following elements of $HH^*(\Lambda)$:

a) $x = \sum_{3\leq i\leq 2n-3} c_i \in HH^0(\Lambda) = Z(\Lambda)$, where c_i is the nonzero cycle of length 4 at i;

b) $y \in HH^1(\Lambda)$ represented by $\tilde{y} : \oplus_{a\in Q_1}\Lambda e_{i(a)}\otimes e_{t(a)}\Lambda \longrightarrow \Lambda$, where $\tilde{y}(e_{i(a)}\otimes e_{t(a)}) = a$, for all $a \in Q_1$;

c) $h \in HH^6(\Lambda)$ represented by the multiplication map $\oplus_{i\in Q_0}\Lambda e_i\otimes e_i\Lambda \longrightarrow \Lambda$; and

d) In case char(K) divides n, the element $v \in HH^5(\Lambda)$ represented by $\tilde{v}$: $\oplus_{i\in Q_0}\Lambda e_{\eta(i)} \otimes e_i\Lambda = \oplus_{i\in Q_0}\Lambda e_{\tau(i)} \otimes e_i\Lambda \longrightarrow \Lambda$, where $\tilde{v}(e_{\tau(i)} \otimes e_i) = \delta_{in}e_n$, for all $i \in Q_0$, and δ_{in} is the Kronecker symbol.

Then we have bideg$(x) = (0, 4)$, bideg$(y) = (1, 0)$, bideg$(h) = (6, -4n)$ and bideg$(v) = (5, -2n - 2)$ and the following assertions hold:

1. If char(K) does not divide n, then $HH^*(\Lambda)$ is the commutative bigraded algebra with generators x, y, h, subject to the relations:

$$x^{\frac{n+1}{2}} = 0, \qquad x^{\frac{n-1}{2}} y = 0, \qquad x^{\frac{n-1}{2}} h = 0 \qquad \text{and} \qquad y^2 = 0.$$

2. If $\mathrm{char}(K)$ divides n, then $HH^*(\Lambda)$ is the commutative bigraded algebra with generators x, y, v, h, subject to the relations

$$x^{\frac{n+1}{2}} = 0, \qquad x^{\frac{n-1}{2}} y = 0, \qquad xv = 0, \qquad y^2 = 0, \qquad yv = 0 \qquad \text{and} \qquad v^2 = 0.$$

When n is even, the result is the following (Theorem 6.7.10)

THEOREM. Let n be even, let $\Lambda = \mathbb{B}_n$ be the generalized preprojective algebra over a field of characteristic $\neq 2$ and let view $HH^*(\Lambda)$ as a bigraded algebra (see Section 2). Consider the following elements of $HH^*(\Lambda)$:

1) $x, x_1, ..., x_{2n-1}$ of $HH^0(\Lambda) = Z(\Lambda)$ given as follows:
   (a) $x = \sum_{3 \leq i \leq 2n-3} c_i$, where c_i is the nonzero cycle of length 4 at i;
   (b) $\{x_1, ..., x_{n-2}\}$ given by $x_{2k} = \omega_{2k}$ and $x_{2k-1} = \sum_{r=1}^{k} \omega_{2r-1}$, for all $0 < k < \frac{n}{2}$;
   (c) $x_{n-1} = \sum_{1 \leq k \leq \frac{n}{2}} (\omega_{2k-1} - \omega_{2n-2k+1})$, $x_n = \omega_n$ and $x_i = \omega_i + \omega_{2n-i}$, for all $n < i \leq 2n-1$.
2) $y \in HH^1(\Lambda)$ represented by $\tilde{y} : \oplus_{a \in Q_1} \Lambda e_{i(a)} \otimes e_{t(a)} \Lambda \longrightarrow \Lambda$, where $\tilde{y}(e_{i(a)} \otimes e_{t(a)}) = a$;
3) $z \in HH^2(\Lambda)$ represented by $\tilde{z} : \oplus_{i \in Q_0} \Lambda e_{\tau(i)} \otimes e_i \Lambda \longrightarrow \Lambda$, where $\tilde{z}(e_{\tau(i)} \otimes e_i) = \delta_{in} e_n$;
4) $t \in HH^3(\Lambda)$ represented by $\tilde{t} : \oplus_{i \in Q_0} \Lambda e_{\tau(i)} \otimes e_i \Lambda \longrightarrow \Lambda$, where $\tilde{t}(e_{\tau(i)} \otimes e_i) = \delta_{in} w_n$;
5) $u \in HH^4(\Lambda)$ represented by $\tilde{u} = \oplus_{a \in Q_1} \Lambda e_{\tau(i(a))} \otimes e_{t(a)} \Lambda \longrightarrow \Lambda$, where $\tilde{u}(e_{\tau(i(a))} \otimes e_{t(a)}) = \frac{1}{2} \delta_{a, \alpha_n} \alpha_n - \frac{1}{2} \delta_{a, \beta_n} \beta_{n-1}$;
6) $v_1, ..., v_{n-2} \in HH^5(\Lambda)$, with each v_j is represented by $\tilde{v_j} = \oplus_{i \in Q_0} \Lambda e_i \otimes e_i \Lambda \longrightarrow \Lambda$, where:
   (a) $\tilde{v_j}(e_i \otimes e_i) = (\delta_{ij} - \delta_{i,2n-j}) e_i$, whenever j is even;
   (b) $\tilde{v_j}(e_i \otimes e_i) = (\delta_{ij} - \delta_{i,j+2} - \delta_{i,2n-j} + \delta_{i,2n-j-2}) e_i$, whenever j is odd.
7) $h \in HH^6(\Lambda)$ represented by the multiplication map $\tilde{h} : \oplus_{i \in Q_0} \Lambda e_i \otimes e_i \Lambda \longrightarrow \Lambda$.

The length degrees of these elements are $\mathrm{ldeg}(x) = 4$, $\mathrm{ldeg}(x_i) = 2n-2$, $\mathrm{ldeg}(y) = 0$, $\mathrm{ldeg}(z) = \mathrm{ldeg}(t) = -2$, $\mathrm{ldeg}(u) = -2n$, $\mathrm{ldeg}(v_j) = -2n-2$ and $\mathrm{ldeg}(h) = -4n$.

Moreover, as an algebra, $HH^*(\Lambda)$ is generated by these elements, subject to the graded commutativity relations with respect to the homological grading plus the following relations:

a) $x^{\frac{n}{2}} = xz = xt = xv_i = 0$

b) $x_i\xi = 0$, for each generator ξ, except in case $i \leq n-2$ and $\xi = v_i$ or $\xi = h$

c) $x_iv_i = x^{\frac{n}{2}-1}yu$, for all $i \leq n-2$.

d) $y^2 = yt = 0$

e) $yz = -nt$

f) $yv_{2k} = 4nx_{2k-1}h$ and $yv_{2k-1} = -4nx_{2k}h$, for all $1 \leq k < \frac{n}{2}$

g) $z^2 = -nx^{\frac{n}{2}-1}u$

h) $zt = x^{\frac{n}{2}-1}yu$

i) $zv_j = 0$, for all $j = 1, ..., n-2$

j) $zu = 0$

k) $t^2 = tu = tv_j = 0$, for all $j = 1, ..., n-2$

l) $u^2 = 0$

m) $uv_j = 0$, for all $j = 1, ..., n-2$

n) $v_{2k}v_j = 4n\delta_{2k-1,j}x^{\frac{n}{2}}uh$, for all $1 \leq k < \frac{n}{2}$ and $j = 1, 2, ..., n-2$.

Notice that Chapters 5 and 6 make clear differences in the behaviour of the Hochschild cohomology ring $HH^*(\Lambda)$ for $\Lambda = \mathbb{L}_n$ and $\mathbb{B}_n$, and, moreover, between the cases n odd and n even of $\mathbb{B}_n$. For instance, for $\mathbb{L}_n$, $HH^*(\Lambda)$ is always commutative, as it is for $\mathbb{B}_n$, when n is odd. However, for $\mathbb{B}_n$ with n even, and unless $\mathrm{Char}(K)$ divides n, the algebra $HH^*(\Lambda)$ is not commutative anymore.

We include at the end of this dissertation the reference list that we have used.

Chapter 1

Pseudo-Frobenius graded algebras with enough idempotents

1.1 Introduction

1.1.1 Motivation

As we mention in the introduction, any finite dimensional mesh algebra (i.e. m-fold mesh algebra) comes from the mesh algebra of an infinite translation quiver, an algebra with enough idempotents, which satisfies the property of being pseudo-Frobenius. It turns out then that m-fold mesh algebras can be basically understood at the level of their corresponding Galois covers. This motivates the study of the pseudo-Frobenius condition on an algebra with enough idempotents, specially in the case when the algebra is also endowed with a gradation.

1.1.2 Outline of the chapter

In Section 1.2 we establish some notation and we briefly review the notion of graded algebra, recalling that it is in correspondence with that of small graded K-category. We will not provide proofs, but we will give classical references where it can be found. In Section 1.3 we introduce and develop the concept of pseudo-Frobenius graded algebra with enough idempotents. We characterize such algebras and, in addition, we guarantee the existence of the so-called Nakayama automorphism. Section 1.4 is devoted to the study of the preservation of the pseudo-Frobenius condition via the usual covering functor. In particular, we give a necessary condition on the Galois cover of any m-fold mesh algebra so that the associated covering preserves the pseudo-Frobenius condition. Furthermore, under the previous hypothesis, we show that the Nakayama automorphism of the m-fold mesh algebra is induced by the Nakayama automorphism of its Galois cover.

1.1.3 Notation

Throughout this chapter, K is a field and the term 'algebra' will mean always an associative K-algebra. Recall that such an algebra A is said to be an *algebra with enough idempotents*, when there is a family $(e_i)_{i\in I}$ of nonzero orthogonal idempotents such that $\oplus_{i\in I}e_iA = A = \oplus_{i\in I}Ae_i$. Any such family $(e_i)_{i\in I}$ will be called a *distinguished family.* From now on in this chapter A is an algebra with enough idempotents on which we fix a distinguished family of orthogonal idempotents.

All considered (left or right) A-modules are supposed to be unital. For a left (resp. right) A-module M, that means that $AM = M$ (resp. $MA = M$) or, equivalently, that $M = \oplus_{i\in I}e_iM$ (resp. $M = \oplus_{i\in I}Me_i$). We denote by $A-\text{Mod}$ and $\text{Mod}-A$ the categories of left and right A-modules, respectively.

The enveloping algebra of A is the algebra $A^e = A \otimes A^{op}$, where if $a, b \in A$ we will denote by $a \otimes b^o$ the corresponding element of A^e. This is also an algebra with enough idempotents. The distinguished family of orthogonal idempotents which we will work with is the family $(e_i \otimes e_j^o)_{(i,j)\in I\times I}$. A left A^e-module M will be identified with an A-bimodule by putting $axb = (a \otimes b^o)x$, for all $x \in M$ and $a, b \in A$. Similarly, a right A^e-module is identified with an A-bimodule by putting $axb = x(b \otimes a^o)$, for all $x \in M$ and $a, b \in A$. In this way, we identify the three categories $A^e-\text{Mod}$, $\text{Mod}-A^e$ and $A-\text{Mod}-A$, where the last one is the category of unitary A-bimodules, which we will simply name 'bimodules'.

1.2 Preliminaries

1.2.1 Graded algebras with enough idempotents

We start by fixing some notation and basic definitions concerning graded algebras with enough idempotents. Let H be an abelian group with additive notation, fixed throughout this paragraph. An *H-graded algebra with enough idempotents* will be an algebra with enough idempotents A, together with an H-grading $A = \oplus_{h\in H}A_h$, such that one can choose a distinguished family of orthogonal idempotents which are homogeneous of degree 0. Such a family $(e_i)_{i\in I}$ will be fixed from now on. We will denote by $A - Gr$ (resp. $Gr - A$) the category (H-)graded (always unital) left (resp. right) modules, where the morphisms are the graded homomorphisms of degree 0. A *locally finite dimensional left (resp. right) graded A-module* is a graded module $M = \oplus_{h\in H}M_h$ such that, for each $i \in I$ and each $h \in H$, the vector space e_iM_h (resp. M_he_i) is finite dimensional. Note that the definition does not depend on the distinguished family (e_i). We will denote by $A - lfdgr$ and $lfdgr - A$ the categories of left and right locally finite dimensional graded modules.

Given a graded left A-module M, we denote by $D(M)$ the subspace of the vector space $\text{Hom}_K(M, K)$ consisting of the linear forms $f : M \longrightarrow K$ such that $f(e_iM_h) = 0$, for all but finitely many $(i, h) \in I \times H$. The K-vector space $D(M)$ has a canonical structure of graded right A-module given as follows. The multiplication $D(M) \times A \longrightarrow D(M)$ takes $(f, a) \rightsquigarrow fa$, where $(fa)(x) = f(ax)$ for all $x \in M$. Note that then one has $fe_i = 0$, for all but finitely many $i \in I$, and $f = \sum_{i\in I} fe_i$. Therefore $D(M)$ is unital. On the other hand, if we put $D(M)_h := \{f \in D(M) : f(M_k) = 0, \text{ for all } k \in H \setminus \{-h\}\}$, we get a decomposition $D(M) = \oplus_{h\in H}D(M)_h$ which makes $D(M)$ into a graded right A-modules. Note that $D(M)_he_i$ can be identified with $\text{Hom}_K(e_iM_{-h}, K)$, for all $(i, h) \in I \times H$. We will call $D(M)$ the *dual graded module* of M.

Recall that if M is a graded A-module and $k \in H$ is any element, then we get a graded module $M[k]$ having the same underlying ungraded A-module as M, but where $M[k]_h =$

M_{k+h} for each $h \in H$. If M and N are graded left A-modules, then $\mathrm{HOM}_A(M,N) := \oplus_{h\in H}\mathrm{Hom}_{A-Gr}(M,N[h])$ has a structure of graded K-vector space, where the homogeneous component of degree h is precisely $\mathrm{HOM}_A(M,N)_h := \mathrm{Hom}_{A-Gr}(M,N[h])$, i.e., $\mathrm{HOM}_A(M,N)_h$ consists of the graded homomorphisms $M \longrightarrow N$ of degree h. The following is an analogue of classical results for associative rings with unit, whose proof can be easily adapted (see, e.g., [8], Section II.3 and [65], Proposition I.2.14 et sqq.).

Proposition 1.2.1. *The assignment $M \rightsquigarrow D(M)$ extends to an exact contravariant K-linear functor $D : A - Gr \longrightarrow Gr - A$ (resp. $D : Gr - A \longrightarrow A - Gr$) satisfying the following properties:*

1. *The maps $\sigma_M : M \longrightarrow D^2(M) := (D \circ D)(M)$, where $\sigma_M(m)(f) = f(m)$ for all $m \in M$ and $f \in D(M)$, are all injective and give a natural transformation $\sigma : 1_{A-Gr} \longrightarrow D^2 := D \circ D$ (resp. $\sigma : 1_{Gr-A} \longrightarrow D^2 := D \circ D$)*
2. *If M is locally finite dimensional then σ_M is an isomorphism*
3. *The restrictions of D to the subcategories of locally finite dimensional graded A-modules define mutually inverse dualities $D : A - lfdgr \stackrel{\cong^{op}}{\longleftrightarrow} lfdgr - A : D$.*
4. *If M and N are a left and a right graded A-module, respectively, then there is an isomorphism of graded K-vector spaces*

$$\eta_{M,N} : HOM_A(M, D(N)) \longrightarrow D(N \otimes_A M),$$

which is natural on both variables.

When $A = \oplus_{h\in H}A_h$ and $B = \oplus_{h\in H}B_h$ are graded algebras with enough idempotents, the tensor algebra $A \otimes B$ inherits a structure of graded H-algebra, where $(A \otimes B)_h = \oplus_{s+t=h}A_s \otimes B_t$. In particular, this applies to the enveloping algebra A^e and, as in the ungraded case, we will identify the categories $A^e - Gr$ (resp. $Gr - A^e$) and $A - Gr - A$ of graded left (resp. right) A-modules and graded A-bimodules. We will denote by $A - lfgr - A$ the full subcategory of $A - Gr - A$ consisting of locally finite dimensional graded A-bimodules.

Remark 1.2.2. If M is a graded A-bimodule and we denote by $D({}_AM)$, $D(M_A)$ and $D({}_AM_A)$, respectively, the duals of M as a left module, right module or bimodule, then $D({}_AM_A) = D({}_AM) \cap D(M_A)$ and, in general, $D({}_AM)$ and $D(M_A)$ need not be the same vector subspace of $\mathrm{Hom}_K(M,K)$. However, they are equal if the following two properties hold:

1. For each $(i,h) \in I \times H$, there are only finitely many $j \in I$ such that $e_iM_he_j \neq 0$
2. For each $(i,h) \in I \times H$, there are only finitely many $j \in I$ such that $e_jM_he_i \neq 0$.

Remark 1.2.3. When $H = 0$, we have $A - Gr = A - \mathrm{Mod}$ and $D(M) = \{f : M \longrightarrow K : f(e_iM) = 0$, for almost all $i \in I\}$.

Definition 1. Let $A = \oplus_{h\in H}A_h$ be a graded algebra with enough idempotents. It will be called *locally finite dimensional* when the regular bimodule ${}_AA_A$ is locally finite dimensional, i.e., when $e_iA_he_j$ is finite dimensional, for all $(i,j,h) \in I \times I \times H$. Such a graded algebra A will be called *graded locally bounded* when the following two conditions hold:

1. For each $(i,h) \in I \times H$, the set $I^{(i,h)} = \{j \in I : e_iA_he_j \neq 0\}$ is finite
2. For each $(i,h) \in I \times H$, the set $I_{(i,h)} = \{j \in I : e_jA_he_i \neq 0\}$ is finite.

Remark 1.2.4. For $H = 0$, the just defined concepts are the familiar ones of locally finite dimensional and locally bounded, introduced in the language of K-categories by Gabriel and collaborators (see, e.g., [13]).

1.2.2 Graded algebras with enough idempotents versus graded K-categories

In this subsection we remind the reader that graded algebras with enough idempotents can be looked at as small graded K-categories, and viceversa.

A category $\mathcal{C}$ is a K-category if $\mathcal{C}(X,Y)$ is a K-vector space, for all objects X, Y, and the composition map $\mathcal{C}(Y,Z) \times \mathcal{C}(X,Y) \to \mathcal{C}(X,Z)$ is K-bilinear, for all $X, Y, Z \in \text{Ob}(\mathcal{C})$. If now H is a fixed additive abelian group, then $\mathcal{C}$ is a $(H-)$ *graded K-category* if $\mathcal{C}(X,Y) = \oplus_{h\in H}\mathcal{C}_h(X,Y)$ is a graded K-vector space, for all $X, Y \in Obj(\mathcal{C})$, and the composition map restricts to a (K-bilinear) map

$$\mathcal{C}_h(Y,Z) \times \mathcal{C}_k(X,Y) \to \mathcal{C}_{h+k}(X,Z)$$

for any $h, k \in H$. There is an obvious definition of *graded functor (of degree zero)* between graded K-categories, namely, an additive functor which induces morphisms of graded K-vector spaces at the level of morphisms.

The prototypical example of graded K-category is $(K,H)-GR = K-GR$. Its objects are the H-graded K-vector spaces and, for the morphisms, we define $\text{Hom}_{K-GR}(V,W) = \oplus_{h\in H}\text{Hom}_{K-Gr}(V, W[h])$, where $\text{Hom}_{K-Gr}(V, W[h])$ is the space of K-linear maps of degree h from V to W. The grading on $\text{Hom}_{K-GR}(V,W)$ is given by putting, for each $h \in H$, $\text{Hom}_{K-GR}(V,W)_h = \text{Hom}_{K-Gr}(V, W[h])$.

If $A = \oplus_{h\in H}A_h$ is a graded algebra with enough idempotents, on which we fix a distinguished family $(e_i)_{i\in I}$ of orthogonal idempotents of degree zero, then we can look at it as a small graded K-category. Indeed we put $\text{Ob}(A) = I$, $A(i,j) = e_iAe_j$ and take as composition map $e_jAe_k \times e_iAe_j \longrightarrow e_iAe_k$ the antimultiplication: $b \circ a := ab$.

Conversely, if $\mathcal{C}$ is a small graded K-category then $R = \oplus_{X,Y\in\text{Ob}(\mathcal{C})}\mathcal{C}(X,Y)$ is a graded K-algebra with enough idempotents, where the family of identity maps $(1_X)_{X\in\text{Ob}(\mathcal{C})}$ is a distinguished family of homogeneous elements of degree zero. We will call R the *functor algebra associated to* $\mathcal{C}$. Let $GrFun(\mathcal{C}, K-GR)$ denote the category of graded K-linear covariant functors, with morphisms the K-linear natural transformations. To each object F in this category, we canonically associate a graded left R-module $\mathcal{M}(F)$ as follows. The underlying graded K-vector space is $\mathcal{M}(F) = \oplus_{C\in\text{Ob}(\mathcal{C})}F(C)$. If $f \in 1_YR1_X = \mathcal{C}(X,Y)$ and $z \in F(Z)$, then we define $f \cdot z = \delta_{XZ}F(f)(x)$, where δ_{XZ} is the Kronecker symbol.

Note that $f \cdot x$ is an element of $F(Y)$, and if f and x are homogeneous elements, then $f \cdot x$ is homogeneous od degree $\deg(f) + \deg(x)$.

Conversely, given a graded left R-module M, we can associate to it a graded functor $F_M : \mathcal{C} \longrightarrow K - GR$ as follows. We define $F_M(X) = 1_X M$, for each $X \in \text{Ob}(\mathcal{C})$, and if $f \in \mathcal{C}(X,Y) = 1_Y R 1_X$ is any morphism, then $F_M(f) : F_M(X) \longrightarrow F_M(Y)$ maps $x \rightsquigarrow fx$.

Given an object X of the graded K-category $\mathcal{C}$, the associated *representable functor* is the functor $\text{Hom}_{\mathcal{C}}(X, -) : \mathcal{C} \longrightarrow K - GR$ which takes $Y \rightsquigarrow \mathcal{C}(X,Y)$, for each $Y \in \text{Ob}(\mathcal{C})$. With an easy adaptation of the proof in the ungraded case (see, e.g., [37][Proposition II.2]), we get:

Proposition 1.2.5. *Let $\mathcal{C}$ be a small (H-)graded K- category and let R be its associated functor algebra. Then the assignments $F \rightsquigarrow \mathcal{M}(F)$ and $M \rightsquigarrow F_M$ extend to mutually quasi-inverse equivalences of categories GrFun$(\mathcal{C}^{op}, K - Gr) \overset{\cong}{\longleftrightarrow} R - Gr$. These equivalences restrict to mutually quasi-inverse equivalences GrFun$(\mathcal{C}^{op}, K - lfdGR) \overset{\cong}{\longleftrightarrow} R - lfdgr$, where $K - lfdGR$ denotes the full graded subcategory of $K - GR$ consisting of the locally finite dimensional graded K-vector spaces.*

These equivalences identify the finitely generated projective R-modules with the direct summands of representable functors.

Due to the contents of this subsection, we will freely move from the language of graded algebras with enough idempotents to that of small graded K-categories and viceversa. In particular, given graded algebras with enough idempotents A and B, we will say that $F : A \longrightarrow B$ is a graded functor if it so when we interpret A and B a small graded K-categories.

1.3 Pseudo-Frobenius graded algebras

1.3.1 Definition and characterization

We still work with a fixed abelian additive group H and all gradings on algebras and modules will be H-grading. For the convenience of the reader we start with the following definition:

Definition 2. A locally finite dimensional graded algebra with enough idempotents $A = \oplus_{h \in H} A_h$ will be called *weakly basic* when it has a distinguished family $(e_i)_{i \in I}$ of orthogonal homogeneous idempotents of degree 0 such that:

1. $e_i A_0 e_i$ is a local algebra, for each $i \in I$
2. $e_i A e_j$ is contained in the graded Jacobson radical $J^{gr}(A)$, for all $i, j \in I$, $i \neq j$.

It will be called *basic* when, in addition, $e_i A_h e_i \subseteq J^{gr}(A)$, for all $i \in I$ and $h \in H \setminus \{0\}$.

We will use also the term '(weakly) basic' to denote any distinguished family $(e_i)_{i \in I}$ of orthogonal idempotents satisfying the above conditions.

A weakly basic graded algebra with enough idempotents will be called *split* when $e_i A_0 e_i / e_i J(A_0) e_i \cong K$, for each $i \in I$.

With the idea of extending the definition of finite dimensional self-injective algebras, or more general, that of pseudo-Frobenius rings (cf. [36], V.24.32) to the present context, it makes sense to define:

Recall that a graded module is *finitely cogenerated* when it is finitely generated and its graded socle is essential as a graded submodule.

Definition 3. A weakly basic graded algebra with enough idempotents will be called *graded pseudo-Frobenius* if the projective finitely generated objects and the injective finitely cogenerated objects coincide in $A-Gr$ (resp. $Gr-A$).

Once we have set a formal definition, it is natural to ask oneself if pseudo-Frobenius graded algebras with enough idempotents behave similarly to finite dimensional self-injective algebras. The answer is given throughout the rest of this subsection and, fortunately, as in the self-injective case, one can define and even guarantee the existence of relevant notions as the Nakayama form and automorphism. We first need some preliminaries.

Proposition 1.3.1. *Let $A=\oplus_{h\in H}A_h$ be a weakly basic locally finite dimensional algebra with enough idempotents and let (e_i) be a weakly basic distinguished family of orthogonal idempotents. The following assertions hold:*

1. *$J^{gr}(A)_0$ is the Jacobson radical of A_0.*
2. *Each indecomposable finitely generated projective graded left A-module is isomorphic to $Ae_i[h]$, for some $(i,h)\in I\times H$. Moreover, if $Ae_i[h]$ and $Ae_j[k]$ are isomorphic in $A-Gr$, then $i=j$ and, in case A is basic, also $h=k$.*
3. *Each finitely generated projective graded left A-module is a finite direct sum of graded modules of the form $Ae_i[h]$, with $(i,h)\in I\times H$*
4. *Each finitely generated graded left A-module has a projective cover in the category $A-Gr$*
5. *Each finitely generated projective graded left A-module is the projective cover of a finite direct sum of graded-simple modules (=simple objects of the category $A-Gr$).*

Moreover, the left-right symmetric versions of these assertions also hold.

Proof. 1) For each left ideal U of A_0 one has $AU\cap A_0=U$. With this in mind, let $\mathbf{m}$ be a maximal graded left ideal of A. Then $\mathbf{m}_0=A_0\cap\mathbf{m}$ is a proper left ideal of A_0 since A_0 contains all the e_i. But if $\mathbf{m}_0\subsetneq U$, for some proper left ideal U of A_0, then $AU+\mathbf{m}$ is a proper graded left ideal of A for its 0-homogeneous component is $U+\mathbf{m}_0=U$. But we have $\mathbf{m}\subsetneq UA+\mathbf{m}$, which contradicts the maximality of $\mathbf{m}$. It follows that U cannot exist, so that $\mathbf{m}_0$ is a maximal left ideal of A_0. From the equality $J^{gr}(A)_0=\cap_{\mathbf{m}}\mathbf{m}_0$, where $\mathbf{m}$ varies on the set of maximal graded left ideals of A, we derive that $J^{gr}(A)_0$ is an intersection of maximal left ideal of A_0. It follows that $J(A_0)\subseteq J^{gr}(A)_0$.

We claim that this inclusion is actually an equality. Suppose not, so that we have $i,j\in I$ such that $e_iJ(A_0)e_j\subsetneq e_iJ^{gr}(A)_0e_j$. If $i\neq j$ then, by definition 2, we have

$e_iA_0e_j = e_iJ^{gr}(A)_0e_j$ so that we have $e_iJ(A_0)e_j \subsetneq e_iA_0e_j$. As in the case of associative unital algebras, this implies that $A_0e_i \cong A_0e_j$ or, equivalently, the existence of $x \in e_iA_0e_j$ and $y \in e_jA_0e_i$ such that $xy = e_i$ and $yx = e_j$. Then the maps $\rho_x : Ae_i \longrightarrow Ae_j$ and $\rho_y : Ae_j \longrightarrow Ae_i$ are mutually inverse isomorphisms of graded left A-modules. This contradicts assertion 2, which is proved below. Therefore we necessarily have $i = j$. But then the fact that $e_iA_0e_i$ is a local algebra forces the equality $e_iJ^{gr}(A)_0e_i = e_iA_0e_i$, which implies that $J^{gr}(A)$ contains the homogeneous idempotent e_i. This is clearly absurd.

The proof of the remaining assertions is entirely similar to the one for semiperfect (ungraded) associative algebras with unit (see, e.g., [52]) and here we only summarize the adaptation, leaving the details to the reader. For assertion 2), suppose that there is an isomorphism $f : Ae_i[h] \xrightarrow{\cong} Ae_j[k]$ in $A - Gr$, with $(i,h),(j,k) \in I \times H$. The map $\rho : e_iA_{k-h}e_j \longrightarrow \mathrm{Hom}_{A-Gr}(Ae_i[h], Ae_j[k])$, given by $\rho(x)(a) = ax$, for all $a \in Ae_i$, is an isomorphism of K-vector spaces, so that $f = \rho_x$, for a unique $x \in e_iA_{k-h}e_j$. Similarly, there is a unique $y \in e_jA_{h-k}e_i$ such that $f^{-1} = \rho_y$. We again get that $yx = e_i$ and $xy = e_j$. If $i \neq j$, this is a contradiction since $e_iAe_j + e_jAe_i \subseteq J^{gr}(A)$. Therefore we necessarily have $i = j$ and, in case A is basic, we also have $h = k$ for otherwise we would have that $yx = e_i \in J^{gr}(A)$, which is absurd.

On the other hand, the map $\rho : e_iA_0e_i \longrightarrow \mathrm{End}_{A-Gr}(Ae_i[h])$ given above is an isomorphism of algebras. Therefore each $Ae_i[h]$ has a local endomorphism algebra in $A - Gr$. Since each finitely generated graded left A-module is an epimorphic image of a finite direct sum of modules of the form $Ae_i[h]$, we conclude that the category $A - grproj$ of finitely generated projective graded left A-module is a Krull-Schmidt one, with any indecomposable object isomorphic to some $Ae_i[h]$. This proves assertion 2 and 3.

As in the ungraded case, the fact that $\mathrm{End}_{A-Gr}(Ae_i[h])$ is a local algebra implies that $J^{gr}(A)e_i[h]$ is the unique maximal graded submodule of $Ae_i[h]$. If $S_i := Ae_i/J^{gr}(A)e_i$, then $S_i[h]$ is a graded-simple module, for each $h \in H$, and all graded-simple left modules are of this form, up to isomorphism. Since the projection $Ae_i[h] \longrightarrow S_i[h]$ is a projective cover in $A - Gr$ we conclude that each graded-simple left A-module has a projective cover in $A - Gr$. From this argument we immediately get assertion 5, while assertion 4 follows as in the ungraded case.

Finally, the definition of weakly basic locally finite dimensional graded algebra is left-right symmetric, so that the last statement of the proposition also follows. □

We look at K as an H-graded algebra such that $K_h = 0$, for $h \neq 0$. If $V = \oplus_{h \in H} V_h$ is a graded K-vector space, then its dual $D(V)$ gets identified with the graded K-vector space $\oplus_{h \in H}\mathrm{Hom}_K(V_h, K)$, with $D(V)_h = \mathrm{Hom}_K(V_{-h}, K)$ for all $h \in H$.

Definition 4. Let $V = \oplus_{h \in H} V_h$ and $W = \oplus_{h \in H} W_h$ be graded K-vector spaces, where the homogeneous components are finite dimensional, and let $d \in H$ be any element. A bilinear form $(-,-) : V \times W \longrightarrow K$ is said to be *of degree d* if $(V_h, W_k) \neq 0$ implies that $h + k = d$. Such a form will called *nondegenerate* when the induced maps $W \longrightarrow D(V)$ $(w \rightsquigarrow (-,w)))$ and $V \longrightarrow D(W)$ $(v \rightsquigarrow (v,-)))$ are bijective.

Note that, in the above situation, if $(-,-) : V \times W \longrightarrow K$ is a nondegenerate bilinear form of degree d, then the bijective map $W \longrightarrow D(V)$ (resp $V \longrightarrow D(W)$) given above gives an isomorphism of graded K-vector spaces $W[d] \xrightarrow{\cong} D(V)$ (resp. $V[d] \xrightarrow{\cong} D(W)$).

The following concept is fundamental for us.

Definition 5. Let $A = \oplus_{h \in H} A_h$ be a weakly basic graded algebra with enough idempotents. A bilinear form $(-,-) : A \times A \longrightarrow K$ is said to be a *graded Nakayama form* when the following assertions hold:

1. $(ab, c) = (a, bc)$, for all $a, b, c \in A$
2. For each $i \in I$ there is a unique $\nu(i) \in I$ such that $(e_i A, Ae_{\nu(i)}) \neq 0$ and the assignment $i \rightsquigarrow \nu(i)$ defines a bijection $\nu : I \longrightarrow I$.
3. There is a map $\mathbf{h} : I \longrightarrow H$ such that the induced map $(-,-) : e_i A e_j \times e_j A e_{\nu(i)} \longrightarrow K$ is a nondegenerated graded bilinear form degree $h_i = \mathbf{h}(i)$, for all $i, j \in I$.

The bijection ν is called the *Nakayama permutation* and $\mathbf{h}$ will be called the *degree map.* When $\mathbf{h}$ is a constant map and $\mathbf{h}(i) = h$, we will say that $(-,-) : A \times A \longrightarrow K$ is a *graded Nakayama form of degree h.*

Definition 6. A graded algebra with enough idempotents $A = \oplus_{h \in H} A_h$ will be called *left (resp. right) locally Noetherian* when Ae_i (resp. $e_i A$) satisfies ACC on graded submodules, for each $i \in I$. We will simply say that it is locally Noetherian when it is left and right locally Noetherian.

Recall that a Quillen exact category $\mathcal{E}$ (e.g. an abelian category) is said to be a *Frobenius category* when it has enough projectives and enough injectives and the projective and the injective objects are the same in $\mathcal{E}$.

The following result characterizes the pseudo-Frobenius graded algebras with enough idempotents.

Theorem 1.3.2. *Let $A = \oplus_{h \in H} A_h$ be a weakly basic graded algebra with enough idempotents. Consider the following assertions:*

1. *$A - Gr$ and $Gr - A$ are Frobenius categories*
2. *$D(_A A)$ and $D(A_A)$ are projective graded A-modules*
3. *A is graded pseudo-Frobenius*
4. *There exists a graded Nakayama form $(-,-) : A \times A \longrightarrow K$.*

Then the following chain of implications holds:

$$1) \Longrightarrow 2) \Longrightarrow 3) \Longleftrightarrow 4).$$

When A is graded locally bounded, also $4) \Longrightarrow 2)$ *holds. Finally, if A is graded locally Noetherian, then the four assertions are equivalent.*

Proof. $1) \Longrightarrow 2)$ By Proposition 1.2.1, we have a natural isomorphism

$$HOM_A(?, D(A_A)) \cong D(A \otimes_A ?) : A - Gr \longrightarrow K - Gr,$$

and the second functor is exact. Then also the first is exact, which is equivalent to saying that $D(A_A)$ is an injective object of $A-Gr$ (see [65][Lemma I.2.4]). A symmetric argument proves that $D(_AA)$ is injective in $Gr - A$. Then both $D(A_A)$ and $D(_AA)$ are projective in $A - Gr$ and $Gr - A$ since these are Frobenius categories.

2) $\Longrightarrow$ 3) The duality $D : A - lfdgr \overset{\cong^{op}}{\longleftrightarrow} lfdgr - A : D$ exchanges projective and injective objects, and, also, simple objects on the left and on the right. Since A is locally finite dimensional all finitely generated left or right graded A-modules are locally finite dimensional. Moreover, our hypotheses guarantee that each finitely generated projective graded A-module P is the projective cover of a finite direct sum of simple graded modules. Then $D(P)$ is the injective envelope in $A - lfdgr$ of a finite direct sum of simple objects. We claim that each injective object E of $A-lfdgr$ is an injective object of $A-Gr$. Indeed if U is a graded left ideal of A, $h \in H$ is any element and $f : U[h] \longrightarrow E$ is morphism in $A - Gr$, then we want to prove that f extends to $A[h]$. By an appropriate use of Zorn Lemma, we can assume without loss of generality that there is no graded submodule V of $A[h]$ such that $U[h] \subsetneq V$ and f is extendable to V. The task is then reduced to prove that $U = A$. Suppose this is not the case, so that there exist $i \in I$ and a homogeneous element $x \in Ae_i$ such that $x \notin U$. But then $Ax + U/U$ is a locally finite dimensional graded A-module since so is Ax. It follows that $\mathrm{Ext}_{A-lfdgr}(\frac{U+Ax}{U}[h], E) = 0$, which implies that $f : U[h] \longrightarrow E$ can be extended to $(U + Ax)[h]$, thus giving a contradiction. Now the obvious graded version of Baer's criterion (see [65][Lema I.2.4]) holds and E is injective in $A - Gr$. In our situation, we conclude that $D(P)$ is a finitely cogenerated injective object of $A - Gr$, for each finitely generated projective object P of $Gr - A$.

Conversely, if S is a simple graded right A-modules and $p : P \longrightarrow D(S)$ is a projective cover, then $D(p) : S \cong DD(S) \longrightarrow D(P)$ is an injective envelope. This proves that the injective envelope in $A - Gr$ of any simple object, and hence any finitely cogenerated injective object of $A - Gr$, is locally finite dimensional.

Let now E be any locally finite dimensional graded left A-module. We then get that E is an injective finitely cogenerated object of $A - Gr$ if, and only if, $E \cong D(P)$ for some finitely generated projective graded right A-module P. This implies that E is isomorphic to a finite direct sum of graded modules of the form $D(e_iA[-h_i]) \cong D(e_iA)[h_i]$, where $h_i \in H$. We then assume, without loss of generality, that $E = D(e_iA)[h]$, for some $i \in I$ and $h \in H$. Since $e_iA[-h]$ is a direct summand of $A[-h]$ in $Gr-A$, assertion 2 implies that E is a projective object in $A - Gr$. Then E is isomorphic to a direct summand of a direct sum of graded modules of the form $Ae_i[h_i]$. From the fact that E has a finitely generated essential graded socle we easily derive that E is a direct summand of $\oplus_{1 \leq k \leq r} Ae_{i_k}[h_{i_k}]$, for some indices $i_k \in I$. Therefore each finitely cogenerated injective object of $A - Gr$ is finitely generated projective. The analogous fact is true for graded right A-modules.

On the other hand, if P is a finitely generated projective graded left A-module, then $D(P)$ is a finitely cogenerated injective objet of $Gr - A$ and, by the previous paragraph, we know that $D(P)$ is finitely generated projective. We then get that $P \cong DD(P)$ is finitely cogenerated in $A - Gr$.

3) $\Longrightarrow$ 4) From assertion 3) we obtain its left-right symmetric statement by applying the duality $D : A - lfdgr \overset{\cong^{op}}{\longleftrightarrow} lfdgr - A : D$, bearing in mind that an injective object in

$lfdgr - A$ is also injective in $Gr - A$. It follows that $D(e_iA)$ is an indecomposable finitely generated projective left A-module, for each $i \in I$. We then get a unique index $\nu(i) \in I$ and $h_i \in H$ such that $D(e_iA) \cong Ae_{\nu(i)}[h_i]$. We then have a map $\nu : I \longrightarrow I$. Similarly, for each $i \in I$, we choose one $h_i \in H$ such that $D(e_iA) \cong Ae_{\nu(i)}[h_i]$ and, in this way, we get the degree map $h : I \longrightarrow H$. By the same reason, given another $j \in I$, we have that $D(Ae_j) \cong e_{\mu(j)}A[k_j]$, for a unique $\mu(j) \in I$ and some $k_j \in H$. We then get

$$e_iA \cong DD(e_iA) \cong D(Ae_{\nu(i)}[h_i]) \cong D(Ae_{\nu(i)})[-h_i] \cong e_{\mu\nu(i)}A[k_{\nu(i)} - h_i],$$

and, by Proposition 1.3.1, we conclude that $\mu\nu(i) = i$, for all $i \in I$. This and its symmetric argument prove that the maps μ and ν are mutually inverse.

We fix an isomorphism of graded left A-modules $f_i : Ae_{\nu(i)}[h_i] \xrightarrow{\cong} D(e_iA)$, for each $i \in I$. Then we get a bilinear map

$$e_iA \times Ae_{\nu(i)} \xrightarrow{1\times f_i} e_iA \times D(e_iA) \xrightarrow{\text{can}} K.$$

Note that we have $(a, cb) = f_i(cb)(a) = [cf_i(b)](a) = f_i(b)(ac) = (ac, b)$, for all $(a, b) \in e_iA\times Ae_{\nu(i)}$ and all $c \in A$. This bilinear form is clearly nondegenerate because e_iA is locally finite dimensional and, due to the duality D, the canonical bilinear form $e_iA \times D(e_iA) \longrightarrow K$ is nondegenerate, and actually graded of degree 0 since $D(e_iA)_k = D(e_iA_{-k}) = \text{Hom}_K(e_iA_{-k}, K)$, for each $k \in H$. On the other hand, if $s, t \in H$ and $a \in e_iA_s$ and $b \in A_te_{\nu(i)}$ are homogeneous elements, then the degree of b in $Ae_{\nu(i)}[h_i]$ is $t - h_i$. We get that $(a, b) \neq 0$ if, and only if, $s + (t - h_i) = 0$. This shows that the given bilinear form is graded of degree h_i.

We then define an obvious bilinear form $(-,-) : A\times A \longrightarrow K$ such that $(e_iA, Ae_j) = 0$, whenever $j \neq \nu(i)$, and whose restriction to $e_iA \times Ae_{\nu(i)}$ is the graded bilinear form of degree h_i given above, for each $i \in I$. Since $(a, b) = \sum_{i,j\in I}(e_ia, be_j) = \sum_{i\in I}(e_ia, be_{\nu(i)})$, we get that $(ac, b) = (a, cb)$, for all $a, b, c \in A$, and, hence, that $(-,-) : A \times A \longrightarrow K$ is a graded Nakayama form.

4) $\Longrightarrow$ 3) Let $(-,-) : A\times A \longrightarrow K$ be a graded Nakayama form and let $\nu : I \longrightarrow I$ and $\mathbf{h} : I \longrightarrow H$ be the maps given in definition 5. We put $\mathbf{h}(i) = h_i$, for each $i \in I$. Since the restriction of $(-,-) : e_iA\times Ae_{\nu(i)} \longrightarrow K$ is a nondegenerate graded bilinear form of degree h_i, we get induced isomorphisms of graded K-vector spaces $f_i : Ae_{\nu(i)}[h_i] \longrightarrow D(e_iA)$ and $g_i : e_{\nu^{-1}(i)}A[h_i] \longrightarrow D(Ae_i)$, where $f_i(b) = (-, b) : x \rightsquigarrow (x, b)$ and $g_i(a) = (a,-) : y \rightsquigarrow (a, y)$. The fact that $(ac, b) = (a, cb)$, for all $a, b, c \in A$ implies that f_i is a morphism in $A-Gr$ and g_i is a morphism in $Gr-A$. Therefore the projective finitely generated objects and the injective finitely cogenerated objects coincide in $A-lfdgr$ and $lfdgr-A$. By our comments about the graded Baer criterion, assertion 3 follows immediately.

3), 4) $\Longrightarrow$ 2) We assume that A is graded locally bounded. The hypotheses imply that the injective finitely cogenerated objects of $A - Gr$ and $Gr - A$ are locally finite dimensional and they coincide with the finitely generated projective modules. But in this case A is locally finite dimensional both as a left and as a right graded A-module. Indeed, given $i \in I$, one has $e_iA_h = \oplus_{j\in I}e_iA_he_j$. By the graded locally bounded condition of A almost all summands of this direct sum are zero. This gives that, for each $(i, h) \in I \times H$, the vector spaces e_iA_h is finite dimensional, whence, that ${}_AA$ is in $A-lfdgr$. Similarly, we get that $A_A \in A-lfdgr$. It follows that $D({}_AA)$ and $D(A_A)$ are locally finite dimensional.

We claim that $D(A_A)$ is isomorphic to $\oplus_{i\in I}D(e_iA)$ which, together with assertion 3, will give that $D(A_A)$ is a projective graded left A-module. This plus its symmetric argument will then finish the proof.

To prove our claim, note that, using the duality D, we know that $D(A_A)$ is the product in the category $A - lfdgr$ of the $D(e_iA)$. It is not clear in principle what this product is since the category $A - lfdgr$ is not closed under taking products in $A - Gr$. What we shall do is to prove that there is an isomorphism of graded left A-modules $\prod_{i\in I} D(e_iA) \cong \oplus_{i\in I}D(e_iA)$, where the product is taken in $A - Gr$. Note that, for each $(j,h) \in I \times H$, we have that $e_j(\oplus_{i\in I}D(e_iA))_h = \oplus_{i\in I}e_jD(e_iA)_h = \oplus_{i\in I}D(e_iA_{-h}e_j)$, and this is a finite dimensional vector space due to the graded locally bounded condition of A. It will follow that $\oplus_{i\in I}D(e_iA)$ is locally finite dimensional and is isomorphic to the product, both in $A - Gr$ and $A - lfdgr$, of the $D(e_iA)$. Our claim will be then settled.

The product of the $D(e_iA)$ in $A -$ Mod is the largest unitary submodule of the cartesian product $\prod_{i\in I} D(e_iA)$. Therefore it is $\oplus_{j\in I}(e_j \prod_{i\in I} D(e_iA)) \cong \oplus_{j\in I} \prod_{i\in I} D(e_iAe_j)$. The product of the $D(e_iA)$ in $A - Gr$ is then $\oplus_{h\in H}(\oplus_{j\in I} \prod_{i\in I} D(e_iAe_j)_h) \cong \oplus_{h\in H} \oplus_{j\in I} \prod_{i\in I} D(e_iA_{-h}e_j))$. The graded locally bounded condition of A implies that this last vector space coincides with $\oplus_{h\in H} \oplus_{j\in I} \oplus_{i\in I}D(e_iA_{-h}e_j))$. This is exactly $\oplus_{i\in I}D(e_iA)$, and so we have an isomorphism $\prod_{i\in I} D(e_iA) \cong \oplus_{i\in I}D(e_iA)$ in $A - Gr$.

$3), 4) \Longrightarrow 1)$ We assume that A is graded locally Noetherian. Then $A - Gr$ and $Gr - A$ are locally Noetherian Grothendieck categories, i.e., the Noetherian objects form a set and generate both categories. Then each injective object in $A - Gr$ or $Gr - A$ is a direct sum of indecomposable injective objects and each direct sum of injective objects is again injective (see [37][Proposition IV.6 and Theorem IV.2]). Since, by hypothesis, Ae_i and e_iA are injective objects in $A - Gr$ and $Gr - A$, respectively, we deduce that each projective object in any of these categories is injective.

On the other hand, Ae_i (resp. e_iA) is a Noetherian object of $A - Gr$ (resp. $Gr - A$), which implies by duality that $D(Ae_i)$ (resp. $D(e_iA)$) is an artinian object of $lfdgr - A$ (resp. $A - lfdgr$), and hence also of $Gr - A$ (resp. $A - Gr$). But we have $D(Ae_i) \cong e_{\nu^{-1}(i)}A[h_{\nu^{-1}(i)}]$ (resp. $D(e_iA) \cong Ae_{\nu(i)}[h_i]$), where ν is the Nakayama permutation. By the bijectivity of ν, we get that all Ae_j and e_jA are Artinian (and Noetherian) objects, whence they have finite length. Therefore $A - Gr$ and $Gr - A$ have a set of generators of finite length, which easily implies that the graded socle of each object in these categories is a graded essential submodule. In particular, each injective object in $A - Gr$ (resp. $Gr - A$) is the injective envelope of its graded socle. But if $\{S_t : t \in T\}$ is a family of simple objects of $A - Gr$ (resp. $Gr - A$) and $\iota_t : S_t \rightarrowtail E(S_t)$ is an injective envelope in $A - Gr$ (resp. $Gr - A$), then the induced map $\iota := \oplus_{t\in T} : \oplus_{t\in T}S_t \rightarrowtail \oplus_{t\in T}E(S_t)$ is an injective envelope in $A - Gr$ (resp. $Gr - A$) since the direct sum of injectives is injective. Since each $E(S_t)$ is finitely cogenerated, whence projective by hypothesis, it follows that each injective object in $A - Gr$ (resp. $Gr - A$) is projective. □

Definition 7. A weakly basic locally finite dimensional graded algebra satisfying condition 1 will be called *graded Quasi-Frobenius.*

Remark 1.3.3. The concepts of pseudo-Frobenius (PF) and Quasi-Frobenius (QF) associative unital algebras (over a commutative ring and not just over a field) are classical (see, e.g., [36], [58] and [69]). Such an algebra A is left PF when ${}_AA$ is an injective cogenerator of $A-\text{Mod}$ while it is QF when $A-\text{Mod}$, or equivalently $\text{Mod}-A$, is what today is called a Frobenius category. Pseudo-Frobenius algebras are the left and right PF algebras and they are characterized by the fact that the finitely generated projective and the finitely cogenerated injective objects coincide in $A-\text{Mod}$ and $\text{Mod}-A$. Although not yet with this name, pseudo-Frobenius algebras already appear in the original work of Morita ([64]).

Examples 1.3.4. The following are examples of graded pseudo-Frobenius algebras over a field K:

1. When $H=0$ and $A=\Lambda$ is a finite dimensional self-injective algebra, which is equivalent to saying that Λ is quasi-Frobenius.
2. When Λ is any finite dimensional split basic algebra and $A=\hat{\Lambda}$ is its repetitive algebra, in the terminology of [46], then A is a (trivially graded) quasi-Frobenius algebra with enough idempotents (see op.cit.[Chapter II]).
3. The $\mathbb{Z}$-graded algebra $A=K[x,x^{-1},y,z]/(y^2,z^2)$, where $\deg(x)=\deg(y)=\deg(z)=1$. Given any integer m, we have a canonical basis $\mathcal{B}_m=\{x^m,x^{m-1}y,x^{m-1}z,x^{m-2}yz\}$ of A_m. Consider the graded bilinear form $A\times A\longrightarrow K$ of degree m identified by the fact that if $f\in A_n$ and $g\in A_{m-n}$, then (f,g) is the coefficient of $x^{m-2}yz$ in the expression of fg as a K-linear combination of the elements of $\mathcal{B}_m$. Then $(-,-)$ is a graded Nakayama form for A, so that A is graded pseudo-Frobenius.

The following result complements Theorem 1.3.2 and gives a handy criterion, in the locally Noetherian case, for A to be graded Quasi-Frobenius.

Corollary 1.3.5. *Let $A=\oplus_{h\in H}A_h$ be a weakly basic locally Noetherian graded algebra with enough idempotents. The following assertions are equivalent:*

1. *The following two conditions hold:*
   - *(a) For each $i\in I$, Ae_i and e_iA have a simple essential socle in $A-Gr$ and $Gr-A$, respectively*
   - *(b) There are bijective maps $\nu,\nu':I\longrightarrow I$ such that $Soc_{gr}(e_iA)\cong\frac{e_{\nu(i)}A}{e_{\nu(i)}J^{gr}(A)}[h_i]$ and $Soc_{gr}(Ae_i)\cong\frac{Ae_{\nu'(i)}}{J^{gr}(A)e_{\nu'(i)}}[h'_i]$, for certain $h_i,h'_i\in H$*
2. *A is graded Quasi-Frobenius*

Proof. We only need to prove 1) $\Longrightarrow$ 2). By definition of weakly basic, A is locally finite dimensional, so that Ae_i and e_iA are locally finite dimensional modules, for all $i\in I$. It then follows by duality that $D(e_iA)$ is an Artinian object of $A-Gr$, for all $i\in I$. By the same reason, we get that $D(e_iA)$ has a unique simple essential quotient, meaning that $D(e_iA)$ has a unique maximal superfluous subobject. By a classical argument, it follows that $D(e_iA)$ has a projective cover in $A-Gr$, which is an epimorphism of the form $p:Ae_j[h]\twoheadrightarrow D(e_iA)$. It follows from this that $D(e_iA)$ is a Noetherian object, whence, an object of finite length in $A-Gr$ since it is a quotient of a Noetherian object. With this and its symmetric argument we get that all finitely cogenerated injective objects in $A-Gr$ and $Gr-A$ have finite length, which implies by duality that also the finitely generated projective objects have finite length.

The fact that $\mathrm{Soc}_{gr}(e_iA)$ is simple-graded implies that the injective envelope of e_iA in $A-Gr$ is of the form $\iota : e_iA \rightarrowtail E \cong D(Ae_j)[h]$, while the projective cover of E is of the form $p : e_kA[h'] \twoheadrightarrow E$. Then ι factors through p yielding a monomorphism $u : e_iA \rightarrowtail e_kA[h']$. But then the graded socles of e_iA and $e_kA[h']$ are isomorphic. By condition 1.b) and the weakly basic condition, this implies that $i = k$. By comparison of graded composition lengths, we get that u is an isomorphism, which in turn implies that both p and ι are also isomorphisms. Therefore all the e_iA, and hence all finitely generated projective objects, are finitely cogenerated injective objects of $A-Gr$. The left-right symmetry of assertion 1 implies that the analogous fact is true in $Gr-A$. Then, applying duality, we get that the finitely generated projective objects and the finitely cogenerated injective objects coincide in $A-Gr$ and $Gr-A$. Then assertion 3 of Theorem 1.3.2 holds, which together with the locally Noetherian hypothesis imply that A is graded Quasi-Frobenius.

□

Due to the previous characterization, we can derive the existence of an automorphism η of the algebra A, not necessarily graded, which plays the role of the Nakayama automorphism. That is, it satisfies the condition that $D(A) \cong {}_1A_\eta$ as A-bimodules. This automorphism will be also called the *Nakayama automorphism* of A and it may be taken to be graded under certain conditions, in which case, we will refer to it as the *graded Nakayama automorphism* of A.

Corollary 1.3.6. *Let $A = \oplus_{h\in H}A_h$ be a graded pseudo-Frobenius algebra and let $(e_i)_{i\in I}$ be a weakly basic distinguished family of orthogonal idempotents. If A is graded locally bounded, then the following assertions hold:*

1. *There is an automorphism of (ungraded) algebras $\eta : A \longrightarrow A$, which permutes the idempotents e_i and maps homogeneous elements onto homogeneous elements, such that ${}_1A_\eta$ is isomorphic to $D(A)$ as an ungraded A-bimodule.*
2. *If the map $\mathbf{h} : I \longrightarrow H$ associated to the Nakayama form $(-,-) : A \times A \longrightarrow K$ takes constant value h, then η can be chosen to be graded and such that $D(A)$ is isomorphic to ${}_1A_\eta[h]$ as graded A-bimodules.*

Proof. Let first note that, by Remark 1.2.2, we have $D({}_AA) = D({}_AA_A) = D(A_A)$ in this case.

1) Let us fix a graded Nakayama form $(-,-) : A \times A \longrightarrow K$ and associated maps $\nu : I \longrightarrow I$ and $h : I \longrightarrow H$. The assignment $b \rightsquigarrow (-,b)$ gives an isomorphism of graded A-modules $Ae_{\nu(i)}[h_i] \xrightarrow{\cong} D(e_iA)$, for each $i \in I$. By taking the direct sum of all these maps, we get an isomorphism of ungraded left A-modules ${}_AA \longrightarrow \oplus_{i\in I}D(e_iA)$. But we have seen in the proof of the implication $3), 4) \Longrightarrow 2)$ in last theorem that $D(A) \cong \oplus_{i\in I}D(e_iA)$ in $A-Gr$. Therefore the assignment $b \rightsquigarrow (-,b)$ actually gives an isomorphism ${}_AA \xrightarrow{\cong} D(A)$. Symmetrically, the assignment $a \rightsquigarrow (a,-)$ gives an isomorphism $A_A \xrightarrow{\cong} D(A)$. It then follows that, given $a \in A$, there is a unique $\eta(a) \in A$ such that $(a,-) = (-,\eta(a))$. This

gives a K-linear map $\eta : A \longrightarrow A$ which, by its own definition, is bijective. Moreover, given $a, b, x \in A$, we get

$$(x, \eta(ab)) = (ab, x) = (a, bx) = (bx, \eta(a)) = (b, x\eta(a)) = (x\eta(a), \eta(b) = (x, \eta(a)\eta(b)),$$

which shows that $\eta(ab) = \eta(a)\eta(b)$, for all $a, b \in A$. Therefore η is an automorphism of A as an ungraded algebra. Moreover if $0 \neq a \in e_iAe_j$, then $(a, -)$ vanishes on all $e_{i'}Ae_{j'}$ except in $e_jAe_{\nu(i)}$. Therefore $(-, \eta(a))$ does the same. By definition of the Nakayama form, we necessarily have $\eta(a) \in e_{\nu(i)}Ae_{\nu(j)}$. We claim that if $a \in e_iAe_j$ is an element of degree h, then $\eta(a)$ is an element of degree $h+h_j-h_i$. Indeed, let $h' \in H$ be such that $\eta(a)_{h'} \neq 0$. Then the $(-, \eta(a)_{h'}) : e_jAe_{\nu(i)} \longrightarrow K$ is a nonzero linear form which vanishes on $e_iA_ke_{\nu(i)}$, for all $k \neq h_j - h'$. Let us pick up $x \in e_iA_{h_j-h'}e_{\nu(i)}$ such that $(x, \eta(a)_{h'}) \neq 0$. Then we have that $(x, \eta(a)) = (x, \eta(a)_{h'}) \neq 0$, due to the fact that $(-, -) : e_jAe_{\nu(i)} \times e_{\nu(i)}Ae_{\nu(j)} \longrightarrow K$ is a graded bilinear form of degree h_j. We then get that $0 \neq (x, \eta(a)) = (a, x)$, which implies that $h + (h_j - h') = h_i$, which implies that $h' = h + (h_j - h_i)$. Then h' is uniquely determined by a, so that $\eta(a)$ is homogeneous of degree $h + h_j - h_i$ as desired.

Putting $a = e_i$ in the previous paragraph, we get that $\eta(e_i) \in e_{\nu(i)}Ae_{\nu(i)}$ has degree 0, and then $\eta(e_i)$ is an idempotent element of the local algebra $e_{\nu(i)}A_0e_{\nu(i)}$. It follows that $\eta(e_i) = e_{\nu(i)}$, for each $i \in I$.

Finally, we consider the K-linear isomorphism $f : A \longrightarrow D(A)$ which maps $b \rightsquigarrow (-, b) = (\eta^{-1}(b), -)$. We readily see that f is a homomorphism of left A-modules. Moreover, we have equalities

$$(a, b\eta(b')) = (ab, \eta(b')) = (b', ab) = (b'a, b) = [f(b)b'](a),$$

which shows that f is a homomorphism of right A-modules $A_\eta \longrightarrow D(A)$. Then f is an isomorphism ${}_1A_\eta \xrightarrow{\cong} D(A)$.

2) The proof of assertion 1 shows that if $\mathbf{h}(i) = h$, for all $i \in I$, then η is a graded automorphism of degree 0. Moreover, the isomorphism $f :_1 A_\eta \xrightarrow{\cong} D(A)$ is the direct sum of the isomorphisms of graded left A-modules $f_i : Ae_{\nu(i)}[h] \xrightarrow{\cong} D(e_iA)$ which map $b \rightsquigarrow (-, b)$. It then follows that f is an isomorphism of graded bimodules ${}_1A_\eta[h] \xrightarrow{\cong} D(A)$. □

To finish this subsection, we will see that if one knows that A is split graded pseudo-Frobenius, then all possible graded Nakayama forms for A come in similar way. Recall that if $V = \oplus_{h \in H} V_h$ is a graded vector space, then its *support*, denoted $\mathrm{Supp}(V)$, is the set of $h \in H$ such that $V_h \neq 0$.

Proposition 1.3.7. *Let A be a split pseudo-Frobenius graded algebra and $(e_i)_{i \in I}$ a weakly basic distinguished family of orthogonal idempotents. The following assertions hold:*

1. *All graded Nakayama forms for A have the same Nakayama permutation. It assigns to each $i \in I$ the unique $\nu(i) \in I$ such that $e_iSoc_{gr}(A)e_{\nu(i)} \neq 0$.*

2. *If $h_i \in Supp(e_iSoc_{gr}(A))$, then $dim(e_iSoc_{gr}(A))_{h_i}) = 1$*

3. *For a bilinear form $(-, -) : A \times A \longrightarrow K$, the following statements are equivalent:*

(a) $(-,-)$ *is a graded Nakayama form for* A

(b) There exists an element $\mathbf{h} = (h_i) \in \prod_{i\in I} Supp(e_i Soc_{gr}(A))$ *and a basis* $\mathcal{B}_i$ *of* $e_i A_{h_i} e_{\nu(i)}$*, for each* $i \in I$*, such that:*

i. $\mathcal{B}_i$ *contains a (unique) element* w_i *of* $e_i Soc_{gr}(A)_{h_i}$

ii. If $a, b \in \bigcup_{i,j} e_i A e_j$ *are homogeneous elements, then* $(e_i A_h, A_k e_j) = 0$ *unless* $j = \nu(i)$ *and* $h + k = h_i$

iii. If $(a, b) \in e_i A_h \times A_{h_i - h} e_{\nu(i)}$*, then* (a, b) *is the coefficient of* w_i *in the expression of* ab *as a linear combination of the elements of* $\mathcal{B}_i$*.*

Proof. 1) Let $(-,-) : A \times A \longrightarrow K$ be a graded Nakayama form for A. We have seen in the proof of the implication 4) $\Longrightarrow$ 3) in Theorem 1.3.2 that then $D(e_i A) \cong Ae_{\nu(i)}[h_i]$. Due to conditions satisfied by the e_i, we get that $\nu(i)$ is independent of $(-,-)$. Moreover, by duality, we get an isomorphism $e_i A \cong D(Ae_{\nu(i)})[-h_i]$, which induces an isomorphism between the graded socles. But the graded socle of $D(Ae_j)$ is isomorphic to $S_j := e_j A / e_j J^{gr}(A)$, for each $j \in I$. We then get that $e_i \mathrm{Soc}^{gr}(A) e_j[-h] \cong \mathrm{Hom}_{A-Gr}(S_j[h], e_i \mathrm{Soc}^{gr}(A)) = 0$, for all $j \neq \nu(i)$ and $h \in H$.

2) Let us fix $h_i \in \mathrm{Supp}(e_i \mathrm{Soc}^{gr}(A)))$ and suppose that $\{x, y\}$ is a linearly independent subset of $e_i \mathrm{Soc}^{gr}(A))_{h_i}$. We then have $xA = yA$ since $e_i \mathrm{Soc}^{gr}(A))$ is graded-simple. We get from this that also $xA_0 = yA_0$. By Proposition 1.3.1, we know that $J(A_0) = J^{gr}(A)_0$ and the split hypothesis on A implies that $A_0 = J(A_0) \oplus (\oplus_{j\in I} K e_j)$. It follows that $Kx = x(\oplus_{j\in I} K e_j) = xA_0 = yA_0 = y(\oplus_{j\in I} K e_j) = Ky$, which contradicts the linear independence of $\{x, y\}$.

3) *b)* $\Longrightarrow$ *a)* By assertion 1), the Nakayama permutation is completely determined by A. The given element $\mathbf{h}$ is then interpreted as a map $I \longrightarrow H$, which will be our degree function. It only remains to check that $(ab, c) = (a, bc)$, for all a, b, c. This easily reduces to the case when a, b, c are homogeneous and there are indices i, j, k such that $a = e_i a e_j$, $b = e_j b e_k$ and $c = e_k c e_{\nu(i)}$. But in this case, we have $(a, bc) = (ab, c) = 0$ when $\deg(a) + \deg(b) + \deg(c) \neq h_i$. On the other hand, by condition b.iii), if $\deg(a) + \deg(b) + \deg(c) = h_i$ then (ab, c) and (a, bc) are both the coefficient of w_i in the expression of abc as linear combination of the elements of $\mathcal{B}_i$. So the equality $(ab, c) = (a, bc)$ holds, for all $a, b, c \in A$.

a) $\Longrightarrow$ *b)* We first take a basis $\mathcal{B}^0$ of A_0 such that $\mathcal{B}^0 = \{e_i : i \in I\} \cup (\mathcal{B}^0 \cap J(A_0))$ and $\mathcal{B}^0 \subseteq \bigcup_{i,j\in I} e_i A_0 e_j$. The graded Nakayama form gives by restriction a nondegenerate bilinear map

$$(-,-) : e_i A_0 e_i \times e_i A_{h_i} e_{\nu(i)} \longrightarrow K.$$

We choose as $\mathcal{B}_i$ the basis of $e_i A_{h_i} e_{\nu(i)}$ which is right orthogonal to $e_i \mathcal{B}^0 e_i$ with respect to this form. As usual, if $b \in e_i \mathcal{B}^0 e_i$, we denote by b^* the element of $\mathcal{B}_i$ such that $(c, b^*) = \delta_{bc}$, where δ_{bc} is the Kronecker symbol. We then claim that $w_i := e_i^*$ is in $e_i \mathrm{Soc}_{gr}(A)$. This will imply that $h_i \in \mathrm{Supp}(\mathrm{Soc}_{gr}(A))$ and, due to assertion 2, we will get also that w_i is the only element of $e_i \mathrm{Soc}_{gr}(A)_{h_i}$ in $\mathcal{B}_i$. Indeed suppose that $w_i \notin e_i \mathrm{Soc}_{gr}(A)$. We then have $a \in J^{gr}(A)$ such that $aw_i \neq 0$. Without loss of generality, we assume that a is homogeneous and that $a = e_j a e_i$, for some $j \in I$. Then $0 \neq aw_i \in e_j A e_{\nu(i)}$, which implies the existence of a homogeneous element $b \in e_i A e_j$ such that $(b, aw_i) \neq 0$ since the induced graded bilinear form $e_iAe_j \times e_jAe_{\nu(i)} \longrightarrow K$ is nondegenerate. But then we have $(ba, w_i) \neq 0$ and $\deg(ba) = 0$ since the induced graded bilinear form $e_iAe_i \times e_iAe_{\nu(i)} \longrightarrow K$ is of degree h_i. But $ba \in e_iJ^{gr}(A)_0e_i = e_iJ(A_0)e_i$ and, by the choice of the basis $\mathcal{B}^0$, each element of $e_iJ(A_0)e_i$ is a linear combination of the elements in $\mathcal{B}^0 \cap e_iJ(A_0)e_i$. By the choice of w_i, we have $(c, w_i) = 0$, for all $c \in \mathcal{B}^0 \cap e_iJ(A_0)e_i$. It then follows that $(ba, w_i) = 0$, which is a contradiction.

It is now clear that conditions b.i and b.ii hold. In order to prove b.iii, take $(a, b) \in e_iA_h \times A_{h_i-h}e_{\nu(i)}$. We then have $(a, b) = (e_i, ab)$, where $ab \in e_iA_{h_i}e_{\nu(i)}$. Put $ab = \sum_{c\in\mathcal{B}_i} \lambda_c c$, where $\lambda_c \in K$ for each $c \in \mathcal{B}_i$. We then get $(a, b) = (e_i, \sum_c \lambda_c c) = \sum_c \lambda_c(e_i, c) = \lambda_{w_i}$, i.e., (a, b) is the coefficient of w_i in the expression $ab = \sum_c \lambda_c c$. □

Definition 8. Let $A = \oplus_{h\in H} A_h$ be a split pseudo-Frobenius graded algebra, with $(e_i)_{i\in I}$ as weakly basic distinguished family of idempotents and $\nu : I \longrightarrow I$ as Nakayama permutation. Given a pair $(\mathcal{B}, \mathbf{h})$ consisting of an element $\mathbf{h} = (h_i)_{i\in I}$ of $\prod_{i\in I} \mathrm{Supp}(e_i\mathrm{Soc}_{gr}(A))$ and a family $\mathcal{B} = (\mathcal{B}_i)_{i\in I}$, where $\mathcal{B}_i$ is a basis of $e_iA_{h_i}e_{\nu(i)}$ containing an element of $e_iSoc_{gr}(A)$, for each $i \in I$, we call *graded Nakayama form associated to* $(\mathcal{B}, \mathbf{h})$ to the bilinear form $(-, -) : A \times A \longrightarrow K$ determined by the conditions b.ii and b.iii of last proposition. When $\mathbf{h}$ is constant, i.e. there is $h \in H$ such that $h_i = h$ for all $i \in I$, we will call $(-, -)$ the graded Nakayama form of A of degree h associated to $\mathcal{B}$.

1.3.2 Graded algebras given by quivers and relations

Recall that a *quiver* or *oriented graph* is a quadruple $Q = (Q_0, Q_1, i, t)$, where Q_0 and Q_1 are sets, whose elements are called *vertices* and *arrows* respectively, and $i, t : Q_1 \longrightarrow Q_0$ are maps. If $a \in Q_1$ then $i(a)$ and $t(a)$ are called the *origin* (or *initial vertex*) and the *terminus* of a.

Given a quiver Q, a path in Q is a concatenation of arrows $p = a_1a_2...a_r$ such that $t(a_k) = i(a_{k+1})$, for all $k = 1, ..., r$. In such case, we put $i(p) = i(a_1)$ and $t(p) = t(a_r)$ and call them the origin and terminus of p. The number r is the *length* of p and we view the vertices of Q as paths of length 0. The *path algebra* of Q, denoted by KQ, is the K-vector space with basis the set of paths, where the multiplication extends by K-linearity the multiplication of paths. This multiplication is defined as $pq = 0$, when $t(p) \neq i(q)$, and pq is the obvious concatenation path, when $t(p) = i(q)$. The algebra KQ is an algebra with enough idempotents, where Q_0 is a a distinguished family of orthogonal idempotents. If $i \in Q_0$ is a vertex, we will write it as e_i when we view it as an element of KQ.

Definition 9. Let H be an abelian group. An *(H-)graded quiver* is a pair $(Q, \deg)$, where Q is a quiver and $\deg : Q_1 \longrightarrow H$ is a map, called the *degree or weight function.* $(Q, \deg)$ will be called *locally finite dimensional* when, for each $(i, j, h) \in Q_0 \times Q_0 \times H$, the set of arrows a such that $(i(a), t(a), \deg(a)) = (i, j, h)$ is finite.

We will simply say that Q is an H-graded quiver, without mention to the degree function which is implicitly understood. Each degree function on a quiver Q induces an H-grading on the algebra KQ, where the degree of a path of positive length is defined as the sum of the degrees of its arrows and $\deg(e_i) = 0$, for all $i \in Q_0$. In the following result, for each natural number n, we denote by $KQ_{\geq n}$ the vector subspace of KQ generated by the paths of length $\geq n$. For each ideal I of an algebra, we put $I^\omega = \bigcap_{n>0} I^n$.

Proposition 1.3.8. *Let $A = \oplus_{h \in H} A_h$ be a split basic locally finite dimensional graded algebra with enough idempotents and let $J = J^{gr}(A)$ be its graded Jacobson radical. There is an H-graded locally finite dimensional quiver Q and a subset $\rho \subset \bigcup_{i,j \in Q_0} e_i KQ_{\geq 2} e_j$, consisting of homogeneous elements with respect to the induced H-grading on KQ, such that A/J^ω is isomorphic to $KQ/ < \rho >$. Moreover Q is unique, up to isomorphism of H-graded quivers.*

Proof. It is an adaptation of the corresponding proof, in more restrictive situations, of the ungraded case (see, e.g., [13][Section 2]). We give the general outline, leaving aside the details.

Let $(e_i)_{i \in I}$ be the basic distinguished family of orthogonal idempotents. The graded quiver Q will have $Q_0 = I$ as its sets of vertices. Whenever $h \in \text{Supp}(\frac{e_i J e_j}{e_i J^2 e_j})$, we will select a subset $Q_1(i,j)_h$ of $e_i J_h e_j$ whose image by the projection $e_i J_h e_j \twoheadrightarrow \frac{e_i J_h e_j}{e_i (J^2)_h e_j}$ gives a basis of $\frac{e_i J_h e_j}{e_i (J^2)_h e_j}$. We will take as arrows of degree h from i to j the elements of $Q_1(i,j)_h$, and then $Q_1 = \bigcup_{i,j \in Q_0; h \in H} Q_1(i,j)_h$. The so-obtained graded quiver gives a grading on KQ and there is an obvious homomorphism of graded algebras $f : KQ \longrightarrow A$ which takes $e_i \rightsquigarrow e_i$ and $a \rightsquigarrow a$, for all $i \in Q_0$ and $a \in Q_1$.

We claim that the composition $KQ \stackrel{f}{\longrightarrow} A \stackrel{p}{\twoheadrightarrow} A/J^\omega$ is surjective or, equivalently, that $\text{Im}(f) + J^\omega = A$. Due to the split basic condition of A, it is easy to see that $A = (\sum_{i \in I} Ke_i) \oplus J$ and the task is then reduced to prove the inclusion $J \subseteq \text{Im}(f) + J^\omega$. Since $e_i A_h e_j$ is finite dimensional, for each triple $(i,h,j) \in I \times H \times I$, there is a smallest natural number $m_{ij}(h)$ such that $e_i(J^n)_h e_j = e_i(J^{n+1})_h e_j$, for all $n \geq m_{ij}(h)$. We will prove, by induction on $k \geq 0$, that $e_i(J^{m_{ij}(h)-k})_h e_j \subset \text{Im}(f) + J^\omega$, for all (i,h,j), and then the inclusion $J \subseteq \text{Im}(f) + J^\omega$ will follow. The case $k = 0$ is trivial, by the definition of $m_{ij}(h)$. So we assume that $k > 0$ in the sequel. Fix any triple $(i,h,j) \in I \times H \times I$ and put $n := m_{ij}(h) - k$. If $x \in e_i(J^n)_h e_j$ then x is a sum of products of the form $x_1 x_2 \cdot ... \cdot x_n$, where x_r is a homogeneous element in $e_{i'} J e_{j'}$, for some pair $(i',j') \in I \times I$. So it is not restrictive to assume that $x = x_1 x_2 \cdot ... \cdot x_r$ is a product as indicated. By definition of the arrows of Q, each x_r admits a decomposition $x_r = y_r + z_r$, where y_r is a linear combination of arrows (of the same degree) and $z_r \in J^2$. It follows that $x = y + z$, where y is a linear combination of paths of length n and $z \in e_i J^{n+1} e_j$. Then $y \in \text{Im}(f)$ and, by the induction hypothesis, we know that $z \in \text{Im}(f) + J^\omega$.

Proving that $\text{Ker}(p \circ f) \subseteq KQ_{\geq 2}$ goes as in the ungraded case, as so does the proof of the uniqueness of Q. We omit it. □

A weakly basic locally finite dimensional algebra A will be called *connected* when, for each pair $(i,j) \in I \times I$ there is a sequence $i = i_0, i_1, ..., i_n = j$ of elements of I such that, for each $k = 1, ..., n$, either $e_{i_{k-1}} A e_{i_k} \neq 0$ or $e_{i_k} A e_{i_{k-1}} \neq 0$. If Q is a graded quiver, we say that Q is a *connected graded quiver* when, for each pair $(i,j) \in Q_0 \times Q_0$, there is a sequence $i = i_0, i_1, ..., i_n = j$ of vertices such that there is an arrow $i_{k-1} \to i_k$ or an arrow $i_k \to i_{k-1}$, for each $k = 1, ..., n$.

Corollary 1.3.9. *Let $A = \oplus_{n \geq 0} A_n$ be a split basic locally finite dimensional positively $\mathbb{Z}$-graded. Then there exists a positively $\mathbb{Z}$-graded quiver Q, uniquely determined up to isomorphism of $\mathbb{Z}$-graded quivers, such that A is isomorphic to KQ/I, for a homogeneous* *ideal I of KQ such that $I \subseteq KQ_{\geq 2}$. If, moreover, A is connected locally bounded, with A_0 semisimple, and the equality $A_n = A_1 \cdot \overset{n}{\cdots} \cdot A_1$ holds for all $n > 0$, then the following assertions are equivalent:*

1. *A is graded pseudo-Frobenius*
2. *There exists a graded Nakayama form $(-,-) : A \times A \longrightarrow K$ with constant degree function.*

In particular, the Nakayama automorphism η is always graded in this case.

Proof. The point here is that if $x \in J^n$ is a homogeneous element, then $\deg(x) \geq n$, which implies that J^ω does not contain homogeneous elements and, hence, that $J^\omega = 0$. Then the first part of the statement is a direct consequence of Proposition 1.3.8. Moreover, one easily sees that the connectedness of A is equivalent in this case to the connectedness of the quiver Q.

As for the second part, we only need to prove that if $(-,-) : A \times A \longrightarrow K$ is a graded Nakayama form, then its associated degree function is constant. The argument is inspired by [63][Proposition 3.2]. We consider that $A = KQ/I$, where Q is connected. The facts that A_0 is semisimple and $A_n = A_1 \cdot \overset{n}{\cdots} \cdot A_1$, for all $n > 0$, then translate into the fact that the degree function $\deg : Q_1 \longrightarrow \mathbb{Z}$ takes constant value 1, so that the induced grading on KQ is the one by path length.

Let now $\eta : A \longrightarrow A$ be the Nakayama automorphism associated to $(-,-)$. If $a : i \to j$ is any arrow in Q, then from Corollary 1.3.6 we get that $\eta(a)$ is a homogeneous element in $e_{\nu(i)} J e_{\nu(j)} = e_{\nu(i)} A e_{\nu(j)}$. Since obviously $\deg(a) \neq 0$, we get that $\deg(\eta(a)) \geq \deg(a)$, which implies that $\deg(\eta(x)) \geq \deg(x)$, for each homogeneous element $x \in A$. Let again $a : i \to j$ be an arrow and put $x = \eta^{-1}(a)$. We claim that x is homogeneous of degree 1. Indeed, we have $x = x_1 + x_2 + ... + x_n$, with $\deg(x_k) = k$, so that $a = \eta(x) = \eta(x_1) + \eta(x')$, where $x' = \sum_{2 \leq k \leq n} x_k$ and, hence, $\eta(x')$ is a sum of homogeneous elements of degrees ≥ 2. It follows that $a = \eta(x_1)$ and $\eta(x') = 0$, which, by the bijective condition of η, gives that $x' = 0$. Therefore $x = x_1$ as desired.

The last paragraph implies that, for each pair $(i,j) \in Q_0 \times Q_0$ such that there is an arrow $i \to j$ in Q, there is a vector subspace V_{ij} of $e_{\nu^{-1}(i)} KQ_1 e_{\nu^{-1}(j)}$ such that $\eta_{|V_{ij}} : V_{ij} \longrightarrow e_i KQ_1 e_j$ is a bijection. Let now $\tilde{Q}$ be the subquiver of Q with the same vertices and with arrows those $a \in Q_1$ such that $\deg(\eta(a)) = 1$. Then $V_{ij} \subseteq e_{\nu^{-1}(i)} K\tilde{Q} e_{\nu^{-1}(j)}$ and $\tilde{A} = \frac{K\tilde{Q}+I}{I}$ is a subalgebra of $A = KQ/I$ such that the image of the restriction map $\eta_{|\tilde{A}} : \tilde{A} \longrightarrow A$ contains the vertices and the arrows (when viewed as elements of A in the obvious way). Note that $\eta_{|\tilde{A}}$ is a homomorphism of graded algebras, which immediately implies that it is surjective and, hence, bijective. But then necessarily $\tilde{A} = A$ for η is an injective map. We will derive from this that $\deg(\eta(a)) = \deg(a)$, for each $a \in Q_1$. Indeed, if $\deg(\eta(a)) > 1$, then $\eta(a) = \eta(x)$, for some homogeneous element $x \in \tilde{A}$ of degree $\deg(x) = \deg(\eta(a))$. By the injective condition of η, we would get that $a = x$, which is a contradiction.

If now $h : Q_0 \longrightarrow \mathbb{Z}$ is the degree function associated to the graded Nakayama form, the proof of Corollary 1.3.6 gives that $h_{i(a)} = h_{t(a)}$, for each $a \in Q_1$. Due to the connectedness of Q, we conclude that h is a constant function. □

1.4 Covering theory and pseudo-Frobenius algebras

1.4.1 Covering theory of graded algebras

In this part we will present the basics of covering theory of graded categories or, equivalently, of graded algebras with enough idempotents. It is an adaptation of the classical theory (see [66], [39], [13]), where we incorporate more recent ideas of [20] and [7], where some of the constraining hypotheses of the initial theory disappear.

Let $A = \oplus_{h\in H}A_h$ and $B = \oplus_{h\in H}B_h$ be two locally finite dimensional graded algebras with enough idempotents, with $(e_i)_{i\in I}$ and $(\epsilon_j)_{j\in J}$ as respective distinguished families of homogeneous orthogonal idempotents of degree 0. Suppose that $F : A \longrightarrow B$ is a graded functor and that it is surjective on objects, i.e., for each $j \in J$ there exists $i \in I$ such that $F(e_i) = \epsilon_j$. To this functor one canonically associates the *pullup or restriction of scalars functor* $F^\rho : B-Gr \longrightarrow A-Gr$. If X is a graded left B-module, then we put $e_iF^\rho(X) = \epsilon_{F(i)}X$, for all $i \in I$, and if $a \in \bigcup_{i,i'\in I} e_iAe_{i'}$ and $x \in F^\rho(X)$, then $ax := F(a)x$. It has a left adjoint $F_\lambda : A-Gr \longrightarrow B-Gr$, called the *pushdown functor*, whose precise definition will be given below in the case that we will need in this work.

The procedure of taking a weak skeleton gives rise to a graded functor as above. Indeed, suppose that A is as above and consider the equivalence relation $\sim$ in I such that $i \sim i'$ if, and only if, Ae_i and $Ae_{i'}$ are isomorphic graded A-modules. If I_0 is a set of representatives under this relation, then we can consider the full graded subcategory of A having as objects the elements of I_0. This amounts to take the graded subalgebra $B = \oplus_{i,i'\in I_0}e_iAe_{i'}$, which will be called the *weak skeleton* of A. If we denote by $[i]$ the unique element of I_0 such that $i \sim [i]$, then there are elements $\xi_i \in e_iA_0e_{[i]}$ and $\xi_i^{-1} \in e_{[i]}A_0e_i$ such that $\xi_i\xi_i^{-1} = e_i$ and $\xi_i^{-1}\xi_i = e_{[i]}$. We fix ξ_i and ξ_i^{-1} from now on. By convention, we assume that $\xi_{[i]} = e_{[i]}$, for each $[i] \in I_0$. Now we get a surjective on objects graded functor $F : A \longrightarrow B$ which takes $i \rightsquigarrow [i]$ on objects and if $a \in e_iAe_{i'}$, then $F(a) = \xi_i^{-1}a\xi_{i'}$. If we take $P = \oplus_{i\in I_0}e_iA$ then P is an H-graded $B-A-$bimodule and the pullup functor is naturally isomorphic to the 'unitarization' of the graded Hom functor, $A\mathrm{HOM}_B(P,-) : B-Gr \longrightarrow A-Gr$ (see Subsection 1.2.1). It is an equivalence of categories and the pushdown functor F_λ gets identified with $P\otimes_A - : A-Gr \longrightarrow B-Gr$, which, up to isomorphism, takes $M \rightsquigarrow \oplus_{i\in I_0}e_iM$.

Definition 10. Let A and B be as above. A graded functor $F : A \longrightarrow B$ will be called a *covering functor* when it is surjective on objects and, for each $(i,j,h) \in I \times J \times H$, the induced maps

$$\oplus_{i'\in F^{-1}(j)}e_iA_he_{i'} \longrightarrow e_{F(i)}B_he_j$$
$$\oplus_{i'\in F^{-1}(j)}e_{i'}A_he_i \longrightarrow e_jB_he_{F(i)}$$

are bijective.

We shall now present the paradigmatic example of covering functor, which is actually the only one that we will need in our work. In the rest of this subsection, $A = \oplus_{h\in H}A_h$ will be a locally finite dimensional graded algebra with a distinguished family $(e_i)_{i\in I}$ of homogeneous orthogonal idempotents of degree 0, fixed from now on. We will assume that G is a group acting on A as a group of graded automorphisms (of degree 0) which permutes the e_i. That is, if $\mathrm{Aut}^{gr}(A)$ denotes the group of graded automorphisms of degree 0 which permute the e_i, then we have a group homomorphism $\varphi : G \longrightarrow \mathrm{Aut}^{gr}(A)$. We will write $a^g = \varphi(g)(a)$, for each $a \in A$ and $g \in G$. In such a case, the *skew group algebra* $A \star G$ has as elements the formal A-linear combinations $\sum_{g\in G} a_g \star g$, with $a_g \in A$ for all $g \in G$.

The multiplication extends by linearity the product $(a \star g)(b \star g') = ab^g \star gg'$, where $a, b \in A$ and $g, g' \in G$. The new algebra inherits an H-grading from A by taking $(A\star G)_h = A_h \star G = \{\sum_{g\in G} a_g \star g \in A \star G : a_g \in A_h, \text{ for all } g \in G\}$. Then $A \star G$ is a graded algebra with enough idempotents for which we fix the family of orthogonal idempotents $(e_i \star 1)_{i\in I}$ where 1 is the unit of G and, clearly, we have a canonical inclusion of H-graded algebras $\iota : A \hookrightarrow A \star G$ which maps $a \rightsquigarrow a \star 1$.

Proposition 1.4.1. *In the situation above, let Λ be the weak skeleton of $A \star G$ and $F : A \star G \longrightarrow \Lambda$ the corresponding functor. Then the composition $A \stackrel{\iota}{\hookrightarrow} A \star G \stackrel{F}{\longrightarrow} \Lambda$ is a covering functor. The corresponding pushdown functor $F_\lambda \circ \iota_\lambda : A - Gr \longrightarrow \Lambda - Gr$ is exact and takes $Ae_i \rightsquigarrow \Lambda e_{[i]}$, for each $i \in I$.*

Proof. The pullup functor is the composition $\Lambda - Gr \stackrel{F^\rho}{\longrightarrow} A\star G - Gr \stackrel{\iota^\rho}{\longrightarrow} A - Gr$, so that the pushdown functor is $F_\lambda \circ \iota_\lambda$. We know that F_λ is an equivalence of categories. On the other hand ι_λ is naturally isomorphic to $A\star G \otimes_A - : A - Gr \longrightarrow A\star G - Gr$ since ι^ρ is the usual restriction of scalars. The exactness of $A\star G \otimes_A -$ implies that of $F_\lambda \circ \iota_\lambda$ and the action of this functor on projective objects takes $Ae_i \rightsquigarrow (A\star G) \otimes_A Ae_i \cong (A\star G)e_i \rightsquigarrow F_\lambda((A\star G)e_i)$. But this latter graded Λ-module is isomorphic to $\Lambda e_{[i]}$ by the explicit definition of the pushdown functor when taking a weak skeleton.

In order to check that $F \circ \iota$ is a covering functor we look at the definition of the weak skeleton. In our case $(A \star G)e_i \cong (A \star G)e_j$ if, and only if, there are $x \in e_i(A \star G)_0 e_j = \oplus_{g\in G} e_i A_0 e_{g(j)} \star g$ and $y \in e_j(A\star G)_0 e_i = \oplus_{g\in G} e_j A_0 e_{g(i)} \star g$ such that $xy = e_i$ and $yx = e_j$. This immediately implies that i and j are in the same G-orbit, i.e., that $e_i^g = e_j$, for some $g \in G$. The converse is also true for we have equalities $e_i \star g = (e_i \star 1)(e_i \star g)(e_{g^{-1}(i)} \star 1)$ and $e_{g^{-1}(i)} \star g^{-1} = (e_{g^{-1}(i)} \star 1)(e_{g^{-1}(i)} \star g^{-1})(e_i \star 1)$, and also $(e_i \star g)(e_{g^{-1}(i)} \star g^{-1}) = e_i \star 1$ and $(e_{g^{-1}(i)} \star g^{-1})(e_i \star g) = e_{g^{-1}(i)} \star 1$, which shows that $(A \star G)e_i \cong (A \star G)e_{g^{-1}(i)}$ for all $g \in G$ and $i \in I$.

What we do now is to take exactly one index $i \in I$ in each G-orbit and in that way we get a subset I_0 of I. Up to graded isomorphism, we have $\Lambda = \oplus_{i,j\in I_0} e_i(A \star G)e_j$. For the explicit definition of F, we put $\xi_{g(i)} = e_{g(i)} \star g$ and $\xi_{g(i)}^{-1} = e_i \star g^{-1}$, for each $i \in I_0$ and $g \in G$. If $g, g' \in G$ and $i, j \in I_0$, then the map $F : e_{g(i)}(A\star G)e_{g'(j)} \longrightarrow e_i \Lambda e_j = e_i(A\star G)e_j$ takes $x \rightsquigarrow \xi_{g(i)}^{-1} x \xi_{g'(j)} = (e_i \star g^{-1})x(e_{g'(j)} \star g')$. Then the composition

$$e_{g(i)} A e_{g'(j)} \stackrel{\iota}{\longrightarrow} e_{g(i)}(A \star G)e_{g'(j)} \stackrel{F}{\longrightarrow} e_i \Lambda e_j = e_i(A \star G)e_j$$

takes $a \rightsquigarrow (e_i \star g^{-1})(a \star 1)(e_{g'(j)} \star g') = a^{g^{-1}} \star g^{-1}g'$.

The proof that $F \circ \iota$ is a covering functor gets then reduced to check that if $i, j \in I_0$ and $h \in H$ then the maps

$$\oplus_{g\in G} e_{g(i)} A_h e_j \longrightarrow e_i \Lambda_h e_j = e_i(A \star G)_h e_j, \qquad (a_g)_{g\in G} \rightsquigarrow \textstyle\sum_{g\in G} a_g^{g^{-1}} \star g^{-1}$$
$$\oplus_{g\in G} e_i A_h e_{g(j)} \longrightarrow e_i \Lambda_h e_j = e_i(A \star G)_h e_j, \qquad (b_g)_{g\in G} \rightsquigarrow \textstyle\sum_{g\in G} b_g \star g$$

are both bijective. But this is clear since $\oplus_{g\in G}(e_{g(i)}A_he_j)^{g^{-1}} \star g^{-1} = e_i(A\star G)_he_j = \oplus_{g\in G}e_iA_he_{g(j)} \star g$. □

Definition 11. If $A = \oplus_{h\in H}A_h$, G and Λ are as above, then the functor $F\circ\iota : A \longrightarrow \Lambda$ will be called a *G-covering* of Λ.

If A and G are as in the setting, we say that G *acts freely on objects* when $g(i) \neq i$, for all $i \in I$ and $g \in G \setminus \{1\}$. In such case we can form the *orbit category* A/G. The objects of this category are the G-orbits $[i]$ of indices $i \in I$ and the morphisms from $[i]$ to $[j]$ are formal sums $\sum_{g\in G}[a_g]$, where $[a_g]$ is the G-orbit of an element $a_g \in e_iAe_{g(j)}$. This definition does not depend on i, j, but just on the orbits $[i], [j]$. The anticomposition of morphisms extends by K-linearity the following rule. If $a, b \in \bigcup_{i,j\in I} e_iAe_j$ and $[a]$, $[b]$ denote the G-orbits of a and b, then $[a]\cdot[b] = 0$, in case $[t(a)] \neq [i(b)]$, and $[a]\cdot[b] = [ab^g]$, in case $[t(a)] = [i(b)]$, where g is the unique element of G such that $g(i(b)) = t(a)$. We have an obvious canonical projection $\pi : A \longrightarrow A/G$ with takes $a \rightsquigarrow [a]$. The following is the classical interpretation of Λ and is implicit in [7].

Corollary 1.4.2. *Let A, G and Λ be as in Proposition 1.4.1 and suppose that G acts freely on objects. There is an equivalence of categories $\Upsilon : \Lambda \xrightarrow{\cong} A/G$ such that $\Upsilon \circ F \circ \iota : A \longrightarrow A/G$ is the canonical projection.*

Proof. Let us fix a set I_0 of representatives of the elements of I under the equivalence relation $\sim$ given by: $i \sim j$ if, and only if, $(A\star G)e_i \cong (A\star G)e_j$ are isomorphic as graded $(A\star G)$-modules. Then, by definition, Λ is the category having as objects the elements of I_0 and $e_i\Lambda e_j = e_i(A\star G)e_j = \oplus_{g\in G}[e_iAe_{g(j)}\star g]$ as space of morphisms from i to j. The functor $\Upsilon : \Lambda \longrightarrow A/G$ is defined as $\Upsilon(i) = [i]$, for each $i \in I_0$, and by $\Upsilon(a\star g) = [a]$, when $g \in G$ and $a \in e_iAe_{g(j)}$, with $i, j \in I_0$.

The functor is clearly dense. On the other hand, if $\Upsilon(\sum_{g\in G} a_g \star g) = \Upsilon(\sum_{g\in G} b_g \star g)$, with $a_g, b_g \in e_iAe_{g(j)}$ for some $i, j \in I_0$, then we have an equality of formal finite sums of orbits $\sum_{g\in G}[a_g] = \sum_{g\in G}[b_g]$. This implies that $[a_g] = [b_g]$, for each $g \in G$, because if there is an element $\sigma \in G$ such that $\sigma(a_g)$ and b_h have the same origin an terminus, for some $h \in H$, then $\sigma = id$ due to the free action on objects. But the equality $[a_g] = [b_g]$ also implies that $a_g = b_g$ since $i(a_g) = i(b_g) = i$. Therefore Υ is a faithful functor. Finally, the orbit of any homogeneous morphism a in A contains an element with origin, say i, in I_0. Then, in order to prove that Υ is full, we can assume that $[a]$ is the orbit of an element $a \in e_iAe_{g(j)}$, for some $i, j \in I_0$ and some $g \in G$. But then $a\star g \in e_i(A\star G)e_j$, and we clearly have $\Upsilon(a\star g) = [a]$.

The equality of functor $\Upsilon \circ F \circ \iota = \pi$ is straightforward. □

1.4.2 Preservation of the pseudo-Frobenius condition

We start with the notion of G-invariant Nakayama form.

Definition 12. Let $A = \oplus_{h\in H}A_h$ be a graded pseudo-Frobenius algebra and G be a group acting on A as graded automorphisms. A graded Nakayama form $(-,-) : A \times A \longrightarrow K$ will be called *G-invariant* when $(a^g, b^g) = (a, b)$, for all $a, b \in A$ and all $g \in G$.

The following result is most important for us. It shows that the pseudo-Frobenius condition is preserved via the pushdown functor $F : A \longrightarrow A/G$, whenever the G-invariant property is satisfied.

Proposition 1.4.3. *Let $A = \oplus_{h\in H} A_h$ be a (split weakly) basic graded locally bounded algebra, with $(e_i)_{i\in I}$ as distinguished family of orthogonal homogeneous idempotents, and let G be a group which acts on A as graded automorphisms which permute the e_i and which acts freely on objects. Suppose that A is graded pseudo-Frobenius admitting a G-invariant graded Nakayama form $(-,-) : A \times A \longrightarrow K$. Then $\Lambda = A/G$ is a (split weakly) basic graded locally bounded pseudo-Frobenius algebra whose graded Nakayama form is induced from $(-,-)$.*

Proof. We put $\pi := F \circ \iota$, where F and ι are as in Proposition 1.4.1. We then know that π is surjective on objects and each (homogeneous) morphism in Λ is a sum of (homogeneous) morphisms of the form $\pi(a)$, with $a \in \bigcup_{i,j\in I} e_iAe_j$. We will put $\pi(i) = [i]$ and $\pi(a) = [a]$, for each $i \in I$ and homogeneous element $a \in \bigcup_{i,j\in I} e_iAe_j$. Note that $[i]$ and $[a]$ can be identified with the G-orbits of i and a (see corollary 1.4.2).

We first check that Λ is weakly basic whenever A is so. The functor F, which is an equivalence of categories, gives an isomorphism of algebras $e_i(A\star G)_0e_i \cong e_{[i]}\Lambda_0e_{[i]}$, for each $i \in I$. But we have $e_i(A\star G)_0e_i = \oplus_{g\in G}e_iA_0e_{g(i)}\star g$. This algebra is finite dimensional due to the graded locally bounded condition of A and the fact that G acts freely on objects. Then all nilpotent elements of $e_i(A\star G)_0e_i$ belong to its Jacobson radical. It follows that $\mathbf{m} := e_iJ(A_0)e_i \oplus (\oplus_{g\neq 1}e_iA_0e_{g(i)}\star g)$ is contained in $J(e_i(A\star G)_0e_i)$ since, due again to the graded locally bounded condition of A and the free action of G, we know that $\mathbf{m}$ consists of nilpotent elements. Since $\frac{e_i(A\star G)_0e_i}{\mathbf{m}} \cong \frac{e_iA_0e_i}{e_iJ(A_0)e_i}$ is a division algebra, we conclude that $\mathbf{m} = J(e_i(A\star G)_0e_i)$ and that $e_{[i]}\Lambda_0e_{[i]} \cong e_i(A\star G)_0e_i$ is a local algebra. Moreover, we have that $\frac{e_{[i]}\Lambda_0e_{[i]}}{e_{[i]}J(\Lambda_0)e_{[i]}} \cong \frac{e_i(A\star G)_0e_i}{e_iJ((A\star G)_0)} \cong \frac{e_iA_0e_i}{e_iJ(A_0)e_i}$, so that Λ is split whenever A is so.

We next prove that $e_{[i]}\Lambda_he_{[j]} \subset J^{gr}(\Lambda)$ whenever $[i] \neq [j]$. But this amounts to prove that $e_i(A\star G)e_j \subset J^{gr}(A\star G)$ whenever $[i] \neq [j]$ since $F : A\star G \longrightarrow \Lambda$ is an equivalence of graded categories. Let us take $x \in e_i(A\star G)_he_j$. Recall that $x \in J^{gr}(A\star G)$ if, and only if, $e_j - yx$ is invertible in $e_j(A\star G)_0e_j$, for each $y \in e_j(A\star G)_{-h}e_i$. Let us fix such an x and assume that $x \notin J^{gr}(A\star G)$. We then get $y \in e_j(A\star G)_{-h}e_i$ such that $e_j - yx$ is not invertible in the algebra $e_j(A\star G)_0e_j$, which is local by the previous paragraph. It follows that $e_j - yx \in J(e_j(A\star G)_0e_j)$, so that yx is invertible in $e_j(A\star G)_0e_j$. By suitable replacement, without loss of generality, we can assume that $yx = e_j = e_j \star 1$. We write $x = \sum_{g\in G} a_g \star g$ and $y = \sum_{g'} b_{g'} \star g'$, where $a_g \in e_iA_he_{g(j)}$ and $b_{g'} \in e_jA_{-h}e_{g'(i)}$. From $yx = e_i$ we get the equality $\sum_{g\in G} b_{g^{-1}}a_g^{g^{-1}} = e_j$ in A. But $b_{g^{-1}} \in e_jAe_{g^{-1}(i)} \subseteq J^{gr}(A)$ because A is weakly basic and $j \neq g^{-1}(i)$. It then follows that $e_j \in J^{gr}(A)$, which is a contradiction. Therefore Λ is weakly basic.

Suppose that A is basic, and let us prove that Λ is also basic. The argument of the previous paragraph is valid, by taking $i = j$ and assuming $h \neq 1$. Using the fact that $e_iA_he_{g^{-1}(i)} \subset J^{gr}(A)$ whenever $g \in G$ and $h \in H \setminus \{1\}$, the argument proves that $e_{[i]}\Lambda_he_{[i]} \subset J^{gr}(\Lambda)$ whenever $h \neq 1$.

We pass to define the graded Nakayama form for Λ. We will define first graded bilinear forms $< -,- >: e_{[i]}\Lambda e_{[j]} \times e_{[k]}\Lambda e_{[l]} \longrightarrow K$, for all objects $[i]$, $[j]$, $[k]$ and $[l]$ of Λ. When

$[j] \neq [k]$ the bilinear form is zero. In case $[j] = [k]$, we need to define $< \pi(a), \pi(b) >$ whenever $a \in \oplus_{g,g' \in G} e_{g(i)} A e_{g'(j)}$ and $b \in \oplus_{g,g' \in G} e_{g(j)} A e_{g'(l)}$. We define $< \pi(a), \pi(b) >$ when $a, b \in \bigcup_{i,l \in I} e_i A e_l$, with $[t(a)] = [i(b)] = [j]$ and then extend by K-bilinearity to the general case. Indeed we define $< [a], [b] >= (a, b^g)$, where $g \in G$ satisfies that $g(i(b)) = t(a)$. Note that g is unique since G acts freely on objects. It is routine to check that $< -, - >: e_{[i]} \Lambda e_{[j]} \times e_{[j]} \Lambda e_{[k]}$ is well-defined. The graded bilinear form $< -, - >: \Lambda \times \Lambda \longrightarrow K$ is defined as the 'direct sum' of the just defined graded bilinear forms.

We next check that it satisfies all the conditions of definition 5. We first check condition 2 in that definition. Let $x, y \in \bigcup_{[i],[j]} e_{[i]} \Lambda e_{[j]}$ be such that $< x, y > \neq 0$. Then we know that there is $j \in I$ such that $t(x) = [j] = i(y)$. Fix such index $j \in I$. Since the functor $\pi : A \longrightarrow \Lambda$ is covering it gives bijections $\oplus_{g \in G} e_{g(i)} A e_j \xrightarrow{\cong} e_{[i]} \Lambda e_{[j]}$ and $\oplus_{g \in G} e_j A e_{g(k)} \xrightarrow{\cong} e_{[j]} \Lambda e_{[k]}$, for all G-orbits of indices $[j]$ and $[k]$. We then put $x = \sum_{g \in G} \pi(a_g)$ and $y = \sum_{g \in G} \pi(b_g)$ such that $a_g \in e_{g(i)} A e_j$ and $b_g \in e_j A e_{g(k)}$, for all $g \in G$. By definition of $< -, - >$, we then have $0 \neq < x, y >= \sum_{g,g' \in G} (a_g, b_{g'})$, which implies that there are $g, g' \in G$ such that $(a_g, b_{g'}) \neq 0$. This implies that $g'(k) = \nu(g(i))$, where ν is the Nakayama permutation associated to $(-,-)$. But, due to the G-invariant condition of $(-,-)$, we have that $\nu(g(i)) = g(\nu(i))$. This shows that $[k] = [\nu(i)]$. It follows that $< e_{[i]} \Lambda, \Lambda e_{[k]} > \neq 0$ implies that $[k] = [\nu(i)]$. Therefore assertion 2 of definition 5 holds, and the bijection $\bar{\nu} : I/G \xrightarrow{\cong} I/G$ maps $[i] \rightsquigarrow [\nu(i)]$.

The G-invariance of $(-,-)$ also implies that if $\mathbf{h} : I \longrightarrow H$ is the degree function associated to $(-,-)$, then $h(g(i)) = h(i)\ \forall i \in I$. As a consequence, the graded bilinear form $< -, - >: e_{[i]} \Lambda \times \Lambda e_{[\nu(i)]} \longrightarrow K$ is of degree $h_i := \mathbf{h}(i)$, for each $i \in I$. Then the map $\bar{\mathbf{h}} : I/G \longrightarrow H$, $[i] \rightsquigarrow h_i$, is the degree function of $< -, - >$.

It remains to check that $< xy, z >=< x, yz >$, for all $x, y, z \in \Lambda$. For that, it is not restrictive to assume that $x = [a]$, $y = [b]$ and $z = [c]$, where a, b, c are homogeneous elements in $\bigcup_{i,j \in I} e_i A e_j$. In such a case, note that if one member of the desired equality $< xy, z >=< x, yz >$ is nonzero, then $t(x) = i(y)$ and $t(y) = i(z)$ or, equivalently, $[t(a)] = [i(b)]$ and $[t(b)] = [i(c)]$. If this holds, then we have

$$< xy, z >=< [a][b], [c] >=< [ab^g], [c] >= (ab^g, c^{g'}),$$

where $g, g' \in G$ are the elements such that $g(i(b)) = t(a)$ and $g'(i(c)) = g(t(b))$. Note that then $(g^{-1}g')(i(c)) = t(b)$ and, hence, we also have

$$< x, yz >=< [a], [b][c] >=< [a], [bc^{g^{-1}g'}] >= (a, (bc^{g^{-1}g'})^g) = (a, b^g c^{g'}).$$

The equality $< xy, z >=< x, yz >$ follows then from the fact that $(-,-) : A \times A \longrightarrow K$ is a graded Nakayama form for A. □

The following result completes the last proposition by showing how to construct G-invariant graded Nakayama forms in the split case.

Corollary 1.4.4. *Let $A = \oplus_{h \in H} A_h$ be a split basic graded pseudo-Frobenius algebra and let G be a group of graded automorphisms of A which permute the e_i and acts freely on objects. There exist an element $\mathbf{h} = (h_i)_{i \in I} \in \prod_{i \in I} Supp(e_i Soc_{gr}(A))$ and basis $\mathcal{B}_i$ of $e_i A_{h_i}$, for each $i \in I$, satisfying the following properties:*

1. *$h_i = h_{g(i)}$, for all $i \in I$*

2. *$g(\mathcal{B}_i) = \mathcal{B}_{g(i)}$ and $\mathcal{B}_i$ contains an element of $e_i Soc_{gr}(A)$, for all $i \in I$*

In such case, letting $\mathcal{B} = \bigcup_{i\in I} \mathcal{B}_i$, the graded Nakayama form associated to the pair $(\mathcal{B}, \mathbf{h})$ (see definition 8) is G-invariant.

Proof. We fix a subset $I_0 \subseteq I$ which is a set of representatives of the G-orbits of objects. Then the assignment $i \rightsquigarrow [i]$ defines a bijection between I_0 and the set of objects of $\Lambda = A/G$. For each $i \in I_0$, we fix an $h_i \in Supp(e_i Soc_{gr}(A))$ and a basis $\mathcal{B}_i$ of $e_i A_{h_i}$ containing an element $w_i \in e_i Soc_{gr}(A)$, for each $i \in I_0$. Note that $g(e_i Soc_{gr}(A)) = e_{g(i)} Soc_{gr}(A)$ since G consists of graded automorphisms. It then follows that $h_i \in Supp(e_{g(i)} Soc_{gr}(A))$. Given $j \in I$, the free action of G on objects implies that there are unique elements $i \in I_0$ and $g \in G$ such that $g(i) = j$. We then define $h_j = h_i$ and $\mathcal{B}_j = g(\mathcal{B}_i)$, whenever $j = g(i)$, with $i \in I_0$. Note that $\mathcal{B}_j$ contains the element $g(w_i)$ of $e_j Soc_{gr}(A)$. It is now clear that $\mathbf{h} = (h_j)_{j\in I}$ is in $\prod_{j\in I} Supp(e_j Soc_{gr}(A))$ and that $\mathcal{B}_j$ is a basis of $e_j A_{h_j}$ containing an element of $e_j Soc_{gr}(A)$, for each $j \in I$. It is also clear that if $\mathcal{B} := \bigcup_{j\in I} \mathcal{B}_j$ then $g(\mathcal{B}) = \mathcal{B}$, for all $g \in G$.

By definition of the graded Nakayama form $(-,-) : A \times A \longrightarrow K$ associated to $(\mathcal{B}, \mathbf{h})$ (see definition 8) and the fact that $w_j = g(w_j) = w_{g(j)}$, for all $g \in G$ and $j \in I$, we easily conclude that $(-,-)$ is G-invariant. □

The following result states that, assuming that the Nakayama form of A is G-invariant, the Nakayama automorphism of the algebra A/G is induced by the Nakayama automorphism of A. It is a direct consequence of Proposition 1.4.3, its proof and the definition of the Nakayama automorphism.

Corollary 1.4.5. *Let $A = \oplus_{h\in H} A_h$ be a weakly basic graded pseudo-Frobenius algebra and let $(-,-) : A \times A \longrightarrow K$ be a G-invariant graded Nakayama. The following assertions hold:*

1. *If $\eta : A \longrightarrow A$ is the Nakayama automorphism associated to $(-,-)$, then $\eta \circ g = g \circ \eta$, for all $g \in G$*

2. *Let $< -,- >: \Lambda \times \Lambda \longrightarrow K$ be the graded Nakayama form induced from $(-,-)$ and let $\bar{\eta} : \Lambda \longrightarrow \Lambda$ be the associated Nakayama automorphism. Then $\bar{\eta}([a]) = [\eta(a)]$ for each $a \in \bigcup_{i,j} e_i A e_j$.*

Chapter 2

The mesh algebra of a Dynkin diagram

2.1 Introduction

2.1.1 Motivation

The relevance of the Nakayama automorphism η of a finite dimensional self-injective algebra Λ becomes clear, at least, in questions related to the symmetry and the Calabi-Yau condition, due to the existence of the isomorphism of Λ-bimodules $D(\Lambda) \cong {}_1\Lambda_\eta$. Despite of the fact that we are actually interested in the class of the m-fold mesh algebras, we approach the problem of determining the Nakayama automorphism of any algebra in this class by investigating that of the corresponding universal Galois cover. In view of Lemma 1.4.5 and, in order to establish a suitable relationship between the associated Nakayama automorphisms, the G-invariant condition has to be required. This turns out to be the main reason for studying in depth the mesh algebra of a Dynkin diagram.

2.1.2 Outline of the chapter

In Section 2.2 we recall the general definition of a stable translation quiver paying special attention to the most interesting example for our purposes, namely, the stable translation quiver $\mathbb{Z}\Delta$ associated to a Dynkin diagram Δ. Concerning Section 2.3, we first introduce the notion and essential properties of the mesh algebra of a Dynkin diagram which is, by definition, related to the stable translation quiver $\mathbb{Z}\Delta$, for some Dynkin quiver Δ. Next, we exhibit the list of the m-fold mesh algebras, which arise as the orbit algebras of the mesh algebras of Dynkin diagrams by factoring out a weakly admissible group of automorphisms G and, for such class of algebras, we introduce the notion of extended type. Finally, we end the section by performing a change of relations which, roughly speaking, transforms sums of paths of length 2 into differences. In Section 2.4 we give the explicit formula, for any choice (Δ, G), of a G-invariant Nakayama automorphism of the mesh algebra associated to Δ.

2.1.3 Notation

Throughout this chapter K will be a fixed field, Δ will be one of the Dynkin quivers $\mathbb{A}_n$, $\mathbb{D}_{n+1}$ ($n \geq 3$) or $\mathbb{E}_n$ ($n = 6, 7, 8$), and $\mathbb{Z}\Delta$ will be the associated translation quiver. Its path algebra will be denoted by $K\mathbb{Z}\Delta$ and we will put $B = K(\mathbb{Z}\Delta)$ for the mesh algebra. The Nakayama automorphism of B will be denoted by η.

2.2 Stable translation quivers

This section is devoted, only and exclusively, to introduce the notion of a stable translation quiver and its associated mesh algebra.

We recall that a *stable translation quiver* is a pair (Γ, τ), where Γ is a locally finite quiver (i.e. given any vertex, there are only finite arrows having it as origin or terminus) and $\tau : \Gamma_0 \to \Gamma_0$ is a bijective map such that for any $x, y \in \Gamma_0$, the number of arrows from x to y is equal to the number of arrows from $\tau(y)$ to x. The map τ will be called the *Auslander-Reiten translation*. Throughout the rest of the work, whenever we have a stable translation quiver, we will also fix a bijection $\sigma : \Gamma_1(x, y) \to \Gamma_1(\tau(y), x)$ called a *polarization* of (Γ, τ). Note that, from the definition of σ, one gets that τ can be extended to a graph automorphism of Γ by setting $\tau(\alpha) = \sigma^2(\alpha)$ $\forall \alpha \in \Gamma_1$. If $K\Gamma$ denotes the path algebra of Γ, then the *mesh algebra* of Γ is $K(\Gamma) = K\Gamma/I$, where I is the ideal of $K\Gamma$ generated by the so-called *mesh relations* r_x, where $r_x = \sum_{a \in \Gamma_1,\, t(a)=x} \sigma(a)a$, for each $x \in \Gamma_0$. Note that, when Γ is viewed as a $\mathbb{Z}$-graded quiver with all arrows having degree 1, then I is homogeneous with respect to the induced grading on $K\Gamma$. Therefore $K(\Gamma)$ is canonically a positively ($\mathbb{Z}$-)graded algebra with enough idempotents and τ becomes a graded automorphism of $K(\Gamma)$.

The typical example of stable translation quiver is the following. Given a locally finite quiver Δ, the stable translation quiver $\mathbb{Z}\Delta$ will have as set of vertices $(\mathbb{Z}\Delta)_0 = \mathbb{Z} \times \Delta_0$. Moreover, for each arrow $\alpha : x \to y$ in Δ_1, we have arrows $(n, \alpha) : (n, x) \to (n, y)$ and $(n, \alpha)' : (n, y) \to (n + 1, x)$ in $(\mathbb{Z}\Delta)_1$. Finally, we define $\tau(n, x) = (n - 1, x)$, for each $(n, x) \in (\mathbb{Z}\Delta)_0$, and $\sigma(n, \alpha) = (n - 1, \alpha)'$ and $\sigma[(n, \alpha)'] = (n, \alpha)$.

In general, different quivers Δ and Δ' with the same underlying graph give non-isomorphic translation quivers $\mathbb{Z}\Delta$ and $\mathbb{Z}\Delta'$. However, when Δ is a tree, e.g., when Δ is any of the Dykin quivers $\mathbb{A}_n, \mathbb{D}_{n+1}, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$, the isoclass of the translation quiver $\mathbb{Z}\Delta$ does not depend on the orientation of the arrows.

A group of automorphism G of a stable translation quiver (Γ, τ) is a group of automorphisms of Γ which commute with τ and σ. Such a group is called *weakly admissible* when $x^+ \cap (gx)^+ = \emptyset$, for each $x \in \Gamma_0$ and $g \in G \backslash \{1\}$, where $x^+ := \{y \in \Gamma_0 : \Gamma_1(x, y) \neq \emptyset\}$. In such a case, when G acts freely on objects, the orbit quiver Γ/G inherits a structure of stable translation quiver, with the AR translation $\bar{\tau}$ mapping $[x] \rightsquigarrow [\tau(x)]$, for each $x \in \Gamma_0 \cup \Gamma_1$. Moreover, the group G can be interpreted as a group of graded automorphisms of the mesh algebra $K(\Gamma)$ and $K(\Gamma)/G$ is canonically isomorphic to the mesh algebra of Γ/G.

2.3 The mesh algebra of a Dynkin diagram

2.3.1 Definition and basic properties

Definition 13. Given a Dynkin diagram Δ, we will say that $B = B(\Delta)$ is the *mesh algebra of the Dynkin diagram* Δ if it is the mesh algebra of the stable translation quiver $\mathbb{Z}\Delta$.

When $\Delta = \mathbb{A}_{2n-1}$, $\mathbb{E}_6$ or $\mathbb{D}_{n+1}$, with $n > 3$, the underlying unoriented graph admits a canonical automorphism ρ of order 2. Similarly, $\mathbb{D}_4$ admits an automorphism of order 3. In each case, the automorphism ρ extends to an automorphism of $\mathbb{Z}\Delta$ with the same order. In the case of $\mathbb{A}_{2n}$ the canonical automorphism of order 2 of the underlying graph extends to an automorphism of $\mathbb{Z}\Delta$, but this automorphism has infinite order. It is still denoted by ρ and it plays, in some sense, a role similar to the other cases. This automorphism of $\mathbb{Z}\mathbb{A}_{2n}$ is obtained by applying the symmetry with respect to the horizontal line and moving half a unit to the right. Note that we have $\rho^2 = \tau^{-1}$. On the contrary, when $\Delta = \mathbb{E}_7$ or $\mathbb{E}_8$ there is no automorphism ρ defined.

Although the orientation in Δ does not change the isomorphism type of $\mathbb{Z}\Delta$, in order to numbering the vertices of $\mathbb{Z}\Delta$ we need to fix an orientation in Δ. Below we fix such an orientation, and then give the corresponding definition of the automorphism ρ of $\mathbb{Z}\Delta$ mentioned above.

1. If $\Delta = \mathbb{A}_{2n}$:

   $$1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow 2n \ ,$$

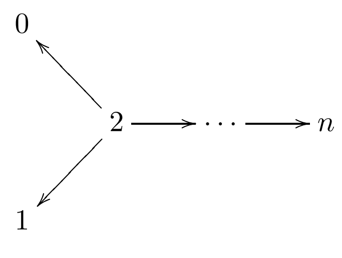
   then $\rho(k, i) = (k + i - n, 2n + 1 - i)$

2. If $\Delta = \mathbb{A}_{2n-1}$:

   $$1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow 2n - 1 \ ,$$

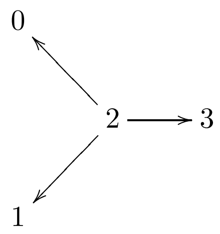
   then $\rho(k, i) = (k + i - n, 2n - i)$

3. $\Delta = \mathbb{D}_{n+1}$:

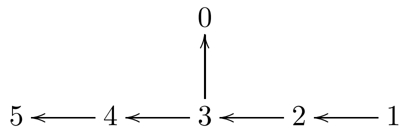
   with $n > 3$, then $\rho(k, 0) = (k, 1)$, $\rho(k, 1) = (k, 0)$ and ρ fixes all vertices (k, i), with $i \neq 0, 1$.

4. If $\Delta = \mathbb{D}_4$:



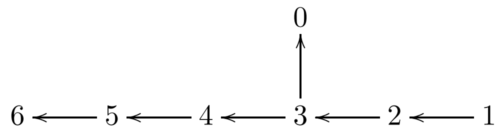
then ρ fixes the vertices $(k, 2)$ and, for k fixed, it applies the 3-cycle (013) to the second component of each vertex (k, i).

5. If $\Delta = \mathbb{E}_6$:

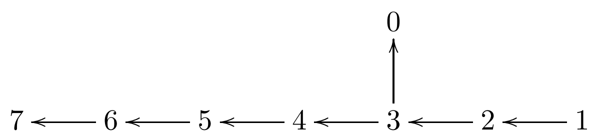


then $\rho(k, i) = (k + i - 3, 6 - i)$ for all $i \neq 0$ and $\rho(k, 0) = (k, 0)$

6. If $\Delta = \mathbb{E}_7$:

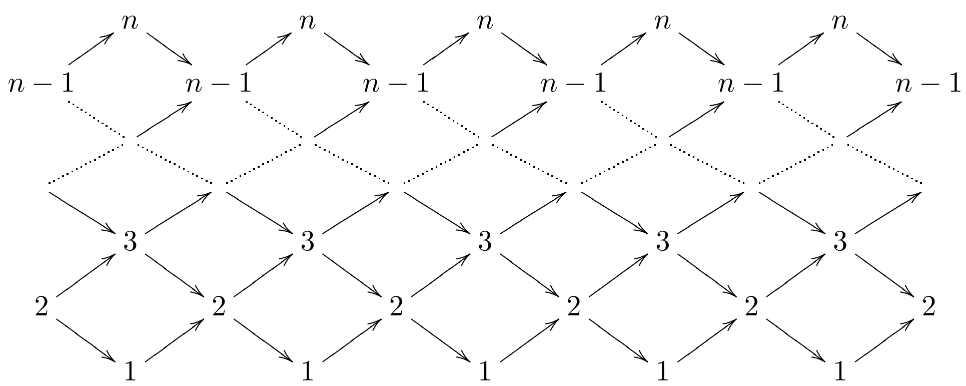


7. If $\Delta = \mathbb{E}_8$:



Following the definition of mesh algebra given in the previous section, we give bellow the quiver and relations of the mesh algebra $B = B(\Delta)$ associated to the stable translation quiver $\mathbb{Z}\Delta$, for $\Delta = \mathbb{A}_n, \mathbb{D}_{n+1}$ and $\mathbb{E}_6$.

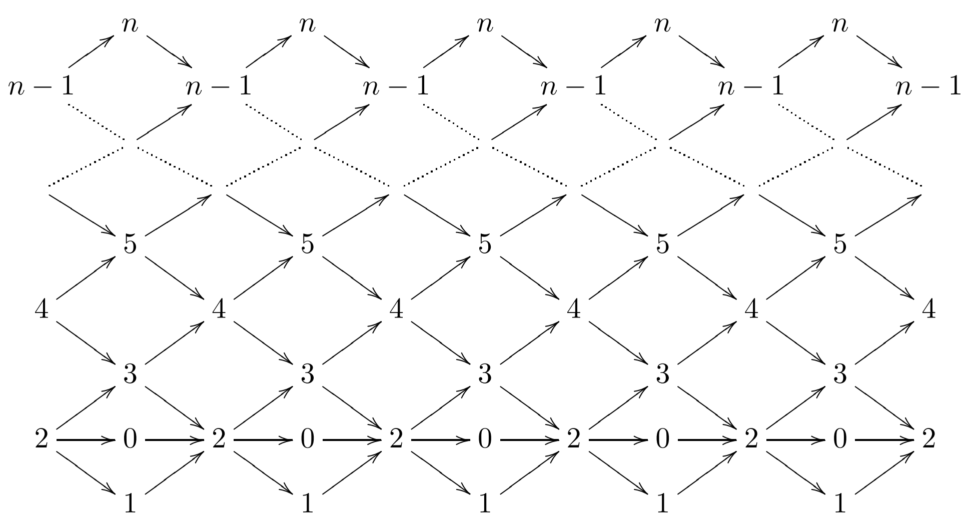
a) If $\Delta = \mathbb{A}_n$, then $\mathbb{Z}\mathbb{A}_n$ has quiver:



Letting $p_i : i \to i+1 \to i$ for $i = 0, \dots, n-1$ and $q_i : i \to i-1 \to i$ for $i = 1, \dots, n$, the relations are given by

$$p_0 = 0 = q_n$$
$$p_i + q_i = 0 \text{ for } i = 1, \dots, n-1$$

b) If $\Delta = \mathbb{D}_{n+1}$, then $\mathbb{Z}\mathbb{D}_{n+1}$ has quiver:

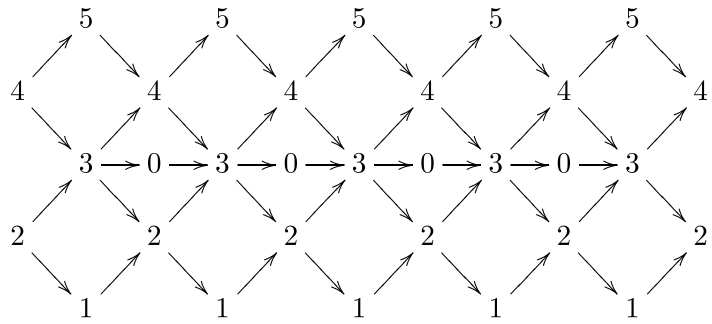


We put $u : 2 \longrightarrow 0 \longrightarrow 2$, $v : 2 \longrightarrow 1 \longrightarrow 2$, $w : 2 \longrightarrow 3 \longrightarrow 2$, $p_0 : 0 \longrightarrow 2 \longrightarrow 0$, and $p_1 : 1 \longrightarrow 2 \longrightarrow 1$.

Then, the relations are:

$$q_n = 0 = p_0 = p_1$$
$$p_i + q_i = 0 \ \forall i = 3, \dots n-1$$
$$u^2 = 0 = v^2$$
$$u + v + w = 0$$

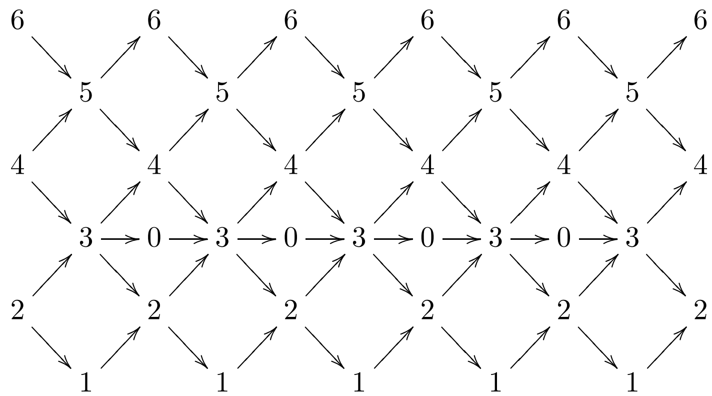
c) If $\Delta = \mathbb{E}_6$, then $\mathbb{Z}\mathbb{E}_6$ has quiver



We put $u : 3 \longrightarrow 0 \longrightarrow 3$, $v : 3 \longrightarrow 4 \longrightarrow 3$, $w : 3 \longrightarrow 2 \longrightarrow 3$, and $p_0 : 0 \longrightarrow 3 \longrightarrow 0$. Then, we have relations

$$p_0 = p_1 = 0 = q_5$$
$$p_i + q_i = 0 \ \forall i = 2, 4$$
$$u + v + w = 0$$

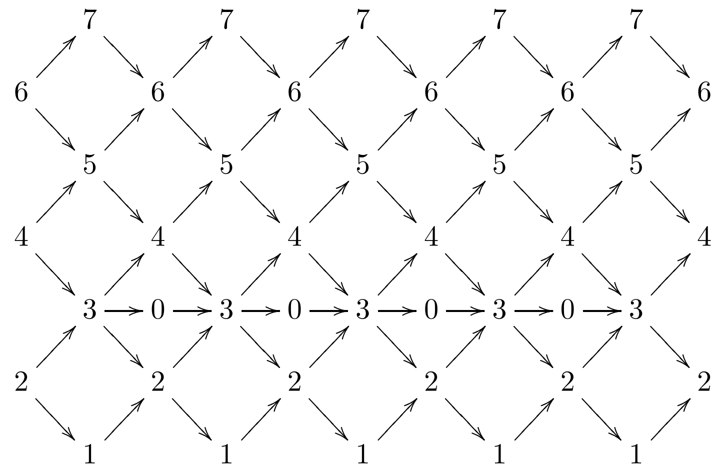
d) If $\Delta = \mathbb{E}_7$, then $\mathbb{Z}\mathbb{E}_7$ has quiver



We put $u : 3 \longrightarrow 0 \longrightarrow 3$, $v : 3 \longrightarrow 4 \longrightarrow 3$, $w : 3 \longrightarrow 2 \longrightarrow 3$, and $p_0 : 0 \longrightarrow 3 \longrightarrow 0$. Then, we have relations

$$p_0 = p_1 = 0 = q_6$$
$$p_i + q_i = 0 \ \forall i = 2, 4, 5$$
$$u + v + w = 0$$

e) If $\Delta = \mathbb{E}_8$, then $\mathbb{Z}\mathbb{E}_8$ has quiver



We put $u : 3 \longrightarrow 0 \longrightarrow 3$, $v : 3 \longrightarrow 4 \longrightarrow 3$, $w : 3 \longrightarrow 2 \longrightarrow 3$, and $p_0 : 0 \longrightarrow 3 \longrightarrow 0$. Then, we have relations

$$p_0 = p_1 = 0 = q_7$$
$$p_i + q_i = 0 \ \forall i = 2, 4, 5, 6$$
$$u + v + w = 0$$

Dynkin diagrams are fundamental in the classification of simple Lie algebras (see [49]). The Weyl group of such an algebra is the subgroup of the isometry group of its root system generated by the simple reflections. The product of these simple reflections is then an element of the Weyl group which is uniquely determined, up to conjugacy. The order of this element is called the *Coxeter number* of the corresponding Dynkin diagram. We will not need to go through the theory of Lie algebras in this work, but we will need the precise value c_Δ of the Coxeter number, for each Dynkin diagram $\Delta = \mathbb{A}_r$, $\mathbb{D}_{n+1}$, $\mathbb{E}_6$, $\mathbb{E}_7$ or $\mathbb{E}_8$. It is included in the next result.

The following facts are well-known (cf. [15][Section 1.1] and [43][Section 6.5]).

Proposition 2.3.1. *Let Δ be a Dynkin quiver, $\bar{\Delta}$ be its associated graph, c_Δ be its Coxeter number and $B = K(\mathbb{Z}\Delta)$ be the mesh algebra of the translation quiver $\mathbb{Z}\Delta$. The following assertions hold:*

1. *Each path of length $> c_\Delta - 2$ in $\mathbb{Z}\Delta$ is zero in B.*

2. *For each $(k, i) \in (\mathbb{Z}\Delta)_0$, there is a unique vertex $\nu(k, i) \in (\mathbb{Z}\Delta)_0$ for which there is a path $(k, i) \to \ldots \to \nu(k, i)$ in $\mathbb{Z}\Delta$ of length $c_\Delta - 2$ which is nonzero in B. This path is unique, up to sign in B.*

3. *If $(k, i) \to \ldots \to (m, j)$ is a nonzero path then there is a path $q : (m, j) \to \ldots \to \nu(k, i)$ such that pq is a nonzero path (of length $c_\Delta - 2$)*

4. *The assignment $(k, i) \rightsquigarrow \nu(k, i)$ gives a bijection $\nu : (\mathbb{Z}\Delta)_0 \longrightarrow (\mathbb{Z}\Delta)_0$, called the* Nakayama permutation.

5. *The vertex* $\nu(k,i)$ *is given as follows:*

   (a) *If* $\Delta = \mathbb{A}_r$, *with* $r = 2n$ *or* $2n-1$, *(hence* $c_\Delta = r+1$*), then* $\nu(k,i) = \rho\tau^{1-n}(k,i) = (k+i-1, r+1-i)$

   (b) *If* $\Delta = \mathbb{D}_{n+1}$ *(hence* $c_\Delta = 2n$*), then*

      i. $\nu(k,i) = \tau^{1-n}(k,i) = (k+n-1, i)$, *in case* $n+1$ *is even*

      ii. $\nu(k,i) = \rho\tau^{1-n}(k,i)$, *in case* $n+1$ *is odd.*

   (c) *If* $\Delta = \mathbb{E}_6$ *(hence* c_Δ*=12), then* $\nu(k,i) = \rho\tau^{-5}(k,i)$.

   (d) *If* $\bar{\Delta} = \mathbb{E}_7$ *(hence* $c_\Delta = 18$*), with any orientation, then* $\nu(k,i) = \tau^{-8}(k,i) = (k+8, i)$

   (e) *If* $\bar{\Delta} = \mathbb{E}_8$ *(hence* $c_\Delta = 30$*), with any orientation, then* $\nu(k,i) = \tau^{-14}(k,i) = (k+14, i)$.

In the following result we prove that mesh algebra B is not only a pseudo-Frobenius graded algebra, but also a Quasi-Frobenius graded algebra. This means, in particular, that it admits a Nakayama automorphism being graded.

Corollary 2.3.2. *B is a split basic graded Quasi-Frobenius algebra admitting a graded Nakayama form whose associate degree function takes constant value* $l = c_\Delta - 2$.

Proof. By last proposition, we know that $Be_{(k,i)}$ and $e_{(k,i)}B$ are finite dimensional graded B-modules. In particular, both are Noetherian, so that B is a locally Noetherian graded algebra. Note that $e_{(k,i)}Be_{(k,i)} \cong K$, for each vertex $(k,i) \in \Gamma_0$, and that $J^{gr}(B) = J(B)$ is the vector subspace generated by the paths of length > 0. Therefore B is clearly split basic. On the other hand, if ν es the Nakayama permutation and we fix a nonzero path $w_{(k,i)} : (k,i) \to ... \to \nu(k,i)$ of length $l = c_\Delta - 2$, then last proposition says that $w_{(k,i)}$ is in the (graded and ungraded) socle of $e_{(k,i)}B$.

By conditions 2 and 3 of Proposition 2.3.1, we have that $\dim(\mathrm{Soc}(e_{(k,i)}B)) = 1$ and that $\mathrm{Soc}(e_{(k,i)}B)$ is an essential (graded and ungraded) submodule of $e_{(k,i)}B)$. Note that B^{op} is the mesh algebra of the opposite Dynkin quiver Δ^{op}, which is again Dynkin of the same type. Then also $Be_{(k,i)}$ has essential simple (graded and ungraded) socle, which is isomorphic to $S_{\nu^{-1}(k,i)}[l]$ as graded left B-module. Then all conditions of Corollary 1.3.5 are satisfied, with $\nu' = \nu^{-1}$.

By Corollary 1.3.9, we know that B admits a graded Nakayama form with constant degree function and, by Proposition 1.3.7 and its proof, we have a unique choice, namely $\mathbf{h}(k,i) = l$ for all $(k,i) \in \Gamma_0$, because the support of $\mathrm{Soc}_{gr}(e_{(k,i)}B)$ is $\{l\}$. □

2.3.2 m-fold mesh algebras

With all our tools in place, we are ready to describe the class of the m-fold mesh algebras. When $\Gamma = \mathbb{Z}\Delta$, with Δ a Dynkin quiver, it is known that each weakly admissible automorphism is infinite cyclic (see [66], [1]) and below is the list of the resulting stable translation quivers $\mathbb{Z}\Delta/G$ that appear, where a generator of G is given in each case (see [25]). In each case, the following automorphism ρ is always that of the list preceding Proposition 2.3.1:

- $\Delta^{(m)} = \mathbb{Z}\Delta/\langle\tau^m\rangle$, for $\Delta = \mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_n$.
- $\mathbb{B}_n^{(m)} = \mathbb{Z}\mathbb{A}_{2n-1}/\langle\rho\tau^m\rangle$.
- $\mathbb{C}_n^{(m)} = \mathbb{Z}\mathbb{D}_{n+1}/\langle\rho\tau^m\rangle$.
- $\mathbb{F}_4^{(m)} = \mathbb{Z}\mathbb{E}_6/\langle\rho\tau^m\rangle$.
- $\mathbb{G}_2^{(m)} = \mathbb{Z}\mathbb{D}_4/\langle\rho\tau^m\rangle$.
- $\mathbb{L}_n^{(m)} = \mathbb{Z}\mathbb{A}_{2n}/\langle\rho\tau^m\rangle$.

As shown by Dugas (see [25][Section 3]), they are the only translation quivers with finite dimensional mesh algebras. These mesh algebras are isomorphic to $\Lambda = B/G$ in each case, where B is the mesh algebra of $\mathbb{Z}\Delta$. Abusing of notation, we will simply write $\Lambda = \mathbb{Z}\Delta/ < \varphi >$. These algebras are called *m-fold mesh algebras* and are known to be self-injective, a fact that can be easily seen by applying Proposition 1.4.3 since the cyclic group G acts freely on the objects, i.e., on $(\mathbb{Z}\Delta)_0$. They are also periodic (see [14]).

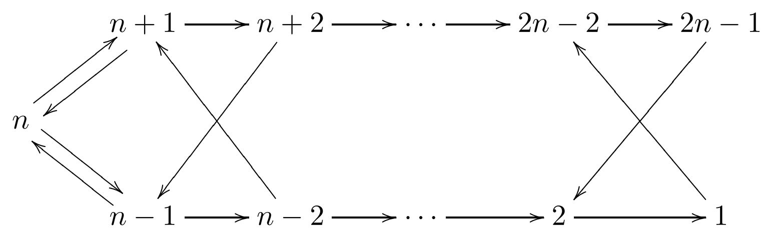
Note that, except for $\mathbb{L}_n^{(m)}$, each generator of the group G in the above list is of the form $\rho\tau^m$, where ρ is an automorphism of order 1 (i.e. $\rho = id_{\mathbb{Z}\Delta}$), 2 or 3. This leads us to introduce the following concept, which will be used later on in this work.

Definition 14. Let $\Lambda = \mathbb{Z}\Delta/ < \rho\tau^m >$ be an m-fold mesh algebra of a Dynkin quiver, possibly with $\rho = id_{\mathbb{Z}\Delta}$. The *extended type* of Λ will be the triple (Δ, m, t), where t is the order of ρ, in case $\Lambda \neq \mathbb{L}_n^{(m)}$, and $t = 2$ when $\Lambda = \mathbb{L}_n^{(m)}$.

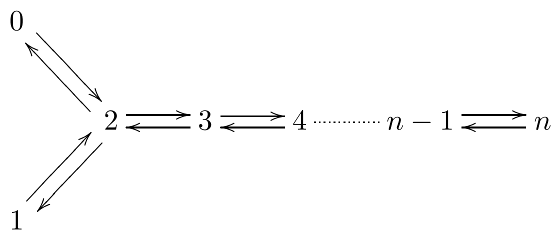
It is well-known that the stable Auslander algebra of any representation-finite self-injective finite dimensional algebra is an m-fold mesh algebra, but the converse is not true (see [25] and [51]). The reader is warned that the commonly used type of such an stable Auslander algebra (see [6], [25],[51]) does not coincide with the here defined extended type.

Examples 2.3.3. The following are the quivers of the m-fold mesh algebras of extended type $(\Delta, 1, 2)$ or $(\mathbb{D}_4, 1, 3)$, usually called *generalized preprojective algebras.*

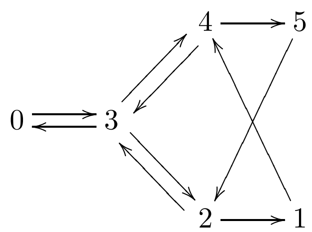
$\mathbb{B}_n$:



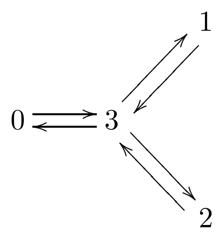
$\mathbb{C}_n$:



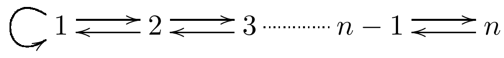
$\mathbb{F}_4$:



$\mathbb{G}_2$:



$\mathbb{L}_n$:



2.3.3 A change of presentation

For calculation purposes, it is convenient to modify the mesh relations. We want that if $(k, i) \in (\mathbb{Z}\Delta)_0$ is a vertex which is the end of exactly two arrows, then the corresponding mesh relation changes from a sum to a difference. When $\Delta = \mathbb{D}_{n+1}$ and we consider the three paths $(k, 2) \to (k, i) \to (k + 1, 2)$ $(i = 0, 1, 3)$, we want that the path going through $(k, 3)$ is the sum of the other two. Finally, when $\Delta = \mathbb{E}_n$ $(n = 6, 7, 8)$ and we consider the three paths $(k, 3) \to (k, i) \to (k + 1, 3)$ $(i = 0, 4)$ and $(k, 3) \to (k + 1, 2) \to (k + 1, 3)$, we want that the one going through $(k, 0)$ is the sum of the other two. This can be done by selecting an appropriate subset $X \subset (\mathbb{Z}\Delta)_1$ and applying the automorphism of $K\mathbb{Z}\Delta$ which fixes the vertices and all the arrows not in X and change the sign of the arrows in X. But we want the same phenomena to pass from B to $\Lambda = B/G$, for any weakly admissible group of automorphisms G of $\mathbb{Z}\Delta$. This forces us to impose the condition that X is G-invariant, i.e., that $g(X) = X$ for each $g \in G$.

Proposition 2.3.4. *Let Δ be a Dynkin quiver, $K\mathbb{Z}\Delta$ be the path algebra of $\mathbb{Z}\Delta$, let I be the ideal of $K\mathbb{Z}\Delta$ generated by the mesh relations and let $\hat{G}$ be the group of automorphisms of $\mathbb{Z}\Delta$ generated by ρ and τ, whenever ρ exists, and just by τ otherwise. Let $X \subset (\mathbb{Z}\Delta)_1$ be the set of arrows constructed as follows:*

1. *If $\Delta \neq \mathbb{A}_{2n-1}, \mathbb{D}_4$ and X' is the set of arrows given in the following list, then X is the union of the $\hat{G}$-orbits of elements of X':*
   - (a) *When $\Delta = \mathbb{A}_{2n}$, $X' = \{(0, i) \to (0, i+1) : 1 \leq i \leq n-1$ and $i \not\equiv n \pmod 2\}$.*
   - (b) *When $\Delta = \mathbb{D}_{n+1}$, with $n > 3$, $X' = \{(0, i) \to (0, i+1) : 2 \leq i \leq n-2$ and $i \equiv 0 \pmod 2\}$.*
   - (c) *When $\Delta = \mathbb{E}_6$, $X' = \{(0, 2) \to (0, 3)\}$.*
   - (d) *When $\Delta = \mathbb{E}_n$ $(n = 7, 8)$, $X' = \{(0, 2) \to (0, 3),\ (0, 4) \to (1, 3),\ (0, 6) \to (1, 5)\}$.*
2. *If $\Delta = \mathbb{D}_4$ and $G =< \tau^m >$, then X is the union of the $< \tau >$-orbits of the arrows $(0, 2) \to (0, 3)$*
3. *If $\Delta = \mathbb{A}_{2n-1}$ and we denote by $< - >$ the 'subgroup generated by', then:*
   - (a) *When $G =< \tau^m >$, X is the union of the $< \tau >$-orbits of arrows in the set $X' = \{(0, i) \to (0, i+1) : 1 \leq i \leq 2n-3$ and $i \not\equiv 0 \pmod 2\}$.*
   - (b) *When $G =< \rho\tau^m >$, with m odd, X is the union of all $< \rho\tau >$-orbits of arrows in the set $X' = \{(0, i) \to (0, i+1) : 1 \leq i \leq n-1\}$.*
   - (c) *When $G =< \rho\tau^m >$, with m even, X is the union of the $< \rho, \tau^2 >$-orbits of arrows in the set $X'_1 = \{(0, i) \to (0, i+1) : 1 \leq i \leq n-2\}$ and the G-orbits of arrows in the set $X'_2 = \{(2r, i) \to (2r, i+1) : 0 \leq 2r < m$ and $i = n-1, n\}$.*

When $\Delta \neq \mathbb{A}_{2n-1}, \mathbb{D}_4$, the given set X is G-invariant, for all choices of the weakly admissible group of automorphisms G. When $\Delta = \mathbb{A}_{2n-1}$, X is G-invariant for the respective group G.

Moreover, let $s : X \longrightarrow \mathbb{Z}_2$ be the signature map*, where $s(a) = 1$ exactly when $a \in X$, and let $\varphi : K\mathbb{Z}\Delta \longrightarrow K\mathbb{Z}\Delta$ be the unique graded algebra automorphism which fixes the vertices and maps $a \rightsquigarrow (-1)^{s(a)}a$, for each $a \in (\mathbb{Z}\Delta)_1$. Then $\varphi(I)$ is the ideal of $K\mathbb{Z}\Delta$ generated by the relations mentioned in the paragraph preceding this proposition.*

Proof. The G-invariance of X is clear. In order to prove that $\varphi(I)$ is as indicated, note that the mesh relation $\sum_{t(a)=(k,i)} \sigma(a)a$ is mapped onto $\sum_{t(a)=(k,i)} (-1)^{s(\sigma(a)a)}\sigma(a)a$, with the signature $s(p)$ of a path defined as the sum of the signature of its arrows. The result will follow from the verification of the following facts, which are routine:

i) If (k, i) is the terminus of exactly two arrows a and b, then the set $X \cap \{a, b, \sigma(a), \sigma(b)\}$ has only one element.

ii) When $\Delta = \mathbb{D}_{n+1}$, with $n > 3$, and $a : (k-1, 3) \to (k, 2)$, $b : (k-1, 0) \to (k, 2)$ and $c : (k-1, 1) \to (k, 2)$ are the three arrows ending at $(k, 2)$, then $X \cap \{a, b, c, \sigma(a), \sigma(b), \sigma(c)\} = \{\sigma(a)\}$

iii) When $\Delta = \mathbb{E}_n$ $(n = 6, 7, 8)$ and $a : (k, 2) \to (k, 3)$, $b : (k-1, 0) \to (k, 3)$ and $c : (k-1, 4) \to (k, 3)$ are the three arrows ending at $(k, 2)$, then $s(\sigma(b)b) \neq 1 = s(\sigma(a)a) = s(\sigma(c)c)$.

□

Corollary 2.3.5. *With the terminology of the previous proposition, the mesh algebra is isomorphic as a graded algebra to $B' := K\mathbb{Z}\Delta/\varphi(I)$ and, in each case, the ideal $\varphi(I)$ is G-invariant. In particular, G may be viewed as group of graded automorphisms of B' and φ induces an isomorphism $B/G \xrightarrow{\cong} B'/G$.*

Proof. Since φ is a graded automorphism of the path algebra $K\mathbb{Z}\Delta$ it induces an isomorphism $B = K\mathbb{Z}\Delta/I \xrightarrow{\cong} K\mathbb{Z}\Delta/\varphi(I) = B'$. If we view G as a group of graded automorphisms of $K\mathbb{Z}\Delta$, then the fact that X is G-invariant implies that $\varphi \circ g = g \circ \varphi$, for each $g \in G$. From this remark the rest of the Corollary is clear. □

Remark 2.3.6. When $\Delta = \mathbb{D}_4$ and $G =< \rho\tau^m >$, one cannot find a G-invariant set of arrows X as in the above proposition guaranteeing that, each $k \in \mathbb{Z}$, the path $(k-1, 2) \to (k-1, 3) \to (k, 2)$ is the sum of the other two paths from $(k-1, 2)$ to $(k, 2)$. This is the reason for the following convention.

Convention 2.3.7. From now on in this dissertation, the term 'mesh algebra' will denote the algebra $K\mathbb{Z}\Delta/\varphi(I)$ given by Corollary 2.3.5, or just $K\mathbb{Z}\mathbb{D}_4/I$ in case $(\Delta, G) = (\mathbb{D}_4, <\rho\tau^m >)$. This 'new' mesh algebra will be still denoted by B.

2.4 The Nakayama automorphism

In this section we focus our attention on the Nakayama automorphism of the mesh algebra B. This is given by the only automorphism η of B satisfying that $(a, b) = (b, \eta(a))$, for all $a, b \in B$, where $(-, -)$ denotes the graded Nakayama form associated to B.

First notice that the quiver $\mathbb{Z}\Delta$ does not have double arrows and, hence, if $a : x \to y$ is an arrow, then there exists exactly one arrow $\nu(x) \to \nu(y)$, where ν is the Nakayama permutation. This allows us to extend ν to an automorphism of the translation quiver $\mathbb{Z}\Delta$ and, hence, also to an automorphism of the path algebra $K\mathbb{Z}\Delta$. Moreover, due to the (new) mesh relations (see Proposition 2.3.4 and the paragraph preceding it), we easily see that if I' is the ideal of $K\mathbb{Z}\Delta$ generated by those mesh relations, then $\nu(I') = I'$. Note also from Proposition 2.3.1 that, as an automorphism of the quiver $\mathbb{Z}\Delta$, we have that $\nu = \tau^k$ or $\nu = \rho\tau^k$, for a suitable natural number k. It follows that if G is any weakly admissible automorphism of $\mathbb{Z}\Delta$, then $\nu \circ g = g \circ \nu$ for all $g \in G$. All these comments prove:

Lemma 2.4.1. *Let Δ be a Dykin quiver, B be its associated mesh algebra and G be a weakly admissible automorphism of $\mathbb{Z}\Delta$. The Nakayama permutation ν extends to a graded automorphism $\nu : B \longrightarrow B$ such that $\nu \circ g = g \circ \nu$, for all $g \in G$.*

The following result, which provides an explicit formula for a G-invariant graded Nakayama automorphism of any mesh algebra, is fundamental for us.

Theorem 2.4.2. *Let Δ be a Dynkin quiver with the labeling of vertices and the orientation of the arrows of Subsection 2.3.1, and let $G =< \varphi >$ be a weakly admissible automorphism of $\mathbb{Z}\Delta$. If η is the graded automorphism of B which acts as the Nakayama permutation on the vertices and acts on the arrows as indicated in the following list, then η is a Nakayama automorphism of B such that $\eta \circ g = g \circ \eta$, for all $g \in G$.*

1. *When $\Delta = \mathbb{A}_n$ and φ is arbitrary, $\eta(\alpha) = \nu(\alpha)$ for all $\alpha \in (\mathbb{Z}\Delta)_1$*
2. *When $\Delta = \mathbb{D}_{n+1}$:*
   (a) *If $n+1 \geq 4$ and $\varphi = \tau^m$ then:*
      i. *$\eta(\alpha) = -\nu(\alpha)$, whenever $\alpha : (k,i) \longrightarrow (k,i+1)$ is an upward arrow with $i \in \{2,...,n-1\}$.*
      ii. *$\eta(\alpha) = \nu(\alpha)$, whenever $\alpha : (k,i) \longrightarrow (k+1,i-1)$ is downward arrow with $i \in \{3,...,n\}$.*
      iii. *$\eta(\varepsilon_i) = (-1)^i \nu(\varepsilon_i)$, for the arrow $\varepsilon_i : (k,2) \longrightarrow (k,i)$ $(i = 0,1)$,*
      iv. *$\eta(\varepsilon'_i) = (-1)^{i+1} \nu(\varepsilon'_i)$, for the arrow $\varepsilon'_i : (k,i) \longrightarrow (k+1,2)$ $(i = 0,1)$.*

   (b) *If $n+1 > 4$ and $\varphi = \rho\tau^m$ then:*
      i. *$\eta(\alpha) = -\nu(\alpha)$, whenever α is an upward arrow as above or $\alpha : (k,i) \longrightarrow (k+1,i-1)$ is downward arrow as above such that $k \equiv -1 \pmod m$.*
      ii. *$\eta(\alpha) = \nu(\alpha)$, whenever $\alpha : (k,i) \longrightarrow (k+1,i-1)$ is downward arrow such that $k \not\equiv -1 \pmod m$*
      iii. *For the remaining arrows, if q and r are the quotient and remainder of dividing k by m, then*
      $\eta(\varepsilon_i) = (-1)^{q+i} \nu(\varepsilon_i)$ $(i = 0,1)$.
      $\eta(\varepsilon'_i) = (-1)^{q+i+1} \nu(\varepsilon'_i)$, when $r \neq m-1$, and $\eta(\varepsilon'_i) = (-1)^{q+i} \nu(\varepsilon'_i)$ otherwise

   (c) *If $n+1 = 4$ and $\varphi = \rho\tau^m$ (see the convention 2.3.7), then:*
      i. *$\eta(\varepsilon_i) = \nu(\varepsilon_i)$, whenever $\varepsilon_i : (k,2) \to (k,i)$ $(i = 0,1,3)$*
      ii. *$\eta(\varepsilon'_i) = -\nu(\varepsilon'_i)$, whenever $\varepsilon'_i : (k,i) \to (k+1,2)$ $(i = 0,1,3)$.*
3. *When $\Delta = \mathbb{E}_6$:*
   (a) *If $\varphi = \tau^m$ then:*
      i. *$\eta(\alpha) = \nu(\alpha)$ and $\eta(\alpha') = -\nu(\alpha')$, where $\alpha : (k,1) \to (k,2)$ and $\alpha' : (k,2) \to (k+1,1)$.*

ii. $\eta(\beta)=\nu(\beta)$ *and* $\eta(\beta^{'})=-\nu(\beta^{'})$, *where* $\beta:(k,2)\to(k,3)$ *and* $\beta^{'}:(k,3)\to(k+1,2)$.

iii. $\eta(\gamma)=\nu(\gamma)$ *and* $\eta(\gamma^{'})=-\nu(\gamma^{'})$, *where* $\gamma:(k,3)\to(k,4)$ *and* $\gamma^{'}:(k,4)\to(k+1,3)$.

iv. $\eta(\delta)=-\nu(\delta)$ *and* $\eta(\delta^{'})=\nu(\delta^{'})$, *where* $\delta:(k,4)\to(k,5)$ *and* $\delta^{'}:(k,5)\to(k+1,4)$.

v. $\eta(\varepsilon)=-\nu(\varepsilon)$ *and* $\eta(\varepsilon^{'})=\nu(\varepsilon^{'})$, *where* $\varepsilon:(k,3)\to(k,0)$ *and* $\varepsilon^{'}:(k,0)\to(k+1,3)$.

(b) *If* $\varphi=\rho\tau^m$, (k,i) *is the origin of the given arrow,* q *and* r *are the quotient and remainder of dividing* k *by* m, *then:*

i. $\eta(\alpha)=\nu(\alpha)$.

ii. $\eta(\alpha^{'})=-\nu(\alpha^{'})$.

iii. $\eta(\beta)=(-1)^q\nu(\beta)$

iv. $\eta(\beta^{'})=(-1)^{q+1}\nu(\beta^{'})$

v. $\eta(\gamma)=(-1)^q\nu(\gamma)$

vi. $\eta(\gamma^{'})=\nu(\gamma^{'})$, *when either* q *is odd and* $r\neq m-1$ *or* q *is even and* $r=m-1$, *and* $\eta(\gamma^{'})=-\nu(\gamma^{'})$ *otherwise.*

vii. $\eta(\delta)=-\nu(\delta)$

viii. $\eta(\delta^{'})=\nu(\delta^{'})$.

ix. $\eta(\varepsilon)=-\nu(\varepsilon)$

x. $\eta(\varepsilon^{'})=-\nu(\varepsilon^{'})$, *when* $r=m-1$, *and* $\eta(\varepsilon^{'})=\nu(\varepsilon^{'})$ *otherwise.*

4. *When* $\Delta=\mathbb{E}_7$, $\varphi=\tau^m$, *and then:*

i $\eta(a)$ *is given as in* 3.(a) *for any arrow* a *contained in the copy of* $\mathbb{E}_6$.

ii $\eta(\zeta)=\nu(\zeta)$ *and* $\eta(\zeta^{'})=-\nu(\zeta^{'})$, *where* $\zeta:(k,5)\to(k,6)$ *and* $\zeta^{'}:(k,6)\to(k+1,5)$.

5. *When* $\Delta=\mathbb{E}_8$, $\varphi=\tau^m$, *and then:*

i $\eta(a)$ *is given as in* 4 *for any arrow* a *contained in the copy of* $\mathbb{E}_7$.

ii $\eta(\theta)=\nu(\theta)$ *and* $\eta(\theta^{'})=-\nu(\theta^{'})$, *where* $\theta:(k,6)\to(k,7)$ *and* $\theta^{'}:(k,7)\to(k+1,6)$.

Proof. Let ν be the Nakayama permutation of the $\mathbb{Z}\Delta$ (see Proposition 2.3.1). By Corollary 2.3.2, we know that $Soc_{gr}(e_{(k,i)}B)=\text{Soc}(e_{(k,i)}B)$ is one-dimensional and concentrated in degree $l=c_\Delta-2$, for each $(k,i)\in\mathbb{Z}\Delta_0$. By applying Corollary 1.4.4, after taking a nonzero element $w_{(k,i)}\in e_{(k,i)}Soc_{gr}(B)$, for each $(k,i)\in(\mathbb{Z}\Delta)_0$, we can take the graded Nakayama form $(-,-):B\times B\longrightarrow K$ of degree l associated to $\mathcal{B}=(\mathcal{B}_{(k,i)})_{(k,i)\in\mathbb{Z}\Delta_0}$ (see definition 8), where $\mathcal{B}_{(k,i)}=\{w_{(k,i)}\}$ is a basis of $e_{(k,i)}B_le_{\nu(k,i)}$, for each $(k,i)\in\mathbb{Z}\Delta_0$. It is clear that the so obtained graded Nakayama form will be G-invariant whenever $\mathcal{B}=\bigcup_{(k,i)\in\mathbb{Z}\Delta_0}\mathcal{B}_{(k,i)}$ is G-invariant. Moreover, in such case the associated Nakayama automorphism η will satisfy that $\eta\circ g=g\circ\eta$, for all $g\in G$ (see Corollary 1.4.5). The canonical way of constructing such a G-invariant basis $\mathcal{B}$ is given in the proof of Corollary 1.4.4. Namely, we select a set I' of representatives of the G-orbits of vertices and a element $0 \neq w_{(k,i)} \in e_{(k,i)}Soc_{gr}(B)$, for each $(k,i) \in I'$. Then $\mathcal{B} = \{g(w_{(k,i)}) : g \in G, (k,i) \in I'\}$ is a G-invariant basis as desired. However, note that if we choose $\mathcal{B}$ to be τ-invariant, then it is G-invariant for $G =< \tau^m >$. So, in order to construct $\mathcal{B}$, we will only need to consider the cases $\varphi = \tau$ and $\varphi = \rho\tau^m$

To construct $\mathcal{B}$ when $\Delta = \mathbb{A}_n$ has no problem, for all paths of length $l = c_\Delta - 2$ from (k,i) to $\nu(k,i)$ are equal in B. So in this case the choice of $w_{(k,i)}$ will be the element of B represented by a path from (k,i) to $\nu(k,i)$ and $\mathcal{B} = \{w_{(k,i)} : (k,i) \in (\mathbb{Z}\Delta)_0\}$ is G-invariant for any choice of φ. So, on what concerns the calculation of $\mathcal{B}$, we assume in the sequel that Δ is either $\mathbb{D}_{n+1}$ or $\mathbb{E}_r$ $(r = 6, 7, 8)$. For these cases, if $\varphi = \tau$ we will take $I' = S$, where $S := \{(0,i) : i \in \Delta_0\}$ is the canonical slice. The desired elements $w_{(0,i)} \in e_{(0,i)}Soc_{gr}(B)$ are the paths given below. If $\varphi = \rho\tau^m$ and $\Delta = \mathbb{D}_{n+1}$, with $n > 3$, we will take $I' = \{(k,i) : i \in \Delta_0 \text{ and } 0 \leq k < m\}$ and we will put $w_{(k,i)} = \tau^{-k}(w_{(0,i)})$, for each $(k,i) \in I'$. On the other hand, if $\varphi = \rho\tau^m$ and $\Delta = \mathbb{E}_6$ we will consider the slice $T = \{(0,i) : i = 0,3,4,5\} \cup \{(1,2),(2,1)\}$, which is ρ-invariant, and then take $I' = \{\tau^{-k}(r,i) : (r,i) \in T \text{ and } 0 \leq k < m\}$. The paths $w_{(0,i)}$ $(i = 0,3,4,5)$ will be as in the case $\varphi = \tau$, and we will define below the paths $w_{(1,2)}$ and $w_{(2,1)}$ below. Then we will take $w_{\tau^{-k}(r,j)} = \tau^{-k}(w_{(r,j)})$, for all $(r,j) \in T$ and $0 \leq k < m$.

When $\Delta = \mathbb{D}_4$ and $\varphi = \rho\tau^m$ (see the convention 2.3.7), we slightly divert from the previous paragraph. We take $w_{(0,0)} = \varepsilon_0'\varepsilon_1\varepsilon_1'\varepsilon_0$ and $w_{(0,2)} = \varepsilon_0\varepsilon_0'\varepsilon_1\varepsilon_1'$. Due to the fact that all nonzero paths from $(0,2)$ to $\nu(0,2) = (2,2)$ are equal, up to sign, in B we know that the action $< \rho >$ on those paths is trivial. The base $\mathcal{B}$ will be the union of the orbits of $w_{(0,0)}$ and $w_{(0,2)}$ under the action of the group of automorphisms generated by ρ and τ.

Suppose that $\Delta = \mathbb{D}_{n+1}$, with $n > 3$ in case $\varphi = \rho\tau^m$. To simplify the notation, we shall denote by u, v and w, respectively, each of the paths of length 2

$$\begin{array}{c}(r,2) \to (r,0) \to (r+1,2)\\ (r,2) \to (r,1) \to (r+1,2)\\ (r,2) \to (r,3) \to (r+1,2),\end{array}$$

with no mention to r. Then a composition of those paths $(r,2) \to (r+1,2) \to ... \to (r+i,2)$ will be denoted as a (noncommutative) monomials in the u, v, w.

We will need also to name the paths that we will use. Concretely:

1. $\gamma_{(k,i)}$ is the downward path $(k,i) \to ... \to (k+i-2,2)$, with the convention that $\gamma_{(k,2)} = e_{(k,2)}$.
2. $\delta_{(m,j)}$ is the upward path $(m,2) \to ... \to (m,j)$, with the convention that $\delta_{(m,2)} = e_{(m,2)}$.
3. $\varepsilon_{(k,j)}$ is the arrow $(k,2) \longrightarrow (k,j)$ and $\varepsilon'_{(k,j)}$ is the arrow $(k,j) \longrightarrow (k+1,2)$, for $j = 0,1$.

Our choice of the $w_{(0,i)}$ is then the following:

(a) $w_{(0,i)} = \gamma_{(0,i)}uvuv...\delta_{(n-1,i)}$ whenever $i = 2,...,n$.

(b) $w_{(0,0)} = \varepsilon'_{(0,0)} vuvu ... \varepsilon_{\nu(0,0)}$

(c) $w_{(0,1)} = \varepsilon'_{(0,1)} uvuv ... \varepsilon_{\nu(0,1)}$

(note that, for $j = 0, 1$, the vertex $\nu(0, j)$ depends on whether $n + 1$ is even or odd).

If $\Delta = \mathbb{E}_n$ with $n = 6, 7, 8$, we name the paths from $(k, 3)$ to $(k + 1, 3)$ as follows:

$$\begin{aligned} u &: (k,3) \to (k,0) \to (k+1,3) \\ v &: (k,3) \to (k,4) \to (k+1,3) \\ w &: (k,3) \to (k+1,2) \to (k+1,3). \end{aligned}$$

Then any path $(k, 3) \to ... \to (k + r, 3)$ is equal in B to a monomial in u, v, w, with the obvious sense of 'monomial'. With the abuse of notation of omitting k when showing a vertex (k, i) in the diagrams below, we then take:

1. When $\Delta = \mathbb{E}_6$

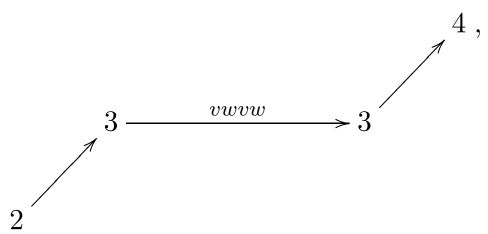
   (a) $w_{(0,3)}$ is the path

$$3 \xrightarrow{vwvwv} 3$$

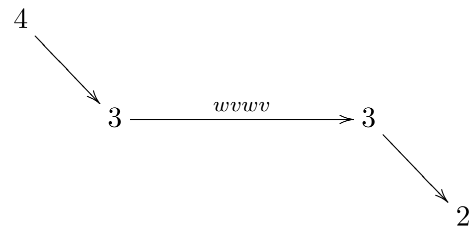
   (b) $w_{(0,0)}$ is the path

$$0 \longrightarrow 3 \xrightarrow{vwvw} 3 \longrightarrow 0$$

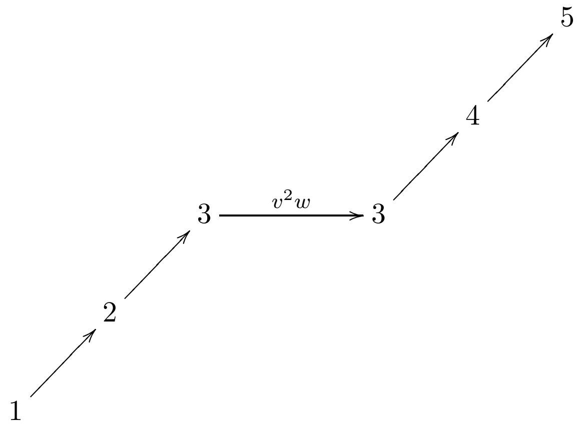
   (c) $w_{(0,2)}$, in case $\varphi = \tau$, and $w_{(1,2)}$, in case $\varphi = \rho\tau^m$, is the path



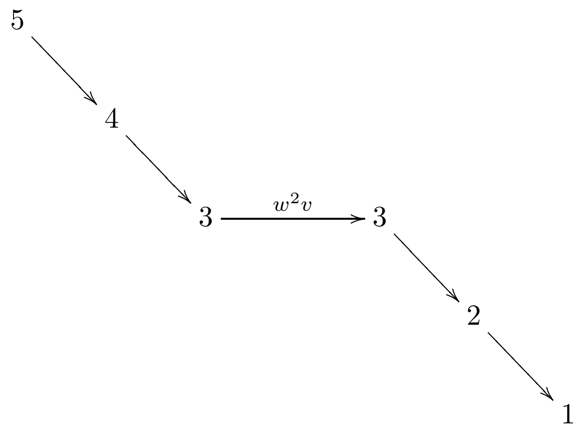
   (d) $w_{(0,4)}$ is the path



(e) $w_{(0,1)}$, in case $\varphi = \tau$, and $w_{(2,1)}$, in case $\varphi = \rho\tau^m$, is the path



(f) $w_{(0,5)}$ is the path



,

2. When $\Delta = \mathbb{E}_7$

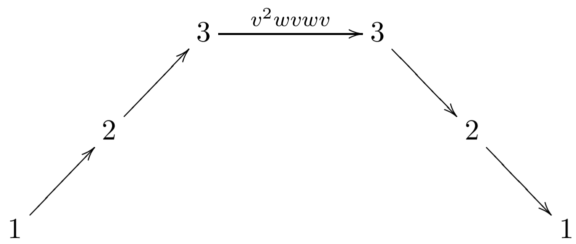
(a) $w_{(0,3)}$ is the path

$$3 \xrightarrow{vwvwvwvw} 3$$

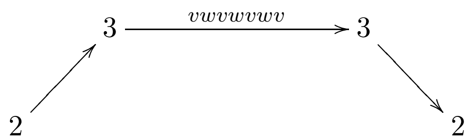
(b) $w_{(0,0)}$ is the path

$$0 \longrightarrow 3 \xrightarrow{vwvwvwv} 3 \longrightarrow 0$$

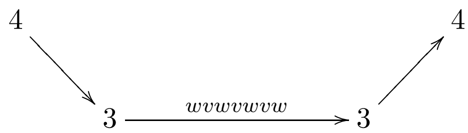
(c) $w_{(0,1)}$ is the path



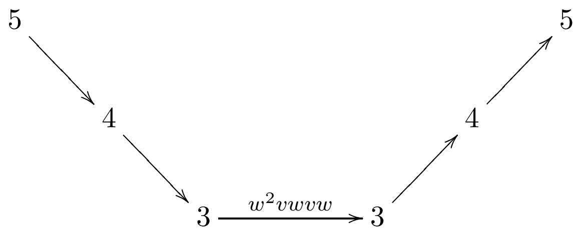
(d) $w_{(0,2)}$ is the path



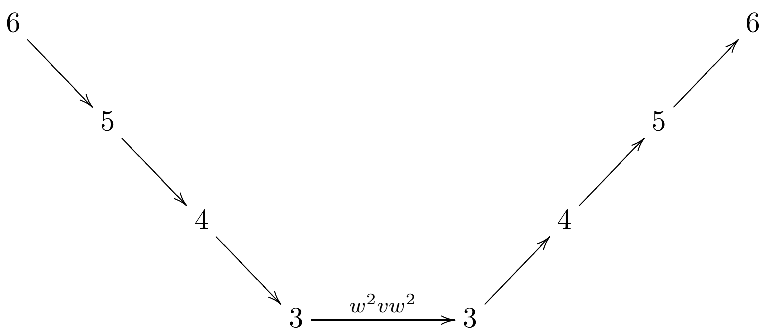
(e) $w_{(0,4)}$ is the path



(f) $w_{(0,5)}$ is the path



(g) $w_{(0,6)}$ is the path



3. When $\Delta = \mathbb{E}_8$

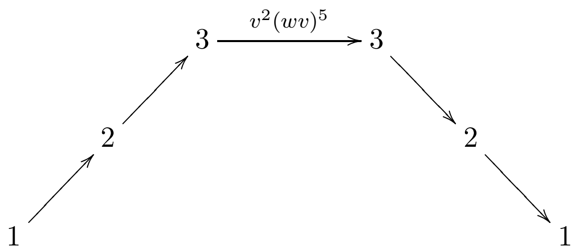
(a) $w_{(0,3)}$ is the path

$$3 \xrightarrow{(vw)^7} 3$$

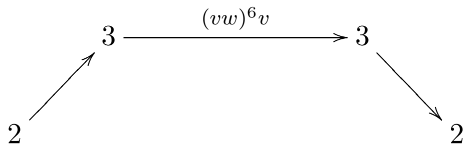
(b) $w_{(0,0)}$ is the path

$$0 \longrightarrow 3 \xrightarrow{(vw)^6 v} 3 \longrightarrow 0$$

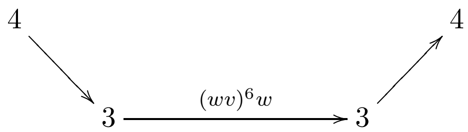
(c) $w_{(0,1)}$ is the path



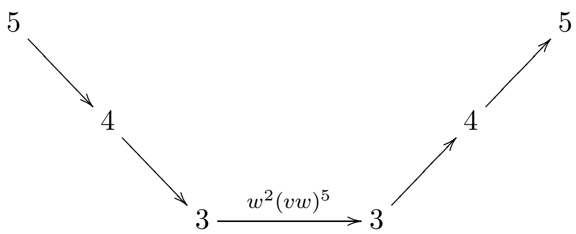
(d) $w_{(0,2)}$ is the path



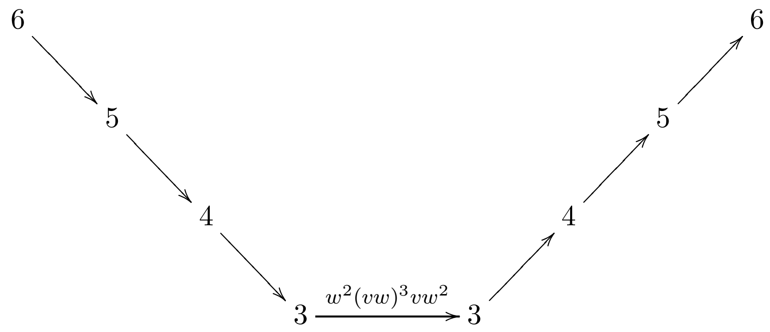
(e) $w_{(0,4)}$ is the path



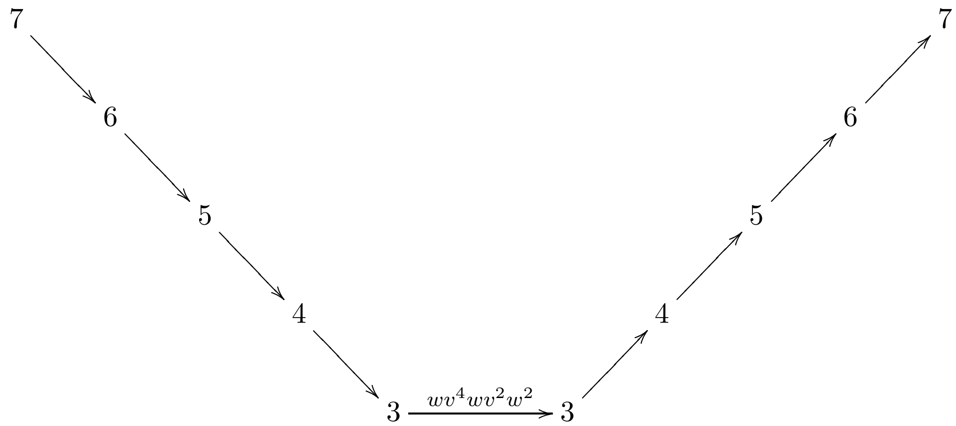
(f) $w_{(0,5)}$ is the path



(g) $w_{(0,6)}$ is the path



(h) $w_{(0,7)}$ is the path



Once the G-invariant basis $\mathcal{B}$ of $Soc_{gr}(B) = \mathrm{Soc}(B)$ has been described, the strategy to identify the action of the associated Nakayama automorphism η on the arrows is very simple. Given an arrow α, we take a path $q : t(\alpha) \to ... \to \nu(i(\alpha))$ of length $l-1$ such that αq is a nonzero path. Then we have $\alpha q = (-1)^{u(\alpha)} w_{i(\alpha)}$, so that, by definition of the graded Nakayama form associated to $\mathcal{B}$, we have an equality $(\alpha, q) = (-1)^{u(\alpha)}$. Since the quiver $\mathbb{Z}\Delta$ does not have double arrows we know that $\eta(\alpha) = \lambda(\alpha)\nu(\alpha)$, for some $\lambda(\alpha) \in K^*$. In particular we know that $q\nu(\alpha)$ is a nonzero path (of length l) because $(q, \eta(\alpha)) = (\alpha, q) \neq 0$. If we have an equality $q\nu(\alpha) = (-1)^{v(\alpha)} w_{t(\alpha)}$ in B, then it follows that $(-1)^{u(\alpha)} = (\alpha, q) = (q, \eta(\alpha)) = \lambda(\alpha)(q, \nu(\alpha)) = \lambda(\alpha)(-1)^{v(\alpha)}$. Then we get $\lambda(\alpha) = (-1)^{u(\alpha)-v(\alpha)}$ and the task is reduced to calculate the exponents $u(\alpha)$ and $v(\alpha)$ in each case. Taking into account that we have $\eta \circ g = g \circ \eta$, for each $g \in G$, it is enough to calculate $u(\alpha)$ and $v(\alpha)$ just for the arrows starting at a vertex of I'.

We pass to consider the situation for each of the three Dynkin quivers:

1) $\underline{\Delta = \mathbb{A}_n}$: This is trivial and we have $\eta(\alpha) = \nu(\alpha)$, for each $\alpha \in (\mathbb{Z}\Delta)_1$.
2) $\underline{\Delta = \mathbb{D}_{n+1}}$:
We still use $\gamma_{(k,i)}$, $\delta_{(m,j)}$, $\varepsilon_{(k,j)}$ and $\varepsilon'_{(k,j)}$ with the same meaning as above. We will use the letter α to denote un upward arrow $(k, i) \to (k, i+1)$, with $i = 2, ..., n-1$, and the letter β to denote a downward arrow $(k, i) \to (k+1, i-1)$ with $i = 3, ..., n$. We will also

consider the arrows $\varepsilon_j := \varepsilon_{(k,j)} : (k,2) \to (k,j)$ and $\varepsilon'_j := \varepsilon'_{(k,j)} : (k,j) \to (k+1,2)$, for $j = 0,1$. In all cases we consider that the origin of each arrow is a vertex of I'. We will now create a table, where, for each of these arrows a, the path p_a will be a path of length $l-1$ from $t(a)$ to $\nu(i(a))$ such that $ap_a \neq 0$ in B. Then, a routine verification shows that $u(a), v(a)$ will be elements of $\mathbb{Z}_2$ such that $ap_a = (-1)^{u(a)} w_{i(a)}$ and $p_a\nu(a) = (-1)^{v(a)} w_{t(a)}$.

a) For the cases when $\varphi = \tau^m$, it is enough to consider that $m = 1$, for if $\eta \circ \tau = \tau \circ \eta$, then $\eta \circ \tau^m = \tau^m \circ \eta$, for all $m \geq 1$. For $\varphi = \tau$:

a	p_a	$u(a)$	$v(a)$
$\alpha : (0,i) \to (0,i+1)$	$\gamma_{(0,i+1)} vuvu...\delta_{(n-1,i)}$	0	1
$\beta : (0,i) \to (1,i-1)$	$\gamma_{(1,i-1)} uvuv...\delta_{(n-1,i)}$	0	0
$\varepsilon'_0 : (0,0) \to (1,2)$	$vuvu...\varepsilon_{\nu(0,0)}$	0	1
$\varepsilon'_1 : (0,1) \to (1,2)$	$uvuv...\varepsilon_{\nu(0,1)})$	0	0
$\varepsilon_0 : (0,2) \to (1,0)$	$\varepsilon'_0 vuv...$	0	0
$\varepsilon_1 : (0,2) \to (0,1)$	$\varepsilon'_1 uvu...$	1	0

and assertion 2.a follows.

b) When $\varphi = \rho\tau^m$ and $n > 3$, for the arrows a starting and ending at a vertex of I', we take p_a as in the table above and $u(a)$ and $v(a)$ take the same values as in that table. In the corresponding table for this case, it is enough to give only the data for the arrows which start at a vertex of I' but end at one not in I':

a	p_a	$u(a)$	$v(a)$
$\beta : (m-1,i) \to (m,i-1)$	$\gamma_{(m,i-1)} uvuv...\delta_{(m+n-2,i)}$	0	1
$\varepsilon'_0 : (m-1,0) \to (m,2)$	$vuvu...\varepsilon_{\nu(m-1,0)}$	0	0
$\varepsilon'_1 : (m-1,1) \to (m,2)$	$uvuv...\varepsilon_{\nu(m-1,1)})$	0	1

These values come from the fact that $w_{(m,i)} = \rho\tau^{-m}(w_{(0,i)}) = \gamma_{(m,i)} vuvu...\delta_{(m+n-1,i)}$, for each $i = 2,...,n$. It is now clear that assertions 2.b.i and 2.b.ii hold. As for 2.b.iii, put $I'(q) = \{(k,i) : qm \leq k < (q+1)m \text{ and } i \in\in \Delta_0\}$, i.e., the set of vertices (k,i) such that the quotient of dividing k by m is q. If $\varepsilon_0 : (k,2) \longrightarrow (k,0)$ has origin (and end) in $I'(q)$, then $\rho\tau^{-m}(\varepsilon_0) = \varepsilon_1 : (k+m,1) \to (k+m,2)$. The symmetric equality is true when exchanging the roles of 0 and 1. It follows that $\eta(\varepsilon_0) = \nu(\varepsilon_0)$ (resp. $\eta(\varepsilon_1) = -\nu(\varepsilon_0)$) when the origin of ε_0 (resp. ε_1) is in $I'(q)$, with q even, and $\eta(\varepsilon_0) = -\nu(\varepsilon_0)$ (resp. $\eta(\varepsilon_1) = \nu(\varepsilon_1)$) otherwise. That is, we have $\eta(\varepsilon_i) = (-1)^{q+i}\nu(\varepsilon_i)$.

A similar argument shows that if $k \not\equiv 1 \pmod{m}$ and $\varepsilon'_j : (k,j) \to (k+1,2)$, then we have $\eta(\varepsilon'_j) = (-1)^{q+j+1}\nu(\varepsilon'_j)$. Finally, if $\varepsilon'_j : ((q+1)m-1, j) \to ((q+1)m, 2)$ we get that $\eta(\varepsilon'_j) = (-1)^{q+j}\nu(\varepsilon'_j)$, which shows that the equalities in 2.b.iii also hold.

c) Suppose now that $\Delta = \mathbb{D}_4$ and $\varphi = \rho\tau^m$, where the mesh arrows are the original ones $r_{(k,i)} = \sum_{t(a)=(k,i)} \sigma(a)a$. Note that if $\varepsilon_i : (k,2) \to (k,i)$ and $\varepsilon'_i : (k,i) \to (k+1,2)$, for $i = 0,1,3$, then we have $w_{(k,i)} = \varepsilon'_i \varepsilon_{\rho(i)} \varepsilon'_{\rho(i)} \varepsilon_i$ and $w_{(k,2)} = \varepsilon_i \varepsilon'_i \varepsilon_{\rho(i)} \varepsilon'_{\rho(i)} = -\varepsilon_{\rho(i)} \varepsilon'_{\rho(i)} \varepsilon_i \varepsilon'_i$, for all $i = 0,1,3$. The corresponding table is then given as

a	p_a	$u(a)$	$v(a)$
ε'_i	$\varepsilon_{\rho(i)} \varepsilon'_{\rho(i)} \varepsilon_i$	0	1
ε_i	$\varepsilon'_i \varepsilon_{\rho(i)} \varepsilon'_{\rho(i)}$	0	0

3) $\underline{\Delta = \mathbb{E}_n}$ $(n = 6, 7, 8)$:

For the sake of simplicity, we will write any path as a composition of arrows in $\{\alpha, \alpha^{'}, \beta, \beta^{'}, \gamma, \gamma^{'}, \delta, \delta^{'}, \zeta, \zeta', \theta, \theta'\varepsilon, \varepsilon^{'}\}$ whenever they exist and assuming that each arrow is considered in the appropriate slice so that the composition makes sense.

Also, we denote by u, w and v, respectively, each of the paths of length 2

$$(r,3) \longrightarrow (r,0) \longrightarrow (r+1,3)$$
$$(r,3) \longrightarrow (r+1,2) \longrightarrow (r+1,3)$$
$$(r,3) \longrightarrow (r,4) \longrightarrow (r+1,3)$$

with no mention to r. Then $\beta^{'}\beta = w$, $\gamma\gamma^{'} = v$ and $\varepsilon\varepsilon^{'} = u$. It is important to keep in mind that $u = v + w$. Also notice that, as with $\mathbb{D}_{n+1}$, for the case when $\varphi = \tau^m$ it is not restrictive to assume that $m = 1$. Then $I' = \{(0,i) : i \in \Delta_0\}$.

1. If $\Delta = \mathbb{E}_6$, using the mesh relations, one gets, among others, the equalities $u^2 = w^3 = v^3 = 0$, $vwv = wvw$, $vw^2v = -vwv^2 - v^2wv$ and $vwvwv = -wvwvw$.

   Then, if $\varphi = \tau^m$, the table is the following:

a	p_a	$u(a)$	$v(a)$
$\alpha : (0,1) \to (0,2)$	$\beta v^2 w\gamma\delta$	0	0
$\beta : (0,2) \to (0,3)$	$vwvw\gamma$	0	0
$\gamma : (0,3) \to (0,4)$	$\gamma' wvwv$	0	0
$\delta : (0,4) \to (0,5)$	$\delta'\gamma' w^2 v\beta'$	1	0
$\varepsilon : (0,3) \to (0,0)$	$\varepsilon' vwvw$	1	0
$\alpha' : (0,2) \to (1,1)$	$\alpha\beta v^2 w\gamma$	1	0
$\beta' : (0,3) \to (1,2)$	$\beta vwvw$	1	0
$\gamma' : (0,4) \to (1,3)$	$wvwv\beta'$	0	1
$\delta' : (0,5) \to (1,4)$	$\gamma' w^2 v\beta'\alpha'$	0	0
$\varepsilon' : (0,0) \to (1,3)$	$vwvw\varepsilon$	0	0

   From this table the equalities in 3.a follow.

   Suppose now that $\varphi = \rho\tau^m$ and recall that in this case we take $I' = \{\tau^{-k}(r,i) = (k+r, i) : (r,i) \in T \text{ and } 0 \leq k < m\}$, where $T = \{(0,i) : i = 0,3,4,5\} \cup \{(1,2),(2,1)\}$. Arguing as in the case of $\mathbb{D}_{n+1}$, we see that the values $u(a)$ and $v(a)$ are the ones in the last table, when $i(a), t(a) \in I'$. We then need only to give those values for the arrows a with origin in I' and terminus not in I'. We have the table:

a	p_a	$u(a)$	$v(a)$
$\alpha : (m+1,1) \to (m+1,2)$	$\beta v^2 w\gamma\delta$	0	0
$\beta : (m,2) \to (m,3)$	$vwvw\gamma$	0	1
$\gamma' : (m-1,4) \to (m,3)$	$wvwv\beta'$	0	0
$\delta' : (m-1,5) \to (m,4)$	$\gamma' w^2 v\beta'\alpha'$	0	0
$\varepsilon' : (m-1,0) \to (m,3)$	$vwvw\varepsilon$	0	1

We have used in the construction of this table the fact that $w_{(k,2)} = \beta vwvw\gamma$ and $w_{(k,4)} = \gamma' wvwv\beta'$, for all $k \in \mathbb{Z}$, while $w_{(2r,3)} = vwvwv$ and $w_{(2r+1,3)} = wvwvw$.

Note that, with the labeling of vertices that we are using, we have that $\rho(k,i) = (k+i-3, 6-i)$ for each $i \neq 0$ and $\rho(k,0) = (k,0)$. For each $q \in \mathbb{Z}$, we put $I'(q) := (\rho\tau^{-m})^q(I')$. When passing from a piece $I'(q)$ to $I'(q+1)$ by applying $\rho\tau^{-m}$, an arrow α is transformed in an arrow δ' and an arrow δ' in an arrow α. From the last two tables we then get that $\eta(\alpha) = \nu(\alpha)$ and $\eta(\delta') = \nu(\delta')$, for all arrows of the type α or δ' in $\mathbb{Z}\Delta$.

The argument of the previous paragraph can be applied to the pair of arrows (γ, β') instead of (α, δ') and we get from the last two tables that $\eta(\gamma) = \nu(\gamma)$ (resp. $\eta(\beta') = -\nu(\beta')$) when γ (resp. β') has its origin in $I'(q)$, with q even, and $\eta(\gamma) = -\nu(\gamma)$ (resp. $\eta(\beta') = \nu(\beta')$) otherwise. From this the formulas in 3.b concerning γ and β' are clear.

We apply the argument next to the pair or arrows (δ, α') and get that $\eta(\delta) = -\nu(\delta)$ (resp. $\eta(\alpha') = -\nu(\alpha')$), for all arrows of type δ or α' in $\mathbb{Z}\Delta$.

An arrow of type ε (resp. ε') is transformed on one of the same type when applying $\rho\tau^{-m}$. It then follows that $\eta(\varepsilon) = -\nu(\varepsilon)$, for any arrow of type ε. It also follows that $\eta(\varepsilon') = -\nu(\varepsilon')$, when the origin of ε' is $(k,0)$ with $k \equiv -1 \pmod{m}$, and $\eta(\varepsilon') = \nu(\varepsilon')$ otherwise.

We finally apply the argument to the pair of arrows (β, γ'). If we look at the two pieces $I'(0)$ and $I'(1)$, then from the last two tables we see that if $\beta : (k,2) \to (k,3)$, with $(k,3) \in I'(0) \cup I'(1)$, then $\eta(\beta) = \nu(\beta)$, when $k \in \{1, 2, ..., m-1, 2m\}$, and $\eta(\beta) = -\nu(\beta)$, when $k \in \{m, m+1, ..., 2m-1\}$. We then get that $\eta(\beta) = (-1)^q\nu(\beta)$, where q is the quotient of dividing k by m. By doing the same with $\gamma' : (k,4) \to (k+1,3)$, we see that $\eta(\gamma') = -\nu(\gamma')$, when $k \in \{0, 1, ..., m-2, 2m-1\}$, and $\eta(\gamma') = \nu(\gamma')$, when $k \in \{m-1, m, ..., 2m-2\}$. If now $k \in \mathbb{Z}$ is arbitrary, then that $\eta(\gamma') = \nu(\gamma')$ if, and only if, $k \notin \bigcup_{t \in \mathbb{Z}} (2tm-2, (2t+1)m-1)$. Equivalently, when q is odd and $r \neq m-1$ or q is even and $r = m-1$.

2. If $\Delta = \mathbb{E}_7$, then we have, among others, the equalities $u^2 = w^3 = v^4 = 0$, $vwv = wvw - v^3$, and $vwvwv = -wvwvw$. Since $\varphi = \tau^m$, we get the following table:

a	p_a	$u(a)$	$v(a)$
$\alpha : (0,1) \to (0,2)$	$\beta v^2 wvwv\beta'\alpha'$	0	0
$\beta : (0,2) \to (0,3)$	$vwvwvwv\beta'$	0	0
$\gamma : (0,3) \to (0,4)$	$\gamma' wvwvwvw$	0	0
$\delta : (0,4) \to (0,5)$	$\delta'\gamma' wv^2 wvw\gamma$	0	1
$\zeta : (0,5) \to (0,6)$	$\zeta'\delta'\gamma' wv^3 w\gamma\delta$	0	0
$\varepsilon : (0,3) \to (0,0)$	$\varepsilon' wvwvwvw$	0	1
$\alpha' : (0,2) \to (1,1)$	$\alpha\beta vwvwv^2\beta'$	0	1
$\beta' : (0,3) \to (1,2)$	$\beta vwvwvwv$	1	0
$\gamma' : (0,4) \to (1,3)$	$wvwvwvw\gamma$	0	1
$\delta' : (0,5) \to (1,4)$	$\gamma' w^2 vwvw\gamma\delta$	0	0
$\zeta' : (0,6) \to (1,5)$	$\delta'\gamma' w^2 vw^2\gamma\delta\zeta$	0	1
$\varepsilon' : (0,0) \to (1,3)$	$vwvwvwv\varepsilon$	0	0

From this table the equalities in 4 follow.

3. If $\Delta = \mathbb{E}_8$, as in the previous case, $\varphi = \tau^m$ and, considering the equalities $u^2 = w^3 = v^5 = 0$, $vwv = wvw - v^3$, $(vw)^3 = (wv)^3 + vwv^4 - v^4wv$, $(vw)^6 = (wv)^6 + (wv)^3 vwv^4 - v^4 wv^2 wv^4$, and $(vw)^7 = -(wv)^7$, we obtain the table below:

a	p_a	$u(a)$	$v(a)$
$\alpha : (0,1) \to (0,2)$	$\beta v^2 (wv)^5 \beta'\alpha'$	0	0
$\beta : (0,2) \to (0,3)$	$(vw)^6 v\beta'$	0	0
$\gamma : (0,3) \to (0,4)$	$\gamma' w(vw)^6$	0	0
$\delta : (0,4) \to (0,5)$	$\delta'\gamma' wv^2 (wv)^4 w\gamma$	0	1
$\zeta : (0,5) \to (0,6)$	$\zeta'\delta'\gamma' wv^3 (wv)^3 w\gamma\delta$	0	0
$\theta : (0,6) \to (0,7)$	$\theta'\zeta'\delta'\gamma' wv^4 wv^2 w^2 \gamma\delta\zeta$	0	0
$\varepsilon : (0,3) \to (0,0)$	$\varepsilon' w(vw)^6$	0	1
$\alpha' : (0,2) \to (1,1)$	$\alpha\beta(vw)^5 v^2\beta'$	0	1
$\beta' : (0,3) \to (1,2)$	$\beta(vw)^6 v$	1	0
$\gamma' : (0,4) \to (1,3)$	$(wv)^6 w\gamma$	0	1
$\delta' : (0,5) \to (1,4)$	$\gamma' w^2 (vw)^5 \gamma\delta$	0	0
$\zeta' : (0,6) \to (1,5)$	$\delta'\gamma' w^2 (vw)^4 w\gamma\delta\zeta$	0	1
$\theta' : (0,7) \to (1,6)$	$\zeta'\delta'\gamma' wv^4 wv^2 w^2 \gamma\delta\zeta\theta$	0	1
$\varepsilon' : (0,0) \to (1,3)$	$(vw)^6 v\varepsilon$	0	0

From this table the equalities in 5 follow.

□

Remark 2.4.3. When $\Delta = \mathbb{E}_6$ and $\varphi = \rho\tau$, then $q = k$ and $r = 0$ in 3.b of the last proposition. The explicit definition of $\eta(\gamma')$ should be clarified. A follow-up of our arguments shows that $\eta(\gamma') = (-1)^k \nu(\gamma')$ in that case.

Chapter 3

m-fold mesh algebras

3.1 Introduction

3.1.1 Motivation

The study of the Calabi-Yau condition on a finite dimensional self-injective algebra has become very popular in recent years (see e.g. [28], [12], [35], [25], [50], [51], ...). So far, concerning the class of m-fold mesh algebras, the problem of determining those which are Calabi-Yau has been solved only for the class of the stable Auslander algebras of a representation-finite self-injective algebra (see [25] and [51]). Related to the Calabi-Yau property, Eu and Schedler defined the notion of Frobenius Calabi-Yau algebra ([35]). It is well known that any Frobenius Calabi-Yau algebra is always Calabi-Yau and, moreover, the Calabi-Yau dimension is always less or equal than the Calabi-Yau Frobenius dimension. However, it is not known, in general, if the equality holds.

On the other hand, although there are many examples of periodic algebras in the literature, the explicit computation of the period turns out to be, in most of the cases, a very hard task. Such was the case that, regarding the class of m-fold algebras, the period has only been calculated for some of the stable Auslander algebras of a standard representation-finite self-injective algebra.

Another homological property that is worth studying when dealing with finite dimensional self-injective algebras is that of being symmetric or weakly symmetric.

The goal of this chapter is to give, for the class of m-fold mesh algebras, an answer to the previous questions. Concretely, we provide:

1. An identification of all weakly symmetric and symmetric algebras in the class (Theorem 3.3.1);

2. An explicit formula for the period of any algebra in the class (Proposition 3.4.8, when $\Delta = \mathbb{A}_2$, and Theorem 3.4.12 for all the other cases).

3. An identification of the precise relation between the stable Calabi-Yau dimension and the Calabi-Yau Frobenius dimension of an m-fold algebra, showing that both dimensions may differ when $\Delta = \mathbb{A}_2$, but are always equal when $\Delta \neq \mathbb{A}_r$, for $r = 1, 2$ (Propositions 3.4.13 and 3.4.14)

4. A criterion for an m-fold mesh algebra to be stably Calabi-Yau, together with the identification in such case of the stable Calabi-Yau dimension (Proposition 3.4.7, for the case $\Delta = \mathbb{A}_2$, Corollary 3.4.18, for characteristic 2, and Theorem 3.4.19 for all other cases).

3.1.2 Outline of the chapter

In Section 3.2 we include two auxiliary results that will be frequently used in the rest of the chapter. In Section 3.3 we identify all the symmetric and the weakly symmetric m-fold mesh algebras. We start Section 3.4 by computing the initial part of a G-invariant minimal projective resolution of B as B-bimodule from which we can deduce, in particular, that the third syzygy of Λ as Λ-bimodule, $\Omega^3_{\Lambda^e}(\Lambda)$, is isomorphic to a twisted bimodule of Λ, ${}_{\bar{\mu}}\Lambda_1$, induced by a an automorphism $\bar{\mu}$ whose formula we determine precisely. The role played by the third syzygy turns out to be essential for the two major purposes of the last section. Next we introduce the definition of stably inner automorphism and we proof that, in most of the cases, the notions of inner and stably inner coincide. We then compute the period of any m-fold mesh algebra. We shall distinguish the algebras having Loewy length 2 from the rest. Finally, we deal with the question on Calabi-Yau dimensions. As in the previous question, we first consider the algebras having Loewy length 2, which is actually the only case where the stable and Frobenius Calabi-Yau dimensions do not coincide. Otherwise, we show that both notions are equivalent, and moreover, their respective Calabi-Yau dimensions are equal. We end the chapter by computing the stable Calabi-Yau dimension, and hence, the Frobenius Calabi-Yau dimension, of any m-fold mesh algebra with Loewy length different from 2.

3.1.3 Notation

Besides the notation fixed in the previous chapters, we still need to establish the following.

Given an algebra A and an automorphism $\sigma \in \text{Aut}(A)$, it is well known that each A-module M admits a twisted version ${}_{\sigma}M$, where the underlying R-module is M and the multiplication by elements of A is given by $a \cdot m = \sigma(a)m$, for all $a \in A$ and $m \in M$. It is also well-known that the assignment $M \mapsto {}_{\sigma}M$ defines an equivalence of categories acting as the identity on morphisms ${}_A\text{Mod} \xrightarrow{\cong} {}_A\text{Mod}$ with quasi-inverse taking M to ${}_{\sigma^{-1}}M$.

Suppose now that $\sigma, \tau \in \text{Aut}(A)$. Then we get an automorphism of the enveloping algebra, $\sigma \otimes \tau^o : A \otimes \Lambda^{op} \longrightarrow A \otimes A^{op}$, which takes $a \otimes b^o$ to $\sigma(a) \otimes \tau(b)^o$. If M is a A-bimodule, which we view as a left A^e-module, the previous paragraph gives a new left A^e-module ${}_{\sigma \otimes \tau^o}M$. In the usual way, we interpret it as a A-bimodule ${}_{\sigma}M_{\tau}$, and then the multiplications by elements of A are given by $a \cdot m \cdot b = \sigma(a)m\tau(b)$. In particular, the assignment $M \mapsto {}_{\sigma}M_{\tau}$ underlies an equivalence of categories ${}_A\text{Mod}_A \xrightarrow{\cong} {}_A\text{Mod}_A$.

In addition, whenever G is a weakly admissible group of automorphisms of a mesh algebra B and f is a G-invariant morphism of B, we will always write $\bar{f}$ for the induced morphism of $\Lambda = B/G$ via the pushdown functor.

All the results on m-fold mesh algebras will be given in terms of its extended type as defined in 14.

3.2 Two important auxiliary results

In this first section we present two results that will be very useful and will simplify our task in the subsequent sections. Before, we remind the definition of an acyclic character of a quiver.

Recall that a *walk* in a quiver Q between the vertices i and j is a finite sequence $i = i_0 \leftrightarrow i_1 \leftrightarrow ...i_{r-1} \leftrightarrow i_r = j$, where each edge $i_{k-1} \leftrightarrow i_k$ is either an arrow $i_{k-1} \rightarrow i_k$ or an arrow $i_k \rightarrow i_{k-1}$. We write such a walk as $\alpha_1^{\epsilon_1}...\alpha_r^{\epsilon_r}$, where α_i are arrows and ϵ_i is 1 or -1, depending on whether the corresponding edge is an arrow pointing to the right or to the left.

We will need the following concept from [45]:

Definition 15. Let Q be a (not necessarily finite) quiver. An *acyclic character* of Q (over the field K) is a map $\chi : Q_1 \longrightarrow K^*$ such that if $p = \alpha_1^{\epsilon_1}...\alpha_r^{\epsilon_r}$ and $q = \beta_1^{\epsilon'_1}...\beta_s^{\epsilon'_s}$ are two walks of length > 0 between any given vertices i and j, then $\prod_{1\leq i\leq r}\chi(\alpha_i)^{\epsilon_i} = \prod_{1\leq j\leq s}\chi(\beta_j)^{\epsilon'_j}$.

The following general result is the first of the two auxiliary lemmas. In particular, it gives us a criterion to determine when two morphisms of the mesh algebra B induce, up to conjugation, the same morphism of the m-fold mesh algebra $\Lambda = B/G$.

If A is a graded algebra with enough idempotents with the fixed family of orthogonal idempotents $(e_i)_{i\in I}$, then, whenever $a \in e_iAe_j$, we will write $i(a) = i$ and $t(a) = j$.

Lemma 3.2.1. *Let $A = \oplus_{n\geq 0}A_n$ be a basic positively $\mathbb{Z}$-graded pseudo-Frobenius algebra with enough idempotents such that $e_iA_0e_i \cong K$, for each $i \in I$, let G be a group of graded automorphisms of A acting freely on objects such that $\Lambda = A/G$ is finite dimensional and let $f, h : A \longrightarrow A$ be graded automorphisms satisfying the following three conditions:*

i) f and h permute the idempotents e_i

ii) $f(e_i) = h(e_i)$, for all $i \in I$

iii) $f \circ g = g \circ f$ and $h \circ g = g \circ h$, for all $g \in G$.

Then the following assertions hold:

1. *The assignment $[a] \rightsquigarrow [f(a)]$, with $a \in \bigcup_{i,j\in I} e_iAe_j$, determines a graded automorphism $\bar{f}$ of $\Lambda = A/G$, and analogously for h.*

2. *For $\bar{f}$ and $\bar{h}$ as in assertion 1, the following assertions are equivalent:*

   (a) $\bar{f}^{-1}\bar{h}$ is an inner automorphism of Λ

   (b) There is a map $\lambda : I \longrightarrow K^*$ such that $h(a) = \lambda(f(i(a)))^{-1}\lambda(f(t(a)))f(a)$ (resp. $f(a) = \lambda(i(a))^{-1}\lambda(t(a))h(a)$), for all $a \in \bigcup_{i,j\in I} e_iAe_j$, and $\lambda \circ g_{|I} = \lambda$, for all $g \in G$

Proof. Assertion 1 is clear. We then prove assertion 2:

a) $\Longrightarrow$ *b)* Let $\lambda : I \longrightarrow K^*$ be any map and $\psi : A \longrightarrow A$ be any graded automorphism. If $\chi_\lambda : A \longrightarrow A$ is the (graded) automorphism which is the identity on objects and maps $a \rightsquigarrow \lambda(i(a))^{-1}\lambda(t(a))a$, for each $a \in \bigcup_{i,j} e_iAe_j$, then the composition $\chi_\lambda \circ \psi$ (resp. $\psi \circ \chi_\lambda$) acts as ψ on objects and maps $a \rightsquigarrow \lambda(\psi(i(a)))^{-1}\lambda(\psi(t(a)))\psi(a)$ (resp. $a \rightsquigarrow \lambda(i(a))^{-1}\lambda(t(a))\psi(a)$), for each $a \in \bigcup_{i,j} e_iAe_j$, with the obvious interpretation of ψ as permutation of the set I.

If now f and h are as in the statement, the goal is to find a map λ as in the previous paragraph such that $\chi_\lambda \circ h = f$ (resp. $h \circ \chi_\lambda = f$) and $\lambda \circ g_{|I} = \lambda$, for all $g \in G$. Replacing f by $f \circ h^{-1}$ (resp. $h^{-1} \circ f$) if necessary, we can assume, without loss of generality, that $h = id_A$ and that f acts as the identity on objects. The task is hence reduced to check that if $\bar{f} : \Lambda \longrightarrow \Lambda$ is inner, then there is a map $\lambda : I \longrightarrow K^*$ such that $f = \chi_\lambda$ and $\lambda \circ g_{|I} = \lambda$, for all $g \in G$.

We know from Proposition 1.4.3 that Λ is a split basic graded algebra. So it is given by a finite graded quiver with relations whose set of vertices is (in bijection with) the set $I/G = \{[i] : i \in I\}$ of G-orbits of elements of I. From [45][Proposition 10 and Theorem 12] we get a map $\bar{\lambda} : I/G \longrightarrow K^*$ such that the assignment $[a] \rightsquigarrow \bar{\lambda}([i(a)])^{-1}\bar{\lambda}([t(a)])[a]$, where $a \in \bigcup_{i,j \in I} e_iAe_j$, is a (graded) inner automorphism u of Λ such that $u^{-1} \circ \bar{f}$ is the inner automorphism $\iota = \iota_{1-x}$ of Λ defined by an element of the form $1 - x$, where $x \in J(\Lambda)$. In our situation, the equality $J(\Lambda) = \oplus_{n>0}\Lambda_n$ holds, so that x is a sum of homogeneous elements of degree > 0. But $\iota = u \circ \bar{f}$ is also a graded automorphism, so that we have that $\iota(\Lambda_n) = (1-x)\Lambda_n(1-x)^{-1} = \Lambda_n$. If $y \in \Lambda_n$ then the n-th homogeneous component of $(1-x)y(1-x)^{-1}$ is y. It follows that ι is the identity on A_n, for each $n \geq 0$. Therefore we have $\iota = id_\Lambda$, so that $\bar{f} = u$.

Let now $\pi : A \longrightarrow \Lambda = A/G$ be the G-covering functor and let λ be the composition map $I \xrightarrow{\pi} I/G \xrightarrow{\bar{\lambda}} K^*$. By definition, we have that $\lambda \circ g = \lambda$, for all $g \in G$. As a consequence, the associated automorphism $\chi_\lambda : A \longrightarrow A$ defined above has the property that $[\chi_\lambda(a)] = u([a]) = \bar{f}([a]) = [f(a)]$, for each $a \in \bigcup_{i,j} e_iAe_j$. Since f is the identity on objects we immediately get that $f = \chi_\lambda$ as desired.

b) $\Longrightarrow$ *a)* The map λ of the hypothesis satisfies that $\chi_\lambda \circ h = f$. It then follows that $\bar{\chi}_\lambda \circ \bar{h} = \bar{f}$, where $\bar{\chi}_\lambda : \Lambda \longrightarrow \Lambda$ maps $[a] \rightsquigarrow \lambda(i(a))^{-1}\lambda(t(a))[a]$, for each $a \in \bigcup_{i,j} e_iAe_j$. Note that $\bar{\chi}_\lambda$ is well-defined because $\lambda \circ g = \lambda$, for all $g \in G$. It turns out that $\bar{\chi}_\lambda$ is the inner automorphism of Λ defined by the element $\sum_{[i] \in I/G} \lambda(i)^{-1}e_{[i]}$.

□

The second result consists of an identification of a subgroup of the integers which is crucial for our purposes.

Proposition 3.2.2. *Let Λ be the m-fold mesh algebra of extended type (Δ, m, t) and let $H(\Delta, m, t)$ be the set of integers s such that $\bar{\eta}^s\bar{\nu}^{-s}$ is an inner automorphism of Λ. Then $H(\Delta, m, t)$ is a subgroup of $\mathbb{Z}$ and the following assertions hold:*

1. *If $char(K) = 2$ or $\Delta = \mathbb{A}_r$ then $H(\Delta, m, t) = \mathbb{Z}$*
2. *If $char(K) \neq 2$ and $\Delta \neq \mathbb{A}_r$, then $H(\Delta, m, t) = \mathbb{Z}$, when $m + t$ is odd, and $H(\Delta, m, t) = 2\mathbb{Z}$ otherwise.*

Proof. The fact that $H(\Delta, m, t)$ is a subgroup of $\mathbb{Z}$ is clear since the subgroup $\mathrm{Inn}(\Lambda)$ of inner automorphisms is normal in $\mathrm{Aut}(\Lambda)$. For the explicit identification of this subgroup, we use the G-invariant graded Nakayama form of the mesh algebra B given by Theorem 2.4.2 and follow the notation of this theorem to name the arrows. For each integer $s > 0$, there is a map $\lambda : \mathbb{Z}\Delta_0 \longrightarrow K^*$ such that $\eta^s(a) = \lambda_{i(a)}^{-1}\lambda_{t(a)}\nu^s(a)$, for each $a \in \mathbb{Z}\Delta_1$. This map is uniquely determined up to multiplication by an element of K^*. According to Lemma 3.2.1, the integer s will be in $H(\Delta, m, t)$ if, and only if, the equality $\lambda \circ g_{|\mathbb{Z}\Delta_0} = \lambda$ holds, for all $g \in G$.

If $\mathrm{char}(K) = 2$ or $\Delta = \mathbb{A}_r$ Theorem 2.4.2 says that $\eta = \nu$ is a graded Nakayama form, and the result is clear in this case. We suppose in the sequel that $\mathrm{char}(K) \neq 2$ and $\Delta \neq \mathbb{A}_r$.

1) Suppose first that $t = 1$. Theorem 2.4.2 gives a formula $\eta(a) = (-1)^{u(a)}\nu(a)$, where $u(a) \in \mathbb{Z}_2$ for each $a \in \mathbb{Z}\Delta_1$. A careful examination of the $u(a)$ shows that the following properties hold in all cases:

i) $u(\sigma(a)) \neq u(a)$;

ii) If $v(a) := u(a) + u(\nu(a))$ then $v(\sigma(a)) = v(a)$,

for all $a \in \mathbb{Z}\Delta_1$. Let now $\lambda : \mathbb{Z}\Delta_0 \longrightarrow K^*$ be the map mentioned above for $s = 1$. Then we have $\lambda_{i(a)}^{-1}\lambda_{t(a)} = (-1)^{u(a)}$, for all $a \in \mathbb{Z}\Delta_1$. Together with property i) above, we then get that $\lambda_{\tau(k,i)} = -\lambda_{(k,i)}$, for all $(k, i) \in \mathbb{Z}\Delta_0$. This implies that $\lambda_{\tau^m(k,i)} = (-1)^m\lambda_{(k,i)}$. Then $s = 1$ is in $H(\Delta, m, t)$ if, and only if, m is even.

On the other hand, we have that $\eta^2(a) = \eta((-1)^{u(a)}\nu(a)) = (-1)^{u(a)+u(\nu(a))}\nu^2(a) = (-1)^{v(a)}\ \nu^2(a)$, for each $a \in \mathbb{Z}\Delta_1$. Let now $\lambda : \mathbb{Z}\Delta_0 \longrightarrow K^*$ be a map such that $\eta^2(a) = \lambda_{i(a)}^{-1}\lambda_{t(a)}\nu^2(a)$, for all $a \in \mathbb{Z}\Delta_1$. We then get that $\lambda_{i(a)}^{-1}\lambda_{t(a)} = (-1)^{v(a)}$. Together with property ii) above, we get that $\lambda_{\tau(k,i)} = \lambda_{(k,i)}$, and so $\lambda_{\tau^m(k,i)} = \lambda_{(k,i)}$, for all $(k, i) \in \mathbb{Z}\Delta_0$. It follows that $s = 2$ is in $H(\Delta, m, t)$, which proves that $H(\Delta, m, t) = 2\mathbb{Z}$ when m is odd.

2)Suppose that $\underline{(\Delta, t) = (\Delta = \mathbb{D}_{n+1}, 2)}$. For any integer k, we define the element $c(k) \in \mathbb{Z}_2$ to be 0, when $k \not\equiv -1 \pmod{m}$, and 1 otherwise. Theorem 2.4.2 gives that $\eta(a) = -\nu(a)$, when $a : (k, i) \to (k, i+1)$ is an upward arrow, and $\eta(a) = (-1)^{c(k)}\nu(a)$, when $a : (k, i) \to (k+1, i-1)$ $(i = 3, ..., n)$ is a downward arrow. If $\lambda : \mathbb{Z}\Delta_0 \longrightarrow K^*$ is the map considered in the first paragraph of this proof for $s = 1$, then we get that $\lambda_{(k+1,i)} = (-1)^{c(k)+1}\lambda_{(k,i)}$, for each $i \neq 0, 1$. It follows from this that $\lambda_{\rho\tau^{-m}(k,i)} = \lambda_{(k+m,i)} = (-1)^{\gamma(k)+m}$, where $\gamma(k) = \sum_{0 \leq j < m} c(k+j)$. But $\gamma(k) = 1$ since there is exactly one summand which is nonzero. We then have $\lambda_{\rho\tau^{-m}(k,i)} = (-1)^{m+1}\lambda_{(k,i)}$. This shows that if $s = 1$ is in $H(\mathbb{D}_{n+1}, m, 2)$ then m is necessarily odd. We claim that the converse is also true, so that $H(\mathbb{D}_{n+1}, m, 2) = \mathbb{Z}$ in this case. Indeed from Theorem 2.4.2 we get equalities $\eta(\varepsilon_i) = (-1)^{q+i}\nu(\varepsilon_i)$ and $\eta(\varepsilon_i') = (-1)^{q+i+1+c(k)}\nu(\varepsilon_i')$, for $i = 0, 1$. Denoting by $q(k)$ and $q(k+1)$ the quotients of dividing k and $k+1$ by m, we then get that $\lambda_{k+1,i} = (-1)^{\psi(k)}\lambda_{k,i}$, where $\psi(k) = q(k) + i + 1 + c(k) + q(k+1) + i$. Let us view $\psi(k)$ as an element of $\mathbb{Z}_2$ and bear in mind that $q(k+1) = q(k)$, unless $k \equiv -1 \pmod{m}$, in which case $q(k+1) = q(k) + 1$. We then see that, for k arbitrary, we always have $\psi(k) = 1$. It then follows:

$$\lambda_{\rho\tau^{-m}(k,i)} = \lambda_{\rho(k+m,i)} = -\lambda_{(k+m,i)} = -(-1)^m\lambda_{(k,i)} = (-1)^{m+1}\lambda_{(k,i)} = \lambda_{(k,i)}$$

since m is odd. Therefore the equality $\lambda_{\rho\tau^{-m}(k,i)} = \lambda_{(k,i)}$ holds, for all $(k,i) \in \mathbb{Z}\Delta_0$, so that $\lambda \circ g_{|\mathbb{Z}\Delta_0} = \lambda$ for all $g \in G$.

Still with the case $\Delta = \mathbb{D}_{n+1}$, suppose now that m is even. Note that we have $\eta^2(a) = \nu^2(a)$, for each upward arrow. Let $a : (k,i) \to (k+1,i-1)$ be any downward arrow. The arrows a and $\nu(a)$ have origins in the slices k and $k+(n-1)$, respectively. It follows that $\eta^2(a) = (-1)^{c(k)+c(k+(n-1))}\nu^2(a)$. If now $\lambda : \mathbb{Z}\Delta_0 \longrightarrow K^*$ is the usual map for $s = 2$, then we get that $\lambda_{(k+1,i)} = (-1)^{c(k)+c(k+(n-1))}\lambda_{(k,i)}$, for each $i = 2,3,...,n$. It follows that $\lambda_{\rho\tau^{-m}(k,i)} = \lambda_{(k+m,i)} = (-1)^{\xi(k)}$, where $\xi(k) = \sum_{0\leq j<m}[c(k+j) + c(k+j+(n-1))] = \gamma(k) + \gamma(k+(n-1))$, which is zero in $\mathbb{Z}_2$. This shows that $\lambda_{\rho\tau^{-m}(k,i)} = \lambda_{(k,i)}$ whenever $i = 2,3,...,n$. On the other hand, taking into account the definition of ν (see Proposition 2.3.1), if $i = 0,1$ we have:

1. When $n+1$ is even: $\eta^2(\varepsilon_i) = \eta((-1)^{q(k)+i}\nu(\varepsilon_i)) = (-1)^{q(k)+i+q(k+(n-1))+i}\nu^2(\varepsilon_i) = (-1)^{q(k)+q(k+(n-1))}\nu^2(\varepsilon_i)$;

2. when $n+1$ is odd: $\eta^2(\varepsilon_i) = \eta((-1)^{q(k)+i}\nu(\varepsilon_i)) = (-1)^{q(k)+i+q(k+(n-1))+i+1}\nu^2(\varepsilon_i) = (-1)^{q(k)+q(k+(n-1))+1}\nu^2(\varepsilon_i)$,

We then get $\lambda_{(k,i)} = (-1)^{u(k,i)}\lambda_{(k,2)}$, where $u(k,i) = q(k) + q(k+(n-1)$ in the first case and $u(k,i) = q(k) + q(k+(n-1)+1$ in the second case. In both cases, we get that $\lambda_{(k,0)} = \lambda_{(k,1)}$. Suppose now that $\eta^2(\varepsilon'_i) = (-1)^{v(k,i)}\nu^2(\varepsilon'_i)$. Then we will have $\lambda_{(k+1,2)} = (-1)^{u(k,i)+v(k,i)}\lambda_{(k,2)}$ which, together with the equality $\lambda_{(k+1,2)} = (-1)^{c(k)+c(k+(n-1))}\lambda_{(k,2)}$ seen above, proves the equality in $\mathbb{Z}_2$: $v(k,i) = u(k,i) + c(k) + c(k+(n-1))$. We then get $\lambda_{(k+1,i)} = (-1)^{v(k,i)}(-1)^{u(k+1,i)}\lambda_{(k,i)} = (-1)^{\chi(k,i)}\lambda_{(k,i)}$, where $\chi(k,i) = u(k,i)+u(k+1,i)+c(k)+c(k+(n-1))$. It follows from this that $\lambda_{\rho\tau^{-m}(k,i)} = \lambda_{(k+m,i)} = (-1)^{\sigma(k,i)}\lambda_{(k,i)}$, where $\sigma(k,i) = \sum_{0\leq j<m}\chi(k+j,i) = \sum_{0\leq j<m}[c(k+j) + c(k+j+(n-1))] + \sum_{0\leq j<m}[u(k,i) + u(k+1,i)]$. The first summand in the last member of this equality has already been shown to be even. But we have an equality in $\mathbb{Z}_2$:

$$\sum_{0\leq j<m}[u(k+j,i) + u(k+j+1,i)] =$$
$$\sum_{0\leq j<m}[q(k+j) + q(k+j+(n-1))] + \sum_{0\leq j<m}[q(k+1+j) + q(k+1+j+(n-1))] =$$
$$\sum_{0\leq j<m}[q(k+j) + q(k+1+j)] + \sum_{0\leq j<m}[q(k+j+(n-1)) + q(k+1+j+(n-1))].$$

As has already been noted, the equality $q(k+r) = q(k+1+r)$ holds, except when $k+r \equiv -1 \pmod m$, in which case $q(k+1+r) = q(k+r)+1$. This comment proves that each summand of the last member in the centered equality is equal to 1 in $\mathbb{Z}_2$. It follows that $\sigma(k,i) = 0$ in $\mathbb{Z}_2$ and, hence, that $\lambda_{\rho\tau^{-m}(k,i)} = \lambda_{(k,i)}$, for all $(k,i) \in \mathbb{Z}\Delta_0$. By the first paragraph of this proof, we conclude that $H(\mathbb{D}_{n+1}, m, 2) = 2\mathbb{Z}$ whenever m is even.

c) Suppose next that $(\Delta, t) = (\mathbb{E}_6, 2)$. If $s > 0$ is any integer, then, by Theorem 2.4.2, we have $\eta^s(a) = \nu^s(a)$, when $a \in \{\alpha, \delta'\}$, and $\eta^s(a) = (-1)^s\nu^s(a)$, when $a \in \{\alpha', \delta\}$. If $\lambda : \mathbb{Z}\Delta_0 \longrightarrow K^*$ is a map such that $\eta^s(a) = \lambda^{-}_{i(a)}\lambda_{t(a)}\nu^s(a)$, for each $a \in \mathbb{Z}\Delta_1$, we then get equalities: $\lambda_{(k,2)} = \lambda_{(k,1)}$, $\lambda_{(k+1,4)} = \lambda_{(k,5)}$, $\lambda_{(k+1,1)} = (-1)^s\lambda_{(k,2)}$ and $\lambda_{(k,5)} = (-1)^s\lambda_{(k,4)}$. It follows that $\lambda_{(k+1,i)} = (-1)^s\lambda_{(k,i)}$, for each $(k,i) \in \mathbb{Z}\Delta_0$ such that $i = 1,2,4,5$.

Note that we have $\eta(\varepsilon') = (-1)^{c(k)}\nu(\epsilon')$, where $c(k)$ is defined as in the case $(\Delta, t) = (\mathbb{D}_{n+1}, 2)$. We then have that $\eta^2(\varepsilon') = (-1)^{c(k)+c(k+5)}\nu^2(\varepsilon')$. Since we also have $\eta^s(\varepsilon) = (-1)^s\nu^s(\varepsilon)$ we get:

1. When $s = 1$, $\lambda_{(k+1,i)} = (-1)^{c(k)+1}\lambda_{(k,i)}$;
2. When $s = 2$, $\lambda_{(k+1,i)} = (-1)^{c(k)+c(k+5)}\lambda_{(k,i)}$

for $i = 0, 3$. We also have:

1. When $s = 1$, $\eta(\gamma) = (-1)^q\nu(\gamma)$ and $\eta(\beta') = (-1)^{q+1}\nu(\beta')$, where $q = q(k)$ is the quotient of dividing k by m;
2. When $s = 2$, $\eta^2(\gamma) = (-1)^{q(k)+q(k+5)+1}\nu^2(\gamma)$ and $\eta^2(\beta') = (-1)^{q(k)+q(k+5)+1}\nu^2(\beta')$.

It follows from this that, in case $s = 1$, we have $\lambda_{(k,4)} = (-1)^q\lambda_{(k,3)}$ and $\lambda_{(k+1,2)} = (-1)^{q+1}\lambda_{(k,3)}$ and, hence, $\lambda_{(k,4)} = -\lambda_{(k+1,2)}$. This, together with the equalities in the previous paragraph, show that $\lambda_{\rho(k,i)} = -\lambda_{(k,i)}$, for all $i = 1, 2, 4, 5$. Therefore, when $s = 1$, we get:

$$\lambda_{\rho\tau^{-m}(k,i)} = -\lambda_{(k+m,i)} = -(-1)^m\lambda_{(k,i)} = (-1)^{m+1}\lambda_{(k,i)}, \text{ for } i \neq 0, 3,$$
and
$$\lambda_{\rho\tau^{-m}(k,i)} = \lambda_{(k+m,i)} = (-1)^{\gamma(k)+m}\lambda_{(k,i)} = (-1)^{m+1}\lambda_{(k,i)}, \text{ for } i = 0, 3, \text{ since}$$
$$\gamma(k) = \textstyle\sum_{0\leq j<m} c(k+j) = 1.$$

By the first paragraph of this proof, we get that $s = 1$ is an element of $H(\mathbb{E}_6, m, 2)$ if, and only if, m is odd.

Suppose now that m is even and that $s = 2$. Then for the corresponding map λ we have that $\lambda_{(k,4)} = (-1)^{q(k)+q(k+5)+1}\lambda_{(k,3)}$ and $\lambda_{(k+1,2)} = (-1)^{q(k)+q(k+5)+1}\lambda_{(k,3)}$, from which we get that $\lambda_{\rho(k,i)} = \lambda_{(k,i)}$, for all $i \in \Delta_0$. From the fact that $\lambda_{(k+1,i)} = \lambda_{(k,i)}$, for $i \neq 0, 3$, and $\lambda_{(k+1,i)} = (-1)^{c(k)+c(k+5)}\lambda_{(k,i)}$, for $i = 0, 3$, we get:

$$\lambda_{\rho\tau^{-m}(k,i)} = \lambda_{(k+m,i)} = \lambda_{(k,i)}, \text{ for } i \neq 0, 3,$$
and
$$\lambda_{\rho\tau^{-m}(k,i)} = \lambda_{(k+m,i)} = (-1)^{\gamma(k)+\gamma(k+5)}\lambda_{(k,i)} = (-1)^2\lambda_{(k,i)} = \lambda_{(k,i)}, \text{ for } i = 0, 3,$$

because $\gamma(k) = \sum_{0\leq j<m} c(k+j) = 1$ for each integer k. Therefore, when m is even, $s = 2$ is an element of $H(\mathbb{E}_6, m, 2)$, thus showing that this group is $2\mathbb{Z}$ in such case.

d) Suppose finally that $(\Delta, t) = (\mathbb{D}_4, 3)$. If $s > 0$ is an integer then $\eta^s(\varepsilon_i) = \nu^s(\epsilon_i)$ and $\eta^s(\varepsilon'_i) = (-1)^s\nu^s(\varepsilon'_i)$ since $\nu = \tau^{-2}$ in this case. If $\lambda : \mathbb{Z}\Delta_0 \longrightarrow K^*$ is the map such that $\eta^s(a) = \lambda_{i(a)}^{-1}\lambda_{t(a)}\nu^s(a)$, for each $a \in \mathbb{Z}\Delta_1$, then we easily get that $\lambda_{\rho(k,i)} = \lambda_{(k,i)}$ and $\lambda_{(k+1,i)} = (-1)^s\lambda_{(k,i)}$, so that $\lambda_{(\rho\tau^m)^{-1}(k,i)} = \lambda_{\rho^2\tau^{-m}(k,i)} = \lambda_{(k+m,i)} = (-1)^{sm}\lambda_{(k,i)}$. It follows that $s = 1$ is in $H(\mathbb{D}_4, m, 3)$ if, and only if, m is even. On the other hand, when m is odd, we have that $2 \in H(\mathbb{D}_4, m, 3)$.

□

3.3 Symmetric and weakly symmetric m-fold mesh algebras

The only result of this subsection identifies all the symmetric and weakly symmetric m-fold mesh algebras. Recall that an algebra is *weakly symmetric* if, and only if, its Nakayama automorphism acts as the identity on vertices. A weakly symmetric algebra is in addition *symmetric* if, and only if, its Nakayama automorphism is inner.

Theorem 3.3.1. *Let Λ be an m-fold mesh algebra of extended type (Δ, m, t). If Λ is weakly symmetric then $t = 1$ or $t = 2$ and, when char$(K) = 2$ or $\Delta = \mathbb{A}_r$, such an algebra is also symmetric. Moreover, the following assertions hold:*

1. *When $t = 1$, Λ is weakly symmetric if, and only if, Δ is $\mathbb{D}_{2r}$, $\mathbb{E}_7$ or $\mathbb{E}_8$ and m is a divisor of $\frac{c_\Delta}{2} - 1$. When char$(K) \neq 2$, such an algebra is symmetric if, and only if, m is even.*

2. *When $t = 2$ and $\Delta \neq \mathbb{A}_{2n}$, Λ is weakly symmetric if, and only if, m divides $\frac{c_\Delta}{2} - 1$ and, moreover, the quotient of the division is odd, in case $\Delta = \mathbb{A}_{2n-1}$, and even, in case $\Delta = \mathbb{D}_{2r}$. When char$(K) \neq 2$, such an algebra is symmetric if, and only if, $\Delta = \mathbb{A}_{2n-1}$ or m is odd.*

3. *When $(\Delta, m, t) = (\mathbb{A}_{2n}, m, 2)$, i.e. $\Lambda = \mathbb{L}_n^{(m)}$, the algebra is (weakly) symmetric if, and only if, $2m - 1$ divides $2n - 1$.*

Proof. The algebra Λ is weakly symmetric if, and only if, the automorphism $\bar{\nu} : \Lambda \longrightarrow \Lambda$ induced by ν is the identity on vertices. We identify the vertices of the quiver of Λ as G-orbits of vertices of $\mathbb{Z}\Delta_0$, where G is the weakly admissible group of automorphism considered in each case. If we take care to choose a vertex (k, i) which is not fixed by ρ, then the equality $\bar{\nu}([(k, i)]) = [(k, i)]$ holds exactly when there is a $g \in G$ such that $\nu(k, i) = g(k, i)$. But if $\hat{G}$ denotes the group of automorphisms generated by ρ and τ, then $\hat{G}$ acts freely on the vertices not fixed by ρ. Since $G \subset \hat{G}$ and $\nu \in \hat{G}$ (see Proposition 2.3.1) the equality $\nu(k, i) = g(k, i)$ implies that $\nu = g$. Therefore the algebra Λ is weakly symmetric if, and only if, ν belongs to G.

On the other hand, Λ is symmetric if, and only if, $\bar{\eta} : \Lambda \longrightarrow \Lambda$ is an inner automorphism. By Lemma 3.2.1, this is equivalent to saying that Λ is weakly symmetric and $\bar{\eta} \circ \bar{\nu}^{-1}$ is an inner automorphism of Λ. That is, Λ is symmetric if, and only if, Λ is weakly symmetric and $H(\Delta, m, t) = \mathbb{Z}$. As a consequence, once the weakly symmetric m-fold mesh algebras have been identified, the part of the theorem referring to symmetric algebras follows directly from Proposition 3.2.2.

If $t = 3$ then $\Delta = \mathbb{D}_4$, $G =< \rho\tau^m >$, with ρ acting on vertices as the 3-cycle (013), and $\nu = \tau^{-2}$. It is impossible to have $\tau^{-2} \in G$ and therefore Λ is never weakly symmetric in this case.

If $t = 1$ then $G =< \tau^m >$. If we assume that $\Delta \neq \mathbb{D}_{2r}, \mathbb{E}_7, \mathbb{E}_8$ then $\nu = \rho\tau^{1-n}$, for some integer n. Again it is impossible that $\nu \in G$ and, hence, Λ cannot be weakly symmetric. On the contrary, suppose that Δ is one of $\mathbb{D}_{2r}, \mathbb{E}_7, \mathbb{E}_8$. Then $\nu = \tau^{1-n}$, with $n = \frac{c_\Delta}{2}$, and ν belongs to G if, and only if, there is an integer r such that $\tau^{1-n} = (\tau^m)^r$, which is equivalent to saying that $n - 1 = -mr$ since τ has infinite order. Then Λ is weakly symmetric in this case if, and only if, m divides $n - 1$.

Suppose now that $t = 2$ and $\Delta \neq \mathbb{A}_{2n}$. Then $G =< \rho\tau^m >$ and, except when $\Delta = \mathbb{D}_{2r}$, we have that $\nu = \rho\tau^{1-n}$, where $n = \frac{c_\Delta}{2}$. Assume that $\Delta \neq \mathbb{D}_{2r}$. Then ν is in G if, and only if, there is an integer r such that $\rho\tau^{1-n} = (\rho\tau^m)^r$. Note that then r is necessarily odd. If follows that Λ is weakly symmetric if, and only if, m divides $n-1$ and the quotient $\frac{n-1}{m}$ is an odd number. But the condition that $\frac{n-1}{m}$ be odd is superfluous when $\Delta = \mathbb{D}_{2r+1}$ or $\mathbb{E}_6$ because n is even in both cases.

Consider now the case in which $(\Delta, t) = (\mathbb{D}_{2r}, 2)$. Then $\nu = \tau^{1-n}$, where $n = \frac{c_\Delta}{2} = 2r-1$. Then ν is in G if, and only if, there is an integer s such that $\tau^{1-n} = (\rho\tau^m)^s$. This forces s to be even. We then get that Λ is weakly symmetric if, and only if, m divides $n-1$ and the quotient $\frac{n-1}{m}$ is even.

Finally, let us consider the case when the extended type is $(\mathbb{A}_{2n}, m, 2)$. In this case $\rho^2 = \tau^{-1}$ and $\nu = \rho\tau^{1-n}$. Then ν is in G if, and only if, there is an integer r such that $\rho\tau^{1-n} = (\rho\tau^m)^r$. This forces $r = 2s+1$ to be odd, and then $\rho\tau^{-s+m(2s+1)} = (\rho\tau^m)^{2s+1} = \rho\tau^{1-n}$. Then Λ is weakly symmetric if, and only if, there is an integer s such that $(2m-1)s = 1-m-n$. That is, if and only if $2m-1$ divides $m+n-1$, which is equivalent to saying that $2m-1$ divides $2(m+n-1)-(2m-1) = 2n-1$. □

3.4 The period and the stable Calabi-Yau dimension of an m-fold mesh algebra

3.4.1 The minimal projective resolution of the regular bimodule

We start this section by pointing out that, for any m-fold mesh algebra $\Lambda = B/G$, we can always guarantee the existence of a basis of its corresponding mesh algebra B inducing, via the pushdown functor, a basis of Λ.

Lemma 3.4.1. *Let Δ be a Dynkin quiver and B be its associated mesh algebra. For any weakly admissible group of automorphisms G of $\mathbb{Z}\Delta$, there is a basis $\mathcal{B}$ of B consisting of paths which is G-invariant (i.e. $g(\mathcal{B}) = \mathcal{B}$ for all $g \in G$).*

Proof. The way of constructing the basis $\mathcal{B}$ is entirely analogous to the way in which a G-invariant basis of $\mathrm{Soc}(B)$ was constructed (see the initial paragraphs of the proof of Theorem 2.4.2). The task is then reduced to find, for each vertex (k,i) in the chosen slice, S or T, a basis of $e_{(k,i)}B$ consisting of paths. Since the existence of this basis is clear the result follows. □

Suppose that $(-,-) : B \times B \longrightarrow K$ is a G-invariant graded Nakayama form for B. Given a basis $\mathcal{B}$ as in last lemma, its *(right) dual basis* with respect to $(-,-)$ will be the basis $\mathcal{B}^* = \bigcup_{(k,i)\in(\mathbb{Z}\Delta)_0} \mathcal{B}^* e_{\nu(k,i)}$, where $\mathcal{B}^* e_{\nu(k,i)}$, is the (right) dual basis of $e_{(k,i)}\mathcal{B}$ with respect to the induced graded bilinear form $(-,-) : e_{(k,i)}B \times Be_{\nu(k,i)} \longrightarrow K$. By the graded condition of this bilinear form, $\mathcal{B}^*$ consists of homogeneous elements. By the G-invariance of $(-,-)$ and $\mathcal{B}$, we immediately get that $\mathcal{B}^*$ is G-invariant. On what concerns the minimal projective resolution of B as a bimodule, we will need to fix a basis $\mathcal{B}$ as given by last lemma and use it and its dual basis to give the desired resolution.

Proposition 3.4.2. *Let Δ be a Dynkin quiver, let $X \subseteq (\mathbb{Z}\Delta)_1$ be the set of arrows given by Proposition 2.3.4, which we assume to be the empty set when $(\Delta, G) = (\mathbb{D}_4, < \rho\tau^m >)$, and let $s : (\mathbb{Z}\Delta)_1 \longrightarrow \mathbb{Z}_2$ be the associated signature map. Denote by τ' the graded automorphism of B which acts as τ on vertices and maps $a \rightsquigarrow (-1)^{s(a)+s(\tau(a))}\tau(a)$, for each $a \in \mathbb{Z}\Delta)_1$. Up to isomorphism, the initial part of the minimal graded projective resolution of B as a B-bimodule is given by*

$$Q^{-2} \xrightarrow{R} Q^{-1} \xrightarrow{\delta} Q^0 \xrightarrow{u} B \to 0,$$

where:

1. *The graded projective B-bimodules are $Q^0 = (\oplus_{(k,i)\in(\mathbb{Z}\Delta)_0} Be_{(k,i)} \otimes e_{(k,i)}B)[0]$, $Q^{-1} = (\oplus_{a\in(\mathbb{Z}\Delta)_1} Be_{i(a)} \otimes e_{t(a)}B)[-1]$ and $Q^{-2} = (\oplus_{(k,i)\in(\mathbb{Z}\Delta)_0} Be_{\tau(k,i)} \otimes e_{(k,i)}B)[-2]$;*
2. *u is the multiplication map;*
3. *δ is the only homomorphism of B-bimodules such that, for all $a \in (\mathbb{Z}\Delta)_1$,*

$$\delta(e_{i(a)} \otimes e_{t(a)}) = a \otimes e_{t(a)} - e_{i(a)} \otimes a;$$

4. *R is the only homomorphism of B-bimodules such that, for all $(k,i) \in (\mathbb{Z}\Delta)_0$,*

$$R(e_{\tau(k,i)} \otimes e_{(k,i)}) = \sum_{t(a)=(k,i)} (-1)^{s(\sigma(a)a)} [\sigma(a) \otimes e_{(k,i)} + e_{\tau(k,i)} \otimes a]$$

where the signature of a path is the sum of the signatures of its arrows.

Moreover, if for each $(k,i) \in (\mathbb{Z}\Delta)_0$ we consider the homogeneous elements of Q^{-2} given by

$$\xi'_{(k,i)} = \sum_{x\in e_{(k,i)}\mathcal{B}} (-1)^{deg(x)}\tau'(x) \otimes x^*,$$

then, $\oplus_{(k,i)\in\mathbb{Z}\Delta_0} B\xi'_{(k,i)} = Ker(R) = \oplus_{(k,i)\in\mathbb{Z}\Delta_0} \xi'_{(k,i)}B$.

Proof. Let B' be the original mesh algebra, i.e., $K\mathbb{Z}\Delta/I$, where I is the ideal generated by $r_{(k,i)} = \sum_{t(a)=(k,i)} \sigma(a)a$, with $(k,i) \in \mathbb{Z}\Delta_0$. By classical argument for unital algebras, also valid here (see, e.g., [11] or [25]), we know that the initial part of the minimal projective resolution of B' as a bimodule is

$$P^{-2} \xrightarrow{R'} P^{-1} \xrightarrow{\delta'} P^0 \xrightarrow{u'} B' \to 0,$$

where:

1. The graded projective B'-bimodules are $P^0 = (\oplus_{(k,i)\in(\mathbb{Z}\Delta)_0} B'e_{(k,i)} \otimes e_{(k,i)}B')[0]$, $P^{-1} = (\oplus_{a\in(\mathbb{Z}\Delta)_1} B'e_{i(a)} \otimes e_{t(a)}B')[-1]$ and $P^{-2} = (\oplus_{(k,i)\in(\mathbb{Z}\Delta)_0} B'e_{\tau(k,i)} \otimes e_{(k,i)}B')[-2]$;
2. u' is the multiplication map;

3. δ' is the only homomorphism of B'-bimodules such that, for all $a \in (\mathbb{Z}\Delta)_1$,

$$\delta'(e_{i(a)} \otimes e_{t(a)}) = a \otimes e_{t(a)} - e_{i(a)} \otimes a;$$

4. R' is the only homomorphism of B'-bimodules such that, for all $(k,i) \in (\mathbb{Z}\Delta)_0$,

$$R'(e_{\tau(k,i)} \otimes e_{(k,i)}) = \sum_{t(a)=(k,i)} (\sigma(a) \otimes e_{(k,i)} + e_{\tau(k,i)} \otimes a)$$

Consider now the canonical algebra isomorphism $\varphi = \varphi^{-1} : K\mathbb{Z}\Delta \xrightarrow{\cong} K\mathbb{Z}\Delta$, given in Proposition 2.3.4, and denote by h the induced isomorphism of graded algebras $B \xrightarrow{\cong} B'$ and by f its inverse. We put $\mathcal{B}' = h(\mathcal{B})$, where $\mathcal{B}$ is the G-invariant basis of B given by the previous lemma. The mentioned classical arguments also show that the elements $\xi_{(k,i)} = \sum_{x \in e_{(k,i)}\mathcal{B}'} (-1)^{\deg(x)}\tau(x) \otimes x^*$, with $(k,i) \in (\mathbb{Z}\Delta)_0$, are in $\mathrm{Ker}(R')$. Note that the argument which proves for unital algebras that the $\xi_{(k,i)}$ generate $\mathrm{Ker}(R')$ cannot be adapted in a straightforward way.

If $(k,i),(m,j) \in (\mathbb{Z}\Delta)_0$ are any vertices then the induced map $f \otimes f : B'e_{(k,i)} \otimes e_{(m,j)}B' \longrightarrow Be_{(k,i)} \otimes e_{(m,j)}B$ gives an isomorphism of (graded projective) B-bimodules ${}_h(B'e_{(k,i)} \otimes e_{(m,j)}B')_h \xrightarrow{\cong} Be_{(k,i)} \otimes e_{(m,j)}B$. It follows that if $\chi' : B'e_{(k,i)} \otimes e_{(m,j)}B' \longrightarrow B'e_{(r,u)} \otimes e_{(t,v)}B'$ is a morphism of graded projective B'-bimodules, then the corresponding morphism of graded projective B-bimodules $\chi : Be_{(k,i)} \otimes e_{(m,j)}B \longrightarrow Be_{(r,u)} \otimes e_{(t,v)}B$ takes $a \otimes b \rightsquigarrow (f \otimes f)(\chi'(f^{-1}(a) \otimes f^{-1}(b)))$. From these considerations it easily follows that, up to isomorphism, the initial part of the minimal projective resolution of B as a B-bimodule is:

$$Q^{-2} \xrightarrow{R''} Q^{-1} \xrightarrow{\delta''} Q^0 \xrightarrow{u} B' \to 0,$$

where:

1. The Q^i are as indicated in the statement
2. u is the multiplication map;
3. δ'' is the only homomorphism of B-bimodules such that, for all $a \in (\mathbb{Z}\Delta)_1$,

$$\delta(e_{i(a)} \otimes e_{t(a)}) = (-1)^{s(a)}(a \otimes e_{t(a)} - e_{i(a)} \otimes a);$$

4. R'' is the only homomorphism of B-bimodules such that, for all $(k,i) \in (\mathbb{Z}\Delta)_0$,

$$R''(e_{\tau(k,i)} \otimes e_{(k,i)}) = \sum_{t(a)=(k,i)} [(-1)^{s(\sigma(a))}\sigma(a) \otimes e_{(k,i)} + (-1)^{s(a)}e_{\tau(k,i)} \otimes a]$$

Let $\psi : \oplus_{a \in (\mathbb{Z}\Delta)_1} Be_{i(a)} \otimes e_{t(a)}B \longrightarrow \oplus_{a \in (\mathbb{Z}\Delta)_1} Be_{i(a)} \otimes e_{t(a)}B$ the only homomorphism of B-bimodules mapping $e_{i(a)} \otimes e_{t(a)} \rightsquigarrow (-1)^{s(a)}e_{i(a)} \otimes e_{t(a)}$, for each $a \in (\mathbb{Z}\Delta)_1$. It is clearly an isomorphism and we have equalities $\delta \circ \psi = \delta''$ and $\psi \circ R'' = R$. Then

$$Q^{-2} \xrightarrow{R} Q^{-1} \xrightarrow{\delta} Q^0 \xrightarrow{u} B' \to 0,$$

is also the initial part of the minimal projective resolution of B as a B-bimodule and we have $L := \mathrm{Ker}(R) = \mathrm{Ker}(R'')$. Moreover from the equalities $f(\tau(x)) = \tau'(f(x))$ and $f(x^*) = f(x)^*$, for all $x \in \mathcal{B}'$, and the fact that $f(\mathcal{B}') = \mathcal{B}$ we immediately get that $\xi'_{(k,i)} = f(\xi_{(k,i)}) = \sum_{y \in e_{(k,i)}\mathcal{B}} (-1)^{\deg(y)} \tau'(y) \otimes y^*$. Therefore the $\xi'_{(k,i)}$ are elements of L.

If $S_{(m,j)} = Be_{(m,j)}/J(B)e_{(m,j)}$ is the simple graded left module concentrated in degree zero associated to the vertex (m,j), then the induced sequence

$$Q^{-2} \otimes_B S_{(m,j)} \xrightarrow{R\otimes 1} Q^{-1} \otimes_B S_{(m,j)} \xrightarrow{\delta\otimes 1} Q^0 \otimes_B S_{(m,j)} \longrightarrow S_{(m,j)} \to 0$$

is the initial part of the minimal projective resolution of $S_{(m,j)}$. It is easy to see that the pushdown functor $F_\lambda : B - Gr \longrightarrow \Lambda - Gr$ preserves and reflects simple objects When applied to the last resolution, we then get the minimal projective resolution of the simple Λ-module $S_{[(m,j)]}$, where Λ is viewed as the orbit category B/G (see Corollary 1.4.2) and where $[(m,j)]$ denotes the G-orbit of (m,j). But we know that $\Omega^3_\Lambda(S_{[(m,j)]})$ is a simple Λ-module (see, e.g., [25]). It follows that $\Omega^3_B(S_{(m,j)})$ is a graded simple left B-module. But we have an isomorphism $Q^{-2} \otimes_B S_{(m,j)} \cong Be_{\tau(m,j)}[-2]$ in $B - Gr$. By definition of the Nakayama permutation, we have that $\mathrm{Soc}_{gr}(Be_{\tau(m,j)}) \cong S_{\nu^{-1}\tau(m,j)}[-c_\Delta + 2]$. Then we have an isomorphism $\Omega^3_B(S_{(m,j)}) \cong S_{\nu^{-1}\tau(m,j)}[-c_\Delta]$, for all $(m,j) \in \mathbb{Z}\Delta_0$. Considering the decomposition $B/J(B) = \oplus_{(m,j)\in\mathbb{Z}\Delta_0} S_{(m,j)}$, we then get that $L/LJ(B) \cong L \otimes_B \frac{B}{J(B)}$ is isomorphic to $B/J(B)[-c_\Delta]$ as a graded left B-module. Due to the fact that $J(B) = J^{gr}(B)$ is nilpotent, we know that every left or right graded B-module has a projective cover. By taking projective covers in $B - Gr$ and bearing in mind that L is projective on the left and on the right, we then get that $L_B \cong B_B[-c_\Delta]$. With a symmetric argument, one also gets that ${}_BL \cong {}_BB[-c_\Delta]$. In particular, ${}_BL = {}_B\Omega^3_{B^e}(B)$ (resp. $L_B = \Omega^3_{B^e}(B)_B$) decomposes as a direct sum of indecomposable projective graded B-modules, all of them with multiplicity 1.

Note now that we have equalities $e_{\tau\nu^{-1}(k,i)}\xi'_{\nu^{-1}(k,i)} = \xi'_{\nu^{-1}(k,i)} = \xi'_{\nu^{-1}(k,i)}e_{(k,i)}$, for all $(k,i) \in \mathbb{Z}\Delta_0$. This gives surjective homomorphisms $Be_{\tau\nu^{-1}(k,i)}[-c_\Delta] \overset{\rho}{\twoheadrightarrow} B\xi'_{\nu^{-1}(k,i)}$ and $e_{(k,i)}B[-c_\Delta] \overset{\lambda}{\twoheadrightarrow} \xi'_{\nu^{-1}(k,i)}B$ of graded left and right B-modules given by right and left multiplication by $\xi'_{\nu^{-1}(k,i)}$. But ρ and λ do not vanish on $\mathrm{Soc}_{gr}(Be_{\tau\nu^{-1}(k,i)})$ and $\mathrm{Soc}_{gr}(e_{(k,i)}B)$, which are simple graded modules, respectively. It follows that ρ and λ are injective and, hence, they are isomorphisms. We then get that $N := \oplus_{(k,i)\in\mathbb{Z}\Delta_0} B\xi'_{\nu^{-1}(k,i)} = \oplus_{(k,i)\in\mathbb{Z}\Delta_0} B\xi'_{(k,i)}$ is a graded submodule of ${}_BL$ isomorphic to ${}_BB \cong {}_BL$ and, hence, it is injective in $B - Gr$ since this category is Frobenius. We then get that N is a direct summand of ${}_BL$ which is isomorphic to ${}_BL$. Since $\mathrm{End}_{B-Gr}(Be_{(k,i)}) \cong K$ for each vertex (k,i), Azumaya's theorem applies (see [2][Theorem 12.6]) and we can conclude that $L = N = \oplus_{(k,i)\in\mathbb{Z}\Delta_0} B\xi'_{(k,i)}$ for otherwise the decomposition of ${}_BL \cong {}_BB$ as a direct sum of indecomposables would contain summands with multiplicity > 1. By a symmetric argument, we get that $L = \oplus_{(k,i)\in\mathbb{Z}\Delta_0} \xi'_{(k,i)}B$.

□

Proposition 3.4.3. *Let Δ be a Dynkin quiver, let G be a weakly admissible group of automorphism of B and fix a G-invariant graded Nakayama form and its associated Nakayama automorphism η (see Theorem 2.4.2). Assume that X is the G-invariant set of arrows given in Proposition 2.3.4, which we assume to be the emptyset when $(\Delta, G) = (\mathbb{D}_4, <\rho\tau^m>)$ and with respect to which we calculate the signature of arrows. Finally, let κ and ϑ be the graded automorphisms of B which fix the vertices and act on arrows as:*

1. $\kappa(a) = -a$
2. $\vartheta(a) = (-1)^{s(\tau^{-1}(a))+s(a)}a$,

for all $a \in (\mathbb{Z}\Delta)_1$. Let us consider $\mu = \kappa \circ \eta \circ \tau^{-1} \circ \vartheta$, for any (Δ, G), or $\mu = \eta \circ \tau^{-1} \circ \vartheta$ whenever $(\Delta, G) \neq (\mathbb{A}_{2n}, <\rho\tau^m>)$. Then $\mu \circ g = g \circ \mu$, for all $g \in G$, and there exists an isomorphism of graded B-bimodules $\Omega^3_{B^e}(B) \cong {}_\mu B_1[-c_\Delta]$.

Proof. We first put $\mu = \kappa \circ \eta \circ \tau^{-1} \circ \vartheta$ in all the cases and will prove that $\Omega^3_{B^e}(B) \cong {}_\mu B_1[-c_\Delta]$, for any choice of (Δ, G). At the end, we will see that κ can be 'deleted' when $(\Delta, G) \neq (\mathbb{A}_{2n}, <\rho\tau^m>)$. Note that, for any of the choices of the set X, the sum $s(\sigma^{-1}(a)) + s(\sigma(a)) + s(\tau^{-1}(a)) + s(a)$ in $\mathbb{Z}_2$ is constant when a varies on the set of arrows ending at a given vertex $(k, i) \in (\mathbb{Z}\Delta)_0$. This implies that ϑ either preserves the relation $\sum_{t(a)=(k,i)} (-1)^{s(\sigma(a)a)}\sigma(a)a$ or multiplies it by -1. Then ϑ is a well-defined automorphism of B. Moreover, the G-invariant condition of the set of arrows X implies that the sum $s(\tau^{-1}(a)) + s(a)$ in $\mathbb{Z}_2$ is G-invariant. This shows that $\vartheta \circ g = g \circ \vartheta$, for all $g \in G$. This implies that $\mu \circ g = g \circ \mu$ since we have $\kappa \circ g = g \circ \kappa$, for all $g \in G$.

All throughout the rest of the proof, a G-invariant basis $\mathcal{B}$ of B consisting of paths in $\mathbb{Z}\Delta$ is fixed, with respect to which the $\xi'_{(k,i)}$ are calculated. We shall prove that $a\xi'_{\tau^{-1}(t(a))} = \xi'_{\tau^{-1}(i(a))}\mu(a)$, for all $a \in (\mathbb{Z}\Delta)_1$. Once this is proved, one easily shows by induction on $\deg(b)$ that if $b \in \bigcup_{(k,i),(m,j) \in (\mathbb{Z}\Delta)_0} e_{(k,i)}Be_{(m,j)}$ is a homogeneous element with respect to the length grading, then the equality $b\xi'_{\tau^{-1}(t(b))} = \xi'_{\tau^{-1}(i(b))}\mu(b)$ holds. It follows from this that the assignment $b \rightsquigarrow b\xi'_{\tau^{-1}(t(b))}$ extends to an isomorphism of B-bimodules ${}_1B_{\mu^{-1}} \xrightarrow{\cong} L$, which actually induces an isomorphism of graded B-bimodules ${}_\mu B_1[-c_\Delta] \cong \Omega^3_{B^e}(B)$, when we view $\Omega^3_{B^e}(B)$ as a graded sub-bimodule of $Q^{-2} = (\otimes_{(k,i) \in (\mathbb{Z}\Delta)_0} Be_{\tau(k,i)} \otimes e_{(k,i)}B)[-2]$.

We have an equality:

$$a\xi'_{\tau^{-1}(t(a))} = \textstyle\sum_{x \in e_{\tau^{-1}(t(a))}\mathcal{B}} (-1)^{\deg(x)} a\tau'(x) \otimes x^*.$$

But we have $\tau'(\tau^{-1}(a)) = (-1)^{s(\tau^{-1}(a))+s(a)}a$, so that

$$a\tau'(x) = (-1)^{s(\tau^{-1}(a))+s(a)}\tau'(\tau^{-1}(a))\tau'(x) = (-1)^{s(\tau^{-1}(a))+s(a)}\tau'(\tau^{-1}(a)x).$$

Note that we have $\tau^{-1}(a)x = \sum_{y \in e_{\tau^{-1}(i(a))}\mathcal{B}} (\tau^{-1}(a)x, y^*)y$ from which we get the equality

$$a\xi'_{\tau^{-1}(t(a))} = \textstyle\sum_{x \in e_{\tau^{-1}(t(a))}\mathcal{B}} \sum_{y \in e_{\tau^{-1}(i(a))}\mathcal{B}} (-1)^{\deg(x)}(-1)^{s(\tau^{-1}(a))+s(a)}(\tau^{-1}(a)x, y^*)\tau'(y) \otimes x^* \quad (!)$$

On the other hand, a direct calculation shows that, for each $a \in (\mathbb{Z}\Delta)_1$, $\mu(a) = (-1)^{1+s(\tau^{-1}(a))+s(a)}(\eta \circ \tau^{-1})(a)$. Then we have another equality

$$\xi'_{\tau^{-1}(i(a))}\mu(a) = \sum_{y \in e_{\tau^{-1}(i(a))}\mathcal{B}'} (-1)^{\deg(y)}(-1)^{s(\tau^{-1}(a))+s(a)+1}\tau'(y) \otimes y^*(\eta \circ \tau^{-1})(a).$$

But we have an equality

$$y^*(\eta \circ \tau^{-1})(a) = \sum_{x \in e_{\tau^{-1}(t(a))}\mathcal{B}}(x, y^*(\eta \circ \tau^{-1})(a))x^* = \sum_{x \in e_{\tau^{-1}(t(a))}\mathcal{B}}(xy^*, \eta(\tau^{-1}(a)))x^* = \sum_{x \in e_{\tau^{-1}(t(a))}\mathcal{B}}(\tau^{-1}(a), xy^*)x^* = \sum_{x \in e_{\tau^{-1}(t(a))}\mathcal{B}}(\tau^{-1}(a)x, y^*)x^*,$$

using that $(-,-)$ is a graded Nakayama form and that η is its associated Nakayama automorphism. We then get

$$\xi'_{\tau^{-1}(i(a))}\mu(a) = \sum_{y \in e_{\tau^{-1}(i(a))}\mathcal{B}'} \sum_{x \in e_{\tau^{-1}(t(a))}\mathcal{B}'} (-1)^{\deg(y)}(-1)^{s(\tau^{-1}(a))+s(a)+1}(\tau^{-1}(a)x, y^*)\tau'(y) \otimes x^* \qquad (!!)$$

Bearing in mind that $\deg(y) = \deg(\tau^{-1}(a)x) = \deg(x) + 1$ whenever $(\tau^{-1}(a)x, y^*) \neq 0$ we readily see that the second members of the equalities (!) and (!!) are equal. We then get $a\xi'_{\tau^{-1}(t(a))} = \xi'_{\tau^{-1}(i(a))}\mu(a)$, as desired.

Finally, suppose that $(\Delta, G) \neq (\mathbb{A}_{2n}, < \rho\tau^m >)$ we put $\mu' := \eta \circ \tau^{-1} \circ \vartheta$ and we shall define an isomorphism of bimodules $\psi : {}_{\mu'}B_1 \xrightarrow{\cong} {}_{\mu}B_1$. To do that, note that it is always possible to choose a map $\lambda : (\mathbb{Z}\Delta)_0 \longrightarrow K^*$, taking values in $\{-1, 1\}$, such that $\lambda_{i(a)} = -\lambda_{t(a)}$, for all $a \in (\mathbb{Z}\Delta)_1$ and $\lambda \circ g_{\mathbb{Z}\Delta_0} = \lambda$, for all $g \in G$. Indeed, when $\Delta \neq \mathbb{D}_{n+1}$, we define $\lambda(k, i) = (-1)^i$ for each $(k, i) \in (\mathbb{Z}\Delta)_0$. When $\Delta = \mathbb{D}_{n+1}$, we put $\lambda(k, i) = (-1)^i$, when $i \neq 0$, and $\lambda(k, 0) = -1$. With this map at hand, the map $\psi : B \longrightarrow B$ taking $b \rightsquigarrow \lambda_{i(b)}b$, for any homogeneous element $b \in \bigcup_{(k,i),(m,j) \in (\mathbb{Z}\Delta)_0} e_{(k,i)}Be_{(m,j)}$, defines the desired isomorphism $\psi : {}_{\mu'}B_1 \xrightarrow{\cong} {}_{\mu}B_1$. It is clearly an isomorphism of right B-modules and the verification that it is also a morphism of left B-modules reduces to check that $\psi(\mu'(a)b) = \mu(a)\psi(b)$, for all homogeneous elements $a, b \in \bigcup_{(k,i),(m,j) \in (\mathbb{Z}\Delta)_0} e_{(k,i)}Be_{(m,j)}$. We use the fact that $\lambda_{t(a)} = (-1)^{\deg(a)}\lambda_{i(a)}$ and $\mu(a) = (-1)^{\deg(a)}\mu'(a)$, for any such a. Assuming that $\nu(t(a)) = i(b)$, which is the only case that we need to consider, we get:

$$\psi(\mu'(a)b) = \lambda_{\nu(i(a))}\mu'(a)b = (-1)^{\deg(a)}\lambda_{\nu(t(a))}\mu'(a)b = (-1)^{\deg(a)}\lambda_{i(b)}\mu'(a)b = [(-1)^{\deg(a)}\mu'(a)] \cdot [\lambda_{i(b)}b] = \mu(a)\psi(b),$$

and the proof is finished. □

Remark 3.4.4. Note that, except when $(\Delta, G) = (\mathbb{A}_{2n-1}, \rho\tau^m)$, the automorphism ϑ of last proposition is the identity since $X = \tau(X)$.

Crucial for our goals is that what has been done in the last two propositions is 'G-invariant'. As a consequence, we obtain the aforementioned identification of the third syzygy.

Corollary 3.4.5. *Let Δ be a Dynkin quiver, B the corresponding mesh algebra, G a weakly admissible group of automorphisms of $\mathbb{Z}\Delta$ and let $\Lambda = B/G$ be the associated m-fold mesh algebra. If μ is the graded automorphism of B of the previous proposition and $\bar{\mu} : \Lambda \longrightarrow \Lambda$ is the induced graded automorphism of Λ, then there is an isomorphism of graded Λ-bimodules $\Omega^3_{\Lambda^e}(\Lambda) \cong {}_{\bar{\mu}}\Lambda_1[-c_\Delta]$, where c_Δ is the Coxeter number.*

Proof. We fix a G-invariant basis of B as in Lemma 3.4.1 and a G-invariant graded Nakayama form $(-,-) : B \times B \longrightarrow K$. If we interpret $\Lambda = B/G$ as the orbit category and $[x]$ denotes the G-orbit of x, for each $x \in \bigcup_{(k,i),(m,j)} e_{(k,i)}Be_{(m,j)}$, note that the G-orbits of elements of $\mathcal{B}$ form a basis $\bar{\mathcal{B}}$ of Λ consisting of homogeneous elements in $\bigcup_{[(k,i)],[(m,j)]\in\mathbb{Z}\Delta_0/G} e_{[(k,i)]}\Lambda e_{[(m,j)]}$. Moreover, if $\mathcal{B}^*$ is the right dual basis of $\mathcal{B}$ with respect $(-,-)$, then $\bar{\mathcal{B}}^* = \{[x^*] : [x] \in \bar{\mathcal{B}}\}$ is the right dual basis of $\bar{\mathcal{B}}$ with respect to the graded Nakayama form $< -,- >: \Lambda \times \Lambda \longrightarrow K$ induced from $(-,-)$ (see Proposition 1.4.3 and its proof).

By taking into account the change of presentation of Λ and [25][Section 4], we see that the initial part of the minimal projective resolution of Λ as a graded Λ-bimodule is of the form

$$P^{-2} \xrightarrow{\bar{R}} P^{-1} \longrightarrow P^0 \longrightarrow \Lambda \to 0,$$

where $P^{-2} = \oplus_{[(k,i)]\in\mathbb{Z}\Delta_0/G} e_{[\tau(k,i)]}\Lambda e_{[(k,i)]}$ and we have equalities $\oplus_{[(k,i)]\in\mathbb{Z}\Delta_0/G}\Lambda\bar{\xi}'_{[(k,i)]} = \text{Ker}(\bar{R}) = \oplus_{[(k,i)]\in\mathbb{Z}\Delta_0/G}\bar{\xi}'_{[(k,i)]}\Lambda$, where $\bar{\xi}'_{[(k,i)]} = \sum_{[x]\in e_{[(k,i)]}\bar{\mathcal{B}}}(-1)^{\deg(x)}[\tau'(x)] \otimes [x^*]$, for each $[(k,i)] \in \mathbb{Z}\Delta_0/G$.

On the other hand, since $\mu \circ g = g \circ \mu$, for all $g \in G$, we get an induced graded automorphism $\bar{\mu} : \Lambda \longrightarrow \Lambda$ which maps $[x] \rightsquigarrow [\mu(x)]$. In case $\mu = k \circ \eta \circ \tau^{-1} \circ \vartheta$, we get the equality $[b]\bar{\xi}'_{[\tau'(i(b))]} = \bar{\xi}'_{[\tau^{-1}(i(b))]}\bar{\mu}([b])$, for each homogeneous element $[b] \in \bigcup_{[(k,i)],[(m,j)]\in\mathbb{Z}\Delta_0/G} e_{[(k,i)]}\Lambda e_{[(m,j)]}$ from the corresponding equality in the proof of the previous proposition, just by replacing the homogeneous elements of B by their orbits. It then follows that the assignment $[b] \rightsquigarrow [b]\bar{\xi}'_{[\tau'(i(b))]}$ gives an isomorphism of graded Λ-bimodules $\Omega^3_{\Lambda^e}(\Lambda) \cong {}_1\Lambda_{\bar{\mu}^{-1}}[-c_\Delta] \cong {}_{\bar{\mu}}\Lambda_1[-c_\Delta]$.

When $(\Delta, G) \neq (\mathbb{A}_{2n}, < \rho\tau^m >)$ and we take $\mu' = \eta \circ \tau^{-1} \circ \vartheta$, we have seen in the proof of the last proposition that there is a map $\lambda : \mathbb{Z}\Delta_0 \longrightarrow K^*$ such that $\lambda \circ g_{\mathbb{Z}\Delta_0} = \lambda$, for all $g \in G$, and such that $\mu(a) = \lambda^{-1}_{i(a)}\lambda_{t(a)}\mu'(a)$, for all homogeneous elements $a \in \bigcup_{(k,i),(m,j)} e_{(k,i)}Be_{(m,j)}$. We then get from Lemma 3.2.1 that $\bar{\mu}^{-1}\bar{\mu}'$ is an inner automorphism of Λ, so that also $\Omega^3_{\Lambda^e}(\Lambda) \cong {}_{\bar{\mu}'}\Lambda_1$.

□

3.4.2 Inner and stably inner automorphisms

Recall from [51] that an automorphism σ of Λ is *stably inner* if the functor ${}_\sigma(-) \cong {}_\sigma\Lambda_1 \otimes_\Lambda - : \Lambda - \underline{\text{mod}} \longrightarrow \Lambda - \underline{\text{mod}}$ is naturally isomorphic to the identity functor. In particular, each inner automorphism is stably inner.

Lemma 3.4.6. *Let $\Lambda = KQ/I$ be a finite dimensional self-injective algebra, where I is a homogeneous ideal of KQ with respect to the grading by path length, and consider the* *induced grading on Λ. Suppose that the Loewy length of Λ is greater or equal than 4. A graded automorphism of Λ is inner if, and only if, it is stably inner.*

Proof. Let φ be a stably inner graded automorphism of Λ. Let l be the Loewy length of Λ. If $J = J(\Lambda) = J^{gr}(\Lambda)$ is the Jacobson radical and $\mathrm{Soc}^n(\Lambda) = \mathrm{Soc}^n_{gr}(\Lambda)$ is the n-socle of Λ (i.e. $\mathrm{Soc}^0(\Lambda) = 0$ and $\mathrm{Soc}^{n+1}(\Lambda)/\mathrm{Soc}^n(\Lambda)$ is the socle of $\Lambda/\mathrm{Soc}^n(\Lambda)$, for all $n \geq 0$), then we have $J^n = \mathrm{Soc}^{l-n}(\Lambda) = \oplus_{k\geq n}\Lambda_k$, for all $n \geq 0$.

We then have $\mathrm{Soc}^2(\Lambda) \subseteq J^2$ since $l \geq 4$. By Corollary 2.11 of [51], we have a map $\lambda : Q_0 \longrightarrow K^*$ such that $\varphi(a) - \lambda^{-1}_{i(a)}\lambda_{t(a)}a \in J(A)^2$, for all $a \in Q_1$. If we define $\chi_\lambda : \Lambda \longrightarrow \Lambda$ as in the proof of Lemma 3.2.1, we get that χ_λ is an inner automorphism of Λ such that $(\varphi \circ \chi_\lambda^{-1})(a) - a \in J(A)^2$, for all $a \in Q_1$. But $\varphi \circ \chi_\lambda^{-1}$ is a graded automorphism since so are φ and χ_λ. It then follows that $(\varphi \circ \chi_\lambda)(a) = a$, for all $a \in Q_1$, which implies that $\varphi \circ \chi_\lambda = id_\Lambda$, and so $\varphi = \chi_\lambda$ is inner. □

Recall that Λ is a *Nakayama algebra* if each left or right indecomposable projective Λ-module is uniserial. We will need the following properties of self-injective algebras of Loewy length 2.

Proposition 3.4.7. *Let $\Lambda = KQ/KQ_{\geq 2}$ be a self-injective algebra such that $J(\Lambda)^2 = 0$ and suppose that Λ does not have any semisimple summand as an algebra. The following assertions hold:*

1. *Λ is a Nakayama algebra and Q is a disjoint union of oriented cycles, with relations all the paths of length 2.*

2. *Λ is a finite direct product of m-fold mesh algebras of Dynkin graph $\Delta = \mathbb{A}_2$.*

3. *A graded automorphism φ of Λ is stably inner if, and only if, it fixes the vertices.*

4. *φ is inner if, and only if, it fixes the vertices and if $\varphi(a) = \chi(a)a$, for each arrow $a \in Q_1$, with $\chi(a) \in K^*$, then the induced map $\chi : Q_1 \longrightarrow K^*$ is an acyclic character of Q.*

5. *If the quiver Q is connected with n vertices (whence an oriented cycle with $Q_0 = \mathbb{Z}_n$), then $\Omega_{\Lambda^e}(\Lambda)$ is isomorphic to the Λ-bimodule ${}_{\bar{\mu}}\Lambda_1$, where $\bar{\mu}$ is the automorphism acting on vertices as the n-cycle $(12...n)$ and on arrows as $\bar{\mu}(a_i) = -a_{i+1}$, where $a_i : i \to i+1$ for each $i \in \mathbb{Z}_n$.*

Proof. Assertion 1 is folklore. But $\mathbb{A}_2^{(m)} = \mathbb{Z}\mathbb{A}_2/ < \tau^m >$ is the connected Nakayama algebra of Loewy length 2 with $2m$ vertices while $\mathbb{L}_1^{(m)} = \mathbb{Z}\mathbb{A}_2/ < \rho\tau^m >$ is the one with $2m - 1$ vertices. Then assertion 2 is clear.

The only indecomposable objects in the stable category $\Lambda - \underline{\mathrm{mod}}$ are the simple modules, all of which have endomorphism algebra isomorphic to K. It follows that each additive self-equivalence $F : \Lambda - \underline{\mathrm{mod}} \xrightarrow{\cong} \Lambda - \underline{\mathrm{mod}}$ such that $F(S) \cong S$, for each simple module S, is naturally isomorphic to the identity. Since each automorphism φ of Λ induces the self-equivalence $F = {}_\varphi(-)$, assertion 3 is clear.

Assertion 4 follows directly from [45][Theorem 12], taking into account that the only inner graded automorphism induced by an element $1-x$, with $x \in J$, is the identity (see the proof of Lemma 3.2.1).

Suppose now that Q is connected and has n vertices, so that Λ is an m-fold mesh algebra of type $\mathbb{A}_2^{(m)}$, and then $n = 2m$, or $\mathbb{L}_1^{(m)}$, and then $n = 2m-1$. By the explicit definition of the minimal projective resolution of Λ as a bimodule (see [25]), we get that $\Omega_{\Lambda^e}(\Lambda)$ is generated as a Λ-bimodule by the elements $x_i = a_i \otimes e_{i+1} - e_i \otimes a_i$ ($i \in \mathbb{Z}_n$). But we have $\oplus_{i\in\mathbb{Z}_n}\Lambda x_i = \Omega_{\Lambda^e}(\Lambda) = \oplus_{i\in\mathbb{Z}_n} x_i\Lambda$. Moreover, if $\bar{\mu}$ is the automorphism mentioned in assertion 5 and $x = \sum_{i\in\mathbb{Z}_n} x_i$, then we have $yx = x\bar{\mu}(y)$, whenever y is either a vertex or an arrow. It then follows that the assignment $y \rightsquigarrow yx$ gives an isomorphism of Λ-bimodules ${}_1\Lambda_{\bar{\mu}^{-1}} \xrightarrow{\cong} \Omega_{\Lambda^e}(\Lambda)$.

□

3.4.3 The period of an m-fold mesh algebra

This section is devoted to compute the Ω-*period* of an m-fold mesh algebra Λ. That is, the smallest of the positive integers r such that $\Omega^r_{\Lambda^e}(\Lambda)$ is isomorphic to Λ as a Λ-bimodule. We need to separate the case of Loewy length 2 from the rest.

Proposition 3.4.8. *Let Λ be a connected self-injective algebra of Loewy length 2. The following assertions hold:*

1. *If char$(K) = 2$ or $\Lambda = \mathbb{A}_2^{(m)}$, i.e. $|Q_0|$ is even, then the period of Λ is $|Q_0|$.*

2. *If char$(K) \neq 2$ and $\Lambda = \mathbb{L}_1^{(m)}$, i.e. $|Q_0|$ is odd, then the period of Λ is $2|Q_0|$.*

Proof. By Proposition 3.4.7, we know that $\Omega_{\Lambda^e}(\Lambda) \cong {}_{\bar{\mu}}\Lambda_1$, where $\bar{\mu}$ is the automorphism which acts on vertices as the n-cycle $(12...n)$ and on arrows by $a_i \rightsquigarrow -a_{i+1}$. The period of Λ is then the smallest of the integers $r > 0$ such that $\bar{\mu}^r$ is inner. But since inner automorphisms fix the vertices each such r is multiple of $n = |Q_0|$. When char$(K) = 2$ or n is even, we have that $\bar{\mu}^n$ fixes the vertices an maps $a_i \rightsquigarrow (-1)^n a_i = a_i$, for each $i \in Q_0$. Then $\bar{\mu}^n = id_\Lambda$ and the period of Λ is n. However, when char$(K) \neq 2$ and n is odd, we have that $\bar{\mu}^n$ is not inner, because the map $Q_1 \xrightarrow{\chi} K^*$ which takes constant value -1 is not an acyclic character of Q, but $\bar{\mu}^{2n} = id_\Lambda$. It follows that the period of Λ is $2n$ in this case. □

We will need the following:

Lemma 3.4.9. *Let $\Lambda = B/G$ be an m-fold mesh algebra, with $\Delta \neq \mathbb{A}_1, \mathbb{A}_2$, and let $r \geq 0$ be an integer. The following assertions hold:*

1. *$dim(\Omega^r_{\Lambda^e}(\Lambda)) = dim(\Lambda)$ if, and only if, $r \in 3\mathbb{Z}$.*

2. *If η is a G-invariant graded Nakayama automorphism of B, then $\bar{\eta} \circ \bar{\tau}^{-1} \circ \bar{\eta}^{-1} \circ \bar{\tau}$ is an inner automorphism of Λ.*

Proof. 1) The 'if' part follows from the fact that $\Omega^3_{\Lambda^e}(\Lambda) \cong {}_{\bar{\mu}}\Lambda_1$, and hence, $\Omega^{3r}_{\Lambda^e}(\Lambda) \cong {}_{\bar{\mu}^r}\Lambda_1$, as Λ-bimodules. For the 'only if' part, note that we have the following formulas for the dimensions of the syzygies:

1. $\dim(\Omega^r_{\Lambda^e}(\Lambda)) = \dim(\oplus_{i\in Q_0}\Lambda e_i \otimes e_i\Lambda) - \dim(\Lambda) = \sum_{i\in Q_0}\dim(\Lambda e_i)(\dim(e_i\Lambda) - 1)$, whenever $r \equiv 1 \pmod 3$

2. $\dim(\Omega^r_{\Lambda^e}(\Lambda)) = \dim(\oplus_{i\in Q_0}\Lambda e_{\tau(i)} \otimes e_i\Lambda) - \dim({}_{\bar{\mu}}\Lambda_1) = \sum_{i\in Q_0}\dim(\Lambda e_{\tau(i)})(\dim(e_i\Lambda) - 1) = \sum_{i\in Q_0}\dim(\Lambda e_i)(\dim(e_i\Lambda) - 1)$, whenever $r \equiv 2 \pmod 3$

For $r \equiv 1, 2 \pmod 3$ the equality $\dim(\Omega^r_{\Lambda^e}(\Lambda)) = \dim(\Lambda)$ can occur if, and only if, $\dim(e_i\Lambda) = 2$, for each $i \in Q_0$. But this can only happen when the Loewy length is 2, which is discarded (see Proposition 3.4.7).

2) There is no loss of generality in assuming that η is the G-invariant graded Nakayama automorphism of B given by Theorem 2.4.2. On the other hand, since ν is either τ^r or $\rho\tau^r$, for some integer r, we know that $\nu \circ g = g \circ \nu$, for all $g \in G$. Moreover, there is a unique map $u : \mathbb{Z}\Delta_1 \longrightarrow K^*$ such that $\chi(a) = (-1)^{u(a)}a$, for all $a \in \mathbb{Z}\Delta_1$, and $\eta = \nu \circ \chi$. It then follows that $\chi \circ g = g \circ \chi$ or, equivalently, $u(a^g) = u(a)$, for all $g \in G$.

Assertion 2 states that the images of $\bar{\eta}$ and $\bar{\tau}^{-1}$ by the canonical projection $\text{Aut}(\Lambda) \twoheadrightarrow \text{Out}(\Lambda) = \frac{\text{Aut}(\Lambda)}{\text{Inn}(\Lambda)}$ commute. Proposition 3.2.2 tells us that $\bar{\eta}$ and $\bar{\nu}$ have the same image by this projection, whenever $\text{char}(K) = 2$, $\Delta = \mathbb{A}_r$ or $m + t$ is odd, where (Δ, m, t) is the extended type of Λ. So in these cases the assertion follows immediately since ν and τ commute.

In order to prove the assertion in the remaining cases, it is enough to prove that $\bar{\chi} \circ \bar{\tau}^{-1}$ and $\bar{\tau}^{-1} \circ \bar{\chi}$ are equal, up to composition by an inner automorphism of Λ, because ν and τ^{-1} commute. Note that $\eta(a) = (\nu \circ \chi)(a) = (-1)^{u(a)}\nu(a)$, for each $a \in (\mathbb{Z}\Delta)_1$, and hence the exponents $u(a)$ are those of Theorem 2.4.2. We now apply Lemma 3.2.1, with $f = \chi \circ \tau^{-1}$ and $h = \tau^{-1} \circ \chi$, using the fact that both automorphisms of B act as τ on vertices. We have $f(a) = (-1)^{u(\tau^{-1}(a))}\tau^{-1}(a)$ and $h(a) = (-1)^{u(a)}\tau^{-1}(a)$, for each $a \in \mathbb{Z}\Delta_1$. If $\lambda : \mathbb{Z}\Delta_0 \longrightarrow K^*$ is a map such that $f(a) = \lambda^{-1}_{i(a)}\lambda_{t(a)}h(a)$, for all $a \in \mathbb{Z}\Delta_1$, then we have that $\lambda_{t(a)} = (-1)^{u(a)+u(\tau^{-1}(a))}\lambda_{i(a)}$. When $t = 1$ or $t = 3$, we have that $u(a) = u(\tau^{-1}(a))$, so that $\lambda_{t(a)} = \lambda_{i(a)}$, for all $a \in \mathbb{Z}\Delta_1$. It follows that λ is a constant map and it clearly satisfies that $\lambda \circ g_{|\mathbb{Z}\Delta_0} = \lambda$, for all $g \in G$. So we assume that $t = 2$, m is even, and also that $\Delta \neq \mathbb{A}_r$ in the sequel.

Consider first the case when $\Delta = \mathbb{D}_{n+1}$. Directly from Theorem 2.4.2 we get the formulas in $\mathbb{Z}_2$:

1. $u(a) + u(\tau^{-1})(a) = 1 + 1 = 0$, whenever $a : (k, i) \to (k, i + 1)$ is an upward arrow;

2. $u(a) + u(\tau^{-1}(a)) = 0 + 0 = 0$, whenever $a : (k, i) \to (k + 1, i - 1)$ is a downward arrow, with $k \not\equiv -2, -1 \pmod m$, and $u(a)+u(\tau^{-1}(a)) = 1$, for any other downward arrow.

3. If $i \in \{0, 1\}$, $\varepsilon_i : (k, 2) \to (k, i)$ and q is the quotient of dividing k by m, then $u(\varepsilon_i) + u(\tau^{-1}(\varepsilon_i))$ is equal to:

   (a) $(q + i) + (q + i) = 0$, when $k \not\equiv -1 \pmod m$,

(b) $(q+i)+(q+1+i)=1$, when $k \equiv -1 \pmod{m}$ since $q+1$ is the quotient of dividing $k+1$ by m

4. If $i \in \{0,1\}$, $\varepsilon'_i : (k,i) \to (k+1,2)$ and q is the quotient of dividing k by m, then $u(\varepsilon'_i)+u(\tau^{-1}(\varepsilon'_i))$ is equal to:

   (a) $(q+i+1)+(q+i+1)=0$, whenever $k \not\equiv -2,-1 \pmod{m}$;

   (b) $(q+i+1)+(q+i)=1$, whenever $k \equiv -2 \pmod{m}$;

   (c) $(q+i)+(q+1+i+1)=0$, whenever $k \equiv -1 \pmod{m}$ since $q+1$ is the quotient of dividing $k+1$ by m.

We then get that if $i \in \{2,3,...,n\}$ then $\lambda_{(k+1,i)} = \lambda_{(k,i)}$, when $k \not\equiv -2,-1 \pmod{m}$, and $\lambda_{(k+1,i)} = -\lambda_{(k,i)}$, when $k \equiv -2,-1 \pmod{m}$. For $i=0,1$ we have that $\lambda_{\rho(k,i)} = \lambda_{(k,i)}$ since the formula for $u(\varepsilon_i)+u(\tau^{-1}(\varepsilon_i))$ does not depend on i. Moreover, from the equality $\lambda_{(k+1,i)} = \lambda^{-1}_{i(\varepsilon'_i)}\lambda_{t(\varepsilon'_i)}\lambda^{-1}_{i(\varepsilon_i)}\lambda_{t(\varepsilon_i)}\lambda_{(k,i)}$ and the equalities 3 and 4 in the above list it follows $\lambda_{(k+1,i)} = \lambda_{(k,i)}$, whenever $i=0,1$. We then get $\lambda_{\rho\tau^m(k,i)} = \lambda_{(k+m,i)} = \lambda_{(k,i)}$, for all $(k,i) \in \mathbb{Z}\Delta_0$, which shows that $\lambda \circ g_{|\mathbb{Z}\Delta_0} = \lambda$, for all $g \in G$.

Let finally assume that $\Delta = \mathbb{E}_6$. The value $u(a)$ is constant on the τ-orbit of the arrow a whenever $a \in \{\alpha,\alpha',\delta,\delta',\varepsilon\}$. Then $u(a)+u(\tau^{-1}(a))=0$ in $\mathbb{Z}_2$ for any of these arrows. We easily derived from this that $\lambda_{(k+i,i)} = \lambda_{(k,i)}$, whenever $i \neq 0,3$. On the other hand, if we take $\varepsilon' : (k,0) \to (k+1,3)$, then $u(\varepsilon')+u(\tau^{-1}(\varepsilon'))=0$, when $k \not\equiv -2,-1 \pmod{m}$, and $u(\varepsilon')+u(\tau^{-1}(\varepsilon'))=1$, when $k \equiv -2 \text{ or} -1 \pmod{m}$. This together with the formula for ε imply that, for $i=0,3$, the equality $\lambda_{(k+1,i)} = \lambda_{(k,i)}$ holds whenever $k \not\equiv -2,-1 \pmod{m}$, and $\lambda_{(k+1,i)} = -\lambda_{(k,i)}$ otherwise. On the other hand, we have that $u(\gamma)+u(\tau^{-1}(\gamma))$ is equal to $q+q=0$, when $k \not\equiv -1 \pmod{m}$, and is equal to $q+(q+1)=1$, when $k \equiv -1 \pmod{m}$. We also have that $u(\beta')+u(\tau^{-1}(\beta'))$ is equal to $(q+1)+(q+1)=0$, when $k \not\equiv -1 \pmod{m}$, and is equal to $(q+1)+(q+1+1)=1$, when $k \equiv -1 \pmod{m}$. It follows that there is a exponent $e(k) \in \{0,1\}$ such that $\lambda_{(k,4)} = (-1)^{e(k)}\lambda_{(k,3)} = \lambda_{(k+1,2)}$, which shows that $\lambda_{(k,4)} = \lambda_{\rho(k,4)}$. We easily derive from this and the earlier formulas that $\lambda_{\rho(k,i)} = \lambda_{(k,i)}$, for all $(k,i) \in \mathbb{Z}\Delta_0$. We then get that $\lambda_{\rho\tau^{-m}(k,i)} = \lambda_{(k+m,i)} = \lambda_{(k,i)}$, for all $(k,i) \in \mathbb{Z}\Delta_0$, so that $\lambda \circ g_{|\mathbb{Z}\Delta_0} = \lambda$, for all $g \in G$. □

By the previous lemma, we know that $\dim(\Omega^r_{\Lambda^e}(\Lambda)) \neq \dim(\Lambda)$ whenever $r \notin 3\mathbb{Z}$. Due to the existence of an automorphism $\bar{\mu}$ of Λ satisfying that $\Omega^3_{\Lambda^e}(\Lambda) \cong_{\bar{\mu}} \Lambda_1$ as Λ-bimodules (see Proposition 3.4.3), in order to calculate the (Ω-)period of Λ, we just need to control the positive integers r such that $\bar{\mu}^r$ is inner. For the sake of simplicity, we shall divide the problem into two steps. We begin by identifying the smallest $u \in \mathbb{N}$ such that $(\bar{\nu} \circ \bar{\tau}^{-1})^u = Id_\Lambda$, that is, the smallest u such that $\bar{\mu}^u$ acts as the identity on vertices. This is the content of the next result.

Lemma 3.4.10. *Let $\Lambda = \mathbb{Z}\Delta / < \varphi >$ be an m-fold mesh algebra of extended type (Δ, m, t) and let us put $u := min\{r \in \mathbb{Z}^+ \mid (\bar{\nu} \circ \bar{\tau}^{-1})^r = Id_\Lambda\}$. The following assertions hold:*

1. *If $t = 1$ then:*

   (a) $u = \frac{2m}{gcd(m,c_\Delta)}$, whenever Δ is $\mathbb{A}_r$, $\mathbb{D}_{2r-1}$ or $\mathbb{E}_6$;

(b) $u = \frac{m}{gcd(m, \frac{c_\Delta}{2})}$, *whenever* Δ *is* $\mathbb{D}_{2r}$, $\mathbb{E}_7$ *or* $\mathbb{E}_8$.

2. *If* $t = 2$ *then:*

   (a) $u = \frac{2m}{gcd(2m, m+\frac{c_\Delta}{2})}$, *whenever* Δ *is* $\mathbb{A}_{2n-1}$, $\mathbb{D}_{2r-1}$ *or* $\mathbb{E}_6$*;*

   (b) $u = \frac{2m}{gcd(2m, \frac{c_\Delta}{2})}$, *whenever* Δ *is* $\mathbb{D}_{2r}$*;*

   (c) $u = \frac{2m-1}{gcd(2m-1, 2n+1)}$, *when* $\Delta = \mathbb{A}_{2n}$

3. *If* $t = 3$ *(hence* $\Lambda = \mathbb{Z}\mathbb{D}_4/ < \rho\tau^m >$*), then* $u = m$.

Proof. The argument that we did for ν in the first paragraph of the proof of Theorem 3.3.1 is also valid for $(\nu \circ \tau^{-1})^r$. Then $(\bar{\nu} \circ \bar{\tau}^{-1})^r = id_\Lambda$ if, and only if, $(\nu \circ \tau^{-1})^r \in G$.

When Δ is $\mathbb{A}_{2n-1}$, $\mathbb{D}_{n+1}$, with $n+1$ odd, or $\mathbb{E}_6$, the Nakayama permutation is $\nu = \rho\tau^{1-n}$, where $n = \frac{c_\Delta}{2}$. Then $(\nu\tau^{-1})^r = \rho^r\tau^{-nr}$. If $t = 1$ this automorphism is in G if, and only if, $r = 2r'$ is even and $\tau^{-nr} = \tau^{-2nr'}$ is equal to $(\tau^m)^v = \tau^{mv}$, for some $v \in \mathbb{Z}$. This happens exactly when $2nr' \in m\mathbb{Z}$ and the smallest r' satisfying this is $u' = \frac{m}{\text{gcd}(m,2n)}$. We then get that $u = 2u' = \frac{2m}{\text{gcd}(m,2n)} = \frac{2m}{\text{gcd}(m,c_\Delta)}$. Suppose that $t = 2$. Then $(\nu\tau^{-1})^r = \rho^r\tau^{-nr}$ is in $G =< \rho\tau^m >$ if, and only if, there is $v \in \mathbb{Z}$ such that $v \equiv r \pmod 2$ and $-nr = mv$. This is equivalent to saying that there is $k \in \mathbb{Z}$ such that $-nr = m(r+2k)$ or, equivalently, that $(m+n)r \in 2m\mathbb{Z}$. The smallest r satisfying this property is $u = \frac{2m}{\text{gcd}(2m,m+n)} = \frac{2m}{\text{gcd}(2m,m+\frac{c_\Delta}{2})}$. This proves 1.a, except for $\Delta = \mathbb{A}_{2n}$, and 2.a.

Suppose next that Δ is $\mathbb{D}_{n+1}$, with $n+1$ even, $\mathbb{E}_7$ or $\mathbb{E}_8$. Then $\nu = \tau^{1-n}$, where $n = \frac{c_\Delta}{2}$, so that $(\nu\tau^{-1})^r = \tau^{-nr}$. When $t = 1$, this automorphism is in $G =< \tau^m >$ if, and only if, $nr \in m\mathbb{Z}$. The smallest r satisfying this property is $u = \frac{m}{\text{gcd}(m,n)} = \frac{m}{\text{gcd}(m,\frac{c_\Delta}{2})}$. On the other hand, if $t = 2$ then τ^{-nr} is in $G =< \rho\tau^m >$ if, and only if, there is $v = 2v' \in 2\mathbb{Z}$ such that $-nr = mv = 2mv'$. The smallest r satisfying this property is $u = \frac{2m}{\text{gcd}(2m,n)} = \frac{2m}{\text{gcd}(2m,\frac{c_\Delta}{2})}$. This proves 1.b and 2.b.

Let now take $\Delta = \mathbb{A}_{2n}$. Then $\nu = \rho\tau^{1-n}$, so that $(\nu\tau^{-1})^r = \rho^r\tau^{-nr}$. If $t = 1$, this automorphism is in $G =< \tau^m >$ if, and only if, $r = 2r'$ is even and there exists $v \in \mathbb{Z}$ such that $\rho^{2r'}\tau^{-2nr'} = \tau^{-(2n+1)r'}$ is equal to τ^{mv}. This is equivalent to saying that $(2n+1)r' \in m\mathbb{Z}$. The smallest r' satisfying this property is $u' = \frac{m}{\text{gcd}(m,2n+1)}$. We then get $u = \frac{2m}{\text{gcd}(m,2n+1)} = \frac{2m}{\text{gcd}(m,c_\Delta)}$, which completes 1.a. When $t = 2$, the automorphism $\rho^r\tau^{-nr}$ is in $G =< \rho\tau^m >$ if, and only if, there exists $v \in \mathbb{Z}$ such that $v \equiv r \pmod 2$ and $\rho^r\tau^{-nr} = \rho^v\tau^{mv}$. This is in turn equivalent to the existence of an integer k such that $\rho^r\tau^{-nr} = \rho^{r+2k}\tau^{m(r+2k)} = \rho^r\tau^{-k}\tau^{mr+2mk}$. That is, if and only if $-nr = (2m-1)k + mr$. This happens exactly when $(m+n)r \in (2m-1)\mathbb{Z}$. The smallest r satisfying this property is $u = \frac{2m-1}{\text{gcd}(m+n,2m-1)}$. But we have that $\text{gcd}(m+n, 2m-1) = \text{gcd}(2m-1, 2n+1)$, so that 2.c holds.

Finally, if $t = 3$, and hence $\Delta = \mathbb{D}_4$, then $\nu = \tau^{-2}$, so that $(\nu\tau^{-1})^r = \tau^{-3r}$ is in $G =< \rho\tau^m >$ if, and only if, there is $v = 3v' \in 3\mathbb{Z}$ such that $-3r = 3mv'$. This happens exactly when $r \in m\mathbb{Z}$, which implies that $u = m$ in this case. □

Lemma 3.4.11. *Let Λ be an m-fold algebra of extended type $(\mathbb{A}_r, m, 2)$ and let T be the subgroup of $\mathbb{Z}$ consisting of the integers s such that $\bar{\mu}^s$ and $(\bar{\nu} \circ \bar{\tau}^{-1})^s$ are equal, up to composition by an inner automorphism of Λ. Then $T = 2\mathbb{Z}$, when char$(K) \neq 2$, and $T = \mathbb{Z}$, when char$(K) = 2$.*

Proof. We fix $s > 0$ all throughout the proof and will use Lemma 3.2.1, with $f = \mu^s$ and $h = (\nu \circ \tau^{-1})^s$.

Suppose first that $\Delta = \mathbb{A}_{2n}$ and let $\lambda : \mathbb{Z}\Delta_0 \longrightarrow K^*$ be any map such that $\mu^s(a) = \lambda_{i(a)}^{-1}\lambda_{t(a)}(\nu \circ \tau^{-1})^s(a)$. In this case $\mu = \kappa \circ \nu \circ \tau^{-1}$, where κ is as in Proposition 3.4.3 (see Remark 3.4.4), and this implies that $\mu^s(a) = (-1)^s(\nu \circ \tau^{-1})^s(a)$, for each $a \in \mathbb{Z}\Delta_1$. We then get that $\lambda_{i(a)}^{-1}\lambda_{t(a)} = (-1)^s$. It follows that $\lambda_{(k,i)} = (-1)^s\lambda_{(k,j)}$, whenever $i \not\equiv j$ (mod 2), and that $\lambda_{\tau(k,i)} = \lambda_{(k+1,i)} = (-1)^{2s}\lambda_{(k,i)} = \lambda_{(k,i)}$, for all $(k,i) \in \mathbb{Z}\Delta_0$. We then get that $\lambda_{\rho\tau^m(k,i)} = \lambda_{\rho(k-m,i)} = \lambda_{(k-m+i-n,2n+1-i)} = (-1)^s\lambda_{(k-m+i-n,i)} = (-1)^s\lambda_{(k,i)}$. As a consequence the equality $\lambda \circ g_{|\mathbb{Z}\Delta_0} = \lambda$ holds, for all $g \in G =< \rho\tau^m >$, if and only if $s \in 2\mathbb{Z}$. That is, we have $T = 2\mathbb{Z}$ in this case.

Suppose next that $\Delta = \mathbb{A}_{2n-1}$, we have $\eta = \nu = \rho\tau^{1-n}$ and $\mu = \eta\circ\tau^{-1}\circ\vartheta = \nu\circ\tau^{-1}\circ\vartheta = \rho\circ\tau^{-n}\circ\vartheta$. We also have $(\nu\circ\tau^{-1})^s = \nu^s\circ\tau^{-s} = (\rho\tau^{1-n})^s\tau^{-s} = \rho^s\circ\tau^{-ns}$. Let us fix from now on a map $\lambda : \mathbb{Z}\Delta_0 \longrightarrow K^*$ such that $(\rho\tau^{-n}\vartheta)^s(a) = \lambda_{i(a)}^{-1}\lambda_{t(a)}\rho^s\tau^{-ns}(a)$.

We first consider the case when m is odd. By the choice of the set X which defines the signature of the arrows (see Proposition 2.3.4), we know that $s(\rho\tau(a)) = s(a)$, for each $a \in (\mathbb{Z}\Delta_1)$. On the other hand, we have an equality $s(\tau^{-1}(a)) + s(a) = s(\tau^{-2}(a)) + s(\tau^{-1}(a))$ in $\mathbb{Z}_2$. These two facts imply that ϑ commute both with $\rho\tau^{-1}$ and τ^{-1}. Therefore we have $\mu^s = (\rho\tau^{-n}\vartheta)^s = \rho^s\tau^{-ns}\vartheta^s$. It follows that $(-1)^{s[s(a)+s(\tau^{-1}(a))]}\rho^s\tau^{-ns}(a) = (\rho^s\tau^{-ns}\vartheta^s)(a) = \mu^s(a) = \lambda_{i(a)}^{-1}\lambda_{t(a)}\rho^s\tau^{-ns}(a)$, so that $\lambda_{t(a)} = (-1)^{s[s(a)+s(\tau^{-1}(a))]}\lambda_{i(a)}$, for all $a \in \mathbb{Z}\Delta_1$. If $(k,i) \in \mathbb{Z}\Delta_0$ and a is any arrow such that $i(a) = (k,i)$, then $\lambda_{(k+1,i)} = (-1)^{s[s(a)+s(\tau^{-1}(a))+s(\sigma^{-1}(a))+s(\tau^{-1}\sigma^{-1}(a))]}\lambda_{(k,i)} = (-1)^s\lambda_{(k,i)}$ since, by the choice of X, we have $s(a) + s(\tau^{-1}(a)) + s(\sigma^{-1}(a)) + s(\tau^{-1}\sigma^{-1}(a)) = 1$, for any $a \in \mathbb{Z}\Delta_1$. Moreover, for each arrow a which is either upward in the 'north hemisphere' or downward in the 'south hemisphere', we have that $s(a)+s(\tau^{-1}(a)) = 0$, and this implies that $\lambda_{(k,n)} = \lambda_{(k,i)}$, for all $i \geq n$, and $\lambda_{(k+j,n-j)} = \lambda_{(k,n)}$, for each $0 \leq j < n$. It follows that $\lambda_{\rho(k,i)} = \lambda_{(k,i)}$, for all $(k,i) \in \mathbb{Z}\Delta_0$. We then get that $\lambda_{\rho\tau^{-m}(k,i)} = \lambda_{(k+m,i)} = (-1)^{sm}\lambda_{(k,i)}$. The equality $\lambda \circ g_{|\mathbb{Z}\Delta_0} = \lambda$, for all $g \in G$, holds in this case if, and only if, s is even. That is, when m is odd, we have $T = 2\mathbb{Z}$.

Suppose now that m is even. Due to the choice of the set of arrows X which defines the signature map (see Proposition 2.3.4), if Y denotes the set of arrows a such that $i(a) \neq (k,n)$ and $t(a) \neq (k,n)$, for all $k \in \mathbb{Z}$, then we know that $s(a) = s(\tau^{-2}(a))$ and $s(a) = s(\rho(a))$, for all $a \in Y$. As a consequence, for all $a \in Y$, we have equalities

$$(\tau^{-1} \circ \vartheta)(a) = (-1)^{s(a)+s(\tau^{-1}(a))}\tau^{-1}(a) = (-1)^{s(\tau^{-1}(a))+s(\tau^{-2}(a))}\tau^{-1}(a) = (\vartheta \circ \tau^{-1})(a);$$
$$(\rho \circ \vartheta)(a) = (-1)^{s(a)+s(\tau^{-1}(a))}\rho(a) = (-1)^{s(\rho(a))+s(\tau^{-1}(\rho(a))}\rho(a) = (\vartheta \circ \rho)(a),$$

It then follows that

$$\mu^s(a) = (\rho^s\tau^{-ns}\vartheta^s)(a) = (-1)^{s[s(a)+s(\tau^{-1}(a))]}(\rho^s\tau^{-ns}(a)) = (-1)^{s[s(a)+s(\tau^{-1}(a))]}(\nu\circ\tau^{-1})^s(a),$$

for each $a \in Y$. We then get that, for all $a \in Y$, $\lambda_{t(a)} = (-1)^{s[s(a)+s(\tau^{-1}(a))]}\lambda_{i(a)}$. If (k,i) is a vertex, with $i \neq n$, and $a \in Y$ is any arrow having (k,i) as its origin, then $\lambda_{(k+1,i)} = (-1)^{s[s(a)+s(\tau^{-1}(a))+s(\sigma^{-1}(a))+s(\tau^{-1}\sigma^{-1}(a))]}\lambda_{(k,i)} = (-1)^s\lambda_{(k,i)}$ since $s(a) + s(\tau^{-1}(a)) + s(\sigma^{-1}(a)) + s(\tau^{-1}\sigma^{-1}(a)) = 1$ in $\mathbb{Z}_2$, for any arrow $a \in Y$.

In order to deal with the vertices (k,n), it is convenient to introduce some terminology. The *k-th node* will consist of the vertex (k,n) and the four arrows having it as origin or terminus. As usual, we will denote by q and r the quotient and rest of dividing k by m. Note that if k is odd, then none of the four arrows in the node is in X. On the contrary, if k is even either the two upward arrows are in X or the two downward ones are in X, and exactly one of these two possibilities occurs. One then sees that if a has origin (k,n), then $s(a) + s(\tau^{-1}(a)) + s(\sigma^{-1}(a)) + s(\tau^{-1}\sigma^{-1}(a)) = 1$, unless $k \equiv -2 \pmod{m}$, a case in which $s(a) + s(\tau^{-1}(a)) + s(\sigma^{-1}(a)) + s(\tau^{-1}\sigma^{-1}(a)) = 0$. This implies that if we take $s = 1$ and λ is the associated map in this case, then $\lambda_{(k+1,n)} = -\lambda_{(k,n)}$, when $k \not\equiv -2 \pmod{m}$, and $\lambda_{(k+1,n)} = \lambda_{(k,n)}$ otherwise. It follows then that $\lambda_{\rho\tau^m(k,n)} = \lambda_{\tau^m(k,n)} = \lambda_{(k-m,n)} = (-1)^{m-1}\lambda_{(k,n)} = -\lambda_{(k,n)}$ and hence 1 is not in the subgroup T.

If $s = 2$ and a is again an arrow in the k-th node, it is convenient to rewrite the formula for $\vartheta(a)$ as follows:

i) $\vartheta(a) = (-1)^{q+1}a$, if a is upward and $k \not\equiv -1 \pmod{m}$ or a is downward and $k \equiv -1 \pmod{m}$.

ii) $\vartheta(a) = (-1)^{q}a$, if a is upward and $k \equiv -1 \pmod{m}$ or a is downward and $k \not\equiv -1 \pmod{m}$.

From these equalities we then get:

iii) $(\rho\tau^{-n}\vartheta)(a) = (-1)^{q+1}(\rho\tau^{-n})(a)$, if a is upward and $k \not\equiv -1 \pmod{m}$ or a is downward and $k \equiv -1 \pmod{m}$.

iv) $(\rho\tau^{-n}\vartheta)(a) = (-1)^{q}(\rho\tau^{-n})(a)$, if a is upward and $k \equiv -1 \pmod{m}$ or a is downward and $k \not\equiv -1 \pmod{m}$.

In order to calculate the $(\rho\tau^{-n}\vartheta)^2(a)$, for any integer r, we will denote by $q(r)$ the quotient of dividing r by m. We also put $c(r) = 0$, when $r \not\equiv -1 \pmod{m}$, and $c(r) = 1$, when $r \equiv -1 \pmod{m}$. Direct calculation, using the formulas iii) and iv) above, gives that $\mu^2(a) = (\rho\tau^{-n}\vartheta)^2(a) = (-1)^{e(a)}(\rho\tau^{-n})^2(a) = (-1)^{e(a)}(\nu\circ\tau^{-1})^2(a)$, where:

v) $e(a) = (q(k)+1)+q(k+n)+c(k)+c(k+n)$, when a is upward and $k \not\equiv -1 \pmod{m}$ or a is downward and $k \equiv -1 \pmod{m}$.

vi) $e(a) = (q(k) + (q(k+n)+1) + c(k) + c(k+n)$, when a is downward and $k \not\equiv -1 \pmod{m}$ or a is upward and $k \equiv -1 \pmod{m}$.

Therefore the exponent $e(a)$ only depends on k and we put $e(k) = e(a)$. We then get that $(\rho\tau^{-n}\theta)^2(a) = (-1)^{e(k)}\tau^{-2n}(a)$, for all arrows a in the k-th node. If $\lambda : \mathbb{Z}\Delta_0 \longrightarrow K^*$ is the associated map for $s = 2$, we get that $\lambda_{(k+1,n)} = (-1)^{e(k)+e(k+1)}\lambda_{(k,n)}$, for each $k \in \mathbb{Z}$. Note that $e(k)+e(k+1) = [q(k)+q(k+n)+q(k+1)+q(k+1+n)] + [c(k)+c(k+n) + c(k+1) + c(k+1+n)] + 2$ and that, for each integer r, one has $\sum_{0\leq j<m} c(r+j) = 1$. It follows that $\sum_{0\leq j<m}[e(k+j) + e(k+j+1)] = \sum_{0\leq j<m}[q(k+j) + q(k+1+j)] +$

$\sum_{0\leq j<m}[q(k+j+n)+q(k+1+j+n)]$ in $\mathbb{Z}_2$. But we always have $q(r)=q(r+1)$, unless $r\equiv -1 \pmod m$, a case in which $q(r+1)=q(r)+1$. It follows that the equality $\sum_{0\leq j<m}[q(k+j)+q(k+1+j)]=1=\sum_{0\leq j<m}[q(k+j+n)+q(k+1+j+n)]$, and hence also $\sum_{0\leq j<m}[e(k+j)+e(k+j+1)]=0$, is true in $\mathbb{Z}_2$. As a consequence, we have that $\lambda_{\rho\tau^{-m}(k,n)}=\lambda_{(k+m,n)}=(-1)^{\sum_{0\leq j<m}[e(k+j)+e(k+j+1)]}\lambda_{(k,n)}=\lambda_{(k,n)}$. It follows that λ is a constant map, so that the equality $\lambda\circ g_{\mathbb{Z}\Delta_0}=\lambda$ holds, for all $g\in G$. Therefore $s=2$ is in the subgroup T.

□

We are now ready to describe explicitly the period of any m-fold mesh algebra.

Theorem 3.4.12. *Let Λ be an m-fold mesh algebra of extended type (Δ,m,t), where $\Delta\neq\mathbb{A}_1,\mathbb{A}_2$, let $\pi=\pi(\Lambda)$ denote the period of Λ and, for each positive integer k, denote by $O_2(k)$ the biggest of the natural numbers r such that 2^r divides k. If $char(K)=2$ then $\pi=3u$, where u is the positive integer of Lemma 3.4.10. When $char(K)\neq 2$, the period of Λ is given as follows:*

1. *If $t=1$ then:*
   - (a) *When Δ is $\mathbb{A}_r$, $\mathbb{D}_{2r-1}$ or $\mathbb{E}_6$, the period is $\pi=\frac{6m}{gcd(m,c_\Delta)}$.*
   - (b) *When Δ is $\mathbb{D}_{2r}$, $\mathbb{E}_7$ or $\mathbb{E}_8$, the period is $\pi=\frac{3m}{gcd(m,\frac{c_\Delta}{2})}$, when m is even, and $\pi=\frac{6m}{gcd(m,\frac{c_\Delta}{2})}$, when m is odd.*
2. *If $t=2$ then:*
   - (a) *When Δ is $\mathbb{A}_{2n-1}$, $\mathbb{D}_{2r-1}$ or $\mathbb{E}_6$, the period is $\frac{6m}{gcd(2m,m+\frac{c_\Delta}{2})}$, when $O_2(m)\neq O_2(\frac{c_\Delta}{2})$, and $\pi=\frac{12m}{gcd(2m,m+\frac{c_\Delta}{2})}$ otherwise.*
   - (b) *When $\Delta=\mathbb{D}_{2r}$, the period is $\frac{6m}{gcd(2m,\frac{c_\Delta}{2})}=\frac{6m}{gcd(2m,2r-1)}$.*
   - (c) *When $\Delta=\mathbb{A}_{2n}$, i.e. $\Lambda=\mathbb{L}_n^{(m)}$, the period is $\pi=\frac{6(2m-1)}{gcd(2m-1,2n+1)}$*
3. *If $t=3$ then $\pi=3m$, when m is even, and $6m$, when m is odd.*

Proof. Let $u>0$ be the integer of Lemma 3.4.10. Then $u\mathbb{Z}$ consists of the integers r such that $\bar{\nu}^r=\bar{\tau}^r$, or equivalently $(\bar{\nu}\circ\bar{\tau}^{-1})^r=id_\Lambda$, as automorphisms of Λ. If π is the period of Λ, then, by Lemma 3.4.9, we know that $\pi=3v$, where v is the smallest of the positive integers s such that $\bar{\mu}^s\in\text{Inn}(\Lambda)$. These integers s obviously form a subgroup $S=S(\Delta,m,t)$ of $\mathbb{Z}$, and then $v\mathbb{Z}=S$. This subgroup is the intersection of $u\mathbb{Z}$ with the subgroup T consisting of the integers r such that $\bar{\mu}^r$ and $(\bar{\nu}\circ\bar{\tau}^{-1})^r$ are equal, up to composition by an inner automorphism of Λ. When $(\Delta,m,t)=(\mathbb{A}_r,m,2)$, by Lemma 3.4.11, we get that $v\mathbb{Z}=u\mathbb{Z}\cap 2\mathbb{Z}$, when $\text{char}(K)\neq 2$, and $v\mathbb{Z}=u\mathbb{Z}\cap\mathbb{Z}=u\mathbb{Z}$, when $\text{char}(K)=2$. This automatically gives 2.c and the part of characteristic 2 in this case. We claim that it also gives the formula in 2.a for $\Delta=\mathbb{A}_{2n-1}$. Indeed, by Lemma 3.4.10, we have $u=\frac{2m}{\text{gcd}(2m,m+n)}$ in this case. But the biggest power of 2 which divides $2m$ is a divisor of $\gcd(2m, m+n)$ if, and only if, $O_2(m) = O_2(n)$. Then the equality 2.a for $\mathbb{A}_{2n-1}$ follows automatically.

When $(\Delta, m, t) \neq (\mathbb{A}_r, m, 2)$, by Proposition 3.4.3 and the subsequent remark, we can take $\bar{\mu} = \bar{\eta} \circ \bar{\tau}^{-1}$. Then condition 2 of Lemma 3.4.9 implies that S consists of the integers s such that $\bar{\eta}^s$ and $\bar{\tau}^s$ are equal, up to composition by an inner automorphism of Λ. We then get that $S = u\mathbb{Z} \cap H(\Delta, m, t)$ (see Proposition 3.2.2). Therefore Proposition 3.2.2 tells us that $v = u$, when either $H(\Delta, m, t) = \mathbb{Z}$ or u is even, and $v = 2u$ otherwise. We next check that this fact together with Proposition 3.2.2 give all the remaining formulas of the theorem and, obviously, it completes the assertion for characteristic 2.

For the quivers Δ in 1.a we always have that $H(\Delta, m, t) = \mathbb{Z}$ when $\Delta = \mathbb{A}_r$, and also in the other two cases when m is even. But if m is odd then automatically $u = \frac{2m}{\gcd(m, c_\Delta)}$ is even.

For the quivers in 1.b, we always have that $n = \frac{c_\Delta}{2}$ is odd. Therefore u is even exactly when m is even.

For the quivers in 2.a which are not $\mathbb{A}_{2n-1}$, we have that $H(\Delta, m, t) = \mathbb{Z}$ exactly when m is odd. But $\frac{c_\Delta}{2}$ is even, so that $O_2(m) \neq O_2(\frac{c_\Delta}{2})$ in that case. As we did above in the case $(\Delta, m, t) = (\mathbb{A}_{2n-1}, m, 2)$, in case m even, we have that $u = \frac{2m}{\gcd(2m, m + \frac{c_\Delta}{2})}$ is odd if, and only if, $O_2(m) = O_2(\frac{c_\Delta}{2})$. Then the formula in 2.a is true also for the cases different from $\mathbb{A}_{2n-1}$.

For 2.b, we have that $\frac{c_\Delta}{2}$ is odd, which implies that u is always even, and then the formula in 2.b is true.

Finally, when $t = 3$, we have that $H(\Delta, m, t) = \mathbb{Z}$, exactly when m is even, and then the formula in 3) is automatic.

□

3.4.4 The stable Calabi-Yau dimension of an m-fold mesh algebra

In case Λ is a self-injective algebra, the Auslander formula (see [8], Chapter IV, Section 4) says that one has a natural isomorphism $D\underline{\mathrm{Hom}}_\Lambda(X, -) \cong \mathrm{Ext}^1_\Lambda(-, \tau X)$, where $\tau : {}_\Lambda\underline{\mathrm{mod}} \longrightarrow {}_\Lambda\underline{\mathrm{mod}}$ is the Auslander-Reiten (AR) translation. Moreover, $\tau = \Omega^2\mathcal{N}$, where $\mathcal{N} = DHom_\Lambda(-, \Lambda) \cong D(\Lambda) \otimes_\Lambda - : {}_\Lambda\mathrm{mod} \longrightarrow {}_\Lambda\mathrm{mod}$ is the Nakayama functor (see [8]). Due to the fact that $\mathrm{Ext}^1_\Lambda(-, Y)$ and $\underline{\mathrm{Hom}}_\Lambda(\Omega(-), Y)$ are naturally isomorphic functors ${}_\Lambda\underline{\mathrm{mod}} \longrightarrow {}_\Lambda\underline{\mathrm{mod}}$ we have that $D\underline{\mathrm{Hom}}_\Lambda(X, -) \cong \underline{\mathrm{Hom}}_\Lambda(-, \Omega\mathcal{N}(X))$. Since $\Sigma = \Omega^{-1}$ is the translation functor of ${}_\Lambda\underline{\mathrm{mod}}$ as a triangulated category we conclude that $\Sigma^n = \Omega^{-n}$ is a Serre functor for ${}_\Lambda\underline{\mathrm{mod}}$ if, and only if, $\underline{\mathrm{Hom}}_\Lambda(-, \Omega\mathcal{N}(X)) \cong \underline{\mathrm{Hom}}_\Lambda(-, \Omega^{-n}(X))$. By Yoneda's Lemma, this is in turn equivalent to saying that $\mathcal{N}(X) \cong \Omega^{-(n+1)}(X)$, via isomorphisms which are natural on X. Bearing in mind that $\mathcal{N} \cong D(\Lambda) \otimes_\Lambda - \cong {}_{\bar{\eta}^{-1}}\Lambda_1 \otimes_\Lambda - \cong {}_{\bar{\eta}^{-1}}(-)$ as shown in [28], we derive that the stable category ${}_\Lambda\underline{\mathrm{mod}}$ has CY-dimension m if and only if m is the smallest natural number such that $\Omega_\Lambda^{-m-1} \cong \mathcal{N} \cong {}_{\bar{\eta}^{-1}}(-)$ (equivalently, $\Omega_\Lambda^{m+1} \cong {}_{\bar{\eta}}(-)$) as triangulated functors ${}_\Lambda\underline{\mathrm{mod}} \longrightarrow {}_\Lambda\underline{\mathrm{mod}}$, where $\bar{\eta}$ is the Nakayama automorphism of Λ. We shall say that Λ is *stably Calabi-Yau* when $\Lambda - \underline{\mathrm{mod}}$ is a Calabi-Yau triangulated category. The minimal number m mentioned above will be then called the *stable Calabi-Yau dimension* of Λ and denoted $CY - \dim(\Lambda)$.

Due to the fact the functor $\Omega^d_\Lambda : \Lambda - \underline{\mathrm{mod}} \longrightarrow \Lambda - \underline{\mathrm{mod}}$ is naturally isomorphic to the functor $\Omega^d_{\Lambda^e}(\Lambda) \otimes_\Lambda -$, for all integers d, a sufficient condition for Λ to be stably Calabi-

Yau is that $\Omega_{\Lambda^e}^{d+1}(\Lambda) \cong {}_{\bar{\eta}}\Lambda_1$ as Λ-bimodules. An algebra satisfying this last condition is called *Calabi-Yau Frobenius* in [35] and the minimal d satisfying this property is called the *Calabi-Yau Frobenius dimension* of Λ. We will denote it here by $CYF - \dim(\Lambda)$. We always have $CY - \dim(\Lambda) \leq CYF - \dim(\Lambda)$, but, in general, it is not known if equality holds. We discuss now this problem for m-fold mesh algebras.

Note that, by [51][Theorem 1.8], the functor $\Omega_{\Lambda}^{k+1} : \Lambda - \underline{\text{mod}} \longrightarrow \Lambda - \underline{\text{mod}}$ is naturally isomorphic to ${}_{\bar{\eta}}(-) : \Lambda - \underline{\text{mod}} \longrightarrow \Lambda - \underline{\text{mod}}$ if, and only if, $\Omega_{\Lambda^e}^{k+1}(\Lambda)$ and ${}_{\varphi\bar{\eta}}\Lambda_1$ are isomorphic Λ-bimodules, for some stably inner automorphism φ of Λ.

We are now able to calculate the stable and Frobenius Calabi-Yau dimension of self-injective algebras of Loewy length 2.

Proposition 3.4.13. *Let Λ be a connected self-injective algebra of Loewy length 2. Then Λ is always a stably Calabi-Yau algebra and the following equalities hold:*

1. *If char$(K) = 2$ or $\Lambda = \mathbb{A}_2^{(m)}$, i.e. $|Q_0|$ is even, then $CY - dim(\Lambda) = CYF - dim(\Lambda) = 0$.*

2. *If char$(K) \neq 2$ and $\Lambda = \mathbb{L}_1^{(m)}$, i.e., $|Q_0|$ odd, then $CY - dim(\Lambda) = 0$ and $CYF - dim(\Lambda) = 2m - 1 = |Q_0|$.*

Proof. By Proposition 3.4.7, we know that $\Omega_{\Lambda^e}^{k+1}(\Lambda)$ is isomorphic to ${}_{\bar{\mu}^{k+1}}\Lambda_1$, for each $k \geq 0$. Then $CY - \dim(\Lambda)$ is the smallest of the natural numbers k such that $\bar{\mu}^{k+1}\bar{\eta}^{-1}$ is stably inner, which is equivalent to saying that $\bar{\mu}^{k+1}\bar{\eta}^{-1}$ fixes the vertices. Similarly, $CYF - \dim(\Lambda)$ is the smallest of the k such that $\bar{\mu}^{k+1}\bar{\eta}^{-1}$ is inner. Due to the fact that Λ is an m-fold mesh algebra of type $\mathbb{A}_2$, a (graded) Nakayama automorphism of Λ is $\nu = \rho\tau^{1-1} = \rho$ (see Theorem 2.4.2 and Proposition 2.3.1). It follows that the graded Nakayama automorphism $\bar{\eta}$ of Λ maps $i \rightsquigarrow i + 1$ and $a_i \rightsquigarrow a_{i+1}$, when we identify $Q_0 = \mathbb{Z}_n$. It follows that $\bar{\mu}\bar{\eta}^{-1}$ fixes the vertices and, hence, it is stably inner. This shows that $CY - \dim(\Lambda) = 0$.

More generally, $\bar{\mu}^{k+1}\bar{\eta}^{-1}$ fixes the vertices if, and only if, $i + k + 1 \equiv i + 1 \pmod{n}$, for each $i \in \mathbb{Z}_n$. That is, if and only if $k \in n\mathbb{Z}$. Suppose that this property holds and consider the map $\chi : Q_1 \longrightarrow K^*$ taking constant value $(-1)^{k+1}$. We clearly have $\bar{\mu}^{k+1}\bar{\eta}^{-1}(a_i) = (-1)^{k+1}a_i = \chi(a_i)a_i$, for each $i \in \mathbb{Z}_n$. But χ is an acyclic character if, and only if, either char$(K) = 2$ or $\prod_{1 \leq i \leq n} \chi(a_i) = (-1)^{(k+1)n}$ is equal to 1. So, when char$(K) = 2$, the automorphism $\bar{\mu}^{k+1}\bar{\eta}^{-1}$ is inner for any value of k. In particular, $CYF - \dim(\Lambda) = 0$ in such case.

Suppose that char$(K) \neq 2$. By Proposition 3.4.7, we get that $\bar{\mu}^{k+1}\bar{\eta}^{-1}$ is an inner automorphism if, and only if, $(k + 1)n$ is even. This is always the case when n is even, and in such case $CYF - \dim(\Lambda) = 0$. If $n = 2m - 1$ is odd then $k + 1$ should be even and the smallest $k \in n\mathbb{Z}$ satisfying this property is $k = n$. Then $CYF - \dim(\Lambda) = n = 2m - 1$ in this case.

□

As regards the algebras with Loewy length greater o equal than 4, we also have:

Proposition 3.4.14. *Let Λ be an m-fold mesh algebra of Dynkin type Δ different from $\mathbb{A}_r$, for $r = 1, 2, 3$. Then Λ is stably Calabi-Yau if, and only if, it is Calabi-Yau Frobenius. In such case the equality $CY - dim(\Lambda) = CYF - dim(\Lambda)$ holds.*

Proof. By Corollary 2.3.2, we know that the Loewy length of Λ is $c_\Delta - 1$, where c_Δ is the Coxeter number. The Dynkin graphs $\Delta = \mathbb{A}_r$, with $r = 1, 2, 3$, are the only ones for which $c_\Delta - 1 \leq 3$. So Λ has Loewy length ≥ 4 in our case. Note that if $\Omega^{k+1}_{\Lambda^e}(\Lambda)$ is isomorphic to a twisted bimodule ${}_{\varphi}\Lambda_1$, then we have $\dim(\Omega^{k+1}_{\Lambda^e}(\Lambda)) = \dim(\Lambda)$. By Lemma 3.4.9, we know that then $k + 1 \in 3\mathbb{Z}$.

If there is a k such that $\Omega^{k+1}_{\Lambda^e}(\Lambda) \cong {}_{\varphi\bar{\eta}}\Lambda_1$, for some inner or stably inner automorphism φ, then $k = 3s - 1$, for some integer $s > 0$. But we know that $\Omega^3_{\Lambda^e}(\Lambda) \cong {}_{\bar{\mu}}\Lambda_1$, where $\bar{\mu}$ is a graded automorphism of Λ. We then have that $\Omega^{3s}_{\Lambda^e}(\Lambda) \cong {}_{\varphi\bar{\eta}}\Lambda_1$, for some stably inner (resp. inner) automorphism φ if, and only if, $\bar{\mu}^s\bar{\eta}^{-1}$ is a stably inner (resp. inner) automorphism of Λ. The proof is finished using Lemma 3.4.6 since $\bar{\mu}^s\bar{\eta}^{-1}$ is a graded automorphism. □

The proof of last proposition shows that if Λ is not of type $\mathbb{A}_r$ ($r = 1, 2$), then the algebra Λ will be stably Calabi-Yau (resp. Calabi-Yau Frobenius) if, and only if, there exists an integer $s > 0$ such that $\bar{\mu}^s\bar{\eta}^{-1}$ is stably inner (resp. inner). A necessary condition for this is that $\bar{\mu}^s\bar{\eta}^{-1}$ fixes the vertices. So, as a first step to characterize the stably Calabi-Yau (resp. Calabi-Yau Frobenius) condition of Λ, we shall identify the positive integers s such that $\bar{\mu}^s$ and $\bar{\eta}$ have the same action on vertices.

Definition 16. Let Λ be an m-fold mesh algebra of type $\Delta \neq \mathbb{A}_1, \mathbb{A}_2$, with quiver Q. We will define the following sets of positive integers:

1. $\mathbb{N}_{CY}(\Lambda)$ consists of the integers $s > 0$ such that $\bar{\mu}^s$ and $\bar{\eta}$ have the same action on vertices.

2. $\hat{\mathbb{N}}_{CY}(\Lambda)$ consists of the integers $s > 0$ such that $\bar{\mu}^s\bar{\eta}^{-1}$ is an inner automorphism. Equivalently, it is the set of integers $s > 0$ such that $\Omega^{3s}_{\Lambda^e}(\Lambda)$ is isomorphic to ${}_{\bar{\eta}}\Lambda_1$ as a Λ-bimodule.

Remark 3.4.15. Under the hypotheses of last definition, we clearly have $\hat{\mathbb{N}}_{CY}(\Lambda) \subseteq \mathbb{N}_{CY}(\Lambda)$. Moreover Λ is Calabi-Yau Frobenius if, and only if, $\hat{\mathbb{N}}_{CY}(\Lambda) \neq \emptyset$. In this latter case we have $CYF - \dim(\Lambda) = 3r - 1$, where $r = \min(\hat{\mathbb{N}}_{CY}(\Lambda))$, and this number is equal to $CY - \dim(\Lambda)$ when $\Delta \neq \mathbb{A}_3$. Note also that if $\hat{\mathbb{N}}_{CY}(\Lambda) = \mathbb{N}_{CY}(\Lambda) \neq \emptyset$ then $CY - \dim(\Lambda) = CYF - \dim(\Lambda)$ since the fact that $\bar{\mu}^s\bar{\eta}^{-1}$ be stably inner implies that $s \in \mathbb{N}_{CY}(\Lambda)$.

We first identify $\mathbb{N}_{CY}(\Lambda)$ for any m-fold mesh algebra of Loewy length > 2.

Proposition 3.4.16. *Let Λ be an m-fold mesh algebra of extended type (Δ, m, t), where $\Delta \neq \mathbb{A}_1, \mathbb{A}_2$. The following assertions hold:*

1. *When $t = 1$, the set $\mathbb{N}_{CY}(\Lambda)$ is nonempty if, and only if, the following condition is true in each case:*

(a) *gcd*$(m, c_\Delta) = 1$, *when* Δ *is* $\mathbb{A}_r$, $\mathbb{D}_{2r-1}$ *or* $\mathbb{E}_6$. *Then* $\mathbb{N}_{CY}(\Lambda) = \{s = 2s' + 1, s' > 0 : c_\Delta s' \equiv -1 \pmod m\}$

(b) *gcd*$(m, \frac{c_\Delta}{2}) = 1$, *when* Δ *is* $\mathbb{D}_{2r}$, $\mathbb{E}_7$ *or* $\mathbb{E}_8$. *Then* $\mathbb{N}_{CY}(\Lambda) = \{s > 0 : \frac{c_\Delta}{2}(s-1) \equiv -1 \pmod m\}$.

2. *When* $t = 2$, *the set* $\mathbb{N}_{CY}(\Lambda)$ *is nonempty if, and only if, the following condition is true in each case:*

(a) *gcd*$(2m, m + \frac{c_\Delta}{2}) = 1$, *when* Δ *is* $\mathbb{A}_{2n-1}$, $\mathbb{D}_{2r-1}$ *or* $\mathbb{E}_6$. *Then* $\mathbb{N}_{CY}(\Lambda) = \{s > 0 : (m + \frac{c_\Delta}{2})(s-1) \equiv -1 \pmod{2m}\}$, *and this set consists of even numbers.*

(b) *gcd*$(m, \frac{c_\Delta}{2}) =$ *gcd*$(m, , 2r-1) = 1$, *when* $\Delta = \mathbb{D}_{2r}$. *Then* $\mathbb{N}_{CY}(\Lambda) = \{s > 0 : (2r-1)(s-1) \equiv -1 \pmod{2m}\}$ *and this set consists of even numbers.*

(c) *gcd*$(2m-1, 2n+1) = 1$, *when* $\Delta = \mathbb{A}_{2n}$. *Then* $\mathbb{N}_{CY}(\Lambda) = \{s > 0 : (m+n)(s-1) \equiv -1 \pmod{2m-1}\}$.

3. *If* $t = 3$ *(and hence* $\Delta = \mathbb{D}_4$*), then* $\mathbb{N}_{CY}(\Lambda) = \emptyset$.

Proof. Note that $\bar{\mu}$ acts on vertices as $\bar{\nu}\bar{\tau}^{-1}$, where ν is the Nakayama permutation and τ the Auslander-Reiten translation of B. Viewing the vertices of the quiver of Λ as G-orbits of vertices in $\mathbb{Z}\Delta$, we get that s is in $\mathbb{N}_{CY}(\Lambda)$ if, and only if, $(\bar{\nu}\bar{\tau}^{-1})^s([(k,i)]) = \bar{\nu}([(k,i)])$, equivalently $\bar{\nu}^{s-1}\bar{\tau}^{-s}([(k,i)]) = [(k,i)]$, for each G-orbit $[(k,i)]$. Now the argument in the first paragraph of the proof of Theorem 3.3.1 can be applied to the automorphism $\nu^{s-1}\tau^{-s}$. We then get that $s \in \mathbb{N}_{CY}(\Lambda)$ if, and only if, $\nu^{s-1}\tau^{-s} \in G$. We use this to identify the set $\mathbb{N}_{CY}(\Lambda)$ for all possible extended types, and the result will be derived from that.

If $t = 3$ and so $\Delta = \mathbb{D}_4$, then we know that $\nu = \tau^{-2}$. It follows that $s \in \mathbb{N}_{CY}(\Lambda)$ if, and only if, $\tau^{-2(s-1)}\tau^{-s} = (\rho\tau^m)^q$, for some $q \in \mathbb{Z}$, where ρ is the automorphism of order 3 of $\mathbb{D}_4$. By the free action of the group $< \rho, \tau >$ on vertices not fixed by ρ, necessarily $q \in 3\mathbb{Z}$ and $2 - 3s = mq$, which is absurd. Then assertion 3 follows.

Suppose first that $\Delta \neq \mathbb{A}_{2n}$. If Δ is $\mathbb{A}_{2n-1}$, $\mathbb{D}_{2r-1}$ or $\mathbb{E}_6$, then $\nu = \rho\tau^{1-n}$, where $n = \frac{c_\Delta}{2}$. Then $\nu^{s-1}\tau^{-s} = \rho^{s-1}\tau^{(1-n)(s-1)}\tau^{-s} = \rho^{s-1}\tau^{-[n(s-1)+1]}$. When $t = 1$, we have that $G =< \tau^m >$ and, hence, the automorphism $\nu^{s-1}\tau^{-s}$ is in G if, and only if, there is $q \in \mathbb{Z}$ such that $\rho^{s-1}\tau^{-[n(s-1)+1]} = (\tau^m)^q$. This happens if, and only if, $s - 1 = 2s'$ is even and there is $q \in \mathbb{Z}$ such that $-2ns' - 1 = -n(s-1) - 1$ is equal to mq. Therefore s exists if, and only if, $\gcd(m, c_\Delta) = \gcd(m, 2n) = 1$. In this case $\mathbb{N}_{CY}(\Lambda) = \{s = 2s' + 1 > 0 : 2ns' \equiv -1 \pmod m\} = \{s = 2s' + 1 : c_\Delta s' \equiv -1 \pmod m)\}$, which gives 1.a, except for the case $\Delta = \mathbb{A}_{2n}$. On the other hand, if $t = 2$, and hence $G =< \rho\tau^m >$, then the automorphism $\nu^{s-1}\tau^{-s}$ is in G if, and only if, there is an integer q such that $q \equiv s - 1 \pmod 2$ and $\rho^{s-1}\tau^{-[n(s-1)+1]} = \rho^q\tau^{mq}$ or, equivalently, $-n(s-1) - 1 = mq$. But this happens if, and only if, there is $k \in \mathbb{Z}$ such that $-n(s-1) - 1 = m(s - 1 + 2k)$, which is equivalent to saying that $(m+n)(s-1) + 2mk + 1 = 0$. Therefore s exists if, and only if, $\gcd(2m, m + \frac{c_\Delta}{2}) = \gcd(2m, m+n) = 1$. In this case $\mathbb{N}_{CY}(\Lambda) = \{s > 0 : (m + \frac{c_\Delta}{2})(s-1) \equiv -1 \pmod{2m}\}$ and this proves 2.a.

Suppose next that Δ is $\mathbb{D}_{2r}$, $\mathbb{E}_7$ or $\mathbb{E}_8$, so that $\nu = \tau^{1-n}$, where $n = \frac{c_\Delta}{2}$. Then $\nu^{s-1}\tau^{-s} = \tau^{(1-n)(s-1)}\tau^{-s} = \tau^{-[n(s-1)+1]}$. When $t = 1$, this automorphism is in $G =< \tau^m >$ if, and only if, there is $q \in \mathbb{Z}$ such that $-n(s-1) - 1 = mq$. Then s exists if, and only if, $\gcd(m, \frac{c_\Delta}{2}) = \gcd(m,n) = 1$. In this case $\mathbb{N}_{CY}(\Lambda) = \{s > 0 : \frac{c_\Delta}{2}(s-1) \equiv -1 \pmod{m}\}$, which proves 1.b. When $t = 2$, whence $\Delta = \mathbb{D}_{2r}$, the automorphism $\nu^{s-1}\tau^{-s}$ is in $G =<\rho\tau^m>$ if, and only if, there is an even integer $q = 2q'$ such that $-n(s-1)-1 = 2mq'$. Then s exists if, and only if, $\gcd(2m,n) = 1$. But $n = 2r-1$ is odd in this case. Then $\gcd(2m,n) = 1$ if, and only if, $\gcd(m, 2r-1) = \gcd(m,n) = 1$. On the other hand, note that $s-1$ is necessarily odd, which implies that $\mathbb{N}_{CY}(\Lambda) \subset 2\mathbb{Z}$. This completes the proof of 2.b.

Suppose now that $\Delta = \mathbb{A}_{2n}$, so that $\rho^2 = \tau^{-1}$. Here $\nu = \rho\tau^{1-n}$ and $\nu^{s-1}\tau^{-s} = \rho^{s-1}\tau^{(1-n)(s-1)-s} = \rho^{s-1}\tau^{-[n(s-1)+1]}$. When $t = 1$, this automorphism is in $G =<\tau^m>$ if, and only if, $s-1 = 2s'$ is even and $\tau^{-s'}\tau^{-(2ns'+1)} = (\tau^m)^q$, for some integer q. That is, s exists if, and only if, there are $s' \geq 0$ and $q \in \mathbb{Z}$ such that $mq + (2n+1)s' + 1 = 0$. Therefore s exists if, and only if, $\gcd(m, c_\Delta) = \gcd(m, 2n+1) = 1$. In this case $s = 2s'+1$, where $s' \geq 0$ and $c_\Delta s' = (2n+1)s' \equiv -1 \pmod{m}$. This completes 1.a. When $t = 2$ the automorphism $\nu^{s-1}\tau^{-s}$ is in $G =<\rho\tau^m>$ if, and only if, there is $q \in \mathbb{Z}$ such that $q \equiv s-1 \pmod{2}$ and $\rho^{s-1}\tau^{-[n(s-1)+1]} = \rho^q\tau^{mq}$. This is equivalent to the existence of an integer k such that $\rho^{s-1}\tau^{-[n(s-1)+1]} = \rho^{s-1+2k}\tau^{m(s-1+2k)}$. Canceling ρ^{s-1}, we see that the condition is equivalent to the existence of an integer k such that $-n(s-1)-1 = m(s-1)+(2m-1)k$ or, equivalently, such that $(m+n)(s-1)+(2m-1)k+1 = 0$. Then s exists if, and only if, $\gcd(m+n, 2m-1) = 1$, which is turn equivalent to saying that $\gcd(2m-1, 2n+1) = 1$ since $(2m-1)+(2n+1) = 2(m+n)$. This proves 2.c and the proof is complete. □

We now want to identify $\hat{\mathbb{N}}_{CY}(\Lambda)$. The following is our crucial tool.

Lemma 3.4.17. *Let Δ be a Dynkin quiver different from $\mathbb{A}_1, \mathbb{A}_2$, B be its associated mesh algebra, $\Lambda = B/G$ be an m-fold mesh algebra of extended type (Δ, m, t) and let η be a G-invariant graded Nakayama automorphism of B. If s is an integer in $\mathbb{N}_{CY}(\Lambda)$, then the following assertions are equivalent:*

1. *s is in $\hat{\mathbb{N}}_{CY}(\Lambda)$ (see definition 16).*
2. *There is a map $\lambda : \mathbb{Z}\Delta_0 \longrightarrow K^*$ such that:*
   (a) *$\mu^s(a) = \lambda_{i(a)}^{-1}\lambda_{t(a)}\eta(\nu^{s-1}\tau^{-s}(a))$, for all $a \in (\mathbb{Z}\Delta)_1$, where μ is the graded automorphism of Proposition 3.4.3.*
   (b) *$\lambda \circ g_{|\mathbb{Z}\Delta_0} = \lambda$, for all $g \in G$.*

   If $(\Delta, m, t) \neq (\mathbb{A}_{2n-1}, m, 2)$, then these conditions are also equivalent to:
3. *There is a map $\lambda : \mathbb{Z}\Delta_0 \longrightarrow K^*$ satisfying condition 2.b and such that $(-1)^s\eta^{s-1}(a) = \lambda_{i(a)}^{-1}\lambda_{t(a)}\nu^{s-1}(a)$, for all $a \in (\mathbb{Z}\Delta)_1$.*

   If $(\Delta, t) \neq (\mathbb{A}_r, 2)$ then the conditions are also equivalent to
4. *$s-1$ is in $H(\Delta, m, t)$ (see Proposition 3.2.2).*

Proof. The first paragraph of the proof of Proposition 3.4.16 says that $s \in \mathbb{N}_{CY}(\Lambda)$ if, and only if, $\nu^{s-1}\tau^{-s} \in G$. The goal is to give necessary and sufficient conditions on such an integer s so that $\bar{\mu}^s$ and $\bar{\eta} = \overline{\eta\nu^{s-1}\tau^{-s}}$ are equal, up to composition by an inner automorphism of Λ. But the actions of $\mu^s = (k \circ \eta \circ \tau^{-1} \circ \vartheta)^s$ and $\eta \circ \nu^{s-1} \circ \tau^{-s}$ on $(\mathbb{Z}\Delta)_0$ are equal. By Lemma 3.2.1, we then get that assertions 1 and 2 are equivalent.

When $(\Delta, m, t) \neq (\mathbb{A}_{2n-1}, m, 2)$, what we know is that $\vartheta = id_B$ and, by Lemma 3.4.9, we know that $\bar{\eta}$ and $\bar{\tau}^{-1}$ commute, up to composition by an inner automorphism of Λ. Then s is in $\hat{\mathbb{N}}_{CY}(\Lambda)$ if, and only if, $\bar{k}^s\bar{\eta}^s\bar{\tau}^{-s}$ and $\bar{\eta}\bar{\nu}^{s-1}\bar{\tau}^{-s}$ are equal up to composition by an inner automorphism of Λ. By Lemma 3.2.1, this last condition is equivalent to saying that there is a map $\lambda : \mathbb{Z}\Delta_0 \longrightarrow K^*$ satisfying 2.b such that $(-1)^s\eta^s(\tau^{-s}(a)) = \lambda_{i(a)}^{-1}\lambda_{t(a)}\eta(\nu^{s-1}\tau^{-s}(a))$, for each $a \in (\mathbb{Z}\Delta)_1$. Putting $b = \tau^{-s}(a)$ and defining $\tilde{\lambda} : (\mathbb{Z}\Delta)_0 \longrightarrow K^*$ by the rule $\tilde{\lambda}(i) = \lambda(\tau^s(i))$, we get that $(-1)^s\eta^{s-1}(b) = \tilde{\lambda}_{i(b)}^{-1}\tilde{\lambda}_{t(b)}\nu^{s-1}(b)$, for all $b \in (\mathbb{Z}\Delta)_1$. Then assertions 2 and 3 are equivalent.

Finally, when $(\Delta, t) \neq (\mathbb{A}_r, 2)$, Proposition 3.4.3 says that we can choose $\mu = \eta \circ \tau^{-1}$ since ϑ is the identity map. Then the proof of the equivalence of assertions 2 and 3, taken for $\kappa = id_B$, shows that assertion 2 holds if, and only if, there is a map $\lambda : \mathbb{Z}\Delta_0 \longrightarrow K^*$ satisfying condition 2.b and such that $\eta^{s-1}(b) = \lambda_{i(b)}^{-1}\lambda_{t(b)}\nu^{s-1}(b)$, for all $b \in (\mathbb{Z}\Delta)_1$. This equivalent to saying that $s - 1 \in H(\Delta, m, t)$. □

The following is now a consequence of Proposition 3.4.16 and the foregoing lemma.

Corollary 3.4.18. *Let Λ be an m-fold mesh algebra over a field of characteristic 2, with $\Delta \neq \mathbb{A}_1$. The algebra is stably Calabi-Yau if, and only if, it is Calabi-Yau Frobenius. When in addition $\Delta \neq \mathbb{A}_2$, this is in turn equivalent to saying that $\mathbb{N}_{CY}(\Lambda) \neq \emptyset$. Moreover, the following assertions hold:*

1. *When the Loewy length of Λ is ≤ 2, i.e. $\Delta = \mathbb{A}_2$, the algebra is always Calabi-Yau Frobenius and $CY - dim(\Lambda) = CYF - dim(\Lambda) = 0$.*

2. *When $\Delta \neq \mathbb{A}_2$, we have $CY - dim(\Lambda) = CYF - dim(\Lambda) = 3m - 1$, where $m = min(\mathbb{N}_{CY}(\Lambda))$ (see Proposition 3.4.16).*

Proof. The case of Loewy length 2 is covered by Proposition 3.4.13. So we assume $\Delta \neq \mathbb{A}_2$ in the sequel. If Λ is stably Calabi-Yau, then $\mathbb{N}_{CY}(\Lambda) \neq \emptyset$. But, when char$(K) = 2$, the G-invariant graded Nakayama automorphism of Theorem 2.4.2 is $\eta = \nu$. In addition, the automorphisms ϑ and κ of Proposition 3.4.3 are the identity. Then, in order to prove the equality $\hat{\mathbb{N}}_{CY}(\Lambda) = \mathbb{N}_{CY}(\Lambda)$, one only need to prove that if $s \in \mathbb{N}_{CY}(\Lambda)$ then condition 2 of last lemma holds. But this is clear, by taking as λ any constant map. □

We are now ready to give, for Char$(K) \neq 2$, the precise criterion for an m-fold mesh algebra to be stably Calabi-Yau, and to calculate $CY - \dim(\Lambda)$ in that case.

Theorem 3.4.19. *Let us assume that char$(K) \neq 2$ and let Λ be the m-fold mesh algebra of extended type (Δ, m, t), where $\Delta \neq \mathbb{A}_1, \mathbb{A}_2$. We adopt the convention that if a, b, k are fixed integers, then $au \equiv b \pmod{k}$ means that u is the smallest positive integer satisfying the congruence. The algebra is Calabi-Yau Frobenius if, and only if, it is stably Calabi-Yau. Moreover, we have $CYF - dim(\Lambda) = CY - dim(\Lambda)$ and the following assertions hold:*

1. *If $t = 1$ then*

(a) *When* Δ *is* $\mathbb{A}_r$, $\mathbb{D}_{2r-1}$ *or* $\mathbb{E}_6$, *the algebra is stably Calabi-Yau if, and only if,* $gcd(m, c_\Delta) = 1$. *Then* $CY - dim(\Lambda) = 6u + 2$, *where* $c_\Delta u \equiv -1 \pmod{m}$.

(b) *When* Δ *is* $\mathbb{D}_{2r}$, $\mathbb{E}_7$ *or* $\mathbb{E}_8$, *the algebra is stably Calabi-Yau if, and only if,* $gcd(m, \frac{c_\Delta}{2}) = 1$. *Then:*

   i. $CY - dim(\Lambda) = 3u + 2$, *where* $\frac{c_\Delta}{2} u \equiv -1 \pmod{m}$, *whenever* m *is even;*

   ii. $CY - dim(\Lambda) = 6u + 2$, *where* $c_\Delta u \equiv -1 \pmod{m}$, *whenever* m *is odd;*

2. *If* $t = 2$ *then*

(a) *When* Δ *is* $\mathbb{A}_{2n-1}$, $\mathbb{D}_{2r-1}$ *or* $\mathbb{E}_6$, *the algebra is stably Calabi-Yau if, and only if,* $gcd(2m, m + \frac{c_\Delta}{2}) = 1$. *Then* $CY - dim(\Lambda) = 3u + 2$, *where* $(m + \frac{c_\Delta}{2})u \equiv -1 \pmod{2m}$.

(b) *When* $\Delta = \mathbb{D}_{2r}$, *the algebra is stably Calabi-Yau if, and only if,* $gcd(m, 2r-1) = 1$ *and* m *is odd. Then* $CY - dim(\Lambda) = 3u+2$, *where* $(2r-1)u \equiv -1 \pmod{2m}$.

(c) *When* $\Delta = \mathbb{A}_{2n}$, *the algebra is stably Calabi-Yau if, and only if,* $gcd(2m-1, 2n+1) = 1$. *Then* $CY - dim(\Lambda) = 6u-1$, *where* $(m+n)(2u-1) \equiv -1 \pmod{2m-1}$

3. *If* $t = 3$ *then the algebra is not stably Calabi-Yau.*

Proof. By Proposition 3.4.14, we know that, when $\Delta \neq \mathbb{A}_3$, the algebra Λ is stably Calabi-Yau if, and only if, it is Calabi-Yau Frobenius and the corresponding dimensions are equal. From our arguments below it will follow that, when $\Delta = \mathbb{A}_3$, we always have $\hat{\mathbb{N}}_{CY}(\Lambda) = \mathbb{N}_{CY}(\Lambda)$, and then $CY - \dim(\Lambda) = CYF - \dim(\Lambda)$ also in this case (see Remark 3.4.15).

Our arguments will give an explicit identification of $\hat{\mathbb{N}}_{CY}(\Lambda)$ in terms of $\mathbb{N}_{CY}(\Lambda)$. Then $CY - \dim(\Lambda)$ will be $3v - 1$, where $v = \min(\hat{\mathbb{N}}_{CY}(\Lambda))$.

From Propositions 3.4.16 and 3.4.14, we know that, when $t = 3$, the algebra is never stably Calabi-Yau. So we assume in the sequel that $t \neq 3$.

Suppose first that $(\Delta, m, t) \neq (\mathbb{A}_r, m, 2)$. Then Lemma 3.4.17 tells us that $\hat{\mathbb{N}}_{CY}(\Lambda) = \mathbb{N}_{CY}(\Lambda) \cap (H(\Delta, m, t) + 1)$, where $H(\Delta, m, t) + 1 = \{s \in \mathbb{Z} : s - 1 \in H := H(\Delta, m, t)\}$. By Proposition 3.2.2, we get in these cases that the equality $\hat{\mathbb{N}}_{CY}(\Lambda) = \mathbb{N}_{CY}(\Lambda)$ holds whenever $m + t$ is odd. We now examine the different cases:

1.a) If $\Delta = \mathbb{A}_r$ then $H = \mathbb{Z}$. When Δ is $\mathbb{D}_{2r-1}$ or $\mathbb{E}_6$, the Coxeter number c_Δ is even. If $\mathbb{N}_{CY}(\Lambda) \neq \emptyset$ then $\gcd(m, c_\Delta) = 1$, so that m is odd and $H = 2\mathbb{Z}$. But then $\hat{\mathbb{N}}_{CY}(\Lambda) = \mathbb{N}_{CY}(\Lambda) \cap (2\mathbb{Z} + 1)$, which is equal to $\mathbb{N}_{CY}(\Lambda)$ due to Proposition 3.4.16. So Λ is stably Calabi-Yau if, and only if, $\gcd(m, c_\Delta) = 1$. Then $CY - \dim(\Lambda) = 3(2u + 1) - 1 = 6u + 2$, where $2u + 1 = \min(\mathbb{N}_{CY}(\Lambda))$.

1.b) We need to consider the case when m is odd. In this case $\hat{\mathbb{N}}_{CY}(\Lambda) = \mathbb{N}_{CY}(\Lambda) \cap (2\mathbb{Z} + 1)$ is properly contained in $\mathbb{N}_{CY}(\Lambda)$. However, we claim that if $\mathbb{N}_{CY}(\Lambda) \neq \emptyset$ then $\hat{\mathbb{N}}_{CY}(\Lambda) \neq \emptyset$, which will prove that Λ is stably Calabi-Yau if, and only if, $\gcd(m, \frac{c_\Delta}{2}) = 1$ using Proposition 3.4.16. Indeed, we need to prove that if $\gcd(m, \frac{c_\Delta}{2}) = 1$, then there is an integer $u' \geq 0$ such that $2u' + 1 \in \mathbb{N}_{CY}(\Lambda)$ or, equivalently, that $\frac{c_\Delta}{2}(2u' + 1 - 1) \equiv -1 \pmod{m}$. But this is clear for if m is odd then also $\gcd(m, c_\Delta) = 1$. Now the formulas in 1.b.i) an 1.b.ii) come directly from putting $s = u + 1$ and $s = 2u + 1$ and use the fact that $\frac{c_\Delta}{2}(s - 1) \equiv -1 \pmod{m}$.

2.a) Suppose first that Δ is $\mathbb{D}_{2r-1}$ or $\mathbb{E}_6$. In this case $\frac{c_\Delta}{2}$ is even. Then $\gcd(2m, m+\frac{c_\Delta}{2}) = 1$ implies that m is odd and, hence, that $H = \mathbb{Z}$. So in this case $\hat{\mathbb{N}}_{CY}(\Lambda) = \mathbb{N}_{CY}(\Lambda)$ and the formula for $CY - \dim(\Lambda)$ comes from putting $s = 1 + u$, where $(m + \frac{c_\Delta}{2})u \equiv -1 \pmod{2m}$.

Suppose next that $(\Delta, m, t) = (\mathbb{A}_{2n-1}, m, 2)$, i.e. $\Lambda = \mathbb{B}_n^{(m)}$. Here $\eta = \nu$. Then condition 2 of Lemma 3.4.17 can be rephrased by saying that $\bar{\mu}^s$ and $(\bar{\nu}\circ\bar{\tau}^{-1})^s$ are equal, up to composition by an inner automorphism of Λ. This proves that $\hat{\mathbb{N}}_{CY}(\Lambda) = \mathbb{N}_{CY}(\Lambda) \cap 2\mathbb{Z}$ due to Lemma 3.4.11. But Proposition 3.4.16 tells us that then $\hat{\mathbb{N}}_{CY}(\Lambda) = \mathbb{N}_{CY}(\Lambda)$. The formula for $CY - \dim(\Lambda)$ is calculated as in the other two quivers of 2.a.

2.b) If $\mathbb{N}_{CY}(\Lambda) \neq \emptyset$ then $\gcd(m, 2r-1) = 1$. If m is odd then $H = \mathbb{Z}$. If m is even, then $H = 2\mathbb{Z}$ which implies that $\hat{\mathbb{N}}_{CY}(\Lambda) = \mathbb{N}_{CY}(\Lambda) \cap (2\mathbb{Z}+1)$. But this is the empty set due to Proposition 3.4.16. The formula for $CY - \dim(\Lambda)$ in the case when m is odd follows again from putting $s - 1 = u$ and $(2r-1)u \equiv -1 \pmod{2m}$.

2.c) It remains to consider the case $(\Delta, m, t) = (\mathbb{A}_{2n}, m, 2)$, i.e. $\Lambda = \mathbb{L}_n^{(m)}$. We use condition 3 of Lemma 3.4.17. If $\lambda : \mathbb{Z}\Delta_0 \longrightarrow K^*$ is any map such that $(-1)^s\eta^{s-1}(a) = \lambda_{i(a)}^{-1}\lambda_{t(a)}\nu^{s-1}(a)$, then $\lambda_{i(a)}^{-1}\lambda_{t(a)} = (-1)^s$ since $\eta^{s-1}(a) = \nu^{s-1}(a)$, for all $a \in (\mathbb{Z}\Delta)_1$. It follows that $\lambda_{(k,i)} = (-1)^s\lambda_{(k,j)}$, whenever $i \not\equiv j \pmod 2$, and that $\lambda_{\tau(k,i)} = \lambda_{(k+1,i)} = (-1)^{2s}\lambda_{(k,i)} = \lambda_{(k,i)}$, for all $(k,i) \in \mathbb{Z}\Delta_0$. We then get that $\lambda_{\rho\tau^m(k,i)} = \lambda_{\rho(k+m,i)} = \lambda_{(k+m+i-n,2n+1-i)} = (-1)^s\lambda_{(k,i)}$. As a consequence the equality $\lambda \circ g_{|\mathbb{Z}\Delta_0} = \lambda$ holds, for all $g \in G = <\rho\tau^m>$, if and only if $s \in 2\mathbb{Z}$. It follows that $\hat{\mathbb{N}}_{CY}(\Lambda) = \mathbb{N}_{CY}(\Lambda) \cap 2\mathbb{Z}$. We claim that if $\mathbb{N}_{CY}(\Lambda) \neq \emptyset$ then $\hat{\mathbb{N}}_{CY}(\Lambda) \neq \emptyset$, which implies that Λ is stably Calabi-Yau exactly when $\gcd(2m-1, 2n+1) = 1$, using Proposition 3.4.16. Indeed, using the description of this last proposition, we need to see that the diophantic equation $(m+n)(2x-1)+(2m-1)y+1$ has a solution. But this is clear since $\gcd(2(m+n), 2m-1) = 1$. The formula for $CY - \dim(\Lambda)$ is now clear. □

Chapter 4

Hochschild cohomology of finite dimensional self-injective algebras

4.1 Introduction

4.1.1 Motivation

The aim of this chapter is to gather the background on algebras over a commutative ring and finite dimensional self-injective algebras that will be needed in the subsequent chapters. Some of the contents concerning self-injective finite dimensional algebras can be directly derived from the results given in Chapter 1 for pseudo-Frobenius graded algebras by considering the grading group $H = 0$. Also, we introduce the notion and essential properties of the Hochschild cohomology ring.

4.1.2 Outline of the chapter

In Section 4.2 we remind the definition of the Yoneda product of extensions and, particularly, we introduce the notion of the Hochschild cohomology ring, denoted by $HH^*(\Lambda)$, of an algebra Λ over a commutative ring R. Then, in Section 4.3, and whenever the algebra is in addition Gorenstein projective as Λ-bimodule, we include the definition of the stable Hochschild cohomology ring, $\underline{HH}^*(\Lambda)$. In Section 4.4 we describe the structure as a bigraded ($\mathbb{Z}\times\mathbb{Z}$-graded) algebra induced by the canonical homological grading as well as by what we call the length grading on Λ. As regards Section 4.5, we show that if Λ is a periodic Frobenius algebra, in the sense of [35], then the stable Hochschild cohomology ring is a graded-commutative which turns out to be a localization of the classical one. Finally, in Section 4.6 we revisit some results concerning self-injective finite dimensional algebras and we also introduce the concept of dualizable basis and give conditions for its existence.

4.1.3 Notation

Throughout this chapter we fix a commutative ground ring R and all algebras are taken over R. For simplicity, we will assume in addition that these algebras are projective as

R-modules, although this assumption is not necessary in some of the statements. Later in the chapter R will be a field and, in order to emphasize that, we will put $R = K$ in that case.

The prototypical example of algebras that we should keep in mind is that of an algebra given by quiver and relations. Suppose that Q is a quiver. If Q_0 is finite, then the associated path algebra RQ is unital and it is free as an R-module. We are interested in quotients $\Lambda = RQ/I$ where I is a two-sided ideal of RQ contained in the (free) R-submodule $RQ_{\geq 2}$ of RQ generated by the paths of length ≥ 1, and having a direct sum complement in RQ as an R-module. We shall only consider quotients of path algebras of this form. Note that $1 = \sum_{i \in Q_0} e_i$, where e_i is the idempotent associated to i.

The convention and notation of the Subsection 1.1.3 are also followed here, but with K replaced by R. In particular, unadorned tensor products are taken over R.

4.2 The Yoneda product of extensions

For the convenience of the reader we recall the definition of $HH^*(\Lambda)$ and of the Yoneda product. By the classical theory of derived functors, for each pair M, N of Λ-modules, one can compute the R-module $\mathrm{Ext}^n_\Lambda(M, N)$ as the n-th cohomology R-module of the complex $Hom_\Lambda(P^\bullet, N)$, where

$$P^\bullet : ...P^{-n-1} \overset{d^{-n-1}}{\longrightarrow} P^{-n} \longrightarrow ... \longrightarrow P^{-1} \overset{d^{-1}}{\longrightarrow} P^0 \twoheadrightarrow M \to 0$$

is a projective resolution of M.

Suppose that L, M, N are Λ-modules, that $P^\bullet$ and $Q^\bullet$ are projective resolutions of L and M, respectively, and that m, n are natural numbers. If $\delta \in \mathrm{Ext}^n_\Lambda(L, M)$ and $\epsilon \in \mathrm{Ext}^m_\Lambda(M, N)$, then we can choose a $\tilde{\delta} \in Hom_\Lambda(P^{-n}, M)$, belonging to the kernel of the transpose map $(d^{-n-1})^* : Hom_\Lambda(P^{-n}, M) \longrightarrow Hom_\Lambda(P^{-n-1}, M)$ of the differential $d^{-n-1} : P^{-n-1} \longrightarrow P^{-n}$ of $P^\bullet$, which represents δ. Similarly, we can choose an $\tilde{\epsilon} \in Hom_\Lambda(Q_m, N)$ which represents ϵ. Due to the projectivity of the P^i, there is a (non-unique) sequence of morphisms of Λ-modules $\delta^{-k} : P^{-n-k} \longrightarrow Q^{-k}$ ($k = 0, 1, ...,$) making the following diagram commute:

$$\begin{array}{ccccccccccccccc}
\cdots \longrightarrow & P^{-n-k} & \longrightarrow \cdots \longrightarrow & P^{-n-1} & \longrightarrow & P^{-n} & & \longrightarrow \cdots \longrightarrow & P^0 & \longrightarrow & L & \longrightarrow & 0 \\
& \downarrow{\scriptstyle \delta^{-k}} & & \downarrow{\scriptstyle \delta^{-1}} & & \downarrow{\scriptstyle \delta^0} & \searrow{\scriptstyle \tilde{\delta}} & & & & & & \\
\cdots \longrightarrow & Q^{-k} & \longrightarrow \cdots \longrightarrow & Q^{-1} & \longrightarrow & Q^0 & \longrightarrow & M & \longrightarrow & 0 & & &
\end{array}$$

Then the composition $\tilde{\epsilon} \circ \delta^{-m} : P^{-m-n} \longrightarrow N$ is in the kernel of $(d^{-m-n})^*$ and, thus, it represents an element of $\mathrm{Ext}^{m+n}_\Lambda(L, N)$. This element is denoted by $\epsilon\delta$ and does not depend on the choices made. It is called the *Yoneda product* of ϵ and δ. It is well-known that the map

$$\mathrm{Ext}^m_\Lambda(M, N) \times \mathrm{Ext}^n_\Lambda(L, M) \longrightarrow \mathrm{Ext}^{m+n}_\Lambda(L, N) \qquad ((\epsilon, \delta) \mapsto \epsilon\delta)$$

is R-bilinear.

When $M = N$ in the above setting, the R-module $\mathrm{Ext}^*_\Lambda(M,M) = \oplus_{i\geq 0}\mathrm{Ext}^i_\Lambda(M,M)$ inherits a structure of graded R-algebra, where the multiplication of homogeneous elements is the Yoneda product. In this thesis we are interested in the particular case where Λ is replaced by Λ^e and M is replaced by Λ, viewed as Λ^e-module (i.e. as a Λ-bimodule). Then $HH^i(\Lambda) := \mathrm{Ext}^i_{\Lambda^e}(\Lambda,\Lambda)$ is called the *i-th Hochschild cohomology R-module*, for each $i \geq 0$. The corresponding graded algebra $Ext^*_{\Lambda^e}(\Lambda,\Lambda)$ is denoted by $HH^*(\Lambda)$ and called the *Hochschild cohomology ring (or algebra)* of Λ. By a celebrated result of Gerstenhaber ([43]), we know that $HH^*(\Lambda)$ is graded commutative. That is, if $\epsilon \in HH^i(\Lambda)$ and $\delta \in HH^j(\Lambda)$ are homogeneous elements, then $\epsilon\delta = (-1)^{ij}\delta\epsilon$.

4.3 Stable and absolute Hochschild (co)homology

In this section, we recall some concepts which allow us to extend Hochschild homology and cohomology to negative degrees.

Definition 17. Let A be an algebra and let $\mathcal{C}A$ denote the category of (cochain) complexes of A-modules. An object $X^\bullet$ of $\mathcal{C}A$ is called a *totally acyclic complex* when it is acyclic and the complex of R-modules $\mathrm{Hom}_A(X^\bullet, Q)$ is acyclic, for each projective A-module Q.

An A-module M is said to be *Gorenstein projective* (see [26]) when there is a totally acyclic complex

$$P^\bullet : \cdots \longrightarrow P^{-2} \xrightarrow{d^{-2}} P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{d^0} P^1 \xrightarrow{d^1} P^2 \longrightarrow \cdots$$

in $\mathcal{C}A$ such that each P^n is a projective A-module and $Z^1 := \mathrm{Ker}(d^1) = M$. In that case $P^\bullet$ is a called a *complete projective resolution* of M and its module of $(1-n)$-cocycles Z^{1-n} will be called the *Gorenstein n-syzygy* of M and denoted by $\Omega^n_A(M)$, for each $n \in \mathbb{Z}$. It is uniquely determined, up to isomorphism, in the stable category modulo projectives and, for $n \geq 0$, it coincides with the usual syzygy.

We summarize in the following proposition a few known properties of Gorenstein projective modules (see [9][Section 4]).

Proposition 4.3.1. *Let A be an algebra. The following assertions hold:*

1. *The complete projective resolution of a Gorenstein projective A-module is unique, up to isomorphism, in the homotopy category $\mathcal{H}A$*

2. *The full subcategory $A - GProj$ of ${}_A Mod$ consisting of the Gorenstein projective A-modules is a Frobenius exact category (in the sense of Quillen) on which the injective (=projective) objects are the projective A-modules*

3. *The stable category $A - \underline{GProj} = \frac{A-GProj}{{}_A Proj}$ is a triangulated category on which the suspension functor takes M to $\Omega^{-1}_A M$.*

4. *Assigning to each $M \in A - \underline{GProj}$ its complete projective resolution induces a fully faithful triangulated functor $\underline{A - GProj} \longrightarrow \mathcal{H}A$.*

Definition 18. Let M, N be left A-modules and X be a right A-module, and suppose that M is Gorenstein projective, with $P^\bullet$ as complete projective resolution. For each $i \in \mathbb{Z}$, we put

1. $\underline{\mathrm{Ext}}^i_A(M,N) = H^i(Hom_A(P^\bullet, N))$
2. $\underline{\mathrm{Tor}}^A_i(X,M) = H^{-i}(X \otimes_A P^\bullet)$,

where $H^i(-)$ denotes the i-th homology R-module of the given complex.

We call $\underline{\mathrm{Ext}}^i_A(-,-)$ and $\underline{\mathrm{Tor}}^A_i(-,-)$ the stable Ext and the stable Tor, respectively. Their definition does not depend on the complete resolution $P^\bullet$ that we choose.

We clearly have $\underline{\mathrm{Ext}}^i_A(M,N) = \mathrm{Ext}^i_A(M,N)$ and $\underline{\mathrm{Tor}}^A_i(X,M) = \mathrm{Tor}^A_i(X,M)$, for all $i > 0$. In particular, we have canonical homomorphisms of graded R-modules

$$\mathrm{Ext}^*_A(M,N) = \oplus_{i\geq 0}\mathrm{Ext}^i_A(M,N) \stackrel{\lambda_{M,N}}{\longrightarrow} \oplus_{i\in\mathbb{Z}}\underline{\mathrm{Ext}}^i_A(M,N) =: \underline{\mathrm{Ext}}^*_A(M,N)$$

and

$$\underline{\mathrm{Tor}}^A_*(X,M) = \oplus_{i\in\mathbb{Z}}\underline{\mathrm{Tor}}^A_i(X,M) \stackrel{\mu_{X,M}}{\longrightarrow} \oplus_{i\geq 0}\mathrm{Tor}^A_i(X,M) = \mathrm{Tor}^A_*(X,M)$$

where $\mathrm{Ker}(\lambda_{M,N})$ and $\mathrm{Coker}(\mu_{M,N})$ are concentrated in degree 0. Actually, $\mathrm{Ker}(\lambda_{M,N}) = \mathcal{P}(M,N) = \{f \in Hom_A(M,N) = \mathrm{Ext}^0_A(M,N) \mid \text{f factors through a projective A-module}\}$ and $\mathrm{Coker}(\mu_{X,N})$ is isomorphic to the image of the morphism $1_X \otimes j_M : X \otimes M \longrightarrow X \otimes_A P^1$, where $j : M = Z^1 \longrightarrow P^1$ is the inclusion of 1-cocycles into 1-cochains. Finally, note that $\underline{\mathrm{Ext}}^i_A(M,N) \cong \underline{\mathrm{Hom}}_A(\Omega^i_A(M), N)$ for all $i \in \mathbb{Z}$ where $\underline{\mathrm{Hom}}_A(?,?)$ denotes the Hom bifunctor in $A-\underline{Mod}$. In particular, for $M = N$ we get a structure of graded algebra on $\underline{\mathrm{Ext}}^*_A(M,M)$ induced from that of $\oplus_{i\in\mathbb{Z}}\underline{\mathrm{Hom}}_A(\Omega^i_A(M),M)$, which is defined by the rule

$$g \cdot f = g \circ \Omega^i_A(f),$$

whenever $f \in \underline{\mathrm{Hom}}_A(\Omega^j_A(M),M)$ and $g \in \underline{\mathrm{Hom}}_A(\Omega^i_A(M),M)$. In particular, the multiplication on $\underline{\mathrm{Ext}}^*_A(M,M)$ extends the Yoneda product defined in Section 4.2. Then, the next result follows in a straightforward way.

Proposition 4.3.2. *Let M, N be left A-modules and suppose that M is Gorenstein projective. The R-module $\underline{Ext}^*_A(M,M)$ has a canonical structure of graded algebra over which $\underline{Ext}^*_A(M,N)$ is a graded right module. Moreover, the map*

$$\lambda_{M,M} : Ext^*_A(M,M) \longrightarrow \underline{Ext}^*_A(M,M)$$

is a homomorphism of graded algebras, and the following diagram is commutative, where the horizontal arrows are the multiplication maps:

$$\begin{array}{ccc} Ext^*_A(M,N) \otimes Ext^*_A(M,M) & \xrightarrow{Yoneda} & Ext^*_A(M,N) \\ \downarrow{\scriptstyle \lambda_{M,N}\otimes\lambda_{M,M}} & & \downarrow{\scriptstyle \lambda_{M,N}} \\ \underline{Ext}^*_A(M,N) \otimes \underline{Ext}^*_A(M,M) & \longrightarrow & \underline{Ext}^*_A(M,N) \end{array}$$

Consider the graded R-module $\underline{\mathrm{Tor}}^A_{-*}(X,M)$ which has the same underlying R-module as $\underline{\mathrm{Tor}}^A_*(X,M)$ but with the grading inverted. If now $P = P^\bullet$ is a fixed totally complete projective resolution of M, then P is canonically a differential graded (dg) A-module, i.e., an object in $\mathcal{C}_{dg}A$, using the terminology of [54]. Then $B = End_{\mathcal{C}_{dg}A}(P)$ is a dg algebra which acts on $X \otimes_A P$ in the obvious way, making $X \otimes_A P$ into a dg left B-module. As a consequence, $H^*(X \otimes_A P) = \underline{\mathrm{Tor}}^A_{-*}(X,M)$ is a graded left module over the cohomology algebra $H^*B = \oplus_{n\in\mathbb{Z}} Hom_{\mathcal{H}A}(P,P[n])$, where $\mathcal{H}A$ is the homotopy category of A (see [54]). But, due to Proposition 4.3.1(4), we have canonical isomorphisms of graded algebras $H^*B \cong \oplus_{n\in\mathbb{Z}} \underline{\mathrm{Hom}}_A(M, \Omega_A^{-n}M) \cong \underline{\mathrm{Ext}}^*_A(M,M)$. It, then, follows that $\underline{\mathrm{Tor}}^A_{-*}(X,M)$ has the structure of a graded left $\underline{\mathrm{Ext}}^*_A(M,M)$-module, as desired.

If we now take the non-negatively graded subalgebra

$$\underline{\mathrm{Ext}}^{\geq 0}_A(M,M) := \oplus_{n\geq 0} \underline{\mathrm{Ext}}^n_A(M,M),$$

of $\underline{\mathrm{Ext}}^*_A(M,M)$, then

$$\underline{\mathrm{Tor}}_{-{>0}}(X,M) = \oplus_{j<0} \underline{\mathrm{Tor}}_j(X,M)$$

is a graded $\underline{\mathrm{Ext}}^{\geq 0}(M,M)$-submodule of $\underline{\mathrm{Tor}}^A_{-*}(X,M)$, and the quotient $\frac{\underline{\mathrm{Tor}}^A_{-*}(X,M)}{\underline{\mathrm{Tor}}^A_{-(>0)}(X,M)}$, which is isomorphic to $\underline{\mathrm{Tor}}^A_{-(\leq 0)}(X,M)$ as a graded R-module, is a graded left $\underline{\mathrm{Ext}}^{\geq 0}_A(M,M)$-module. That is, $\oplus_{i\geq 0}\underline{\mathrm{Tor}}^A_i(X,M)$ has a canonical structure of graded left $\underline{\mathrm{Ext}}^{\geq 0}_A(M,M)$-module, where $\underline{\mathrm{Tor}}^A_i(X,M)$ is the component of degree $-i$, for all $i \geq 0$. Since we have a surjective morphism of graded algebras $\mathrm{Ext}^*_A(M,M) \twoheadrightarrow \underline{\mathrm{Ext}}^{\geq 0}_A(M,M)$, we get a structure of graded left $\mathrm{Ext}^*_A(M,M)$-module on $\underline{\mathrm{Tor}}^A_{-(\leq 0)}(X,M)$.

We can now provide $\mathrm{Tor}^A_{-*}(X,M)$ (i.e., just $\mathrm{Tor}^A_*(X,M)$, but with $\mathrm{Tor}^A_i(X,M)$ in degree $-i$, for all $i \geq 0$) with a structure of graded left $\mathrm{Ext}^*_A(M,M)$-module of which $\underline{\mathrm{Tor}}^A_{-(\leq 0)}(X,M)$ is a graded submodule. Indeed, we have that the product $\mathrm{Ext}^i_A(M,M) \cdot \mathrm{Tor}^A_{-(-j)}(X,M) = \mathrm{Ext}^i_A(M,M) \cdot \underline{\mathrm{Tor}}^A_{-(-j)}(X,M)$ is given by the preceding product using the isomorphism $\mathrm{Tor}^A_j(-,-) \cong \underline{\mathrm{Tor}}^A_j(-,-)$ when $j > 0$ and $i \leq j$, together with the natural map $\underline{\mathrm{Tor}}^A_{j-i}(-,-) \to \mathrm{Tor}^A_{j-i}(-,-)$. If $i > j$, this product is zero. For $j = 0$ we put

$$\mathrm{Ext}^i_A(M,M) \cdot \mathrm{Tor}^A_0(X,M) = 0 \text{ if } i > 0,$$

and for $i = 0$ the multiplication is given by the following diagram,

$$\begin{array}{ccc}
\mathrm{Ext}^0_A(M,M) \times \mathrm{Tor}^A_0(X,M) & \longrightarrow & \mathrm{Tor}^A_0(X,M) \\
\downarrow \wr\!\wr & & \downarrow \wr\!\wr \\
End_A(M) \times (X \otimes_A M) & \longrightarrow & X \otimes_A M,
\end{array}$$

where the bottom horizontal arrow is the canonical map $(f, x \otimes m) \mapsto x \otimes f(m)$.

These comments prove the following analogue of Proposition 4.3.2 for Tor.

Proposition 4.3.3. *Let X and M be a right and a left A-modules, respectively, and suppose that M is Gorenstein projective. Then $\underline{Tor}^A_{-*}(X,M)$ (resp. $Tor^A_{-*}(X,M)$) has a* *canonical structure of graded left* $\underline{Ext}_A^*(M,M)$*- (resp.* $Ext_A^*(M,M)$*-)module. Moreover, the following diagram is commutative:*

$$\begin{array}{ccc}
\mathbf{Ext}_A^*(M,M)\times\underline{\mathbf{Tor}}_{-*}^A(X,M) \xrightarrow{\lambda_{M,M}\times 1} \underline{\mathbf{Ext}}_A^*(M,M)\times\underline{\mathbf{Tor}}_{-*}^A(X,M) & \xrightarrow{mult.} & \underline{\mathbf{Tor}}_{-*}^A(X,M) \\
\downarrow 1\times\mu_{X,M} & & \downarrow \mu_{X,M} \\
\mathbf{Ext}_A^*(M,M)\times\mathbf{Tor}_{-*}^A(X,M) & \xrightarrow[mult.]{} & \mathbf{Tor}_{-*}^A(X,M)
\end{array}$$

We are especially interested in the particular case of the two previous propositions in which $A=\Lambda^e=\Lambda\otimes\Lambda^{op}$, where Λ is an R-algebra which is Gorenstein projective as Λ-bimodule and $M=\Lambda$ viewed as Λ^e-module. In that case, we put $\underline{HH}^n(\Lambda,N)=\underline{\mathrm{Ext}}_{\Lambda^e}^n(\Lambda,N)$ and $\underline{HH}_n(\Lambda,N)=\underline{\mathrm{Tor}}_n^{\Lambda^e}(\Lambda,N)$ and call them the n-th stable Hochschild cohomology and homology R-modules of Λ with coefficients in N, respectively. Putting $\underline{HH}^*(\Lambda,N)=\oplus_{n\in\mathbb{Z}}\underline{HH}^n(\Lambda,N)$, $\underline{HH}^*(\Lambda)=\underline{HH}^*(\Lambda,\Lambda)$, $\underline{HH}_*(\Lambda,N)=\oplus_{n\in\mathbb{Z}}\underline{HH}_n(\Lambda,N)$, and $\underline{HH}_*(\Lambda)=\underline{HH}_*(\Lambda,\Lambda)$, we have the following straightforward consequence of Propositions 4.3.2 and 4.3.3.

Corollary 4.3.4. *In the situation above,* $\underline{HH}^*(\Lambda)$ *(resp.* $HH^*(\Lambda)$*) has a canonical structure of graded algebra over which* $\underline{HH}^*(\Lambda,N)$ *(resp.* $HH^*(\Lambda,N)$*) is a graded right module and* $\underline{HH}_{-*}(\Lambda,N)$ *(resp.* $HH_{-*}(\Lambda,N)$*) is a graded left module. Moreover, the graded algebra structure on* $HH^*(\Lambda)$ *and the graded module structures on* $HH^*(\Lambda,N)$ *and* $HH_{-*}(\Lambda,N)$ *are determined by their stable analogues, except in degree zero.*

4.4 Bigrading on the stable and absolute Hochschild (co)homology

Suppose that A admits a positive grading $A=\oplus_{n\geq 0}A_n$ such that each A_n is finitely generated (projective) as an R-module. We denote by $A-Gr$ the category of ($\mathbb{Z}$-)graded modules and, for each $n\in\mathbb{Z}$ and each $M\in A-Gr$, we denote by $M[n]$ the graded A-module which has the same underlying A-module as M, but with grading given by $M[n]_i=M_{n+i}$, for all $i\in\mathbb{Z}$.

With the obvious adaptation, recall from Subsection 1.2.1 that if M and N are objects of $A-Gr$, then $HOM_A(M,N)=\oplus_{n\in\mathbb{Z}}Hom_{A-Gr}(M,N[n])$ is a graded R-module and we have an inclusion $HOM_A(M,N)\subseteq Hom_A(M,N)$. This inclusion is an equality when M is finitely generated (see [65], Section I.2). We then get an induced functor $Hom_A(-,N):A-gr\longrightarrow R-Gr$, where $A-gr$ denotes the full subcategory of $A-Gr$ whose objects are the finitely generated graded modules. Note that we have an isomorphism $Hom_A(M[-n],N)\cong Hom_A(M,N)[n]$ in $R-Gr$.

If we assume that M has a projective resolution $P^\bullet$ in $A-Gr$ consisting of finitely generated terms (e.g. if A and M are finitely generated projective as R-modules), then $Hom_A(P^{-r},N)$ is a graded R-module, for each $r\geq 0$, and the differentials of the complex $Hom_A(P^\bullet,N)$ are morphisms of graded R-modules. As a consequence, $\mathrm{Ext}^r_A(M,N)$ inherits a structure of graded R-module. Then $\mathrm{Ext}^*_A(M,N)$ becomes canonically a bigraded (= $\mathbb{Z}\times\mathbb{Z}$-graded) R-module where the homogeneous component of bidegree (m,n) is precisely $\mathrm{Ext}^m_A(M,N)_n$.

We call a complex of projective graded A-modules $P^\bullet$ totally acyclic when it is totally acyclic as a complex of ungraded A-modules. Then a complete graded projective resolution of $M\in A-Gr$ is just a totally acyclic complex $P^\bullet$ of graded projective A-modules such that $Z^1=M$. The scheme of the preceding paragraph can be essentially copied so that, when $M,N\in A-Gr$ and M admits a complete graded projective resolution consisting of finitely generated terms, the R-module $\underline{\mathrm{Ext}}^*_A(M,N)$ is bigraded.

It is not hard to see that when M,N,T are graded A-modules and M and N admit (complete) projective resolutions with finitely generated terms, then the Yoneda products

$$\mathrm{Ext}^*_A(N,T)\times\mathrm{Ext}^*_A(M,N)\longrightarrow\mathrm{Ext}^*_A(M,T)\quad\text{and}$$

$$\underline{\mathrm{Ext}}^*_A(N,T)\times\underline{\mathrm{Ext}}^*_A(M,N)\longrightarrow\underline{\mathrm{Ext}}^*_A(M,T)$$

are compatible with the bigrading, i.e., if $\mathrm{bideg}(\epsilon)=(m,n)$ and $\mathrm{bideg}(\delta)=(p,q)$, then $\mathrm{bideg}(\epsilon\delta)=(m+p,n+q)$ whenever $\epsilon\delta\neq 0$, where bideg denotes the bidegree. It follows that the algebras $\mathrm{Ext}^*_A(M,M)$ and $\underline{\mathrm{Ext}}^*_A(M,M)$ are bigraded and that $\mathrm{Ext}^*_A(M,N)$ and $\underline{\mathrm{Ext}}^*_A(M,N)$ are bigraded right modules over them, respectively.

The particular case that is most interesting for us is the one in which $A=\Lambda^e=\Lambda\otimes\Lambda^{op}$ is the enveloping algebra of a graded algebra $\Lambda=\oplus_{n\geq 0}\Lambda_n$ which admits a complete projective resolution as a graded Λ-bimodule whose terms are finitely generated, with $\Lambda_{>0}=\oplus_{n\geq 0}\Lambda_n$ nilpotent and Λ_0 isomorphic as an algebra to a finite direct product of copies of R. In that case $A=\Lambda^e$ inherits a grading $A_n=\oplus_{r+s=n}\Lambda_r\otimes\Lambda^{op}_s$ satisfying the requirements of the preceding paragraphs. Now taking $M=\Lambda$, with its canonical structure of left Λ^e-module, we see that it is graded. We conclude that $HH^*(\Lambda)$ and $\underline{HH}^*(\Lambda)$ are bigraded algebras and that, whenever N is a graded Λ-bimodule, $HH^*(\Lambda,N)$ and $\underline{HH}^*(\Lambda,N)$ are bigraded right modules over them, respectively.

If Q is a finite quiver, then its path algebra RQ has a natural grading, where the degree of each path is its length. If in the above setting $\Lambda=RQ/I$ is a quotient of a path algebra, where I is homogeneous with respect to this grading of RQ, then the induced grading on Λ will be called the *length grading* and the same term will be used to name the induced grading on $HH^*(\Lambda)$ or on each $HH^n(\Lambda)$ (resp. on $\underline{HH}^*(\Lambda)$ or $\underline{HH}^n(\Lambda)$, when Λ is also Frobenius). Sometimes we use the term *homological grading* for the canonical grading $HH^*(\Lambda)=\oplus_{n\geq 0}HH^n(\Lambda)$ (resp. $\underline{HH}^*(\Lambda)=\oplus_{n\in\mathbb{Z}}\underline{HH}^n(\Lambda)$) in order to distinguish it from the length grading.

4.5 Frobenius algebras (following Eu and Schedler)

In this section we introduce a class of algebras studied by Eu and Schedler [35], which generalize the class of self-injective finite dimensional algebras and to which, with the suitable adaptation, one can apply some of the results of chapter 1. We will give a brief account of the results of [op.cit] which are more useful to us in this paper.

Definition 19. An algebra A is a *Frobenius algebra* if it is finitely generated projective as an R-module and admits a nondegenerate R-bilinear form $(-,-) : A \times A \longrightarrow R$ satisfying:

i) $(a, bc) = (ab, c)$, for all $a, b, c \in A$

ii) The map $A \longrightarrow D(A) := \mathrm{Hom}_R(A, R)$, $b \to (-, b)$, is bijective.

In such a case, if $P^\bullet = (P^\bullet, d^\bullet)$ is any acyclic complex of finitely generated projective A-modules with the property that each map $P^i \longrightarrow \mathrm{Im}(d^i)$ is a retraction (=split epimorphism) of R-modules, then it is totally acyclic. Each finitely generated A-module M which is projective as an R-module is the module of 1-cocycles of such an acyclic complex, and, hence, it is Gorenstein projective (see [35]).

In the particular case when $R = K$ is a field, a Frobenius algebra is just a self-injective finite dimensional algebra (see Example 4.5.1 below), and then 'acyclic' and 'totally acyclic' are synonymous terms for complexes of (arbitrary) projective A-modules. Each A-module is Gorenstein projective in this case.

When Λ is a Frobenius algebra, it is of common use to denote by $\Lambda - \mathrm{mod}$ (resp. $\mathrm{mod} - \Lambda$) the category of left (resp. right) Λ which are finitely generated projective as R-modules. These modules turn out to be Gorenstein projective, so that the homological theory of the previous subsections apply to them. The category $\Lambda - mod$ is Frobenius exact category, although it need not be abelian. By [46], Chapter I, we then know that the associated stable category $\Lambda - \underline{mod}$ is a triangulated category with the cosyzygy functor $\Omega_\Lambda^{-1} : \Lambda - \underline{mod} \longrightarrow \Lambda - \underline{mod}$ as suspension. The contravariant functor $\mathrm{Hom}_R(-, R)$ clearly induces an 'involutive' duality $D : \Lambda - mod \overset{\cong^{op}}{\longleftrightarrow} mod - \Lambda$.

Example 4.5.1. If $R = K$ is a field, then one easily sees, using Theorem 1.3.2 with the grading group $H = 0$, that the following assertions are equivalent for a K-algebra Λ:

1. Λ is a Frobenius K-algebra in the sense of [35]

2. Λ is Quasi-Frobenius

3. Λ is pseudo-Frobenius and finite dimensional

4. Λ is self-injective and finite dimensional

In such case $\Lambda - \mathrm{mod}$ is the category of <u>all</u> finitely generated Λ-modules.

As shown by Eu and Schedler, many developments of Chapter 1 apply to Frobenius algebras. If $(-,-) : \Lambda \times \Lambda \longrightarrow R$ is an R-bilinear form as in Definition 3, which we will call *Nakayama form* in the sequel, then there is a unique automorphism η of Λ, called the *Nakayama automorphism*, such that $(a, b) = (b, \eta(a))$ for all $a, b \in \Lambda$, and hence $D(\Lambda)$ is isomorphic to ${}_1\Lambda_\eta$ as Λ-bimodule. Conversely, any isomorphism $f : {}_1\Lambda_\eta \overset{\cong}{\longrightarrow} D(\Lambda)$ of Λ-bimodules, where η is an automorphism of Λ, gives rise to a Nakayama form for Λ given by $(a, b) = f(b)(a)$, for all $(a, b) \in \Lambda \times \Lambda$.

An important consequence of Definition 19 is that if A and B are Frobenius algebras, then also $A \otimes B$ is a Frobenius algebra. Indeed it is clear that $A \otimes B$ is projetive as an R-module. Moreover, if $(-,-)_A : A \times A \longrightarrow R$ and $(-,-)_B : B \otimes B \longrightarrow R$ are Nakayama forms, then the map $< -,- >: (A \otimes B) \times (A \otimes B) \longrightarrow R$ identified by the equality $< a \otimes b, a' \otimes b' >= (a,a') \cdot (b,b')$ is a Nakayama form for $A \otimes B$. In particular, if Λ is our Frobenius algebra, then its enveloping algebra $\Lambda^e = \Lambda \otimes \Lambda^{op}$ is a Frobenius algebra and, since Λ is finitely generated and projective over R, we have that Λ is Gorenstein-projective. In particular, the stable Hochschild cohomology ring $\underline{HH}^*(\Lambda)$ is well-defined. The following fact was proved in [Eu-Schedler, Theorem 2.1.15] and gives the 'missing point' of Corollary 4.3.4.

Proposition 4.5.2. *If Λ is a Frobenius algebra, then $\underline{HH}^*(\Lambda)$ is graded-commutative.*

In the context of Frobenius algebras, some of the concepts studied in chapters 2 and 3 for the m-fold mesh algebras or for arbitrary self-injective finite dimensional algebras are similarly defined. For instance Λ is said to be periodic when $\Omega^m_{\Lambda^e}(\Lambda)$ is isomorphic to Λ as a Λ-bimodule, for some integer $m > 0$, and the smallest of these m is called the period of m. When $\Lambda - \underline{mod}$ is Calabi-Yau, we will say that Λ is *stably Calabi-Yau (of CY-dimension* m) when $\Lambda - \underline{mod}$ is a Calabi-Yau triangulated category (of dimension m). We will say that Λ is *Calabi-Yau Frobenius (of dimension m)* when there is an integer $r > 0$ such that $\Omega^{-r-1}_{\Lambda^e}(\Lambda)$ is isomorphic to $D(\Lambda)$ in the stable category $\Lambda^e - \underline{mod}$ (and m is the smallest such r). For our purposes it is worth noting that if Λ is a symmetric periodic Frobenius algebra of period m, then it is Calabi-Yau of CY-dimension $\leq m-1$.

Remark 4.5.3. The definition of Calabi-Yau Frobenius algebra given above is the one given in [35]. In case $K = R$ is a field, so that Λ is a self-injective finite dimensional algebra, the category $\Lambda^e - \text{mod} = \Lambda - \text{mod} - \Lambda$ of finitely generated Λ-bimodule is Krull-Schmidt (i.e., each object is a finite direct sum of objects with local endomorphism rings). Moreover, each object has a minimal (complete) projective resolution. It follows that $\Omega^{-r-1}_{\Lambda^e}(\Lambda)$ is uniquely determined in $\Lambda - \text{mod} - \Lambda$, up to isomorphism, when using the minimal projective resolution of Λ. As a consequence, whenever Λ does not have semisimple direct summands, there is an isomorphism $\Omega^{-r-1}_{\Lambda^e}(\Lambda) \cong D(\Lambda)$ in $\Lambda - \text{mod} - \Lambda$ if, and only if, that isomorphism exists in the stable category $\Lambda^e - \underline{mod}$. Therefore the concept of Calabi-Yau Frobenius algebra agrees with the one that we have used in Chapter 3.

Remark 4.5.4. If $B = \oplus_{i \in \mathbb{Z}} B_i$ is a graded commutative algebra, then any graded left B-module $V = \oplus_{i \in \mathbb{Z}} V_i$ may be viewed as a graded right B-module by defining $vb = (-1)^{\deg(b)\deg(v)} bv$, for all homogeneous elements $b \in B$ and $v \in V$. In particular, we shall view in this way $HH_{-*}(\Lambda, M)$ as graded right $\underline{HH}^*(\Lambda)$-module, for each Λ-bimodule M. We proceed similarly with $\underline{HH}_{-*}(\Lambda, M)$ over $\underline{HH}^*(\Lambda)$.

Remark 4.5.5. The following holds, for each Λ-bimodule M in the latter subcategory:

1. $\underline{HH}_{-*}(\Lambda, M) \cong D(\underline{HH}^*(\Lambda, D(M)))$ as graded $\underline{HH}^*(\Lambda)$-modules.
2. $HH_{-*}(\Lambda, M) \cong D(HH^*(\Lambda, D(M)))$ as graded $HH^*(\Lambda)$-modules.

Indeed, note that if Q and M are a projective and an arbitrary Λ-bimodule, then $D(Q \otimes_{\Lambda^e} M) \cong Hom_{\Lambda^e}(Q, D(M))$ by adjunction. If now $P = P^\bullet$ is a complete projective resolution of Λ as a bimodule (equivalently, as a right Λ^e-module) which consists of finitely generated terms, then we have an isomorphism of complexes $D(P \otimes_{\Lambda^e} M) \cong Hom_{\Lambda^e}(P, D(M)) = Hom_{\mathcal{C}_{dg}\Lambda^e}(P, D(M))$, using the convention that $D(T)^i = D(T^{-i})$ for each complex (or each graded R-module) T and each $i \in \mathbb{Z}$. It is straightforward to see that the last isomorphism preserves the structures of right dg modules over the dg algebra $B := End_{\mathcal{C}_{dg}\Lambda^e}(P, P)$.

In the following definition ${}_\Lambda\underline{\mathrm{Mod}}_\Lambda = \frac{{}_\Lambda\mathrm{Mod}_\Lambda}{{}_\Lambda Proj_\Lambda}$ denotes the stable category modulo projectives of the category of Λ-bimodules.

In case H is a graded commutative ring and $f \in H$ is a homogeneous element which is not nilpotent, we will denote by $H_{(f)}$ the localization of H with respect to the multiplicative subset $\{1, f, f^2, \dots\}$. It is a graded commutative ring where $\deg(\frac{g}{f^n}) = \deg(g) - n \cdot \deg(f)$, for all homogeneous elements $g \in H$ and all $n \geq 0$. If M is a graded H-module, we will denote by $M_{(f)}$ the localization of M at $\{1, f, f^2, \dots\}$.

Proposition 4.5.6. *Let us assume that 2 is invertible in R, let Λ be a Frobenius R-algebra which is periodic of period s and let $h \in HH^s(\Lambda)$ be any element represented by an isomorphism $\Omega^s_{\Lambda^e}(\Lambda) \xrightarrow{\sim} \Lambda$ in ${}_\Lambda\underline{Mod}_\Lambda$. Suppose that M is a Λ-bimodule. The following assertions hold:*

1. $\underline{HH}^*(\Lambda, M) \cong \underline{HH}^*(\Lambda, M)[s]$ *and* $\underline{HH}_{-*}(\Lambda, M) \cong \underline{HH}_{-*}(\Lambda, M)[s]$ *as graded* $\underline{HH}^*(\Lambda)$*-modules.*
2. *h is an element of $HH^*(\Lambda)$ which is not nilpotent and $\underline{HH}^*(\Lambda)$ is isomorphic, as a graded algebra, to $HH^*(\Lambda)_{(h)}$.*
3. $\underline{HH}^*(\Lambda, M)$ *is isomorphic to* $HH^*(\Lambda, M)_{(h)}$ *as a graded* $\underline{HH}^*(\Lambda)$*-module.*

Proof. We have already seen in the previous comments that $\underline{HH}^*(\Lambda)$ is isomorphic to the graded algebra $\oplus_{n \in \mathbb{Z}} \underline{\mathrm{Hom}}_{\Lambda^e}(\Omega^n_{\Lambda^e}(\Lambda, \Lambda))$, where the multiplication of homogeneous elements on this algebra is given by $g \cdot f = g \circ \Omega^n_{\Lambda^e}(f)$. If now $\widehat{h} : \Omega^s_{\Lambda^e}(\Lambda) \xrightarrow{\sim} \Lambda$ is an isomorphism representing h, then $\Omega^{-s}_{\Lambda^e}(\widehat{h}^{-1}) : \Omega^{-s}_{\Lambda^e}(\Lambda) \longrightarrow \Lambda$ represents an element $h' \in \underline{HH}^{-s}(\Lambda)$. But then $h' \cdot h = 1$ since $h' \cdot h$ is represented by $\Omega^{-s}_{\Lambda^e}(\widehat{h}^{-1}) \circ \Omega^{-s}_{\Lambda^e}(\widehat{h}) = \Omega^{-s}_{\Lambda^e}(\widehat{h}^{-1}\widehat{h}) = \Omega^{-s}_{\Lambda^e}(1_{\Omega^s_{\Lambda^e}(\Lambda)}) = 1_\Lambda$.

The above paragraph shows that h is invertible (of degree s) in $\underline{HH}^*(\Lambda)$, from which it follows that the multiplication by h gives an isomorphism $Y \xrightarrow{\sim} Y[s]$, for each graded $\underline{HH}^*(\Lambda)$-module Y (here we have used that, when $\underline{HH}^0(\Lambda) = Z(\Lambda)/\mathcal{P}(\Lambda, \Lambda)$ is a ring of characteristic $\neq 2$, in particular when 2 is invertible in R, the period s is even, cf. [35] Theorem 2.3.47]).

Since the multiplication of homogeneous elements of degree > 0 is the same in $HH^*(\Lambda)$ and in $\underline{HH}^*(\Lambda)$ and h in invertible in the latter algebra, it follows that h is not nilpotent in $HH^*(\Lambda)$. On the other hand, the universal property of the module of quotients gives a unique morphism of graded $HH^*(\Lambda)$-modules

$$\Phi : HH^*(\Lambda, M)_{(h)} \longrightarrow \underline{HH}^*(\Lambda, M)$$

which takes the fraction $\frac{\eta}{h^n}$ to $h'^n\eta$, where h' is the inverse of h in $\underline{HH}^*(\Lambda)$. It is clear that the homogeneous elements of degree ≥ 0 are in the image of Φ. On the other hand, if $\xi \in \underline{HH}^{-j}(\Lambda)$, with $j > 0$, then there is a $k > 0$ such that $ks > j$. Fixing such a k, we have that $\eta := h^k\xi \in \underline{HH}^{ks-j}(\Lambda, M) = HH^{ks-j}(\Lambda, M)$ and, clearly, the equality $\Phi(\frac{\eta}{h^k}) = \xi$ holds. Therefore Φ is surjective. Moreover, $\mathrm{Ker}(\Phi)$ consists of those fractions $\frac{\eta}{h^n}$ such that $h'^n\eta = 0$ in $\underline{HH}^*(\Lambda, M)$. This is, in turn, equivalent to saying that $\eta = 0$ in $\underline{HH}^*(\Lambda, M)$ for h' is invertible in $\underline{HH}^*(\Lambda)$. That is, η is in the kernel of the canonical map $\lambda_{\Lambda,M} : HH^*(\Lambda, M) \longrightarrow \underline{HH}^*(\Lambda, M)$. Hence, we get that $\eta \in \mathcal{P}(\Lambda, M) = \{f \in Hom_\Lambda(\Lambda, M) :$ f factors through a projective Λ-module$\}$, which implies that $h\eta = 0$ in $HH^*(\Lambda, M)$. It follows that $\frac{\eta}{h^n} = \frac{h\eta}{h^{n+1}} = 0$, and so Φ is also injective. Finally, in case $\Lambda = M$, the map Φ is a homomorphism of graded algebras, and the proof is complete. □

The following is a result by Eu and Schedler ([35], Theorem 2.3.27):

Theorem 4.5.7. *(Eu-Schedler) Let Λ be a Calabi-Yau Frobenius algebra of dimension m and let M be any Λ-bimodule. There are isomorphisms of graded right $\underline{HH}^*(\Lambda)$-modules:*

1. $\underline{HH}_{-*}(\Lambda, M)[-m] \cong \underline{HH}^*(\Lambda, M)$
2. $\underline{HH}^*(\Lambda, M) \cong D(\underline{HH}^*(\Lambda, D(M)))[-m] = D(\underline{HH}^*(\Lambda, D(M))[m])$
3. $\underline{HH}^*(\Lambda) \cong D(\underline{HH}^*(\Lambda))[-2m-1] = D(\underline{HH}^*(\Lambda)[2m+1])$

In particular $\underline{HH}^*(\Lambda)$ is a pseudo-Frobenius graded algebra.

Note that, if Λ is symmetric, then Λ is periodic of period s exactly when it is $(s-1)$-Calabi-Yau Frobenius. However, we remind the reader that, with the convention of [35], when Λ is symmetric but not periodic, it is said to be (-1)-Calabi-Yau Frobenius.

Corollary 4.5.8. *If Λ is a symmetric periodic algebra of period s and M is a Λ-bimodule, then:*

1. *The multiplicative structure of $\underline{HH}^*(\Lambda)$ is determined by that of $HH^*(\Lambda)$.*
2. *The structures of $\underline{HH}^*(\Lambda, M)$ and $\underline{HH}_{-*}(\Lambda, M)$ as graded $\underline{HH}^*(\Lambda)$-modules and the structure of $HH_{-*}(\Lambda, M)$ as graded $HH^*(\Lambda)$-module are determined by the structure of $HH^*(\Lambda, M)$ as graded $HH^*(\Lambda)$-module.*

Proof. Since Λ is CY Frobenius, the two assertions are a direct consequence of the theorem and of Proposition 4.5.6. □

The following result was given in [35] for $A = \underline{HH}^*(\Lambda)$, when Λ was a the (classical) preprojective algebra of a Dynkin quiver, i.e., a m-fold mesh algebra of extended type $(\Delta, 1, 1)$. According to Theorem 4.5.7 it also applies to the stable Hochschild cohomology ring of any Calabi-Yau Frobenius algebra over a field.

Lemma 4.5.9. *Let $A = \oplus_{n\in\mathbb{Z}} A_n$ be a connected $\mathbb{Z}$-graded pseudo-Frobenius algebra over a field admitting a graded Nakayama form $(-,-) : A \times A \longrightarrow K$ of constant degree function m. Suppose that A is graded commutative and let $a, b, c \in A$ be any homogeneous elements. The following equalities hold:*

$$(ab,c) = (-1)^{deg(b)\,deg(c)}(ac,b) = (-1)^{deg(a)[deg(b)+deg(c)]}(bc,a).$$

Proof. Note that since A is connected and graded commutative the only nonzero homogeneous idempotent is 1. Then we have $\{1\}$ as distinguished family of orthogonal homogeneous idempotents in the sense of subsection 1.2. Suppose now that $x, y \in A$ are two homogeneous elements such that $\deg(x) + \deg(y) = m$. Then, using the properties of the graded Nakayama form and the graded commutativity of A, we have an equality

$$(x,y) = (1,xy) = (1,(-1)^{\deg(x)\deg(y)}yx) = (-1)^{\deg(x)\deg(y)}(1,yx) = (-1)^{\deg(x)\deg(y)}(y,x).$$

When $\deg(a) + \deg(b) + \deg(c) \neq m$, the three terms in the proposed equalities are zero and there is nothing to prove. We then assume that $\deg(a) + \deg(b) + \deg(c) = m$. We then have equalities:

$$(ab,c) = (a,bc) = (a,(-1)^{\deg(b)\deg(c)}cb) = (-1)^{\deg(b)\deg(c)}(a,cb) = (-1)^{\deg(b)\deg(c)}(ac,b)$$

and

$$(ab,c) = (a,bc) = (-1)^{\deg(a)\deg(bc)}(bc,a) = (-1)^{\deg(a)[\deg(b)+\deg(c)]}(bc,a).$$

□

4.6 Some facts on self-injective algebras

Throughout this Section we will assume that $R = K$ is a field and, for simplicity, we will assume that the algebra $\Lambda = KQ/I$ is a finite dimensional quotient of the path algebra of a finite quiver Q, with $I \subseteq KQ_{\geq 2}$. By Example 4.5.1, we can apply to Λ the results about Frobenius algebras from the previous subsection and also the results of chapter 1 on graded pseudo-Frobenius algebras, simply by assuming that the grading abelian group is $H = 0$.

Given a Nakayama form $(-,-) : \Lambda \times \Lambda \longrightarrow K$ and any basis B of Λ, one obtains a right (resp. left) dual basis $B^* = \{b^* : b \in B\}$ (resp. ${}^*B = \{{}^*b : b \in B\}$) identified by the property that $(b,c^*) = \delta_{bc}$ (resp. $(c^*,b) = \delta_{cb}$), for all $b,c \in B$, where δ_{bc} is the Kronecker symbol.

Taking $H = 0$ and $I = Q_0$, the following is a straightforward consequence of Proposition 1.3.7:

Proposition 4.6.1. *Let Λ be a self-injective algebra which contains no nonzero semisimple summand as an algebra, and let $(-,-) : \Lambda \times \Lambda \longrightarrow K$ be a bilinear form. The following assertions are equivalent:*

1. *$(-,-)$ is a Nakayama form*
2. *There is a basis $B = \bigcup_{i,j\in Q_0} e_iBe_j$ of Λ which contains the vertices and also contains a basis $\{\omega_i : i \in Q_0\}$ of $Soc(\Lambda)$ such that $(x,y) = \sum_{i\in Q_0}\lambda_i$ for all $x,y \in \Lambda$, where λ_i is the coefficient of ω_i in the expression of xy as a linear combination of the elements of B.*

Given a basis $\mathcal{B} \subseteq \bigcup_{i,j \in Q_0}$ containing the idempotents e_i and an socle element $w_i \in \mathrm{Soc}(e_i\Lambda)$, for each $i \in Q_0$, adapting the terminology from chapter 1, the bilinear form given by condition 2 in last proposition is called the *Nakayama form associated to* $\mathcal{B}$

Definition 20. A basis $\mathcal{B}$ as above is called *dualizable* when its associated Nakayama form is symmetric.

We look now at the case when I is a homogeneous ideal of KQ with respect to the length grading on KQ and consider the induced length grading $\Lambda = \oplus_{n \geq 0} \Lambda_n$. The following lemma gives a handy criterion for a basis to be dualizable.

Lemma 4.6.2. *Let $\Lambda = \frac{KQ}{I}$ be a length graded self-injective algebra such that its Nakayama permutation is the identity. Let B be a basis of Λ consisting of paths and negatives of paths and which contains the vertices and also contains a basis $\{\omega_i : i \in \mathcal{Q}_0\}$ of Soc(Λ), with $w_i \in e_i\Lambda$ for each $i \in Q_0$. If $(-,-) : \Lambda \times \Lambda \longrightarrow K$ is the Nakayama form associated to B, consider the following assertions:*

1) $a^*a = \omega_{t(a)}$, for all $a \in Q_1$.

2) $b^{**} = b$, for each $b \in B$.

3) $(-,-)$ is symmetric, i.e., B is a dualizable basis.

Then 2) *and* 3) *are equivalent and they always imply* 1). *Moreover, if we have that dim($e_i\Lambda_n e_j$) ≤ 1, for all $i, j \in Q_0$ and all natural numbers n, then the three conditions are equivalent.*

Proof. 2) $\Longleftrightarrow$ 3)) Given any finite basis B for a vector space V with a nondegenerate bilinear form, then the form is symmetric if, and only if, $(b', b^*) = (b^*, b')$ for all $b, b' \in B$. But the latter condition is equivalent to $b = b^{**}$ for all $b \in B$ and thus, $2 \Longleftrightarrow 3 \Longrightarrow 1$).

1) $\Longrightarrow$ 2) To show that 1) implies 2), assume that $\dim(e_i\Lambda_n e_j) \leq 1$, for all $i, j \in Q_0$ and all natural numbers n.

First observe that our hypotheses guarantee that the nonzero homogeneous elements in $\bigcup_{i,j \in Q_0} e_i\Lambda e_j$ are precisely the scalar multiples of the elements of B. We denote by H the set of these nonzero homogeneous elements. Therefore, an alternative description of b^* is that it is the unique element of H such that $bb^* = \omega_{i(b)}$. We can extend $(-)^*$ to a bijective map $(-)^* : H \longrightarrow H$ so that h^* is the unique element of H such that $hh^* = \omega_{i(h)}$. It is then clear that $(\lambda h)^* = \lambda^{-1}h^*$, for all $h \in H$ and $\lambda \in K^*$.

Observe that if $h_1, h_2 \in H$ are such that $h_1h_2(h_1h_2)^* = \omega_{i(h_1)}$, then $h_2(h_1h_2)^* = h_1^*$.

We next prove that if $a \in Q_1$ and $h \in H$ are such that $ah \neq 0$, then $(ah)^*a = h^*$. We proceed by induction on deg(h). The case deg(h) $= 0$ is a direct consequence of the hypothesis. Since h is a scalar multiple of an element of B, we can assume without loss of generality that h is a path in Q, say, $h = \alpha_1 \cdots \alpha_r$. Then we have

$$h[(ah)^*a] = \alpha_1 \cdots \alpha_r(a\alpha_1 \cdots \alpha_r)^*a = \alpha_1 \cdots \alpha_{r-1}[\alpha_r(a\alpha_1 \cdots \alpha_{r-1}\alpha_r)^*]a =$$

$$\alpha_1 \cdots \alpha_{r-1}(a\alpha_1 \cdots \alpha_{r-1})^*a$$

By the induction hypothesis, the last term is equal to $\alpha_1 \cdots \alpha_{r-1}(\alpha_1 \cdots \alpha_{r-1})^* = \omega_{i(h)}$. It follows that $(ah)^* a = h^*$.

We finally prove by induction on $\deg(h)$ that $h^* h = \omega_{t(h)}$ for all $h \in H$ which implies that $h^{**} = h$ for all $h \in H$. This will complete the proof. The cases of $\deg(h) = 0, 1$ are clear. So we assume that $\deg(h) > 1$ and, again, assume that $h = \alpha_1 \cdots \alpha_r$ is a path $(r > 1)$. Then

$$h^* h = [\alpha_1(\alpha_2 \cdots \alpha_r)]^* \alpha_1 \alpha_2 \cdots \alpha_r = (\alpha_2 \cdots \alpha_r)^* \alpha_2 \cdots \alpha_r,$$

and, by the induction hypothesis, the last term is equal to $\omega_{t(\alpha_r)} = \omega_{t(h)}$. □

Chapter 5

The Hochschild cohomology ring of $\mathbb{L}_n$

5.1 Introduction

5.1.1 Motivation

As mentioned in detail in the introduction, the Hochschild cohomology ring $HH^*(\Lambda)$ of a self-injective finite dimensional algebra Λ is of great interest in connection with the study of varieties of modules and with questions about its relationship with the Yoneda algebra of Λ. Also, a related intriguing open problem is that of determining whether Ω-periodicity of Λ/J as a Λ-module, where J denotes the Jacobson radical of Λ, implies the periodicity of Λ.

The aim of this chapter is to determine the structure of the classical and stable Hochschild cohomology rings of the generalized preprojective algebra $\mathbb{L}_n$ over any commutative ring where 2 is invertible, as well as the structure of the classical and stable Hochschild homology groups as graded modules over the respective Hochschild cohomology rings. We warn the reader that the algebra $\mathbb{L}_n$ is also known as the preprojective algebra of generalized Dynkin type $\mathbb{L}_n$. A very well-known fact, which is proved in fact in Theorem 3.4.12, is that $\mathbb{L}_n$ has ($\Omega-$) period 6 (here Ω denotes Heller's syzygy operator) and, consequently, the Hochschild cohomology spaces also have period 6. Our approach to the proof of the main theorem is to first prove it when R is a field, and then to deduce from this the general statement. Also, let us point out that the results given here were obtained before the first part of this thesis and, hence, we use the original mesh relations as defined in Chapter 2, Section 2.2. The results in this chapter appear in [3] and [4] for the cases when the characteristic of the field K is different from 2 or equal to 2, respectively.

5.1.2 Outline of the chapter

We start with Section 5.2 where we study the generalized preprojective algebra $\Lambda = \mathbb{L}_n$ over a filed K. By Theorem 3.3.1, we know that Λ is symmetric. We explicitly give a dualizable basis of Λ. We then give a minimal projective resolution of Λ as a Λ-bimodule which induces in turn the concrete cochain complex of graded K-vector spaces which computes the Hochschild cohomology. In Section 5.3 we explicitly calculate the dimensions of the Hochschild cohomology and homology spaces and also those of the cyclic homology spaces in characteristic zero. We finally give a canonical basis of each $HH^i(\Lambda)$ consisting of homogeneous elements with respect to the length grading. This is done by identifying previously the structure of each $HH^i(\Lambda)$ as a module over $Z(\Lambda)$. Section 5.4 studies the multiplication in $HH^*(\Lambda)$ and, at the end of it, we give the proof of the main theorem , first over a field of characteristic $\neq 2$ and then in the general case. We also derive a presentation of the stable Hochschild cohomology ring of Λ, $\underline{HH}^*(\Lambda)$, by generators and relations. In the final Section 5.5 we deal with the case when the base field is of characteristic $= 2$.

Remark 5.1.1. After we had developed the contents of this chapter, we learnt about the preprint [34], where the multiplicative and the Batalin-Vilkovisky structure of $HH^*(\Lambda)$ is calculated over a field of characteristic zero (actually over the complex numbers). Note that Eu uses the term 'type T' instead of 'type L'. We do not look at the Gerstenhaber bracket in this chapter. Regarding the multiplicative structure, in the case of a ground field, the techniques used in our work are valid for all characteristics $\neq 2$ and detect a subtle difference of behavior between the cases when $\mathrm{Char}(K)$ divides $2n+1$ or not, where n is the number of vertices. We will comment throughout the text on similarities and dissimilarities between our work and [34].

5.1.3 Notation

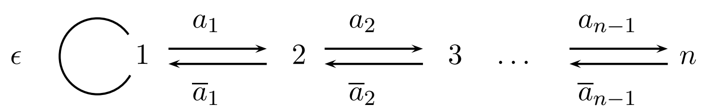
In this chapter, unless otherwise stated, Λ will be the generalized preprojective algebra $\mathbb{L}_n$ over a field K. We will follow the notation and terminology given in Chapter 4.

5.2 The generalized preprojective algebra $\mathbb{L}_n$

generalized preprojective algebra

5.2.1 Definition

We start by defining explicitly the generalized preprojective algebra $\mathbb{L}_n$. Recall that it was already introduced in Chapter 2, Subsection 2.3.2 as an orbit algebra of the mesh algebra associated to the Dynkin diagram $\mathbb{A}_{2n}$. The quiver quiver Q of $\mathbb{L}_n$ is



and the relations are $\epsilon^2 + a_1\bar{a}_1 = 0$, $a_i\bar{a}_i + \bar{a}_{i-1}a_{i-1} = 0$ $(i = 2, ..., n-1)$ and $\bar{a}_{n-1}a_{n-1} = 0$.

5.2.2 A dualizable basis

In [11] the authors used the fact that Λ is self-injective to prove that Λ is a periodic algebra. Note that the path algebra KQ admits an obvious involutive anti-isomorphism $(-)^{-} : KQ \longrightarrow KQ$ $(x \mapsto \bar{x})$ which fixes the vertices and the arrow ϵ and maps a_i to $\bar{a}_i$ and $\bar{a}_i$ to a_i, for all $i = 1, ..., n-1$. It clearly preserves the relations for Λ, and hence, it induces another involutive anti-isomorphism $(-)^{-} : \Lambda \longrightarrow \Lambda$. We shall call it the *canonical (involutive) antiautomorphism* of Λ. The next proposition shows that we can apply to Λ the results of the previous chapter. It also fixes the basis of Λ, with which we shall work throughout the chapter.

Proposition 5.2.1. *Let Λ be the generalized preprojective algebra $\mathbb{L}_n$ and put $B = \bigcup_{i,j} e_iBe_j$, where*

a) $e_1Be_1 = \{e_1, \epsilon, \epsilon^2, \dots, \epsilon^{2n-1}\}$

b) $e_1Be_j = \{a_1 \cdots a_{j-1}, \epsilon a_1 \cdots a_{j-1}, \epsilon^2 a_1 \cdots a_{j-1}, \dots, \epsilon^{2(n-j)+1} a_1 \cdots a_{j-1}\}$ *in case* $j \neq 1$

c) $e_iBe_j = \{a_i \cdots a_{j-1}, a_i \cdots a_j\bar{a}_j, \dots, a_i \cdots a_{n-1}\bar{a}_{n-1} \cdots \bar{a}_j\} \bigcup$
$\{\bar{a}_{i-1} \cdots \bar{a}_1 \epsilon a_1 \cdots a_{j-1}, \bar{a}_{i-1} \cdots \bar{a}_1 \epsilon^3 a_1 \cdots a_{j-1}, \dots,$
$(-1)^{s_{ij}} \bar{a}_{i-1} \cdots \bar{a}_1 \epsilon^{2(n-j)+1} a_1 \cdots a_{j-1}\}$

where $s_{ij} = 0$ for $i \neq j$ and $s_{ii} = \frac{i(i-1)}{2}$, whenever $1 < i \leq j \leq n$ (here adopt the convention that $a_i ... a_{j-1} = e_i$ in case $i = j$).

d) $e_iBe_j = \{\bar{b} : b \in e_jBe_i\}$ *in case* $i > j$,

Then B is a dualizable basis of Λ.

Proof. Note that e_iBe_j contains, at most, one element of a given degree. In order to see that B is a basis we just need to prove that all paths in e_iBe_j are nonzero and that they generate $e_i\Lambda e_j$ as a K-vector space. If so, we will get that $dim(e_i\Lambda_k e_j) \leq 1$, for all $i, j \in Q_0$ and all natural numbers k, and Lemma 4.6.2 can be applied. Let then assume that $i, j \in Q_0$ and $i \leq j$. The antiautomorphism $(-)^{-}$ given before guarantees that if e_iBe_j is a basis of $e_i\Lambda e_j$, then a basis e_jBe_i of $e_j\Lambda e_i$ is obtained by adding bars to the monomials in e_iBe_j.

Observe that for each vertex $i \neq 1$ we have, up to sign, a unique cycle of minimum length, namely $a_i\bar{a}_i$. However for the vertex $i = 1$ we do not only have the cycle $a_1\bar{a}_1 = -\epsilon^2$, but also the loop ϵ.

Let $0 \neq b$ be a monomial of a fixed length starting at i and ending at $i + s$. The previous comment tells us that b contains either an even number or an odd number of arrows of type ϵ.

In the first case, the equality $(\bar{a}_{i-1}a_{i-1})a_i \cdots a_{i+s-1} = (-1)^s a_i \cdots a_{i+s-1}(a_{i+s}\bar{a}_{i+s})$ shows that b has at most $n - i$ non-bar letters and $n - (s + i)$ bar letters. Thus we can set as a basis element the non-zero path $b = a_i \cdots a_{i+s+j}\bar{a}_{i+s+j} \cdots \bar{a}_{i+s}$ $(j \leq n - 1 - i - s)$, that is, where all the bar letters are to the right.

On the contrary, if b contains and odd number of ϵ-arrows, we have that

$$b = (a_i\bar{a}_i)(\bar{a}_{i-1}\cdots\bar{a}_1\epsilon^{2t-1}a_1\cdots a_{i+s-1}) = (-1)^i(\bar{a}_{i-1}\cdots\bar{a}_1\epsilon^{2t+1}a_1\cdots a_{i+s-1})$$

which is, up to sign, equal to

$$\bar{a}_{i-1}\cdots\bar{a}_1\epsilon(a_1\cdots a_t\bar{a}_t\cdots\bar{a}_1a_1\cdots a_{i+s-1})$$

But notice that the arrows between brackets form a path with an even number of ϵ arrows which is in time, up to sign, equal to $a_1\cdots a_{t+i+s-1}\bar{a}_{t+i+s-1}\cdots\bar{a}_{i+s}$. Hence, we can conclude that $\bar{a}_{i-1}\cdots\bar{a}_1\epsilon^{2t+1}a_1\cdots a_{i+s-1}$ is a nonzero path if and only if $0\leq t\leq n-(s+i)$. Thus, the sets given in the statement are in fact a basis of Λ.

It remains to prove that B is a dualizable basis. This task is reduced to prove that $a^*a=\omega_{t(a)}$, for each $a\in Q_1$. We have $\omega_{i(\epsilon)}=\epsilon^{2n-1}$, hence $\epsilon^*=\epsilon^{2n-2}$ and we clearly have $\epsilon^*\epsilon=\omega_{i(\epsilon)}$.

For a_i (i=1, …, n-1) we have

$$a_i[\bar{a}_i\cdots\bar{a}_1\epsilon^{2(n-i-1)+1}a_1\cdots a_{i-1}] = (-1)^i\bar{a}_{i-1}\cdots\bar{a}_1\epsilon^{2(n-i)+1}a_1\cdots a_{i-1} =$$

$$(-1)^i(-1)^{\frac{i(i-1)}{2}}\omega_i = (-1)^{\frac{i(i+1)}{2}}\omega_i = (-1)^{\frac{i(i+1)}{2}}\omega_{i(a_i)}.$$

Then $a_i^* = (-1)^{\frac{i(i+1)}{2}}\bar{a}_i\cdots\bar{a}_1\epsilon^{2(n-i-1)+1}a_1\cdots a_{i-1}$ and therefore

$$a_i^*a_i = (-1)^{\frac{i(i+1)}{2}}\bar{a}_i\cdots\bar{a}_1\epsilon^{2(n-i-1)+1}a_1\cdots a_{i-1}a_i = \omega_{i+1} = \omega_{t(a_i)}$$

The argument is symmetric for the arrows $\bar{a}_i$ and, therefore, the basis B is dualizable. □

Remark 5.2.2. Notice that if one modifies the basis B of Proposition 5.2.1, by putting $\omega_i = \bar{a}_{i-1}\cdots\bar{a}_1\epsilon^{2(n-i)+1}\ a_1\cdots a_{i-1}$, for all $i=1,\ldots,n$, then the resulting basis is no longer dualizable. Indeed, the proof of the lemma shows that, in the new situation, $a_i^* = (-1)^i\bar{a}_i\bar{a}_{i-1}\cdots\bar{a}_1\epsilon^{2(n-i-1)+1}a_1\cdots a_{i-1}$, and then $a_i^*a_i=(-1)^i\omega_{i+1}$.

By [11] (see also Corollary 3.4.5), we know that the third syzygy of Λ as a bimodule is isomorphic to ${}_1\Lambda_\mu$, for some $\mu\in\mathrm{Aut}(\Lambda)$ such that $\mu^2=id_\Lambda$. Our emphasis on choosing a dualizable basis on Λ comes from the fact that it allows a very precise determination of μ. In fact, combining results of [11] and [32], we know that if B is a dualizable basis, then the initial part of the minimal projective resolution of Λ as a bimodule is:

$$0\longrightarrow N\overset{\iota}{\hookrightarrow}P\overset{R}{\longrightarrow}Q\overset{\delta}{\longrightarrow}P\overset{u}{\longrightarrow}\Lambda\to 0,$$

where $P=\oplus_{i\in Q_0}\Lambda e_i\otimes e_i\Lambda$, $Q=\oplus_{a\in Q_1}\Lambda e_{i(a)}\otimes e_{t(a)}\Lambda$ and $N=\oplus_{i\in Q_0}\Lambda\xi_i=\oplus_{i\in Q_0}\xi_i\Lambda$, where $\xi_i=\sum_{x\in e_iB}(-1)^{\deg(x)}x\otimes x^*$, where $\mathcal{B}$ is any given basis of Λ consisting of paths and negative of paths which contains the vertices, the arrows and a basis of $\mathrm{Soc}(\Lambda)$. Here ι is the inclusion, u is the multiplication map and R and δ are as in Proposition 5.2.7 below.

The following result was proved in [11].

Lemma 5.2.3 (see [11], Proposition 2.3)**.** *Let $\mathcal{B}$ be a dualizable basis of Λ, let N be the Λ-bimodule mentioned above and let $\mu \in Aut(\Lambda)$ be the only automorphism of Λ such that $\mu(e_i) = e_i$ and $\mu(a) = -a$, for all $i \in Q_0$ and $a \in Q_1$. There is an isomorphism of Λ-bimodules $\phi : {}_1\Lambda_\mu \xrightarrow{\cong} N$ mapping e_i to ξ_i, for each $i \in Q_0$.*

Remark 5.2.4. The dualizable basis hypothesis does not appear in the statement of Proposition 2.3 in [11]. However, it is implicitly used in the proof of [11][Lemma 2.4]. From our work with examples it seems that, without that extra hypothesis, the element $\sum_{x\in e_i\mathcal{B}}(-1)^{\deg(x)}x \otimes x^*$ need not be in $\mathrm{Ker}(R)$.

The dualizable hypothesis seems to be implicitly used also in the argument of [34], Section 7.1, where the corresponding result (with the automorphism μ conveniently modified) is proved. In both cases, the crucial point is to guarantee that if $x \in \mathcal{B}$ is a homogeneous element of the basis $\mathcal{B}$ of degree > 0, then, for any arrow $a \in Q_1$, the element ax^* (resp. x^*a) should again be of the form y^*, for some $y \in \mathcal{B}$, whenever the product is nonzero. This follows immediately in case one has $a(ya)^* = y^*$ and $(ay)^*a = y$, for all $y \in \mathcal{B}$ and $a \in Q_1$. This is precisely the statement of Lemma 2.4 in [11] and is implicit in the argument of [34][Section 7.1].

Essentially, by the proof of our Lemma 4.6.2, we see that the mentioned crucial point is tantamount to require that $\mathcal{B}$ is a dualizable basis and that $(-,-)$ is its associated Nakayama form. If, as in the spirit of [34][Section 6.3], one has a symmetric Nakayama form $(-,-)$ from the very beginning and finds a basis $\mathcal{B}$ consisting of homogeneous elements which contains the vertices and has the property that the dual elements $\{\omega_i := e_i^* : i \in Q_0\}$ (in $\mathcal{B}^*$) belong to $\mathcal{B} \cap \mathrm{Soc}(\Lambda)$, then one readily sees that $\mathcal{B}$ is dualizable and $(-,-)$ is its associated Nakayama form.

In the rest of the chapter, the basis $\mathcal{B}$ will always be that of Proposition 5.2.1. The following properties can be derived in a routine way. We will omit the proof.

Recall that the *Cartan matrix* of a finite dimensional algebra A is given by the $Q_0 \times Q_0$ matrix $C_A := (c_{ij})$, where $c_{ij} = \dim_k(e_iAe_j)$.

Corollary 5.2.5. *Let $i, j \in Q_0$ be vertices. The following holds:*

1. *The set of degrees of the elements in e_iBe_j is*

$$\{j-i, j-i+2, j-i+4, \dots, j-i+2(n-j) = 2n-(i+j)\}$$
$$\bigcup\{j+i-1, j+i+1, j+i+3, \dots, j+i+2(n-max(i,j))-1\}$$

2. *If $\bar{a}_{i-1}\cdots\bar{a}_1\epsilon^{2k}a_1\cdots a_{j-1}$ is a nonzero element of Λ, then $k \le n-i-j+1$.*
3. $a_1\cdots a_{j-1}\bar{a}_{j-1}\cdots\bar{a}_1 = (-1)^{\frac{j(j-1)}{2}}\epsilon^{2(j-1)}$ *for $j = 2, \dots, n$.*
4. $a_1\cdots a_{j-1}\bar{a}_{j-1} = (-1)^{j-1}\epsilon^2 a_1\cdots a_{j-2}$
5. $\bar{a}_i a_i ... a_j = (-1)^{j-i+1}a_{i+1}...a_{j+1}\bar{a}_{j+1}$ *whenever $i \le j < n$ (using the convention that $a_n = 0$).*

6. *dim* $(Hom_{\Lambda^e}(P,\Lambda)) = \sum_{i=1}^{n} dim\ (e_i\Lambda e_i) = \sum_{i=1}^{n}[2(n-i)+2] = n^2+n$

7. *dim* $(Hom_{\Lambda^e}(Q,\Lambda)) = dim\ (e_1\Lambda e_1) + 2\sum_{i=1}^{n-1}(e_i\Lambda e_{i+1}) = 2n + 2\sum_{i=1}^{n-1}[2(n-i-1)+1] = 2n^2$

8. *The Cartan matrix of* Λ *is given by:*

$$\mathcal{C}_{\mathbb{L}_n} = \left(\begin{array}{c|cccc} 2n & 2(n\text{-}1) & 2(n\text{-}2) & \cdots & 2 \\ \hline 2(n\text{-}1) & & & & \\ \vdots & & \mathcal{C}_{\mathbb{L}_{n-1}} & & \\ 2 & & & & \end{array}\right)$$

where

$$\mathcal{C}_{\mathbb{L}_2} = \begin{pmatrix} 4 & 2 \\ 2 & 2 \end{pmatrix}$$

Its determinant is $det(\mathcal{C}_{\mathbb{L}_n}) = 2^n$ *(see Remark 3.3 in [48]).*

5.2.3 The minimal projective resolution

As we will do in Chapter 6 with $\mathbb{B}_n$, this projective resolution can be derived from Propositions 3.4.2 and 3.4.3. However, due to the fact that we are using the original mesh relations in this chapter, we prefer to give the explicit argument adapted to this case.

Let us first denote by $F_\tau : {}_\Lambda\text{Mod}_\Lambda \xrightarrow{\cong} {}_\Lambda\text{Mod}_\Lambda$ the equivalence taking M to ${}_1M_\tau$. We will need an alternative description of the self-equivalence of categories induced by F_τ on the full subcategory ${}_\Lambda Proj_\Lambda$ of ${}_\Lambda\text{Mod}_\Lambda$ consisting of the projective Λ-bimodules. We still denote by $F_\tau : {}_\Lambda Proj_\Lambda \xrightarrow{\cong} {}_\Lambda Proj_\Lambda$ the mentioned self-equivalence.

Lemma 5.2.6. *Let* $\Lambda = RQ/I$ *be a quotient of path algebra, let* $\tau \in Aut(\Lambda)$ *be an automorphism which fixes the vertices, and consider the R-linear functor* $G_\tau : {}_\Lambda Proj_\Lambda \longrightarrow {}_\Lambda Proj_\Lambda$ *identified by the following data:*

1. $G_\tau(P) = P$*, for each projective* Λ*-bimodule* P

2. G_τ *preserves coproducts*

3. *If* $f : \Lambda e_i \otimes e_j\Lambda \longrightarrow \Lambda e_k \otimes e_l\Lambda$ *is a morphism in* ${}_\Lambda Proj_\Lambda$*, then* $f_\tau := G_\tau(f)$ *is the only morphism of* Λ*-bimodules* $f_\tau : \Lambda e_i \otimes e_j\Lambda \longrightarrow \Lambda e_k \otimes e_l\Lambda$ *taking* $e_i \otimes e_j$ *to*$\sum_{1\leq r\leq m} a_r \otimes \tau^{-1}(b_r)$*, where* $f(e_i \otimes e_j) = \sum_{1\leq k\leq r} a_r \otimes b_r$*.*

Then G_τ *is naturally isomorphic to the self-equivalence* $F_\tau = {}_1(-)_\tau : {}_\Lambda Proj_\Lambda \longrightarrow {}_\Lambda Proj_\Lambda$*.*

Proof. Due to the fact that each projective Λ-bimodule is isomorphic to a direct summand of a coproduct of bimodules of the form $\Lambda e_i \otimes e_j\Lambda$, the given conditions determine a unique R-linear functor $G_\tau : {}_\Lambda Proj_\Lambda \longrightarrow {}_\Lambda Proj_\Lambda$. In order to give the desired natural isomorphism $\psi : G_\tau \xrightarrow{\cong} F_\tau$, it will be enough to define it on Λ-bimodules of the form $P = \Lambda e_i \otimes e_j \Lambda$. Indeed, for such a P, we define $\psi_P : G_\tau(P) = P \longrightarrow {}_1P_\tau = F_\tau(P)$ by the rule $\psi_P(a \otimes b) = a \otimes \tau(b)$. It is clear that ψ_P is an isomorphism of Λ-bimodules. Finally, it is straightforward to verify that, if $f : P = \Lambda e_i \otimes e_j \Lambda \longrightarrow Q = \Lambda e_k \otimes e_l \Lambda$ is a morphism of Λ-bimodules, then

$$F_\tau(f) \circ \psi_P = f \circ \psi_P = \psi_Q \circ f_\tau = \psi_Q \circ G_\tau(f),$$

which shows that the ψ_P define a natural isomorphism $\psi : G_\tau \xrightarrow{\cong} F_\tau$ as desired. □

We are now ready to give all the modules and maps of the minimal projective resolution $\Lambda = \mathbb{L}_n$ as a length-graded bimodule. We remark that the automorphism μ is involutive, which implies that ${}_\mu\Lambda_1 \cong {}_1\Lambda_\mu$. Recall from Section 4.4 the definition of the shifted graded module $M[n]$, which is used in the next result.

Proposition 5.2.7. *Let Λ be the generalized preprojective algebra $\mathbb{L}_n$, let $\mathcal{B}$ be the dualizable basis of Proposition 5.2.1 and let $\mu \in Aut(\Lambda)$ be the algebra automorphism that fixes the vertices and satisfies that $\mu(a) = -a$, for all $a \in Q_1$. The chain complex $\dots P^{-2} \xrightarrow{d^{-2}} P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{u} \Lambda \longrightarrow 0$ identified by the following properties is a minimal projective resolution of Λ as a length-graded bimodule:*

a) $P^{-r} = Q := \bigoplus_{a \in Q_1} \Lambda e_{i(a)} \otimes e_{t(a)} \Lambda$ if $r \equiv 1 \pmod 3$ and $P^{-r} = P := \bigoplus_{i \in Q_0} \Lambda e_i \otimes e_i \Lambda$ otherwise, as ungraded Λ-bimodules.

b) u is the multiplication map, $d^m = (d^r)_\mu$ whenever $m - r = \pm 3$ and the initial differentials $d^{-1} =: \delta$, $d^{-2} =: R$ and $d^{-3} =: k$ are the only homomorphisms of Λ-bimodules satisfying:

i) $\delta(e_{i(a)} \otimes e_{t(a)}) = a \otimes e_{t(a)} - e_{i(a)} \otimes a$

ii) $R(e_i \otimes e_i) = \sum_{a \in Q_1 i(a)=i} e_{i(a)} \otimes \bar{a} + a \otimes e_{i(a)}$

iii) $k(e_i \otimes e_i) = \sum_{x \in e_i B} (-1)^{deg(x)} x \otimes x^*$

for all $a \in Q_1$ and $i \in Q_0$.

c) When P and Q are given their canonical length grading, then, as graded Λ-bimodules, the equalities $P^{-3k-t} = P[-(2n+1)k - t]$, for $t = 0, 2$, and $P^{-3k-1} = Q[-(2n+1)k - 1]$, hold for all $k \geq 0$.

Proof. By Lemma 5.2.3 (see [11], Prop. 2.3), we have an exact sequence of Λ-bimodules:

$$0 \longrightarrow {}_1\Lambda_\mu \xrightarrow{j} P \xrightarrow{R} Q \xrightarrow{\delta} P \xrightarrow{u} \Lambda \longrightarrow 0,$$

where the map j satisfies that $j(e_i) = \sum_{x \in e_i \mathcal{B}} (-1)^{\deg(x)} x \otimes x^*$ for each $i \in Q_0$.

Applying the self-equivalence $F_\mu : {}_\Lambda\mathrm{Mod}_\Lambda \longrightarrow {}_\Lambda\mathrm{Mod}_\Lambda$, which acts as the identity on morphisms, and bearing in mind that $\mu^2 = 1_\Lambda$, we get an exact sequence

$$0 \longrightarrow \Lambda \xrightarrow{j} {}_1P_\mu \xrightarrow{R} {}_1Q_\mu \xrightarrow{\delta} {}_1P_\delta \xrightarrow{u} {}_1\Lambda_\mu \longrightarrow 0$$

By Lemma 5.2.6, we then get an exact sequence of Λ-bimodules

$$0 \longrightarrow \Lambda \xrightarrow{\tilde{j}} P \xrightarrow{R_\mu} Q \xrightarrow{\delta_\mu} P \xrightarrow{\tilde{u}} {}_1\Lambda_\mu \longrightarrow 0,$$

where, if $\psi : G_\mu \xrightarrow{\cong} F_\mu$ denotes the natural isomorphism of Lemma 5.2.6, then $\tilde{u} = u \circ \psi_P : a \otimes b \mapsto a\mu(b)$ and $\tilde{j} = \psi_P^{-1} \circ j$ which takes e_i to $-\sum_{x \in e_i \mathcal{B}} x \otimes x^*$.

The composition $P \xrightarrow{\tilde{u}}_1 \Lambda_\mu \xrightarrow{j} P$ takes $e_i \otimes e_i \sum_{x \in e_i \mathcal{B}} (-1)^{\deg(x)} x \otimes x^*$ and, hence, coincides with the morphism k given in the statement. Finally, the composition $P \xrightarrow{u} \Lambda \xrightarrow{\tilde{j}} P$ takes $e_i \otimes e_i$ to $-\sum_{x \in e_i \mathcal{B}} x \otimes x^* = \sum_{x \in e_i \mathcal{B}} (-1)^{\deg(x)} x \otimes \mu(x^*) = k_\mu(e_i \otimes e_i)$. Therefore we have the equality $\tilde{j} \circ u = k_\mu$, from which the exactness of the cochain complex $P^\bullet$ follows.

We finally see that $P^\bullet$ can be made into a (minimal) projective resolution in the category of length-graded Λ-bimodules, using the grading determined by condition c) in the statement. Indeed, when considering each P^{-r} with its canonical length grading, the differential d^{-r} is a graded morphism of degree 1, when $r \equiv 1, 2 \pmod 3$, and of degree 2n-1, when $r \equiv 0 \pmod 3$. It follows that if we shift the gradings and put $P^{-3k-t} = P[-(2n+1)k - t]$, for $t = 0, 2$, and $P^{-3k-1} = Q[-(2n+1) - 1]$, for all $k \geq 0$, then all the differentials in $P^\bullet$ become graded maps of degree 0, i.e. morphisms in the category of graded Λ-bimodules. It follows that the resulting complex is a (minimal) projective resolution of Λ in this category. □

Remark 5.2.8. The action of δ_μ, R_μ and k_μ is given as follows:

i) $\delta_\mu(e_{i(a)} \otimes e_{t(a)}) = a \otimes e_{t(a)} + e_{i(a)} \otimes a$

ii) $R_\mu(e_i \otimes e_i) = \sum_{a \in Q_1 i(a) = i} (a \otimes e_{i(a)} - e_{i(a)} \otimes \bar{a})$

iii) $k_\mu(e_i \otimes e_i) = -\sum_{x \in e_i \mathcal{B}} x \otimes x^*$

for all $a \in Q_1$ and $i \in Q_0$.

5.2.4 A cochain complex giving the Hochschild cohomology

Note that if M is a graded Λ-bimodule and r is any integer, then we have an isomorphism of graded K-vector spaces $\mathrm{Hom}_{\Lambda^e}(M[-r], \Lambda) \cong \mathrm{Hom}_{\Lambda^e}(M, \Lambda)[r]$ (see Section 4.4 for further details). In the particular case that $M = \Lambda e_i \otimes e_j \Lambda$, the grading on $e_i \Lambda e_j$ derived from the isomorphism $\mathrm{Hom}_{\Lambda^e}(\Lambda e_i \otimes e_j \Lambda, \Lambda) \cong e_i \Lambda e_j$ and the length grading on $\Lambda e_i \otimes e_j \Lambda$ is just the usual length grading.

If $f : \oplus_{s=1}^m \Lambda e_{i_s} \otimes e_{j_s} \Lambda[-r] \longrightarrow \oplus_{t=1}^p \Lambda e_{k_t} \otimes e_{l_t} \Lambda[-m]$ $(i_s, j_s, k_t, l_t \in Q_0)$ is a morphism of projective graded Λ-bimodules, an application of the contravariant functor $\mathrm{Hom}_{\Lambda^e}(, \Lambda) : {}_\Lambda\mathrm{Mod}_\Lambda \longrightarrow {}_k\mathrm{Mod}$ gives a morphism of graded K-vector spaces

$$f^* : Hom_{\Lambda^e}(\oplus_{t=1}^p \Lambda e_{k_t} \otimes e_{l_t}, \Lambda)[m] \longrightarrow Hom_{\Lambda^e}(\oplus_{s=1}^m \Lambda e_{i_s} \otimes e_{j_s} \Lambda, \Lambda)[r].$$

Using the isomorphism mentioned in the previous paragraph, we get an induced morphism, still denoted the same $f^* : \oplus_{t=1}^p e_{k_t} \Lambda e_{l_t}[m] \longrightarrow \oplus_{s=1}^m e_{i_s} \Lambda e_{j_s}[r]$.

As usual, we will also denote by $J = J(\Lambda)$ the Jacobson radical of Λ. With this terminology, we get:

Proposition 5.2.9. *Let Λ be taken with its length grading and consider the complex of K-vector spaces*

$$V^{\bullet}: \cdots 0 \longrightarrow \oplus_{i\in Q_0} e_i\Lambda e_i \xrightarrow{\delta^*} \oplus_{a\in Q_1} e_{i(a)}\Lambda e_{t(a)} \xrightarrow{R^*} \oplus_{i\in Q_0} e_i\Lambda e_i \xrightarrow{k^*} \oplus_{i\in Q_0} e_i\Lambda e_i$$

$$\xrightarrow{\delta^*_\mu} \oplus_{a\in Q_1} e_{i(a)}\Lambda e_{t(a)} \xrightarrow{R^*_\mu} \oplus_{i\in Q_0} e_i\Lambda e_i \xrightarrow{k^*_\mu} \oplus_{i\in Q_0} e_i\Lambda e_i \xrightarrow{\delta^*} \oplus_{a\in Q_1} e_{i(a)}\Lambda e_{t(a)} \cdots$$

where $V^0 = \sum_{i\in Q_0} e_i\Lambda e_i$ and $V^n = 0$ $\forall n < 0$. We view $V^{\bullet}$ as a complex of graded K-vector spaces by putting $V^{3k+t} = \oplus_{i\in Q_0} e_i\Lambda e_i[(2n+1)k+t]$, when $t = 0, 2$, and $V^{3k+1} = \oplus_{a\in Q_1} e_{i(a)}\Lambda e_{t(a)}[(2n+1)k+1]$, for all $k \geq 0$. Then $HH^n(\Lambda)$ is the n-th cohomology graded K-vector space of $V^{\bullet}$, for each $n \geq 0$.

Moreover, viewing $\oplus_{i\in Q_0} e_i\Lambda e_i$ and $\oplus_{a\in Q_1} e_{i(a)}\Lambda e_{t(a)}$ as subspaces of Λ, the differentials of $V^{\bullet}$ act as follows, for each oriented cycle c at i and each path $p : i(a) \to \cdots \to t(a)$:

a) $\delta^*(c) = a_{i-1}c - c\bar{a}_{i-1} + \bar{a}_i c - ca_i$

b) $R^*(p) = p\bar{a} + \bar{a}p$

c) $k^*(c) = 0$ *(i.e. k^* is the zero map)*

d) $\delta^*_\mu(c) = a_{i-1}c + c\bar{a}_{i-1} + \bar{a}_i c + ca_i$

e) $R^*_\mu(p) = p\bar{a} - \bar{a}p$

f) $k^*_\mu(c) = 0$ *if $c \in e_iJe_i$, and* $k^*_\mu(e_i) = -\sum_{j\in Q_0} dim\ (e_i\Lambda e_j)\omega_j$

using the convention that $a_0 = \bar{a}_0 = \epsilon$ and $a_n = \bar{a}_n = 0$

Proof. $HH^n(\Lambda)$ is the n-th cohomology graded space of the complex obtained by applying $Hom_{\Lambda^e}(\ -, \Lambda)$ to the minimal projective resolution of Λ as a graded Λ-bimodule. The graded K-vector spaces of that complex are precisely those of $V^{\bullet}$ (see the comments preceding this proposition) and the only nontrivial part is the explicit definition of its differentials.

We have two canonical isomorphisms of k-vector spaces:

$$\oplus_{j\in Q_0} e_j\Lambda e_j \xrightarrow{\sim} Hom_{\Lambda^e}(\oplus_{j\in Q_0}\Lambda e_j \otimes e_j\Lambda, \Lambda)$$

$$\oplus_{a\in Q_1} e_{i(a)}\Lambda e_{t(a)} \xrightarrow{\sim} Hom_{\Lambda^e}(\oplus_{a\in Q_1}\Lambda e_{i(a)} \otimes e_{t(a)}\Lambda, \Lambda)$$

The first one identifies a nonzero oriented cycle c at i with the morphism of Λ-bimodules $\oplus_{j\in Q_0}\Lambda e_j \otimes e_j\Lambda \xrightarrow{\tilde{c}} \Lambda$ taking $e_j \otimes e_j$ to $\delta_{ij}c$, where δ is the Kronecker symbol. Similarly, a nonzero path $p : i(a) \to \cdots \to t(a)$ is identified by the second isomorphism with the morphism of Λ-bimodules $\oplus_{b\in Q_1}\Lambda e_{i(b)} \otimes e_{t(b)}\Lambda \longrightarrow \Lambda$ taking $e_{i(b)} \otimes e_{t(b)}$ to $\delta_{ab}p$. Using these identifications, it is straightforward to verify that the given formulas for the differentials are correct. We only give a few sample computations:

a) $\delta^*(c)$ is the element of $\oplus_{b\in Q_1} e_{i(b)}\Lambda e_{t(b)}$ identified with $\tilde{c}\circ\delta \in Hom_{\Lambda^e}(\oplus_{b\in Q_1}\Lambda e_{i(b)} \otimes e_{t(b)}\Lambda, \Lambda)$. Then

$$\delta^*(c) = \sum_{b\in Q_1} (\tilde{c}\circ\delta)(e_{i(b)} \otimes e_{t(b)}) = \sum_{b\in Q_1} \tilde{c}(b\otimes e_{t(b)} - e_{i(b)} \otimes b) =$$

$$\sum_{b\in Q_1} [b\tilde{c}(e_{t(b)} \otimes e_{t(b)}) - \tilde{c}(e_{t(b)} \otimes e_{i(b)})b] = \sum_{b\in Q_1, t(b)=i} bc \quad - \sum_{b\in Q_1, i(b)=i} cb =$$

$$a_{i-1}c + \bar{a}_i c - ca_i - c\bar{a}_{i-1}$$

c) $k^*(c)$ is the element of $\oplus_{j\in Q_0} e_j\Lambda e_j$ identified with $\tilde{c}\circ k \in Hom_{\Lambda^e}(\oplus_{j\in Q_0}\Lambda e_j \otimes e_j\Lambda, \Lambda)$. Then

$$k^*(c) = \sum_{j\in Q_0} (\tilde{c}\circ k)(e_j \otimes e_j) = \sum_{j\in Q_0} \tilde{c}(\sum_{x\in e_j\mathcal{B}} (-1)^{\deg(x)} x\otimes x^*) = \sum_{j\in Q_0}\sum_{x\in e_j\mathcal{B}e_i} (-1)^{\deg(x)} xcx^*$$

But $xcx^* = 0$ in case $\deg(c) > 0$ because then $\deg(xcx^*) > 2n-1 = c_\Delta - 2$. In case $c = e_j$ we have $k^*(e_j) = \sum_{j\in Q_0}\sum_{x\in e_j\mathcal{B}e_i} (-1)^{\deg(x)} xx^*$. Bearing in mind that $xx^* = \omega_j$ for each $x \in e_j\mathcal{B}e_i$ and that the number of elements in $e_j\mathcal{B}e_i$ with even degree is the same as the number of those with odd degree, we conclude that also $k^*(e_i) = 0$. Since k^* vanishes on all nonzero oriented cycles it follows that $k^* = 0$.

f) Arguing similarly with k_μ^* we get that
$k_\mu^*(c) = 0$ if $\deg(c) > 0$ and
$k_\mu^*(e_i) = -\sum_{j\in Q_0}\sum_{x\in e_j\mathcal{B}e_i} xx^* = -\sum_{j\in Q_0} \dim\,(e_j\Lambda e_i)\omega_j$ □

Remark 5.2.10. With the adequate change of presentation of the algebra, the complex $V^\bullet$ should correspond to the sequence of morphisms in [34][Section 7.4], although the differentials defined there do not seem to make it into a complex.

Corollary 5.2.11. Λ *is a symmetric periodic algebra of period* 6 *and* $\mathcal{P}(\Lambda,\Lambda) = Soc(\Lambda)$ *when we view the isomorphism* $HH^0(\Lambda) \cong Z(\Lambda)$ *as an identification.*

Proof. By Proposition 3.3.1, we know that Λ is symmetric, and, by Proposition 3.4.12, Λ is periodic of period 6.

To see that the isomorphism $HH^0(\Lambda) \cong Z(\Lambda)$ identifies $\mathcal{P}(\Lambda,\Lambda)$ with $\mathrm{Soc}(\Lambda) = \mathrm{Soc}(Z(\Lambda))$, note that from Proposition 5.2.7, with the same terminology, one obtains a complete projective resolution of Λ as a length-graded Λ-bimodule, by putting $P^{-3k-t} = P[-(2n+1)k-t]$, if $t = 0, 2$, and $P^{-3k-1} = Q[-(2n+1)k-1]$, for all $k \in \mathbb{Z}$,

$$\cdots P^{-2} \longrightarrow P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{d^0} P^1 \xrightarrow{d^1} P^2 \longrightarrow \cdots,$$

and the differentials are given by $d^m = (d^n)_\mu$ whenever $m \equiv n \pmod 3$ and $d^{-1} = \delta$, $d^{-2} = R$ and $d^{-3} = k$. It follows that $\underline{HH}^*(\Lambda)$ is the cohomology of the complex

$$\cdots V^{-2} \xrightarrow{R_\mu^*} V^{-1} \xrightarrow{k_\mu^*} V^0 \xrightarrow{\delta^*} V^1 \xrightarrow{R^*} V^2 \longrightarrow \cdots,$$

where $V^i = V^{i+6}$, for all $i \in \mathbb{Z}$. In particular, we have $\underline{HH}^0(\Lambda) = \frac{\text{Ker}(\delta^*)}{\text{Im}(k_\mu^*)}$. But $\text{Ker}(\delta^*) = HH^0(\Lambda) = Z(\Lambda)$ while $\text{Im}(k_\mu^*) = \text{Soc}(\Lambda)$ since the Cartan matrix of Λ is invertible. Note that the isomorphism $Z(\Lambda) \cong End_{\Lambda^e}(\Lambda)$ identifies $\text{Im}(k_\mu^*)$ with $\mathcal{P}(\Lambda, \Lambda)$. □

Corollary 5.2.12. *There are isomorphisms of graded $\underline{HH}^*(\Lambda)$-modules:*

$$\underline{HH}^*(\Lambda) \cong \underline{HH}^*(\Lambda)[6]$$

$$\underline{HH}_{-*}(\Lambda) \cong D(\underline{HH}^*(\Lambda)) \cong \underline{HH}^*(\Lambda)[5]$$

and isomorphisms of graded $HH^*(\Lambda)$-modules $HH_{-*}(\Lambda) \cong D(HH^*(\Lambda))$.

Moreover, $\underline{HH}^*(\Lambda)$ is a pseudo-Frobenius graded algebra admitting a graded Nakayama form of constant degree function equal to 5*.*

Proof. $\underline{HH}^*(\Lambda) \cong \underline{HH}^*(\Lambda)[6]$ since Λ is periodic of period 6. On the other hand, Λ is 5-CY Frobenius and, by Theorem 4.5.7, we have

$$D(\underline{HH}^*(\Lambda)) \cong \underline{HH}^*(\Lambda)[11]$$

$$\underline{HH}_{-*}(\Lambda) \cong \underline{HH}^*(\Lambda)[5]$$

Then, the isomorphisms in the statement follow. The graded pseudo-Frobenius condition of $\underline{HH}^*(\Lambda)$ also follows from Theorem 4.5.7 and the isomorphism $D(\underline{HH}^*(\Lambda)) \cong \underline{HH}^*(\Lambda)[5]$ imply that we can choose a graded Nakayama form for $\underline{HH}^*(\Lambda)$ of constant degree function equal to 5.

On the other hand, due to Remark 4.5.5 and the fact that $D(\Lambda) \cong \Lambda$, we have an isomorphism $HH_{-*}(\Lambda) \cong D(HH^*(\Lambda, D(\Lambda))) \cong D(HH^*(\Lambda))$. □

5.3 The Hochschild cohomology spaces

Except in the last section, we assume that $\text{Char}(K) \neq 2$.

In this section we will use the complex $V^\bullet$ of Proposition 5.2.9 to calculate the dimension and an appropriate basis of each space $HH^i(\Lambda)$. In the proof of the following lemma and in the rest of the chapter, the matrix of a linear map is always written by columns.

Lemma 5.3.1. *The equality $Im(R^*) = \oplus_{i \in Q_0} e_i J e_i$ holds and $Im(R_\mu^*)$ is a subspace of codimension n in $\oplus_{i \in Q_0} e_i J e_i$. In particular, we have:*

$$dim\ (Im(R^*)) = n^2$$

$$dim\ (Im(R_\mu^*)) = n^2 - n.$$

Proof. We put $V = \oplus_{a\in Q_1} e_{i(a)}\Lambda e_{t(a)}$ and $W = \oplus_{i\in Q_0} e_i J e_i$ for simplicity and view R^* and R^*_μ as K-linear maps $V \longrightarrow W$. For each $0 \leq k < 2n$ we denote by V_k (resp. W_k) the vector subspace consisting of the elements of degree k. Since both R^* and R^*_μ are graded maps of degree 1 we have induced K-linear maps

$$R^*, R^*_\mu : V_{k-1} \longrightarrow W_k$$

for $k = 1, \dots, 2n-1$.

It is important now to notice that the canonical anti-automorphism of Λ, $x \mapsto \bar{x}$, is the identity on W. Moreover, we have equalities $R^*(\bar{p}) = \overline{R^*(p)}$ and $R^*_\mu(\bar{p}) = -\overline{R^*_\mu(p)}$. We then get $R^*(\bar{p}) = R^*(p)$ and $R^*_\mu(\bar{p}) = -R^*_\mu(p)$. This tells us that the images of the maps $R^*, R^*_\mu : V \longrightarrow W$ are the same as those of their restrictions to $V^+ = V \bigcap (\oplus_{j=0}^{n-1} e_{i(a_j)}\Lambda e_{t(a_j)})$ (using the convention that $a_0 = \epsilon$). Those images are in turn the direct sum of the images of the induced maps

$$R^*, R^*_\mu : V^+_{k-1} \longrightarrow W_k \qquad (k = 1, \dots 2n-1).$$

These are the ones that we shall calculate.

Let us denote by b^t_i the only element in $e_i B e_i$ of degree t.

We start by considering the case when $k = 2m$ is even ($1 \leq m \leq n-1$). In that situation, a basis of W_{2m} is given by $\{b^{2m}_1, b^{2m}_2, \dots, b^{2m}_{n-m}\}$ while a basis of V^+_{2m-1} is $\{v_\epsilon, v_{a_1}, \dots, v_{a_{n-m}}\}$ where $v_\epsilon = \epsilon^{2m-1}$ and $v_{a_i} = a_i \cdots a_{i+m-1}\bar{a}_{i+m-1} \cdots \bar{a}_{i+1}$ for $i = 1, \dots, n-m$. In particular dim $(V^+_{2m-1}) = n-m+1$ and dim $(W_{2m}) = n-m$. Direct computation, using Remark 5.2.5, shows that

i) $R^*(v_\epsilon) = 2b^{2m}_1, \qquad R^*_\mu(v_\epsilon) = 0$

ii) $R^*(v_{a_1}) = (-1)^{\frac{(m+1)m}{2}} b^{2m}_1 + (-1)^m b^{2m}_2$

$R^*_\mu(v_{a_1}) = (-1)^{\frac{(m+1)m}{2}} b^{2m}_1 + (-1)^{m+1} b^{2m}_2$

iii) $R^*(v_{a_i}) = b^{2m}_i + (-1)^m b^{2m}_{i+1}$

$R^*_\mu(v_{a_i}) = b^{2m}_i + (-1)^{m+1} b^{2m}_{i+1}$

(using the convention that $b^{2m}_j = 0$ if $j > n-m$)

Then in the matrices of R^* and R^*_μ with respect to the given bases of V^+_{2m-1} and W_{2m}, which are both of size $(n-m)\times(n-m+1)$, the columns from the 2^{nd} to the $(n-m+1)-th$ are linearly independent. Therefore, we get that the maps $R^*, R^*_\mu : V_{2m-1} \longrightarrow W_{2m}$ are both surjective for each $m = 1, \dots n-1$.

We now deal with the case when $k = 2m-1$ is odd, in which case a basis of W_{2m-1} is $\{b^{2m-1}_1, \dots b^{2m-1}_m\}$. On the other hand, a basis of V^+_{2m-2} is given by $\{v'_\epsilon, v'_{a_1}, \dots v'_{a_{m-1}}\}$, where $v'_\epsilon = \epsilon^{2m-2}$ and $v'_{a_i} = \bar{a}_{i-1} \cdots \bar{a}_1 \epsilon^{2(m-i)-1} a_1 \cdots a_i$ for $i = 1, \dots, m-1$. Direct calculation, using again Remark 5.2.5, shows the following:

i) $R^*(v'_\epsilon) = 2b^{2m-1}_1, \qquad R^*_\mu(v'_\epsilon) = 0$

ii) If $m \neq n$ then
$R^*(v'_{a_i}) = (-1)^i b_i^{2m-1} + b_{i+1}^{2m-1}$
$R^*_\mu(v'_{a_i}) = (-1)^i b_i^{2m-1} - b_{i+1}^{2m-1}$

iii) If $m = n$ then
$R^*(v'_{a_i}) = (-1)^{\frac{(i+1)i}{2}} b_i^{2n-1} + (-1)^{\frac{(i+1)i}{2}} b_{i+1}^{2n-1} = (-1)^{\frac{(i+1)i}{2}}(\omega_i + \omega_{i+1})$
$R^*_\mu(v'_{a_i}) = (-1)^{\frac{(i+1)i}{2}} b_i^{2n-1} - (-1)^{\frac{(i+1)i}{2}} b_{i+1}^{2n-1} = (-1)^{\frac{(i+1)i}{2}}(\omega_i - \omega_{i+1})$

Therefore, the square matrices of R^* and R^*_μ with respect to the given bases of V^+_{2m-2} and W_{2m-1} are upper triangular. In the case of R^* all its diagonal entries are nonzero while in the case of R^*_μ only the entry $(1,1)$ is zero. It follows:

a) The map $R^* : V_{2m-2} \longrightarrow W_{2m-1}$ is surjective for all $m = 1, \dots n$.

b) The image of the map $R^*_\mu : V_{2m-2} \longrightarrow W_{2m-1}$ has codimension 1 in W_{2m-2} for all $m = 1, \dots, n$.

The final conclusion is that the map $R^* : V \longrightarrow W$ is surjective while the image of $R^*_\mu : V \longrightarrow W$ has codimension exactly the number of odd numbers in $\{1, 2, \dots 2n-1\}$. That is, dim (W) − dim $(\mathrm{Im}(R^*_\mu)) = n$. □

Remark 5.3.2. The proof of Lemma 5.3.1 gives that if ω_j is viewed as an element of $\mathrm{Ker}(k^*_\mu)$ $\forall j \in Q_0$, then $\omega_j - \omega_{j+1} \in \mathrm{Im}(R^*_\mu)$ $\forall j = 1, 2, \dots, n-1$.

The following result describes the structure of the center $Z(\Lambda)$ of Λ, .

Proposition 5.3.3. *The center of Λ is isomorphic to $\frac{K[x_0, x_1, \dots x_n]}{I}$, where I is the ideal of $K[x_0, x_1, \dots x_n]$ generated by x_0^n and all the products $x_i x_j$ with $(i,j) \neq (0,0)$. In particular, dim $(HH^0(\Lambda)) = 2n$.*

Proof. It is well-known that $Z(\Lambda) \subseteq \oplus_{i \in Q_0} e_i \Lambda e_i$, that $J(Z(\Lambda)) = Z(\Lambda) \bigcap J(\Lambda)$ and $\frac{Z(\Lambda)}{J(\Lambda)} = K \cdot 1 = K(e_1 + \cdots e_n)$. Since Λ is graded, one readily sees that the grading on Λ gives by restriction a grading on $Z(\Lambda)$.

We claim that if $z \in Z(\Lambda)_{2m-1}$ is an element of odd degree $2m-1$, then $m = n$ and z is a linear combination of the socle elements $\omega_1, \dots, \omega_n$. Indeed, we have $z = \sum_{i=1}^r \lambda_i b_i^{2m-1}$, with $\lambda_r \neq 0$, for some integer $1 \leq r \leq m$. If $r < n$ then $\lambda_r b_r^{2m-1} a_r = z a_r = a_r z = 0$, and hence $0 = b_r^{2m-1} a_r = \bar{a}_{r-1} \cdots \bar{a}_1 \epsilon^{2(m-r)+1} a_1 \cdots a_{r-1} a_r$. This only happens when $m = n$, in which case $b_r^{2m-1} = b_r^{2n-1} = \omega_r$. On the other hand, if $r = n$ then $n = m$ and we are also over in this case.

The previous paragraph shows that $Z(\Lambda)_{odd} := \oplus_{m>0} Z(\Lambda)_{2m-1} = \sum_{i \in Q_0} K\omega_i = \mathrm{Soc}(\Lambda)$ since $\omega_i \in Z(\Lambda)_{2n-1}$ for each $i \in Q_0$. We now want to identify $Z(\Lambda)^+_{even} := \oplus_{m>0} Z(\Lambda)_{2m}$. One easily checks that $x_0 = \sum_{i=1}^{n-1} (-1)^i a_i \bar{a}_i = b_1^2 + \sum_{i=2}^{n-1} (-1)^i b_i^2$ is an element of $Z(\Lambda)_2$. Moreover, $(b_i^2)^m \neq 0$ if and only if $1 \leq i \leq n-m$ and $m < n$. In this case we necessarily have an equality $(b_i^2)^m = (-1)^{t_i} b_i^{2m}$, for some integer exponent t_i. In particular $x_0^m \neq 0$ and $x_0^m = \sum_{i=0}^{n-m} \lambda_i b_i^{2m}$, with scalars λ_i all nonzero. We claim that if

$0 \neq z \in Z(\Lambda)_{2m}$ and we write it as a K-linear combination $z = \sum_{i=1}^{n-m} \mu_i b_i^{2m}$, then $\mu_i \neq 0$ for all $i = 1, \ldots, n-m$. Suppose that it is not the case. We first prove that if $\mu_j = 0$ then $\mu_i = 0$ for each $i \leq j$. For that purpose, we can assume $j > 1$ and then we have

$$0 = \mu_j a_{j-1} b_j^{2m} = a_{j-1} z = z a_{j-1} = \mu_{j-1} b_{j-1}^{2m} a_{j-1}$$

But $b_{j-1}^{2m} a_{j-1} \neq 0$ since $j \leq n-m \leq n-1$ and so $j-1 < n-m$. It follows that $\mu_{j-1} = 0$ and, by iterating the process, that $\mu_i = 0 \ \forall i \leq j$.

We can then write $z = \sum_{i=r}^{n-m} \mu_i b_i^{2m}$ for some $1 \leq r \leq n-m$ and $\mu_i \neq 0 \ \forall i = r, \ldots, n-m$. We prove that $r = 1$ and our claim will be settled. Indeed, if $r > 1$ then we have

$$\mu_r a_{r-1} b_r^{2m} = a_{r-1} z = z a_{r-1} = 0$$

which implies that $\mu_r = 0$ since $a_{r-1} b_r^{2m} \neq 0$. This is a contradiction.

Once we know that if $z \in Z(\Lambda)_{2m} \backslash 0\}$ and $z = \sum_{i=1}^{n-m} \mu_i b_i^{2m}$ then $\mu_i \neq 0 \ \forall i = 1, \ldots n-m$, we easily conclude that any such z is a scalar multiple of x_0^m. Then $Z(\Lambda)_{2m} = K x_0^m$, for each $m > 0$.

Putting now $x_i = \omega_i \ \forall i = 1, \ldots, n$ we clearly have that $x_0, x_1, \ldots x_n$ generate $Z(\Lambda)$ as an algebra and they are subject to the relations $x_0^n = 0$ and $x_i x_j = 0$ for $(i,j) \neq (0,0)$. □

We are now ready to prove the main result of this section which provides the dimension of each Hochschild cohomology space $HH^i(\Lambda)$ as a K-vector space:

Theorem 5.3.4. *Let us assume that Char(K) $\neq 2$ and let Λ be the generalized preprojective algebra $\mathbb{L}_n$ over K. Then dim $(HH^0(\Lambda)) =$ dim $(HH_0(\Lambda)) = 2n$ and dim $(HH^i(\Lambda)) =$ dim $(HH_i(\Lambda)) = n$ for all $i > 0$.*

Proof. By the isomorphism $HH_{-*}(\Lambda) \cong D(HH^*(\Lambda))$ (see Remark 4.5.5), it is enough to calculate the dimensions of the Hochschild cohomology spaces.

On the other hand, by Corollary 5.2.12, we have and isomorphism $\underline{HH}^*(\Lambda) \cong \underline{HH}^*(\Lambda)[6]$. We then get isomorphisms of K-vector spaces

$$HH^{6k}(\Lambda) \cong \underline{HH}^0(\Lambda) = \frac{HH^0(\Lambda)}{\mathcal{P}(\Lambda,\Lambda)} = \frac{Z(\Lambda)}{\mathrm{Soc}(\Lambda)}$$
$$HH^{6k+i}(\Lambda) \cong HH^i(\Lambda),$$

for all $k > 0$ and $i = 1, 2, 3, 4, 5$.

By the same corollary, we have an isomorphism $D(\underline{HH}^*(\Lambda)) \cong \underline{HH}^*(\Lambda)[5]$, which gives isomorphisms of K-vector spaces:

$$D(\underline{HH}^0(\Lambda)) \cong HH^5(\Lambda)$$
$$D(HH^1(\Lambda)) \cong HH^4(\Lambda)$$
$$D(HH^2(\Lambda)) \cong HH^3(\Lambda).$$

Bearing in mind Lemma 4.3, the proof is reduced to check that

$$\dim \left(\frac{Z(\Lambda)}{\mathrm{Soc}(\Lambda)}\right) = \dim \left(HH^1(\Lambda)\right) = \dim \left(HH^2(\Lambda)\right) = n.$$

That dim $(\frac{Z(\Lambda)}{\mathrm{Soc}(\Lambda)}) = n$ follows directly from Proposition 5.3.3 and its proof. Moreover, we have two exact sequences

$$0 \longrightarrow \mathrm{Ker}(R^*) \hookrightarrow \oplus_{a\in Q_1} e_{i(a)}\Lambda e_{t(a)} \xrightarrow{R^*} \oplus_{i\in Q_0} e_i J e_i \longrightarrow 0$$

$$0 \longrightarrow Z(\Lambda) \hookrightarrow \oplus_{i\in Q_=} e_i \Lambda e_i \longrightarrow \mathrm{Im}(\delta^*) \longrightarrow 0$$

From the first one we get dim $(\mathrm{Ker}(R^*)) = 2n^2 - n^2 = n^2$, using Lemma 5.3.1 and Corollary 5.2.5. From the second sequence we get dim $(\mathrm{Im}(\delta^*)) = (n^2+n) - 2n = n^2 - n$, using Lemma 5.3.3. It follows that dim $(HH^1(\Lambda)) = n$.

We also have that $HH^2(\Lambda) \cong \mathrm{Coker}(R^*)$ since $k^* = 0$. But $\mathrm{Im}(R^*) = \oplus_{i\in Q_0} e_i J e_i$ by Lemma 5.3.1. It follows that dim $(HH^2(\Lambda)) =$ dim $(\oplus_{i\in Q_0} \frac{e_i\Lambda e_i}{e_i J e_i}) = n$. □

Once we have computed the dimensions of the Hochschild (co)homology spaces of Λ, we can do the same for its cyclic homology spaces in characteristic zero, denoted by $HC_i(\Lambda)$ following the notation used in [59]. We start by recalling the following fact about graded algebras.

Proposition 5.3.5. *Suppose Char$(K) = 0$ and let $A = \oplus_{i\geq 0} A_i$ be a positively graded algebra such that A_0 is a semisimple algebra. The following assertions hold:*

1. *As K-vector spaces,* $HC_i(A_0) \cong \begin{cases} 0 & \text{if } i \text{ is odd} \\ A_0 & \text{if } i \text{ is even} \end{cases}$

2. *Connes' boundary map B induces an exact sequence*

$$0 \longrightarrow A_0 \longrightarrow HH_0(A) \xrightarrow{B} HH_1(A) \xrightarrow{B} HH_2(A) \longrightarrow \cdots$$

such that the image of $B : HH_i(A) \longrightarrow HH_{i+1}(A)$ is isomorphic to $\frac{HC_i(A)}{HC_i(A_0)}$, for all $n \geq 0$.

Proof. Assertion 1 is well-known, and is a direct consequence of Connes' periodicity exact sequence ([59], Theorem 2.2.1) and the fact that $HH_i(A_0) = 0$, for all $i > 0$.

On the other hand, by [59], Theorem 4.1.13, we know that Connes' periodicity exact sequence gives exact sequences:

$$0 \longrightarrow \frac{HC_{i-1}(A)}{HC_{i-1}(A_0)} \xrightarrow{B} \frac{HH_i(A)}{HH_i(A_0)} \xrightarrow{I} \frac{HC_i(A)}{HC_i(A_0)} \longrightarrow 0$$

for all $i \geq 0$. Since $HH_i(A_0) = 0$, for $i > 0$, we get an induced K-linear map $B \circ I : HH_i(A) \longrightarrow HH_{i+1}(A)$ such that $\mathrm{Im}(B \circ I) = Im(B) \cong \frac{HC_i(A)}{HC_i(A_0)}$. □

Corollary 5.3.6. *If Λ is the generalized preprojective algebra $\mathbb{L}_n$, then*

$$dim\ HC_i(\Lambda) = \begin{cases} 0 & \text{if } i \text{ is odd} \\ 2n & \text{if } i \text{ is even} \end{cases}$$

Proof. Put $B^i := \mathrm{Im}(HH_i(\Lambda) \xrightarrow{B} HH_{i+1}(\Lambda))$ where B is Connes' map. From the previous theorem, we have

$$\dim\,(B^0) = \dim\, HH_0(\Lambda) - \dim\,(KQ_0) = 2n - n = n$$

and

$$\dim\,(B^i) = \dim\, HH_i(\Lambda) - \dim\,(B^{i-1}) = n - \dim\,(B^{i-1}),$$

for all $i > 0$.

It follows that $\dim\,(B^i) = n$, when i is even and zero otherwise.

Then we have

$$\dim\, HC_i(\Lambda) - \dim\, HC_i(KQ_0) = \begin{cases} n & \text{if i is even} \\ 0 & \text{if i is odd} \end{cases}$$

From this the result follows using the foregoing proposition. □

Remark 5.3.7. In [34][Section 7.5] the author calculates the reduced cyclic homology spaces $\overline{HC}_i(\Lambda)$ using Connes' sequence (see Proposition 5.3.5(2)) and, as a byproduct, he also calculates the absolute cyclic homology spaces. However, he states that the equality $HC_i(\Lambda) = \overline{HC}_i(\Lambda)$ holds, for all $i > 0$. This is not true since $\overline{HC}_i(\Lambda) = \frac{HC_i(\Lambda)}{HC_i(\Lambda_0)}$, for all $i > 0$. Therefore, the description of the $HC_i(\Lambda)$ in [34], page 22, is not correct.

Remark 5.3.8. Due to the fact that Λ is a Λ^e- $Z(\Lambda)$-bimodule, for each Λ-bimodule M, the K-vector space $Hom_{\Lambda^e}(M,\Lambda)$ inherits a structure of $Z(\Lambda)$-module. In particular, via the isomorphisms,

$$\oplus_{i\in Q_0} e_i \Lambda e_i \xrightarrow{\sim} Hom_{\Lambda^e}(P,\Lambda)$$

$$\oplus_{a\in Q_1} e_{i(a)} \Lambda e_{t(a)} \xrightarrow{\sim} Hom_{\Lambda^e}(Q,\Lambda)$$

both $\oplus_{i\in Q_0} e_i\Lambda e_i$ and $\oplus_{a\in Q_1} e_{i(a)}\Lambda e_{t(a)}$ have a structure of $Z(\Lambda)$-modules. It is easy to see that these structures are given by the multiplication in Λ and that the differentials of the complex $V^\bullet$ in Proposition 5.2.9 are all morphisms of $Z(\Lambda)$-modules.

Next, we identify the structure of each $HH^i(\Lambda)$ as a module over the center of Λ.

Lemma 5.3.9. *We view $Soc(\Lambda)$ as an ideal of $Z(\Lambda)$. The following assertions hold.*

1) $Soc(\Lambda)HH^j(\Lambda) = 0$ for all $j > 0$.

2) $HH^j(\Lambda)$ is a semisimple $Z(\Lambda)$-module for all $j \equiv 2,3 \pmod 6$

3) $HH^j(\Lambda)$ is isomorphic to $\frac{Z(\Lambda)}{Soc(\Lambda)}$ as a $Z(\Lambda)$-module for all $j > 0$, $j \not\equiv 2,3 \pmod 6$

Proof. 1) is a direct consequence of the fact that $\mathcal{P}(\Lambda,\Lambda) = \mathrm{Soc}(\Lambda)$ and $HH^j(\Lambda) \cong \underline{\mathrm{Hom}}_{\Lambda^e}(\Omega^j_{\Lambda^e}(\Lambda),\Lambda)$ for all $j > 0$.

2) If $x_0 = \sum_{i=0}^{n-1}(-1)^i a_i\bar{a}_i$ as in Proposition 5.3.3, then $x_0 HH^j(\Lambda) = x_0 \cdot (\oplus \frac{e_i\Lambda e_i}{e_iJe_i}) = 0$ when $j \equiv 2 \pmod 6$ and $x_0 HH^j(\Lambda) = x_0 \cdot \mathrm{Soc}(\Lambda) = 0$ when $j \equiv 3 \pmod 6$.

3) We clearly have an isomorphism $HH^j(\Lambda) \cong HH^{j+6}(\Lambda)$ for all $j > 0$, so we only need to prove the claim for $j = 1, 4, 5, 6$.

   From Corollary 5.2.12 and the fact that $Hom_K(\frac{Z(\Lambda)}{\mathrm{Soc}(\Lambda)}, K) \cong \frac{Z(\Lambda)}{\mathrm{Soc}(\Lambda)}$ as $Z(\Lambda)$-modules we get

$$HH^5(\Lambda) = \underline{HH}^5(\Lambda) \cong D(\underline{HH}^0(\Lambda)) \cong D(\underline{HH}^6(\Lambda))$$
$$HH^4(\Lambda) = \underline{HH}^4(\Lambda) \cong D(\underline{HH}^1(\Lambda)) \cong D(HH^1(\Lambda))$$

   Then, the proof reduces to check the cases $j = 1, 6$.

   For $j = 6$, we take $h = 1 + \mathrm{Im}(k_\mu^*) \in \frac{\mathrm{Ker}(\delta^*)}{\mathrm{Im}(k_\mu^*)} = \frac{Z(\Lambda)}{\mathrm{Soc}(\Lambda)}$ and one obviously has that $Z(\Lambda)h = \frac{Z(\Lambda)}{\mathrm{Soc}(\Lambda)} = HH^6(\Lambda)$.

   For $j = 1$ we take the element $\widehat{y} = \sum_{a\in Q_1} a \in \oplus_{a\in Q_1} e_{i(a)}\Lambda e_{t(a)}$. In a routine way, one can check that $R^*(\widehat{y}) = 0$. We then get an element $y = \widehat{y} + \mathrm{Im}(\delta^*) \in HH^1(\Lambda) = \frac{\mathrm{Ker}(R^*)}{\mathrm{Im}(\delta^*)}$.

   We now take the induced morphism of $Z(\Lambda)$-modules

$$\frac{K[x_0]}{(x_0^n)} \cong \frac{Z(\Lambda)}{\mathrm{Soc}(\Lambda)} \longrightarrow Z(\Lambda)y$$
$$\bar{x} \quad \mapsto \quad xy$$

   Its kernel is an ideal of $\frac{K[x_0]}{(x_0^n)}$, and, hence, it is of the form $\frac{(x_0^k)}{(x_0^n)}$, for some $k \leq n$.

   We claim that if $k < n$ then $x_0^k y \neq 0$. That will imply that $\frac{K[x_0]}{(x_0^n)} \cong Z(\Lambda)y$ so that $Z(\Lambda)y = HH^1(\Lambda)$ by a dimension argument.

   Suppose that $k < n$ and $yx_0^k = 0$. Then $\widehat{y}x_0^k \in \mathrm{Im}(\delta^*)$. Since δ^* is a graded map of degree 1 (with respect to length grading) we will have an element $x = \sum_{i=1}^{n-k} \mu_i b_i^{2k}$ of length-degree $2k$ in $\oplus_{i\in Q_0} e_i\Lambda e_i$ such that $\delta^*(x) = \widehat{y}x_0^k$. Since this element belongs to $\oplus_{a\in Q_1} e_{i(a)}\Lambda e_{t(a)}$ we can look at its ϵ-component:

$$\delta^*(x)_\epsilon = \epsilon x - \epsilon x = \mu_1 \epsilon b_1^{2k} - \mu_1 b_1^{2k}\epsilon = 0$$

$$(\widehat{y}x_0^k)_\epsilon = \lambda_1 \epsilon b_1^{2k},$$

   where λ_1 is the coefficient of b_i^{2k} in the expression $x_0^k = \sum_{i=1}^{n-k} \lambda_i b_i^{2k}$. We know from the proof of Proposition 5.3.3 that $\lambda_1 \neq 0$, which gives a contradiction since $\epsilon b_1^{2k} = \epsilon^{2k+1} \neq 0$.

□

The following result is now a consequence of Corollary 5.2.12 and Lemma 4.5.9
If

$$...P^{-i} \xrightarrow{d^{-i}} P^{-i+1} \longrightarrow ... \longrightarrow P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{u} \Lambda \to 0$$

is the minimal projective resolution of Λ (see Proposition 5.2.7) then, by definition, we have $HH^i(\Lambda) = \frac{\mathrm{Ker}((d^{-i-1})^*)}{\mathrm{Im}((d^{-i})^*)} \subseteq \frac{(P^{-i})^*}{\mathrm{Im}((d^{-i})^*)}$ for each $i > 0$. Thus, any element $\eta \in HH^i(\Lambda)$ is of the form $\eta = \tilde{\eta} + \mathrm{Im}((d^{-i})^*)$, for some $\tilde{\eta} \in Hom_{\Lambda^e}(P^{-i}, \Lambda)$ such that $\tilde{\eta} \circ d^{-i-1} = 0$. We will say that $\tilde{\eta}$ *represents* η or that η *is represented by* $\tilde{\eta}$.

In the statement of the following result we denote by ldeg$(-)$ the length-degree of any element in $HH^i(\Lambda)$.

Proposition 5.3.10. *The following are bases of the $HH^i(\Lambda)$ consisting of homogeneous elements with respect to the length grading, for each $i = 0, 1, ..., 6$:*

1. *For $HH^0(\Lambda) = Z(\Lambda)$: $\{x_0, x_0^2, \dots, x_0^{n-1}, x_1, \dots, x_n\}$, where $x_0 = \sum_{i=1}^{n-1}(-1)^i a_i \bar{a}_i$ and $x_i = \omega_i$, with ldeg$(x_0) = 2$ and ldeg$(x_i) = 2n - 1$, for all $i = 1, \dots n$.*

2. *For $HH^1(\Lambda) = \frac{Ker(R^*)}{Im(\delta^*)}$: $\{y, x_0 y, x_0^2 y, \dots x_0^{n-1} y\}$, where $y = \sum_{a \in Q_1} a + Im(\delta^*)$ and ldeg$(y) = 0$.*

   The element y is represented by the only morphism $\tilde{y} : Q \longrightarrow \Lambda$ such that, for each $a \in Q_1$, $\tilde{y}(e_{i(a)} \otimes e_{t(a)}) = a$.

3. *For $HH^2(\Lambda) = \frac{Ker(k^*)}{Im(R^*)}$: $\{z_1, \dots, z_n\}$, where $z_k = e_k + Im(R^*)$ and ldeg$(z_k) = -2$, for each $k \in Q_0$.*

   The element z_k is represented by the only morphism $\tilde{z}_k : P \longrightarrow \Lambda$ such that, for each $i \in Q_0$, $\tilde{z}_k(e_i \otimes e_i) = \delta_{ik} e_k$.

4. *For $HH^3(\Lambda) = \frac{Ker(\delta_\mu^*)}{Im(k^*)} = Ker(\delta_\mu^*)$: $\{t_1, \dots, t_n\}$, where $t_k = \omega_k$ and ldeg$(t_k) = -2$, for each $k \in Q_0$.*

   The element t_k is represented by the only morphism $\tilde{t}_k : P \longrightarrow \Lambda$ such that, for each $i \in Q_0$, $\tilde{t}_k(e_i \otimes e_i) = \delta_{ki} \omega_k$.

5. *For $HH^4(\Lambda) = \frac{Ker(R_\mu^*)}{Im(\delta_\mu^*)}$: $\{x_0^{n-1}\gamma, \dots, x_0\gamma, \gamma\}$, where $\gamma = e_1 + Im(\delta_\mu^*)$ and ldeg$(\gamma) = -2n - 2$.*

   The element γ is represented by the only morphism $\tilde{\gamma} : Q \longrightarrow \Lambda$ such that, for each $a \in Q_1$, $\tilde{\gamma}(e_{i(a)} \otimes e_{t(a)}) = \delta_{\epsilon a} e_1$.

6. *For $HH^5(\Lambda) = \frac{Ker(k_\mu^*)}{Im(R_\mu^*)}$: $\{x_0^{n-1} y\gamma, \dots, x_0 y\gamma, y\gamma\}$.*

7. *For $HH^6(\Lambda) = \frac{Ker(\delta^*)}{Im(k_\mu^*)}$: $\{h, x_0 h, \dots, x_0^{n-1} h\}$, where $h = 1 + Im(k_\mu^*)$ and ldeg$(h) = -4n - 2$.*

   The element h is represented by the multiplication map $\tilde{h} = u : \oplus_{i \in Q_0} \Lambda e_i \otimes e_i \Lambda \longrightarrow \Lambda$.

Proof. We leave for the end the part of the assertions concerning the length degree. Omitting that part, we have that assertion 1 follows from Proposition 5.3.3 and its proof. Assertions 2 and 7 follow from the proof of Lemma 5.3.9. Assertion 3 follows from Lemma 5.3.1 and the fact that $k^* = 0$ (see Proposition 5.2.9). Assertion 4 follows from the fact that $\text{Soc}(\Lambda) \subseteq \text{Ker}(\delta_\mu *)$ and $\text{dim}HH^3(\Lambda) = n = \dim(\text{Soc}(\Lambda))$ (see Theorem 5.3.4).

To check that in assertion 5 the given set is a basis of $HH^4(\Lambda)$, note that $R^*_\mu(e_1) = e_1\epsilon - \epsilon e_1 = 0$ and that $\overline{\delta^*_\mu(c)} = \delta^*_\mu(c)$, which implies that $\overline{\delta^*_\mu(x)} = \delta^*_\mu(x)$ for $x \in \oplus_{i\in Q_0} e_i\Lambda e_i$. We proceed as in the proof of the case $j = 1$ of Lemma 5.3.9, and check that if $x_0^k e_1 \in \text{Im}(\delta^*_\mu)$ and $k \leq n$ then $k = n$. Indeed, if $x_0^k e_1 = \epsilon^{2k} \in \text{Im}(\delta^*_\mu)$ then there is $1 \leq r \leq k$ and $\mu_1, \ldots, \mu_r \in K$, with $\mu_r \neq 0$, such that $\delta^*_\mu(\sum_{i=1}^r \mu_i b_i^{2k-1}) = \epsilon^{2k}$. We look now at the a_r-component of both members of the equality (i.e. at their image by applying the projection $\oplus_{a\in Q_1} e_{i(a)}\Lambda e_{t(a)} \longrightarrow e_{i(a_r)}\Lambda e_{t(a_r)}$). Therefore, we get $b_r^{2k-1}a_r = 0$, which is only possible in case $r = n$, and hence $k = n$. As in the proof of Lemma 5.3.9, we conclude that $HH^4(\Lambda) = \frac{\text{Ker}(R^*_\mu)}{\text{Im}(\delta^*_\mu)} = Z(\Lambda)\gamma$, where $\gamma := e_1 + \text{Im}(\delta^*_\mu)$. Then $\{\gamma, x_0\gamma, \ldots, x_0^{n-1}\gamma\}$ is a basis of $HH^4(\Lambda)$.

We will prove now that the map $\varphi_y : HH^4(\Lambda) \longrightarrow HH^5(\Lambda)$, given by $f \to yf$, is bijective. This will imply that the set of assertion 6 is a basis of $HH^5(\Lambda)$. We just need to check φ_y is injective since $\text{dim}HH^4(\Lambda) = \text{dim}HH^5(\Lambda)$. Note that $\underline{HH}^0(\Lambda) = \frac{Z(\Lambda)}{\text{Soc}(\Lambda)}$ has $\{x_0^k : k = 0, 1, ..., n-1\}$ as a basis. Fix now any graded Nakayama form $HH^*(\Lambda) \times HH^*(\Lambda) \overset{(-,-)}{\longrightarrow} K$ of constant degree function equal to 5. If now $f \in HH^4(\Lambda)$, then, using Lemma 4.5.9 and the relation between the multiplications in $HH^*(\Lambda)$ and $\underline{HH}^*(\Lambda)$ (see Section 2.1.3), we have that $yf = 0$ if and only if $0 = (yf, x_0^k) = (x_0^k y, f)$, for all $k = 0, 1, ..., n-1$. From assertion 2 and the nondegeneracy of $(-,-)$ we conclude that $f = 0$ and, hence, φ_y is injective.

It remains to calculate the length-degrees of the given elements. For that, we look at their explicit definition and take into account the gradings of the V^r in the complex $V^\bullet$ of Proposition 5.2.9. Just as a sample, note that each $a \in Q_1$ is an element of degree 0 in $V^1 = \oplus_{a\in Q_1} e_{i(a)}\Lambda e_{t(a)}[1]$, which implies that $\text{ldeg}(y) = 0$. Similarly each socle element w_k has degree -2 in $V^3 = \oplus_{i\in Q_0} e_i\Lambda e_i[2n+1]$, which implies that $\text{ldeg}(t_k) = -2$, for each $k = 1, 2, ..., n$. It is routine to check the remaining cases. □

The bases of the $HH^i(\Lambda)$ given in the above proposition will be called *canonical bases.*

Remark 5.3.11. In [34] the author uses the length grading on Λ and looks at the minimal projective resolution of Λ in the category of graded Λ-bimodules. Then, he calculates this graded structure in terms of three seminal graded vector spaces R, U and K (see Theorems 4.0.13 and 4.0.14 in [34]). In our terminology, $R = KQ_0$ (concentrated in degree 0), $U = \frac{Z(\Lambda)}{Soc(\Lambda)}[2]$ (with the length grading on $\frac{Z(\Lambda)}{Soc(\Lambda)}$) and $K = HH^2(\Lambda)[2]$, which is concentrated in degree 0 since $HH^2(\Lambda)$ is concentrated in degree -2.

His strategy to prove the mentioned theorems is based on the use of Connes' exact sequence (see Proposition 5.3.5(2)) and the description of the graded structure of each $HH_i(\Lambda)$ and, using dualities between the Hochschild homology and cohomology graded spaces obtained in [35], the author also gets the graded structure of each $HH^i(\Lambda)$.

Due to the fact that the dimension of R, U and K is n, the dimensions of the $HH_i(\Lambda)$ and the $HH^i(\Lambda)$ can be read off from the mentioned Theorems 4.0.13 and 4.0.14 of [34], even if they were not explicitly stated in a proposition or corollary. After that and before calculating the ring structure of $HH^*(\Lambda)$, Eu gives explicit bases of the $HH^i(\Lambda)$ using the corresponding of our complex $V^\bullet$ (see Section 8 in [34]).

Recall from Proposition 1.3.7 and Definition 8 that the graded Nakayama form of $HH^*(\Lambda)$ associated to the basis $\mathcal{B}_5 = \{x_0^{n-1}y\gamma, \dots, xy\gamma, y\gamma\}$ of $HH^5(\Lambda)$ acts on pairs of homogeneous elements as follows:

Proposition 5.3.12. *The basis* $\{1, x_0, ..., x_0^{n-1}\}$ *of* $\underline{HH}^0(\Lambda)$ *and the bases of the* $\underline{HH}^i(\Lambda) = HH^i(\Lambda)$ *given in Proposition 5.3.10, for* $i \neq 0$, *are orthogonal by pairs with respect to the restrictions*

$$(-,-) : \underline{HH}^i(\Lambda) \times \underline{HH}^{5-i}(\Lambda) \longrightarrow K$$

of the graded Nakayama form. More specifically, we have:

$$(x_0^k, x_0^{n-1-j}y\gamma) = (x_0^k y, x_0^{n-1-j}\gamma) = (z_k, t_j) = \delta_{kj},$$

where δ_{kj} *is the Kronecker symbol.*

Proof. If we have $i = 6m+j$, with $j \in \{0, 1, ..., 5\}$, then the pairing $\underline{HH}^i(\Lambda) \times \underline{HH}^{5-i}(\Lambda) \longrightarrow K$ is completely determined by the corresponding pairing with i replaced by j. Due to this and the graded commutativity of $\underline{HH}^*(\Lambda)$, the proof is reduced to check the equalities in the statement. By our definition of the graded Nakayama form, it is clear that we have $(x_0^k, x_0^{n-1-j}y\gamma) = (x_0^k y, x_0^{n-1-j}\gamma) = \delta_{kj}$, for all $k, j = 0, 1, ..., n-1$.

It only remains to check that $z_k t_j = \delta_{kj} x_0^{n-1} y\gamma$, for all $k, j = 1, ..., n$. Consider the projective Λ-bimodules P and Q of Proposition 5.2.7(a). We consider the following morphisms of Λ-bimodules:

a) $g_j = \hat{t}_j : P \longrightarrow P$, identified by the equality $g_j(e_i \otimes e_i) = \frac{1}{2}\delta_{ij}(\omega_j \otimes e_j + e_j \otimes \omega_j)$, for all $i, j \in Q_0$.

b) $f_j : Q \longrightarrow Q$, identified by the following rules:
In case $(j, a) \neq (1, \epsilon)$, we have:

$$f_j(e_{i(a)} \otimes e_{t(a)}) = \begin{cases} 0 \text{ if } j \notin \{i(a), t(a)\} \\ \frac{1}{2}e_{i(a)} \otimes \omega_{t(a)} \text{ if } j = t(a) \\ -\frac{1}{2}\omega_{i(a)} \otimes e_{t(a)} \text{ if } j = i(a) \end{cases}$$

and, for $(j, a) = (1, \epsilon)$, we have:

$$f_1(e_{i(\epsilon)} \otimes e_{t(\epsilon)}) = \frac{1}{2}(e_{i(\epsilon)} \otimes \omega_{t(\epsilon)} - \omega_{i(\epsilon)} \otimes e_{t(\epsilon)})$$

c) $\tilde{z}_k : P \longrightarrow \Lambda$ as in Proposition 5.3.10.

It is easy to check that the following diagram is commutative:

$$\begin{array}{ccccc} P & \xrightarrow{R_\mu} & Q & \xrightarrow{\delta_\mu} & P \\ \downarrow{\scriptstyle g_j} & & \downarrow{\scriptstyle f_j} & & \downarrow{\scriptstyle \widehat{t}_j} \\ P & \xrightarrow[R]{} & Q & \xrightarrow[\delta]{} & P \end{array}$$

It follows that the element $z_k t_j \in HH^5(\Lambda)$ is represented by the composition

$$P \xrightarrow{g_j} P \xrightarrow{\tilde{z}_k} \Lambda$$

Due to the fact that $g_j(e_i \otimes e_i) = 0$ for $i \neq j$ and $g_j(\Lambda e_j \otimes e_j \Lambda) \subseteq \Lambda e_j \otimes e_j \Lambda$, we readily see that $\tilde{z}_k \circ g_j = 0$ when $j \neq k$. Moreover, in case $j = k$, we have

$$(\tilde{z}_j \circ g_j)(e_i \otimes e_i) = \frac{1}{2}\delta_{ij}(\omega_j + \omega_j) = \delta_{ij}\omega_j$$

From Remark 5.3.2 we know that, when we view ω_j as an element of $\mathrm{Ker}(k_\mu^*) = \mathrm{Soc}(\Lambda)$, we have $\omega_j + \mathrm{Im}(R_\mu^*) = \omega_1 + \mathrm{Im}(R_\mu^*) = \epsilon^{2n-1} + \mathrm{Im}(R_\mu^*)$ for all $j = 2, \dots, n-1$. Therefore we have that $z_k t_j = \delta_{kj}(\epsilon^{2n-1} + \mathrm{Im}(R_\mu^*))$.

We now check the equality $x_0^{n-1} y\gamma = \epsilon^{2n-1} + \mathrm{Im}(R_\mu^*)$ in $HH^5(\Lambda)$ which will end the proof. Note that if $\tilde{\gamma} : Q \longrightarrow \Lambda$ is as in Proposition 5.3.10, then a lifting of it is the morphism of Λ-bimodules $\hat{\gamma} : Q \longrightarrow P$ taking $e_{i(a)} \otimes e_{t(a)} \to \delta_{a\epsilon} e_1 \otimes e_1$. If now $g : P \longrightarrow Q$ is the morphism of Λ-bimodules which vanishes on all $e_i \otimes e_i$, for $i \neq 0$, and takes $e_1 \otimes e_1 \to e_{i(\epsilon)} \otimes e_{t(\epsilon)}$, we readily see that $\hat{\gamma} \circ R_\mu = \delta \circ g$, where R_μ and δ are the maps in the minimal projective resolution of Λ (see Proposition 5.2.7). The definition of the Yoneda product implies that the element $y\gamma$ of $HH^5(\Lambda)$ is represented by the composition $P \xrightarrow{g} P \xrightarrow{\tilde{y}} \Lambda$. This composition vanishes on $e_i \otimes e_i$, for $i \neq 1$, and takes $e_1 \otimes e_1 \to \epsilon$. Then we have $y\gamma = \epsilon + \mathrm{Im}(R_\mu^*)$, and hence $x_0^{n-1} y\gamma = x_0^{n-1}(\epsilon + \mathrm{Im}(R_\mu^*)) = \epsilon^{2n-2}(\epsilon + \mathrm{Im}(R_\mu^*)) = \epsilon^{2n-1} + \mathrm{Im}(R_\mu^*)$.

□

5.4 The ring structure of the Hochschild cohomology ring

The main result of the chapter is the following, from which we deduce the structure of all other objects of interest, namely, the stable Hochschild cohomology as well as the stable and classical Hochschild homology (see Corollaries 5.2.12 and 5.4.9 and Remark 5.4.10).

Theorem 5.4.1. *Let Λ be the generalized preprojective algebra $\mathbb{L}_n$ over a commutative ring R on which 2 is invertible. The following assertions hold for the Hochschild cohomology ring $HH^*(\Lambda)$:*

1. *$HH^*(\Lambda)$ is the commutative bigraded R-algebra given by*

   a) <u>Generators</u>: $x_0, x_1, \dots, x_n, y, z_1, \dots, z_n, t_1, t_2, \dots t_{n-1}, \gamma, h$

b) Relations:

i) $x_i\xi = 0$ *for each* $i = 1, \ldots, n$ *and each generator* ξ.

ii) $x_0^n = y^2 = x_0 z_j = x_0 t_i = y t_i = t_i t_k = 0$, $(j = 1, \ldots, n \quad i, k = 1, \ldots n-1)$

iii) $z_j z_k = (-1)^{k-j+1}(2j-1)(n-k+1)x_0^{n-1}\gamma$, *for* $1 \leq j \leq k \leq n$.

iv) $z_j\gamma = (-1)^j(n-j+1)x_0^{n-1}h$, *for* $j = 1, ..., n$

v) $\gamma^2 = z_1 h$

vi) $yz_j = (2n+1)\sum_{1\leq k\leq j-1}(-1)^{j-k}(j-k)t_k + (-1)^{j-1}(2j-1)yz_1$, *for* $j = 2, ..., n$

vii) $z_k t_j = \delta_{jk} x_0^{n-1} y\gamma$, *for* $k = 1, \ldots, n \quad j = 1 \ldots, n-1$

viii) $t_j\gamma = \delta_{1j} x_0^{n-1} yh$, *for* $j = 1, \ldots, n-1$.

2. *The homological grading on* $HH^*(\Lambda)$ *is determined by the equalities* $deg(x_i) = 0$, $deg(y) = 1$, $deg(z_j) = 2$, $deg(t_k) = 3$, $deg(\gamma) = 4$ *and* $deg(h) = 6$.

3. *The length grading on* $HH^*(\Lambda)$ *is determined by the equalities* $ldeg(x_0) = 2$, $ldeg(x_i) = 2n-1$, *for* $i \neq 0$, $ldeg(y) = 0$, $ldeg(z_j) = -2$, $ldeg(t_k) = -2$, $ldeg(\gamma) = -2n-2$ *and* $ldeg(h) = -4n-2$.

4. *Multiplication by h gives an isomorphism* $HH^i(\Lambda) \xrightarrow{\cong} HH^{i+6}(\Lambda)$, *for each* $i > 0$.

5. *All* $HH^i(\Lambda)$ *are free R-modules, and the following are bases for them (see Proposition 5.3.10):*

   (a) For $HH^0(\Lambda)$: $\{x_0, x_0^2, \ldots, x_0^{n-1}, x_1, \ldots, x_n\}$.
   (b) For $HH^1(\Lambda)$: $\{y, x_0 y, x_0^2 y, \ldots x_0^{n-1} y\}$.
   (c) For $HH^2(\Lambda)$: $\{z_1, \ldots, z_n\}$.
   (d) For $HH^3(\Lambda)$: $\{t_1, \ldots, t_{n-1}, yz_1\}$.
   (e) For $HH^4(\Lambda)$: $\{x_0^{n-1}\gamma, \ldots, x_0\gamma, \gamma\}$.
   (f) For $HH^5(\Lambda)$: $\{x_0^{n-1}y\gamma, \ldots, x_0 y\gamma, y\gamma\}$.
   (g) For $HH^6(\Lambda)$: $\{h, x_0 h, \ldots, x_0^{n-1} h\}$.

   In particular $dim(HH^0(\Lambda)) = 2n$ *and* $dim(HH^i(\Lambda)) = n$, *for all* $i > 0$, *where* $dim(-)$ *denotes the rank as a free R-module.*

Remark 5.4.2. Note that if R is a field (of characteristic $\neq 2$) in the above theorem, then $2n+1$ is either zero or an invertible element. If R is a commutative ring as in the theorem satisfying either of these two conditions, the description of $HH^*(\Lambda)$ by generators and relations simplifies. First, if $char(R)$ divides $2n+1$ then the relations vi) become

$$\text{vi')} \qquad yz_j = (-1)^{j-1}(2j-1)yz_1, \text{ for } j = 2, ..., n.$$

If $2(2n+1)$ is invertible in R, then we can express each t_i $(i = 1, ..., n-1)$ as a R-linear combination of the yz_j, using the relations vi). Then the following is an immediate consequence of the theorem.

Corollary 5.4.3. *Let R be a commutative ring on which $2(2n+1)$ is invertible. If Λ is the generalized preprojective R-algebra $\mathbb{L}_n$, then $HH^*(\Lambda)$ is the commutative algebra given by*

a) Generators: $x_0, x_1, \dots x_n, y, z_1, \dots, z_n, \gamma, h$ with

b) Relations:

- *i) $x_i\xi = 0$, for each $i = 1, \dots, n$ and each generator ξ.*
- *ii) $x_0^n = y^2 = x_0 z_j = 0$ $(j = 1, \dots, n)$*
- *iii) $z_j z_k = (-1)^{k-j+1}(2j-1)(n-k+1)x_0^{n-1}\gamma$ for $1 \le j \le k \le n$.*
- *iv) $z_j\gamma = (-1)^j(n-j+1)x_0^{n-1}h$ $(j = 1, \dots, n)$*
- *v) $\gamma^2 = z_1 h$*

on which the homological grading is determined by the equalities $deg(x_i) = 0$, $deg(y) = 1$, $deg(z_j) = 2$, $deg(\gamma) = 4$ and $deg(h) = 6$, while the length grading is determined by the equalities $ldeg(x_0) = 2$, $ldeg(x_i) = 2n-1$ $(i \neq 0)$, $ldeg(y) = 0$, $ldeg(z_j) = -2$, $ldeg(\gamma) = -2n-2$ and $ldeg(h) = -4n-2$

The rest of the chapter, except the last section, is devoted to give a proof of the theorem stated above.

5.4.1 When the ground commutative ring is a field

Throughout this subsection, we assume that the ground commutative ring is a field K of odd characteristic.

We start by studying the map $\phi_y : HH^2(\Lambda) \longrightarrow HH^3(\Lambda)$ given by $\phi_y(u) = yu$ for all $u \in HH^2(\Lambda)$

Lemma 5.4.4. *If $C = (C_{kj})$ is the matrix of ϕ_y with respect to the canonical bases of $HH^2(\Lambda)$ and $HH^3(\Lambda)$, then the following conditions hold:*

1) C is a symmetric integer matrix.

2) $C_{jk} = (-1)^{k-j+1}(2j-1)(n-k+1)$ whenever $1 \le j \le k \le n$.

3) $rank(C) = n$, when $Char(K)$ does not divide $2n+1$, and $rank(C) = 1$, when $Char(K)$ divides $2n+1$.

Proof. Let $x = \alpha_1 \cdots \alpha_r$ $(r > 0)$ be any path in $e_j KQ e_k$ which does not belong to the ideal I. We put

$$h_x = \alpha_1 \cdots \alpha_{r-1} \otimes x^* + \alpha_1 \cdots \alpha_{r-2} \otimes \alpha_r x^* + \cdots + e_j \otimes \alpha_2 \cdots \alpha_r x^*,$$

which is an element of $\oplus_{a\in Q_1}\Lambda e_{i(a)} \otimes e_{t(a)}\Lambda$. In case $j = k$ we put $h_{e_j} = 0$ and if $p_j = \bar{a}_{j-1}...\bar{a}_1\epsilon^{2(n-j)+1}a_1...a_{j-1}$ we also define $h_{\omega_j} = (-1)^{\frac{j(j-1)}{2}}h_{p_j}$ (recall that $\omega_j = (-1)^{\frac{j(j-1)}{2}}p_j$). In this way, we have defined h_x for each $x \in e_j\mathcal{B}e_k$ and for all $j, k \in Q_0$.

Direct calculation shows that $\delta(h_x) = x \otimes x^* - e_j \otimes \omega_j$, and hence

$$\delta(\sum_{x \in e_j \mathcal{B} e_k} (-1)^{\deg(x)} h_x) = \sum_{x \in e_j \mathcal{B} e_k} (-1)^{\deg(x)} (x \otimes x^* - e_j \otimes \omega_j) = \sum_{x \in e_j \mathcal{B} e_k} (-1)^{\deg(x)} x \otimes x^*,$$

bearing in mind that in $e_j B e_k$, there are exactly the same number of elements of odd and even length-degree.

Now consider $\tilde{z}_k : P \longrightarrow \Lambda$ as in Proposition 5.3.10. It is clear that the morphism of Λ-bimodules $\hat{z}_k : P \longrightarrow P$ determined by the rule $\hat{z}_k(e_i \otimes e_i) = \delta_{ik} e_k \otimes e_k$ is a lifting of $\tilde{z}_k$ (i.e. $\tilde{z}_k = u \circ \hat{z}_k$).

If now $f_k : P \longrightarrow Q$ is the morphism of Λ-bimodules determined by the rule $f_k(e_j \otimes e_j) = \sum_{x \in e_j \mathcal{B} e_k} (-1)^{\deg(x)} h_x$, then we have a commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{k} & P \\ {\scriptstyle f_k}\downarrow & & \downarrow{\scriptstyle \hat{z}_k} \\ Q & \xrightarrow{\delta} & P \end{array}$$

and hence yz_k is represented by the morphism

$$\tilde{y} \circ f_k : P \longrightarrow \Lambda,\ (e_j \otimes e_j \mapsto \sum_{x \in e_j \mathcal{B} e_k} (-1)^{\deg(x)} \deg(x) \omega_j.$$

That means that if we put $C_{jk} = \sum_{x \in e_j \mathcal{B} e_k} (-1)^{\deg(x)} \deg(x)$ for all $j, k \in Q_0$, then we have $yz_k = \sum_{j \in Q_0} C_{jk} t_j$ (notation as in Proposition 5.3.10). Therefore, $C := (C_{jk})$ is the matrix of $\phi_y : HH^2(\Lambda) \longrightarrow HH^3(\Lambda)$ with respect to the canonical bases of $HH^2(\Lambda)$ and $HH^3(\Lambda)$.

That C is a symmetric integer matrix is clear since the anti-isomorphism $x \mapsto \bar{x}$ gives a bijection between $e_j \mathcal{B} e_k$ and $e_k \mathcal{B} e_j$, which preserves the term $(-1)^{\deg(x)} \deg(x)$. We then proceed to calculate the entries of this matrix. To do that, we should recall the possible degrees of elements in $e_j \mathcal{B} e_k$ (see Remark 5.2.5), for $1 \leq j \leq k \leq n$. There are two possibilities.

i) $k \equiv j \pmod 2$: Then the sum of even degrees is $\frac{[(k-j)+(k-j)+2(n-k)](n-k+1)}{2} = (n-j)(n-k+1)$, while the sum of odd degrees is $\frac{[(k+j-1)+(k+j-1)+2(n-k)](n-k+1)}{2} = (n+j-1)(n-k+1)$. Therefore, we have $C_{jk} = (n-j)(n-k+1) - (n+j-1)(n-k+1) = (1-2j)(n-k+1)$

ii) $k \not\equiv j \pmod 2$: In this case C_{jk} is the negative of the number above, i.e., $C_{jk} = (2j-1)(n-k+1)$.

It finally remains to calculate $\text{rank}(C)$. We view each $n \times n$ matrix as a n-tuple, whose components are its rows. By elementary row transformation one passes from $C = (C_1, \ldots, C_n)$ to

$$C' = (C_1, C_2 + 3C_1, \ldots, C_j + (-1)^j (2j-1) C_1, \ldots, C_n + (-1)^n (2n-1) C_1)$$

so that $\text{rank}(C) = \text{rank}(C')$. We look at the j-th row $C'_j = C_j + (-1)^j(2j-1)C_1$ of C'. It is straightforward to check that for $j \leq k$, one has $C'_{jk} = 0$, and for $j > k$, one has $C'_{jk} = (-1)^{j-k+1}(k-j)(2n+1)$.

Therefore, in case $\text{Char}(K)$ divides $2n+1$, all rows of C' except the first one are zero. On the other hand, we have $C'_{1n} = C_{1n} = (-1)^{n-1+1}(2 \cdot 1 - 1)(n - n + 1) = (-1)^n$. It follows that $\text{rank}(C) = 1$ in case $\text{Char}(K)/2n+1$. In case $\text{Char}(K)$ does not divide $2n+1$, if we apply the n-cycle $(1\ n\ n-1 \cdots\ 2)$ to the rows of C', we obtain a lower triangular matrix with diagonal entries $C'_{21}, C'_{32}, \ldots, C'_{n,n-1}, C'_{1n}$.

We have $C'_{k+1,k} = (-1)^{(k+1)-k+1}(k-(k+1))(2n+1) = -(2n+1)$ for $k = 2, \ldots, n$ and $C'_{1n} = (-1)^n$. It follows that $det(C) = det(C') = (-1)^{2n-1}(2n+1)^{n-1} \neq 0$. Therefore, $\text{rank}(C) = n$ in this case. □

Remark 5.4.5. Given a graph Γ without double edges, its *adjacency matrix* $D = D_\Gamma$ is the symmetric matrix $D = (d_{ij})_{i,j \in \Gamma_0}$ having $d_{ij} = 1$, in case there is an edge i — j, and $d_{ij} = 0$ otherwise. In particular, for the graph $\mathbb{L}_n$, one has $d_{11} = 1$, $d_{i,i+1} = d_{i+1,i} = 1$ for $i = 1, ..., n-1$, and $d_{ij} = 0$ otherwise. Direct computation shows that the matrix C of Lemma 5.4.4 satisfies the equality $-C(2I_n + D) = (2n+1)I_n$, where I_n is the identity $n \times n$ matrix. Therefore, when $char(K)$ does not divide $2n+1$, an alternative description of the matrix C is $C = -(2n+1)(2I_n + D)^{-1}$. Up to signs forced by the different presentation of Λ and the different choice of the exceptional vertex of $\mathbb{L}_n$, the last equality is that of [34][Proposition 9.3.1] (see also [34][Theorem 4.0.16]).

Taking into account also the case when $\text{Char}(K)$ divides $2n+1$ is fundamental for the difference of presentations in our two main theorems and is the part of our work where the arguments of [34] cannot be applied.

Lemma 5.4.6. *The following equalities hold in the ring* $HH^*(\Lambda)$*:*

1. $HH^{2r+1}(\Lambda) \cdot HH^{2s+1}(\Lambda) = 0$*, for all integers* r, s
2. $t_j\gamma = \delta_{1j}x_0^{n-1}yh$*, where* δ_{1j} *is the Kronecker symbol*
3. $\gamma^2 = z_1 h$
4. $z_j\gamma = (-1)^j(n-j+1)x_0^{n-1}h.$

Proof. 1) From Eu-Schedler formula (see Lemma 4.5.9), using the nondegenerancy of $(-,-) : HH^*(\Lambda) \times HH^*(\Lambda) \longrightarrow K$, we get that $\underline{HH}^i(\Lambda) \cdot \underline{HH}^j(\Lambda) = 0$ if, and only if, $\underline{HH}^i(\Lambda) \cdot \underline{HH}^{5-i-j}(\Lambda) = 0$. This gives that $HH^1(\Lambda) \cdot HH^1(\Lambda) = 0$ if, and only if, $HH^1(\Lambda) \cdot HH^3(\Lambda) = 0$. Similarly, it gives that $HH^3(\Lambda) \cdot HH^3(\Lambda) = 0$ if, and only if, $HH^3(\Lambda) \cdot \underline{HH}^{-1}(\Lambda) = 0$ which, by the 6-periodicity of $\underline{HH}^*(\Lambda)$, is equivalent to saying that $HH^3(\Lambda) \cdot HH^5(\Lambda) = 0$. But, by Proposition 5.3.10, we have an equality $HH^1(\Lambda) \cdot HH^4(\Lambda) = HH^5(\Lambda)$. These considerations together with the fact that $HH^*(\Lambda)$ is periodic of period 6 imply that, in order to prove assertion 1, we just need to check that $HH^1(\Lambda) \cdot HH^1(\Lambda) = 0$, for which it is enough to see that $y^2 = 0$. But this follows by considering length-degrees since the length-degree of any nonzero homogeneous element of $HH^2(\Lambda)$ is -2 while $\text{ldeg}(y^2) = 0$.

2) We consider the following diagram, for each $j \in Q_0$:

$$\begin{array}{ccccccccc}
Q & \xrightarrow{\delta} & P & \xrightarrow{k_\mu} & P & \xrightarrow{R_\mu} & Q & \xrightarrow{\delta_\mu} & P \\
\downarrow{\scriptstyle l_j} & & \downarrow{\scriptstyle h_j} & & \downarrow{\scriptstyle g_j} & & \downarrow{\scriptstyle f_j} & & \downarrow{\scriptstyle \widehat{t_j}} \\
Q & \xrightarrow[\delta_\mu]{} & P & \xrightarrow[k]{} & P & \xrightarrow[R]{} & Q & \xrightarrow[\delta]{} & P
\end{array}$$

where f_j and $g_j = \hat{t}$ are as in the proof of Proposition 5.3.12 and h_j and l_j are the only morphisms of Λ-bimodules satisfying the following properties:

a) $h_j(e_i \otimes e_i) = \frac{1}{2}\delta_{ij}(e_j \otimes \omega_j - \omega_j \otimes e_j)$

b) In case $(j, a) \neq (1, \epsilon)$ we have

$$l_j(e_{i(a)} \otimes e_{t(a)}) = \begin{cases} 0 \text{ if } j \notin \{i(a), t(a)\} \\ \frac{1}{2}e_{i(a)} \otimes \omega_{t(a)} \text{ if } j = t(a) \\ \frac{1}{2}\omega_{i(a)} \otimes e_{t(a)} \text{ if } j = i(a) \end{cases}$$

and, in case $(j, a) = (1, \epsilon)$, we have:

$$l_1(e_{i(\epsilon)} \otimes e_{t(\epsilon)}) = \frac{1}{2}(e_{i(\epsilon)} \otimes \omega_{t(\epsilon)} + \omega_{i(\epsilon)} \otimes e_{t(\epsilon)})$$

It is routine to check that the two squares on the left of the diagram are commutative, which will imply that the whole diagram is commutative.

Graded commutativity of $HH^*(\Lambda)$ gives that $t_j\gamma = \gamma t_j$ and the element $\gamma t_j \in HH^7(\Lambda)$ is represented by the composition

$$Q \xrightarrow{l_j} Q \xrightarrow{\tilde{\gamma}} \Lambda.$$

Note that $l_j(e_{i(a)} \otimes e_{t(a)}) \in \Lambda e_{i(a)} \otimes e_{t(a)}\Lambda$, from which we deduce that $\tilde{\gamma} \circ l_j = 0$ for $j \in Q_0/\{1\}$. And for $j = 1$ we have

$$(\tilde{\gamma} \circ l_1)(e_{i(a)} \otimes e_{t(a)}) = \begin{cases} 0 \text{ if } a \neq \epsilon \\ \frac{1}{2}(\omega_1 + \omega_1) = \omega_1 = \epsilon^{2n-1} \text{ if } a = \epsilon \end{cases}$$

But, due to the identification $HH^1(\Lambda) = HH^7(\Lambda)$, which is just multiplication by h, and the proof of Proposition 5.3.12, we know that $x_0^{n-1}yh$ is precisely the element $\epsilon^{2n-1} + \mathrm{Im}(\delta^*) \in \frac{\mathrm{Ker}(R^*)}{\mathrm{Im}(\delta^*)}$. Therefore, we get $t_j\gamma = \gamma t_j = \delta_{1j}x_0^{n-1}yh$, for all $j \in Q_0$.

3) By Lemma 4.5.9, we have $(\gamma^2, t_jh^{-1}) = (\gamma t_jh^{-1}, \gamma))$. By the equality 2 and the graded commutativity of $\underline{HH}^*(\Lambda)$, we then have

$$(\gamma^2, t_jh^{-1}) = \delta_{1j}(x_0^{n-1}yhh^{-1}, \gamma) = \delta_{1j}(x_0^{n-1}y, \gamma).$$

By definition of $(-,-)$ and by Proposition 5.3.12, we then have $(\gamma^2, t_j h^{-1}) = \delta_{1j} = (z_1, t_j) = (z_1 h, t_j h^{-1})$, for all $j = 1, ..., n$. It follows that $\gamma^2 = z_1 h$ since $\{t_j h^{-1} : j = 1, ..., n\}$ is a basis of $\underline{HH}^{-3}(\Lambda)$ and the form $(-,-)$ is nondegenerate.

4) By Lemma 4.5.9, we have that $(z_j\gamma, x_0^r y\gamma h^{-1}) = (x_0^r y\gamma^2 h^{-1}, z_j)$, using also the graded commutativity of $\underline{HH}^*(\Lambda)$. Now, by the equality 3, we get that $(z_j\gamma, x_0^r y\gamma h^{-1}) = (x_0^r y z_1 h h^{-1}, z_j) = (x_0^r y z_1, z_j))$. Note that $x_0^r y z_1 = 0$, for $r > 0$, because $HH^3(\Lambda)$ is a semisimple $Z(\Lambda)$-module (see Lemma 5.3.9). On the other hand, by Lemma 5.4.4, we have $yz_1 = \sum_{k=1}^n c_{k1} t_k$ and, by Proposition 5.3.12, we get an equality $(z_j\gamma, y\gamma h^{-1}) = \sum_{k=1}^n c_{k1}(t_k, z_j) = \sum_{k=1}^n \delta_{kj} c_{k1} = c_{j1} = (-1)^j(n-j+1)$. Therefore $(z_j\gamma, x_0^r y\gamma h^{-1}) = 0$, when $r > 0$, and $(z_j\gamma, y\gamma h^{-1}) = (-1)^j(n-j+1)$.

On the other hand, by definition of $(-,-)$, we also have the equality

$$x_0^{r+n-1} y\gamma = (x_0^{n-1} h, x_0^r y\gamma h^{-1}) x_0^{n-1} y\gamma, \quad \text{for all } r = 0, 1, ..., n-1.$$

This shows that $(x_0^{n-1} h, x_0^r y\gamma h^{-1}) = 0$, for $r > 0$, and $(x_0^{n-1} h, y\gamma h^{-1}) = 1$. Taking into account that $\{x_0^r y\gamma h^{-1} : r = 0, 1, ..., n-1\}$ is a basis of $\underline{HH}^{-1}(\Lambda)$, the nondegeneracy of $(-,-)$ gives that $z_j\gamma = (-1)^j(n-j+1)x_0^{n-1} h$.

□

We are now ready to give:

PROOF OF THEOREM 5.4.1 WHEN $R = K$ IS A FIELD:

Assertion 4 follows from Corollary 5.2.12 and assertions 2, 3 and 5 follow from Proposition 5.3.10.

To prove assertion 1, note that any graded commutative algebra admitting, apart from the graded commutativity ones, the given list of relations, is necessarily commutative since the product of generators of odd degree is always zero.

With the notation used until now, we know from Proposition 5.3.10 that the set

$$\{x_0, x_1, \ldots, x_n, y, z_1, \ldots, z_n, t_1, \ldots t_n, \gamma, h\}$$

generates $HH^*(\Lambda)$ as an algebra.

We now look at Lemma 5.4.4. The coordinate vector of yz_j with respect to the canonical basis $\{t_1, ..., t_n\}$ of $HH^3(\Lambda)$ is precisely the j-th column of the matrix $C = (c_{kj})$. It follows in particular that $\{t_1, ..., t_{n-1}, yz_1\}$ is a basis of $HH^3(\Lambda)$ because $c_{n1} = c_{1n} = (-1)^n$. Therefore, we can delete t_n from the given list of generators.

On the other hand, bearing in mind that C is symmetric, we also get from the proof of Lemma 5.4.4 that the coordinate vector of $yz_j + (-1)^j(2j-1)yz_1$ with respect to the canonical basis is the j-th column of the transpose C'^T of the matrix C' considered there, for each $j = 2, ..., n$. But that column is precisely the j-th row of C', which has entry $c'_{jk} = 0$, for $k \geq j$, and $c'_{jk} = (-1)^{j-k+1}(k-j)(2n+1) = (-1)^{j-k}(j-k)(2n+1)$, for $k < j$. Then, for each $j > 1$, we get an equality

$$yz_j + (-1)^j(2j-1)yz_1 = (2n+1)\sum_{1\leq k\leq j-1}(-1)^{j-k}(j-k)t_k.$$

from which the relations vi) follow.

From the fact that $\mathrm{Soc}(\Lambda)HH^j(\Lambda) = 0 \ \forall j > 0$ one readily obtains the relations in *i*). From lemmas 5.3.3, 5.3.9 and 5.4.6(1) we obtain all the relations in *ii*).

The relations in *iv*), *v*) and *viii*) are included in lemma 5.4.6.

From the equalities in Proposition 5.3.12 and the definition of the Nakayama form $(-,-)$ the relations *vii*) follow. We use them to prove that also the relations in *iii*) hold. Note that the proof of Proposition 5.3.10 gives an isomorphism of $Z(\Lambda)$-modules $\varphi_y : HH^4(\Lambda) \xrightarrow{\sim} HH^5(\Lambda)$ $(f \mapsto yf)$. What we shall prove is the equality

$$z_j(yz_k) = (-1)^{k-j+1}(2j-1)(n-k+1)x_0^{n-1}y\gamma,$$

from which the desired equality will follow.

Indeed, by Lemma 5.4.4, we have that $yz_k = \sum_{l=1}^{n} c_{lk}t_l$ and hence $z_j(yz_k) = \sum_{l=1}^{n} c_{lk}z_jt_l$. Now, using the relations *vii*), we get

$$z_j(yz_k) = c_{jk}z_jt_j = (-1)^{k-j+1}(2j-1)(n-k+1)x_0^{n-1}y\gamma$$

The previous paragraphs show that there is a surjective homomorphism of bigraded algebras from the commutative algebra given by the mentioned generators and relations to the algebra $HH^*(\Lambda)$. By looking at the dimensions in each homological degree, it is not difficult to see that the homomorphism is actually an isomorphism.

Remark 5.4.7. In [34][Section 9] the graded ring structure of $HH^*(\Lambda)$ was calculated taking **C** as ground field. However, the arguments and calculations appear to be valid whenever $\mathrm{Char}(K) \neq 2$ and $\mathrm{Char}(K)$ does not divide $2n+1$. Then, with the suitable changes derived from the different presentations of the algebra, Corollary 5.4.3 could be derived from Eu's work.

Eu's methods use sometimes direct calculation of the products $HH^i(\Lambda) \cdot HH^j(\Lambda)$, other times the graded condition of the minimal projective resolution of Λ (see 9.2) and, on other occasions, the matrix Hilbert series $H_\Lambda(t)$ (see Definition 2.5.2) together with the equality $H_\Lambda(t) = (1+t^{2n+1})((1+t^2)I_n - Dt)^{-1}$ proved in [62], where D is the adjacency matrix of $\mathbb{L}_n$ (see the proof of Lemma 9.3.3 and Section 6.2 in [34]).

We have not used the matrix Hilbert series in this work. We have directly calculated all products $HH^i(\Lambda)\cdot HH^j(\Lambda)$, using Lemma 4.5.9 and working with the bases of Proposition 5.3.10, in which some products have already been included.

5.4.2 The general case

In this final subsection, we assume that R is any ground commutative ring on which 2 is invertible. We shall derive the main theorem from the particular case when R is a field, which was done in the previous subsection. The crucial point is the following result of Commutative Algebra.

Lemma 5.4.8. *Let R be a reduced commutative ring, M be a finitely generated R module, $B = \{x_1, ..., x_r\}$ be a finite set of elements of M and, for each $\mathbf{p} \in Spec(R)$, denote by $k(\mathbf{p})$ the residue field at $\mathbf{p}$. The following assertions hold:*

1. *If the image of B by the map $f_{\mathbf{p}} : M \longrightarrow M \otimes_R k(\mathbf{p})$, $m \to m \otimes 1$, is $k(\mathbf{p})$-linearly independent in $M \otimes_R k(\mathbf{p})$, for each $\mathbf{p} \in Spec(R)$, then B is an R-linearly independent subset of M.*

2. *B generates M as an R-module if, and only if, $f_{\mathbf{p}}(B)$ generates $M \otimes_R k(\mathbf{p})$ as a $k(\mathbf{p})$-vector space, for each $\mathbf{p} \in Spec(R)$.*

3. *B is a basis of M as a free R-module if, and only if, $f_{\mathbf{p}}(B)$ is a basis of $M \otimes_R k(\mathbf{p})$ as a $k(\mathbf{p})$-vector space, for each $\mathbf{p} \in Spec(R)$.*

Proof. 1) If $a_1, ..., a_r \in R$ are elements such that $\sum_{1 \leq i \leq r} a_i x_i = 0$, then the hypothesis says that the image of each a_i by the canonical ring homomorphism $u_{\mathbf{p}} : R \longrightarrow k(\mathbf{p})$ is zero, for each $\mathbf{p} \in \mathrm{Spec}(R)$. Fix i and put $a := a_i$ to simplify the notation. When the tensor functor $? \otimes_R k(\mathbf{p})$ is applied to the exact sequence

$$0 \to Ra \longrightarrow R \longrightarrow R/Ra \to 0,$$

we get an exact sequence

$$Ra \otimes_R k(\mathbf{p}) \stackrel{0}{\longrightarrow} k(\mathbf{p}) \longrightarrow R/Ra \otimes_R k(\mathbf{p}) \to 0.$$

This shows that the support of R/Ra, which is $V(Ra) = \{p \in \mathrm{Spec}(R) \mid Ra \subseteq P\}$, is all $\mathrm{Spec}(R)$. Thus is equivalent to saying that Ra is contained $\bigcap_{\mathbf{p} \in \mathrm{Spec}(R)} \mathbf{p}$ or, equivalently, that a is nilpotent (see [57][Propositions III.4.6 and I.4.5]). By the reduced condition of R, we get $a = 0$. Therefore, B is R-linearly independent.

2) The 'only if' part of the assertion is clear. For the 'if' part, note that if $N = \sum_{1 \leq i \leq r} Rx_i$ and $\iota : N \longrightarrow M$ is the inclusion, then the hypothesis implies that map $\iota \otimes 1_{k(\mathbf{p})} : N \otimes_R k(\mathbf{p}) \longrightarrow M \otimes_R k(\mathbf{p})$ is surjective or, equivalently, that $\frac{M}{N} \otimes_R k(\mathbf{p}) = 0$, for all $\mathbf{p} \in \mathrm{Spec}(R)$. Then, the support of M/N is empty and, hence, we have $M/N = 0$.

3) This assertion follows from 1) and 2). □

PROOF OF THEOREM 5.4.1: We first prove that if $R = \mathbb{Z}[\frac{1}{2}]$ is the localization of $\mathbb{Z}$ at the multiplicative subset $\{2^k : k = 0, 1, ...\}$, and $\Lambda = \mathbb{L}_n$ is the associated R-algebra, then the following conditions hold:

a) The subset B of Proposition 5.2.1 is a basis of Λ as an R-module

b) The complex $P^\bullet$ of Proposition 5.2.7 is still a graded projective resolution of Λ and the canonical map $P^{-i} \longrightarrow \Omega^i_{\Lambda^e}(\Lambda)$ is a retraction (=split epimorphism) in the category of R-modules

c) The complex $V^\bullet$ of Proposition 5.2.9 has as cohomology R-modules the $HH^i(\Lambda)$. Moreover the kernels of its differentials and the $HH^i(\Lambda)$ are free R-modules of finite rank.

Condition a) follows immediately from the previous lemma and, as a consequence, Λe_i, $e_j\Lambda$ and $e_i\Lambda e_j$ are free R-modules of finite rank, for all $i, j \in Q_0$.

To prove b), note that the differentials make sense over R, and then it immediately follows that $P^\bullet$ is again a complex. Now we put $P^1 = \Lambda$ and prove, by decreasing induction on $i \leq 1$, that the complex is exact at P^i and that the surjective map $P^{i-1} \longrightarrow \mathrm{Ker}(d^i)$ splits as a morphism of R-modules. Indeed, in the pass from $i+1$ to i, one has the R-split exact sequence

$$0 \to \mathrm{Ker}(d^i) \longrightarrow P^i \longrightarrow \mathrm{Ker}(d^{i+1}) \to 0,$$

which implies $\mathrm{Ker}(d^i)$ is finitely generated projective as an R-module and that, for each $\mathbf{p} \in \mathrm{Spec}(R)$, the kernel of the differential $d^i \otimes 1 : P^i \otimes_R k(\mathbf{p}) \longrightarrow P^{i+1} \otimes_R k(\mathbf{p})$ is $\mathrm{Ker}(d^i) \otimes k(\mathbf{p})$. From that it easily follows that $\frac{\mathrm{Ker}(d^i)}{\mathrm{Im}(d^{i-1})} \otimes_R k(\mathbf{p}) \cong \frac{\mathrm{Ker}(d^i \otimes 1_{k(\mathbf{p})})}{\mathrm{Im}(d^{i-1} \otimes 1_{k(\mathbf{p})})}$, which is zero since, by Proposition 5.2.7, we know that $P^\bullet \otimes_R k(\mathbf{p})$ is the minimal projective resolution of $\Lambda \otimes_R k(\mathbf{p})$. It follows that $\mathrm{Ker}(d^i)/\mathrm{Im}(d^{i-1})$ is finitely generated R-module with empty support and, hence, it is zero.

c) If P is a finitely generated projective Λ-bimodule, then $\mathrm{Hom}_{\Lambda^e}(P, \Lambda)$ is projective as an R-module and, as a consequence, the canonical map

$$\mathrm{Hom}_{\Lambda^e}(P, \Lambda) \otimes_R k(\mathbf{p}) \longrightarrow \mathrm{Hom}_{(\Lambda \otimes_R k(\mathbf{p}))^e}(P \otimes_R k(\mathbf{p}), \Lambda \otimes_R k(\mathbf{p}))$$

is bijective, for all $\mathbf{p} \in \mathrm{Spec}(R)$. Note that the complex $V^\bullet$ of Proposition 5.2.9 is also isomorphic to $\mathrm{Hom}_{\Lambda^e}(P^\bullet, \Lambda)$ in our case, where $P^\bullet$ is as in Proposition 5.2.7. Therefore $H^i(V^\bullet) \cong HH^i(\Lambda)$ (resp. $H^i(V^\bullet \otimes_R k(\mathbf{p})) \cong HH^i(\Lambda \otimes_R k(\mathbf{p})))$ as a graded R-module (resp. $k(\mathbf{p})$-vector space), for each $i \geq 0$.

On the other hand, according to condition b, the canonical epimorphism $\pi_i : P^{-i} \longrightarrow \Omega^i_{\Lambda^e}(\Lambda) = \mathrm{Ker}(d^{-i+1})$ is a retraction in the category of R-modules. It follows that the induced map $\mathrm{Hom}_{\Lambda^e}(\,\Omega^i_{\Lambda^e}(\Lambda), \Lambda) \longrightarrow \mathrm{Hom}_{\Lambda^e}(P^{-i}, \Lambda)$ is a section (=split monomorphism) in the category of R-modules. This implies that the kernel of each differential $\delta^i : V^i \longrightarrow V^{i+1}$ of the complex $V^\bullet$ is a direct summand of V^i, thus finitely generated projective, as an R-module. It follows that each $HH^i(\Lambda)$ is a finitely generated R-module, and Lemma 5.4.8 can be applied. Moreover, the application of the functor $? \otimes_R k(\mathbf{p})$ gives an isomorphism $HH^i(\Lambda) \otimes_R k(\mathbf{p}) \cong HH^i(\Lambda \otimes_R k(\mathbf{p}))$ due to the fact that this functor keeps exact all the (R-split) exact sequences

$$0 \to \mathrm{Ker}(\delta^i) \longrightarrow V^i \longrightarrow \mathrm{Im}(\delta^i) \to 0.$$

Using Proposition 5.3.10 and lemma 5.4.8, we then get that all the $HH^i(\Lambda)$ are free R-modules with bases as indicated in that proposition.

We finally prove that the relations i)-viii) of Theorem 5.4.1 also hold over R. Indeed, each prime ideal of R is of the form pR and its residue field is $k(p) = R/pR \cong \mathbb{F}_p$, where $p \neq 2$ is a prime integer. This implies that if $v \in HH^i(\Lambda)$ is in the kernel of the canonical map $HH^i(\Lambda) \longrightarrow HH^i(\Lambda \otimes_R k(\mathbf{p})) \cong HH^i(\Lambda) \otimes_R k(\mathbf{p})$, for all $\mathbf{p} \in \mathrm{Spec}(R)$, then $v \in \bigcap_{p \neq 2} pHH^i(\Lambda)$. But this implies that $v = 0$ since $HH^i(\Lambda)$ is a free R-module of finite rank. We apply this argument to each relator (=substraction of the two members of a relation) in the list i)-viii).

The last paragraph provides a morphism of bigraded R-algebras $g : H \longrightarrow HH^*(\Lambda)$, where H is the algebra given by generators and relations as in the statement of Theorem

5.4.1. Moreover $g \otimes 1_{k(\mathbf{p})} : H \otimes_R k(\mathbf{p}) \xrightarrow{\cong} HH^*(\Lambda) \otimes_R k(\mathbf{p}) \cong HH^*(\Lambda \otimes_R k(\mathbf{p}))$ is an isomorphism, for each $\mathbf{p} \in \mathrm{Spec}(R)$, because the statement of the theorem is true for fields. It follows that g is an isomorphism of graded R-algebras since the homogeneous components with respect to the homological grading, H^i and $HH^i(\Lambda)$, are both finitely generated, and $HH^i(\Lambda)$ is free, as R-modules.

We finally consider the general case, in which R is an arbitrary commutative ring on which 2 is invertible. Then we have a structural ring homomorphism $\mathbb{Z}[\frac{1}{2}] \longrightarrow R$ making R into a $\mathbb{Z}[\frac{1}{2}]$-module. But conditions a), b) and c) proved above for $\mathbb{Z}[\frac{1}{2}]$ are kept when applying the functor $? \otimes_{\mathbb{Z}[\frac{1}{2}]} R$. Denoting now by $\Lambda_{\mathbb{Z}[\frac{1}{2}]}$ and Λ_R the respective generalized preprojective algebras of type L, we have an isomorphism $\Lambda_{\mathbb{Z}[\frac{1}{2}]} \otimes_{\mathbb{Z}[\frac{1}{2}]} R \cong \Lambda_R$, which induces an isomorphism of graded algebras $HH^*(\Lambda_{\mathbb{Z}[\frac{1}{2}]}) \otimes_{\mathbb{Z}[\frac{1}{2}]} R \cong HH^*(\Lambda_R)$. It follows that the presentation by generators and relations is also valid for Λ_R.

Corollary 5.4.9. *Let us fix the presentation of $HH^*(\Lambda)$ given by Theorem 5.4.1. A presentation of $\underline{HH}^*(\Lambda)$ is obtained from it by doing the following:*

1. *Replace the generators $x_1, ..., x_n$ by a new generator h' of degree -6*
2. *Replace the relations i) in the list by a new relation $hh' = 1$.*
3. *Leave the remaining generators and relations unchanged.*

Proof. It is clear that the commutative algebra given by the just described generators and relations is isomorphic to $HH^*(\Lambda)_{(h)}$, therefore isomorphic to $\underline{HH}^*(\Lambda)$ (see Proposition 4.5.6). □

Remark 5.4.10. Using a process of passing from a ground field of $\mathrm{Char}(K) \neq 2$ to a ground commutative R on which 2 is invertible, similar to the one used in the proof Theorem 5.4.1, it follows that Corollary 5.2.12 is also true over such a ring R. In particular all $HH_i(\Lambda)$ and $\underline{HH}_i(\Lambda)$ are free R-modules and the formulas of Theorem 5.3.4 still hold, where $\dim(-)$ denotes the rank as free R-module.

5.5 Case when char$(K) = 2$

In this final section we complete the results given in the previous section by dealing with the classical and stable Hochschild cohomology ring of the generalized preprojective algebra $\mathbb{L}_n$ over a field of characteristic 2. We also study the structure as graded modules over it of the classical and stable Hochschild homology.

Throughout this section, K will be an algebraically closed field of characteristic 2. and Λ will be the generalized preprojective algebra $\mathbb{L}_n$ over K.

All the desired structures are obtained from the following main result:

Theorem 5.5.1. *Let us assume that Char$(K) = 2$ and $\Lambda = \mathbb{L}_n$. Then $HH^*(\Lambda)$ is the commutative bigraded algebra, given by:*

a) <u>Generators</u>: $x_0, x_1, \ldots, x_n, y_0, y_1, z_2, \ldots, z_n, h$.

b) *Relations* (given in ascending degree):

i) $x_0^n = x_ix_j = 0$ $(i, j = 0, \ldots, n$ and $(i, j) \neq (0, 0))$
ii) $x_iy_0 = \delta_{i1}x_0^{n-1}y_1$
$x_iy_1 = 0$
iii) $x_iz_k = \delta_{ik}x_0^{n-1}y_0y_1$
$x_0y_0^2 = 0 = y_1^2$
iv) $y_0^3 = nx_0^{n-1}h$
$y_0^2y_1 = \sum_{j=1}^n (n - j + 1)x_jh$
$y_0z_k = (n - k + 1)x_0^{n-1}h$
$y_1z_k = \sum_{j=1}^n (n - max(j, k) + 1)x_jh$
v) $z_kz_l = (n - max(l, k) + 1)x_0^{n-1}y_0h,$

where $i = 1, ..., n$ *and* $k, l = 2, ..., n$ *in the relations ii)-v).*

c) *The homological grading on* $HH^*(\Lambda)$ *is determined by the equalities* $deg(x_i) = 0$, $deg(y_j) = 1$, $deg(z_k) = 2$ *and* $deg(h) = 3$.

d) *The length grading on* $HH^*(\Lambda)$ *is determined by the equalities* $ldeg(x_0) = 2$, $ldeg(x_i) = 2n - 1$, *for* $i \neq 0$, $ldeg(y_0) = -1$, $ldeg(y_1) = 0$, $ldeg(z_k) = -2$ *and* $ldeg(h) = -2n - 1$.

5.5.1 Preliminaries

We will follow the notation used in the previous sections. For the convenience of the reader we restate the needed results with the shape they take in characteristic 2.

The following is a re-statement of Proposition 5.2.1. It fixes the basis of Λ with which we shall work throughout this section.

Proposition 5.5.2. *Let* Λ *be the generalized preprojective algebra* $\mathbb{L}_n$ *and put* $B = \bigcup_{i,j} e_iBe_j$, *where*

a) $e_1Be_1 = \{e_1, \epsilon, \epsilon^2, \ldots, \epsilon^{2n-1}\}$

b) $e_1Be_j = \{a_1 \cdots a_{j-1}, \epsilon a_1 \cdots a_{j-1}, \epsilon^2 a_1 \cdots a_{j-1}, \ldots, \epsilon^{2(n-j)+1}a_1 \cdots a_{j-1}\}$ *in case* $j \neq 1$

c) $e_iBe_j = \{a_i \cdots a_{j-1}, a_i \cdots a_j\bar{a}_j, \ldots, a_i \cdots a_{n-1}\bar{a}_{n-1} \cdots \bar{a}_j\} \bigcup$
$\{\bar{a}_{i-1} \cdots \bar{a}_1\epsilon a_1 \cdots a_{j-1}, \bar{a}_{i-1} \cdots \bar{a}_1\epsilon^3 a_1 \cdots a_{j-1}, \ldots, \bar{a}_{i-1} \cdots \bar{a}_1\epsilon^{2(n-j)+1}a_1 \cdots a_{j-1}\}$
(here we convene that $a_i...a_{j-1} = e_i$ *in case* $i = j$*).*

d) $e_iBe_j = \{\bar{b} : b \in e_jBe_i\}$ *in case* $i > j$,

then B *is a basis of* Λ *such that* $B = B^*$ *and* $b^{**} = b$, *for all* $b \in B$.

Using the above basis and adapting the statement of Proposition 5.2.7, we have the following description of a minimal projective resolution of Λ as a bimodule.

Proposition 5.5.3. *Let B be the basis of Proposition 5.5.2. The chain complex* $\dots P^{-2} \xrightarrow{d^{-2}} P^{-1} \xrightarrow{d^{-1}} P_0 \xrightarrow{u} \Lambda \longrightarrow 0$ *identified by the following properties is a minimal projective resolution of* Λ *as a length-graded bimodule:*

a) $P^{-k} = Q := \bigoplus_{a\in Q_1} \Lambda e_{i(a)} \otimes e_{t(a)}\Lambda$ *if* $k \equiv -1 \pmod 3$ *and* $P^{-k} = P := \bigoplus_{i\in Q_0} \Lambda e_i \otimes e_i \Lambda$ *otherwise.*

b) u *is the multiplication map,* $d^m = d^n$ *whenever* $m \equiv n \pmod 3$ *and the initial differentials* $d^{-1} =: \delta$, $d^{-2} =: R$ *and* $d^{-3} =: k$ *are the only homomorphisms of* Λ*-bimodules satisfying:*

   i) $\delta(e_{i(a)} \otimes e_{t(a)}) = a \otimes e_{t(a)} + e_{i(a)} \otimes a$

   ii) $R(e_i \otimes e_i) = \sum_{a\in Q_1 i(a)=i} e_{i(a)} \otimes \bar{a} + a \otimes e_{i(a)}$

   iii) $k(e_i \otimes e_i) = \sum_{x\in e_i B} x \otimes x^*$

   for all $a \in Q_1$ *and* $i \in Q_0$.

c) *When* P *and* Q *are given their canonical length grading, then, as graded* Λ*-bimodules, the equalities* $P^{-3k-t} = P[-(2n+1)k-t]$, *for* $t = 0, 2$, *and* $P^{-3k-1} = Q[-(2n+1)k-1]$, *hold for all* $k \geq 0$.

Finally, we re-state Proposition 5.2.9:

Proposition 5.5.4. *Let* Λ *be taken with its length grading and consider the complex of* K*-vector spaces*

$$V^\bullet : \cdots 0 \longrightarrow \oplus_{i\in Q_0} e_i\Lambda e_i \xrightarrow{\delta^*} \oplus_{a\in Q_1} e_{i(a)}\Lambda e_{t(a)} \xrightarrow{R^*} \oplus_{i\in Q_0} e_i\Lambda e_i \xrightarrow{k^*} \oplus_{i\in Q_0} e_i\Lambda e_i$$

$$\xrightarrow{\delta^*} \oplus_{a\in Q_1} e_{i(a)}\Lambda e_{t(a)} \xrightarrow{R^*} \oplus_{i\in Q_0} e_i\Lambda e_i \xrightarrow{k^*} \oplus_{i\in Q_0} e_i\Lambda e_i \xrightarrow{\delta^*} \oplus_{a\in Q_1} e_{i(a)}\Lambda e_{t(a)} \cdots$$

where $V^0 = \sum_{i\in Q_0} e_i\Lambda e_i$ *and* $V^n = 0$ $\forall n < 0$. *We view* $V^\bullet$ *as a complex of graded* K*-vector spaces by putting* $V^{3k+t} = \oplus_{i\in Q_0} e_i\Lambda e_i[(2n+1)k+t]$, *when* $t = 0, 2$, *and* $V^{3k+1} = \oplus_{a\in Q_1} e_{i(a)}\Lambda e_{t(a)}[(2n+1)k+1]$, *for all* $k \geq 0$. *Then the space* $HH^n(\Lambda)$ *is isomorphic, as a (length)graded* K*-vector space, to the* n*-th cohomology space of the complex* $V^\bullet$, *for each* $n \geq 0$.

Moreover, viewing $\oplus_{i\in Q=0} e_i\Lambda e_i$ *and* $\oplus_{a\in Q_1} e_{i(a)}\Lambda e_{t(a)}$ *as subspaces of* Λ, *the differentials of* $V^\bullet$ *act as follows for each oriented cycle* c *at* i *and each path* $p : i(a) \to \cdots \to t(a)$:

a) $\delta^*(c) = a_{i-1}c + c\bar{a}_{i-1} + \bar{a}_i c + ca_i$

b) $R^*(p) = p\bar{a} + \bar{a}p$

c) $k^*(c) = 0$ *(i.e.* k^* *is the zero map)*

where we convene that $a_0 = \bar{a}_0 = \epsilon$ *and* $a_n = \bar{a}_n = 0$

As a consequence we get that $HH^{i+3}(\Lambda) \cong HH^i(\Lambda)\ \forall i \geq 0$ via an isomorphism $h : \Omega^3_{\Lambda^e}(\Lambda) \xrightarrow{\sim} \Lambda$. Note that the isomorphism $HH^0(\Lambda) \cong HH^3(\Lambda)$ is due to the fact that $k^* = 0$ and hence $HH^3(\Lambda) = \mathrm{Ker}(\delta^*) = HH^0(\Lambda)$. This is contrast to the case of $\mathrm{Char}(K) \neq 2$, where $HH^0(\Lambda) \ncong HH^6(\Lambda)$.

The following result gives us the description of the structure of the classical and stable Hochschild in terms of the Hochschild cohomology.

Corollary 5.5.5. *There are isomorphisms of graded $\underline{HH}^*(\Lambda)$-modules:*

$$\underline{HH}^*(\Lambda) \cong \underline{HH}^*(\Lambda)[3]$$

$$\underline{HH}_{-*}(\Lambda) \cong D(\underline{HH}^*(\Lambda))$$

and isomorphisms of graded $HH^*(\Lambda)$-modules $HH_{-*}(\Lambda) \cong D(HH^*(\Lambda))$.

Proof. $\underline{HH}^*(\Lambda) \cong \underline{HH}^*(\Lambda)[3]$ since Λ is periodic of period 3. On the other hand, Λ is 2-CY Frobenius and using a result by Eu and Schedler (see [35] and also Theorem 4.5.7 of this thesis), we have

$$D(\underline{HH}^*(\Lambda)) \cong \underline{HH}^*(\Lambda)[5]$$

$$\underline{HH}_{-*}(\Lambda) \cong \underline{HH}^*(\Lambda)[2]$$

Then the isomorphisms in the statement follow. The graded Frobenius condition of $\underline{HH}^*(\Lambda)$ follows from the theorem mentioned before.

On the other hand, due to the fact that $D(\Lambda) \cong \Lambda$, we have an isomorphism $HH_{-*}(\Lambda) \cong D(HH^*(\Lambda, D(\Lambda))) \cong D(HH^*(\Lambda))$.

□

5.5.2 The ring structure of $HH^*(\Lambda)$

In this subsection we determine explicitly the dimension and appropriate basis of each space $HH^i(\Lambda)$ using the complex $V^\bullet$ given above. We start with two lemmas which are the correspondents of Lemma 5.3.1 and Proposition 5.3.3 for the case when $\mathrm{Char}(K) = 2$. The proof is identical, with the suitable adaptation. For instance $R^*_\mu = R^*$ and $x_0 = \sum_{i=0}^{n-1} a_i \bar{a}_i$ in the new situation.

Lemma 5.5.6. *$Im(R^*)$ is a subspace of codimension n in $\oplus_{i \in Q_0} e_i J e_i$. In particular, we have:*

$$dim(Im(R^*)) = n^2 - n.$$

Proposition 5.5.7. *The center of Λ is isomorphic to $\frac{K[x_0, x_1, \dots x_n]}{I}$, where I is the ideal of $K[x_0, x_1, \dots x_n]$ generated by x_0^n and all the products $x_i x_j$ with $(i, j) \neq (0, 0)$. In particular, $dim(HH^0(\Lambda)) = 2n$.*

We are now ready to give the dimension of all cohomology spaces. Note the difference with respect to case of characteristic $\neq 2$.

Proposition 5.5.8. *The equality* $dim(HH^i(\Lambda)) = dim(HH_i(\Lambda)) = dim(\underline{HH}^j(\Lambda)) = dim(\underline{HH}_j(\Lambda)) = 2n$ *holds for all* $i \geq 0$ *and* $j \in \mathbb{Z}$.

Proof. Using Corollary 5.5.5, the computation of the dimensions of the Hochschild homology spaces can be directly deduced from the computation of the Hochschild cohomology spaces.

On the other hand, the proof of Corollary 5.2.11, which is still valid in characteristic 2, shows that $P(\Lambda, \Lambda) = \mathrm{Im}(k_\mu^*)$. Since $k_\mu^* = k^* = 0$ in our case, we conclude that $P(\Lambda, \Lambda) = 0$ and hence $\underline{HH}^i(\Lambda) = HH^i(\Lambda)$ for every $i \geq 0$. Since $\underline{HH}^i(\Lambda) = \underline{HH}^{i+3}(\Lambda)$, for every integer $i \in \mathbb{Z}$, it is enough to prove that $\dim(HH^i(\Lambda)) = 2n$, for $i = 0, 1, 2$.

By the previous lemma, $\dim(HH^0(\Lambda)) = 2n$. To deal with $HH^1(\Lambda)$ we consider

$$0 \longrightarrow \mathrm{Ker}(R^*) \longrightarrow \oplus_{a \in Q_1} e_{i(a)} \Lambda e_{t(a)} \xrightarrow{R^*} \mathrm{Im}(R^*) \longrightarrow 0$$

$$0 \longrightarrow Z(\Lambda) \hookrightarrow \oplus_{i \in Q_0} e_i \Lambda e_i \longrightarrow \mathrm{Im}(\delta^*) \longrightarrow 0$$

Since $HH^1(\Lambda) \cong \frac{\mathrm{Ker}(R^*)}{\mathrm{Im}(\delta^*)}$, we then get that $\dim(HH^1(\Lambda)) = \dim(\oplus_{a \in Q_1} e_{i(a)} \Lambda e_{t(a)}) - \dim(\mathrm{Im}(R^*)) - (\dim(\oplus_{i \in Q_0} e_i \Lambda e_i) - \dim(Z(\Lambda))) = [2n^2 - (n^2 - n)] - [(n^2 + n) - 2n] = (n^2 + n) - (n^2 - n) = 2n$, using Corollary 5.2.5 for the formulas for the dimensions of $\mathrm{Hom}_{\Lambda^e}(P, \Lambda)$ and $\mathrm{Hom}_{\Lambda^e}(Q, \Lambda)$ which are still valid here.

Finally, $HH^2(\Lambda) \cong \mathrm{Coker}(R^*) \cong \frac{\oplus_{i \in Q_0} e_i \Lambda e_i}{\mathrm{Im}(R^*)}$ and thus $\dim(HH^2(\Lambda)) = n^2 + n - (n^2 - n) = 2n$.

□

We next identify the structure of the $HH^i(\Lambda)$ as $Z(\Lambda)$-modules. In the statement and proof the elements x_i which appear are as in Proposition 5.5.7.

Proposition 5.5.9. *As* $Z(\Lambda)$*-modules,* $HH^j(\Lambda)$ *is isomorphic to* $HH^{j+3}(\Lambda)$ *for all* $j \geq 0$. *In particular* $HH^{3k}(\Lambda)$ *is a free* $Z(\Lambda)$*-module of rank* 1*, for all* $k \geq 0$. *Moreover:*

1. *The* $Z(\Lambda)$*-module* $HH^1(\Lambda)$ *has two generators* y_0 *and* y_1*, subject to the relations* $x_1 y_0 = x_0^{n-1} y_1$ *and* $x_i y_j = 0$*, for* $i = 1, ..., n$*,* $j = 0, 1$ *and* $(i, j) \neq (1, 0)$.
2. *The* $Z(\Lambda)$*-module* $HH^2(\Lambda)$ *has* $n + 1$ *generators* $z, z_1, ..., z_n$ *subject to the relations* $x_i z = 0$*,* $x_0 z_i = 0$ *and* $x_j z_i = \delta_{ij} x_0^{n-1} z$*, for all* $i, j = 1, ..., n$ *(here* δ_{ij} *is the Kronecker symbol).*

Proof. The initial statement is a direct consequence of Proposition 5.5.4 and the equality $HH^0(\Lambda) = Z(\Lambda)$.

In order to prove assertion 1, we put $y_0 = e_1 + \mathrm{Im}(\delta^*), y_1 = \epsilon + \mathrm{Im}(\delta^*) \in HH^1(\Lambda)$. Viewing $\mathrm{Soc}(\Lambda)$ as an ideal of $Z(\Lambda)$, we have $\mathrm{Soc}(\Lambda) y_1 = 0$ (equivalently $x_i y_1 = 0$ for all $i = 1, ..., n$) since $\mathrm{Soc}(\Lambda)\epsilon = 0$. Then the assignment $a \rightsquigarrow a y_1$ gives a surjective morphism $\frac{K[x_0]}{(x_0^n)} \cong \frac{Z(\Lambda)}{\mathrm{Soc}(\Lambda)} \twoheadrightarrow Z(\Lambda) y_1$ of $Z(\Lambda)$-modules. We shall prove that it is injective. Note that the equality $x_0^k y_1 = 0$ is equivalent to saying that $\epsilon^{2k+1} \in \mathrm{Im}(\delta^*)$. So the injectivity will follow from something more general that we shall prove, namely that $e_{i(\epsilon)} \Lambda e_{t(\epsilon)} \cap \mathrm{Im}(\delta^*) = 0$. Indeed take the projection $\oplus_{a \in Q_1} e_{i(a)} \Lambda e_{t(a)} \xrightarrow{\pi_\epsilon} e_{i(\epsilon)} \Lambda e_{t(\epsilon)}$ and consider the composition

$$\oplus_{i\in Q_0}e_i\Lambda e_i \xrightarrow{\delta^*} \oplus_{a\in Q_1}e_{i(a)}\Lambda e_{t(a)} \xrightarrow{\pi_\epsilon} e_{i(\epsilon)}\Lambda e_{t(\epsilon)}$$

This map vanishes on $e_i\Lambda e_i$, for $i \neq 1$, and takes $c \rightsquigarrow \epsilon c + c\epsilon$, for each cycle at 1. But $\epsilon c + c\epsilon = 0$ since $e_1\Lambda e_1$ is a commutative algebra and $\mathrm{Char}(K) = 2$.

On the other hand, since $x_i = w_i$ we readily get that $x_iy_0 = 0$ for $i \neq 1$. As for $i = 1$ we have $x_1y_0 = \epsilon^{2n-1} + \mathrm{Im}(\delta^*) = x_0^{n-1}y_1$. It follows that the $Z(\Lambda)y_0 + Z(\Lambda)y_1$ is generated as a K-vector space by $\mathcal{B} := \{y_0, x_0y_0, \ldots, x_0^{n-1}y_0\} \cup \{y_1, x_0y_1, \ldots x_0^{n-1}y_1\}$. But if $\sum_{0\leq k<n}\lambda_k x_0^k y_0 + \sum_{0\leq l<n}\mu_l x_0^l y_1 = 0$, with $\lambda_k, \mu_l \in K$, then $\sum_{0\leq k<n}\lambda_k\epsilon^{2k} + \sum_{0\leq l<n}\mu_l\epsilon^{2l+1} \in \mathrm{Im}(\delta^*)$ and the previous paragraph shows that then $\sum_{0\leq k<n}\lambda_k\epsilon^{2k} + \sum_{0\leq l<n}\mu_l\epsilon^{2l+1} = 0$. This can only happen if $\lambda_k = 0 = \mu_l$, for all $0 \leq k, l < n$. It follows that $\mathcal{B}$ is a basis of $Z(\Lambda)y_0 + Z(\Lambda)y_1$ and, by Proposition 5.5.8, we then have $Z(\Lambda)y_0 + Z(\Lambda)y_1 = HH^1(\Lambda)$.

We finally prove assertion 2. We freely use the part of the proof of Lemma 5.3.1 concerning R_μ^*, which is valid here for R^* since $R^* = R_\mu^*$ in characteristic 2. By Lemma 5.5.6, we have a chain

$$\mathrm{Im}(R^*) \subsetneq \oplus_{i\in Q_0}e_iJe_i \subsetneq \oplus_{i\in Q_0}e_i\Lambda e_i,$$

with each term of codimension n in the next. This implies that a basis of $HH^2(\Lambda) = \frac{\mathrm{Ker}(k^*)}{\mathrm{Im}(R^*)} = \frac{\oplus_{i\in Q_0}e_i\Lambda e_i}{\mathrm{Im}(R^*)}$ is given by $\mathcal{B}' = \{e_i + \mathrm{Im}(R^*) : i \in Q_0\} \cup \mathcal{B}''$, where $\mathcal{B}''$ is a basis of $\frac{\oplus_{i\in Q_0}e_iJe_i}{\mathrm{Im}(R^*)}$.

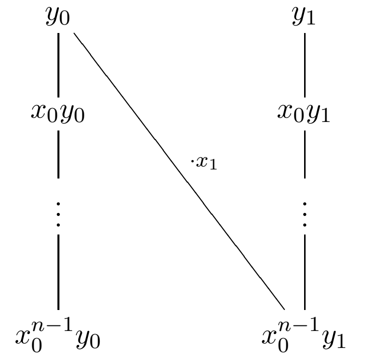
By looking at the proof of Lemma 5.3.1, we know that the matrix of the induced map $R^* = R_\mu^* : V_{2m-2}^+ \longrightarrow W_{2m-1}$ with respect to the there given bases is an upper triangular square matrix whose first column is zero. This implies that $\epsilon^{2m-1} \notin \mathrm{Im}(R^*)$, for all $m = 1, ..., n$. Furthermore, due to the fact that R^* is a graded map, it implies that $\{\epsilon + \mathrm{Im}(R^*), \epsilon^3 + \mathrm{Im}(R^*), ..., \epsilon^{2n-1} + \mathrm{Im}(R^*)\} =: \mathcal{B}''$ is a basis of $\frac{\oplus_{i\in Q_0}e_iJe_i}{\mathrm{Im}(R^*)}$.

We now put $z = \epsilon + \mathrm{Im}(R^*)$, and then $\epsilon^{2m-1} + \mathrm{Im}(R^*) = x_0^m z$ in $HH^2(\Lambda)$. We also put $z_i = e_i + \mathrm{Im}(R^*)$. Then $\mathcal{B}' = \{z_1, ..., z_n\} \cup \{z, x_0z, ..., x_0^{n-1}z\}$ is a basis of $HH^2(\Lambda)$. We immediately get that $x_iz = 0$ since $w_i\epsilon = 0$ in Λ. On the other hand $x_jz_i = w_je_i + \mathrm{Im}(R^*) = 0$, unless $j = i$, a case in which we have $x_jz_j = w_j + \mathrm{Im}(R^*) = w_1 + \mathrm{Im}(R^*) = \epsilon^{2n-1} + \mathrm{Im}(R^*)$ (see Remark 3.2 of [1]). But we have $\epsilon^{2n-1} + \mathrm{Im}(R^*) = x_0^{n-1}z$ so that $x_jz_i = \delta_{ij}x_0^{n-1}z$, for all $i, j \in \{1, ..., n\}$.

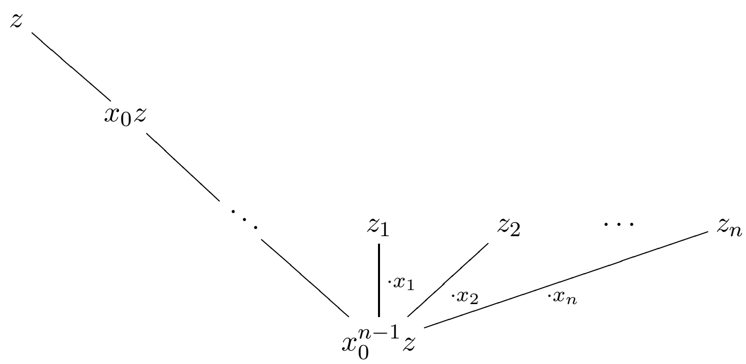
Finally, the induced map $R^* = R_\mu^* : V_{2k-1} \longrightarrow W_{2k}$ is surjective, for all $k > 0$, which implies that $x_0^k z_i = (a_i\bar{a}_i)^k + \mathrm{Im}(R^*) = 0 + \mathrm{Im}(R^*)$, and hence $x_0^k z_i = 0$, for all $i = 1, ..., n$ and $k > 0$. □

Remark 5.5.10. The following diagrams depict the structure of $HH^1(\Lambda)$ and $HH^2(\Lambda)$ as $Z(\Lambda)$-modules:

$HH^1(\Lambda)$:



$HH^2(\Lambda)$:



We have now the following correspondent of Proposition 5.3.10, which is an immediate consequence of the proof of the previous lemma:

Proposition 5.5.11. *The following are bases for the $HH^i(\Lambda)$ $(i = 0, 1, 2)$:*

1. *For $HH^0(\Lambda) = Z(\Lambda)$:* $\{1, x_0, ..., x_0^{n-1}\} \cup \{x_1, ..., x_n\}$, *where* $x_0 = \sum_{i=0}^{n-1} a_i\bar{a}_i$ *and* $x_k = w_k$, *for all* $k = 1, ..., n$.

2. *For $HH^1(\Lambda) = \frac{Ker(R^*)}{Im(\delta^*)}$:* $\{y_0, x_0y_0, ..., x_0^{n-1}y_0\} \cup \{y_1, x_0y_1, ..., x_0^{n-1}y_1\}$, *where* $y_k = \epsilon^k + Im(\delta^*)$ *(convening that* $\epsilon^0 = e_1$*).*

   The element y_k is represented by the only morphism of Λ-bimodules $\tilde{y}_k : Q \longrightarrow \Lambda$ such that $\tilde{y}_k(e_{i(a)} \otimes e_{t(a)}) = \delta_{a\epsilon}\epsilon^k$.

3. *For $HH^2(\Lambda) = \frac{Ker(k^*)}{Im(R^*)}$:* $\{z, x_0z, ..., x_0^{n-1}z\} \cup \{z_1, ..., z_n\}$, *where the* $z_k = e_k + Im(R^*)$ $(k = 1, ..., n)$ *and* $z = \epsilon + Im(R^*)$*. In this case:*

   (a) *The element z_k is represented by the only morphism $\tilde{z}_k : P \longrightarrow \Lambda$ such that $\tilde{z}_k(e_i \otimes e_i) = \delta_{ik}e_k$, for all $i \in Q_0$.*

(b) *The element* z *is represented by the only morphism* $\tilde{z} : P \longrightarrow \Lambda$ *such that* $\tilde{z}(e_i \otimes e_i) = \delta_{1i}\epsilon$, *for all* $i \in Q_0$.

The above bases will be called the *canonical bases* of the $HH^j(\Lambda)$, for $j = 0, 1, 2$. Given the 'equality' $HH^3(\Lambda) = HH^0(\Lambda) = Z(\Lambda)$, we denote $1 \in Z(\Lambda)$ by h when view it as an element of $HH^3(\Lambda)$. With that notation, multiplication by h^k in $HH^*(\Lambda)$ gives an isomorphism $HH^j(\Lambda) \xrightarrow{\cong} HH^{3k+j}(\Lambda)$ for $j = 0, 1, 2$. The canonical basis of HH^{3k+j} will be, by definition, the image of the canonical basis of $HH^j(\Lambda)$ by this isomorphism.

Lemma 5.5.12. *In the algebra* $HH^*(\Lambda)$, *we have equalities*

$$y_0^2 = z_1 \quad y_0y_1 = z \quad y_1^2 = 0.$$

Moreover, multiplication by y_1 *gives an epimorphism of* $Z(\Lambda)$*-modules* $\rho : Z(\Lambda)y_0 \twoheadrightarrow Z(\Lambda)z$ *whose kernel is* Kx_1y_0.

Proof. The morphism $\widehat{y_k} : \oplus_{a\in Q_1}\Lambda e_{i(a)} \otimes e_{t(a)}\Lambda \longrightarrow \oplus_{i\in Q_0}\Lambda e_i \otimes e_i\Lambda$ determined by the rule $\widehat{y_k}(e_{i(a)} \otimes e_{t(a)}) = \delta_{\epsilon a}\epsilon^k \otimes e_1$ satisfies that $u \circ \widehat{y_k} = \tilde{y_k}$, for $k = 0, 1$. The equalities in the statement will follow from commutativity of the next diagram:

$$\begin{array}{ccc} P & \xrightarrow{R} & Q \\ {\scriptstyle \alpha_k}\downarrow & & \downarrow{\scriptstyle \widehat{y_k}} \\ Q & \xrightarrow[\delta]{} & P \end{array},$$

where $\alpha_k : \oplus_{i\in Q_0}\Lambda e_i \otimes e_i\Lambda \longrightarrow \oplus_{a\in Q_1}\Lambda e_{i(a)} \otimes e_{t(a)}\Lambda$ is the only morphism of Λ-bimodules which takes $e_i \otimes e_i$ to $\delta_{i1}\epsilon^k \otimes e_1$. Then y_0^2 is represented by the composition

$$\oplus_{i\in Q_0}\Lambda e_i \otimes e_i\Lambda \xrightarrow{\alpha_0} \oplus_{a\in Q_1}\Lambda e_{i(a)} \otimes e_{t(a)}\Lambda \xrightarrow{\tilde{y}_0} \Lambda.$$

But we have $\tilde{y}_0 \circ \alpha_0(e_i \otimes e_i) = \delta_{i1}e_1 = \tilde{z}_1(e_i \otimes e_i)$, for each $i \in Q_0$, and so $y_0^2 = z_1$.

Similarly we get that y_1y_0 is represented by morphism of Λ-bimodules $\tilde{y}_1 \circ \alpha_0 : \oplus_{i\in Q_0}\Lambda e_i \otimes e_i\Lambda \longrightarrow \Lambda$, which acts as $\tilde{y}_1 \circ \alpha_0(e_i \otimes e_i) = \tilde{y}_1(\delta_{i1}(e_1 \otimes e_1)) = \delta_{i1}\epsilon$ while y_1^2 is represented by the only morphism $\tilde{y}_1 \circ \alpha_1 : \oplus_{i\in Q_0}\Lambda e_i \otimes e_i\Lambda \longrightarrow \Lambda$ which acts as $\tilde{y}_1 \circ \alpha_1(e_i \otimes e_i) = \tilde{y}_1\delta_{i1}\epsilon(e_i \otimes e_i) = \delta_{i1}\epsilon^2$. The first one is clearly represents $z \in HH^2(\Lambda)$, so that $y_1y_0 = z$, while $y_1^2 = 0$ since $\epsilon^2 \in \mathrm{Im}(R^*)$ by the proof of Lemma 5.3.1.

The existence of the epimorphism ρ of the final statement follows from the equality $y_0y_1 = z$. From the proof of Lemma 5.5.9 we know that $\dim(Z(\Lambda)z) = n$ while a basis of $Z(\Lambda)y_0$ as K-vector space is $\{y_0, x_0y_0, ..., x_0^{n-1}y_0, x_1y_0\}$. Since $\rho(x_1y_0) = \rho(x_0^{n-1}y_1) = x_0^{n-1}y_1^2 = 0$ it follows that $\mathrm{Ker}(\rho) = Kx_1y_0$. □

Lemma 5.5.13. *The following equalities hold in the ring* $HH^*(\Lambda)$:

$$y_0z = \sum_{j=1}^{n}(n-j+1)x_jh \quad y_0z_k = (n-k+1)x_0^{n-1}h \quad y_1z_k = \sum_{j=1}^{n}(n-max(k,j)+1)x_jh.$$

Proof. The morphism $\widehat{z}_k : \oplus_{i\in Q_0}\Lambda e_i \otimes e_i\Lambda \longrightarrow \oplus_{i\in Q_0}\Lambda e_i \otimes e_i\Lambda$ given by $\widehat{z}_k(e_i \otimes e_i) = \delta_{ik}(e_k \otimes e_k)$ satisfies that $u \circ \widehat{z}_k = \tilde{z}_k$, for $k = 1, ..., n$. Bearing in mind that Char$(K) = 2$, the proof of Lemma 5.3.9 gives a commutative diagram:

$$\begin{array}{ccc} P & \xrightarrow{k} & P \\ {\scriptstyle f_k}\downarrow & & \downarrow{\scriptstyle \widehat{z}_k} \\ Q & \xrightarrow[\delta]{} & P \end{array} ,$$

where $f_k(e_i \otimes e_i) = \sum_{x\in e_iBe_k} h_x$ for all $k, i \in Q_0$, where $h_{e_i} = 0$, for all $i \in Q_0$. Here the h_x are as in the proof of Lemma 5.4.4

Since $\tilde{y}_0$ is a graded map of degree 0 we get that $\tilde{y}_0 \circ f_k(e_i \otimes e_i)$ is a linear combination of elements of degree $2n - 2$ in $e_i\Lambda e_i$. This implies that $\tilde{y}_0 \circ f_k(e_i \otimes e_i) = 0$ $\forall i \neq 1$. On the contrary, for $i = 1$ we note that if $x = \epsilon^j a_1...a_{k-1} \in e_1Be_k$, then $\tilde{y}_0(h_x) = j\epsilon^{2n-2}$. It follows that

$$\tilde{y}_0 f_k(e_1 \otimes e_1) = \tilde{y}_0(\sum_{x\in e_1Be_k} h_x) = \sum_{x\in e_1Be_k} \tilde{y}_0(h_x) = \sum_{j=1}^{2(n-k)+1} j\epsilon^{2(n-2)}$$

$$= (n - k + 1)[2(n - k) + 1]\epsilon^{2(n-2)} = (n - k + 1)\epsilon^{2(n-2)}.$$

If we now consider $\tilde{y}_1$, it is clear that for each $x \in e_iBe_k$, $\tilde{y}_1(h_x) = m\omega_i$ where m is the number of times that ϵ appears in x. On one hand, if $i = 1$ then, arguing as in the foregoing paragraph, we have $\sum_{x\in e_1Be_k} \tilde{y}_1(h_x) = (n - k + 1)\omega_1$. On the other hand, if $i \neq 1$, then

$$\sum_{x\in e_iBe_k} \tilde{y}_1(h_x) = \sum_{t=0}^{n-max(i,k)} \tilde{y}_1(h_{\bar{a}_{i-1}\cdots\bar{a}_1}\epsilon^{2t+1}a_1 \cdots a_{k-1}) = \sum_{t=0}^{n-max(i,k)} (2t + 1)\omega_i =$$

$$(n - max(i, k) + 1)\omega_i.$$

Therefore we conclude that

$$y_0z_k = (n - k + 1)x_0^{n-1}h \quad \text{and} \quad y_1z_k = \sum_{i=1}^{n}(n - max(i, k) + 1)x_ih.$$

Finally, the lat equality together with Lemma 5.5.12 give

$$y_0z = y_0^2y_1 = z_1y_1 = y_1z_1 = \sum_{i=1}^{n}(n - i + 1)x_ih.$$

□

We are now ready to prove the main theorem of the paper.

PROOF OF THE THEOREM 5.5.1: Proceeding as in the last paragraph of Proposition 5.3.10, one easily sees that the length degrees of the proposed generators are as indicated. On the other hand, "graded commutativity" and "commutativity" are synonymous in characteristic 2. Moreover, since Λ is periodic of period 3 it follows that $HH^*(\Lambda)$ is generated as an algebra by $(\bigcup_{0\leq i\leq 2} HH^i(\Lambda)) \cup \{h\}$, where $h \in HH^3(\Lambda)$ is the element given by a fixed isomorphism $\Omega^3_{\Lambda^2}(\Lambda) \xrightarrow{\cong} \Lambda$. From these comments and Proposition 5.5.9 we get that $\{x_0, x_1, ..., x_n, y_0, y_1, z, z_1, ..., z_n, h\}$ is a set of homogeneous generators of $HH^*(\Lambda)$ as an algebra. But, due to Lemma 5.5.12, we can drop z and z_1 from the list. In that way, one gets the set of generators and their degrees as in the statement.

By Proposition 5.5.7 and Proposition 5.5.9, the generators satisfy the relations in i) and ii). Also Proposition 5.5.9 and Lemma 5.5.12 give the relations in iii). On the other hand, using this latter lemma and Lemma 5.5.13, we see that

$$y_0^3 = z_1 y_0 = n x_0^{n-1} h$$

and

$$y_0^2 y_1 = z_1 y_1 = y_1 z_1 = \sum_{1\leq j\leq n} (n-j+1) x_j h.$$

The other two relations in iv) follow from Lemma 5.5.13.

It remains to check the relations in v), the rest of the proof is routine. From Lemma 5.5.9, we easily get that $\mathrm{Soc}(HH^1(\Lambda)) = Kx_0^{n-1}y_0 + Kx_0^{n-1}y_1 = Kx_0^{n-1}y_0 + Kx_1y_0$ (socle as $Z(\Lambda)$-module). This implies that $\mathrm{Soc}(HH^4(\Lambda)) = Kx_0^{n-1}y_0h + Kx_1y_0h$. We claim that this socle contains every product $z_k z_l$, for which it will be enough to see that $x_i z_k z_l = 0$ for all indices i, k, l. This is a direct consequence of Proposition 5.5.9 except, perhaps, for the case $i = k = l$. But in this case, this same proposition gives $x_k z_k z_k = x_0^{n-1} z z_k = x_0^{n-1} z_k z = 0$.

We put $N = \sum_{2\leq k,l\leq n} K z_k z_l$. We shall prove that $(N + Kx_0^{n-1}y_0h) \cap Kx_1y_0h = 0$. For that we need to give a morphism of Λ-bimodules representing each product $z_k z_l \in HH^4(\Lambda)$. Following the notation of the proof of Lemma 5.5.13, we consider the commutative diagram

$$\begin{array}{ccccc} Q & \xrightarrow{\delta} & P & \xrightarrow{k} & P \\ \downarrow{\scriptstyle g_k} & & \downarrow{\scriptstyle f_k} & & \downarrow{\scriptstyle \widehat{z}_k} \\ P & \xrightarrow[R]{} & Q & \xrightarrow[\delta]{} & P \end{array}$$

that is needed to compute $z_l z_k$. Although we do not know g_k explicitly, recall that $f_k(e_i \otimes e_i) = \sum_{x\in e_iBe_k} h_x$, so that f_k is a graded map of degree $2n-2$. Since δ and R are both graded maps of degree 1 and the left square is required to be commutative we can choose g_k to be graded of degree $2n-2$, so that $g_k(e_{i(a)} \otimes e_{t(a)}) = \sum_{j=1}^n \gamma_j \otimes \mu_j$ where $\gamma_j \in e_{i(a)}\Lambda$ and $\mu_j \in \Lambda e_{t(a)}$ are elements such that $\deg(\gamma_j) + \deg(\mu_j) = 2n-2$.

Since the product $z_l z_k$ will be represented by the composition

$$\oplus_{a\in Q_1} \Lambda e_{i(a)} \otimes e_{t(a)}\Lambda \xrightarrow{g_k} \oplus_{i\in Q_0} \Lambda e_i \otimes e_i \Lambda \xrightarrow{\tilde{z}_l} \Lambda$$

we have $\tilde{z}_l \circ g_k(e_{i(a)} \otimes e_{t(a)}) = \sum_{j=1}^n \gamma_j \tilde{z}_l(e_{t(\gamma_j)} \otimes e_{i(\mu_j)})\mu_j$, which is a linear combination of paths of length $2n-2$ in $e_{i(a)}\Lambda e_{t(a)}$. Therefore, it is zero whenever $a \neq \epsilon$. Putting

$v_{kl} = (\tilde{z}_l \circ g_k)(e_{i(\epsilon)} \otimes e_{t(\epsilon)})$, which is an element of degree $2n-2$ of $e_1\Lambda e_1$, we get the equality

$$\sum_{2\leq k,l\leq n} \lambda_{kl} z_k z_l + \lambda x_0^{n-1} y_0 h = \sum_{2\leq k,l\leq n} \lambda_{kl} v_{kl} + \lambda \epsilon^{2n-2} + \mathrm{Im}(\delta^*)$$

in $\frac{\mathrm{Ker}(R^*)}{\mathrm{Im}(\delta^*)} = HH^4(\Lambda)$, for all λ_{kl} and λ in K. That implies that $\sum_{2\leq k,l\leq n} \lambda_{kl} z_k z_l + \lambda x_0^{n-1} y_0 h = v + \mathrm{Im}(\delta^*)$, for some element $v \in e_1\Lambda e_1$ of degree $2n-2$.

On the other hand, for any $\mu \in K$, we have $\mu x_1 y_0 h = \mu \epsilon^{2n-1} + \mathrm{Im}(\delta^*)$. If we have an equality

$$\sum_{1\leq k,l\leq n} \lambda_{kl} z_k z_l + \lambda x_0^{n-1} y_0 h = \mu x_1 y_0 h,$$

then the element $v - \mu\epsilon^{2n-1} \in \mathrm{Im}(\delta^*)$. Since δ^* is a graded map and v and $\mu\epsilon^{2n-1}$ are homogeneous elements of different length degree it follows that $v \in \mathrm{Im}(\delta^*)$ and $\mu\epsilon^{2n-1} \in \mathrm{Im}(\delta^*)$. The latter is equivalent to saying that $\mu x_1 y_0 h = 0$ in $HH^4(\Lambda)$, which can only happens if $\mu = 0$. This proves the desired equality $(N + Kx_0^{n-1}y_0 h) \cap Kx_1 y_0 h = 0$.

The last two paragraphs together with Lemma 5.5.12 and the canonical isomorphism $HH^1(\Lambda) \xrightarrow{\cong} HH^4(\Lambda)$ show that if $\rho : Z(\Lambda)y_0 h \twoheadrightarrow Z(\Lambda)zh$ is the epimorphism of $Z(\Lambda)$-modules given by $u \rightsquigarrow y_1 u$, then the composition

$$\tilde{\rho} : N + Kx_0^{n-1} y_0 h \hookrightarrow Z(\Lambda)y_0 h \overset{\rho}{\twoheadrightarrow} Z(\Lambda)zh$$

is a monomorphism. But, using the commutativity of $HH^*(\Lambda)$, Lemma 5.5.13 and Proposition 5.5.9, we have an equality

$$\begin{aligned}
\tilde{\rho}(z_k z_l - (n - max(k,l) + 1)x_0^{n-1} y_0 h) &= y_1 z_k z_l - (n - max(k,l) + 1)x_0^{n-1} y_0 y_1 h \\
&= \sum_{1\leq j\leq n} (n - max(j,k) + 1)x_j h z_l - (n - max(k,l) + 1)x_0^{n-1} zh \\
&= \sum_{1\leq j\leq n} (n - max(j,k) + 1)x_j z_l h - (n - max(k,l) + 1)x_0^{n-1} zh \\
&= (n - max(l,k) + 1)x_0^{n-1} zh - (n - max(k,l) + 1)x_0^{n-1} zh = 0.
\end{aligned}$$

It follows that $z_k z_l - (n - max(k,l) + 1)x_0^{n-1} y_0 h = 0$, for all $k, l = 1, ..., n$, from which the relations v) in the statement follow.

Corollary 5.5.14. *Let us fix the presentation of $HH^*(\Lambda)$ given by Theorem 5.5.1. A presentation of $\underline{HH}^*(\Lambda)$ is obtaining from it by doing the following:*

1. *Add a new generator h' of degree -3 and the relation $hh' = 1$.*
2. *Leave all the other generators and relations unchanged.*

Proof. It is immediately seen that the graded commutative algebra given by the just described generators and relations is isomorphic to $HH^*(\Lambda)_{(h)}$, whence isomorphic to $\underline{HH}^*(\Lambda)$ (see Proposition 4.5.6, which is also valid when Char$(K) = 2$). □

Chapter 6

The Hochschild cohomology ring of $\mathbb{B}_n$

6.1 Introduction

6.2 Motivation

We conclude this dissertation by describing the multiplicative structure of the Hochschild cohomology ring $HH^*(\Lambda)$ of the generalized preprojective algebra $\Lambda = \mathbb{B}_n$ over a field of characteristic $\neq 2$. This algebra together with the algebra $\mathbb{L}_n$, studied in the previous chapter, form the generalized preprojective algebras of type $\mathbb{A}_n$. Thus, the results in the present chapter complete the problem of determining the multiplicative structure of $HH^*(\Lambda)$ for the preprojective and generalized preprojective algebras Λ of type $\mathbb{A}_n$.

As for $\mathbb{L}_n$, it is known and follows from Theorem 3.4.12 that $\mathbb{B}_n$ has (Ω -)period 6 and, hence, the Hochschild cohomology spaces also have period 6.

Our results show the surprisingly different behavior of $HH^*(\Lambda)$ when n is odd and when n is even. In case n is odd, many Hochschild cohomology groups vanish and, when $\mathrm{char}(K)$ does not divide n, the ring $HH^*(\Lambda)$ can be generated by just three elements. The situation is rather different in case n is even, where a minimal set of generators of $HH^*(\Lambda)$ involves all the degrees from 0 to 6. Moreover, when n is odd, $HH^*(\Lambda)$ is always commutative, while, when n is even, it is only commutative when $\mathrm{char}(K)$ divides n.

6.3 Outline of the chapter

The organization of this chapter is similar to that of Chapter 5. However, unlike the previous chapter, we will use here the full strength of covering theory and the results from Chapters 1-3. In Section 6.5 we look at the generalized preprojective algebra $\mathbb{B}_n$. We begin this section by providing some more information of the mesh algebra B of the stable translation quiver $\mathbb{Z}\mathbb{A}_{2n-1}$. Specifically, we provide a combinatorial criterion to determine when $e_{(k,i)}Be_{(m,j)} \neq 0$, for any two given vertices (k,i) and (m,j) of $\mathbb{Z}\mathbb{A}_{2n-1}$. In the rest of this section we study the algebra $\Lambda = \mathbb{B}_n$ and, in particular, we provide the cochain complex of graded K-vector spaces which is later used to compute the Hochschild cohomology of $\mathbb{B}_n$. In Section 6.6 we give the structure of the Hochschild cohomology spaces $HH^i(\Lambda)$ as modules over $Z(\Lambda) = HH^0(\Lambda)$. Finally, Section 6.7 is devoted to compute the structure of the Hochschild cohomology ring $HH^*(\Lambda)$ as a bigraded algebra over a field of characteristic $\neq 2$ by means of generators and relations, distinguishing the cases when n is either even or odd.

6.4 Notation

In this chapter, unless otherwise stated, Λ will be the generalized preprojective algebra $\mathbb{B}_n$ over a field K with characteristic $\neq 2$. We will follow the notation and terminology given in Chapter 4.

6.5 The generalized preprojective algebra $\mathbb{B}_n$

6.5.1 The mesh algebra B of $\mathbb{Z}\mathbb{A}_{2n-1}$

In this section we present some results on the Galois cover of $\mathbb{B}_n$, that is, the mesh algebra B of the stable translation quiver $\mathbb{Z}\mathbb{A}_{2n-1}$. For its definition and basic properties the reader is referred to Chapter 2, Section 2.3. We also point out that in this chapter we will use the change of relations given in Chapter 2, Section 2.3.3. Then, the mesh algebra B of $\mathbb{Z}\mathbb{A}_{2n-1}$ is given by the stable translation quiver $\mathbb{Z}\mathbb{A}_{2n-1}$ and, for each vertex $(k,i) \in (\mathbb{Z}\mathbb{A}_{2n-1})_0$, we have a relation $r_{(k,i)}$ where

$$r(k,i) = \sum_{t(a)=(k,i)} (-1)^{s(\sigma(a)a)} \sigma(a)a$$

and s is the corresponding signature map (see Chapter 2, Section 2.3.3) From now on in this chapter these $r(k,i)$ will be the *mesh relations* for us. Our next goal is to identify those pairs $[(k,i),(m,j)]$ of vertices of $\mathbb{Z}\mathbb{A}_{2n-1}$ for which $e_{(k,i)}Be_{(m,j)} \neq 0$, identifying the dimension of this space. Notice that when $(k,i),(m,j) \in (\mathbb{Z}\mathbb{A}_{2n-1})_0$ are any two vertices, if there are any paths between them, they all go in just one direction and have the same length. For that reason, it is appropriate to call *distance* between the two vertices, denoted $d[(k,i),(m,j)]$, to the common length of those paths, when they exist. In what follows, the orientation of the arrows is the one given in Subsection 2.3.1. That is, the canonical k-slice of $\mathbb{Z}\mathbb{A}_{2n-1}$ is $(k,1) \longrightarrow (k,2) \longrightarrow \cdots \longrightarrow (k,2n-1)$.

Proposition 6.5.1. *Suppose that (k,i), (m,j) are any two vertices in $\mathbb{Z}\mathbb{A}_{2n-1}$. Then, there is a path $(k,i) \to \dots \to (m,j)$ in $\mathbb{Z}\mathbb{A}_{2n-1}$ if, and only if, max$\{0, i-j\} \leq m-k$. Moreover, in this case, $d[(k,i),(m,j)] = 2(m-k)+(j-i)$.*

Proof. Let $p = (k,i) \to \dots \to (m,j)$ be a path in $\mathbb{Z}\mathbb{A}_{2n-1}$. We prove by induction on its length l that $\max\{0, i-j\} \leq m-k$. For $l = 0$ it is clear, so we assume that $l > 0$. Since there are at most two arrows starting at (k,i), namely $(k,i) \longrightarrow (k,i+1)$ and $(k,i) \to (k+1,i-1)$, we can consider that there are either a path $(k,i+1) \to \dots \to (m,j)$ or a path $(k+1,i-1) \to \dots \to (m,j)$, each one of length $l-1$. The induction hypothesis says that $\max\{0, i+1-j\} \leq m-k$, in the first case, and $\max\{0, i-1-j\} \leq m-k-1$, in the second one. From both of them it follows that $\max\{0, i-j\} \leq m-k$, and hence this inequality follows from the existence of the path p.

Conversely, the inequality $\max\{0, i-j\} \leq m-k$ is equivalent to say that $k \leq m$ and either $i \leq j$ or $i > j$ and $i-j \leq m-k$. We then assume that $k \leq m$. If $i \leq j$ then we have an obvious path $(k,i) \longrightarrow ... \rightarrow (k,j) \rightarrow ... \rightarrow (m,j)$. If $i > j$ and $i-j \leq m-k$, then we have a path $(k,i) \rightarrow (k+1, i-1) \rightarrow ... \rightarrow (k+i-j, j) \rightarrow ... \rightarrow (m,j)$, because $k+i-j \leq m$. Moreover, either one of these two paths has length $2(m-k)+(j-i)$ and, hence, this is the distance between (k,i) and (m,j).

□

Definition 21. Let us look at $\Delta = \mathbb{A}_{2n-1}$ as a subquiver of $\mathbb{A}_{\infty}^{\infty} : ... \rightarrow -1 \rightarrow 0 \rightarrow 1 \rightarrow ...$ and, hence, at $\mathbb{Z}\Delta$ as a subquiver of $\mathbb{Z}\mathbb{A}_{\infty}^{\infty}$. Given $(k,i), (m,j) \in (\mathbb{Z}\Delta)_0$ such that there is a path in $\mathbb{Z}\Delta$ from (k,i) to (m,j) (i.e. $max\{0, i-j\} \leq m-k$), we call *imaginary rectangle based on* (k,i) *and* (m,j) to the rectangle in $\mathbb{Z}\mathbb{A}_{\infty}^{\infty}$ with vertices (k,i), $(m, k-m+i)$, (m,j) and $(k, m-k+j)$. We will say that the *top (resp. bottom) of the rectangle can be built in* $\mathbb{Z}\mathbb{A}_{2n-1}$ when the vertex $(k, m-k+j)$ (resp, $(m, k-m+i)$) is in $\mathbb{Z}\mathbb{A}_{2n-1}$. That is, when $m-k+j \leq n$ (resp. $k-m+i \geq 1$). We will say that the rectangle *can be constructed within* $\mathbb{Z}\mathbb{A}_{2n-1}$ when the top and the bottom of the imaginary triangle can be built in $\mathbb{Z}\mathbb{A}_{2n-1}$.

The following lemma admits a symmetric one. Both proofs are very easy and we will omit them.

Lemma 6.5.2. *Let* $q : (k,i) \rightarrow (k+1, i-1) \rightarrow ... \rightarrow (m, k-m+i)$ *and* $p : (m, k-m+i) \rightarrow ... \rightarrow (m,j)$ *paths in* $\Gamma = \mathbb{Z}\mathbb{A}_{2n-1}$ *of positive length consisting of downward arrows and upward arrows, respectively. The following assertions are equivalent:*

1. *The product* qp *is nonzero in the mesh algebra* $B = K\Gamma/I$
2. *The top of the rectangle based on* (k,i) *and* (m,j) *can be built in* $\mathbb{Z}\mathbb{A}_{2n-1}$

In that case, all paths from (k,i) *to* (m,j) *in* Γ *are equal modulo the mesh relations.*

The following result is a criterion to determine when there exists a path between any two given vertices in B. Moreover, in such case, the path is unique up to sign, in B.

Theorem 6.5.3. *Let* $\Delta = \mathbb{A}_{2n-1}$*, let* $\Gamma = \mathbb{Z}\Delta$ *its associated translation quiver and* B *the associated mesh algebra. Let* (k,i) *and* (m,j) *be vertices in* Γ*. Then, the space* $e_{(k,i)}Be_{(m,j)}$ *is nonzero if, and only if,* $max\{0, i-j\} \leq m-k \leq min\{i-1, 2n-1-j\}$*. In this case one has* $dim(e_{(k,i)}Be_{(m,j)}) = 1$*.*

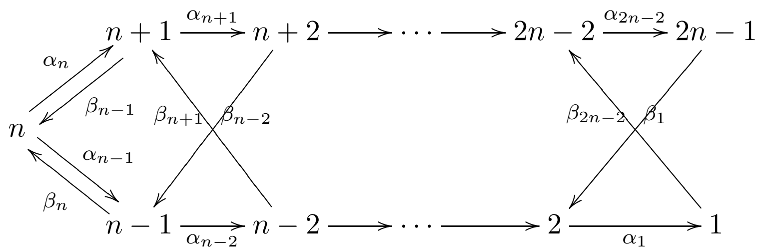
Proof. Given any two vertices $(k,i), (m,j) \in \Gamma_0$, any path from (k,i) to (m,j) in $\mathbb{Z}\Delta$ is a product of paths $q_1p_1...q_rp_r$, where the q_j consist only of downward arrows, the p_i consist only of upward arrows, and one has length$(p_i) > 0$, for $i = 1, ..., r-1$, and length$(q_j) > 0$, for $j = 2, ..., r$. If such a path exists then, by Proposition 6.5.1, $\max\{0, i-j\} \leq m-k$. On the other hand, by using the previous lemma and its symmetric, we see that if the path is nonzero in B then the imaginary rectangle based on (k,i) and (m,j) can be built in $\mathbb{Z}\mathbb{A}_{2n-1}$. Then we have $m-k+j \leq 2n-1$ and $k-m+i \geq 1$, which is equivalent to say that $m-k \leq \min\{i-1, 2n-1-j\}$. Therefore we get that $\max\{0, i-j\} \leq m-k \leq \min\{i-1, 2n-1-j\}$.

Conversely, if $\max\{0, i-j\} \leq m-k \leq \min\{i-1, 2n-1-j\}$ then we have a path $(k,i) \to ... \to (m,j)$ in $\mathbb{Z}\Delta$, due to Proposition 6.5.1. Moreover, by definition 21, the imaginary rectangle based on (k,i) and (m,j) can be built in $\mathbb{Z}\mathbb{A}_{2n-1}$. By Lemma 6.5.2, we get a path $(k,i) \to ... \to (m,j)$ which is nonzero in B and is unique modulo the mesh relations.

□

6.5.2 Definition of $\mathbb{B}_n$

In the sequel, Λ will be generalized preprojective algebra $\mathbb{B}_n$ over a field K of characteristic different from 2, and Q_0 and Q_1 will denote the sets of vertices and arrows in Λ, respectively. We will see Λ as the orbit algebra $\Lambda = B/ < \rho\tau >$ where B, ρ and τ are the mesh algebra and the automorphisms of $\mathbb{Z}\mathbb{A}_{2n-1}$ described in Chapter 2, Section 2.3.1. We will divert from the notation in [25] and we will put $i = [(1,i)]$, for $n \leq i \leq 2n-1$, and $i = [(1+n-i, i)]$, for $1 \leq i \leq n$, where $[(k,i)]$ denotes the $\rho\tau$-orbit of the vertex (k,i). Then, the *generalized preprojective algebra* $\mathbb{B}_n$ is the finite dimensional algebra given, up to isomorphism, by the quiver



and subject to all the commutativity relations and the monomial relations of length 2 ending at the extreme vertices 1 and $2n-1$.

Recall from [25] that the quiver Q of Λ is also a translation quiver whose AR translation and polarization are induced from those of $\mathbb{Z}\mathbb{A}_{2n-1}$. Precisely, the action of the Auslander-Reiten translation of Λ, also denoted by τ, is given by $\tau(i) = 2n-i$ for all $i \in Q_0$. As for the arrows, it is given by $\tau(\alpha_i) = \alpha_{2n-1-i}$ and $\tau(\beta_i) = \beta_{2n-1-i}$ for all $i \in Q_0$. In regard to the polarization σ, we have that $\sigma(\alpha_i) = \beta_i$ and $\sigma(\beta_i) = \alpha_{2n-1-i}$ for all $i \in Q_0$. Observe then that the equality $\sigma^2(a) = \tau(a)$ holds for each arrow $a \in Q_1$. Also note that if $X \subseteq (\mathbb{Z}\mathbb{A}_{2n-1})_1$ is the set of arrows defining the signature map, then its image $\bar{X}$ by the pushdown functor $B \longrightarrow \Lambda$ consists of the arrows in Q that appear in the 'crossing path'

$$2n-1 \to 2 \to 2n-3 \to \cdots \to n-1 \to n \qquad \text{when } n \text{ is odd}$$

and

$$1 \to 2n-2 \to 3 \to \cdots \to n-1 \to n \qquad \text{when } n \text{ is even}$$

However, for unification purposes (see for instance Lemma 6.5.9 and Proposition 6.5.10), whenever n is even we will consider $\tau(X)$ instead of X and, in such case, $\bar{X}$ consists of the arrows appearing in the following path

$$2n-1 \to 2 \to 2n-3 \to \cdots \to n+1 \to n$$

Notice that, also for the new considered $\bar{X}$, the relations for Λ are those of the form $\sum_{t(a)=i}(-1)^{s(\sigma(a)a)}\sigma(a)a$. Up to sign, they are exactly the commutativity relations plus the zero relations $\beta_{2n-2}\alpha_{2n-2}$ and $\beta_1\alpha_1$. For the sake of simplicity, the arrows β_i will be called *crossing arrows* and the arrows α_i will be called *noncrossing arrows*.

Recall from Proposition 3.4.3 and Corollary 3.4.5 that, with their canonical gradings, both B and Λ are graded pseudo-Frobenius and we have isomorphisms of graded bimodules $D(B) \cong {}_1B_\eta[2n-2]$ and $D(\Lambda) \cong {}_1\Lambda_\eta[2n-2]$, where η is the respective Nakayama automorphism in each case. In particular, we get:

Remark 6.5.4. In the above setting, if A is either B or Λ, then the duality isomorphism $D(A) \cong {}_1A_\eta[2n-2]$ gives an isomorphism of K-vector spaces

$$e_iA_{2n-2-d}e_{\eta(j)} \cong e_i\left({}_1A_\eta[2n-2]\right)_{-d}e_j \cong e_iD(A)_{-d}e_j \cong D(e_jA_de_i),$$

for all vertices $i, j \in \Gamma_0$ and natural number d.

6.5.3 The Cartan matrix

Due to the fact that B is the universal Galois covering of Λ, if (k,i) and (m,j) are vertices of $\mathbb{Z}\mathbb{A}_{2n-1}$ and d is a natural number, then $e_{[(k,i)]}\Lambda_de_{[(m,j)]} \neq 0$ if, and only if, there is a (unique) integer r such that $d[(\rho\tau)^r(k,i),(m,j)] = d$ and $e_{(\rho\tau)^r(k,i)}Be_{(m,j)} \neq 0$. We then introduce the following rather useful sets of integers.

Definition 22. Let $i, j \in Q_0$ be vertices. If $i, j \geq n$, the set $X^+(i,j)$ (resp. $X^-(i,j)$) will consist of the integers s such that $e_{(\rho\tau)^{2s}(1,i)}Be_{(1,j)} \neq 0$ (resp. $e_{(\rho\tau)^{2s+1}(1,i)}Be_{(1,j)} \neq 0$).

Similarly, if $i \geq n \geq j$ then $Z^+(i,j)$ (resp. $Z^-(i,j)$) will denote the set of integers s such that $e_{(\rho\tau)^{2s}(1,i)}Be_{(1+n-j,j)} \neq 0$ (resp. $e_{(\rho\tau)^{2s+1}(1,i)}Be_{(1+n-j,j)} \neq 0$).

Lemma 6.5.5. *Let $i, j \in Q_0$ be vertices and d be a natural number. The following assertions hold:*

1. *When $i, j \geq n$, the inequality $e_i\Lambda_de_j \neq 0$ holds if, and only if, $d = 4s + (j-i)$, for some integer $s \in X^+(i,j)$, or $d = 4s+2+(j-i)$, for some integer $s \in X^-(i,j)$. In particular, the dimension of $e_i\Lambda e_j$ is $|X^+(i,j)| + |X^-(i,j)|$, where $|-|$ denotes the cardinal of a set.*

2. *When $i \geq n \geq j$, the inequality $e_i\Lambda_de_j \neq 0$ holds if, and only if, $d = 4s+(2n-i-j)$, for some integer $s \in Z^+(i,j)$, or $d = 4s+2+(2n-i-j)$, for some integer $s \in Z^-(i,j)$. In particular the dimension of $e_i\Lambda e_j$ is $|Z^+(i,j)| + |Z^-(i,j)|$.*

Proof. Recall that we have the equality $\rho\tau(k,i) = (k+i-n-1, 2n-i)$. Bearing in mind that $(\rho\tau)^{2s} = \tau^{2s}$, we readily check the following two formulas:

$$(\rho\tau)^{2s}(k,i) = (k-2s,i)$$
$$\text{and } (\rho\tau)^{2s+1}(k,i) = (k+i-2s-n-1, 2n-i),$$

for all $s \in \mathbb{Z}$.

A direct calculation shows that we have the following formulas for the distance between two vertices of $\mathbb{Z}\mathbb{A}_{2n-1}$, whenever there is a path in $\mathbb{Z}\mathbb{A}_{2n-1}$ between the indicated vertices (see Proposition 6.5.1). If $i,j \geq n$ then $d[\rho\tau)^r(1,i),(1,j)] = 2r + (j-i)$, and if $i \geq n \geq j$ then $d[\rho\tau)^r(1,i),(1+n-j,j)] = 2(n+r) - (i+j)$.

For assertion 1, suppose that $i,j \geq n$, so that we have $i = [(1,i)]$ and $j = [(1,j)]$. Then $e_i\Lambda_d e_j \neq 0$ if, and only if, there is an integer $r \in \mathbb{Z}$ such that $e_{(\rho\tau)^r(1,i)}Be_{(1,j)} \neq 0$ and $d[\rho\tau)^r(1,i),(1,j)] = d$. But $e_{(\rho\tau)^r(1,i)}Be_{(1,j)} \neq 0$ if, and only if, $r \in 2X^+(i,j) \cup (2X^-(i,j)+1)$.

2) Bearing in mind that $j = [(1+n-j,j)]$, whenever $j \leq n$, a similar argument works when $i \geq n \geq j$, replacing $X^+(i,j)$ and $X^-(i,j)$ by $Z^+(i,j)$ and $Z^-(i,j)$, respectively.

For the formulas of the dimensions of $e_i\Lambda e_j$, note that if $(k,i),(m,j) \in \mathbb{Z}\mathbb{A}_{2n-1}$ are any two vertices then $\dim(e_{(\rho\tau)^r(k,i)}Be_{(m,j)}$ is either 0 or 1 since all paths from $(\rho\tau)^r(k,i)$ to (m,j) are equal modulo the mesh relations. If now $[(k,i)]$ and $[(m,j)]$ denote their $\rho\tau$-orbits, viewed as vertices of Q, then $\dim(e_{[(k,i)]})\Lambda_d e_{[(m,j)]})$ is 1 or 0, depending on whether there exists an integer r such that $e_{(\rho\tau)^r(k,i)}Be_{(m,j)} \neq 0$ and $d[(\rho\tau)^r(k,i),(m,j)] = d$, or not. The formula for $\dim(e_i\Lambda e_j)$ is now clear. □

We next include a technical lemma which will be very useful.

Lemma 6.5.6. *The following assertions hold:*

1. *If $i,j \geq n$ then:*
   (a) *s is in $X^+(i,j)$ if, and only if, $max\{0, \frac{i-j}{2}\} \leq s \leq n - \frac{j+1}{2}$;*
   (b) *s is in $X^-(i,j)$ if, and only if, $\frac{i-n-1}{2} \leq s \leq \min\{\frac{n-2}{2}, \frac{n-2+i-j}{2}\}$.*

2. *If $i \geq n \geq j$ then:*
   (a) *s is in $Z^+(i,j)$ if, and only if, $\frac{i-n}{2} \leq s \leq min\{\frac{i+j-n-1}{2}, \frac{n-1}{2}\}$;*
   (b) *s is in $Z^-(i,j)$ if, and only if, $max\{-\frac{1}{2}, \frac{i+j-2n-1}{2}\} \leq s \leq \frac{j-2}{2}$.*

Proof. We use the formulas for $(\rho\tau)^r(k,i)$ given in the first paragraph of the proof of Lemma 6.5.5. Also, we will denote by ν the Nakayama permutation of $\mathbb{Z}\mathbb{A}_{2n-1}$ (see Proposition 2.3.1).

1) By Theorem 6.5.3, we have the following chain of double implications for an integer s:

$$s \in X^+(i,j) \Longleftrightarrow e_{(1-2s,i)}Be_{(1,j)} \neq 0 \Longleftrightarrow$$
$$\max\{0, i-j\} \leq 2s \leq \min\{i-1, 2n-1-j\} = 2n-1-j \Longleftrightarrow \max\{0, \tfrac{i-j}{2}\} \leq s \leq n - \tfrac{j+1}{2},$$

and similarly, by using the formula of $(\rho\tau)^{2s+1}$ and the duality D of B (see Remark 6.5.4), we get:

$$s \in X^-(i,j) \iff e_{(i-2s-n,2n-i)}Be_{(1,j)} \neq 0 \iff e_{(1,j)}Be_{\nu(i-2s-n,2n-i)} \neq 0 \iff$$
$$e_{(1,j)}Be_{(n-2s-1,i)} \neq 0 \iff \max\{0, j-i\} \leq n-2s-2 \leq \min\{j-1, 2n-1-i\} = 2n-1-i$$
$$\iff \min\{n-2, i-j+n-2\} \geq 2s \geq i-1-n \iff \frac{i-n-1}{2} \leq s \leq \min\{\frac{n-2}{2}, \frac{n-2+i-j}{2}\}.$$

2) We use an analogous argument. We have a chain of double implications:

$$s \in Z^+(i,j) \iff e_{(1-2s,i)}Be_{(1+n-j,j)} \neq 0 \iff$$
$$i-j = \max\{0, i-j\} \leq n+2s-j \leq \min\{i-1, 2n-1-j\} \iff$$
$$i-n \leq 2s \leq \min\{i+j-n-1, n-1\} \iff \frac{i-n}{2} \leq s \leq \min\{\frac{i+j-n-1}{2}, \frac{n-1}{2}\},$$

and similarly:

$$s \in Z^-(i,j) \iff e_{(\rho\tau)^{2s+1}(1,i)}Be_{(1+n-j,j)} \neq 0 \iff e_{(i-2s-n,2n-i)}Be_{(1+n-j,j)} \neq 0 \iff$$
$$e_{(1+n-j,j)}Be_{\nu(i-2s-n,2n-i)} \neq 0 \iff e_{(1+n-j,j)}Be_{(n-2s-1,i)} \neq 0 \iff$$
$$0 = \max\{0, j-i\} \leq j-2s-2 \leq \min\{j-1, 2n-1-i\} \iff$$
$$j-2 \geq 2s \geq \max\{-1, i+j-2n-1\} \iff \max\{\frac{-1}{2}, \frac{i+j-2n-1}{2}\} \leq s \leq \frac{j-2}{2}.$$

□

We conclude this section by describing the *Cartan matrix* C_Λ of Λ. This is the $|Q_0| \times |Q_0|$ matrix whose entry (i,j) is $c_{ij} := \dim_K(Hom_\Lambda(\Lambda e_i, \Lambda e_j)) = \dim_K(e_i \Lambda e_j)$. To describe C_Λ in an easy-to-remember way, given a vertex $i \in Q_0$, the *distance to the extremes* of i will be the number $d(i) = \min\{i, 2n-i\}$.

Proposition 6.5.7. *The Cartan matrix $C_\Lambda = (c_{ij})$ of Λ is given as follows:*

1. *If n is even, then:*

   (a) *$c_{ij} = min\{d(i), d(j)\}$, if either i or j is even;*

   (b) *$c_{ij} = min\{d(i), d(j)\} - 1$, if both i and j are odd and exactly one of i, j is $< n$;*

   (c) *$c_{ij} = min\{d(i), d(j)\} + 1$ otherwise.*

   In this case the rank of C_Λ is $n+1$.

2. *If n is odd and $\xi : Q_0 \times Q_0 \longrightarrow Q_0 \times Q_0$ is the bijection given by $(i,j) \rightsquigarrow (j, 2n-i)$, then C_Λ is the only integer $(2n-1) \times (2n-1)$-matrix satisfying the following conditions:*

   (a) *$c_{ij} = c_{j,2n-i}$, for all $i, j \in Q_0$*

   (b) *$c_{ij} = min\{d(i), d(j)\}$, when i and j are either both even or both odd*

   (c) *$c_{ij} = min\{d(i), d(j)\} + 1$, whenever $(i,j) \in \bigcup_{r \in \mathbb{N}} \xi^r(X)$, where X consists of those pairs (k,l) such that k is even, l is odd and $n \leq k \leq l$*

   (d) *$c_{ij} = min\{d(i), d(j)\} - 1$ otherwise.*

Proof. First step: We calculate the entry $c_{ij} = dim(e_i \Lambda e_j)$, for each pair $(i, j) \in Q_0 \times Q_0$ such that $i, j \geq n$.

Note that once this is done, we will have calculated all the entries of C_Λ, in case n is odd, and all the entries c_{ij} with the property that either $i, j \geq n$ or $i, j \leq n$, in case n is even. For the case of n being even, this is clear since we have an equality $c_{ij} = \dim(e_i \Lambda e_j) = \dim(\tau(e_i \Lambda e_j)) = \dim(e_{\tau(i)} \Lambda e_{\tau(j)}) = c_{2n-i,2n-j}$. Let then assume that n is odd in the rest of the paragraph. By applying the successive powers of the bijection $\xi(i,j) = (j, 2n-i)$ to the set $Y = \{(i,j) \in Q_0 \times Q_0 \colon i, j \geq n\}$, we get the whole set $Q_0 \times Q_0$. The Nakayama automorphism is $\eta = \tau$ (see Proposition 6.5.10), which acts on vertices as $\tau(i) = 2n - i$. Therefore, by applying the duality D, we get an equality

$$c_{ij} = \dim(e_i \Lambda e_j) = \dim(D(e_i \Lambda e_j)) = \dim(e_j \Lambda e_{2n-i}) = c_{j,2n-i},$$

and an iteration of this formula starting with a pair $(i, j) \in Y$ gives all the entries of $C := C_\Lambda$.

By Lemma 6.5.5, we know that $c_{ij} = |X^+(i,j)| + |X^-(i,j)|$. Then, by using Lemma 6.5.6, we get the following two tables, distinguishing the case when $n \leq i \leq j$ (and n arbitrary) from the case when n is odd and $n \leq j \leq i$.

Case $n \leq i \leq j$	$\lvert X^+(i,j)\rvert$	$\lvert X^-(i,j)\rvert$	c_{ij}
$i \equiv n \pmod 2$; j odd	$n - \frac{j-1}{2}$	$n - \frac{j+1}{2}$	2n-j
$i \equiv n \pmod 2$; j even	$n - \frac{j}{2}$	$n - \frac{j}{2}$	2n-j
$i \not\equiv n \pmod 2$; j odd	$n - \frac{j-1}{2}$	$n - \frac{j-1}{2}$	2n-j+1
$i \not\equiv n \pmod 2$; j even	$n - \frac{j}{2}$	$n - \frac{j}{2}$	2n-j

Case (n odd; $n \leq j \leq i$)	$\lvert X^+(i,j)\rvert$	$\lvert X^-(i,j)\rvert$	c_{ij}
i, j even	$n - \frac{i}{2}$	$n - \frac{i}{2}$	2n-i
i odd; j even	$n - \frac{i+1}{2}$	$n - \frac{i+1}{2}$	2n-i-1
i even; j odd	$n - \frac{i}{2}$	$n - \frac{i}{2}$	2n-i
i, j odd	$n - \frac{i-1}{2}$	$n - \frac{i+1}{2}$	2n-i

From the first table we get that if n is even and $n \leq i \leq j$, then $c_{ij} = c_{ji} = 2n - j = \min\{d(i), d(j)\}$, except in the case when i and j are both odd. In this latter case, one has $c_{ij} = c_{ji} = \min\{d(i), d(j)\} + 1$. Note that $c_{ij} = \dim(e_i \Lambda e_j) = \dim[\tau(e_i \Lambda e_j)] = e_{2n-i} \Lambda e_{2n-j}$ and that $d(2n - i) = d(i)$ for every $i \in Q_0$. We conclude that if n is even and either $i, j \geq n$ or $i, j \leq n$, then $c_{ij} = \min\{d(i), d(j)\}$, when at least one of i and j is even, and $c_{ij} = \min\{d(i), d(j)\} + 1$, when i and j are both odd.

For the case in which n is odd, the first table says that if $n \leq i \leq j$, then $c_{ij} = \min\{d(i), d(j)\}$, except when i is even and j is odd. In this latter case, one has $c_{ij} = \min\{d(i), d(j)\} + 1$. On the other hand, the second table says that, when n is odd and $n \leq j \leq i$, the entry c_{ij} is equal to $2n - i = \min\{d(i), d(j)\}$, except when i is odd and j is even. In this latter case $c_{ij} = \min\{d(i), d(j)\} - 1$. This complete the identification of c_{ij}, for all pairs $(i, j) \in Y$, and, by the comments above, also the identification of all the entries of C in case n is odd. It is an easy exercise to check that C is the unique integer $(2n - 1) \times (2n - 1)$-matrix satisfying the conditions of assertion 2.

Second step: calculation, for n even, of the entries c_{ij}, for $i \geq n \geq j$:

Note that once this step is completed, using the equality $c_{ij} = c_{2n-i,2n-j}$, we will have identified all the entries of C for the case in which n is even. Here we have $i = [(1, i)]$ and $j = [(1 + n - j, j)] = [(1 - j, j)]$ and it is enough to calculate c_{ij} when $j \geq 2n - i$ (so that $\min\{d(i), d(j)\} = 2n - i$). Indeed, if $j < 2n - i$ then the symmetric condition of C gives an equality $c_{ij} = c_{2n-i,2n-j} = c_{2n-j,2n-i}$, and the pair $(2n - j, 2n - i)$ has the property that $2n - j \geq n \geq 2n - i$ and $2n - i \geq 2n - (2n - j)$.

So, from now on in this second step, we assume that $i \geq n \geq j$ and $j \geq 2n - i$. We know from Lemma 6.5.5 that $c_{ij} = |Z^+(i, j)| + |Z^-(i, j)|$. Using now Lemma 6.5.6, we get the following table

Case (n even; $j \geq 2n - i$)	$\vert Z^+(i, j)\vert$	$\vert Z^-(i, j)\vert$	c_{ij}
i, j even	$n - \frac{i}{2}$	$n - \frac{i}{2}$	2n-i
i even; j odd	$n - \frac{i}{2}$	$n - \frac{i}{2}$	2n-i
i odd; j even	$n - \frac{i+1}{2}$	$n - \frac{i-1}{2}$	2n-i
i, j odd	$n - \frac{i+1}{2}$	$n - \frac{i+1}{2}$	2n-i-1

Then $c_{ij} = 2n - i = \min\{d(i), d(j)\}$, except when i and j are both odd. In this latter case $c_{ij} = \min\{d(i), d(j)\} - 1$. Considering the first and second step for n even, we see that C is the matrix given in assertion 1.

Third step: calculation of the rank of C in case n is even:

Let S_{2n-1} be the symmetric group on $2n - 1$ indices and, by considering its action on vertices, let us interpret the AR translation τ as an element of S_{2n-1}. Then τ is the product of the transpositions $(i, 2n - i)$ $(i = 1, ..., n - 1)$. Let $P = P_\tau$ be the associated permutation matrix, i.e., we have $p_{i\tau(i)} = 1$ and $p_{ij} = 0$, for $j \neq \tau(i)$. The fact that $c_{ij} = c_{(2n-i,2n-j)} = c_{\tau(i)\tau(j)}$, for all $i, j \in Q_0$, implies that $PC = CP$. For simplicity, let us denote by f (resp. g) the K-linear map $V := K^{2n-1} \longrightarrow V$ which takes a column-vector v to Cv (resp. Pv). If $\mathcal{B} = \{v_1, ..., v_{2n-1}\}$ is the canonical basis of V, then $\{v_1 + v_{2n-1}, ..., v_{n-1} + v_{n+1}, v_n\}$ and $\{v_1 - v_{2n-1}, ..., v_{n-1} - v_{n+1}\}$ are linearly independent sets of eigenvectors of g associated to the eigenvalues 1 and -1, respectively. Then we have a decomposition $V = \mathrm{Ker}(g - 1_V) \oplus \mathrm{Ker}(g + 1_V)$, where both summands are invariant by f due to the equality $g \circ f = f \circ g$.

We then get a decomposition $\mathrm{Ker}(f) = (\mathrm{Ker}(f) \cap \mathrm{Ker}(g - 1_V)) \oplus (\mathrm{Ker}(f) \cap \mathrm{Ker}(g - 1_V))$. We claim that $\mathrm{Ker}(f) \cap \mathrm{Ker}(g - 1_V) = 0$. Indeed, let us take $v \in \mathrm{Ker}(f) \cap \mathrm{Ker}(g - 1_V)$, so that we have scalars $a_1, ..., a_n \in K$ such that $v = a_1(v_1 + v_{2n-1}) + ... + a_{n-1}(v_{n-1} + v_{n+1}) + a_n v_n$. We apply f to this equality and get an equality of column-vectors

$$a_1(C^1 + C^{2n-1}) + ... + a_{n-1}(C^{n-1} + C^{n+1}) + a_n C^n = 0, \qquad (*)$$

where C^j denotes the j-th column of C. Note that if $i, j \leq n$ then assertion 1 gives that $c_{ij} = c_{i,2n-j} = \min\{d(i), d(j)\}$, in case either i or j is even, and that $\{c_{ij}, c_{i,2n-j}\} = \{\min\{d(i), d(j)\} + 1, \min\{d(i), d(j)\} - 1\}$, when i and j are odd. It follows that if we put $m_{ij} = \min\{d(i), d(j)\}$, then we always have $c_{ij} + c_{i,2n-j} = 2m_{ij}$. For later use, note also that if $i, j \leq n$ then $c_{ij} - c_{i,2n-j} = 0$, when either i or j is even, and $c_{ij} - c_{i,2n-j} = 2$ otherwise.

Taking the i-th component of the vector equality (*), for each $i = 1, 2, ..., n$ we get an equality $\sum_{1\leq j\leq n-1} 2m_{ij}a_j + ia_n = 0$. It follows that the column-vector $(a_1, ..., a_{n-1}, a_n)^T$ is a solution of the homogeneous system of linear equations with associated $n \times n$-matrix

$$X_n = \begin{pmatrix} 2 & 2 & 2 & . & . & . & 2 & 1 \\ 2 & 4 & 4 & . & . & . & 4 & 2 \\ 2 & 4 & 6 & . & . & . & 6 & 3 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 2 & 4 & 6 & . & . & . & 2(n-1) & n-1 \\ 2 & 4 & 6 & . & . & . & 2(n-1) & n \end{pmatrix}$$

By substracting the first row from all the others, we get a block decomposition of the form $\begin{pmatrix} 2 & * \\ 0 & X_{n-1} \end{pmatrix}$, from which we get that $\det(X_n) = 2\det(X_{n-1})$ and, by induction, that $\det(X_n) = 2^{n-1}$. It follows that X_n is invertible and, hence, that $(a_1, ..., a_n) = (0, ..., 0)$ and that $v = 0$. This proves our claim.

We then have $\mathrm{Ker}(f) = \mathrm{Ker}(f) \cap \mathrm{Ker}(g + 1_V)$. In particular, each $v \in \mathrm{Ker}(f)$ can be written in the form $v = b_1(v_1 - v_{2n-1}) + ... + b_{n-1}(v_{n-1} - v_{2n-1})$, with $b_j \in K$ for $j = 1, ..., n-1$. By applying f to this equality, we get a second equality of column-vectors

$$b_1(C^1 - C^{2n-1}) + ... + b_{n-1}(C^{n-1} - C^{n+1}) = 0. \qquad (**)$$

When taking the i-th component and the $(2n - i)$-th component in this equality, we see that the first members of the two equalities are equal up to sign. It follows that $\mathrm{Ker}(f)$ is isomorphic to the subspace of K^{n-1} consisting of those $(n-1)$-tuples $(b_1, ..., b_{n-1})$ such that $\sum_{1\leq j<n} b_j(c_{ij} - c_{i,2n-j})$, for $i = 1, 2, ..., n$. By comments made above about the difference $c_{ij} - c_{i,2n-j}$, these $(n-1)$-tuples are precisely the ones which satisfy the equality

$$2b_1 + 2b_3 + ... + 2b_{n-1} = 0,$$

which form a hyperplane in K^{n-1}. It follows that $\dim(\mathrm{Ker}(f)) = n-2$, and so $\mathrm{rank}(C) = (2n-1) - (n-2) = n+1$. □

Remark 6.5.8. The calculation of $\mathrm{rank}(C)$ in the proof of last proposition also gives an explicit basis of $\mathrm{Ker}(f)$, namely $\{v_{2k} - v_{2n-2k}\colon k = 1, ..., \frac{n}{2} - 1\} \cup \{v_{2k-1} - v_{2n-2k+1} - v_{2k+1} + v_{2n-2k-1}\colon k = 1, ..., \frac{n}{2} - 1\}$.

6.5.4 The minimal projective resolution

In Theorem 2.4.2 we explicitly gave a graded Nakayama automorphism η_B of the mesh algebra $B = K\mathbb{Z}\mathbb{A}_{2n-1}$ such that $\eta_B \circ g = g \circ \eta_B$, for all $g \in G =< \rho\tau >$. Also, in Proposition 3.4.3, we gave a graded automorphism μ_B of B such that $\Omega^3_{B^e}(B) \cong_{\mu_B} B_1$ and $\mu_B \circ g = g \circ \mu_B$, for all $g \in G =< \rho\tau >$. Since Λ may be viewed as the orbit category of B under the action of the group G, we then get induced graded automorphisms $\eta := \bar{\eta}_B : \Lambda = B/G \longrightarrow B/G = \Lambda$ and $\mu := \bar{\mu}_B : \Lambda = B/G \longrightarrow B/G = \Lambda$. The first one is a graded Nakayama automorphism of Λ and the second one has the property that $\Omega^3_{\Lambda^e}(\Lambda)$ is isomorphic to ${}_\mu\Lambda_1$ as a bimodule.

Lemma 6.5.9. *The following assertions identify η and μ:*

1. *When n is even, η is the identity map. In this case, the automorphism μ acts as τ on vertices and crossing arrows, and maps $a \rightsquigarrow -\tau(a)$, for each noncrossing arrow $a \in Q_1$.*

2. *When n is odd, we have $\eta = \tau$. In this case, the automorphism μ fixes the vertices and the crossing arrows, and changes the sign of the noncrossing arrows.*

Proof. We start by considering the change of signs given by the original set of arrows $\bar{X}$. Since $G =< \rho\tau_B >$ it follows that ρ and τ_B induce the same automorphism $\bar{\rho} = \bar{\tau}_B : \Lambda \longrightarrow \Lambda$. Clearly, this automorphism is induced by the canonical symmetry of the quiver Q. We will denote it also by τ.

By Theorem 2.4.2, we have that $\eta_B = \nu$, where $\nu = \rho\tau_B^{1-n}$. When n is even, we have $\rho\tau_B^{1-n} = (\rho\tau_B)^{1-n} \in G$ and then $\eta = id_\Lambda$ since it acts as the identity on G-orbits. When n is odd $\rho\tau_B^{1-n} = \rho(\rho\tau_B)^{1-n}$. This automorphism acts as $\bar{\rho} = \bar{\tau}_B$ on G-orbits and, hence, we have $\eta = \tau$ in this case.

On the other hand, by Proposition 3.4.3, if $\vartheta : B \longrightarrow B$ is the automorphism which fixes the vertices and acts on arrows as $\vartheta(a) = (-1)^{s(\tau_B^{-1}(a))+s(a)}a$, and κ is the automorphism of B which fixes the vertices and changes the sign of the arrows, then $\mu_B = \kappa \circ \eta_B \circ \tau_B^{-1} \circ \vartheta$. Bearing in mind that $\tau^2 = id_\Lambda$, we easily get that μ acts as $\eta\tau^{-1} = \eta\tau$ on vertices, and maps $a \rightsquigarrow (-1)^{1+s(\tau(a))+s(a)}(\eta \circ \tau)(a)$, for each $a \in Q_1$. Here the signature map $s : Q_1 \longrightarrow \mathbb{Z}_2$ is the one induced by the set $\bar{X}$ (see Subsection 2.3.3). It is then clear that the sum $1 + s(\tau(a)) + s(a)$ is zero in $\mathbb{Z}_2$ if, and only if, the arrow a is crossing.

Finally, observe that the arguments given above also follows when considering the set $\bar{X}$ as fixed in the beginning of Subsection 6.5.2. □

We are ready to give the minimal projective resolution of Λ as a graded bimodule. To do that, we fix a basis $\mathcal{B}$ of Λ consisting of paths, and w_i will denote the only nonzero path from i to $\nu(i)$, which is an element of $\mathcal{B}$. Recall that $\mathrm{Soc}(e_i\Lambda) = Kw_i$ and we have a graded Nakayama form $(-,-) : \Lambda \times \Lambda \longrightarrow K$ of degree $2n-2$ defined as follows. It vanishes on $e_i\Lambda e_j \times e_r\Lambda e_s$, whenever either $j \neq r$ or $s \neq \nu(i)$, and the induced graded bilinear map $(-,-) : e_i\Lambda e_j \times e_j\Lambda e_{\nu(i)} \longrightarrow K$ vanishes on all pairs (a,b) of homogeneous elements such $\deg(a) + \deg(b) \neq 2n-2$. If instead $(a,b) \in e_i\Lambda e_j \times e_j\Lambda e_{\nu(i)}$ and $\deg(a)+\deg(b) = 2n-2$, then one has $(a,b)w_i = ab$. We will denote by $\mathcal{B}^*$ the right dual basis with respect to this Nakayama form. Note that if $x \in \mathcal{B}$ is any element, then the number on noncrossing arrows in any path in Q which represents x only depends on x.

In the following result we derive a minimal projective resolution of Λ as a Λ-bimodule from that of B as a B-bimodule.

Proposition 6.5.10. *The minimal projective resolution of Λ as a graded bimodule has the property that $\Omega^6_{\Lambda^e}(\Lambda)$ is isomorphic to $\Lambda[-4n]$ as a graded Λ-bimodule. Its initial part is given by:*

$$\ldots \oplus_{i\in Q_0} \Lambda e_i \otimes e_i\Lambda[-4n] \xrightarrow{\mu k} \oplus_{i\in Q_0}\Lambda e_{\eta(i)} \otimes e_i\Lambda[-2n-2] \xrightarrow{\mu R}$$
$$\oplus_{a\in Q_1}\Lambda e_{\mu(i(a))} \otimes e_{t(a)}\Lambda[-2n-1] \xrightarrow{\mu \delta} \oplus_{i\in Q_0}\Lambda e_{\mu(i)} \otimes e_i\Lambda[-2n] \xrightarrow{k}$$
$$\oplus_{i\in Q_0}\Lambda e_{\tau(i)} \otimes e_i\Lambda[-2] \xrightarrow{R} \oplus_{a\in Q_1}\Lambda e_{i(a)} \otimes e_{t(a)}\Lambda[-1] \xrightarrow{\delta} \oplus_{i\in Q_0}\Lambda e_i \otimes e_i\Lambda \xrightarrow{u} \Lambda \to 0,$$

where the maps are the only morphisms of graded Λ*-bimodules determined by the following facts:*

1. *u is the multiplication map;*
2. $\delta(e_{i(a)} \otimes e_{t(a)}) = a \otimes e_{t(a)} - e_{i(a)} \otimes a$*;*
3. $R(e_{\tau(i)} \otimes e_i) = \sum_{t(a)=i}(-1)^{s(\sigma(a)a)}[\sigma(a) \otimes e_i + e_{\tau(i)} \otimes a]$*;*
4. $k(e_{\mu(i)} \otimes e_i) = \sum_{x\in e_{\eta(i)}\mathcal{B}}(-1)^{c(x)}\tau(x) \otimes x^*$*, where $c(x)$ is the number of noncrossing arrows in x;*
5. ${}_{\mu}\delta(e_{\mu(i(a))} \otimes e_{t(a)}) = \mu(a) \otimes e_{t(a)} - e_{\mu(i(a))} \otimes a$*;*
6. ${}_{\mu}R(e_{\eta(i)} \otimes e_i) = \sum_{t(a)=i}(-1)^{s(\sigma(a)a)}[\mu(\sigma(a)) \otimes e_i + e_{\eta(i)} \otimes a]$*;*
7. ${}_{\mu}k(e_i \otimes e_i) = \sum_{x\in e_{\eta(i)}\mathcal{B}} \eta(x) \otimes x^*$.

Proof. The minimal projective resolution of Λ as a graded bimodule is obtained from that of B by replacing elements of B by G-orbits, as indicated in the proof of Corollary 3.4.5. Let us also point out that, regarding this resolution, the only difference when considering the new set $\bar{X}$ instead of the original one is that the map R (and hence the map ${}_{\mu}R$) becomes $-R$ (resp. $-{}_{\mu}R$) which is a fact that certainly can be ignored.

Alternatively, one can adapt the argument in [25][Section 4], bearing in mind the change of relations. In either way, one gets that the initial part of the minimal graded projective resolution of Λ is $P^{-2} \xrightarrow{R} P^{-1} \xrightarrow{\delta} P^0 \xrightarrow{u} \Lambda \to 0$, where the graded modules are $P^0 = \oplus_{i\in Q_0}\Lambda e_i \otimes e_i\Lambda$, $P^{-1} = \oplus_{a\in Q_1}\Lambda e_{i(a)} \otimes e_{t(a)}\Lambda[-1]$ and $P^{-2} = \oplus_{i\in Q_0}\Lambda e_{\tau(i)} \otimes e_i\Lambda[-2]$ and the maps are given as in the statement of this proposition.

From Subsection 3.4.1 we see that $\Omega^3_{\Lambda^e}(\Lambda) = \mathrm{Ker}(R)$ is generated, both as a left and as a right Λ-module, by the elements $\xi'_i = \sum_{x\in e_i\mathcal{B}}(-1)^{\deg(x)}\tau'(x) \otimes x^*$, where $\tau' : \Lambda \longrightarrow \Lambda$ is the graded automorphism acting as τ on vertices and mapping $a \rightsquigarrow (-1)^{s(\tau^{-1}(a))+s(a)}\tau(a)$, for each $a \in Q_1$. Then $\tau'(a) = \tau(a)$, when a is a noncrossing arrow, while $\tau'(a) = -\tau(a)$, when a is a crossing one. If $x \in \mathcal{B}$, then the number of crossing arrows in (any path representing) x is exactly $\deg(x) - c(x)$. It then follows that $\xi'_i = \sum_{x\in e_i\mathcal{B}}(-1)^{c(x)}\tau(x) \otimes x^*$, for each $i \in Q_0$. It was also proved in Corollary 3.4.5. that the assignment $b \rightsquigarrow b\xi'_{\tau^{-1}(t(b))}$ gives an isomorphism of graded bimodules ${}_1\Lambda_{\mu^{-1}}[-2n] \xrightarrow{\cong} \Omega^3_{\Lambda^e}(\Lambda)$. This in turn implies that the assignment $a \rightsquigarrow \mu^{-1}(a)\xi'_{\tau^{-1}\mu^{-1}(t(a))}$ gives an isomorphism $h : {}_{\mu}\Lambda_1[-2n] \xrightarrow{\cong} \Omega^3_{\Lambda^e}(\Lambda)$. But note that $\tau^{-1}\mu^{-1}(i) = \eta^{-1}(i) = \eta(i)$, for all $i \in Q_0$, because $\mu^2 = id_{\Lambda} = \eta^2$. Due to this involutive condition of μ, there is a unique morphism of graded Λ-bimodules $\oplus_{i\in Q_0}\Lambda e_{\mu(i)} \otimes e_i\Lambda[-2n] \longrightarrow {}_{\mu}\Lambda_1[-2n]$ which maps $e_{\mu(i)} \otimes e_i \rightsquigarrow e_i$. Composing this morphism with h, we get a morphism of graded Λ-bimodules $k : \oplus_{i\in Q_0}\Lambda e_{\mu(i)} \otimes e_i\Lambda[-2n] \longrightarrow \oplus_{i\in Q_0}\Lambda e_{\tau(i)} \otimes e_i\Lambda$, which maps $e_{\mu(i)} \otimes e_i \rightsquigarrow \xi'_{\eta(i)} = \sum_{x \in e_{\eta(i)}\mathcal{B}} (-1)^{c(x)}\tau(x) \otimes x^*$ and whose image is $\mathrm{Ker}(R) = \Omega^3_{\Lambda^e}(\Lambda)$.

Once the differentials δ, R and k are known, one easily obtain the differentials ${}_\mu\delta$ and ${}_\mu R$, by applying the canonical equivalence ${}_\mu(-)_1 : {}_\Lambda Gr_\Lambda \xrightarrow{\cong} {}_\Lambda Gr_\Lambda$, where ${}_\Lambda Gr_\Lambda$ is the category of graded Λ-bimodules, and taking into account that, for all $i, j \in Q_0$, we have an isomorphism $\Lambda e_{\mu(i)} \otimes e_j\Lambda \xrightarrow{\cong} {}_\mu(\Lambda e_i \otimes e_j\Lambda)_1$ of graded Λ-bimodules which maps $a \otimes b \rightsquigarrow \mu(a) \otimes b$. We can apply the same argument to obtain ${}_\mu k$ from k, but it is convenient to use the formula $k(e_{\mu(i)} \otimes e_i) = \sum_{x \in e_{\eta(i)}\mathcal{B}} (-1)^{\deg(x)}\tau'(x) \otimes x^*$ which is another form of expressing the element $\xi'_{\eta(i)}$. Indeed from the equality $\mu_B = \kappa \circ \eta_B \circ \tau_B^{-1} \circ \vartheta$ of automorphisms of B one obtains the equality $\mu \circ \tau' = \bar{\kappa} \circ \eta \circ \tau' \circ \tau' = \bar{\kappa} \circ \eta$ of automorphisms of Λ, where $\bar{\kappa}$ is the identity on vertices and changes the sign of all arrows of Q. It then follows that $(-1)^{\deg(x)}\mu\tau'(x) = (-1)^{\deg(x)}(\bar{\kappa} \circ \eta(x)) = (-1)^{\deg(x)+\deg(\eta(x))}\eta(x) = \eta(x)$. With this in mind one easily gets the desired formula ${}_\mu k(e_i \otimes e_i) = \sum_{x \in e_{\eta(i)}\mathcal{B}} \eta(x) \otimes x^*$.

□

6.5.5 The complex which calculates the Hochschild cohomology

Proposition 6.5.11. *Consider the complex $V^\cdot$ of graded K-vector spaces, concentrated in degrees ≥ 0, given as follows:*

1. *For each $i \geq 0$, the differential $V^{i+6} \longrightarrow V^{i+7}$ is obtained from the differential $V^i \longrightarrow V^{i+1}$ by applying the shift equivalence $?[4n] : K - Gr \longrightarrow K - Gr$.*

2. *The initial part of the complex is:*

$$...0 \to \oplus_{i \in Q_0} e_i \Lambda e_i \xrightarrow{\delta^*} \oplus_{a \in Q_1} e_{i(a)} \Lambda e_{t(a)}[1] \xrightarrow{R^*} \oplus_{i \in Q_0} e_{\tau(i)} \Lambda e_i[2] \xrightarrow{k^*}$$
$$\oplus_{i \in Q_0} e_{\mu(i)} \Lambda e_i[2n] \xrightarrow{{}_\mu\delta^*} \oplus_{a \in Q_1} e_{\mu(i(a))} \Lambda e_{t(a)}[2n+1] \xrightarrow{{}_\mu R^*} \oplus_{i \in Q_0} e_{\eta(i)} \Lambda e_i[2n+2] \xrightarrow{{}_\mu k^*}$$
$$\oplus_{i \in Q_0} e_i \Lambda e_i[4n] \xrightarrow{\delta^*} ...,$$

where the differentials are:

(a) If c is a cycle at i, then $\delta^*(c) = \sum_{t(a)=i} ac - \sum_{i(a)=i} ca$

(b) If $p : i(a) \to t(a)$ is a path, then
$R^*(p) = (-1)^{s(\sigma(a)a)}\sigma(a)p + (-1)^{s(a\sigma^{-1}(a))}p\sigma^{-1}(a)$

(c) When n is even, k^* is the zero map. When n is odd, $k^*(e_n) = w_n$ and k^* vanishes on $(\oplus_{i \neq n} e_{\tau(i)} \Lambda e_i) \oplus e_n J(\Lambda) e_n$

(d) If $p : \mu(i) \to i$ is a path, then ${}_\mu\delta^*(p) = \sum_{t(a)=i} \mu(a)p - \sum_{i(a)=i} pa$

(e) If $p : \mu(i(a)) \to t(a)$ is a path, then
${}_\mu R^*(p) = (-1)^{s(\sigma(a)a)}\mu(\sigma(a))p + (-1)^{s(a\sigma^{-1}(a))}p\sigma^{-1}(a)$

(f) The map ${}_\mu k^*$ satisfies the following properties:

i. When n is even, ${}_\mu k^*(e_j) = \sum_{i \in Q_0} dim(e_i \Lambda e_j) w_i$, for each $j \in Q_0$, and ${}_\mu k^*$ vanishes on $\oplus_{i \in Q_0} e_i J(\Lambda) e_i$.

ii. When n is odd, ${}_\mu k^*(e_n) = nw_n$ and ${}_\mu k^*$ vanishes on $(\oplus_{i\neq n} e_{\eta(i)}\Lambda e_i) \oplus e_n J(\Lambda) e_n$

Then $HH^n(\Lambda)$ is isomorphic, as a (length-)graded K-vector space, to the n-th cohomology space of the complex $V^\cdot$, for each integer $n \geq 0$. Moreover, $V^\cdot$ is a complex of $Z(\Lambda)$-modules.

Proof. Except for k^* and ${}_\mu k^*$, the formulas for the other differentials are easily derived by applying the contravariant functor $\mathrm{Hom}_{\Lambda^e}(-,\Lambda) : {}_\Lambda Gr_\Lambda \longrightarrow K - Gr$ to the minimal projective resolution of Λ given in Proposition 6.5.10, and by using the isomorphism of graded K-vector spaces $\mathrm{Hom}_{\Lambda^e}(\Lambda e_i \otimes e_j \Lambda[-k], \Lambda) \xrightarrow{\cong} e_i \Lambda e_j[k]$ $(f \rightsquigarrow f(e_i \otimes e_j))$, for all $i, j \in Q_0$ and $k \in \mathbb{Z}$.

Applying the same contravariant functor to k, we see that if $p : \tau(j) \to j$ is any path, then $k^*(p) = \sum_{i\in Q_0} \sum_{x\in e_{\eta(i)}\mathcal{B}e_j} (-1)^{c(x)} \tau(x) p x^*$. Whenever length$(p) > 0$ the product $\tau(x)px^*$ is zero for its length degree exceeds $2n - 2 = c_\Delta - 2$. But if length$(p) = 0$, then $\tau(j) = j$ and, hence, necessarily $j = n$. On the other hand, if $x \in e_{\eta(i)}\mathcal{B}e_n$, $p = e_n$ and the product $\tau(x)px^* = \tau(x)x^*$ is nonzero, then necessarily $i = n$, which shows that $k^*(e_n) = \sum_{x\in e_n\mathcal{B}e_n} (-1)^{c(x)} \tau(x) x^* = \sum_{x\in e_n\mathcal{B}e_n} (-1)^{c(x)} x x^* = (\sum_{x\in e_n\mathcal{B}e_n} (-1)^{c(x)}) w_n$. By Lemma 6.5.5, we know that the elements of $e_n\mathcal{B}e_n$ have even degrees $0, 2, ..., 2n - 2$. Since each element of $e_n\mathcal{B}e_n$ is a power of the cycle of length 2 at n we get that $c(x) = \frac{1}{2}\deg(x)$, for each $x \in e_n\mathcal{B}e_n$. The sum $(\sum_{x\in e_n\mathcal{B}e_n} (-1)^{c(x)})$ is then 0 or 1, depending on whether n is even or odd.

A procedure similar to that of the previous paragraph is followed for ${}_\mu k^*$. If $p : \eta(j) \to j$ is any nonzero path, then, by the usual method, we get ${}_\mu k^*(p) = \sum_{i\in Q_0} \sum_{x\in e_{\eta(i)}\mathcal{B}e_j} \eta(x) p x^*$. We clearly have ${}_\mu k^*(p) = 0$ when length$(p) > 0$. On the other hand, if length$(p) = 0$ then we necessarily have $\eta(j) = j$. When n is odd, this implies that $j = n$, while j can be arbitrary when n is even (see Proposition 6.5.10). We distinguish the two situations:

a) When n is odd, the comments above show that ${}_\mu k^*$ vanishes on $(\oplus_{i\neq n} e_{\eta(i)}\Lambda e_i) \oplus e_n J(\Lambda) e_n$. On the other hand, we have ${}_\mu k^*(e_n) = \sum_{i\in Q_0} \sum_{x\in e_{\eta(i)}\mathcal{B}e_n} \eta(x) x^*$. As for k^*, we see that if the product $\eta(x)x^*$ is nonzero, then $i = n$ and so ${}_\mu k^*(e_n) = \sum_{x\in e_n B e_n} \eta(x) x^*$. But we have $\eta(x) = \tau(x) = x$, for each $x \in e_n\mathcal{B}e_n$, and this implies that ${}_\mu k^*(e_n) = \sum_{x\in e_n\mathcal{B}e_n} x x^* = \dim(e_n\Lambda e_n) w_n = n w_n$ (see Proposition 6.5.7).

b) When n is even, we have $\eta = id_\Lambda$. Then, when $p = e_j$ is of zero length, we have:

$${}_\mu k^*(e_j) = \sum_{i\in Q_0} \sum_{x\in e_{\eta(i)}\mathcal{B}e_j} x x^* = \sum_{i\in Q_0} \sum_{x\in e_i\mathcal{B}e_j} x x^* = \sum_{i\in Q_0} \dim(e_i\Lambda e_j) w_i.$$

Finally, bearing in mind that Λ is a $\Lambda^e - Z(\Lambda)$-module, we have that, for each Λ-module M, the K-vector space $Hom_{\Lambda^e}(M, \Lambda)$ is in fact a $Z(\Lambda)$-module. Now, using the isomorphism $e_i\Lambda e_j \cong Hom_{\Lambda^e}(\Lambda e_i \otimes e_j \Lambda, \Lambda)$, it is clear that, $\forall i, j \in Q_0$, $e_i\Lambda e_j$ has a structure of $Z(\Lambda)$-module given by the multiplication in Λ. But then, it is easy to check that the differentials involved in the complex $V^\cdot$ are also morphisms of $Z(\Lambda)$-modules.

□

6.6 The Hochschild cohomology spaces

With all our tools in place, we move on to study the structure as a $Z(\Lambda)$-module of each cohomology space $HH^i(\Lambda)$. Note that each term of the complex of Proposition 6.5.11 is a graded vector subspace of Λ, when this algebra is considered with its canonical length grading. Given a natural number d and any term V^r of the mentioned complex, we will need to calculate the dimension of the homogeneous component V^r_d. The following is a first result in this direction.

Proposition 6.6.1. *Let $d \in \mathbb{N}$ be a natural number. There exists a vertex $i \in Q_0$ such that $e_i\Lambda_d e_i \neq 0$ if, and only if, d is even and $d \leq 2n-2$. In such a case the following holds:*

1. *If $d \equiv 0 \pmod 4$, then $\dim(\oplus_{i\in Q_0} e_i\Lambda_d e_i) = 2n-d-1$.*
2. *If $d \equiv 2 \pmod 4$, then $\dim(\oplus_{i\in Q_0} e_i\Lambda_d e_i) = d+1$.*

In particular, the dimension of $\oplus_{i\in Q_0} e_i\Lambda_d e_i$ is n^2+n, when n is even, and n^2, when n is odd.

Proof. Since the maximal length of a nonzero path is $2n-2$, if $e_i\Lambda_d e_i \neq 0$, for some $i \in Q_0$, then $d \leq 2n-2$. By Lemma 6.5.5, we also know that d is even. Conversely, consider an even natural number $d \leq 2n-2$. Then either $d = 4s$, with $s \leq \frac{n-1}{2}$, or $d = 4s+2$, with $s \leq \frac{n-2}{2}$. Due to Lemma 6.5.6, we know that in the first case $s \in X^+(n,n)$, while in the second case $s \in X^-(n,n)$. Then, Lemma 6.5.5 tells us that $e_n\Lambda_d e_n \neq 0$ in both cases.

In the rest of the proof we assume that $d \leq 2n-2$ is an even natural number. Suppose first that $d \equiv 0 \pmod 4$ and put $d = 4s$, with $s \in \mathbb{N}$. Given a vertex $i \geq n$, the Lemmas 6.5.6 and 6.5.5 give the following chain of double implications

$$e_i\Lambda_d e_i \neq 0 \iff s \in X^+(i,i) \iff s \leq n - \tfrac{i+1}{2} \iff i \leq 2n-2s-1.$$

By applying the symmetry τ, we conclude that $e_i\Lambda_d e_i \neq 0$ if, and only if, $2s+1 \leq i \leq 2n-2s-1$. There are exactly $2(n-2s-1)+1 = 2n-4s-1 = 2n-d-1$ vertices in this list, and that is precisely the dimension of $\oplus_{i\in Q_0} e_i\Lambda_d e_i$.

Suppose now that $d \equiv 2 \pmod 4$ and put $d = 4s+2$, with $s \in \mathbb{N}$ and $s \leq \frac{n-2}{2}$. The two mentioned lemmas yield a chain of double implications for a vertex $i \geq n$:

$$e_i\Lambda_d e_i \neq 0 \iff s \in X^-(i,i) \iff \tfrac{i-n-1}{2} \leq s \iff i \leq n+2s+1.$$

Arguing as in the other case, we get that $\dim(\oplus_{i\in Q_0} e_i\Lambda_d e_i) = 2(2s+1)+1 = d+1$.

For the global formulas, we put $\delta_r = \dim(\oplus_{d\equiv r \pmod 4} \oplus_{i\in Q_0} e_i\Lambda_d e_i)$, for $r = 0,2$. In case n is even, we have that $2n-2 \equiv 2 \pmod 4$ and then $\{d \in \mathbb{N} : d \leq 2n-2 \text{ and } d \equiv 0 \pmod 4\} = \{0,4,...,2n-4\} = \{4s : 0 \leq s \leq \frac{n}{2}-1\}$, while $\{d \in \mathbb{N} : d \leq 2n-2 \text{ and } d \equiv 2 \pmod 4\} = \{2,6,...,2n-2\} = \{4s+2 : 0 \leq s \leq \frac{n}{2}-1\}$. We then get equalities:

$$\delta_0 = \textstyle\sum_{0\leq s\leq \frac{n}{2}-1}(2n-1-4s) = (2n-1)\frac{n}{2} - 4\sum_{0\leq s\leq \frac{n}{2}-1} s = \frac{n(n+1)}{2}$$

and

$$\delta_2 = \textstyle\sum_{0\leq s\leq \frac{n}{2}-1}(4s+3) = 3\frac{n}{2} + 4\sum_{0\leq s\leq \frac{n}{2}-1} s = \frac{n(n+1)}{2}.$$

It follows that $\dim(\oplus_{i\in Q_0} e_i\Lambda e_i) = \delta_0 + \delta_2 = n^2 + n$.

In case n is odd, we have that $2n-2 \equiv 0 \pmod 4$, so that $\{d \in \mathbb{N} : d \leq 2n-2 \text{ and } d \equiv 0 \pmod 4\} = \{0, 4, ..., 2n-2\} = \{4s : 0 \leq s \leq \frac{n-1}{2}\}$, while $\{d \in \mathbb{N} : d \leq 2n-2 \text{ and } d \equiv 2 \pmod 4\} = \{2, 6, ..., 2n-4\} = \{4s+2 : 0 \leq s \leq \frac{n-3}{2}\}$. We then get:

$$\delta_0 = \sum_{0\leq s\leq \frac{n-1}{2}} (2n-1-4s) = (2n-1)\frac{n+1}{2} - 4\sum_{0\leq s\leq \frac{n-1}{2}} s = \frac{n(n+1)}{2}$$
and
$$\delta_2 = \sum_{0\leq s\leq \frac{n-3}{2}} (4s+3) = 3\frac{n-1}{2} + 4\sum_{0\leq s\leq \frac{n-3}{2}} s = \frac{n(n-1)}{2}.$$

Therefore we have $\dim(\oplus_{i\in Q_0} e_i\Lambda e_i) = \delta_0 + \delta_2 = n^2$ in this case.

□

We are ready to identify the center of Λ.

Proposition 6.6.2. *Let us consider $\Lambda = B_n$. For each $i \in Q_0$, let w_i be a nonzero element in $Soc(e_i\Lambda)$, and let us put $x = \sum_{3\leq i\leq 2n-3} c_i$, where c_i is the (unique up to relations) cycle of length 4 at i. The following assertions hold:*

1. *If n is odd then the center $Z(\Lambda)$ is the subalgebra of Λ generated by x and it is isomorphic to $K[x]/(x^{\frac{n+1}{2}})$. In particular, the dimension of $Z(\Lambda)$ is $\frac{n+1}{2}$.*
2. *If n is even, then $Z(\Lambda)$ is the subalgebra generated by x and the w_i. It is isomorphic to the commutative algebra with generators $x, y_1, ..., y_{2n-1}$ subject to the relations:*
   - *(a) $x^{\frac{n}{2}} = 0$*
   - *(b) $xy_i = 0$, for $i = 1, 2, ..., 2n-1$*
   - *(c) $y_iy_j = 0$, for all $i, j = 1, 2, ..., 2n-1$.*

   In particular, the dimension of $Z(\Lambda)$ is $\frac{5n}{2} - 1$.

Proof. When $n = 2$, we have $c_i = 0$, for each $i \in Q_0$, and hence $x = 0$. Then, the result in this case is that $Z(\Lambda) = K \oplus \mathrm{Soc}(\Lambda)$.

We assume in the sequel that $n > 2$. Let $i \in Q_0$ be such that $n \leq i \leq 2n-3$. By Lemma 6.5.5, we have $e_i\Lambda_4 e_i \neq 0$ because $s = 1 \in X^+(i,i)$ (see Lemma 6.5.6). Then, up to relations, there is a unique nonzero cycle of length 4 at i. By applying τ, the same is true for each $i \in Q_0$ such that $3 \leq i \leq n$.

We next claim that the following sets of arrows in Q coincide:

(i) The set U of arrows a such that $e_{i(a)}\Lambda_5 e_{t(a)} \neq 0$

(ii) The set V of arrows a such that $3 \leq i(a), t(a) \leq 2n-3$.

To check our claim we distinguish the noncrossing and the crossing arrows. Let first do the noncrossing ones. Note that, due to the symmetry τ, it is enough to check that the arrows $\alpha_i : i \to i+1$, with $i \geq n$, which are in U coincide with those in V. This amount to prove, for a vertex $i \geq n$, that $e_i\Lambda_5 e_{i+1} \neq 0$ if, and only if, $i \leq 2n-4$. By Lemma 6.5.5, the inequality $e_i\Lambda_5 e_{i+1} \neq 0$ holds if, and only if, $1 \in X^+(i, i+1)$. By Lemma 6.5.6, we know that $1 \in X^+(i, i+1)$ if, and only if, $1 \leq n - \frac{i+2}{2}$, which is equivalent to say that $i \leq 2n - 4$.

We next check our claim for crossing arrows. Again, by the symmetry τ, we can assume that $i(a) \geq n$. Then our task consists in proving that if $i \geq n$ then the inequality $e_i \Lambda_5 e_{2n-i+1} \neq 0$ holds if, and only if, $i \leq 2n - 3$. By Lemma 6.5.5, we have that $e_i \Lambda_5 e_{2n-i+1} \neq 0$ holds if, and only if, $1 \in Z^-(i, 2n - i + 1)$. By Lemma 6.5.6 this happens exactly when $1 \leq \frac{2n-1-i}{2}$ or, equivalently, when $i \leq 2n - 3$. So our claim is settled.

Once we know that the sets of arrows U and V coincide, we easily see that $x = \sum_{3 \leq i \leq 2n-3} c_i$ is in $Z(\Lambda)$. Note that we just need to prove that $xa = ax$, for each $a \in Q_1$. Indeed we have that $xa = 0$ if, and only if, $a \notin U = V$ if, and only if, $ax = 0$. By negation, we have that $xa \neq 0$ if, and only if, $ax \neq 0$. But $xa = c_{i(a)} a$ and $ax = a c_{t(a)}$ are then nonzero elements of $e_{i(a)} \Lambda_5 e_{t(a)}$. They are both equal in Λ since all paths of a given length between two vertices are the same modulo relations. It follows that $xa = ax$, for all $a \in Q_1$.

Note now that $x^t = \sum_{3 \leq i \leq 2n-3} c_i^t$, for each integer $t > 0$. It follows that $x^t \neq 0$ if, and only if, $4t \leq 2n - 2$. This is because c_n^t is a cycle at n of length $\leq \text{length}(w_n) = 2n - 2$. Then the subalgebra of Λ generated by x is isomorphic to $K[x]/(x^m)$, where $m = \frac{n+1}{2}$, in case n is odd, and $m = \frac{n}{2}$, in case n is even.

We denote by A the subalgebra of Λ generated by x. We shall prove that each homogeneous element in $Z(\Lambda)$ of degree $4t$ is a scalar multiple of a power of x. We first claim that $c_i^t \neq 0$ if, and only if, $2t + 1 \leq i \leq 2n - 2t - 1$. Note that, due to the symmetry τ, it is enough to prove that if $i \geq n$ then $c_i^t \neq 0$ exactly when $i \leq 2n - 2t - 1$. Using Lemmas 6.5.5 and 6.5.6, we get the following chain of double implications:

$$c_i^t \neq 0 \iff e_i \Lambda_{4t} e_i \neq 0 \iff t \in X^+(i, i) \iff t \leq n - \tfrac{i+1}{2} \iff i \leq 2n - 2t - 1$$

and the claim is settled.

We then get that any element in $Z(\Lambda)_{4t}$ can be written as a linear combination of the form $y = \sum_{2t+1 \leq i \leq 2n-2t-1} \lambda_i c_i^t$, with $\lambda_i \in K$ for all i. We have $c_{i-1}^t \neq 0 \neq c_i^t$. We claim that then $c_{i-1}^t \alpha_i \neq 0 \neq \alpha_{i-1} c_i^t$, for which we just need to prove that $e_{i-1} \Lambda_{4t+1} e_i \neq 0$. But this is a consequence of Lemma 6.5.5 and the fact that $t \in X^+(i-1, i)$ (see Lemma 6.5.6).

Since y is in the center we have an equality

$$\lambda_{i-1} c_{i-1}^t \alpha_{i-1} = y \alpha_{i-1} = \alpha_{i-1} y = \lambda_i \alpha_{i-1} c_i^t,$$

from which we get that $\lambda_{i-1} = \lambda_i$. It follows that $\lambda_n = \lambda_{n+1} = \ldots = \lambda_{2n-2t-1}$ and, by applying the symmetry τ, also that $\lambda_{2t+1} = \ldots = \lambda_{n-1} = \lambda_n$. We then get that $y = \lambda \sum_{2t+1 \leq i \leq 2n-2t-1} c_i^t = \lambda x^t$, for some $\lambda \in K$.

Since the degrees of homogeneous elements in $\oplus_{i \in Q_0} e_i \Lambda e_i$ are all even, in order to finish de proof, it will be enough to check that if $4t + 2 < 2n - 2$ then $Z(\Lambda)_{4t+2} = 0$. Indeed, if this is proved then the case in which n is odd will be settled, because $2n - 2 \in 4\mathbb{Z}$ and then we will have $Z(\Lambda) = \oplus_{t \geq 0} Z(\Lambda)_{4t}$. By the previous paragraph, we conclude that $Z(\Lambda)$ is the subalgebra of Λ generated by x. On the other hand, if n is even, then $\oplus_{i \in Q_0} e_i \Lambda_{2n-2} e_i = \text{Soc}(\Lambda) \subseteq Z(\Lambda)$, thus showing that $Z(\Lambda) = \text{Soc}(\Lambda) \oplus (\oplus_{t \geq 0} Z(\Lambda)_{4t})$. From the previous paragraphs assertion 2 of the theorem follows easily by taking $y_i = w_i$, for all $i \in Q_0$.

Let us finally prove that if $4t+2 < 2n-2$ then $Z(\Lambda)_{4t+2} = 0$. Given a vertex $i \geq n$, by Lemma 6.5.5 and Lemma 6.5.6, we have the following chain of double implications:

$$e_i\Lambda_{4t+2}e_i \neq 0 \Longleftrightarrow t \in X^-(i,i) \Longleftrightarrow \frac{i-n-1}{2} \leq t \leq \frac{n-2}{2} \Longleftrightarrow i \leq n+2t+1.$$

The last double implication is due to the fact that we are assuming that $4t+2 < 2n-2$, i.e., that $t < \frac{n-2}{2}$. By using the symmetry τ, if $i \leq n$ then we get that $e_i\Lambda_{4t+2}e_i \neq 0$ if, and only if, $i \geq n-2t-1$. So we get, for any vertex $i \in Q_0$:

$$e_i\Lambda_{4t+2}e_i \neq 0 \Longleftrightarrow n-2t-1 \leq i \leq n+2t+1.$$

For any such vertex i, we fix a nonzero cycle z_i at i of length $4t+2$. Then any element of $Z(\Lambda)_{4t+2}$ will be a linear combination of the form $z = \sum_{n-2t-1 \leq i \leq n+2t+1} \lambda_i z_i$, with the λ_i in K. Suppose that there is a vertex $n \leq j \leq n+2t+1$ such that $\lambda_j \neq 0$ and choose this j to be maximal. Note that $n+2t+1 < 2n-1$ because we are assuming $4t+2 < 2n-2$. This implies that the arrow $\alpha_j : j \to j+1$ exists and we have $\lambda_j z_j \alpha_j = z\alpha_j = \alpha_j z = 0$. But the fact that $4t+2 < 2n-2$ implies that $4t+2 \leq 2n-4$ or, equivalently, that $t \leq \frac{n-3}{2}$. It follows that $t \in X^-(j, j+1)$ (see Lemma 6.5.6) and, by Lemma 6.5.5, we get that $e_j\Lambda_{4t+3}e_{j+1} \neq 0$. As a consequence, we have that $z_j\alpha_j \neq 0$ and, hence, that $\lambda_j = 0$. This contradicts our choice of j. □

We now continue with the calculation of the dimensions of the spaces V^r of Proposition 6.5.11.

Proposition 6.6.3. *Let $d \in \mathbb{N}$ be a natural number. There is an arrow $a \in Q_1$ such that $e_{i(a)}\Lambda_d e_{t(a)} \neq 0$ if, and only if, $d < 2n-2$ and d is odd. In such a case, the following holds:*

1. *If $d \equiv 1 \pmod 4$ then $dim(\oplus_{a\in Q_1} e_{i(a)}\Lambda_d e_{t(a)}) = 4n-2d-2$;*
2. *If $d \equiv 3 \pmod 4$ then $dim(\oplus_{a\in Q_1} e_{i(a)}\Lambda_d e_{t(a)}) = 2d+2$.*

In particular, the dimension of $\oplus_{a\in Q_1} e_{i(a)}\Lambda e_{t(a)}$ is equal to $2(n^2-n)$, when n is even, and to $2(n^2-1)$, when n is odd.

Proof. We first prove the last assertion, assuming that the formulas for the dimensions of the space $\oplus_{a\in Q_1} e_{i(a)}\Lambda_d e_{t(a)}$ are correct. For simplification, call a natural number d efficient when $e_{i(a)}\Lambda_d e_{t(a)} \neq 0$, for some $a \in Q_1$. Suppose first that n is even. Then we have $2n-3 \equiv 1 \pmod 4$, so that $\{1, 5, ..., 2n-3\}$ is the set of efficient natural numbers d such that $d \equiv 1 \pmod 4$. Then we have an equality, putting in the last part $d = 4s+1$:

$$d_1 := \sum_{d\equiv 1 \pmod 4, a\in Q_1} \dim(e_{i(a)}\Lambda_d e_{t(a)}) = \sum_{d\equiv 1 \pmod 4, 1\leq d\leq 2n-3}(4n-2d-2) = \sum_{0\leq s\leq \frac{n}{2}-1}(4n-8s-4) = n^2.$$

On the other hand, if d is efficient and $d \equiv 3 \pmod 4$ then $d \leq 2n-5$, and so $\{3, 7, ..., 2n-5\}$ is the set of these efficient natural numbers. Then we have an equality:

$$d_3 := \sum_{d\equiv 3 \pmod 4, a\in Q_1} \dim(e_{i(a)}\Lambda_d e_{t(a)}) = \sum_{d\equiv 3 \pmod 4, 3\leq d\leq 2n-5}(2d+2) = 8+16+...+(4n-8) = 8[1+2+...+(\tfrac{n}{2}-1)] = n^2-2n.$$

It follows that $\dim(\oplus_{a\in Q_1} e_{i(a)}\Lambda e_{t(a)}) = 2(n^2-n)$ in case n is even.

Suppose next that n is odd. Then $2n-3 \equiv 3 \pmod 4$. It follows that a natural number d such that $d \equiv 1 \pmod 4$ is efficient if, and only if, it is in the set $\{1,5,...,2n-5\}$. The formula for d_1 in this case is:

$$d_1 := \sum_{d\equiv 1 \pmod 4, a\in Q_1} \dim(e_{i(a)}\Lambda_d e_{t(a)}) = \sum_{d\equiv 1 \pmod 4, 1\le d\le 2n-5}(4n-2d-2) = \sum_{0\le s\le \frac{n-3}{2}}(4n-8s-4) = n^2-1.$$

On the other hand, $\{3,7,...,2n-3\}$ is the set of efficient natural numbers d such that $d \equiv 3 \pmod 4$. The formula for d_3 is in this case:

$$d_3 := \sum_{d\equiv 3 \pmod 4, a\in Q_1} \dim(e_{i(a)}\Lambda_d e_{t(a)}) = \sum_{d\equiv 3 \pmod 4, 3\le d\le 2n-3}(2d+2) = 8+16+...+(4n-4) = 8[1+2+...+(\tfrac{n-1}{2})] = n^2-1.$$

It follows that $\dim(\oplus_{a\in Q_1} e_{i(a)}\Lambda e_{t(a)}) = 2(n^2-1)$ in case n is odd.

We now pass to prove the initial part of the proposition. Due to the symmetry τ, one readily sees that a natural number d is efficient if, and only if, $d \le 2n-2$ and $e_i\Lambda_d e_{i+1} \neq 0$, for some vertex $i \ge n$, or $e_i\Lambda_d e_{2n-i+1} \neq 0$, for some $i > n$. But due to Lemma 6.5.5, we know that in such a case d is odd and, as a consequence, $d < 2n-2$. From the same lemma we get that $e_n\Lambda_d e_{n+1} \neq 0$ if, and only if, either $d \in 4X^+(n,n+1)+1$ or $d \in 4X^-(n,n+1)+3$. Looking at Lemma 6.5.6, we see that:

1. $4X^+(n,n+1)+1 = \{1,5,...,2n-3\}$, when n is even, and $4X^+(n,n+1)+1 = \{1,5,...,2n-5\}$, when n is odd;
2. $4X^-(n,n+1)+3 = \{3,7,...,2n-5\}$, when n is even, and $4X^-(n,n+1)+3 = \{3,7,...,2n-3\}$, when n is odd.

We then see that each natural number $d < 2n-2$ is efficient, which proves the first assertion of the proposition.

Let us now fix an efficient natural number of the form $d = 4s+1$. Note that we have that $\dim(\oplus_{a\in Q_1} e_{i(a)}\Lambda_d e_{t(a)})$ is equal to the number of elements in the set $\{a \in Q_1 : e_{i(a)}\Lambda_d e_{t(a)} \neq 0\}$. Using the symmetry τ and Lemma 6.5.5, we then have:

$$\dim(\oplus_{a\in Q_1} e_{i(a)}\Lambda_d e_{t(a)}) = 2|A_s| + 2|B_s|,$$

where $A_s = \{i \ge n : e_i\Lambda_{4s+1}e_{i+1} \neq 0\} = \{i \ge n : s \in X^+(i,i+1)\}$ and $B_s = \{i > n : e_i\Lambda_{4s+1}e_{2n-i+1} \neq 0\} = \{i > n : s \in Z^-(i,2n-i+1)\}$. Using Lemma 6.5.6, we see that, for $i \ge n$, one has that $i \in A_s$ if, and only if, $i \le 2n-2s-2$. Similarly, for $i > n$, one has that $i \in B_s$ if, and only if, $i \le 2n-2s-1$. It follows that $|A_s| = |B_s| = n-2s-1$, so that

$$\dim(\oplus_{a\in Q_1} e_{i(a)}\Lambda_d e_{t(a)}) = 4(n-2s-1) = 4n-8s-4 = 4n-2d-2.$$

Suppose now that d is an efficient natural number of the form $d = 4s+3$. An argument analogous to that of the previous paragraph shows that if $i \ge n$ then $e_i\Lambda_{4s+3}e_{i+1} \neq 0$ if, and only if, $s \in X^-(i,i+1)$, while if $i > n$ then $e_i\Lambda_{4s+3}e_{2n-i+1} \neq 0$ if, and only if, $s+1 \in Z^+(i,2n-i+1)$. Putting $C_s = \{i \ge n : s \in X^-(i,i+1)\}$ and $D_s = \{i > n : s+1 \in Z^+(i,2n-i+1)\}$ and applying Lemma 6.5.6, we readily see that $C_s = \{n, n+1, ..., n+2s+1\}$ and $D_s = \{n+1, n+2, ..., n+2s+2\}$. It follows that

$$\dim(\oplus_{a\in Q_1} e_{i(a)}\Lambda_d e_{t(a)}) = \dim(\oplus_{a\in Q_1} e_{i(a)}\Lambda_{4s+3} e_{t(a)}) = 2(|C_s| + |D_s|) = 2[(2s+2)+(2s+2)] = 2(4s+4) = 2d+2.$$

□

Proposition 6.6.4. *Let $d \in \mathbb{N}$ be a natural number. There is a vertex $i \in Q_0$ such that $e_{\tau(i)}\Lambda_d e_i \neq 0$ if, and only if, $d \leq 2n-2$ and d is even. In such a case, the following holds:*

1. *If $d \equiv 0 \pmod 4$ then $dim(\oplus_{i\in Q_0} e_{\tau(i)}\Lambda_d e_i) = d+1$;*

2. *If $d \equiv 2 \pmod 4$ then $dim(\oplus_{i\in Q_0} e_{\tau(i)}\Lambda_d e_i) = 2n-d-1$.*

In particular, the dimension of $\oplus_{i\in Q_0} e_{\tau(i)}\Lambda e_i$ is equal to n^2-n, when n is even, and to n^2, when n is odd.

Proof. The proof goes along the lines of that of the foregoing proposition. We readily see that $e_{\tau(n)}\Lambda_d e_n = e_n\Lambda_d e_n \neq 0$, for each even number $0 \leq d \leq 2n-2$. Bearing in mind that $\tau(i) = 2n-i \equiv i \pmod 2$, the first assertion of the proposition follows now directly from Lemma 6.5.5.

Let us fix an even natural number $0 \leq d \leq 2n-2$ in the rest of the proof. If $i \geq n$ and $d = 4s$ then, by Lemma 6.5.5 again, we know that $e_i\Lambda_d e_{\tau^{-1}(i)} = e_i\Lambda_d e_{2n-i} \neq 0$ if, and only if, $s \in Z^+(i, 2n-i)$. Then we have:

$$\{i \geq n : e_i\Lambda_d e_{\tau^{-1}(i)} \neq 0\} = \{n, n+1, ..., n+2s\},$$

which implies, by applying the symmetry τ, that $\{i \in Q_0 : e_i\Lambda_d e_{\tau^{-1}(i)} \neq 0\} = \{n-2s, ..., n-1, n, n+1, ..., n+2s\}$. It follows that

$$\dim(\oplus_{i\in Q_0} e_{\tau(i)}\Lambda_d e_i) = 4s+1 = d+1.$$

On the other hand, if $i \geq n$ and $d = 4s+2$ then, by Lemma 6.5.5, we get that $e_i\Lambda_d e_{2n-i} \neq 0$ if, and only if, $s \in Z^-(i, 2n-i)$. Then we have:

$$\{i \geq n : e_i\Lambda_d e_{\tau^{-1}(i)} \neq 0\} = \{n, n+1, ..., 2n-2s-2\}.$$

An argument similar to the one in the previous paragraph shows that then

$$\dim(\oplus_{i\in Q_0} e_{\tau(i)}\Lambda_d e_i) = 2(n-2s-2)+1 = 2n-d-1.$$

When n is odd, using Remark 6.5.4 and Lemma 6.5.9, we get that $D(e_{\tau(i)}\Lambda_d e_i) \cong e_i\Lambda_{2n-2-d} e_i$. Then the formula for $\dim(\oplus_{i\in Q_0})e_{\tau(i)}\Lambda e_i)$ follows from Proposition 6.6.1 in this case. Suppose that n is even in the rest of the proof. We have $2n-2 \equiv 2 \pmod 4$ so that we have the following equalities of sets:

$$\{d \in \mathbb{N} : e_{\tau(i)}\Lambda_d e_i \neq 0, \text{ for some } i \in Q_0 \text{ and } d \equiv 0 \pmod 4\} = \{0, 4, ..., 2n-4\} = \{d \in \mathbb{N} : d = 4s, \text{ with } 0 \leq s \leq \tfrac{n}{2}-1\}$$

and

$$\{d \in \mathbb{N} : e_{\tau(i)}\Lambda_d e_i \neq 0, \text{ for some } i \in Q_0 \text{ and } d \equiv 2 \pmod 4\} = \{2, 6, ..., 2n-2\}.$$

It follows that

$$\dim(\oplus_{i\in Q_0, d\equiv 0 \pmod 4} e_{\tau(i)}\Lambda_d e_i) = \sum_{0\leq s\leq \frac{n}{2}-1}(4s+1) = \frac{n^2-n}{2}$$

and

$$\dim(\oplus_{i\in Q_0, d\equiv 2 \pmod 4} e_{\tau(i)}\Lambda_d e_i) = (2n-3)+(2n-7)+...+5+1 = \sum_{0\leq t\leq \frac{n}{2}-1}(4t+1) = \frac{n^2-n}{2}.$$

Therefore, when n is even, we have $\dim(\oplus_{i\in Q_0} e_{\tau(i)}\Lambda e_i) = n^2 - n$.

□

Remark 6.6.5. The global formulas given in Propositions 6.6.3 and 6.6.4, that is, the formulas for $\dim(\oplus_{a\in Q_1} e_{i(a)}\Lambda e_{t(a)})$ and $\dim(\oplus_{i\in Q_0} e_{\tau(i)}\Lambda e_i)$ can be alternatively obtained from the Cartan matrix, which has been identified in Proposition 6.5.7.

The dimension of the last space of codimensions in $V^{\cdot}$ that we need to calculate is the following.

Proposition 6.6.6. *Let n be even and let d be an odd natural number such that $0 < d < 2n-2$. The following assertions hold:*

1. *If $d \equiv 1 \pmod 4$ then dim$(\oplus_{a\in Q_1} e_{\tau(i(a))}\Lambda_d e_{t(a)}) = 2d+2$*
2. *If $d \equiv 3 \pmod 4$ then dim$(\oplus_{a\in Q_1} e_{\tau(i(a))}\Lambda_d e_{t(a)}) = 4n-2d-2$.*

In particular, the dimension of $\oplus_{a\in Q_1} e_{\tau(i(a))}\Lambda e_{t(a)}$ is $2(n^2-n)$.

Proof. Throughout the proof, fix an odd natural number d such that $0 < d < 2n-2$. If $i < n$ and $a = \alpha_i$, then $\tau(i(a)) = \tau(i+1) = 2n-i-1$ while $t(a) = i$. But, using Remark 6.5.4, we get that $e_{2n-i-1}\Lambda_d e_i \neq 0 \iff e_i\Lambda_{2n-d-2}e_{2n-i-1} \neq 0$.

Note that $i = i(\beta_{2n-i-1})$ and $2n-i-1 = t(\beta_{2n-i-1})$ and that $\beta_{2n-i-1} = \sigma^{-1}(\alpha_i)$. Let us denote by Q_1^+ and Q_1^- the subsets of Q_1 consisting of the non-crossing arrows and of the crossing arrows, respectively. The previous comments together with an application of the symmetry τ shows that the bijection $\sigma^{-1} : Q_1^+ \longrightarrow Q_1^-$ has the property that $e_{\tau(i(a))}\Lambda_d e_{t(a)} \neq 0$ if, and only if, $e_{i(\sigma^{-1}(a))}\Lambda_{2n-2-d}e_{t(\sigma^{-1}(a))} \neq 0$.

On the other hand, if $i \geq n$ and $a = \beta_i$ then $\tau(i(a)) = i+1 = t(\alpha_i) = t(\sigma^{-1}(a))$ and $t(a) = i = i(\sigma^{-1}(a))$. An argument as in the previous paragraph shows that also the bijection $\sigma^{-1} : Q_1^- \longrightarrow Q_1^+$ has the property that $e_{\tau(i(a))}\Lambda_d e_{t(a)} \neq 0$ if, and only if, $e_{i(\sigma^{-1}(a))}\Lambda_{2n-2-d}e_{t(\sigma^{-1}(a))} \neq 0$.

Frow the two last paragraphs and Proposition 6.6.3 the result follows immediately. □

We are ready to calculate the dimension of the kernel and image of R^* in each degree.

Proposition 6.6.7. *Let d be an odd natural number such that $d \leq 2n-3$. The following assertions hold:*

a) If $f : \oplus_{a\in Q_1} e_{i(a)}\Lambda_d e_{t(a)} \longrightarrow \oplus_{i\in Q_0} e_{\tau(i)}\Lambda_{d+1}e_i$ is either the restriction of R^* or, when n is odd, the restriction of ${}_{\mu}R^*$, then f is surjective.

b) If n is even, $d \equiv 1 \pmod 4$ and $h : \oplus_{a\in Q_1} e_{\tau(i(a))}\Lambda_d e_{t(a)} \longrightarrow \oplus_{i\in Q_0} e_i\Lambda_{d+1}e_i$ is the restriction of ${}_{\mu}R^*$, then Im(h) has codimension 1 in its codomain.

Moreover, the following formulae hold:

1. *If* $d \equiv 1 \pmod 4$*, then dim*$(Ker(f)) = 2n - d$

2. *If* $d \equiv 3 \pmod 4$*, then dim*$(Ker(f)) = d$

3. *If* n *is even and* $d \equiv 1 \pmod 4$*, then dim*$(Ker(h)) = d + 1$.

In particular, we have $Im(R^*) = e_n J(\Lambda) e_n \oplus (\oplus_{i \neq n} e_{\tau(i)} \Lambda e_i)$ *and, when* n *is odd, also* $Im({}_\mu R^*) = e_n J(\Lambda) e_n \oplus (\oplus_{i \neq n} e_{\tau(i)} \Lambda e_i)$.

Proof. If f is surjective then the two formulae for the dimension of $\mathrm{Ker}(f)$ follow from Propositions 6.6.3 and 6.6.4. On the other hand, we have that $e_n J(\Lambda) e_n \oplus (\oplus_{i \neq n} e_{\tau(i)} \Lambda e_i) = \oplus_{1 \leq d \leq 2n-3,\ d \text{ odd}} (\oplus_{i \in Q_0} e_{\tau(i)} \Lambda_{d+1} e_i)$. From the surjectivity of f, we get that $e_n J(\Lambda) e_n \oplus (\oplus_{i \neq n} e_{\tau(i)} \Lambda e_i) \subseteq \mathrm{Im}(R^*)$ and, when n is odd, that $e_n J(\Lambda) e_n \oplus (\oplus_{i \neq n} e_{\tau(i)} \Lambda e_i) \subseteq \mathrm{Im}({}_\mu R^*)$. The converse inclusions are clear. On the other hand, if $\mathrm{Im}(h)$ has codimension 1 in its codomain, then the equality $\dim(\mathrm{Ker}(h)) = d+1$ follows from Propositions 6.6.1 and 6.6.6.

We next prove that f is surjective. Suppose first that $d \equiv 1 \pmod 4$ and put $d = 4s+1$, with $s \in \mathbb{N}$. From the proof of Proposition 6.6.3 we get that $e_{i(a)} \Lambda_d e_{t(a)} \neq 0$ if, and only if, $a = \alpha_i$ or $a = \beta_i$, for some index i such that $2s + 1 \leq i \leq 2n - 2s - 2$. On the other hand, from the proof of Proposition 6.6.4 we get that $e_{\tau(i)} \Lambda_{d+1} e_i \neq 0$ if, and only if $2s + 2 \leq i \leq 2n - 2s - 2$.

When $d \equiv 3 \pmod 4$, and hence $d = 4s + 3$, we get corresponding conclusions to those in the last paragraph. Concretely, from the proof of Proposition 6.6.3 we get that $e_{i(a)} \Lambda_d e_{t(a)} \neq 0$ if, and only if, $a = \alpha_i$ or $a = \beta_i$, for some index i such that $n - 2s - 2 \leq i \leq n + 2s + 1$. Bearing in mind that $d + 1 = 4(s + 1)$, we get from the proof of 6.6.4 that $e_{\tau(i)} \Lambda_{d+1} e_i \neq 0$ if, and only if, $n - 2s - 2 \leq i \leq n + 2s + 2$.

We fix $d < 2n - 2$ odd in the sequel. Whenever $e_{i(a)} \Lambda_d e_{t(a)} \neq 0$, we fix a path p_a of length d from $i(a)$ to $t(a)$, so that $e_{i(a)} \Lambda_d e_{t(a)} = K p_a$. Analogously, if $e_{\tau(i)} \Lambda_{d+1} e_i \neq 0$ we fix a path q_i of length $d+1$ from $\tau(i)$ to i, so that $e_{\tau(i)} \Lambda_{d+1} e_i = K q_i$. We now look at the action of f on the p_a. Note that if $a \in Q_1$ then $(-1)^{s(\sigma(a)a)}$ and $(-1)^{s(a\sigma^{-1}(a))}$ are equal, when a is crossing, and are opposite to each other when a is noncrossing. Since changing the sign of a column in a matrix does not alter the rank of the matrix we can assume, without loss of generality, that:

1. When f is the restriction of R^*: i) $f(p_a) = \sigma(a) p_a + p_a \sigma^{-1}(a) = q_{t(a)} + q_{\tau(i(a))}$, when a is crossing; ii) $f(p_a) = \sigma(a) p_a - p_a \sigma^{-1}(a) = q_{t(a)} - q_{\tau(i(a))}$, when a is noncrossing.

2. When n is odd and f is the restriction of ${}_\mu R^*$, then $f(a) = \sigma(a) p_a - p_a \sigma^{-1}(a) = q_{t(a)} - q_{\tau(i(a))}$, for all $a \in Q_1$.

Suppose first that $d = 4s + 1 \equiv 1 \pmod 4$. For each $j = 2s + 1, ..., 2n - 2s - 2$, the induced map $\tilde{f} : \oplus_{2s+1 \leq i \leq 2n-2s-2} K p_{\beta_i} \longrightarrow \oplus_{2s+2 \leq i \leq 2n-2s-2} K q_i = e_{\tau(i)} \Lambda e_i$ takes $p_{\beta_j} \rightsquigarrow q_j + q_{j+1}$, when $\tilde{f}$ is the restriction of R^*, and, with an appropriate change of signs if necessary, it maps $p_{\beta_j} \rightsquigarrow q_j - q_{j+1}$, when $\tilde{f}$ is the restriction of ${}_\mu R^*$. Here we adopt the convention that $q_{2s+1} = 0 = q_{2n-2s-1}$. The matrix of $\tilde{f}$ with respect to the obvious bases of its domain and codomain is then of size $m \times (m + 1)$, where $m = 2(2n - 2s - 2) + 1$.

The submatrix obtained from it by deleting the first column is lower triangular with 1 in all its diagonal entries. Therefore $\tilde{f}$, and hence f, is surjective in this case.

Suppose now that $d = 4s+3 \equiv 3 \pmod 4$. We consider $V = (\oplus_{n-2s-2\leq i\leq n+2s+1} Kp_{\beta_i}) \oplus Kp_{\alpha_{n-2s-2}}$ and denote by $g : V \longrightarrow \oplus_{n-2s-2\leq i\leq n+2s+2} Kq_i = e_{\tau(i)}\Lambda_{d+1}e_i$ the restriction of f. The matrix of g, denoted (λ_{ij}) in the sequel, is now of size $r\times r$, where $r = 2(2s+2)+1$. Suppose that $1 \leq j < r$ and let us look at its j-th column. When f is the restriction of R^*, we have $\lambda_{jj} = \lambda_{j+1,j} = 1$ and $\lambda_{ij} = 0$, for $i \neq j, j+1$. When f is the restriction of ${}_\mu R^*$, after changing the sign of the column if necessary, we have $\lambda_{jj} = -\lambda_{j+1,j} = 1$ and $\lambda_{ij} = 0$, for $i \neq j, j+1$. Finally, both when f is the restriction of R^* and when it is the restriction of ${}_\mu R^*$, we have that $f(p_{\alpha_{n-2s-2}}) = q_{n-2s-2} - q_{n+2s+1}$, so that the last column has $\lambda_{1r} = 1$, $\lambda_{r-1,r} = -1$ and $\lambda_{ij} = 0$, for all $i \neq 1, r-1$. Direct calculation using the fact that r is odd, shows that the determinant of this matrix is 2. Therefore g, and hence, f is also surjective in this case.

It remains to check that $\mathrm{Im}(h)$ has codimension 1 in its codomain. Put $d = 4s+1$. From the proof of Proposition 6.6.6, for any arrow a, we have that $e_{\tau(i(a))}\Lambda_d e_{t(a)} \neq 0$ if, and only if, $e_{i(\sigma^{-1}(a))}\Lambda_{2n-2-d}e_{t(\sigma^{-1}(a))} \neq 0$. Putting $t := \frac{n}{2} - s - 1$ and looking at the proof of Proposition 6.6.3, we then get:

$$e_{\tau(i(a))}\Lambda_d e_{t(a)} \neq 0 \Longleftrightarrow e_{i(\sigma^{-1}(a))}\Lambda_{4t+1}e_{t(\sigma^{-1}(a))} \neq 0 \Longleftrightarrow \sigma^{-1}(a) \in \{\alpha_i, \beta_i : 2t+1 \leq i \leq 2n-2t-2\} \Longleftrightarrow a \in \{\alpha_i, \beta_i : 2t+1 \leq i \leq 2n-2t-2\} \Longleftrightarrow a \in \{\alpha_i, \beta_i : n-2s-2 \leq i \leq n+2s+1\}.$$

On the other hand, by the proof of Proposition 6.6.1, we know that $e_i\Lambda_{d+1}e_i \neq 0$ if, and only if, $n-2s-1 \leq i \leq n+2s+1$.

For any arrow $a \in \{\alpha_i, \beta_i : n-2s-2 \leq i \leq n+2s+1\}$, denote by u_a a path of length d from $\tau(i(a))$ to $t(a)$, and, for any vertex $n-2s-1 \leq i \leq n+2s+1$, denote by c_i a cycle of length $d+1$ at i. We then have that $V := \oplus_{a\in Q_1} e_{\tau(i(a))}\Lambda_d e_{t(a)} = \oplus_{n-2s-2\leq i\leq n+2s+1}(Ku_{\alpha_i} \oplus Ku_{\beta_i})$ and $W := \oplus_{i\in Q_0} e_i\Lambda_{d+1}e_i = \oplus_{n-2s-1\leq i\leq n+2s+1} Kc_i$ are the domain and codomain of h, respectively.

Arguing as in the case of f, we can assume without loss of generality that $h(u_a) = \tau(\sigma(a))u_a - u_a\sigma^{-1}(a) = c_{t(a)} - c_{\tau(i(a))}$. This gives:

1. $h(u_{\alpha_i}) = c_{i+1} - c_{2n-i}$, when $i \geq n$, and $h(u_{\alpha_i}) = c_i - c_{2n-i-1}$, when $i < n$;
2. $h(u_{\beta_i}) = c_i - c_{i+1}$, when $i \geq n$, and $h(u_{\beta_i}) = c_{i+1} - c_i$, when $i < n$.

It follows from this that all the cycles c_i $(n-2s-1 \leq i \leq n+2s+1)$ are equal modulo $\mathrm{Im}(h)$, but none of them is in $\mathrm{Im}(h)$. Hence $\mathrm{Im}(h)$ has codimension 1 in W.

□

Lemma 6.6.8. *Let $d < 2n-2$ be an even natural number and let us consider that $f : \oplus_{i\in Q_0} e_{\mu(i)}\Lambda_d e_i \longrightarrow \oplus_{a\in Q_1} e_{\mu(i(a))}\Lambda_{d+1}e_{t(a)}$ is the restriction of ${}_\mu\delta^*$. Then f is injective and $Ker({}_\mu\delta^*) = Soc(\Lambda) \cap (\oplus_{i\in Q_0} e_{\mu(i)}\Lambda e_i) = Kw_n$.*

Proof. Note first that if n is even, then $\mu(i) = \tau(i)$, for all $i \in Q_0$, and then $\mathrm{Soc}(\Lambda) \cap (\oplus_{i\in Q_0} e_{\mu(i)}\Lambda e_i) = \mathrm{Soc}(\Lambda) \cap (\oplus_{i\in Q_0} e_{\tau(i)}\Lambda e_i) = Kw_n$. On the other hand, if n is odd then $\mu(i) = i$, for all $i \in Q_0$, and then $\mathrm{Soc}(\Lambda)\cap(\oplus_{i\in Q_0} e_{\mu(i)}\Lambda e_i) = \mathrm{Soc}(\Lambda)\cap(\oplus_{i\in Q_0} e_i\Lambda e_i) = Kw_n$.

We have an exact sequence of Λ-bimodules

$$\oplus_{a\in Q_1}\Lambda e_{\mu(i(a))}\otimes e_{t(a)}\Lambda \xrightarrow{\mu\delta} \oplus_{i\in Q_0}\Lambda e_{\mu(i)}\otimes e_i\Lambda \longrightarrow {}_\mu\Lambda_1\to 0.$$

By applying the contravariant functor $\mathrm{Hom}_{\Lambda^e}(-,\Lambda)$, we get that $\mathrm{Ker}({}_\mu\delta^*)\cong \mathrm{Hom}_{\Lambda^e}({}_\mu\Lambda_1,\Lambda)$. But this is a graded vector space isomorphic to $W := \{x\in \oplus_{i\in Q_0}e_{\mu(i)}\Lambda e_i : \mu^{-1}(a)x = xa \text{ for all } a\in Q_1\} = \{x\in \oplus_{i\in Q_0}e_{\mu(i)}\Lambda e_i : \mu(a)x = xa \text{ for all } a\in Q_1\}$. Let then $x\in \oplus_{i\in Q_0}e_{\mu(i)}\Lambda_d e_i$, where $d<2n-2$ is an even natural number. We have that $x=\sum_{i\in Q_0}\lambda_i c_i$, where, for all $i\in Q_0$, $\lambda_i\in K$ and c_i is either the (possibly zero) cycle at i of length d when n is odd, or the (possibly zero) path $\tau(i)\longrightarrow\cdots\longrightarrow i$ of length d when n is even. We get that $xa=\lambda_{i(a)}c_{i(a)}a=\lambda_{i(a)}p_a$, where p_a is either the (possibly zero) path of length $d+1$ from $i(a)$ to $t(a)$ when n is odd, or the (possibly zero) path of length $d+1$ from $\tau(i(a))$ to $t(a)$ when n is even. By Lemma 6.5.9, we also have $\mu(a)x=\lambda_{t(a)}p_a$, when a is a crossing arrow, and $\mu(a)x=-\lambda_{t(a)}p_a$, when a is noncrossing. It follows that if $p_a\neq 0$ or, equivalently, if $e_{\mu(i(a))}\Lambda_{d+1}e_{t(a)}\neq 0$, then $\lambda_{i(a)}=\lambda_{t(a)}$, when a is crossing, and $\lambda_{i(a)}=-\lambda_{t(a)}$, when a is noncrossing.

Note that, regardless of whether n is either even or odd, for $a\in\{\alpha_{n-1},\beta_n\}$, we know that $p_a\neq 0$. It follows that $\lambda_n=\lambda_{i(\alpha_{n-1})}=-\lambda_{n-1}=-\lambda_{t(\alpha_{n-1})}$ while $\lambda_n=\lambda_{t(\beta_n)}=\lambda_{n-1}=\lambda_{i(\beta_n)}$. It then follows that $\lambda_n=\lambda_{n-1}=0$ and, by an easy induction argument, one gets that $\lambda_i=0$ for all i such that $c_i\neq 0$. Hence, we get that $\mathrm{Ker}(f)=0$.

Due to Propositions 6.6.1 and 6.6.4, the previous paragraph shows that $\mathrm{Ker}({}_\mu\delta^*)$ lives in (length) degree $2n-2$. But we have $\oplus_{i\in Q_0}e_{\mu(i)}\Lambda_{2n-2}e_i=Kw_n$ and ${}_\mu\delta^*(w_n)=0$. □

We are now able to give the structure of the Hochschild cohomology spaces as modules over $Z(\Lambda)=HH^0(\Lambda)$. Recall that, due to the 6-periodicity of Λ, we have that $HH^{6k}(\Lambda)\cong \underline{HH}^0(\Lambda)=\frac{HH^0(\Lambda)}{\mathrm{Im}({}_\mu k^*)}$, for all $k>0$, and $HH^{6k+i}(\Lambda)\cong HH^i(\Lambda)$, for all $i>0$ and $k\geq 0$.

Remark 6.6.9. We will adopt the following convention. Suppose that $d^i: V^i\longrightarrow V^{i+1}$ is the differential of the complex $V^\cdot$ of Proposition 6.5.11. Considering Λ with its usual grading and looking at V^i as a graded subspace of Λ, for each $d\in\mathbb{N}$, we denote by $\mathrm{Ker}(d^i)_d$ the kernel of the restriction of d^i to V^i_d and put $HH^i(\Lambda)_d=\frac{\mathrm{Ker}(d^i)_d}{\mathrm{Ker}(d^i)_d\cap\mathrm{Im}(d^{i-1})}$. The reader is warned that, generally, an element $x\in HH^i(\Lambda)_d$ <u>does not</u> have length degree equal to d.

Theorem 6.6.10. *Let Λ be the generalized preprojective algebra $\mathbb{B}_n$ over a field K with char$(K)\neq 2$. The following assertions hold:*

1. *When n is even, we have isomorphisms of $Z(\Lambda)$-modules:*
   - (a) *$HH^i(\Lambda)\cong Z(\Lambda)/I$, for all $i>0$ and $i\equiv 0 \pmod 6$, where I is the subspace of $Z(\Lambda)$ generated by the set $\{w_i+w_{2n-i} : 1\leq i<n\}\cup\{w_n\}\cup\{v\}$, with $v=\sum_{1\leq k\leq\frac{n}{2}}(w_{2k-1}-w_{2n-2k+1})$;*
   - (b) *$HH^i(\Lambda)\cong Z(\Lambda)/Soc(\Lambda)$, for all $i\equiv 1$ or $4 \pmod 6$;*
   - (c) *$HH^i(\Lambda)\cong Z(\Lambda)/J(Z(\Lambda))=K$, for all $i\equiv 2$ or $3 \pmod 6$;*
   - (d) *$HH^i(\Lambda)\cong D(Z(\Lambda)/I)$, for all $i\equiv 5 \pmod 6$;*

2. *When n is odd, we have isomorphisms of $Z(\Lambda)$-modules:*

(a) *If char* (K) *does not divide* n*, then:*

i. $HH^i(\Lambda) \cong \frac{Z(\Lambda)}{Soc(\Lambda)\cap Z(\Lambda)}$*, for all* $i > 0$ *and* $i \equiv 0$ *or* $1 \pmod 6$*;*

ii. $HH^i(\Lambda) = 0$ *otherwise.*

(b) *If char* (K) *divides* n*, then:*

i. $HH^i(\Lambda) \cong Z(\Lambda)$*, for all* $i \equiv 0 \pmod 6$*;*

ii. $HH^i(\Lambda) \cong \frac{Z(\Lambda)}{Soc(\Lambda)\cap Z(\Lambda)}$*, for all* $i \equiv 1 \pmod 6$*;*

iii. $HH^i(\Lambda) \cong Z(\Lambda)/J(Z(\Lambda)) = K$*, for all* $i \equiv 5 \pmod 6$*;*

iv. $HH^i(\Lambda) = 0$ *otherwise.*

Proof. If $\underline{HH}^*(\Lambda)$ is the stable Hochschild cohomology ring then, in case n is even, we know from Eu-Schedler [35] that we have a correspondent of Corollary 5.2.12 and, hence, that there is an isomorphism of graded $\underline{HH}^*(\Lambda)$-modules $D(\underline{HH}^*(\Lambda)) \cong \underline{HH}^*(\Lambda)[5]$. This and the 6-periodicity of Λ yield isomorphisms of modules over $HH^0(\Lambda) = Z(\Lambda)$

$$\begin{aligned} HH^6(\Lambda) &\cong \underline{HH}^0(\Lambda) \\ D(\underline{HH}^0(\Lambda)) &\cong HH^5(\Lambda) \\ D(HH^1(\Lambda)) &\cong HH^4(\Lambda) \\ D(HH^2(\Lambda)) &\cong HH^3(\Lambda). \end{aligned}$$

On the other hand, the $Z(\Lambda)$-module $Z(\Lambda)/\mathrm{Soc}(\Lambda)$ is isomorphic to $K[x]/(x^{\frac{n}{2}})$, with the terminology of Proposition 6.6.2. Then we have an isomorphism $D(Z(\Lambda)/\mathrm{Soc}(\Lambda)) \cong Z(\Lambda)/\mathrm{Soc}(\Lambda)$ and, of course, we have another one $D(Z(\Lambda)/J(Z(\Lambda))) \cong Z(\Lambda)/J(Z(\Lambda))$ since $\frac{Z(\Lambda)}{J(Z(\Lambda))} \cong K$. Then, in order to prove assertion 1, we just need to prove 1.a and, in 1.b and 1.c, just the cases $i = 1$ and $i = 2$, respectively.

We know that if $i > 0$ and $i \equiv 0 \pmod 6$, then there is an isomorphism of $Z(\Lambda)$-modules $HH^i(\Lambda) \cong \underline{HH}^0(\Lambda) = \frac{Z(\Lambda)}{\mathcal{P}(\Lambda,\Lambda)}$. Since $\Lambda \cong \Omega^6_{\Lambda^e}(\Lambda)$, we readily get that $\mathcal{P}(\Lambda,\Lambda) = \mathrm{Im}({}_{\mu}k^*)$. In case n is odd, Proposition 6.5.11 gives that $\mathcal{P}(\Lambda,\Lambda) = Kw_n$ when $\mathrm{char}(K)$ does not divide n and that $\mathcal{P}(\Lambda,\Lambda) = 0$, when $\mathrm{char}(K)$ divides n. Then the formulas 2.a.i and 2.b.i, when $i \equiv 0 \pmod 6$, are automatic since $Z(\Lambda) \cap \mathrm{Soc}(\Lambda) = Kw_n$ in this case.

In case n is even, using again Proposition 6.5.11, we can assume without loss of generality that ${}_{\mu}k^*$ is a map $KQ_0 \longrightarrow \mathrm{Soc}(\Lambda)$ whose associated matrix with respect to the canonical bases of KQ_0 and $\mathrm{Soc}(\Lambda)$ is the Cartan matrix C_Λ. Following the calculation of $\mathrm{rank}(C_\Lambda)$ done in the proof of Proposition 6.5.7, we put $f = {}_{\mu}k^*$ and $V = KQ_0$ for simplicity. It was shown in that proof that the induced map $\mathrm{Ker}(\tau - 1_V) \longrightarrow \mathrm{Ker}(\tau - 1_V)$ is bijective. Then $\{w_i + w_{2n-i} : 1 \leq i < n\} \cup \{w_n\}$ is a basis of $f(\mathrm{Ker}(\tau - 1_V))$.

On the other hand, also in the proof of Proposition 6.5.7, it is shown that the restriction of f to $\mathrm{Ker}(\tau + 1_V)$ has a kernel of codimension 1 in $\mathrm{Ker}(\tau + 1_V)$. Then $f(\mathrm{Ker}(\tau + 1_V))$ has dimension 1. From the mentioned proof it follows that $f(e_{2k-1}) - f(e_{2n-2k+1}) = f(e_1) - f(e_{2n-1})$, for all $k = 1, ..., \frac{n}{2}$, because $(e_1 - e_{2n-1}) - (e_{2k-1} - e_{2n-2k+1})$ is always in the kernel of f (see Remark 6.5.8). But $f(e_1) - f(e_{2n-1}) = \sum_{1 \leq i \leq 2n-1}(c_{i1} - c_{i,2n-1})w_i$, and by Proposition 6.5.7 we have

$$c_{i1} - c_{i,2n-1} = \begin{cases} 0 & \text{if} \quad i \text{ is even} \\ 2 & \text{if} \quad i \text{ is odd and } i \leq n \\ -2 & \text{if} \quad i \text{ is odd and } i > n \end{cases}$$

It follows that $f(e_1) - f(e_{2n-1}) = 2\sum_{1\leq k\leq \frac{n}{2}}(w_{2k-1} - w_{2n-2k+1})$. From this and the previous paragraph we get 1.a.

We next find the structure as a $Z(\Lambda)$-module of the space $HH^1(\Lambda)$. For that we consider $HH^1(\Lambda)$ as a graded vector space, by using length degrees, and look at the exact sequence

$$0 \to Z(\Lambda)_d \longrightarrow \oplus_{i\in Q_0} e_i\Lambda_d e_i \longrightarrow \mathrm{Im}(\delta^*)_{d+1} \to 0,$$

for any even degree $0 \leq d < 2n-2$. Recall from the proof of Proposition 6.6.2 that if $d \equiv 2 \pmod 4$ then $Z(\Lambda)_d = 0$, which implies that $\dim(\mathrm{Im}(\delta^*)_{d+1}) = \dim(\oplus_{i\in Q_0} e_i\Lambda_d e_i) = d+1$ (see Proposition 6.6.1). On the other hand, by Proposition 6.6.7 we get $\dim(\mathrm{Ker}(R^*)_{d+1}) = d+1$. It follows that $HH^1(\Lambda)_e = 0$ whenever $0 < e < 2n-2$ and $e \equiv 3 \pmod 4$.

When $0 \leq d < 2n-2$ and $d \equiv 0 \pmod 4$, we know from the proof of Proposition 6.6.2 that $\dim(Z(\Lambda)_d) = 1$. It then follows that $\dim(\mathrm{Im}(\delta^*)_{d+1}) = \dim(\oplus_{i\in Q_0} e_i\Lambda_d e_i) - 1 = (2n-d-1)-1 = 2n-d-2$. Moreover, from Proposition 6.6.7 we get that $\dim(\mathrm{Ker}(R^*)_{d+1}) = 2n-(d+1) = 2n-d-1$. It follows that $\dim(HH^1(\Lambda)_e) = 1$ whenever $0 < e < 2n-2$ and $e \equiv 1 \pmod 4$. We will now pick up an element $0 \neq y \in HH^1(\Lambda)_1$ and will prove that the induced morphism of $Z(\Lambda)$-modules $Z(\Lambda) \longrightarrow HH^1(\Lambda)$ $(a \rightsquigarrow ay)$ vanishes on $Z(\Lambda)\cap \mathrm{Soc}(\Lambda)$ and induces an isomorphism of $Z(\Lambda)$-modules $\frac{Z(\Lambda)}{Z(\Lambda)\cap\mathrm{Soc}(\Lambda)} \xrightarrow{\cong} HH^1(\Lambda)$. The formula 1.b for $i \equiv 1 \pmod 6$ and the formula 2.b.ii will immediately follow from this.

Fix a natural number d such that $d \equiv 0 \pmod 4$ and $d < 2n-2$. Extending the notation of the proof of Proposition 6.6.7, we denote by p_a the possibly zero (unique up to relations) path of length $d+1$ from $i(a)$ to $t(a)$. Note that the element $v_{d+1} = \sum_{a\in Q_1} p_a$ is in $\mathrm{Ker}(R^*)$. Indeed $R^*(v_{d+1}) = \sum_{a\in Q_1}[(-1)^{s(\sigma(a)a)}\sigma(a)p_a + (-1)^{s(a\sigma^{-1}(a))}p_a\sigma^{-1}(a)]$. Let now $i \in Q_0$ and $i > n$ be any vertex such that $e_{\tau(i)}\Lambda_{d+2}e_i \neq 0$ and denote by q_i the (unique up to relations) nonzero path of length $d+2$ from $\tau(i)$ to i. Due to the equalities $q_i = \beta_{i-1}p_{\alpha_{i-1}} = p_{\beta_{i-1}}\alpha_{i-1} = p_{\alpha_{2n-i-1}}\beta_i = \alpha_{2n-i-1}p_{\beta_i}$ and the fact that $s(a) = 0$, for each noncrossing arrow a, we see that q_i in the last summatory appears with coefficient $[(-1)^{s(\beta_{i-1}\alpha_{i-1})} + (-1)^{s(\beta_{i-1}\alpha_{i-1})} + (-1)^{s(\alpha_{2n-i-1}\beta_i)} + (-1)^{s(\alpha_{2n-i-1}\beta_i)}] = [(-1)^{s(\beta_{i-1})} + (-1)^{s(\beta_{i-1})} + (-1)^{s(\beta_i)} + (-1)^{s(\beta_i)}$. This coefficient is zero because exactly one of the arrows β_{i-1}, β_i is in $\tilde{X}$ and, hence, $s(\beta_{i-1}) \neq s(\beta_i)$. A similar argument shows that q_n appears with coefficient $(-1)^{s(\beta_{n-1})} + (-1)^{s(\beta_{n-1})} + (-1)^{s(\beta_n)} + (-1)^{s(\beta_n)} = 0$ and, by application of the symmetry τ, we conclude that the coefficient of q_i in the summatory is zero, for all $i \in Q_0$ such that $e_{\tau(i)}\Lambda_{d+2}e_i \neq 0$. Therefore we have $R^*(v_{d+1}) = 0$.

On the other hand, we claim that $v_{d+1} \notin \mathrm{Im}(\delta)$. To see that, note that the proof of Proposition 6.6.1 shows that if c_i denotes the cycle of length d at i, for each $i \in Q_0$, then $\{c_i : 2s+1 \leq i \leq 2n-2s-1\}$ is a basis of $\oplus_{i\in Q_0} e_i\Lambda_d e_i$, where $4s = d$. If $v_{d+1} \in \mathrm{Im}(\delta^*)$ then there is a linear combination $u = \sum_{2s+1\leq i\leq 2n-2s-1}\lambda_i c_i$ such that $\delta^*(u) = v_{d+1}$. Note that the arrows a such that $p_a \neq 0$, equivalently those for which $e_{i(a)}\Lambda_{d+1}e_{t(a)} \neq 0$, are precisely the ones such that $i(a), t(a) \in \{i \in Q_0 : 2s+1 \leq i \leq 2n-2s-1\}$ (see the proof of Proposition 6.6.3). Denoting this set of arrows by H, we readily see that $\delta^*(u) = \sum_{a\in H}(\lambda_{t(a)} - \lambda_{i(a)})p_a$. The equality $\delta^*(u) = v_{d+1}$ gives that the coefficient of p_{α_n} in this expression is $\lambda_{n+1} - \lambda_n = 1$ while the coefficient of $p_{\beta_{n-1}}$ is $\lambda_n - \lambda_{n+1} = 1$. This is absurd.

We write $\bar{v}_{d+1} = v_{d+1} + \text{Im}(\delta^*) \in HH^1(\Lambda)_{d+1}$ and put $y = \bar{v}_1 \in HH^1(\Lambda)_1$. Taking the element $x \in Z(\Lambda)$ as in Proposition 6.6.2 and putting $d = 4s$, we readily see that the equality $\bar{v}_{d+1} = x^s y$ holds in $HH^1(\Lambda)$ since the multiplication by elements of $Z(\Lambda)$ is induced by the multiplication in Λ. It then follows that $[Z(\Lambda)y]_d = HH^1(\Lambda)_{d+1}$, for all natural number $d \equiv 0 \pmod 4$. But then we have $HH^1(\Lambda) = Z(\Lambda)y$ since $HH^1(\Lambda) = \oplus_{d\equiv 0 \pmod 4} HH^1(\Lambda)_{d+1}$. It is clear that $\text{Soc}(\Lambda) \cap Z(\Lambda) \subseteq \text{ann}_{Z(\Lambda)}(y)$, so that $HH^1(\Lambda)$ has a canonical structure of module over $\overline{Z(\Lambda)} = \frac{Z(\Lambda)}{\text{Soc}(\Lambda)\cap Z(\Lambda)}$. But $\overline{Z(\Lambda)}$ is isomorphic to $K[x]/(x^{\frac{n}{2}})$ or $K[x]/(x^{\frac{n-1}{2}})$, depending on whether n is even or odd. If s is a natural number such that $s < \frac{n-1}{2}$ then, both when n is even and when n is odd, we have that $x^s y = \bar{v}_{4s+1}$, which is nonzero since $4s + 1 \leq 2n - 3$. It follows that the morphism $Z(\Lambda) \longrightarrow HH^1(\Lambda)$ which takes $a \rightsquigarrow ay$ gives an isomorphism $\frac{Z(\Lambda)}{\text{Soc}(\Lambda)\bigcap Z(\Lambda)} \cong HH^1(\Lambda)$, as desired.

From Propositions 6.5.11 and 6.6.7 we immediately get the formula in 1.c and the fact that $HH^2(\Lambda) = 0$ when n is odd.

Since the proof of assertion 1 is now complete, we assume in the rest of the proof that n is odd. Then combining both Lemma 6.6.8 and Proposition 6.5.11, we then have $\text{Ker}({}_\mu\delta^*) = \text{Im}(k^*)$ and so $HH^3(\Lambda) = 0$.

Lemma 6.6.8 shows that, if $d < 2n - 2$, the induced map ${}_\mu\delta^* : \oplus_{i\in Q_0} e_i\Lambda_d e_i = \oplus_{i\in Q_0} e_{\mu(i)}\Lambda_d e_i \longrightarrow \oplus_{a\in Q_1} e_{i(a)}\Lambda_{d+1}e_{t(a)} = \oplus_{a\in Q_1} e_{\mu(i(a))}\Lambda_{d+1}e_{t(a)}$ is injective. Putting $r = d + 1$ and using Proposition 6.6.1, we then have that $\dim(\text{Im}({}_\mu\delta^*)_r) = \dim(\oplus_{i\in Q_0} e_i\Lambda_d e_i)$ is equal to $2n - d - 1 = 2n - r$, when $r \equiv 1 \pmod 4$, and equal to $d + 1 = r$, when $r \equiv 3 \pmod 4$. By Proposition 6.6.7, we conclude that $\text{Ker}({}_\mu R^*)_r = \text{Im}({}_\mu\delta^*)_r$, for all odd natural number $r \leq 2n - 3$. We then get $HH^4(\Lambda) = 0$.

Finally, from Propositions 6.5.11 and 6.6.7, we easily get that $HH^5(\Lambda) = 0$, when $\text{char}(K)$ does not divide n, and $HH^5(\Lambda)$ is 1-dimensional when $\text{char}(K)$ divides n. Then the proof is complete. □

As a straightforward consequence, we get:

Corollary 6.6.11. *The following are the dimensions of the $HH^i(\Lambda)$ as K-vector spaces:*

1. *When n is even:*

   (a) *dim$(HH^0(\Lambda)) = \frac{5n}{2} - 1$.*
   (b) *dim$(HH^i(\Lambda)) = \frac{3n}{2} - 2$ for all $i > 0$ and $i \equiv 0$ or $5 \pmod 6$.*
   (c) *dim$(HH^i(\Lambda)) = \frac{n}{2}$ for all $i > 0$ and $i \equiv 1$ or $4 \pmod 6$.*
   (d) *dim$(HH^i(\Lambda)) = 1$ for all $i > 0$ and $i \equiv 2$ or $3 \pmod 6$.*

2. *When n is odd, dim$(HH^0(\Lambda)) = \frac{n+1}{2}$ and, for $i > 0$, we have*

   (a) *If Char(K) does not divide n, then:*

i. *dim*$(HH^i(\Lambda)) = \frac{n-1}{2}$, *when* $i \equiv 0$ *or* $1 \pmod 6$.
ii. *dim*$(HH^i(\Lambda)) = 0$ *otherwise.*

(b) *If Char*(K) *divides* n, *then:*

i. *dim*$(HH^i(\Lambda)) = \frac{n+1}{2}$ *for all* $i \equiv 0 \pmod 6$.
ii. *dim*$(HH^i(\Lambda)) = \frac{n-1}{2}$ *for all* $i \equiv 1 \pmod 6$.
iii. *dim*$(HH^i(\Lambda)) = 1$ *for all* $i \equiv 5 \pmod 6$.
iv. *dim*$(HH^i(\Lambda)) = 0$ *otherwise.*

6.7 The ring structure of the Hochschild cohomology ring

Finally, in this section we compute the structure of the Hochschild cohomology ring $HH^*(\Lambda)$ of the generalized preprojective algebra $\Lambda = \mathbb{B}_n$ as a bigraded algebra over a field of characteristic different from 2. As done in Chapter 4, we will follow a convention for the statements of the two main theorems. Let us denote by $P^{\cdot} : ...P^{-n} \to ... \to P^{-1} \to P^0 \to \Lambda \to 0$ the minimal projective resolution of Λ as a graded bimodule (see Proposition 6.5.10). In the statement of the theorems, if a generator g is in $HH^n(\Lambda)$, we will give a morphism of Λ-bimodules $\tilde{g} : P^{-n} \longrightarrow \Lambda$ such that $\tilde{g}$ is in the kernel of the transpose map $(d^{-n-1})^* : \mathrm{Hom}_{\Lambda^e}(P^{-n}, \Lambda) = (P^{-n})^* \longrightarrow (P^{-n-1})^* = \mathrm{Hom}_{\Lambda^e}(P^{-n-1}, \Lambda)$ and g is the image of $\tilde{g}$ by the canonical projection $\mathrm{Ker}(d^{-n-1})^* \twoheadrightarrow HH^n(\Lambda)$. We will say that g is represented by $\tilde{g}$ and $\tilde{g}$ will be identified simply by its action on the canonical generators $e_i \otimes e_j$ of P^{-n}.

We start with the simplest case: when n is odd.

6.7.1 When n is odd

Theorem 6.7.1. *Let* n *be odd and let* Λ *be the generalized preprojective algebra* $\mathbb{B}_n$ *over a field of characteristic* $\neq 2$ *and let view* $HH^*(\Lambda)$ *as a bigraded algebra (see Section 2). Consider the following elements of* $HH^*(\Lambda)$*:*

a) $x = \sum_{3 \leq i \leq 2n-3} c_i \in HH^0(\Lambda) = Z(\Lambda)$, *where* c_i *is the cycle of length* 4 *at* i*;*

b) $y \in HH^1(\Lambda)$ *represented by* $\tilde{y} : \oplus_{a \in Q_1} \Lambda e_{i(a)} \otimes e_{t(a)} \Lambda \longrightarrow \Lambda$, *where* $\tilde{y}(e_{i(a)} \otimes e_{t(a)}) = a$, *for all* $a \in Q_1$*;*

c) $h \in HH^6(\Lambda)$ *represented by the multiplication map* $\oplus_{i \in Q_0} \Lambda e_i \otimes e_i \Lambda \longrightarrow \Lambda$*; and*

d) *In case char*(K) *divides* n, *the element* $v \in HH^5(\Lambda)$ *represented by* $\tilde{v} : \oplus_{i \in Q_0} \Lambda e_{\eta(i)} \otimes e_i \Lambda = \oplus_{i \in Q_0} \Lambda e_{\tau(i)} \otimes e_i \Lambda \longrightarrow \Lambda$, *where* $\tilde{v}(e_{\tau(i)} \otimes e_i) = \delta_{in} e_n$, *for all* $i \in Q_0$, *and* δ_{in} *is the Kronecker symbol.*

Then we have bideg$(x) = (0, 4)$, *bideg*$(y) = (1, 0)$, *bideg*$(h) = (6, -4n)$ *and bideg*$(v) = (5, -2n-2)$ *and the following assertions hold:*

1. *If char*(K) *does not divide* n, *then* $HH^*(\Lambda)$ *is the commutative bigraded algebra with generators* x, y, h, *subject to the relations:*

$$x^{\frac{n+1}{2}} = 0, \quad x^{\frac{n-1}{2}}y = 0, \quad x^{\frac{n-1}{2}}h = 0 \quad \text{and} \quad y^2 = 0.$$

2. *If char(K) divides n, then $HH^*(\Lambda)$ is the commutative bigraded algebra with generators x, y, v, h, subject to the relations*

$$x^{\frac{n+1}{2}} = 0, \quad x^{\frac{n-1}{2}}y = 0, \quad xv = 0, \quad y^2 = 0, \quad yv = 0 \quad \text{and} \quad v^2 = 0.$$

Proof. We know that y is the image of $\sum_{a\in Q_1} a$ by the projection $\mathrm{Ker}(R^*) \twoheadrightarrow HH^1(\Lambda)$. But the length degree of $\sum_{a\in Q_1} a$ in $\oplus_{a\in Q_1} e_{i(a)}\Lambda e_{t(a)}[1]$ is equal to 0, so that $\mathrm{bideg}(y) = (1,0)$. On the other hand h is the image of 1 by the projection $\mathrm{Ker}(\delta^*) \twoheadrightarrow HH^6(\Lambda)$. But the length degree of 1 in $\oplus_{i\in Q_0} e_i\Lambda e_i[4n]$ is $-4n$, so that $\mathrm{bideg}(h) = (6,-4n)$. When char$(K)$ divides n, we have that v is the image of e_n by the projection $\mathrm{Ker}({}_\mu k^*) \twoheadrightarrow HH^5(\Lambda)$. But the length degree of e_n in $\oplus_{\eta(i)}\Lambda e_i[2n+2]$ is $-2n-2$, so that $\mathrm{bideg}(v) = (5,-2n-2)$.

By the multiplicative structure of the center (see Proposition 6.6.2), we know that $x^{\frac{n+1}{2}} = 0$. By Theorem 6.6.10 and its proof, we know that the map $Z(\Lambda) \longrightarrow HH^1(\Lambda)$ $(a \rightsquigarrow ay)$ is surjective with kernel $\mathrm{Soc}(\Lambda)\cap Z(\Lambda)$, which is the ideal of $Z(\Lambda)$ generated by $w_n = x^{\frac{n-1}{2}}$. We also know that the map $Z(\Lambda) \longrightarrow HH^6(\Lambda)$ $(a \rightsquigarrow ah)$ is surjective, with zero kernel in case char(K) divides n, and with kernel $\mathrm{Soc}(\Lambda)\cap Z(\Lambda) = Kx^{\frac{n-1}{2}}$ otherwise. Moreover, we have $y^2 = 0$ because $HH^2(\Lambda) = 0$. When char(K) does not divide n, this together with periodicity and the fact that $HH^3(\Lambda) = 0 = HH^5(\Lambda)$ imply that $HH^*(\Lambda)$ is a commutative algebra since all products of homogeneous elements of odd homological degree are zero. We conclude that, when char(K) does not divide n, the bigraded algebra $HH^*(\Lambda)$ is given by generators and relations as indicated in the statement of the theorem.

Suppose in the rest of the proof that char(K) divides n. Then $v^2 = 0$ since $v^2 \in HH^{10}(\Lambda) = 0$. That $xv = 0$ follows easily from taking bidegrees. Indeed $HH^5(\Lambda)$ is 1-dimensional and its nonzero elements are all scalar multiples of v, thus of bidgree $(5,-2n-2)$, while if we had $xv \neq 0$ we would have $\mathrm{bideg}(xv) = (0,4)+(5,-2n-2) = (5,-2n+2)$. Note that we cannot apply an analogous argument to prove that $yv = 0$ since $\mathrm{bideg}(yv) = \mathrm{bideg}(x^{\frac{n-1}{2}}h)$ and $x^{\frac{n-1}{2}}h \neq 0$. Instead, given $i \in Q_0$ and a nonzero element $0 \neq z \in e_{\tau(i)}\Lambda e_n$ which is equal in Λ to a path from $\tau(i)$ to n, we fix such a path, say $a_1a_2...a_r$, where the a_i are arrows. We now imitate an argument used in the proof of Lemma 5.4.4 and take the element

$$\xi_z = \tau(a_1)...\tau(a_{r-1}) \otimes z^* + \tau(a_1)...\tau(a_{r-2}) \otimes \tau(a_r)z^* + ... + \tau(a_1) \otimes \tau(a_3)...\tau(a_r)z^* + e_i \otimes \tau(a_2)...\tau(a_r)z^*.$$

Note that ξ_z is an element of $\oplus_{a\in Q_1}\Lambda e_{i(a)} \otimes e_{t(a)}\Lambda$ which depends on the path chosen to represent z. However, direct computation shows that $\delta(\xi_z) = \tau(z)\otimes z^* - e_i \otimes \tau(z)z^*$, and this element does not depend on that choice. When $i \neq n$, we always have $\tau(z)z^* = 0$ in Λ, so that $\delta(\xi_z) = \tau(z)\otimes z^*$. Fixing now a basis $e_{\tau(i)}\mathcal{B}e_n$ of $e_{\tau(i)}\Lambda e_n$ consisting of paths, we get that $\delta(\sum_{z\in e_{\tau(i)}\mathcal{B}e_n} \xi_z) = \sum_{z\in e_{\tau(i)}\mathcal{B}e_n} \tau(z)\otimes z^*$. But for $i = n$, with the convention that $\xi_{e_n} = 0$ and so $\delta(\xi_{e_n}) = e_n \otimes w_n - e_n \otimes w_n$, we also get:

$$\delta(\textstyle\sum_{z\in e_{\tau(n)}\mathcal{B}e_n} \xi_z) = \sum_{z\in e_n\mathcal{B}e_n}[\tau(z)\otimes z^* - e_n\otimes w_n] = (\sum_{z\in e_n\mathcal{B}e_n} \tau(z)\otimes z^*) - \dim(e_n\Lambda e_n)(e_n\otimes w_n) = (\sum_{z\in e_n\mathcal{B}e_n} \tau(z)\otimes z^*) - n(e_n\otimes w_n) = \sum_{z\in e_n\mathcal{B}e_n} \tau(z)\otimes z^*,$$

because $\dim(e_n\Lambda e_n) = n$ (see Proposition 6.5.7) and $\text{char}(K)$ divides n.

We now explicitly calculate the product yv in $HH^*(\Lambda)$. The morphism $\tilde{v} : \oplus_{i\in Q_0}\Lambda e_{\tau(i)}\otimes e_i\Lambda \longrightarrow \Lambda$, which represents v, admits an obvious lifting $\hat{v} : \oplus_{i\in Q_0}\Lambda e_{\tau(i)} \otimes e_i\Lambda \longrightarrow \oplus_{i\in Q_0}\Lambda e_i \otimes e_i\Lambda$, which is the unique morphism of Λ-bimodules which maps $e_{\tau(i)} \otimes e_i \rightsquigarrow \delta_{in}e_n \otimes e_n$. Bearing in mind that $\eta = \tau$ in this case, we then get

$$(\hat{v} \circ {}_\mu k)(e_i \otimes e_i) = \hat{v}(\textstyle\sum_{z\in e_{\tau(i)}\mathcal{B}} \tau(z) \otimes z^*) = \sum_{z\in e_{\tau(i)}\mathcal{B}e_n} \tau(z) \otimes z^* = \delta(\sum_{z\in e_{\tau(i)}\mathcal{B}e_n} \xi_z).$$

This implies that if we take the unique morphism $\oplus_{i\in Q_0}e_i\Lambda e_i \xrightarrow{f} \oplus_{a\in Q_1}\Lambda e_{i(a)} \otimes e_{t(a)}\Lambda$ of Λ-bimodules which maps $e_i \otimes e_i \rightsquigarrow \sum_{z\in e_{\tau(i)}\mathcal{B}e_n} \xi_z$, then the composition $\oplus_{i\in Q_0}e_i\Lambda e_i \xrightarrow{f} \oplus_{a\in Q_1}\Lambda e_{i(a)}\otimes e_{t(a)}\Lambda \xrightarrow{\tilde{y}} \Lambda$ represents the product yv. One readily sees that $(\tilde{y}\circ f)(e_i\otimes e_i) = 0$, for all vertices $i \neq n$. On the other hand, if $z \in e_n\mathcal{B}e_n$ then $\tilde{y}(\xi_z) = \text{length}(z)\tau(z)z^* = \text{length}(z)w_n$. It follows that

$$(\tilde{y}\circ f)(e_n \otimes e_n) = \textstyle\sum_{z\in e_n\mathcal{B}e_n} \text{length}(z)w_n = (0+2+4+...+(2n-2))w_n = n(n-1)w_n = 0.$$

Then we have $yv = 0$ in $HH^*(\Lambda)$. □

6.7.2 When n is even

In the rest of the thesis, we assume that n is even. We will adopt the convention of Remark 6.6.9.

Lemma 6.7.2. *Let n be even and d be an odd natural number such that $0 < d < 2n-2$. Then $HH^4(\Lambda)_d \neq 0$ if, and only if, $d \equiv 1 \pmod 4$. In such a case one has $\dim(HH^4(\Lambda)_d) = 1$.*

Proof. By Proposition 6.6.7, we know that the kernel of the map ${}_\mu R^* : \oplus_{a\in Q_1}e_{\tau(i(a))}\Lambda_d e_{t(a)} \longrightarrow \oplus_{i\in Q_0}e_i\Lambda_{d+1}e_i$ has dimension $d+1$. On the other hand, from Lemma 6.6.8 we know that ${}_\mu\delta^* : \oplus_{i\in Q_0}e_{\tau(i)}\Lambda_{d-1}e_i \longrightarrow \oplus_{a\in Q_1}e_{\tau(i(a))}\Lambda_d e_{t(a)}$ is injective. Then we have the equalities $\dim(\text{Im}({}_\mu\delta^*)) = \dim(\oplus_{i\in Q_0}e_{\tau(i)}\Lambda_{d-1}e_i) = d$ (see Proposition 6.6.4). It follows that $\dim HH^*(\Lambda)_d = 1$. Using Corollary 6.6.11, we then get:

$$\frac{n}{2} = \dim(HH^4(\Lambda)) \geq \sum_{d<2n-2, d\equiv 1 \pmod 4} \dim(HH^4(\Lambda)_d) = \frac{n}{2}$$

From this the result follows immediately. □

Lemma 6.7.3. *For each $i, j \in Q_0$, let G_{ij} be the set of natural numbers d such that $e_i\Lambda_d e_j \neq 0$ and put $\Upsilon_{ij} = \sum_{d\in G_{ij}} d - \sum_{e\in G_{i,2n-j}} e$. The following assertions hold:*

1. *If $i, j \leq n$ then:*
   (a) *If either $i = n$ or $j = n$ or i and j are both even, then $\Upsilon_{ij} = 0$*
   (b) *If both i and j are odd, then $\Upsilon_{ij} = 2n-2$*
   (c) *If i is odd and j is even, then:*

i. $\Upsilon_{ij} = -2j$, *when* $i > j$
ii. $\Upsilon_{ij} = 2(n-j)$, *when* $i < j$

2. *The following relations hold:*

   (a) $\Upsilon_{ij} = \Upsilon_{2n-i,2n-j}$
   (b) $-\Upsilon_{i,2n-j} = \Upsilon_{ij} = -\Upsilon_{2n-i,j}$
   (c) $\Upsilon_{ij} = -\Upsilon_{ji}$ *whenever* i *or* j *is even*
   (d) $\Upsilon_{ij} = 4n - 4 - \Upsilon_{ji}$ *whenever* i *and* j *are both odd and smaller than* n.

Proof. We first prove assertion 2). The equality 2.a follows from the fact that, due to the symmetry τ, one has that $G_{ij} = G_{2n-i,2n-j}$. By definition of Υ_{ij} and $\Upsilon_{i,2n-j}$, we get that $\Upsilon_{ij} = -\Upsilon_{i,2n-j}$. But then, using 2.a, we get that $\Upsilon_{ij} = -\Upsilon_{2n-i,j}$ and so 2.b follows.

We next prove 2.c and 2.d at once. By Remark 6.5.4, we have that $d \in G_{ij}$ if, and only if, $2n-2-d \in G_{ji}$. This together with the equality of sets $G_{2n-j,i} = G_{j,2n-i}$ gives:

$$\Upsilon_{ij} = \sum_{d\in G_{ij}} d - \sum_{e\in G_{i,2n-j}} e = \sum_{d'\in G_{ji}} (2n-2-d') - \sum_{e'\in G_{2n-j,i}} (2n-2-e') = (2n-2)|G_{ji}| - (2n-2)|G_{j,2n-i}| - \Upsilon_{ji} = (2n-2)(c_{ji} - c_{j,2n-i}) - \Upsilon_{ji}.$$

From Proposition 6.5.7 and its proof we know that $c_{ji} - c_{j,2n-i} = 0$, when i or j is even, and $c_{ji} - c_{j,2n-i} = 2$, when i and j are both odd and smaller than n.

We clearly have that $\Upsilon_{in} = 0$, and from 2.c we also get $\Upsilon_{ni} = 0$. In order to prove the rest of assertion 1, let us note that, by Lemma 6.5.5, we have the formula:

$$\Upsilon_{ij} = \sum_{s\in X^+(i,j)} [4s + (j-i)] + \sum_{s\in X^-(i,j)} [4s + 2 + (j-i)] - \sum_{t\in Z^+(i,2n-j)} [4t + (j-i)] - \sum_{t\in Z^-(i,2n-j)} [4t + 2 + (j-i)] = 4a_{ij} + 4b_{ij} + 2u_{ij} + (j-i)v_{ij},$$

where $a_{ij} = \sum_{s\in X^+(i,j)} s - \sum_{t\in Z^-(i,2n-j)} t$, $b_{ij} = \sum_{s\in X^-(i,j)} s - \sum_{t\in Z^+(i,2n-j)} t$, $u_{ij} = |X^-(i,j)| - |Z^-(i,2n-j)|$ and $v_{ij} = c_{ij} - c_{i,2n-j}$. Here $C_\Lambda = (c_{ij})$ is the Cartan matrix. In order to exploit the formula above, let us recall (cf. Lemma 6.5.6), that if $i,j > n$ then we have set equalities:

$$X^+(i,j) = \{s \in \mathbb{N} : max(0, \tfrac{i-j}{2}) \le s \le n - \tfrac{j+1}{2}\}$$
$$Z^-(i,2n-j) = \{t \in \mathbb{N} : max(0, \tfrac{i-j-1}{2}) \le t \le n - \tfrac{j+2}{2}\}$$
$$X^-(i,j) = \{s \in \mathbb{N} : \tfrac{i-n-1}{2} \le s \le min(\tfrac{n-2}{2}, \tfrac{n+i-j-2}{2})\}$$
$$Z^+(i,2n-j) = \{\in \mathbb{N} : \tfrac{i-n}{2} \le t \le min(\tfrac{n-1}{2}, \tfrac{n+i-j-1}{2}))\}.$$

Using these equalities and Proposition 6.5.7, we then get the following table, whenever $n < i \le j$:

Condition	a_{ij}	b_{ij}	u_{ij}	v_{ij}
i and j even	0	0	$(n-\frac{j}{2}) - (n-\frac{j}{2}) = 0$	0
i and j odd	$n - \frac{j+1}{2}$	$\frac{i-n-1}{2}$	$(n-\frac{j-1}{2}) - (n-\frac{j+1}{2}) = 1$	2
i odd, j even	0	$-\|X^-(i,j)\| = -(n-\frac{j}{2})$	$(n-\frac{j}{2}) - (n-\frac{j}{2}) = 0$	0
i even, j odd	$n - \frac{j+1}{2}$	$-\frac{n+i-j-1}{2}$	$(n-\frac{j+1}{2}) - (n-\frac{j+1}{2}) = 0$	0

From this table, using 2.a, we get immediately 1.a and 1.b. Finally note that, by 2.a again, proving 1.c is equivalent to proving that if $i, j \geq n$, with i odd and j even, then $\Upsilon_{ij} = -2(2n - j)$, when $i < j$, and $\Upsilon_{ij} = -2(n - j)$, when $i > j$. The first equality follows immediately from the pre-last row of the table. The second formula follows from the last row of the table, by using 2.c.

□

Notation 6.7.4. Given vertices $i, j \in Q_0$, we shall denote by $N(j, i)$ the set of natural numbers d such that $e_j \Lambda_d e_{\tau(i)} \neq 0 \neq e_i \Lambda_{2n-2-d} e_j$ or, equivalently, such that $e_j \Lambda_d e_{\tau(i)} \neq 0 \neq e_j \Lambda_d e_i$. We then put $\tilde{\mathcal{B}} = \bigcup_{i,j \in Q_0} e_j \tilde{\mathcal{B}} e_{\tau(i)}$, where $e_j \tilde{\mathcal{B}} e_{\tau(i)}$ denotes the subset of $e_j \Lambda e_{\tau(i)}$ consisting of the classes x modulo relations of paths from j to $\tau(i)$ such that $\deg(x) \in N(j, i)$. Finally, for each $x \in e_j \tilde{\mathcal{B}} e_{\tau(i)}$, we shall denote by $x^\diamond$ the class modulo relations of any path from i to j of length $2n - 2 - \deg(x)$.

In the following lemma, for each $j \in Q_0$, we shall denote by $x(j)$ and $y(j)$ the shortest paths from j to n and from n to j, respectively, which we view as elements of Λ. Note that both of them have length $|n - j|$.

Lemma 6.7.5. *Let $i, j \in Q_0$ be any vertices. The following assertions hold:*

1. *$N(j, i) = N(j, 2n - i) = N(2n - j, i) = N(2n - j, 2n - i)$ and the assignment $d \rightsquigarrow 2n - 2 - d$ defines a bijection $N(j, i) \xrightarrow{\cong} N(i, j)$.*

2. *If $i, j \geq n$ the set $N(j, i)$ is nonempty exactly when $i + j \leq 3n - 1$. In such case $e_j \tilde{\mathcal{B}} e_{\tau(i)} = \{x(j) c^r y(\tau(i)) : 0 \leq r \leq 3n - 1 - (i + j)\}$, where c is the cycle of length 2 at n.*

3. *If $j \neq n$ then, for each pair $(0, 0) \neq (\lambda, \mu) \in (K \times \{0\}) \cup (\{0\} \times K)$, there is a unique map $h : e_j \tilde{\mathcal{B}} \longrightarrow K$ such that $(h(x(j)), h(x(j)\alpha_{n-1})) = (\lambda, \mu)$ and h satisfies the following three conditions, for each $x \in e_j \tilde{\mathcal{B}}$:*

   (a) $h(xa) = h(x)$, whenever a is a crossing arrow $\neq \beta_{n-1}$ and $xa \in e_j \tilde{\mathcal{B}}$

   (b) $h(xa) = -h(x)$, whenever a is a noncrossing arrow $\neq \alpha_{n-1}$ and $xa \in e_j \tilde{\mathcal{B}}$

   (c) If $a \in \{\alpha_{n-1}, \beta_{n-1}\}$ and $xa \in e_j \tilde{\mathcal{B}}$, then exactly one of the values $h(x)$ and $h(xa)$ is nonzero.

4. *If $j \neq n$, $h : e_j \tilde{\mathcal{B}} \longrightarrow K$ is the map satisfying the conditions 3.a-c) and $x, xc \in e_j \tilde{\mathcal{B}}$, then exactly one of $h(x)$ and $h(xc)$ is nonzero. If, in addition, also xc^2 is in $e_j \tilde{\mathcal{B}}$, then $h(x) = h(xc^2)$.*

Proof. 1) This assertion is a direct consequence of the definition of $N(j, i)$ and of the isomorphisms $D(e_j \Lambda_d e_k) \cong D(e_k \Lambda_{2n-2-d} e_j)$ and $e_j \Lambda_d e_k \cong e_{2n-j} \Lambda_d e_{2n-k}$, which hold for all $j, k \in Q_0$.

2) Let d be any natural number. From Lemma 6.5.5 we know that $e_j \Lambda_d e_{2n-i} \neq 0$ (resp. $e_j \Lambda_d e_i \neq 0$) if, and only if, there is a $s \in Z^+(j, 2n - i)$ (resp. $x \in X^+(j, i)$) such that $d = 4s + i - j$ or there is a $s \in Z^-(j, 2n - i)$ (resp. $x \in X^-(j, i)$) such that

$d = 4s + 2 + i - j$. It follows that $d \in N(j,i)$ if, and only if, either $d = 4s + i - j$, with $s \in X^+(j,i) \cap Z^+(j, 2n-i)$, or $d = 4s + 2 + i - j$, with $s \in X^-(j,i) \cap Z^-(j, 2n-i)$. By Lemma 6.5.6, we have the following set equalities:

$$X^+(j,i) \cap Z^+(j, 2n-i) = \{s \in \mathbb{N} : \tfrac{j-n}{2} \leq s \leq n - \tfrac{i+1}{2}\}$$
$$X^-(j,i) \cap Z^-(j, 2n-i) = \{s \in \mathbb{N} : \tfrac{j-n-1}{2} \leq s \leq n - \tfrac{i+2}{2}\}.$$

It follows that $N(j,i)$ consists of the natural numbers d such that $d \equiv i - j$ or $d \equiv i - j + 2$ (mod 4) and $i + j - 2n \leq d \leq 4n - 2 - (i + j)$. This set is nonempty precisely when $i + j - 2n \leq 4n - 2 - (i + j)$, which is equivalent to saying that $i + j \leq 3n - 1$.

Note that we have actually proved that $N(j,i) = \{i + j - 2n + 2r : r = 0, 1, ..., 3n - 1 - i - j\}$. But the elements of this set are precisely the lengths of the paths $x(j)c^r y(\tau(i))$, which proves that $e_j\tilde{\mathcal{B}}e_{\tau(i)} = \{x(j)c^r y(\tau(i)) : 0 \leq r \leq 3n - 1 - (i + j)\}$.

3) We first suppose that $\mu = 0$. By assertions 1 and 2, each $x \in e_j\tilde{\mathcal{B}}$ can be written in the form $x = x(j)c^r y(i)$, for uniquely determined vertices $i, j \in Q_0$ and natural number r, where r is bounded above by a formula depending on i, j (e.g. if $i, j \geq n$ then $0 \leq r \leq 3n - 1 - (i + j)$). We define the map $h : e_j\tilde{\mathcal{B}} \longrightarrow K$ as follows:

i) If either $i \geq n$ and r is odd or $i < n$ and r is even, we put $h(x) = 0$;

ii) If $i \geq n$ and r is even, we put $h(x) = (-1)^{i-n}\lambda$;

iii) If $i < n$ and r is odd, we put $h(x) = (-1)^{i-n+1}\lambda$.

Let us check that this map satisfies the required conditions. On one hand, we clearly have $(h(x(j)), h(x(j)\alpha_{n-1}) = (\lambda, 0) = (\lambda, \mu)$. On the other hand, let a be a noncrossing arrow, with $a \neq \alpha_{n-1}$, such that also $xa \in e_j\tilde{\mathcal{B}}$. Then we have $i(a) = i$. If $i \geq n$ then $a = \alpha_i$ and $y(i)a = y(i + 1)$, so that $xa = x(j)c^r y(i + 1)$. It follows from this that $h(xa) = -h(x)$ and a similar argument works in case $i < n$.

Suppose that a is a crossing arrow such that $xa \in e_j\tilde{\mathcal{B}}$. We do the case when $i > n$, leaving to the reader the case $i < n$. We then get that $a = \beta_{2n-i}$ and, by the relations, we have an equality $y(i)\beta_{2n-i} = c\alpha_{n-1}...\alpha_{2n-i+1} = cy(2n - i + 1)$. We then get $xa = x(j)c^{r+1}y(2n - i + 1)$. By the definition of h, if r is odd then we have $h(x) = 0 = h(xa)$. If r is even then $h(x) = (-1)^{i-n}\lambda$, while $h(xa) = (-1)^{[n-(2n-i+1)]+1}\lambda$. It follows that $h(xa) = h(x)$.

If $x\alpha_{n-1} \in e_j\tilde{\mathcal{B}}$, which implies that $i = n$ and $x = x(j)c^r$, then $h(x\alpha_{n-1})$ is equal to 0 or λ, depending on wether r is even or r is odd. But in the first case $h(x) \neq 0$ while in the second case $h(x) = 0$. It then follows that exactly one of $h(x)$ and $h(x\alpha_{n-1})$ is nonzero. Suppose finally that $x\beta_{n-1} \in e_j\tilde{\mathcal{B}}$, which implies that $i = n + 1$ and $x = x(j)c^r\alpha_n$. Note that then $x\beta_{n-1} = x(j)c^{r+1}$. If r is even,then we have $h(x) = -h(x(j)c^r) = -\lambda \neq 0 = h(x\beta_{n-1}) = h(x(j)c^{r+1})$. On the contrary, if r is odd then we have $h(x) = -h(x(j)c^r) = 0 \neq h(x(j)c^{r+1}) = h(x\beta_{n-1})$. We then have that exactly one of $h(x)$ and $h(x\beta_{n-1})$ is nonzero.

We now prove that h is unique. Indeed, suppose that $h' : e_j\tilde{\mathcal{B}} \longrightarrow K$ is a map such that $h' \neq h$ and h' satisfies the requirements. Note that $e_j\tilde{\mathcal{B}}$ admits the divisibility partial order. That is, $x \leq y$ if and only if there is a homogeneous element $z \in \Lambda$ such that $xz = y$.

We choose $x_0 \in e_j\tilde{\mathcal{B}}$ such that $h'(x_0) \neq h(x_0)$ and x_0 is minimal with this property. By the properties of h and h', we know that $x_0 \notin \{x(j), x(j)\alpha_{n-1}\}$. But, due to the relations, any element in $e_j\tilde{\mathcal{B}} \setminus \{x(j), x(j)\alpha_{n-1}\}$ can be written in the form xa, where $x \in e_j\tilde{\mathcal{B}}$ and a is an arrow different from α_{n-1} and β_{n-1}. In particular, we can write $x_0 = xa$. We then get $h(x) = h(xa) \neq h'(xa) = h'(x)$, when a is crossing, and $h(x) = -h(xa) \neq -h'(xa) = h'(x)$, when a is noncrossing. In both cases, we get that $h(x) \neq h'(x)$, which is a contradiction since $x < x_0$.

With a very similar argument, the reader will have no difficulty in checking that if $(\lambda, \mu) = (0, \mu)$, then the desired map h is given as follows:

i) If either $i \geq n$ and r is even or $i < n$ and r is odd, then $h(x) = 0$;

ii) If $i \geq n$ and r is odd, then $h(x) = (-1)^{i-n}\mu$;

iii) If $i < n$ and r is even, then $h(x) = (-1)^{i-n+1}\mu$.

4) This assertion follows immediately from the explicit definition of h given in the proof of assertion 3. □

Definition 23. Let us put $\tilde{\mathcal{B}} = \bigcup_{j \in Q_0} e_j\tilde{\mathcal{B}}$. Then we call *the coefficient map*, denoted in what follows by h, the map $h : \tilde{\mathcal{B}} \longrightarrow K$ identified by its restriction to the $e_j\tilde{\mathcal{B}}$ as follows:

1. If $x \in e_n\tilde{\mathcal{B}} = e_n\mathcal{B}$ and we put $x = c^r y(i)$, where $y(i)$ denotes the shortest path from n to i, then $h(x) = \frac{(-1)^{\deg(y(i))}}{4}$, when $i \geq n$, and $h(x) = \frac{(-1)^{\deg(y(i))-1}}{4}$, when $i < n$.
2. When either $j > n$ and j is odd or $j < n$ and j is even, $h_{|e_j\tilde{\mathcal{B}}} : e_j\tilde{\mathcal{B}} \longrightarrow K$ is the unique map such that $(h(x(j), h(x(j)\alpha_{n-1})) = (0, \frac{1}{2})$ and satisfies conditions 3.a-c of Lemma 6.7.5 ;
3. When either $j < n$ and j is odd or $j > n$ and j is even, $h_{|e_j\tilde{\mathcal{B}}}$ is the unique map such that $(h(x(j), h(x(j)\alpha_{n-1})) = (\frac{1}{2}, 0)$ and satisfies conditions 3.a-c of Lemma 6.7.5

Lemma 6.7.6. *Let a and j be an arrow and a vertex in Q, respectively, and let us denote by $x(j)$ and $y(j)$ the shortest paths from j to n and from to n to j, respectively. Then the following assertions hold:*

1. *If $x \in e_j\widetilde{B}e_{\tau(t(a))}$ is such that $x\sigma(a) \otimes x^\diamond \neq 0$, then $x^\diamond \in \sigma^{-1}(a)\Lambda$ unless $a = \beta_i$, $x^\diamond = x(t(a))y(j)$ and one of the following three conditions is satisfied:*

   (a) $j = n$

   (b) i is odd and either $i, j > n$ or $i < n < j$

   (c) i is even and either $i, j < n$ or $j < n \leq i$

   Moreover, in the last two cases and whenever $i \neq n$, the coefficient map satisfies that $h(x) = 0$.

2. *If* $y \in e_j\widetilde{B}e_{i(a)}$ *is such that* $y \otimes \sigma^{-1}(a)y^{\diamond} \neq 0$, *then* $y \in \Lambda\sigma(a)$ *unless* $a = \alpha_i$, $y = x(j)y(i(a))$ *and one of the following three conditions is satisfied:*

   (a) $j = n$

   (b) j *is odd and either* $i, j \geq n$ *or* $i, j < n$

   (c) j *is even and either* $i < n < j$ *or* $j < n \leq i$

   Moreover, in the last two cases and whenever $i \neq n-1$, *the coefficient map satisfies* $h(y) = 0$.

Proof. We shall deal with the last statement of both assertions 1) and 2) at the end of the proof. We then begin with the first part of each of them.

1) We know from Lemma 6.7.5 that $x^{\diamond} = x(t(a))c^s y(j)$ where $s \in \mathbb{N}$. Due to the symmetry τ, there is no loss of generality in assuming that either $a = \alpha_{i-1} : i-1 \longrightarrow i$ with $i \geq n+1$ or $a = \beta_i : 2n-i-1 \longrightarrow i$ with $i \geq n$.

Suppose first that $a = \alpha_{i-1}$ with $i \geq n+1$, so that, $x^{\diamond} = x(i)c^s y(j)$. But then, it is clear that the first arrow of $x(i)$ is precisely $\beta_{2n-i} = \sigma^{-1}(\alpha_{i-1})$ and we immediately get that $x^{\diamond} \in \sigma^{-1}(a)\Lambda$.

Suppose next that $a = \beta_i : 2n-i-1 \longrightarrow i$ with $i > n$. Then $x^{\diamond} = x(i)c^s y(j)$ where $x(i) = \beta_{2n-i}\beta_{i-2}\cdots\beta_{n-2}\beta_n$ if i is even and $x(i) = \beta_{2n-i}\beta_{i-2}\cdots\beta_{n+1}\beta_{n-1}$ if i is odd. Note that if i is even we get that $x(i)c = \beta_{2n-i}\beta_{i-2}\cdots\beta_{n-2}\beta_n\alpha_n\beta_{n-1} = \alpha_i\beta_{2n-i-1}\cdots\beta_{n+1}$ while if i is odd we obtain that $x(i)c = \beta_{2n-i}\beta_{i-2}\cdots\beta_{n+1}\beta_{n-1}\alpha_{n-1}\beta_n = \alpha_i\beta_{2n-i-1}\cdots\beta_n$. Therefore $x^{\diamond} \in \sigma^{-1}(\beta_i)\Lambda = \alpha_i\Lambda$ whenever $s \geq 1$. We know assume that $s = 0$, that is, $x^{\diamond} = x(i)y(j)$. It is clear that if $j = n$, since $y(j) = e_j$, we have that $x^{\diamond} = x(i) \notin \alpha_i\Lambda$. Now suppose that $j \neq n$. Following the description of $x(i)$ we distinguish two cases. First, let us assume that i is odd. Then, if $j > n$ we have that $x(i)y(j) = \beta_{2n-i}\beta_{i-2}\cdots\beta_{n+1}\beta_{n-1}\alpha_{n-1}\cdots\alpha_j = \alpha_i\beta_{2n-i-1}\cdots\beta_{n-2}\alpha_{n-2}\cdots\alpha_j$ which belongs to $\alpha_i\Lambda$. The result for the case when i is even is very similar to that of when i is odd so it is left to the reader. For the case when $i = n$ and, consequently, $x^{\diamond} = y(j)$, we obviously have that the first arrow of $y(j)$ is $\sigma^{-1}(\beta_n) = \alpha_n$ if, and only if, $j > n$.

2) We proceed similarly to assertion 1). Once again, due to Lemma 6.7.5 we know that $y = x(j)c^s y(i(a))$ for some $s \in \mathbb{N}$ and that we can assume without loss of generality that either $a = \alpha_i : i \longrightarrow i+1$ with $i \geq n$ or $a = \beta_i : 2n-i-1 \longrightarrow i$ with $i \geq n$.

We first suppose that $a = \beta_i : 2n-i-1 \longrightarrow i$ with $i \geq n$ and then, $y = x(j)c^s y(2n-i-1)$ with $i \geq n$ and $\Lambda\sigma(a) = \Lambda\alpha_{2n-i-1}$. This is equivalent to saying that $y = x(j)c^s y(i)$ with $i < n$ and $\Lambda\sigma(a) = \Lambda\alpha_i$. But notice that $y(i) = \alpha_{n-1}\alpha_{n-2}\cdots\alpha_i \in \Lambda\alpha_i$, so we are done.

Finally suppose that $a = \alpha_i \longrightarrow i+1$ with $i \geq n$ and hence, $y = x(j)c^s y(i)$ for some $s \in \mathbb{N}$. Since $i \geq n$, we get that $y(i) = \alpha_n\cdots\alpha_{i-1}$ (here $\alpha_n\cdots\alpha_{i-1} = e_i$ in case $i = n$) which implies that $cy(i) = \alpha_{n-1}\beta_n\alpha_n\cdots\alpha_{i-1} = \alpha_{n-1}\cdots\alpha_{2n-i-1}\beta_i \in \Lambda\sigma(\alpha_i) = \Lambda\beta_i$. Therefore we can assume that $s = 0$, in which case, $y = x(j)y(i)$. If $j = n$ we obviously obtain that $y = y(i) \notin \Lambda\beta_i$. So, from now on, assume that $j \neq n$. Note that the last arrow of $x(j)$ is β_{n-1} if, and only if, either $j < n$ and j is even or $j > n$ and j is odd. Otherwise, the last arrow of $x(j)$ is the arrow β_n. But, since $y(i) \in \alpha_n\Lambda$ if $i \neq n$ and $y(i) = e_i$ if $i = n$ we clearly have that $y = x(j)y(i) \notin \Lambda\beta_i$ if, and only if, the last arrow of $x(j)$ is β_{n-1}, or equivalently, if, and only if, either $j < n$ and j is even or $j > n$ and j is odd.

It only remains to deal with the statements involving the coefficient map h. We first look at the one given in assertion 2). Using definition 23 and the explicit definition of $h_{|e_j\widetilde{B}}$ given in the proof of Lemma 6.7.5, we readily see that if $j \neq n$ and $y = x(j)c^r y(i)$ is an element of $e_j\widetilde{B}e_i$, then coefficient map vanishes on y if, and only if, the triple (i, r, j) satisfies one of the conditions of the following table:

i	r	j
$\geq n$	even	odd $> n$ or even $< n$
$\geq n$	odd	odd $< n$ or even $> n$
$< n$	even	odd $< n$ or even $> n$
$< n$	odd	odd $> n$ or even $< n$

Taking now $a = \alpha_i$ with $i \neq n-1$, the reader will have no difficulty in deducing from this table that if $y = x(j)y(i(a))$ satisfies conditions $2(b)$ and $2(c)$, then h vanishes on y. This finishes assertion 2).

On the other hand, note that if $x = x(j)c^{r(i,j)}y(\tau(i))$ is an element of $e_j\widetilde{B}$ such that $x^\diamond = x(i)y(j)$, then the equality $\deg(x) + \deg(x^\diamond) = 2n-2$ gives that $r(i,j) = n-1-|j-n|-|i-n|$ which is clearly congruent with $i+j+1 \pmod 2$. If now $a \in Q_1$ is any arrow and $x = x(j)c^{r(j,t(a))}y(\tau(t(a)))$, then we deduce from the last table that h vanishes on x exactly when one of the conditions in the following table holds:

j	$t(a)$
$> n$	even $\leq n$
$> n$	odd $> n$
$< n$	even $> n$
$< n$	odd $< n$

If now $a = \beta_i$ is a crossing arrow with $i \neq n$ and x is such that $x^\diamond = x(t(a))y(j)$ with $j \neq n$ and satisfying one of the conditions $1(b)$ or $1(c)$, then a simple verification shows that it satisfies some of the conditions in the last table and, hence, $h(x) = 0$. This finishes the proof.

□

Lemma 6.7.7. *The following diagrams of morphisms of Λ-bimodules are commutative:*

1.

$$\begin{array}{ccccc}
\oplus_{i\in Q_0}\Lambda e_i \otimes e_i\Lambda & \xrightarrow{\mu k} & \oplus_{i\in Q_0}\Lambda e_i \otimes e_i\Lambda & \xrightarrow{\mu R} & \oplus_{a\in Q_1}\Lambda e_{\tau(i(a))} \otimes e_{t(a)}\Lambda \\
\downarrow \xi & & \downarrow \eta & & \downarrow \widehat{u} \\
\oplus_{i\in Q_0}\Lambda e_{\tau(i)} \otimes e_i\Lambda & \xrightarrow[R]{} & \oplus_{a\in Q_1}\Lambda e_{i(a)} \otimes e_{t(a)}\Lambda & \xrightarrow[\delta]{} & \oplus_{i\in Q_0}\Lambda e_i \otimes e_i\Lambda
\end{array}$$

2.

$$\begin{array}{ccccc}
\oplus_{i\in Q_0}\Lambda e_{\tau(i)} \otimes e_i\Lambda & \xrightarrow{k} & \oplus_{i\in Q_0}\Lambda e_{\tau(i)} \otimes e_i\Lambda & \xrightarrow{R} & \oplus_{a\in Q_1}\Lambda e_{i(a)} \otimes e_{t(a)}\Lambda \\
\downarrow g & & \downarrow f & & \downarrow \widehat{y} \\
\oplus_{i\in Q_0}\Lambda e_{\tau(i)} \otimes e_i\Lambda & \xrightarrow[R]{} & \oplus_{a\in Q_1}\Lambda e_{i(a)} \otimes e_{t(a)}\Lambda & \xrightarrow[\delta]{} & \oplus_{i\in Q_0}\Lambda e_i \otimes e_i\Lambda
\end{array}$$

3.

$$\begin{array}{ccccc}
\oplus_{i\in Q_0}\Lambda e_i\otimes e_i\Lambda & \xrightarrow{{}_\mu R} & \oplus_{a\in Q_1}\Lambda e_{\tau(i(a))}\otimes e_{t(a)}\Lambda & \xrightarrow{{}_\mu \delta} & \oplus_{i\in Q_0}\Lambda e_{\tau(i)}\otimes e_i\Lambda\ , \\
\downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \widehat{t}} \\
\oplus_{i\in Q_0}\Lambda e_{\tau(i)}\otimes e_i\Lambda & \xrightarrow[R]{} & \oplus_{a\in Q_1}\Lambda e_{i(a)}\otimes e_{t(a)}\Lambda & \xrightarrow[\delta]{} & \oplus_{i\in Q_0}\Lambda e_i\otimes e_i\Lambda
\end{array}$$

where the vertical morphisms are determined by their action on the generators of the domain as follows:

1. $\hat{u}(e_{\tau(i(a))}\otimes e_{t(a)})=\frac{1}{2}(\delta_{a,\alpha_n}e_n\otimes\alpha_n-\delta_{\alpha,\beta_n}\beta_{n-1}\otimes e_n)$, $\eta(e_i\otimes e_i)=\frac{1}{2}\delta_{in}(e_n\otimes\beta_n+\alpha_{n-1}\otimes e_n)$ *and* $\xi(e_j\otimes e_j)=\sum_{x\in e_j\tilde{\mathcal{B}}}h(x)x\otimes x^\diamond$, *where* $h:\tilde{\mathcal{B}}\longrightarrow K$ *is the coefficient map;*

2. $\hat{y}(e_{i(a)}\otimes e_{t(a)})=\frac{1}{2}(a\otimes e_{t(a)}+e_{i(a)}\otimes a)$, $f(e_{\tau(i)}\otimes e_i)=-\frac{1}{2}\sum_{t(a)=i}(-1)^{s(\sigma(a)a)}(\sigma(a)\otimes e_i-e_{\tau(i)}\otimes a)$ *and* $g(e_{\tau(i)}\otimes e_i)=\sum_{x\in e_i\mathcal{B}}(-1)^{c(x)}deg(x)\tau(x)\otimes x^*$, *where* $c(x)$ *denotes the number of noncrossing arrows in* x*;*

3. $\hat{t}(e_{\tau(i)}\otimes e_i)=\delta_{in}e_n\otimes w_n$, $\varphi(e_{\tau(i(a))}\otimes e_{t(a)})=(\delta_{a,\beta_n}+\delta_{a,\beta_{n-1}})e_{\tau(i(a))}\otimes w_n$ *and* $\psi(e_i\otimes e_i)=\delta_{in}e_n\otimes w_n$.

Proof. 1) We first prove that $\delta\circ\eta=\hat{u}\circ{}_\mu R$. An easy direct computation shows that $(\delta\circ\eta)(e_i\otimes e_i)=\frac{1}{2}(\alpha_{n-1}\beta_n\otimes e_n-e_n\otimes\alpha_{n-1}\beta_n)$. It will be convenient to write ${}_\mu R(e_i\otimes e_i)=\sum_{a\in Q_1}x_a(i)$, with $x_a(i)\in\Lambda e_{\tau(i(a))}\otimes e_{t(a)}\Lambda$, for all $a\in Q_1$. One readily sees from the definition of ${}_\mu R$ that $x_a(i)=(-1)^{s(\sigma(a)a)}\mu(\sigma(a))\otimes e_{t(a)}$, when $t(a)=i$, that $x_a(i)=(-1)^{s(a\sigma^{-1}(a))}e_{\tau(i(a))}\otimes\sigma^{-1}(a)$, when $i(a)=\tau(i)$, and that $x_a(i)=0$ otherwise. But, due to the fact that $\hat{u}$ vanishes on $\Lambda e_{\tau(i(a))}\otimes e_{t(a)}\Lambda$, for $a\neq\alpha_n,\beta_n$, we get that $(\hat{u}\circ{}_\mu R)(e_i\otimes e_i)=\sum_{a\in Q_1}\hat{u}(x_a(i))=\hat{u}(x_{\alpha_n}(i))+\hat{u}(x_{\beta_n}(i))$. But one has $\{t(\alpha_n),t(\beta_n)\}=\{n+1,n\}$ while $\{i\in Q_0:\tau(i)\in\{i(\alpha_n),i(\beta_n)\}\}=\{n,n+1\}$. It follows that $(\hat{u}\circ{}_\mu R)(e_i\otimes e_i)=0$, for all vertices $i\neq n,n+1$.

For $i=n+1$, we have $x_{\alpha_n}(n+1)=(-1)^{s(\sigma(\alpha_n)\alpha_n)}\mu(\sigma(\alpha_n))\otimes e_{t(\alpha_n)}=\tau(\beta_n)\otimes e_{n+1}=\beta_{n-1}\otimes e_{n+1}$ and $x_{\beta_n}(n+1)=(-1)^{s(\beta_n\sigma^{-1}(\beta_n))}e_{\tau(i(\beta_n))}\otimes\sigma^{-1}(\beta_n)=e_{n+1}\otimes\alpha_n$, using Lemma 6.5.9 and the fact that $s(\alpha_n)=s(\beta_n)=0$. It follows that

$$(\hat{u}\circ{}_\mu R)(e_{n+1}\otimes e_{n+1})=\hat{u}(\beta_{n-1}\otimes e_{n+1})+\hat{u}(e_{n+1}\otimes\alpha_n)=\tfrac{1}{2}\beta_{n-1}\otimes\alpha_n-\tfrac{1}{2}\beta_{n-1}\otimes\alpha_n=0.$$

On the other hand, we have the equality $x_{\alpha_n}(n)=(-1)^{s(\alpha_n\sigma^{-1}(\alpha_n))}e_{\tau(i(\alpha_n))}\otimes\sigma^{-1}(\alpha_n)=(-1)^{s(\alpha_n\beta_{n-1})}e_n\otimes\beta_{n-1}=-e_n\otimes\beta_{n-1}$, while $x_{\beta_n}(n)=(-1)^{s(\sigma(\beta_n)\beta_n)}\mu(\sigma(\beta_n))\otimes e_{t(\beta_n)}=(-1)^{s(\alpha_{n-1}\beta_n)}[-\tau(\alpha_{n-1})\otimes e_n=-\alpha_n\otimes e_n$, using again Lemma 6.5.9 and the fact that $0=s(\alpha_n)=s(\alpha_{n-1})=s(\beta_n)\neq s(\beta_{n-1})$. We then get

$$(\hat{u}\circ{}_\mu R)(e_n\otimes e_n)=-\hat{u}(\alpha_n\otimes e_n)-\hat{u}(e_n\otimes\beta_{n-1})=\tfrac{1}{2}\alpha_n\beta_{n-1}\otimes e_n-\tfrac{1}{2}e_n\otimes\alpha_n\beta_{n-1}.$$

This proves the desired equality $\delta\circ\eta=\hat{u}\circ{}_\mu R$ since $\alpha_n\beta_{n-1}=\alpha_{n-1}\beta_n$ in Λ.

We next prove the equality $\eta\circ{}_\mu k=R\circ\xi$. Again, an easy direct computation shows that $(\eta\circ{}_\mu k)(e_j\otimes e_j)=\frac{1}{2}\sum_{x\in e_j\mathcal{B}e_n}(x\otimes\beta_n x^*+x\alpha_{n-1}\otimes x^*)$. On the other hand, we have:

$$(R \circ \xi)(e_j \otimes e_j) = R(\sum_{i \in Q_0} \sum_{x \in e_j \tilde{\mathcal{B}}} h(x) x \otimes x^\diamond) =$$
$$\sum_{x \in e_j \tilde{\mathcal{B}} e_{\tau(i)}} h(x) x [\sum_{t(a)=i} (-1)^{s(\sigma(a)a)} (\sigma(a) \otimes e_i + e_{\tau(i)} \otimes a)] x^\diamond =$$
$$\sum_{a \in Q_1} \sum_{x \in e_j \tilde{\mathcal{B}} e_{\tau(t(a))}} (-1)^{s(\sigma(a)a)} h(x)(x\sigma(a) \otimes x^\diamond + x \otimes a x^\diamond).$$

This is an element of $\otimes_{a \in Q_1} \Lambda e_{i(a)} \otimes e_{t(a)} \Lambda$. Given $b \in Q_1$, the component of this element along the summand $\Lambda e_{i(b)} \otimes e_{t(b)} \Lambda$ is

$$\Psi_b(j) := \sum_{x \in e_j \tilde{\mathcal{B}} e_{\tau(t(b))}} (-1)^{s(\sigma(b)b)} h(x) x\sigma(b) \otimes x^\diamond +$$
$$\sum_{y \in e_j \tilde{\mathcal{B}} e_{i(b)}} (-1)^{s(b\sigma^{-1}(b))} h(y) y \otimes \sigma^{-1}(b). \qquad (*)$$

Note that if $x \in e_j \tilde{\mathcal{B}} e_{\tau(t(b))}$, then $x\sigma(b) \otimes x^\diamond \in e_j \Lambda_{d+1} e_{\tau(i(b))} \otimes e_{t(b)} \Lambda_{2n-2-d} e_j$, where $d = \deg(x)$. If $x\sigma(b) \otimes x^\diamond \neq 0$ is such that $x^\diamond \in \sigma^{-1}(b)\Lambda$, we will have that $x^\diamond = \sigma^{-1}(b)z$, for some nonzero path z from $\tau(i(b))$ to j. But, putting $y := x\sigma(b)$, we necessarily have that $y^\diamond = z$, so that $x\sigma(b) \otimes x^\diamond = y \otimes \sigma^{-1}(b) y^\diamond$. This means that $x\sigma(b) \otimes x^\diamond$ appears in the second summation $\Psi_b(j)$, although with coefficient $h(x\sigma(b))$. A symmetric argument shows that if $y \otimes \sigma^{-1}(b) y^\diamond \neq 0$ is such that $y \in \Lambda\sigma(b)$, then $y \otimes \sigma^{-1}(b) y^\diamond$ also appears in the first summation of the expression. So, under these hypothesis, the coefficient of $x\sigma(b) \otimes x^\diamond = (x\sigma(b)) \otimes \sigma^{-1}(b)(x\sigma(b))^\diamond$ in $\Psi_b(j)$ will be $(-1)^{s(\sigma(b)b)} h(x) + (-1)^{s(b\sigma^{-1}(b))} h(x\sigma(b))$ (**). Moreover, notice that $(-1)^{s(\sigma(b)b)}$ and $(-1)^{s(b\sigma^{-1}(b))}$ are equal, when b is a crossing arrow, and opposite to each other, when b is noncrossing. Then , whenever $b \neq \alpha_{n-1}, \beta_n$, the expression is zero, because the map h satisfies the conditions 3.a and 3.b of Lemma 6.7.5, even when $j = n$.

We claim that $\Psi_b(j) = 0$ for all $j \in Q_0$ and $b \in Q_1 \backslash \{\alpha_{n-1}, \beta_n\}$. Assume first that b is a noncrossing arrow $\neq \alpha_{n-1}$ in the formula (*). Then Lemma 6.7.6 tells us that the argument of the last paragraph works for any element $x \in e_j \widetilde{B} e_{\tau(t(b))}$ and any element $y \in e_j \widetilde{B} e_{i(b)}$ except when $y = x(j) y(i(b))$ and one of the three conditions $2(a)$, $2(b)$ or $2(c)$ in Lemma 6.7.6 is satisfied. Note that when $j = n$, we have that $y = y(i(b))$ and $y \otimes \sigma^{-1}(b)(y)^\diamond = 0$ since $\sigma^{-1}(b)(y)^\diamond = 0$. Hence, also in this case it is true that any nonzero term of the form $y \otimes \sigma^{-1}(b) y^\diamond$, with $y \in e_n \tilde{\mathcal{B}} e_{i(b)}$, is equal to one of the form $x\sigma(b) \otimes x^\diamond$. But, if we assume that $j \neq n$ and $y = x(j) y(i(b)) \neq 0$ satisfies one of the conditions $2(b)$ or $2(c)$ then, using Lemma 6.7.6 (2), we have in both cases that $h(y) = 0$. This means that the summand $h(y) y \otimes \sigma^{-1}(b) y^\diamond$ is zero and hence it follows that $\Psi_b(j) = 0$ for any $j \in Q_0$ and any noncrossing arrow b different from α_{n-1}.

Suppose next that b is a crossing arrow $\neq \beta_n$ in the formula (*). Then, by Lemma 6.7.6, we know that each nonzero term $y \otimes \sigma^{-1}(b) y^\diamond$ in the formula of $\Psi_b(j)$ is of the form $x\sigma(b) \otimes x^\diamond$. Proceeding as in the previous paragraph, we see that the only case in which a term $x\sigma(b) \otimes x^\diamond$ is not of the form $y \otimes \sigma^{-1}(b) y^\diamond$ is precisely when $x^\diamond = x(t(a)) y(j)$ and one of the conditions $1(b)$ or $1(c)$ is satisfied. But then Lemma 6.7.6 tells us that $h(x) = 0$. Therefore we also have $\Psi_b(j) = 0$ in this case.

We next deal with the cases $b = \alpha_{n-1}$ and $b = \beta_n$. Let us consider first that $b = \alpha_{n-1}$ and assume that $j \neq n$. Given $x \in e_j \widetilde{B} e_{n+1}$ such that $x\sigma(\alpha_{n-1}) \otimes x^\diamond \neq 0$, we will need to know the pair $(h(x), h(x\sigma(\alpha_{n-1}))) = (h(x), h(x\beta_{n-1})$. Putting $x = x(j) c^r \alpha_{n+1}$, we have that $x\beta_{n-1} = x(j) c^{r+1}$. Looking at definition 23 and the explicit definition of $h_{|e_j \tilde{B}}$ given in the proof of Lemma 6.7.5, and putting $a := s(\beta_{n-1}\alpha_{n-1})$ and $b := s(\alpha_{n-1}\beta_n)$,we have the following table:

Condition	r	$(h(x), h(x\beta_{n-1})$	$(-1)^a h(x) + (-1)^b h(x\beta_{n-1}))$
$j > n$ odd or $j < n$ even	even	$(0, \frac{1}{2})$	$\frac{1}{2}$
$j > n$ odd or $j < n$ even	odd	$(-\frac{1}{2}, 0)$	$\frac{1}{2}$
$j < n$ odd or $j > n$ even	odd	$(0, \frac{1}{2})$	$\frac{1}{2}$
$j < n$ odd or $j > n$ even	even	$(-\frac{1}{2}, 0)$	$\frac{1}{2}$

By Lemma 6.7.6, we know that any nonzero summand in (*) of the form $x\sigma(\alpha_{n-1}) \otimes x^\diamond$ with $x \in e_j\widetilde{B}e_{n+1}$ is of the form $y \otimes \sigma^{-1}(\alpha_n)y^\diamond = y \otimes \beta_n y^*$ for some $y \in e_j\widetilde{B}e_n$. That summand appears then with coefficient $(-1)^{s(\beta_{n-1}\alpha_{n-1})}h(x) + (-1)^{s(\alpha_{n-1}\beta_n)}h(x\beta_{n-1}) = \frac{1}{2}$ in $\Psi_{\alpha_{n-1}}(j)$. However, if $y = x(j)$ and one of the conditions 2.a or 2.b in Lemma 6.7.6 is satisfied, then the summand $y \otimes \sigma^{-1}(\alpha_{n-1})y^\diamond$ is not of the form $x\sigma(\alpha_{n-1}) \otimes x^\diamond$. Note that in this case either $j < n$ and j odd or $j > n$ and j even, which implies that $h(y) = h(x(j)) = \frac{1}{2}$, by definition 23. We then conclude that $\Psi_{\alpha_{n-1}}(j) = \frac{1}{2}\sum_{y \in e_j\tilde{B}e_n} y \otimes \sigma^{-1}(\alpha_{n-1})y^\diamond = \frac{1}{2}\sum_{y \in e_jBe_n} y \otimes \beta_n y^*$.

To finish the case when $b = \alpha_{n-1}$, suppose finally that $j = n$. If $y = x(n) = e_n$, which is not in $\Lambda\beta_{n-1}$, then $e_n \otimes \sigma^{-1}(\alpha_{n-1})e_n^\diamond = e_n \otimes \beta_n w_n = 0$ and there is nothing left to prove. On the contrary, any $x\sigma(\alpha_{n-1}) \otimes x^\diamond$ with $x \in e_n\widetilde{B}e_{n+1}$ is of the form $y \otimes \sigma^{-1}(\alpha_{n-1})y^\diamond$ for some $y \in e_n\widetilde{B}e_n$, and viceversa. So we can write $x = c^r\alpha_n$ and $x\beta_{n-1} = c^{r+1}$ and the definition of h implies that $(h(x), h(x\beta_{n-1})) = (-\frac{1}{4}, \frac{1}{4})$. It follows that if $y \in e_n\mathcal{B}e_n \setminus \{e_n\}$, then we always have a decomposition $y = x\beta_{n-1}$, with $x \in e_n\tilde{\mathcal{B}}e_{n+1}$, and the coefficient of $y \otimes \sigma^{-1}(\alpha_{n-1})y^* = x\sigma(\alpha_{n-1}) \otimes x^\diamond$ in the expression of $\Psi_{\alpha_{n-1}}(n)$ is $-h(x) + h(x\beta_{n-1}) = \frac{1}{2}$. Then we have $\Psi_{\alpha_{n-1}}(n) = \frac{1}{2}\sum_{y \in e_nBe_n} y \otimes \beta_n y^*$.

Finally suppose that $b = \beta_n$ and consider that $j \neq n$. By Lemma 6.7.6, we know that if $y \in e_j\tilde{\mathcal{B}}e_{n-1}$ and the tensor $y \otimes \sigma^{-1}(\beta_{n-1})y^\diamond = y \otimes \alpha_{n-1}y^\diamond$ is nonzero, then we always have $y = x\sigma(\beta_n) = x\alpha_{n-1}$, for some $x \in e_j\mathcal{B}e_n$. It follows that the term $y \otimes \sigma^{-1}(\beta_n)y^\diamond$ of the formula (*) is equal to one of the form $x\sigma(\beta_n) \otimes x^\diamond$. Lemma 6.7.6 says that the converse is also true, except in case $x^\diamond = x(t(a))y(j) = x(n)y(j) = y(j)$ and conditions 1.b or $1(c)$ in the mentioned lemma are satisfied. But condition $1(b)$ is discarded since n is even and, hence, this exceptional situation appears exactly when $j < n$ and $x^\diamond = y(j)$. We next put $x = x(j)c^r$, and so $x\sigma(\beta_n) = x(j)c^r\alpha_{n-1}$. Using the explicit definition of h and putting $c := s(\alpha_{n-1}\beta_n)$ and $d := s(\beta_n\alpha_n)$, we have the following table in the non-exceptional cases, when $j \neq n$:

Condition	r	$(h(x), h(x\alpha_{n-1})$	$(-1)^c h(x) + (-1)^d h(x\alpha_{n-1}))$
$j > n$ odd or $j < n$ even	even	$(0, \frac{1}{2})$	$\frac{1}{2}$
$j > n$ odd or $j < n$ even	odd	$(\frac{1}{2}, 0)$	$\frac{1}{2}$
$j < n$ odd or $j > n$ even	odd	$(0, \frac{1}{2})$	$\frac{1}{2}$
$j < n$ odd or $j > n$ even	even	$(\frac{1}{2}, 0)$	$\frac{1}{2}$

which shows that, except for the case when $j < n$ and $x^\diamond = y(j)$, the coefficient of $x\sigma(\beta_n) \otimes x^\diamond$ in $\Psi_{\beta_n}(j)$ is $\frac{1}{2}$. But, when $x^\diamond = y(j)$ and $j < n$, we have that $x = x(j)c^r$, where $r = n - 1 - |n - j|$. It is clear that $r \equiv j + 1 \pmod 2$. Looking again at the explicit definition of h, we get that $h(x) = \frac{1}{2}$ in this case. It follows that $\Psi_{\beta_n}(j) = \frac{1}{2}\sum_{x\in e_j\mathcal{B}e_{\tau(t(\beta_n))}} x\sigma(\beta_n)\otimes x^\diamond = \frac{1}{2}\sum_{x\in e_j\mathcal{B}e_n} x\alpha_{n-1}\otimes x^*$.

Still with $b = \beta_n$, suppose now that $j = n$. Each nonzero term $y\otimes\sigma^{-1}(\beta_n)y^\diamond$ in $\Psi_{\beta_n}(n)$ is of the form $x\sigma(\beta_n)\otimes x^\diamond$. The converse is also true because the only possible exceptional case would be when $x^\diamond = y(n) = e_n$, in which case $x\sigma(\beta_n) = w_n\alpha_{n-1} = 0$. Then the coefficient of a nonzero tensor $x\sigma(\beta_n)\otimes x^\diamond$ is $h(x) + h(x\alpha_{n-1})$. Note that we have $x = c^r$ and $x\sigma(\beta_n) = c^r\alpha_{n-1}$, where $0\leq r < n-1$. By definition 23, we have that $(h(x), h(x\alpha_{n-1}) = (\frac{1}{4},\frac{1}{4})$, for each $x\in e_n\mathcal{B}\setminus\{w_n\}$. We then get that $\Psi_{\beta_n}(n) = \frac{1}{2}\sum_{x\in e_n\mathcal{B}e_n} x\alpha_{n-1}\otimes x^*$.

From all the previous paragraphs we get that, for any $j\in Q_0$, the following equality holds:

$$\begin{gathered}(R\circ\xi)(e_j\otimes e_j) = \textstyle\sum_{b\in Q_1}\Psi_b(j) = \Psi_{\alpha_{n-1}}(j) + \Psi_{\beta_n}(j) = \\ \frac{1}{2}\textstyle\sum_{x\in e_j\mathcal{B}e_n} x\otimes\beta_n x^* + \frac{1}{2}\sum_{x\in e_j\mathcal{B}e_n} x\alpha_{n-1}\otimes x^* = (\eta\circ {}_\mu k)(e_j\otimes e_j).\end{gathered}$$

This ends the proof of the commutativity of the first diagram in the statement.

2) It is easy to check the commutativity of the right square in the second diagram. For the left square, note that we have:

$$\begin{gathered}(f\circ k)(e_{\tau(i)}\otimes e_i) = \textstyle\sum_{j\in Q_0}\sum_{x\in e_i\mathcal{B}e_j}(-1)^{c(x)}\tau(x)f(e_{\tau(j)}\otimes e_j)x^* = \\ -\frac{1}{2}\textstyle\sum_{j\in Q_0}\sum_{x\in e_i\mathcal{B}e_j}(-1)^{c(x)}\tau(x)[\sum_{t(a)=j}(-1)^{s(\sigma(a)a)}(\sigma(a)\otimes e_j - e_{\tau(j)}\otimes a)]x^* = \\ -\frac{1}{2}(\textstyle\sum_{a\in Q_1}S^+(i,a) - \sum_{a\in Q_1}S^-(i,a)) = -\frac{1}{2}\sum_{a\in Q_1}[S^+(i,a) - S^-(i,\sigma^{-1}(a)],\end{gathered}$$

where

$$\begin{gathered}S^+(i,a) = \textstyle\sum_{x\in e_i\mathcal{B}e_{t(a)}}(-1)^{c(x)+s(\sigma(a)a)}\tau(x)\sigma(a)\otimes x^* \\ S^-(i,a) = \textstyle\sum_{x\in e_i\mathcal{B}e_{t(a)}}(-1)^{c(x)+s(\sigma(a)a)}\tau(x)\otimes ax^*,\end{gathered}$$

for each $i\in Q_0$ and $a\in Q_1$. Let us fix $a\in Q_1$ and let $x\in e_i\mathcal{B}e_{t(a)}$ be an element such that $\tau(x)\sigma(a)\otimes x^*\neq 0$. If we put $y = x\sigma^{-1}a$ then $\sigma^{-1}(a)y^* = x^*$ and we have $\tau(y)\otimes\sigma^{-1}(a)y^* = \tau(x)\sigma(a)\otimes x^*$. This element appears then with nonzero coefficient both as a summand of $S^+(i,a)$ and as a summand of $S^-(i,\sigma^{-1}(a))$. It then follows that

$$\begin{gathered}S^+(i,a) - S^-(i,\sigma^{-1}(a)) = \\ \textstyle\sum_{x\in e_i\mathcal{B}e_{t(a)}}[(-1)^{c(x)+s(\sigma(a)a)} - (-1)^{c(x\sigma^{-1}(a))+s(a\sigma^{-1}(a))}]\tau(x)\sigma(a)\otimes x^*,\end{gathered}$$

and hence

$$\begin{gathered}(f\circ k)(e_{\tau(i)}\otimes e_i) = \\ -\frac{1}{2}\textstyle\sum_{a\in Q_1}\sum_{x\in e_i\mathcal{B}e_{t(a)}}[(-1)^{c(x)+s(\sigma(a)a)} - (-1)^{c(x\sigma^{-1}(a))+s(a\sigma^{-1}(a))}]\tau(x)\sigma(a)\otimes x^*.\end{gathered}$$

We use the definition of $c(x)$ and the fact that $s(\sigma(a)a) = s(a\sigma^{-1}(a))$, for a crossing, and $s(\sigma(a)a)\neq s(a\sigma^{-1}(a))$, when a is noncrossing, to see that $(-1)^{c(x)+s(\sigma(a)a)}$ and $(-1)^{c(x\sigma^{-1}(a))+s(a\sigma^{-1}(a))}$ always have opposite signs. Then we get:

$$(f\circ k)(e_{\tau(i)}\otimes e_i) = -\textstyle\sum_{a\in Q_1}\sum_{x\in e_i\mathcal{B}e_{t(a)}}[(-1)^{c(x)+s(\sigma(a)a)}\tau(x)\sigma(a)\otimes x^*.$$

On the other hand, we have

$$(R \circ g)(e_{\tau(i)} \otimes e_i) = \sum_{j \in Q_0} \sum_{x \in e_i \mathcal{B} e_j} (-1)^{c(x)} \deg(x) \tau(x) R(e_{\tau(j)} \otimes e_j) x^* =$$

$$\sum_{j \in Q_0} \sum_{x \in e_i \mathcal{B} e_j} (-1)^{c(x)} \deg(x) \tau(x) [\sum_{t(a)=j} (-1)^{s(\sigma(a)a)} (\sigma(a) \otimes e_j + e_{\tau(j)} \otimes a)] x^* =$$

$$\sum_{a \in Q_1} T^+(i,a) + \sum_{a \in Q_1} T^-(i,a) = \sum_{a \in Q_1} (T^+(i,a) + T^-(i, \sigma^{-1}(a))),$$

where

$$T^+(i,a) = \textstyle\sum_{x \in e_i \mathcal{B} e_{t(a)}} (-1)^{c(x)+s(\sigma(a)a)} \deg(x) \tau(x) \sigma(a) \otimes x^*$$
$$T^-(i,a) = \textstyle\sum_{x \in e_i \mathcal{B} e_{t(a)}} (-1)^{c(x)+s(\sigma(a)a)} \deg(x) \tau(x) \otimes a x^*.$$

Proceeding as in the case of the $S(i,a)$, we get

$$T^+(i,a) + T^-(i, \sigma^{-1}(a)) =$$
$$\textstyle\sum_{x \in e_i \mathcal{B} e_{t(a)}} [(-1)^{c(x)+s(\sigma(a)a)} \deg(x) + (-1)^{c(x\sigma^{-1}(a))+s(a\sigma^{-1}(a))} \deg(x\sigma^{-1}(a))] \tau(x) \sigma(a) \otimes x^*$$
$$= \textstyle\sum_{x \in e_i \mathcal{B} e_{t(a)}} [(-1)^{c(x)+s(\sigma(a)a)} \deg(x) + (-1)^{c(x\sigma^{-1}(a))+s(a\sigma^{-1}(a))} (\deg(x)+1)] \tau(x) \sigma(a) \otimes x^* =$$
$$= \textstyle\sum_{x \in e_i \mathcal{B} e_{t(a)}} (-1)^{c(x\sigma^{-1}(a))+s(a\sigma^{-1}(a))} \tau(x) \sigma(a) \otimes x^*,$$

because $(-1)^{c(x)+s(\sigma(a)a)}$ and $(-1)^{c(x\sigma^{-1}(a))+s(a\sigma^{-1}(a))}$ have opposite signs. But this same reason proves the equality

$$(R \circ g)(e_{\tau(i)} \otimes e_i) = \textstyle\sum_{a \in Q_1} (T^+(i,a) + T^-(i, \sigma^{-1}(a))) =$$
$$\textstyle\sum_{a \in Q_1} \sum_{x \in e_i \mathcal{B} e_{t(a)}} (-1)^{c(x\sigma^{-1}(a))+s(a\sigma^{-1}(a))} \tau(x) \sigma(a) \otimes x^* =$$
$$- \textstyle\sum_{a \in Q_1} \sum_{x \in e_i \mathcal{B} e_{t(a)}} [(-1)^{c(x)+s(\sigma(a)a)} \tau(x) \sigma(a) \otimes x^* = (f \circ k)(e_{\tau(i)} \otimes e_i),$$

which gives the desired commutativity of the left square of the diagram.

3) In order to prove the commutativity of the third diagram, note that $(\hat{t} \circ {}_\mu \delta)(e_{\tau(i(a))} \otimes e_{t(a)}) = \hat{t}(\mu(a) \otimes e_{t(a)}) - \hat{t}(e_{\tau(i(a))} \otimes a)$ has its second summand equal to zero, due to the definition of $\hat{t}$. Then $(\hat{t} \circ {}_\mu \delta)(e_{\tau(i(a))} \otimes e_{t(a)}) \neq 0$ implies that $t(a) = n$, whence, that $a = \beta_n$ or $a = \beta_{n-1}$. An easy computation shows that $(\hat{t} \circ {}_\mu \delta)(e_{\tau(i(\beta_n))} \otimes e_{t(\beta_n)}) = \beta_{n-1} \otimes w_n$ while $(\hat{t} \circ {}_\mu \delta)(e_{\tau(i(\beta_{n-1}))} \otimes e_{t(\beta_{n-1})}) = \beta_n \otimes w_n$. On the other hand, by definition of φ, we know that $\delta \circ \varphi$ vanishes on $e_{\tau(i(a))} \otimes e_{t(a)}$, for all arrows $a \neq \beta_n, \beta_{n-1}$. Moreover, $(\delta \circ \varphi)((e_{\tau(i(\beta_n))} \otimes e_{t(\beta_n)}) = \delta(e_{n+1} \otimes w_n) = (\beta_{n-1} \otimes e_{t(\beta_{n-1})} - e_{i(\beta_{n-1})} \otimes \beta_{n-1}) w_n = \beta_{n-1} \otimes w_n$. An analogous argument for β_{n-1} shows that $(\delta \circ \varphi)((e_{\tau(i(\beta_{n-1}))} \otimes e_{t(\beta_{n-1})}) = \beta_n \otimes w_n$ and, hence, the right square of the diagram above is commutative.

For the commutativity of the left square, note that $\varphi \circ {}_\mu R$ vanishes on $e_i \otimes e_i$, for $i \neq n$, and an easy computation using Lemma 6.5.9 shows that $(\varphi \circ {}_\mu R)(e_n \otimes e_n) = -\alpha_n \otimes w_n + \alpha_{n-1} \otimes w_n$. On the other hand, $R \circ \psi$ also vanishes on all $e_i \otimes e_i$, with $i \neq n$, and we have an equality

$(R \circ \psi)(e_n \otimes e_n) = R(e_n \otimes w_n) = R(e_{\tau(n)} \otimes e_n)w_n = [\sum_{t(a)=n} (-1)^{s(\sigma(a)a)}(\sigma(a) \otimes e_n) + e_n \otimes a)]w_n = (-1)^{s(\alpha_{n-1}\beta_n)}\alpha_{n-1} \otimes w_n + (-1)^{s(\alpha_n\beta_{n-1})}\alpha_n \otimes w_n = \alpha_{n-1} \otimes w_n - \alpha_n \otimes w_n.$

□

From our results given in Chapter 4, we know that $\underline{HH}^*(\Lambda)$ is a graded pseudo-Frobenius algebra admitting a graded Nakayama form whose degree function takes constant value 5.

In Proposition 1.3.7 there was given an explicit form of constructing a graded Nakayama form for $\underline{HH}^*(\Lambda)$. Note that, due to the graded commutativity, the only nonzero idempotent of $\underline{HH}^*(\Lambda)$ is 1 and so we apply the mentioned proposition of [op.cit.] with $I = \{1\}$ and $\nu = id_I$. Let $\mathcal{B}_5$ be a basis of $HH^5(\Lambda) = \underline{HH}^5(\Lambda)$ containing a nonzero element, say ω, of $\text{Soc}_{gr}(\underline{HH}^*(\Lambda))^5$, the homogeneous component of (homological) degree 5 of the graded socle of $\text{Soc}_{gr}(\underline{HH}^*(\Lambda))$. Then the *graded Nakayama form associated to* $\mathcal{B}_5$ is the graded bilinear form $(-,-) : \underline{HH}^*(\Lambda) \times \underline{HH}^*(\Lambda) \longrightarrow K$ which acts on pairs of homogeneous elements as follows:

1. $(f, g) = 0$ whenever $\deg(f) + \deg(g) \neq 5$
2. (f, g) is the coefficient of w in the expression of fg as a K-linear combination of the elements of $\mathcal{B}_5$, whenever $\deg(f) + \deg(g) = 5$.

Lemma 6.7.8. *Let us put $\hat{x} = x + Im(_\mu R^*)$, for each $x \in \oplus_{i \in Q_0} e_i \Lambda e_i$. The following assertions hold:*

1. *If c is the cycle of length 2 at the vertex n and x is as in Proposition 6.6.2, then $c \in Ker(_\mu k^*)$ and $x^{\frac{n}{2}-1}\hat{c} = \hat{w}_i \neq 0$ in $HH^5(\Lambda)$, for all $i \in Q_0$. If this element is denoted by $\hat{w}$ then $Soc_{gr}(\underline{HH}^*(\Lambda))^5 = K\hat{w}$.*
2. *Let us consider the elements*
   (a) $v_{2k} = \hat{e}_{2k} - \hat{e}_{2n-2k}$
   (b) $v_{2k-1} = \hat{e}_{2k-1} - \hat{e}_{2k+1} - \hat{e}_{2n-2k+1} + \hat{e}_{2n-2k-1}$,

   for all $0 < k < \frac{n}{2}$. Then $v_j \in HH^5(\Lambda)$, for all $j = 1, ..., n-2$, and $\mathcal{B}_5 = \{x^{\frac{n}{2}-1}\hat{c}, ..., x\hat{c}, \hat{c}, v_1, ..., v_{n-2}\}$ is a basis of $HH^5(\Lambda)$.
3. *Let us put $x_{2k} = w_{2k} + I$ and $x_{2k-1} = \sum_{1 \leq r \leq k} w_{2r-1} + I$, for all $0 < k < \frac{n}{2}$, and let $(-,-) : \underline{HH}^*(\Lambda) \times \underline{HH}^*(\Lambda) \longrightarrow K$ be the graded Nakayama form associated to $\mathcal{B}_5$. Then $\bar{\mathcal{B}}_0 = \{1, x, ..., x^{\frac{n}{2}-1}, x_1, ..., x_{n-2}\}$ is a basis of $\underline{HH}^0(\Lambda) = Z(\Lambda)/I$ which is dual of $\mathcal{B}_5$ with respect to restriction of $(-,-)$ to $\underline{HH}^0(\Lambda) \times \underline{HH}^5(\Lambda)$.*

Proof. 1) and 2): By Proposition 6.5.11, we know that $c \in \text{Ker}(_\mu k^*)$, and then $x^{\frac{n}{2}-1}\hat{c} = \hat{w}_n$ since multiplication by elements of $Z(\Lambda)$ is done as in Λ. Bearing in mind that $\omega_i = c_i$ is the cycle of length $2n-2$ at i, in the last part of the proof of Proposition 6.6.7 it has been proved that $\omega_i - \omega_j \in \text{Im}(_\mu R^*)$, for all $i, j \in Q_0$, and that none of the w_i is in $\text{Im}(_\mu R^*)$. Note that we have proved also that $x^r\hat{c} \neq 0$ in $HH^5(\Lambda)$, for each integer $0 \leq r < \frac{n}{2}$.

On the other hand, we have ${}_\mu k^*(e_{2k} - e_{2n-2k}) = \sum_{i \in Q_0} c_{i,2k} - c_{i,2n-2k}$, where c_{ij} is the (i,j)-entry of the Cartan matrix of Λ. By Proposition 6.5.7, we know that $e_{2k} - e_{2n-2k} \in \mathrm{Ker}({}_\mu k^*)$. On the other hand, as mentioned in the proof of the same proposition, we have that $c_{i,2k-1} - c_{i,2n-2k+1} = 2$, for all $i \in Q_0$ and $0 < k < \frac{n}{2}$. We then get ${}_\mu k^*(e_{2k-1} - e_{2k+1} - e_{2n-2k+11} + e_{2n-2k-1}) = \sum_{i \in Q_0}(c_{i,2k-1} - c_{i,2k+1} - c_{i,2n-2k+1} + c_{i,2n-2k-1}) = 0$. Then the v_j $(j = 1, 2, ..., n-2)$ are well-defined elements of $HH^5(\Lambda)$.

Note now that the degrees of c and e_j as elements of the graded vector space $\oplus_{i \in Q_0} e_i \Lambda e_i$ $[2n+2]$ are $-2n$ and $-2n-2$, respectively. Then, with respect to the induced length degree in $\underline{HH}^*(\Lambda)$, we get that $\mathrm{ldeg}(x^r \hat{c}) = 4r - 2n$, for all $0 \leq r < \frac{n}{2}$, and $\mathrm{ldeg}(v_j) = -2n-2$, for all $1 \leq j \leq n-2$. Since these degrees are all different, in order to prove the linear independence of $\mathcal{B}_5$ it is enough to prove that $\{v_1, ..., v_{n-2}\}$ is linearly independent. But this is clear since $\mathrm{Im}({}_\mu R^*) \subseteq \oplus_{i \in Q_0} e_i J(\Lambda) e_i$ and non-trivial linear combination of the e_i can be in $J(\Lambda)$. That $\mathcal{B}_5$ is a basis of $HH^5(\Lambda)$ follows then from Corollary 6.6.11.

3) That $\bar{\mathcal{B}}_0$ is a basis of $\underline{HH}^0(\Lambda)$ follows directly from theorem 6.6.10 and proposition 6.6.2. On the other hand, the length degrees of $x^r v_j$ and $x_j x^r \hat{c}$ do not coincide with the length degrees of the elements of $\mathcal{B}_5$, which implies that $x^r v_j = 0 = x_j x^r \hat{c}$, and hence that $(x^r, v_j) = 0 = (x_j, x^r \hat{c})$ due to the definition of the graded Nakayama form associated to $\mathcal{B}_5$ (see the lines preceding this lemma). Moreover, if $0 \leq r, s < \frac{n}{2}$ then the coefficient of $\hat{w} = x^{\frac{n}{2}-1}\hat{c}$ in the expression of $x^r(x^s \hat{c}) = x^{r+s}\hat{c}$ as a linear combination of the elements of $\mathcal{B}_5$ is the Kronecker symbol $\delta_{r, \frac{n}{2}-1-s}$, which shows that $(x^r, x^s \hat{c}) = \delta_{r, \frac{n}{2}-1-s}$.

In order to end the proof, we just need to check that $(x_i, v_j) = \delta_{ij}$, for which it is enough to check that $x_i v_j = 0$, when $i \neq j$, and $x_j v_j = \hat{w}$ for all $j = 1, ..., n-2$. Since multiplication by elements of $Z(\Lambda)$ (and in particular by elements of $\mathrm{Soc}(\Lambda)$) is done as in Λ, we clearly have that $x_i v_j = 0$, when i and j are neither both odd nor both even. Moreover, if i and j are even, then we have $x_i v_j = \delta_{ij} \hat{w}_j = \delta_{ij} \hat{w}$. Finally, the product $x_{2k-1} v_{2t-1}$ is clearly zero when $t \neq k, k-1$. Moreover, we have $x_{2k-1} v_{2k-3} = \hat{w}_{2k-3} - \hat{w}_{2k-1} = 0$ and $x_{2k-1} v_{2k-1} = \hat{w}_{2k-1} = \hat{w}$.

□

In the sequel, we just use the graded Nakayama form associated to the basis $\mathcal{B}_5$ given by last lemma. As shown in Lemma 4.5.9, the following result holds.

Lemma 6.7.9. *Let f, g and h be homogeneous elements of $\underline{HH}^*(\Lambda)$ such that their homological degrees satisfy that $deg(f) + deg(g) + deg(h) = 5$. Then the following formula holds:*

$$(fg, h) = (-1)^{deg(g)deg(h)}(fh, g) = (-1)^{deg(f)(deg(g)+deg(h))}(gh, f).$$

We are ready to give the multiplicative structure of $HH^*(\Lambda)$ when n is even.

Theorem 6.7.10. *Let n be even, let Λ be the generalized preprojective algebra $\mathbb{B}_n$ over a field of characteristic $\neq 2$ and let view $HH^*(\Lambda)$ as a bigraded algebra (see Section 2). Let w_i be the cycle of length $2n-2$ at i, viewed as an element of $Soc(e_i\Lambda)$, for each $i \in Q_0$, and consider the following elements of $HH^*(\Lambda)$:*

1) $x, x_1, ..., x_{2n-1}$ of $HH^0(\Lambda) = Z(\Lambda)$ given as follows:

(a) $x = \sum_{3\leq i\leq 2n-3} c_i$, *where* c_i *is the nonzero cycle of length* 4 *at* i*;*

(b) $x_{2k} = w_{2k}$ *and* $x_{2k-1} = \sum_{1\leq r\leq k} w_{2r-1}$, *for all* $0 < k < \frac{n}{2}$*;*

(c) $\{x_{n-1}, x_n, ..., x_{2n-1}\}$ *is any basis of the subspace* I *of* Λ *generated by* $\{w_i + w_{2n-i} : 1 \leq i < n\} \cup \{w_n\} \cup \{v\}$, *with* $v = \sum_{1\leq k\leq \frac{n}{2}} (w_{2k-1} - w_{2n-2k+1})$.

2) $y \in HH^1(\Lambda)$ *represented by* $\tilde{y} : \oplus_{a\in Q_1}\Lambda e_{i(a)} \otimes e_{t(a)}\Lambda \longrightarrow \Lambda$, *where* $\tilde{y}(e_{i(a)} \otimes e_{t(a)}) = a$*;*

3) $z \in HH^2(\Lambda)$ *represented by* $\tilde{z} : \oplus_{i\in Q_0}\Lambda e_{\tau(i)} \otimes e_i\Lambda \longrightarrow \Lambda$, *where* $\tilde{z}(e_{\tau(i)} \otimes e_i) = \delta_{in}e_n$*;*

4) $t \in HH^3(\Lambda)$ *represented by* $\tilde{t} : \oplus_{i\in Q_0}\Lambda e_{\tau(i)} \otimes e_i\Lambda \longrightarrow \Lambda$, *where* $\tilde{t}(e_{\tau(i)} \otimes e_i) = \delta_{in}w_n$*;*

5) $u \in HH^4(\Lambda)$ *represented by* $\tilde{u} = \oplus_{a\in Q_1}\Lambda e_{\tau(i(a))}\otimes e_{t(a)}\Lambda \longrightarrow \Lambda$, *where* $\tilde{u}(e_{\tau(i(a))}\otimes e_{t(a)}) = \frac{1}{2}\delta_{a,\alpha_n}\alpha_n - \frac{1}{2}\delta_{a,\beta_n}\beta_{n-1}$*;*

6) $v_1, ..., v_{n-2} \in\in HH^5(\Lambda)$, *with each* v_j *is represented by* $\tilde{v}_j = \oplus_{i\in Q_0}\Lambda e_i \otimes e_i\Lambda \longrightarrow \Lambda$, *where:*

(a) $\tilde{v}_j(e_i \otimes e_i) = (\delta_{ij} - \delta_{i,2n-j})e_i$, *whenever* j *is even;*

(b) $\tilde{v}_j(e_i \otimes e_i) = (\delta_{ij} - \delta_{i,j+2} - \delta_{i,2n-j} + \delta_{i,2n-j-2})e_i$, *whenever* j *is odd.*

7) $h \in HH^6(\Lambda)$ *represented by the multiplication map* $\tilde{h} : \oplus_{i\in Q_0}\Lambda e_i \otimes e_i\Lambda \longrightarrow \Lambda$.

The length degrees of these elements are $ldeg(x) = 4$, $ldeg(x_i) = 2n - 2$, $ldeg(y) = 0$, $ldeg(z) = ldeg(t) = -2$, $ldeg(u) = -2n$, $ldeg(v_j) = -2n - 2$ *and* $ldeg(h) = -4n$.

Moreover, as an algebra, $HH^*(\Lambda)$ *is generated by these elements, subject to the graded commutativity relations with respect to the homological grading plus the following relations:*

a) $x^{\frac{n}{2}} = xz = xt = xv_i = 0$

b) $x_i\xi = 0$, *for each generator* ξ, *except in case* $i \leq n - 2$ *and* $\xi = v_i$ *or* $\xi = h$

c) $x_iv_i = x^{\frac{n}{2}-1}yu$, *for all* $i \leq n - 2$.

d) $y^2 = yt = 0$

e) $yz = -nt$

f) $yv_{2k} = 4nx_{2k-1}h$ *and* $yv_{2k-1} = -4nx_{2k}h$, *for all* $1 \leq k < \frac{n}{2}$

g) $z^2 = -nx^{\frac{n}{2}-1}u$

h) $zt = x^{\frac{n}{2}-1}yu$

i) $zv_j = 0$, *for all* $j = 1, ..., n - 2$

j) $zu = 0$

k) $t^2 = tu = tv_j = 0$, *for all* $j = 1, ..., n - 2$

l) $u^2 = 0$

m) $uv_j = 0$, *for all* $j = 1, ..., n-2$

n) $v_{2k}v_j = 4n\delta_{2k-1,j}x^{\frac{n}{2}}uh$, *for all* $1 \leq k < \frac{n}{2}$ *and* $j = 1, 2, ..., n-2$.

Proof. Let $P^{\cdot}$ be the minimal graded projective resolution of Λ. One gets the zero map when composing any of the morphisms $\tilde{y}$, $\tilde{z}$, $\tilde{t}$, $\tilde{v}_i$ $(i = 1, ..., n-2)$ and $\tilde{h}$ with the corresponding differential of $P^{\cdot}$. Then y, z, t, v_i $(i = 1, ..., n-2)$ and h are well-defined elements of $HH^*(\Lambda)$, which are easily seen to be nonzero. As for u, note that $\alpha_n \in e_{\mu(i(\alpha_n))}\Lambda e_{t(\alpha_n)}$ while $\beta_{n-1} \in e_{\mu(i(\beta_n))}\Lambda e_{t(\beta_n)}$, so that $\alpha_n - \beta_{n-1} \in \oplus_{a\in Q_1} e_{\mu(i(a))}\Lambda_1 e_{t(a)}$. We then have:

$$
{}_{\mu}R^*(\frac{1}{2}(\alpha_n - \beta_{n-1})) =
$$

$$
\tfrac{1}{2}[(-1)^{s(\beta_n\alpha_n)}\mu(\beta_n)\alpha_n + (-1)^{s(\alpha_n\beta_{n-1})}\alpha_n\beta_{n-1} - (-1)^{s(\alpha_{n-1}\beta_n)}\mu(\alpha_{n-1}\beta_{n-1}) - (-1)^{s(\beta_n\alpha_n)}\beta_{n-1}\alpha_n] =
$$

$$
\frac{1}{2}[\beta_{n-1}\alpha_n - \alpha_n\beta_{n-1}] - \frac{1}{2}[-\alpha_n\beta_{n-1} + \beta_{n-1}\alpha_n] = 0
$$

Then $\frac{1}{2}(\alpha_n - \beta_{n-1}) \in \mathrm{Ker}({}_{\mu}R^*)$ and so $\frac{1}{2}(\alpha_n - \beta_{n-1}) + \mathrm{Im}({}_{\mu}\delta^*)$ is an element of $HH^4(\Lambda)_1$, which clearly coincides with u. Moreover, it is a nonzero element since, by definition of ${}_{\mu}\delta^*$ (see Proposition 6.5.11), the image of the induced map ${}_{\mu}\delta^* : \oplus_{i\in Q_0} e_{\tau(i)}\Lambda_0 e_i = Ke_n \longrightarrow \oplus_{a\in Q_1} e_{\tau(i(a))}\Lambda_1 e_{t(a)}$ is the vector space generated by ${}_{\mu}\delta^*(e_n) = \mu(\beta_{n-1}) + \mu(\beta_n) - \alpha_n - \alpha_{n-1} = \beta_{n-1} + \beta_n - \alpha_n - \alpha_{n-1}$, which does not contain $\frac{1}{2}(\alpha_n - \beta_{n-1})$.

Arguing as in the initial part of the proof of Theorem 6.7.1, one easily gets that the length degrees of the proposed generators are as indicated. In order to see that they generate $HH^*(\Lambda)$ as an algebra, we first give a canonical basis $\mathcal{B}_i$ for each space $HH^i(\Lambda)$ $(i = 0, 1, ..., 6)$:

1. $\mathcal{B}_0 = \{1, x, ..., x^{\frac{n}{2}-1}, x_1, ..., x_{2n-1}\}$
2. $\mathcal{B}_1 = \{x^k y : k = 0, 1, ..., \frac{n}{2} - 1\}$
3. $\mathcal{B}_2 = \{z\}$
4. $\mathcal{B}_3 = \{t\}$
5. $\mathcal{B}_4 = \{x^{\frac{n}{2}-1}u, \ldots, xu, u\}$
6. $\mathcal{B}_5 = \{x^{\frac{n}{2}-1}\hat{c}, \ldots, x\hat{c}, \hat{c}\} \cup \{v_1, ..., v_{n-2}\}$, where $\hat{c} = c + \mathrm{Im}({}_{\mu}R^*)$ and c is the cycle of length 2 at n
7. $\mathcal{B}_6 = \{x^k h : k = 0, 1, ..., \frac{n}{2} - 1\} \cup \{x_i h : i = 1, ..., n-2\}$.

Indeed, by Proposition 6.6.2, we know that $\mathcal{B}_0$ is a basis of $HH^0(\Lambda)$ and, by lemma , that $\mathcal{B}_5$ is a basis of $HH^5(\Lambda)$ since the v_i are exactly those in that lemma. From this same lemma and the isomorphism $\underline{HH}^0(\Lambda) \xrightarrow{\cong} HH^6(\Lambda)$ given by multiplication by h, we also get that $\mathcal{B}_6$ is a basis of $HH^6(\Lambda)$. That $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$ are bases of the respective spaces follows from Theorem 6.6.10 and its proof. On the other hand, once we know that $u \in HH^4(\Lambda)_1$, this same theorem plus Lemma 6.7.2 give that $\mathcal{B}_4$ is a basis of $HH^4(\Lambda)$.

Once we have the bases $\mathcal{B}_i$ at hand, the 6-periodicity of $HH^*(\Lambda)$ implies that $\{x, x_1, ..., x_{2n-1}, y, z, u, v_1, ..., v_{n-2}, \hat{c}, h\}$ is a set of homogeneous generators of $HH^*(\Lambda)$ as a bigraded algebra. We claim that $\hat{c} = yu$, and this will prove that the proposed set of generators given in the statement is a valid one. Indeed, by the commutativity of the diagram in 6.7.7(1), we get that the element yu of $HH^5(\Lambda)$ is represented by the composition $\oplus_{i\in Q_0}\Lambda e_i \otimes e_i\Lambda \xrightarrow{\eta} \oplus_{a\in Q_0}\Lambda e_{i(a)} \otimes e_{t(a)}\Lambda \xrightarrow{\tilde{y}} \Lambda$. Clearly, $\tilde{y} \circ \eta$ vanishes on all $e_i \otimes e_i$, for $i \neq n$, and takes $e_n \otimes e_n \rightsquigarrow \alpha_{n-1}\beta_n = c$. It then follows that $yu = \hat{c}$, as claimed.

We are now in a position to verify the relations. Note that $\mathcal{B}_5 = \{x^{\frac{n}{2}-1}yu, ...xyu, yu, v_1, ..., v_{n-2}\}$.

Relations in a), b) and c):

Since the x_i are linear combinations of the socle elements $\{w_1, ..., w_{n-2}\}$, the relations in b) follow immediately. As for the relations in a), the equality $x^{\frac{n}{2}} = 0$ follows from Proposition 6.6.2 and $xt = 0$ is clear. On the other hand, we have $xz = c_n + \text{Im}(R^*)$, where c_n is the cycle of length 4 at n. By Proposition 6.6.7, we get that $xz = 0$ in $HH^2(\Lambda)$. Moreover, by the condition of dual bases of $\bar{\mathcal{B}}_0$ and $\mathcal{B}_5$ (see Lemma 6.7.2), we immediately get that $xv_i = 0$ and that $x_iv_i = x^{\frac{n}{2}-1}yu$, for all $i = 1, ..., n-2$.

Relations in d), k) and l): All follow by taking length degrees. Indeed, in all cases the length degree of the element in the first member of the desired equality is different of the length degrees of the elements in the basis $\mathcal{B}_i$ of the corresponding $HH^i(\Lambda)$.

Relation in e): From the commutative diagram in 6.7.7(2) we get that $yz = zy$ is represented by the composition $\oplus_{i\in Q_0}\Lambda e_{\tau(i)} \otimes e_i\Lambda \xrightarrow{g} \oplus_{i\in Q_0}\Lambda e_{\tau(i)} \otimes e_i\Lambda \xrightarrow{\tilde{z}} \Lambda$, where g is as in that diagram.

Then we have

$$(\tilde{z} \circ g)(e_{\tau(i)} \otimes e_i) = \tilde{z}(\textstyle\sum_{j\in Q_0} \sum_{x\in e_i\mathcal{B}e_j} (-1)^{c(x)}\deg(x)\tau(x) \otimes x^*) =$$
$$\textstyle\sum_{x\in e_i\mathcal{B}e_n} (-1)^{c(x)}\deg(x)\tau(x)\tilde{z}(e_n \otimes e_n)x^* = \sum_{x\in e_i\mathcal{B}e_n} (-1)^{c(x)}\deg(x)\tau(x)x^*$$

since $\tilde{z}(e_{\tau(j)} \otimes e_j) = 0$, for all $j \neq n$. We then get $(\tilde{z} \circ g)(e_{\tau(i)} \otimes e_i) = 0$, for $i \neq n$, and $(\tilde{z} \circ g)(e_{\tau(n)} \otimes e_n) = \sum_{x\in e_n\mathcal{B}e_n} (-1)^{c(x)}\deg(x)xx^* = (\sum_{x\in e_n\mathcal{B}e_n} (-1)^{c(x)}\deg(x))w_n$ since $\tau(x) = x$, for each $x \in e_n\mathcal{B}e_n$. Note that each $x \in e_n\mathcal{B}e_n$ is a power of the cycle of length 2 at n, which implies that $c(x) = \frac{1}{2}\deg(x)$ and, hence, that $c(x) \equiv 0 \pmod 2$, when $\deg(x) \equiv 0 \pmod 4$, and $c(x) \equiv 1 \pmod 2$, when $\deg(x) \equiv 2 \pmod 4$. We then get:

$$(\tilde{z} \circ g)(e_n \otimes e_n) = (\textstyle\sum_{0\leq k<\frac{n}{2}} (4k) - \sum_{0\leq k<\frac{n}{2}} (4k+2))w_n = -2\frac{n}{2}w_n = -nw_n.$$

This gives the relation e).

Relations in f) and n): We put them together because the relations n) follow from those in f). To see that, let us calculate the products v_iv_j. Due to length grading, we have $v_iv_j = \lambda_{ij}x^{\frac{n}{2}-1}uh$, for some $\lambda_{ij} \in K$. On the other hand, we have $(x^{\frac{n}{2}-1}u, y) = 1$ since $x^{\frac{n}{2}-1}yu = \hat{w}$ (see Lemma 6.7.2 and the proof of the relations in c)). Then, in case the relations in f) are assumed to be true, the graded Nakayama form of $\underline{HH}^*(\Lambda)$ and Eu-Schedler formula (see Lemma 6.7.9) give:

$$\lambda_{2k,j} = (\lambda_{2k,j}x^{\frac{n}{2}-1}u, y) = (\lambda_{2k,j}x^{\frac{n}{2}-1}uh, h^{-1}y) = (v_{2k}v_j, h^{-1}y) = -(v_{2k}h^{-1}y, v_j) =$$
$$(yv_{2k}h^{-1}, v_j) = (4nx_{2k-1}hh^{-1}, v_j) = 4n(x_{2k-1}, v_j) = 4n\delta_{2k-1,j}$$

We then prove the relations in f). Let us consider the only morphism of Λ-bimodules $\tilde{v}_{2k} : \oplus_{i\in Q_0}\Lambda e_i \otimes e_i\Lambda \longrightarrow \Lambda$ identified by the fact that $\tilde{v}_{2k}(e_i \otimes e_i) = (\delta_{i,2k} - \delta_{i,2n-2k})e_i$, where δ_{rs} denotes the Kronecker symbol. It represents v_{2k} and an obvious lifting of it is the morphism $\hat{v}_{2k} : \oplus_{i\in Q_0}\Lambda e_i \otimes e_i\Lambda \longrightarrow \oplus_{i\in Q_0}\Lambda e_i \otimes e_i\Lambda$ such that $\hat{v}_{2k}(e_i \otimes e_i) = (\delta_{i,2k} - \delta_{i,2n-2k})e_i \otimes e_i$.

As in the proof of ([3], Lemma 5.1), we consider, for each path $x = a_1...a_r$ with origin $i = i(x)$, the element $h_x \in \oplus_{a\in Q_1}\Lambda e_{i(a)} \otimes e_{t(a)}\Lambda$ given by

$$h_x = a_1...a_{r-1} \otimes x^* + a_1...a_{r-2} \otimes a_{r-1}x^* + ... + e_i \otimes a_2...a_r x^*.$$

By fixing a path representing each $x \in e_i\mathcal{B}$, we consider the only morphism of Λ-bimodules $\varphi_{2k} : \oplus_{i\in Q_0}\Lambda e_i \otimes e_i\Lambda \longrightarrow \oplus_{a\in Q_1}\Lambda e_{i(a)} \otimes e_{t(a)}\Lambda$ such that $\varphi_{2k}(e_i \otimes e_i) = \sum_{x\in e_i\mathcal{B}e_{2k}} h_x - \sum_{z\in e_i\mathcal{B}e_{2n-2k}} h_z$. It is easy to see that $\delta(h_x) = x \otimes x^* - e_i \otimes e_i$ and from this and the fact that $c_{i,2k} = c_{i,2n-k}$, one readily sees that $\hat{v}_{2k} \circ {}_{\mu}k = \delta \circ \varphi_{2k}$. Then the element yv_{2k} of $HH^6(\Lambda)$ is represented by the morphism $\oplus_{i\in Q_0}\Lambda e_i \otimes e_i\Lambda \xrightarrow{\varphi_{2k}} \oplus_{a\in Q_1}\Lambda e_{i(a)} \otimes e_{t(a)}\Lambda \xrightarrow{\tilde{y}} \Lambda$. Note that $\tilde{y}(h_x) = \deg(x)xx^* = \deg(x)w_{i(x)}$ and, hence, we get that $(\tilde{y} \circ \varphi_{2k})(e_i \otimes e_i) = (\sum_{x\in e_i\mathcal{B}e_{2k}} \deg(x))w_i - (\sum_{z\in e_i\mathcal{B}e_{2n-2k}} \deg(z))w_i = \Upsilon_{i,2k}w_i$.

Multiplication by h gives an isomorphism of $Z(\Lambda)$-modules $Z(\Lambda)/I \xrightarrow{\cong} HH^6(\Lambda)$. The last paragraph says that $yv_{2k} = \sum_{i\in Q_0} \Upsilon_{i,2k}\bar{w}_i h$, where $\bar{w}_i = w_i + I$. By Lemmas 6.7.3 and 6.7.2, we then have

$$\begin{gathered} yv_{2k} = \textstyle\sum_{1\leq r\leq \frac{n}{2}} \Upsilon_{2r-1,2k}(\bar{w}_{2r-1} - \bar{w}_{2n-2r+1})h = \\ \textstyle\sum_{1\leq r\leq k} 2(n-2k)(\bar{w}_{2r-1} - \bar{w}_{2n-2r+1})h + \sum_{k+1\leq r\leq \frac{n}{2}} 2(-2k)(\bar{w}_{2r-1} - \bar{w}_{2n-2r+1})h = \\ (2n-4k)h\textstyle\sum_{1\leq r\leq k}(\bar{w}_{2r-1} - \bar{w}_{2n-2r+1}) - 4kh\sum_{k+1\leq r\leq \frac{n}{2}}(\bar{w}_{2r-1} - \bar{w}_{2n-2r+1}). \end{gathered}$$

Bearing in mind that $\sum_{1\leq k\leq \frac{n}{2}}(w_{2k-1} - w_{2n-2k-1}) \in I$, we get:

$$\begin{gathered} yv_{2k} = (2n-4k)h\textstyle\sum_{1\leq r\leq k}(\bar{w}_{2r-1} - \bar{w}_{2n-2r+1}) + 4kh\sum_{1\leq r\leq k}(\bar{w}_{2r-1} - \bar{w}_{2n-2r+1}) = \\ 2nh\textstyle\sum_{1\leq r\leq k}(\bar{w}_{2r-1} - \bar{w}_{2n-2r+1}). \end{gathered}$$

But we have that $\bar{w}_{2r-1} = -\bar{w}_{2n-2r+1}$, because $w_{2r-1}+w_{2n-2r+1} \in I$ (see Theorem 6.6.10). Then, by the definition of x_{2k-1}, we get the desired equality:

$$yv_{2k} = 2nh\textstyle\sum_{1\leq r\leq k}(2\bar{w}_{2r-1}) = 4nh\sum_{1\leq r\leq k}\bar{w}_{2r-1} = 4nhx_{2k-1} = 4nx_{2k-1}h.$$

We next calculate yv_{2k-1}, where $1 \leq k < \frac{n}{2}$. The proof follows the lines of the calculation of yv_{2k}. In this case the morphism $\tilde{v}_{2k-1} : \oplus_{i\in Q_0}\Lambda e_i \otimes e_i\Lambda \longrightarrow \Lambda$ is identified by the fact that $\tilde{v}_{2k-1}(e_i \otimes e_i) = (\delta_{i,2k-1} - \delta_{i,2n-2k+1} - \delta_{i,2k+1} + \delta_{i,2n-2k-1})e_i$, so that the obvious lifting $\oplus_{i\in Q_0}\Lambda e_i \otimes e_i\Lambda \longrightarrow \oplus_{i\in Q_0}\Lambda e_i \otimes e_i\Lambda$ maps $e_i \otimes e_i \rightsquigarrow (\delta_{i,2k-1} - \delta_{i,2n-2k+1} - \delta_{i,2k+1} + \delta_{i,2n-2k-1})e_i \otimes e_i$. Then the role of φ_{2k} in the case of yv_{2k} will be played here by the morphism $\psi_{2k-1} : \oplus_{i\in Q_0}\Lambda e_i \otimes e_i\Lambda \longrightarrow \oplus_{a\in Q_1}\Lambda e_{i(a)} \otimes e_{t(a)}\Lambda$, identified by the equality:

$$\psi_{2k-1}(e_i \otimes e_i) = \textstyle\sum_{x\in e_i\mathcal{B}e_{2k-1}} h_x - \sum_{x\in e_i\mathcal{B}e_{2n-2k+1}} h_x - \sum_{x\in e_i\mathcal{B}e_{2k+1}} h_x + \sum_{x\in e_i\mathcal{B}e_{2n-2k-1}} h_x.$$

Then we have that the element yv_{2k-1} is represented by the composition $\oplus_{i\in Q_0}\Lambda e_i \otimes e_i\Lambda \xrightarrow{\psi_{2k-1}} \oplus_{a\in Q_1}\Lambda e_{i(a)} \otimes e_{t(a)}\Lambda \xrightarrow{\tilde{y}} \Lambda$, which maps $e_i \otimes e_i \rightsquigarrow (\Upsilon_{i,2k-1} - \Upsilon_{i,2k+1})w_i$. Using 2.b of Lemma 6.7.3, it follows that $yv_{2k-1} = \sum_{1\leq r<\frac{n}{2}}(\Upsilon_{2r,2k-1} - \Upsilon_{2r,2k+1})(\bar{w}_{2r} - \bar{w}_{2n-2r})h$ since $\Upsilon_{i,2k-1} - \Upsilon_{i,2k+1} = 0$ when i is odd. The fact that $\bar{w}_{2r} = -\bar{w}_{2n-2r}$ in $Z(\Lambda)/I$ implies that $yv_{2k-1} = \sum_{1\leq r<\frac{n}{2}} 2(\Upsilon_{2r,2k-1} - \Upsilon_{2r,2k+1})\bar{w}_{2r}h$.

We look now at the term $\Upsilon_{2r,2k-1} - \Upsilon_{2r,2k+1}$. By Lemma 6.7.3, we know that $\Upsilon_{2r,2k-1} = -\Upsilon_{2k-1,2r}$. This is equal to $2r$, when $2r < 2k-1$, and to $-2(n-2r)$, when $2r > 2k-1$. Similarly, we have $\Upsilon_{2r,2k+1} = 2r$, when $2r < 2k+1$ and $\Upsilon_{2r,2k+1} = -2(n-2r)$. We then get that $\Upsilon_{2r,2k-1} - \Upsilon_{2r,2k+1} = 0$, except for $r = k$. In this case we have $\Upsilon_{2k,2k-1} - \Upsilon_{2k,2k+1} = -\Upsilon_{2k-1,2k} + \Upsilon_{2k+1,2k} = -2(n-2k) - 2(2k) = -2n$. From this and the definition of x_{2k} it follows that $yv_{2k-1} = -4n\bar{w}_{2k}h = -4nx_{2k}h$.

Relations in g) and h): The relation in g) follows from those in e) and h). Indeed, the length degree of z^2 is -4 which is also the length degree of $x^{\frac{n}{2}-1}u$, and this is a the only element in $\mathcal{B}_4$ with that length degree. Then one gets $z^2 = \lambda x^{\frac{n}{2}-1}u$, for some $\lambda \in K$. On the other hand, by the relations in e) and h), we get that $-nx^{\frac{n}{2}-1}yu = -ntz = yz^2 = \lambda x^{\frac{n}{2}-1}yu$, which implies that $\lambda = -n$ since $x^{\frac{n}{2}-1}yu \neq 0$.

We then prove the relation in h). Using the commutative diagram in Lemma 6.7.7(3), we get that the element zt of $HH^5(\Lambda)$ is represented by the composition $\oplus_{i\in Q_0}\Lambda e_i \otimes e_i\Lambda \xrightarrow{\psi} \oplus_{i\in Q_0}\Lambda e_{\tau(i)} \otimes e_i\Lambda \xrightarrow{\tilde{z}} \Lambda$ where ψ is as in that diagram. This composition maps $e_i \otimes e_i \rightsquigarrow \delta_{in}w_n$, which proves that $zt = w_n + \mathrm{Im}({}_{\mu}R^*) = \hat{w} = x^{\frac{n}{2}-1}yu$.

It remains to prove the relations i), j) and m), for which it is enough to check j). Indeed, if we had $zv_j \neq 0$, we would have that $\mathrm{ldeg}(zv_j) = \mathrm{ldeg}(x^{\frac{n}{2}-1}yh) = -2n-4$ and there is no other element in the basis $\{x^r yh : r = 0,1,...,\frac{n}{2}-1\}$ of $HH^7(\Lambda)$ with length degree $-4n-2$. It follows that, for each $j = 1,...,n-2$, there is a unique $\lambda_j \in K$ such that $zv_j = \lambda_j x^{\frac{n}{2}-1}yh$. Then Eu-Schedler formula gives the equality:

$$(uv_j, h^{-1}z) = (v_jh^{-1}z, u) = (zv_jh^{-1}, u) = (\lambda_j x^{\frac{n}{2}-1}yhh^{-1}, u) = \lambda_j(x^{\frac{n}{2}-1}y, u) = \lambda_j,$$

from which we get that $uv_j = \lambda_j th$, bearing in mind that $\mathrm{ldeg}(th) = -4n-2 = \mathrm{ldeg}(uv_j)$. Finally, we have $\mathrm{ldeg}(zu) = -2n-2 = \mathrm{ldeg}(x_jh)$, for all $j = 1,...,n-2$, and there is no other element in $\mathcal{B}_6$ of length degree $-2n-2$. It follows that $zu = \sum_{1\leq j\leq n-2}\mu_j x_j h$. Using Eu-Schedler formula, we then get:

$$\lambda_j = (\lambda_j x^{\frac{n}{2}}yhh^{-1}, u) = (zv_jh^{-1}, u) = (h^{-1}zu, v_j) = (\textstyle\sum_{1\leq i\leq n-2}\mu_i x_i hh^{-1}, v_j) = \sum_{1\leq i\leq n-2}\mu_i(x_i, v_j) = \mu_j.$$

It follows that $zu = \sum_{1\leq j\leq n-2}\lambda_j x_j h$. If we prove the relation in j) it will follow that $\lambda_j = 0$, for all $j = 1,...,n-2$, and then the relations in i) and m) will come automatically.

In order to calculate zu, we use the first commutative diagram of Lemma 6.7.7 and the fact that multiplication by h yields an isomorphism of $Z(\Lambda)$-modules $\underline{HH}^0(\Lambda) = \frac{Z(\Lambda)}{I} \xrightarrow{\cong} HH^6(\Lambda)$. By the mentioned diagram, we know that the element zu is represented by morphism of Λ-bimodules $\oplus_{i\in Q_0}\Lambda e_i \otimes e_i\Lambda \xrightarrow{\xi} \oplus_{i\in Q_0}\Lambda e_{\tau(i)} \otimes e_i\Lambda \xrightarrow{\tilde{z}} \Lambda$. Since $\tilde{z}$ vanishes on each summand $\Lambda e_{\tau(i)} \otimes e_i\Lambda$, with $i \neq n$, we get that

$$(\tilde{z} \circ \xi)(e_j \otimes e_j) = \sum_{x \in e_j \tilde{\mathcal{B}} e_n} h(x)xx^{\diamond} = \sum_{x \in e_j \mathcal{B} e_n} h(x)xx^* = (\sum_{x \in e_j \mathcal{B} e_n} h(x))w_j,$$

for all $j \in Q_0$. Note that $w_n \in I$, and so we only need to deal with the cases when $j \neq n$. Note also that any element $x \in e_j \mathcal{B} e_n$ is of the form $x = x(j)c^r$, where $x(j)$ is the shortest path from j to n and $r \in \mathbb{N}$. By looking at the explicit definition of h (see definition 23), we see that if $j > n$ then $h(x(j)c^r) = \frac{1}{2}$, when $r + j \equiv 0 \pmod 2$, and $h(x(j)c^r) = 0$ otherwise. Similarly, if $j < n$, then $h(x(j)c^r) = \frac{1}{2}$, when $r + j \not\equiv 0 \pmod 2$, and $h(x(j)c^r) = 0$ otherwise. Bearing in mind that $0 \leq r \leq 3n - 1 - (n + j) = 2n - 1 - j$, when $j > n$, and that $0 \leq r \leq 3n - 1 - [(2n - j) + n] = j - 1$, when $j < n$, we get the following table:

Condition	Value of $\sum_{x \in e_j \mathcal{B} e_n} h(x)$
$j > n$ and j even	$\frac{n}{2} - \frac{j}{4}$
$j > n$ and j odd	$\frac{n}{2} - \frac{j+1}{4}$
$j < n$ and j even	$\frac{j}{4}$
$j < n$ and j odd	$\frac{j+1}{4}$

Suppose that $j < n$ in the rest of the proof. Recall that $w_j + w_{2n-j} \in I$, for all $j < n$. Denoting $\bar{w} = w + I$, for any socle element w, we see that $zu = \sum_{1 \leq j < n} \mu_j \bar{w}_j h$, where $\mu_j = \frac{j}{4} - [\frac{n}{2} - \frac{2n-j}{4}] = 0$, when j is even, and $\mu_j = \frac{j+1}{4} - [\frac{n}{2} - \frac{2n-j+1}{4}] = \frac{2}{4} = \frac{1}{2}$, when j is odd. But we also have that $\bar{w}_{n-1} = -\sum_{1 \leq k < \frac{n}{2}} \bar{w}_{2k-1}$ (see Lemma 6.7.2), so that $zu = \frac{1}{2}(\sum_{1 \leq k \leq \frac{n}{2}} \bar{w}_{2k-1})h = 0$.

What we have done so far proves that if $A = \oplus_{(m,n) \in \mathbb{Z} \times \mathbb{Z}} A_{(m,n)}$ is the bigraded algebra given by homogeneous generators and relations as indicated in the statement, we have a canonical surjective homomorphism of bigraded algebras $A \twoheadrightarrow HH^*(\Lambda)$. It is easy to see that the homogeneous components have the same dimension, so that A and $HH^*(\Lambda)$ are isomorphic. □

As a final consequence of the two main theorems of this section, we get:

Corollary 6.7.11. *Let $n > 1$ be an integer, put $\Lambda = \mathbb{B}_n$ and suppose that char$(K) \neq 2$. The algebra $HH^*(\Lambda)$ is commutative if, and only if, either i) n is odd; or ii) n is even and char(K) divides n.*

Proof. By Theorem 6.7.1, we know that $HH^*(\Lambda)$ is commutative when n is odd. On the other hand, by the graded commutativity relations and the relations f) and n) of Theorem 6.7.10, we know that, when n is even, the algebra $HH^*(\Lambda)$ is commutative exactly when $n = 0$ in K. □

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