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On the perturbed fast rotating satellite. Closed form vs. series expansion approach.

Sobre el satélite en rotación rápida. Comparación forma cerrada vs. desarrollos.

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Resumen

La dinámica roto-traslatoria del problema completo de los dos cuerpos continúa siendo uno de los problemas más desafiantes en Astronomía y Astronáutica. Debido a su complejidad, los modelos propuestos hasta la fecha se basan en una serie de suposiciones que simplifican el problema con objeto de poder abordarlo tanto analítica como numéricamente (véase por ejemplo [Scheeres (2011), Kopeikin *et al.* (2008), Benettin *et al.* (2008), Ferrer & Lara (2013)]).

Bajo este escenario, esta Memoria se centra en el estudio de la dinámica roto-traslatoria de un satélite alrededor de un cuerpo mucho más masivo y prácticamente esférico del Sistema Solar. En concreto, se asume en primer lugar que las dimensiones del satélite son pequeñas en comparación con la distancia que lo separa del cuerpo atractor. En segundo lugar supondremos también que objeto bajo estudio orbita a dicho cuerpo atractor siguiendo una trayectoria circular, de ahí que el movimiento de traslación ya sea conocido y venga dado de forma trivial. En resumen, este trabajo aborda el estudio de la dinámica de actitud de un sólido rígido triaxial bajo la acción de lo que se conoce como *gravity-gradient torque*. El sistema así constituido resulta ser uno de los modelos básicos no integrables válidos para analizar el movimiento tanto de satélites artificiales [Chernousko (1963), Beletskii (1966)] como de otros cuerpos naturales [Kinoshita (1972), Kinoshita (1977)].

Por tanto, en esta Memoria se estudia el sistema definido por la función Hamiltoniana

$$\mathcal{H}(x, X; v) = \mathcal{H}_0 + \epsilon\, \mathcal{H}_1,$$

cuyo flujo asociado viene dado por el sistema de ecuaciones diferenciales

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial X}, \qquad \dot{X} = -\frac{\partial \mathcal{H}}{\partial x},$$

donde (x, X) representan respectivamente las posiciones y los momentos canónicos conjugados de los diferentes conjuntos de variables que pueden emplearse para analizar el problema; v representa el vector de parámetros del modelo, entre ellos $\{A, B, C\}$ son los tres momentos principales de inercia de un sólido rígido triaxial; y ϵ es el pequeo parámetro que nos permite tratar el problema por métodos de perturbaciones.

En efecto, según Poincaré [Poincaré (1892)], la mayoría de los problemas de Mecánica Celeste que tienen que ver con dinámica orbital o rotacional pueden formularse como sistemas perturbados, lo cual implica que el Hamiltoniano anterior puede ser dividido en dos partes donde $\mathcal{H}_0$ define un sistema integrable y $\epsilon\, \mathcal{H}_1$ es la perturbación. En nuestro caso $\mathcal{H}_0$ corresponde a la energía de un sólido rígido libre y hay que señalar que en algunos problemas la presencia del pequeño parámetro puede venir dada tanto de forma explícita como implícita. Por su parte, algunos autores se refieren a $\mathcal{H}_1$ como *función de perturbación*. Conviene también remarcar que, aunque estas

perturbaciones son generalmente débiles y el movimiento de rotación del cuerpo considerado es casi el de un sólido libre, la acción de estas fuerzas externas puede provocar cambios notables en la dinámica de largo periodo.

Los estudios más precisos de dinámica de actitud de satélites artificiales están comúnmente aproximados de forma numérica, sin embargo, los primeros lanzamientos de estos satélites motivaron nuevos esfuerzos analíticos con objeto de obtener un mayor control de la dinámica proporcionado por soluciones analíticas aproximadas [Cochran (1972), Zanardi (1986)]. Existen incluso grupos en la actualidad que continúan investigando sobre métodos numéricos aplicados al movimiento del sólido rígido, los cuales basan en parte su trabajo en estas aproximaciones analíticas (ver [Celledoni & Zanna (2010)] y sus referencias).

Una forma muy común de abordar tales aproximaciones es llevar a cabo lo que se conoce como la *reducción completa* de la parte integrable considerada como orden cero. Esta técnica consiste en encontrar una transformación canónica que nos permita expesar dicho orden cero como una función que depende solo de los momentos. En ese sentido se han propuesto en la literatura distintos conjuntos de variables tomando como base las variables de Andoyer [Andoyer (1923), Deprit (1967)]. Así, el principal objetivo de este trabajo es mostrar el comportamiento de dos conjuntos diferentes de estas variables que, permitiendo ambas llevar a cabo la reducción completa del problema, pueden presentar ciertas propiedades que las hacen más o menos adecuadas para el estudio de una pertubación. El primer conjunto, conocido como variables de ángulo-acción, fue introducido por Sadov [Sadov (1970a)], el cual será comparado con un nuevo conjunto propuesto recientemente por Ferrer y Lara [Lara & Ferrer (2010b)].

Manejando funciones elípticas. De la integración a las transformaciones simplécticas

El uso de estos nuevos conjuntos de variables para el sólido rígido conlleva la aparición ineludible de funciones integrales elípticas. Como se sabe, este tipo de funciones especiales aparece cuando se resuelven gran cantidad de problemas de dinámica no lineal, incluso algunos tan sencillos como el péndulo simple. En nuestro caso, la solución general del sólido libre viene dada también en términos de funciones e integrales elípticas tanto si se formula el problema en variables de Euler [Whittaker (1937), Heard (2006)] como de Andoyer [Jacobi (1851), Andoyer (1923)]. Sin embargo, cuando el cuerpo posee simetría axial, la solución puede expresarse en términos de funciones trigonométricas cuya manipulación es claramente más sencilla en el estudio de cuerpos casi esféricos.

En otros casos, el denominado coeficiente de triaxialidad es pequeo y el uso de funciones elípticas puede evitarse separando el Hamiltoniano del sólido libre en una parte axial-simétrica y en otra triaxial que contiene la pertubación. La primera parte se toma como orden cero mientras la segunda se aade a la función de pertubación. Existe aún una reordenación más drástica tomando el rotor esférico como el orden cero (véase [Ferrer & Lara (2010a)]). Sin embargo, este esquema no es válido cuando la triaxialidad no es tan pequea, de ahí que el movimiento de muchos satélites no pueda analizarse correctamente bajo esta aproximación.

Debido a esta situación, el manejo de funciones elípticas es esencial para el estudio de la dinámica rotacional de un cuerpo triaxial ya que, no sólo la solución del movimiento libre viene dada en términos de estas funciones especiales, sino que tambien la mayoría de las pertubaciones vendrán expresadas en términos de funciones elípticas. Además conviene recordar que no todas las pertubaciones pueden tratarse por métodos de pertubaciones debido a que las funciones e integrales elípticas no poseen un álgebra cerrada con respecto a la derivación e integración.

El nacimiento de manipuladores algebraicos como Mathematica o Maple ha hecho más sencilla la manipulación de funciones integrales elípticas. En nuestro trabajo todos los cálculos presentados han sido evaluados con Mathematica [Wolfram Mathematica 9.0]. Además, para completar nuestro conocimiento sobre funciones elípticas hemos hecho uso de libros de referencia clásicos [Byrd & Friedman (1971), Abramowitz & Stegun (1972)] y la nueva versión digitalizada de esta última referencia [Olver *et al.* (2012)]. Sin embargo, dado que la literatura sobre funciones elípticas es bastante amplia, hemos recogido en un apéndice parte de los conocimientos necesarios para una mejor comprensión de las manipulaciones aquí presentadas.

Además, dado que llevaremos a cabo un esquema perturbativo por series de Lie, esta Memoria presenta una forma de abordar dicho esquema manipulando las funciones integrales elípticas que intervienen en el proceso de obtención de nuestra teoría analítica aproximada.

Sobre la estructura y resultados contenidos en esta Memoria

De acuerdo con los comentarios y referencias anteriores, y excluyendo el capítulo introductorio, el resto de este trabajo queda organizado de la siguiente manera. En el Capítulo 2 se realiza una revisión del sólido libre donde se recuerda las relaciones y definciones clásicas entre las variables involucradas en el problema además de algunos detalles sobre cinemática y dinámica. La integración del problema no perturbado se da en variables de Andoyer, las cuales se emplearán para llevar a cabo la reducción completa del problema del sólido libre. Por otra parte, este Capítulo 2 recoge en detalle la idea proporcionada en [Molero *et al.*(2013)] de llevar a cabo la intregración del problema realizando una regularización consistente en un cambio de la variable independiente (el tiempo). En ambas integraciones se ofrecen detalles sobre la manipulación de las funciones elípticas involucradas, especialmente cuando se considera la integral elíptica de tercera especie (ver también Apéndice). El capítulo finaliza con una sección donde se recoge el estudio de las fases del problema [Ferrer & Molero (2013)], esto es, las expresiones que determinan cuánto ha girado uno de los ángulos de Andoyer cuando el otro ha rotado 2π radianes. En concreto, se recogen dos demostraciones de la fórmula de Montgomery a través de las variables de Andoyer y se propone el uso de la fase alternativa a la propuesta por Montgomery, es decir, conocer cuánto ha girado la variable ν cuando μ ha dado una vuelta completa.

El Capítulo 3 muestra cómo se lleva a cabo la reducción completa del problema resolviendo la ecuación de Hamilton-Jacobi a la Poincaré. Sin embargo seguimos un esquema general donde los dos conjuntos de nuevas variables empleados en este trabajo no se encuentran eligiendo un nuevo Hamiltoniano sino haciendo algunas selecciones particulares sobre ciertas funciones donde se encuentra implicado el *módulo elíptico*. En efecto, se observa que la elección que conduce a las variables de ángulo-acción de Sadov contiene integrales elípticas pero, por el contrario, la elección que conduce a las variables de Ferrer-Lara consiste en proponer que las funciones del módulo elíptico comentadas anteriormente sean constantes, lo cual hace que el nuevo Hamiltoniano reducido quede expresado trivialmente como una función cuadrática de los nuevos momentos canónicos. Este capítulo proporciona las ecuaciones de transformación directa e inversa para ambos tipos de variables, que dependen sólo de un ángulo como variable intermedia o auxiliar para resolver las cuadraturas que aparecen cuando se aplica el método de Hamilton-Jacobi-Poincaré. Dado que existen distintas variables intermedias que permiten reducir las anteriores cuadraturas, utilizamos una diferente a la propuesta por Sadov para deducir un conjunto alternativo de variables de ángulo-acción cuya bondad dependerá del tipo de perturbación que se esté manejando. Finalmente, al igual que en [Sadov (1970a), Vallejo (1995)], se presentan las ecuaciones de transformación expresadas en términos de funciones Theta de Jacobi.

A continuación, en el Capítulo 4 se realiza una aproximación en forma cerrada de primer orden del problema perturbado donde se muestra, no sólo el modo en que se manejan las funciones elípticas bajo el método de perturbación, sino también las diferencias existentes cuando éstas se analizan empleando los dos conjuntos diferentes de variables estudiados en esta Memoria. A su vez, el Capítulo 5 emula al Capítulo 4 con el objetivo de comparar las soluciones analíticas y numéricas dadas por un desarrollo en serie de la función pertubación. En este sentido, dado que anteriores trabajos han llevado a cabo desarrollos en serie de Fourier (véase [Vallejo (1995)]), en esta Memoria exploramos la posibilidad de llevar a cabo series de Taylor de funciones elípticas previamente expresadas en términos de funciones Theta de Jacobi [Sadov (1970a)]. Tal como se comentó anteriormente, la presencia de funciones elípticas hace que la aplicación de un método de pertubación basado en el algoritmo de Lie-Deprit no sea trivial, dando cuenta, además, de la dificultad de extender la teoría a órdenes superiores. En cualquier caso, este problema ha sido comúnmente evitado expresando la pertubación desarrollada en serie de potencias, lo cual elimina la presencia explícita de funciones elípticas en favor de las funciones trigonométricas, aunque ello suponga la obtención de soluciones menos precisas. Sin embargo, como demuestra el Capítulo 4, algunas pertubaciones como la aquí estudiada, pueden tratarse mediante métodos de pertubación sin la necesidad de realizar ningún desarrollo en serie, escenario que denominamos *forma cerrada*.

Conclusiones

El problema de un satélite en rotación rápida ya fue investigado a comienzos de los años 70 por [Hitzl & Breakwell (1971)]. Sin embargo, en su trabajo, Hitzl y Breakwell sólo proponen y analizan un modelo basado en términos de largo periodo. En esta Memoria se realiza una revisión de la dinámica de actitud del problema del satélite en rotación rápida mejorada mediante la adición de los términos de corto periodo. Además, a pesar de que restringimos el movimiento a una órbita circular, los procedimientos matemáticos empleados no quedan limitados en aplicación a este modelo tan simple, de modo que la teoría puede extenderse a modelos más reales incluyendo, por ejemplo, órbitas elípticas y el efecto del achatamiento terrestre.

De una forma u otra, la elección de las variables que representan el movimiento es crucial de cara al esquema perturbativo, de ahí que se estudie el comportamiento de los dos conjuntos diferentes de variables aplicados al mismo problema. Así, revisando la literatura se usan primero las variables de Sadov que, como se sabe, tienen el inconveniente de requerir la inversión de funciones implícitas cuando se desea computar el valor del módulo elíptico como función de los nuevos momentos. En concreto, una vez que el Hamiltoniano del sólido libre ha sido reducido mediante el método de Hamilton-Jacobi-Poincaré, el módulo elíptico no puede expresarse explícitamente como función de los dos nuevos momentos, lo cual hace más difícil la posterior manipulación analítica de sus derivadas parciales además del tiempo que se necesita para invertir las ecuaciones desde un punto de vista computacional.

A diferencia de Sadov y más cercana a la aproximación de Hitzl y Breakwell, para evitar el inconveniente anterior Ferrer y Lara han propuesto recientemente un nuevo conjunto de variables [Lara & Ferrer (2010b)] que nos permite tratar el problema de forma explícita. En efecto el nuevo Hamiltoniano queda expresado de forma cuadrática en los dos nuevos momentos haciendo su manipulación más sencilla.

A pesar de estas diferencias ambos conjuntos de variables presentan una característica común: la función generatriz que hace posible el promedio de la nueva variable asociada a la variable ν de Andoyer se expresa en términos de la función Zeta de Jacobi. Además de que se pueden

prever dificultades no triviales si se intenta extender la integración en forma cerrada a órdenes superiores, la presencia de esta función Zeta produce diferentes resultados dependiendo de las variables utilizadas (ángulo-acción o Ferrer-Lara). En efecto, la derivada parcial de la función Zeta con respecto al módulo elíptico es diferente para ambos conjuntos de variables. En concreto, mientras esta derivada es periódica en variables de ángulo-acción, en variables de Ferrer-Lara es no periódica, ya que la derivada introduce términos mixtos cuyos efectos son menos significativos cuanto menor es el valor de la triaxialidad y viceversa. La aparición de términos mixtos produce un indeseado efecto en forma de rizado creciente cuando se computa el movimiento en variables Ferrer-Lara, ya sea en forma cerrada o mediante desarrollo en serie. Además, se demuestra también que aunque este efecto indesado puede no ser demasiado importante en la escala de tiempo empleada en Astrodinámica, las variables de ángulo-acción no presentan el citado efecto.

Con respecto a los desarrollos de Taylor de la función perturbación, el hecho de usar funciones Theta de Jacobi como paso intermedio para estos desarrollos tiene algunas ventajas e inconvenientes. Por un lado, el uso de las funciones Theta permite introducir la figura de la noma como la pequeña cantidad sobre la cual acometer los desarrollos en serie buscando una rápida convergencia, ya que generalmente presenta un valor menor que cualquier otra cantidad del problema como por ejemplo el módulo elíptico. Por otro lado, la presencia de estas funciones implica tratar con argumentos en forma compleja cuyos desarrollos en serie introducen funciones hiperbólicas no acotadas. Por tanto, la velocidad de convergencia dependerá principalmente de la forma de cada cuerpo objeto de estudio. En particular, como se muestra en el Capítulo 5, cuanto mayor es la triaxialidad mayor es el orden del desarrollo de Taylor que se necesita, lo cual produce un notable aumento en la complejidad de las funciones a manejar.

Bajo este escenario, la existencia de una teoría análitica en forma cerrada queda claramente justificada. De hecho, la principal ventaja del uso de funciones elípticas en forma cerrada es la validez de las expresiones independientemente del objeto estudiado. Sin embargo, esta aproximación tiene el inconveniente de presentar una velocidad de cálculo más lenta. Así, si se necesita una aplicación en tiempo real, resulta más aconsejable el uso de desarrollos en serie a pesar de la pérdida de precisión en los cálculos que ello conlleve. Por el contrario, si la aplicación requiere una gran precisión, la formulación en forma cerrada será en muchos casos la mejor solución a pesar de un más que probable aumento del tiempo de cálculo.

Contents

1 Introduction **1**

1.1 A brief historical note 2

1.2 Dealing with elliptic functions. From the integration to the symplectic transformation 3

1.3 The techniques 4

1.3.1 Hamilton-Jacobi-Poincaré's transformation 4

1.3.2 Perturbation theory. The Lie-Deprit algorithm 5

1.4 On the structure and results contained in this Memoir 5

2 The rigid body and its unperturbed dynamics **7**

2.1 Space, body and nodal reference frames 7

2.1.1 Classical relations. Euler angles 7

2.1.2 Introducing the Nodal frame by an intermediary plane 9

2.2 On the kinematics and dynamics of the free rigid body 11

2.2.1 Inertia metrics 11

2.2.2 The geometry and the first integrals of the free rigid body 12

2.3 The symplectic character of the Andoyer variables 14

2.4 On the integration of the problem in Andoyer variables 15

2.4.1 Circulation 16

2.4.2 Libration 18

2.5 On the integration of the free rigid body by regularization 19

2.5.1 The Euler equations with a "new time" 19

2.5.2 Completing the integration 21

2.6 On the phases of the free rigid body 22
2.6.1 Spherical area defined by the integral surfaces 22
2.6.2 The trajectory in configuration space 22
2.6.3 The periods of the Andoyer variables 23
2.6.4 Recovering Montgomery formula via equations of integration 24
2.6.5 Recovering Montgomery formula via the trajectory $\mu(\nu)$ 25
2.6.6 A new phase for the free rigid body 26

3 On the complete reduction of the free rigid body 27

3.1 A General Form of the Hamilton-Jacobi Equation 28
3.2 The Hamilton-Jacobi-Poincaré in Andoyer variables 30
3.3 Legendre elliptic integrals and the Standard Hamiltonian 32
3.4 Action-angle variables. Sadov transformation 36
3.4.1 The original transformation 36
3.4.2 New expression for Sadov transformation 38
3.4.3 The transformation in Jacobi Theta functions 39
3.5 Ferrer-Lara transformation 42

4 A perturbed rigid body. Closed form approach 45

4.1 Disturbing function in new variables 46
4.2 Rotating frame 48
4.3 Perturbation approach 49
4.4 Perturbation approach in action-angle variables 50
4.4.1 Transformation equations of the first averaging 50
4.4.2 Average over ϕ 52
4.4.3 Transformation equations of the second averaging 52
4.4.4 Secular terms 53
4.4.5 Results 53
4.5 Perturbation approach in Ferrer-Lara variables 54
4.5.1 Transformation equations of the first averaging 55
4.5.2 Average over ϕ 57

4.5.3 Transformation equations of the second averaging 58
4.5.4 Secular terms 58
4.5.5 Results 59
4.6 Concluding remarks and comparisons 59

5 A perturbed rigid body. Expansions approach 65

5.1 On series expansions of the perturbation 66
5.2 Series expansions in the case of a fast rotating satellite 67
5.2.1 Disturbing function in Jacobi Theta functions (action-angle variables) . . 68
5.2.2 Taylor expansion of the disturbing function (action-angle variables) 70
5.2.3 On the bounding of the Taylor expansion of the disturbing function . . . 73
5.2.4 Disturbing function in Ferrer-Lara variables 73
5.3 Perturbation approach in action-angle variables 74
5.3.1 Average over φ_ν 74
5.3.2 Transformation equations of the first averaging 76
5.3.3 Average over ϕ 78
5.3.4 Transformation equations of the second averaging 79
5.3.5 Secular terms 80
5.3.6 Results 81
5.4 Perturbation approach in Ferrer-Lara variables 82
5.4.1 Average over ℓ 82
5.4.2 Transformation equations of the first averaging 84
5.4.3 Average over ϕ 86
5.4.4 Transformation equations of the second averaging 87
5.4.5 Secular terms 88
5.4.6 Results 88
5.5 Concluding remarks and comparisons 90

A Elliptic integrals and functions 99

A.1 Canonical forms 99
A.2 On elliptic integrals and functions with $\{m, n\} \notin [0, 1]$ 101

A.2.1 Elliptic integrals with $\{m, n\} < 0$ 101
A.2.2 Elliptic integrals with $\{m, n\} > 1$ 102
A.2.3 Elliptic functions with $m < 0$ 102
A.2.4 Elliptic functions with $m > 1$ 102
A.3 Jacobi Zeta function 103
A.3.1 On the derivative of the Jacobi Zeta function 104
A.4 On the elliptic integral of the third kind 104
A.4.1 Computing Π: The Z and Θ Jacobi integral functions 104
A.4.2 Comparing with Vallejo expressions 106
A.4.3 Dealing with formulas of Byrd and Friedman. 107
A.5 Derivative of the Jacobi Zeta function 107
A.5.1 Derivative in Ferrer-Lara variables 108
A.5.2 Derivative in action-angle variables 109
A.6 On Jacobi Theta functions 110
A.6.1 On Jacobi Nome 110
A.6.2 On Jacobi Theta functions and their use to compute elliptic functions . . 110
A.7 On Fourier series of elliptic functions 113

Bibliography **117**

Chapter 1

Introduction

The roto-translatory dynamics of the full two-body problem is still one of the challenging problems both in Astronomy and Astronautics. Due to its complexity, the proposed models rely on a number of assumptions. The range of models extend from collision-bounded-unbounded conditions [Scheeres (2011), Scheeres (2012)] to the high precision involved going back to the Moon based on lunar laser ranging measurements [Kopeikin *et al.* (2008)], including studies with a more theoretical approach [Benettin *et al.* (2008), Ferrer & Lara (2013)].

In this frame, the Memoir focuses on the roto-translatory dynamics of a satellite around a nearly spherical body of the Solar system. Thus, we first assume that the overall dimension of the rigid body is small when compared with the distance to the attracting center. The other simplification consists of assuming a circular orbital motion, hence the translational motion is already integrated. Therefore, our work deals with the attitude dynamics of a triaxial rigid body under gravity-gradient torque, which is considered one of the basic nonintegrable models to analyze the attitude propagation of artificial satellites [Chernousko (1963), Beletskii (1966)], although this approximation is also valid to describe the motion of natural bodies [Kinoshita (1972), Kinoshita (1977)].

In short, in this Memoir we study a system defined by the Hamiltonian function

$$\mathcal{H}(x, X; v) = \mathcal{H}_0 + \epsilon\, \mathcal{H}_1, \tag{1.1}$$

whose associated flow is given by the system of differential equations

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial X}, \qquad \dot{X} = -\frac{\partial \mathcal{H}}{\partial x}, \tag{1.2}$$

where (x, X) represent respectively positions and conjugate momenta of the different sets of variables by which the problem may be treated; v stands for the parameters of the model, among them $\{A, B, C\}$ are the three principal moments of inertia of a triaxial rigid body and ϵ accounts for a small quantity allowing us to manage the problem by means of a perturbative scheme.

According to Poincaré [Poincaré (1892)], most of the problems in Celestial Mechanics concerning orbital or rotational motion can be formulated as a perturbed integrated system. This implies that the Hamiltonian, as we have done above, may be split in two parts where $\mathcal{H}_0$ defines an integrable system and $\epsilon\mathcal{H}_1$ is called the perturbation. In our case, $\mathcal{H}_0$ corresponds to the energy

of the *free rigid body*[1]. Note that in some problems the presence of the small parameter may come in an implicit or explicit form. Some authors refer to $\mathcal{H}_1$ as the perturbing function. Indeed, this is the frame of our problem where, although these perturbations are generally weak and the rotation of a satellite is nearly a torque-free motion, the action of external torques may induce notable changes in its long-term dynamics.

The accurate propagation of the satellite's attitude is commonly approached numerically, however, the early launches of artificial satellites motivated analytical efforts to get a greater insight on the dynamics provided by approximate analytical solutions [Cochran (1972), Zanardi (1986)]. In fact, some groups are still investigating on numerical methods applied to the rigid body motion, to which these analytical approximations provide inspiration (see [Celledoni & Zanna (2010)] and references therein).

A common way to tackle such approximations is to accomplish the *complete reduction* of the integrable part considered as the zero order. It consists of finding a canonical transformation allowing us to express the zero order as a function only of the momenta. Different sets of variables have been proposed in the literature in order to address such analytical approximation, most of them starting from Andoyer variables [Andoyer (1923), Deprit (1967)]. Thus, the main goal of this work is to show the behaviour of two different sets of these variables which, although both allow us to carry out the complete reduction of the free rigid body, they may present a number of properties which can make them more or less suitable for the study of a perturbation. The first set, well known as *action-angle variables*, was introduced by Sadov [Sadov (1970a)] and we will compare it with a new set recently proposed by Ferrer and Lara [Lara & Ferrer (2010b)].

1.1 A brief historical note

As it is well known, the rigid body problem has been studied along the history by several of the most prominent scientifics. The dynamics of a rigid body system is defined by its equations of motion, whose solution shows how the system changes as a function of time. Thus, one of the most important contributions was made in the 18th Century by the Swiss mathematician and physicist Leonhard Euler. Indeed, as we will see in Chapter 2, since the problem requires the use of two different reference frames, one of them fixed in space and the other one attached to the body, Euler first solved the system of differential equations leading to obtain the three components of the angular velocity vector as functions of time. The integration of these equations involves dealing with elliptic functions, as it can be found in [Whittaker (1937), Golubew (1960), Landau & Lifshitz (1976)]. Once obtained the solution of the mobile system, the connection with the fixed system was given by three rotations around the three well known Euler angles (ϕ, ψ, θ), where the use of elliptic integrals is needed.

Like all great contribution, the Euler's approach was followed and used for a long time (even nowadays) until the year 1923, when a French astronomer and mathematician managed to simplify the problem by introducing some new angles arising when the plane associated to the angular momentum vector is considered. Indeed, Henri Andoyer's contribution [Andoyer (1923)], whose ideas were already given by Jacobi [Jacobi (1851)], involved a revolution on how to deal with the problem since, in Hamiltonian mechanics, it was reduced to a 1-DOF system plus a

[1] In the literature there are equivalent ways to refer to this system, namely: Euler-Poinsot problem, torque-free motion and unperturbed model. We take the liberty of using them along the Memoir.

quadrature. In other words, using these new variables, the Hamiltonian of the free rigid body depends only on one angle and two momenta, unlike Euler formulation where only the angle ϕ is verified to be cyclic. Nevertheless, these new variables known as *Andoyer's variables* were not widely used until 1967 when André Deprit rediscovered them [Deprit (1967)].

Later, with the birth of the space travels and space missions, analysis of perturbations affecting satellites or asteroids started to be required to obtain a greater accuracy in the calculations. However, looking for an easier analysis of these perturbations, a more simplified Hamiltonian could be used in order to deal with the free rigid body motion as the zero order within a perturbative scheme. Indeed, a complete reduction of the problem can be performed by finding canonical transformations allowing us to have the Hamiltonian depending only on two of the new momenta. In this sense, authors started to use the Deprit-Andoyer's work with the aim of giving complete reductions of the rigid body problem to develop a more simplified version of the equations of perturbed problems using *perturbation theory*. Obviously, performing this task is easier by using Andoyer variables, since only one angle must be removed. We join these authors hence Andoyer variables will be used as the first stage for this work.

1.2 Dealing with elliptic functions. From the integration to the symplectic transformation

Elliptic integrals and elliptic functions arise when solving most problems on nonlinear dynamics, even some of the easier ones like the simple pendulum. In our case, the general solution of the torque-free motion of a rigid body is also given in terms of elliptic functions and elliptic integrals when formulated either in Euler variables [Whittaker (1937), Heard (2006)] or Andoyer variables (see [Jacobi (1851), Andoyer (1923)]). However, when the body has axial simmetry, then the solution can be expressed in terms of trigonometric functions, whose manipulation is clearly easier for the study of nearly spherical bodies.

In other cases, the *triaxiality coefficient* is small and the use of elliptic functions can be avoided by splitting the free rigid body Hamiltonian into an axisymmetric part and a triaxiality perturbation. The first is taken as the zero order Hamiltonian while the latter is added to the disturbing function. Still there is a more drastic reordering taking the spherical rotor as the zero-order Hamiltonian [Ferrer & Lara (2010a)]. Nevertheless, this scheme is not valid when the triaxiality is not small, hence a lot of satellites can not be well analyzed under this approximation.

Due to this situation, dealing with elliptic functions is essential for the study of the rotational motion of triaxial satellites because, not only the solution of the torque-free motion is given by them, but also the majority of the perturbations will be expressed in terms of these elliptic functions. Moreover, not all the perturbations can be managed by perturbation methods since elliptic functions and integrals do not have a closed algebra with respect to the derivation and integration.

The birth of the algebraic manipulators like Mathematica or Maple has made easier the manipulation of elliptic functions and integrals. In our case, all the calculations here presented have been evaluated by Mathematica 8.0 (although some of the expressions have been checked in [Wolfram Mathematica 9.0]). Furthermore, to complete our knowledge on elliptic functions we have used classical reference books [Byrd & Friedman (1971), Abramowitz & Stegun (1972)]

and the new digitalized version of the last one [Olver *et al.* (2012)]. However, since the literature on elliptic functions is quite large, we have collected in Appendix A some of the knowledge necessary to a better understanding of the manipulations here presented.

1.3 The techniques

1.3.1 Hamilton-Jacobi-Poincaré's transformation

Canonical transformations can be used as a general procedure for solving mechanical problems. In particular, the Hamilton-Jacobi (H-J-P) equation is one of the classic tools in Mechanics [Goldstein (2002)] allowing us to look for variables in order to reduce the systems to simpler versions. Following recent work by Ferrer and Lara [Ferrer & Lara (2010b)], here we approach a problem dealing with Hamiltonian systems given by (1.1). The standard approach in treating perturbed Hamiltonian systems is to define a symplectic transformation, which is built based on $\mathcal{H}_0$, using H-J equation such that the new Hamiltonian depends only on the new momenta.

We recall that the transformations $\mathcal{T}_\Phi : (x_0, x, X_0, X) \to (y_0, y, Y_0, Y)$ are defined by

$$X_i = \frac{\partial \mathcal{W}}{\partial x_i}, \qquad y_i = \frac{\partial \mathcal{W}}{\partial Y_i}, \qquad i = 0, \ldots, n \tag{1.3}$$

derived from a generating function $\mathcal{W} = \mathcal{W}(x_0, x, Y_0, Y, \mu)$ that is a complete solution of the generalized H-J equation

$$\left[\frac{\partial \mathcal{W}}{\partial x_0} + \mathcal{H}_0\left(x, \frac{\partial \mathcal{W}}{\partial x}\right)\right] \chi = \Phi(Y_0, Y) \tag{1.4}$$

where Φ is the new unperturbed Hamiltonian, which may be a function of all or several of the new momenta, and χ is the regularizing factor.

Although Sussman and Wisdom [Sussman & Wisdom (2001)] deserve credit for having presented H-J equation in this form, Sadov [Sadov (1970a)] was the first to obtain such a complete reduction in action-angle variables. By the same time, Hitzl and Breakwell [Hitzl & Breakwell (1971)] propose a different set of non-action-angle variables by reducing the Euler-Poinsot Hamiltonian in Andoyer variables to its averaged form. Later, Kinoshita [Kinoshita (1972)] bases on the work of Hitzl and Breakwell as an intermediary step in his own derivation of the action-angle variables.

Nevertheless, Ferrer and Lara [Ferrer & Lara (2010b)] state that the reduction can be done formally without need of choosing in advance any specific form of the reduced Hamiltonian, what makes that the secular terms of the problem, as well as the generating function, can be computed in a general form that either does not need the previous selection of the canonical variables used in the procedure. They therefore suggest that all the previous solutions of the Euler-Poinsot problem, as well as other appearing in the literature, may pertain to a general family of transformations based on the Hamilton-Jacobi reduction. In fact, the two sets of variables analyzed in this work can be obtained by making specific choices over the general family of transformations.

1.3.2 Perturbation theory. The Lie-Deprit algorithm

The Lie-Deprit algorithm [Deprit (1969), Meyer, Hall & Offin (2009)] defines a canonical transformation depending on a small parameter: $\Psi : (y, Y, \epsilon) \Rightarrow (x, X)$ given by the solution of a system of differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}\epsilon} = \frac{\partial W}{\partial X}, \qquad x(y, Y, \epsilon = 0) = y, \tag{1.5}$$

$$\frac{\mathrm{d}X}{\mathrm{d}\epsilon} = -\frac{\partial W}{\partial x}, \qquad X(y, Y, \epsilon = 0) = Y, \tag{1.6}$$

where $x = x(y, Y; \epsilon)$; $X = X(y, Y; \epsilon)$, being (y, Y) the initial conditions; ϵ the independent variable and W the generating function.

In a more practical point of view, since our problem has the Hamiltonian structure $\mathcal{H} = \mathcal{H}_0 + \epsilon\, \mathcal{H}_1$, we generically start from this Hamiltonian expanded as a power series of a small parameter ϵ and we seek to reduce it to its secular terms $\mathcal{K}$, this is

$$\mathcal{H}(x, X; \epsilon) = \sum_{i \geq 0} \frac{\epsilon^i}{i!}\, \mathcal{H}_{i,0}(x, X) \longrightarrow \mathcal{K}(y, Y; \epsilon) = \sum_{i \geq 0} \frac{\epsilon^i}{i!}\, \mathcal{K}_{0,i}(y, Y), \tag{1.7}$$

where the transformation is computed from a generating function given by

$$W = W(x, X; \epsilon) = \sum_{i \geq 0} \frac{\epsilon^i}{i!} W_{i+1}(x, X). \tag{1.8}$$

This is done in a stepwise procedure that is usually known as "filling the Lie triangle", whose recursive equation is given by

$$\mathcal{H}_{i,j} = \mathcal{H}_{i+1,j-1} + \sum_{0 \leq m \leq i} \binom{i}{m} \left\{ \mathcal{H}_{i-m,j-1}; W_{m+1} \right\}, \tag{1.9}$$

where $\{\mathcal{H}; W\}$ is the Poisson bracket of $\mathcal{H}$ and W.

As observed, the new Hamiltonian $\mathcal{K}_{0,i}$ is selected from an averaging of previous terms while the corresponding order of the generating function W_i is solved from a partial differential equation generically given by

$$\{W_i; \mathcal{H}_{0,0}\} + \mathcal{K}_{0,i} = \mathcal{H}_{i,0}, \tag{1.10}$$

which is called *homological equation*.

Keeping in mind the almost identity character of the Lie transformations, they are usually denote by $\mathcal{T}_\Psi : (x, X) \to (x', X')$, i.e., using a superscript prime. In Chapter 4 all these aspects are taken into account and develop in detail.

1.4 On the structure and results contained in this Memoir

According to the previous comments and references and excluding this introductory chapter, the rest of this Memoir is organized as follows. In Chapter 2 the free rigid body dynamics is revisited where we recall classical definitions and relations among the variables as well as some details

on kinematics and dynamics. The integration of the torque-free motion is given in Andoyer variables, which will be used to accomplish the complete reduction of the torque-free motion. Some details on the manipulation of the involved elliptic functions are also given, specially in the case of the elliptic integral of the third kind (see also Appendix A.4). Furthermore, a different way to address the integration of the free rigid body problem is carried out by a regularization of time. Finally, due to a renewal of interest in geometric aspects of the rigid body dynamics, a study of the phases of the problem is also included, that is, to know how much one of the angles has rotated when the other angle has rotated 2π.

Chapter 3 shows how the complete reduction is carried out by solving the Hamilton-Jacobi-Poincaé equation (Section 1.3.1). However we follow a general scheme where the two sets of new variables used in this work are not found by choosing a new Hamiltonian but rather by making some particular choices more related to expressions where the elliptic modulus is involved. Indeed, it will be observed that the choice leading to Sadov action-angle variables contains elliptic integrals but, on the contrary, the choice leading to get Ferrer-Lara variables consists of assuming some functions as constant values, which makes trivial the expression of the new reduced Hamiltonian. Direct and inverse transformation equations are provided in both cases depending only on one angle as an intermediary variable to reduce the quadratures arising when applying Hamilton-Jacobi-Poincaré method. The expression of the new Hamiltonian is strongly affected by these choices and it is important to remark that the later application of a pertubation method involves calculations whose existence depends entirely on the expression of the zero order. Moreover, an alternative intermediary variable is used to build up a new set of action-angle variables which may be utilized for the study of a number of perturbations.

Next, a first-order closed form solution of the perturbed problem is presented in Chapter 4 where it is shown not only the way to handle the elliptic functions under a perturbation method but also the existing differences when analyzing them using the two different sets of variables given by Sadov and Ferrer-Lara. In turn, Chapter 5 emulates Chapter 4 in order to compare the analytical and numerical solutions given by a series expansion of the perturbing function. In this sense, since other previous works have carried out expansions as Fourier Series [Vallejo (1995)], in this work we explore the possibility of developing Taylor expansions of the elliptic functions previously expressed in terms of Jacobi Theta functions [Sadov (1970a)] (see also Appendix A.6). As commented before, the presence of elliptic functions makes non-trivial the application of a perturbation method based on Lie-Deprit algorithm and, of course, enhances the difficulty to extend the theory to higher orders. Thus, this problem has been commonly avoided by developing the perturbation as a power series expansion, which removes the explicit presence of the elliptic functions in favour of the trigonometric functions, despite giving less accurate solutions. Notwithstanding, some perturbations can be treated by perturbation methods with no need of a previous series expansion, so this scenario will be called *closed form*.

Chapter 2

The rigid body and its unperturbed dynamics

The torque-free motion of a rigid body is an essential part of the full two-body problem [Scheeres (2011)], which continues to be a fundamental problem in rotational dynamics due to several reasons. As Ferrer and Lara have pointed out in [Ferrer & Lara (2010a)], on one side, depite the analytical solution of the so-called Euler-Poinsot problem is well known since the times of Jacobi [Jacobi (1851)] and it has been studied in detail in most classical mechanics textbooks [Whittaker (1937), Landau & Lifshitz (1976), Goldstein (2002)], its closed form solution involves elliptic integrals and functions, whose difficult evaluation has motivated efforts to improve its computation [Fukushima (2009a), Fukushima (2009b), Fukushima (2010), Celledoni & Zanna (2010)]. On the other side, this solution is of difficult application to perturbed problems, and the search for canonical variables that expedite the application of perturbation methods is also an active field of research.

The most intuitive description of the attitude of a rigid body is provided by the Euler angles, but they do not reflect all the dynamical symmetries of the torque-free motion. On the contrary, when using Andoyer variables the problem is trivially reduced to a 1-DOF system plus a quadrature. Thus, this set of variables will be used to give the solution of the torque-free motion.

In this chapter we revisit the rigid body problem by setting first all the definitions of the variables over which the rest of the chapters are based on. Next some details on kinematics and dynamics are provided and we will finally give the full integration of the problem in Andoyer variables.

2.1 Space, body and nodal reference frames

2.1.1 Classical relations. Euler angles

As shown in Fig. 2.1, let us consider two right oriented orthonormal reference frames where one of them is fixed in the space $\mathcal{S} = \{\boldsymbol{s_1}, \boldsymbol{s_2}, \boldsymbol{s_3}\}$ and the other one introduces a certain rotation

with respect to the first one $\mathcal{B} = \{\boldsymbol{b_1}, \boldsymbol{b_2}, \boldsymbol{b_3}\}$, both with the same origin[1] the point O. Under these assumptions, one has different ways to express the vectors of the second reference frame with respect to the space frame. This depends on the way we define the three basic rotations which are needed for connecting both frames.

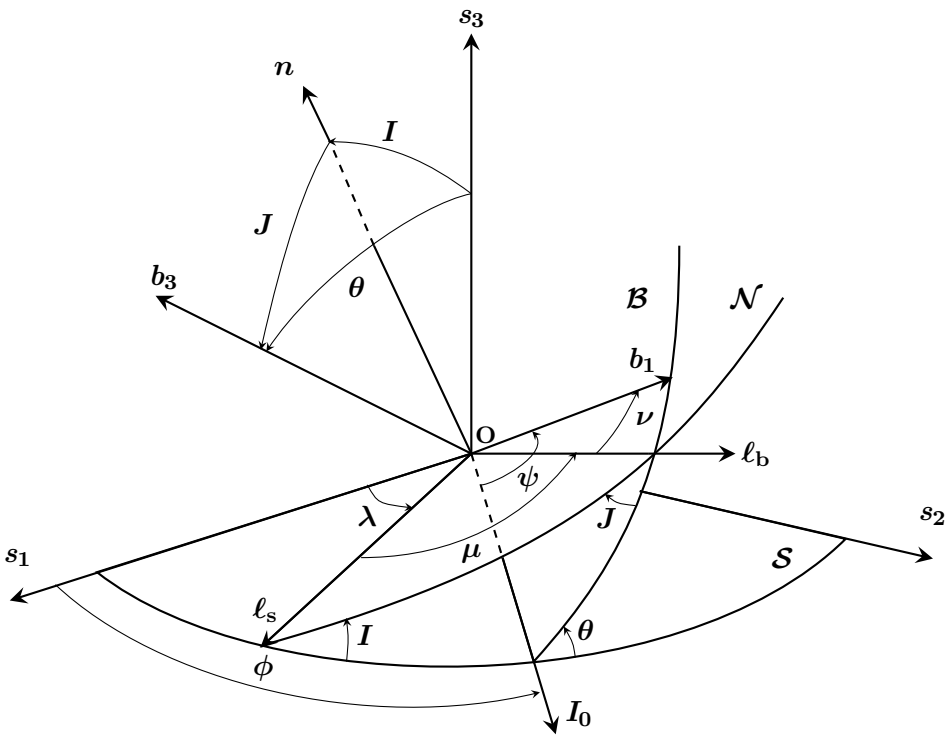


Figure 2.1: Definition of the Andoyer variables.

Let us assume $\boldsymbol{s_3}$ and $\boldsymbol{b_3}$ are not parallel. Then, among the triads named as *Euler angles* (for other choices see Goldstein *et al.* [Goldstein (2002)]) let θ be the angle between them (measured always counterclockwise), i.e.

$$\boldsymbol{s_3} \cdot \boldsymbol{b_3} = \cos\theta \qquad \text{with} \qquad 0 < \theta < \pi. \tag{2.1}$$

According to this geometry, the characteristic planes to these vectors through the origin O intersect giving a straight-line common to both planes. Then, the *ascending node vector* $\boldsymbol{I_0}$ is defined as

$$\boldsymbol{s_3} \times \boldsymbol{b_3} = \boldsymbol{I_0} \sin\theta, \qquad \|\boldsymbol{I_0}\| = 1. \tag{2.2}$$

If we define ϕ to be the angle between $\boldsymbol{s_1}$ and $\boldsymbol{I_0}$, we have

$$\boldsymbol{I_0} = \boldsymbol{s_1}\cos\phi + \boldsymbol{s_2}\sin\phi \qquad \text{with} \qquad 0 \le \phi \le 2\pi. \tag{2.3}$$

Then we may introduce another orthonormal reference frame: $\{\boldsymbol{I_0}, \boldsymbol{b_3} \times \boldsymbol{I_0}, \boldsymbol{b_3}\}$. Thus, if we further define ψ to be the angle on the plane $\{\boldsymbol{I_0}, \boldsymbol{b_3} \times \boldsymbol{I_0}\}$ between $\boldsymbol{I_0}$ and $\boldsymbol{b_1}$, we may also express

$$\boldsymbol{b_1} = \boldsymbol{I_0}\cos\psi + (\boldsymbol{b_3} \times \boldsymbol{I_0})\sin\psi \qquad \text{with} \qquad 0 \le \psi \le 2\pi. \tag{2.4}$$

[1]Later, studying the dynamics of the rigid body, we will make a particular choice of $\mathcal{B}$, the frame attached to the body, taking into account its structure measured by the *moments of inertia*.

Once defined these angles and vectors, the position of the frame $\mathcal{B}$ can be given by a matrix $A^* \in SO(3)$ as follows

$$A^* = R_3(\psi)R_1(\theta)R_3(\phi), \tag{2.5}$$

hence the components of the vectors of the frame $\mathcal{B}$ in the frame $\mathcal{S}$ are found to be

$$(b_1)_1^{\mathcal{S}} = \cos\phi\cos\psi - \cos\theta\sin\phi\sin\psi, \tag{2.6}$$

$$(b_1)_2^{\mathcal{S}} = \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi, \tag{2.7}$$

$$(b_1)_3^{\mathcal{S}} = \sin\theta\sin\psi, \tag{2.8}$$

together with

$$(b_2)_1^{\mathcal{S}} = -\cos\theta\cos\psi\sin\phi - \cos\phi\sin\psi, \tag{2.9}$$

$$(b_2)_2^{\mathcal{S}} = \cos\theta\cos\phi\cos\psi - \sin\phi\sin\psi, \tag{2.10}$$

$$(b_2)_3^{\mathcal{S}} = \sin\theta\cos\psi, \tag{2.11}$$

and

$$(b_3)_1^{\mathcal{S}} = \sin\phi\sin\theta, \tag{2.12}$$

$$(b_3)_2^{\mathcal{S}} = -\cos\phi\sin\theta, \tag{2.13}$$

$$(b_3)_3^{\mathcal{S}} = \cos\theta. \tag{2.14}$$

2.1.2 Introducing the Nodal frame by an intermediary plane

Let us now consider a unit vector $\boldsymbol{n}$ (different from the triads defining the previous frames), and the associated *characteristic plane* through the origin O. In particular it intersects the plane $\boldsymbol{s_1 s_2}$ of $\mathcal{S}$ and the plane $\boldsymbol{b_1 b_2}$ of $\mathcal{B}$. These straight lines are called *nodes*. We will express vector $\boldsymbol{n}$ in reference frames $\mathcal{S}$ and $\mathcal{B}$. (What we have to do is just to proceed with $\boldsymbol{n}$ the same way we have made previously with $\boldsymbol{b_3}$).

• With respect to $\mathcal{S}$ this new vector allows to define the angle I

$$\boldsymbol{s_3} \cdot \boldsymbol{n} = \cos I \qquad \text{with} \qquad 0 \leq I \leq \pi. \tag{2.15}$$

where I is the angle between $\boldsymbol{s_3}$ and $\boldsymbol{n}$, reckoned from $\boldsymbol{s_3}$.

Then, the *ascending node* $\boldsymbol{\ell_s}$ of the characteristic plane to $\boldsymbol{n}$ on the reference plane $\boldsymbol{s_1}$,$\boldsymbol{s_2}$ is defined to be

$$\boldsymbol{s}_3 \times \boldsymbol{n} = \boldsymbol{\ell_s} \sin I, \qquad \|\boldsymbol{\ell_s}\| = 1 \tag{2.16}$$

and it is given by

$$\boldsymbol{\ell_s} = \boldsymbol{s_1}\cos\lambda + \boldsymbol{s_2}\sin\lambda \qquad \text{with} \qquad 0 \leq \lambda \leq 2\pi, \tag{2.17}$$

where λ is the angle between $\boldsymbol{s_1}$ and $\boldsymbol{\ell_s}$.

Then, the components of $\boldsymbol{n}$ in the space frame are

$$n_1^{\mathcal{S}} = \sin\lambda\sin I, \quad n_2^{\mathcal{S}} = -\cos\lambda\sin I, \quad n_3^{\mathcal{S}} = \cos I. \tag{2.18}$$

Note that a new orthonormal basis $\mathcal{N} = \{\boldsymbol{\ell_s}, \boldsymbol{m}, \boldsymbol{n}\}$ can be completed by defining vector $\boldsymbol{m}$ as

$$\boldsymbol{m} = \boldsymbol{n} \times \boldsymbol{\ell_s}. \tag{2.19}$$

• As before, but now with respect to $\mathcal{B}$ we may also define the angle J such that

$$\boldsymbol{n} \cdot \boldsymbol{b_3} = \cos J \qquad \text{with} \qquad 0 \leq J \leq \pi. \tag{2.20}$$

where J is the angle between $\boldsymbol{b_3}$ and $\boldsymbol{n}$, reckoned from $\boldsymbol{b_3}$.

Introducing the node $\boldsymbol{\ell_b}$ as

$$\boldsymbol{n} \times \boldsymbol{b_3} = \boldsymbol{\ell_b} \sin J, \qquad \|\boldsymbol{\ell_b}\| = 1 \tag{2.21}$$

hence the node $\boldsymbol{\ell_b}$ in the plane $\boldsymbol{b_1}$,$\boldsymbol{b_2}$ is given by

$$\boldsymbol{\ell_b} = \boldsymbol{b_1} \cos\delta + \boldsymbol{b_2} \sin\delta \qquad \text{with} \qquad 0 \leq \delta \leq 2\pi, \tag{2.22}$$

where δ is the angle between $\boldsymbol{b_1}$ and $\boldsymbol{\ell_b}$.

Note the node $\boldsymbol{\ell_b}$ may be given by the vectors $\boldsymbol{\ell_s}, \boldsymbol{m}$ of the nodal frame

$$\boldsymbol{\ell_b} = \boldsymbol{\ell_s} \cos\mu + \boldsymbol{m} \sin\mu \qquad \text{with} \qquad 0 \leq \mu \leq 2\pi. \tag{2.23}$$

• Alternatively, we may introduce an angle $\nu = 2\pi - \delta$ which gives the orientation of $\boldsymbol{b_1}$ with respect to the ascending node $\boldsymbol{\ell_b}$

$$\boldsymbol{b_1} = \boldsymbol{\ell_b} \cos\nu + (\boldsymbol{b_3} \times \boldsymbol{\ell_b}) \sin\nu \qquad \text{with} \qquad 0 \leq \nu \leq 2\pi. \tag{2.24}$$

where $\nu = 2\pi - \delta$.

Finally, taking into account all the previous definitions one may get the components in the frame $\mathcal{S}$ of the vectors in the frame $\mathcal{B}$, thus, the first vector can be put into the form

$$\begin{aligned}(b_1)_1^{\mathcal{S}} &= -\cos\nu\cos I\sin\mu\sin\lambda + \cos\mu(\cos\lambda\cos\nu - \cos I\cos J\sin\lambda\sin\nu) \\ &\quad + \sin\nu(-\cos\lambda\cos J\sin\mu + \sin\lambda\sin I\sin J),\end{aligned} \tag{2.25}$$

$$\begin{aligned}(b_1)_2^{\mathcal{S}} &= \sin\lambda(\cos\mu\cos\nu - \cos J\sin\mu\sin\nu) \\ &\quad + \cos\lambda\big[\cos\nu\cos I\sin\mu + \sin\nu(\cos\mu\cos I\cos J - \sin I\sin J)\big],\end{aligned} \tag{2.26}$$

$$(b_1)_3^{\mathcal{S}} = \cos\nu\sin\mu\sin I + \sin\nu(\cos\mu\cos J\sin I + \cos I\sin J); \tag{2.27}$$

the second vector is

$$\begin{aligned}(b_2)_1^{\mathcal{S}} &= -\cos\lambda(\cos\nu\cos J\sin\mu + \cos\mu\sin\nu) \\ &\quad + \sin\lambda(-\cos\mu\cos\nu\cos I\cos J + \cos I\sin\mu\sin\nu + \cos\nu\sin I\sin J),\end{aligned} \tag{2.28}$$

$$\begin{aligned}(b_2)_2^{\mathcal{S}} &= -\cos\lambda\cos I\sin\mu\sin\nu + \cos\mu(\cos\lambda\cos\nu\cos I\cos J - \sin\lambda\sin\nu) \\ &\quad - \cos\nu(\cos J\sin\mu\sin\lambda + \cos\lambda\sin I\sin J),\end{aligned} \tag{2.29}$$

$$(b_2)_3^{\mathcal{S}} = \cos\mu\cos\nu\cos J\sin I - \sin\mu\sin\nu\sin I + \cos\nu\cos I\sin J; \tag{2.30}$$

and the third vector is

$$(b_3)_1^{\mathcal{S}} = \cos J\sin\lambda\sin I + \sin J(\cos\lambda\sin\mu + \cos\mu\cos I\sin\lambda), \tag{2.31}$$

$$(b_3)_2^{\mathcal{S}} = \sin\mu\sin\lambda\sin J - \cos\lambda(\cos J\sin I + \cos\mu\cos I\sin J), \tag{2.32}$$

$$(b_3)_3^{\mathcal{S}} = \cos I\cos J - \cos\mu\sin I\sin J. \tag{2.33}$$

Moreover, the vector of the ascending node $\boldsymbol{\ell_b}$ can explicitly be given also in the frame $\mathcal{S}$ by

$$(\ell_b)_1^{\mathcal{S}} = \cos\mu\cos\lambda - \cos I\sin\mu\sin\lambda, \tag{2.34}$$

$$(\ell_b)_2^{\mathcal{S}} = \cos\lambda\cos I\sin\mu + \cos\mu\sin\lambda, \tag{2.35}$$

$$(\ell_b)_3^{\mathcal{S}} = \sin\mu\sin I. \tag{2.36}$$

2.2 On the kinematics and dynamics of the free rigid body

2.2.1 Inertia metrics

As known, to describe the rotation of a rigid body, we need a magnitude indicating the mass distribution around a point or a rotation axis. This magnitude, known as inertia tensor ($\mathbb{I}$), accounts for the rotational inertia of the body. Once defined the inertia tensor, the rotation kinetic energy in the frame $\mathcal{B}$ may be expressed as

$$E = \frac{1}{2}\boldsymbol{\omega}^{\mathcal{B}} \cdot \mathbb{I} \cdot \boldsymbol{\omega}^{\mathcal{B}} \tag{2.37}$$

where $\boldsymbol{\omega}^{\mathcal{B}}$ is the angular velocity vector in the frame $\mathcal{B}$ and it is known that, in this frame, the inertia tensor is diagonal and then

$$E = \frac{1}{2}\left[A(\omega_1^{\mathcal{B}})^2 + B(\omega_2^{\mathcal{B}})^2 + C(\omega_3^{\mathcal{B}})^2\right], \tag{2.38}$$

where $\{A, B, C\}$ are the principal moments of inertia.

On this moments of inertia, it is important to remark that by choosing $A < B < C$ (with no loss of generality), by construction any rigid body must further satisfy that

$$A + B > C \tag{2.39}$$

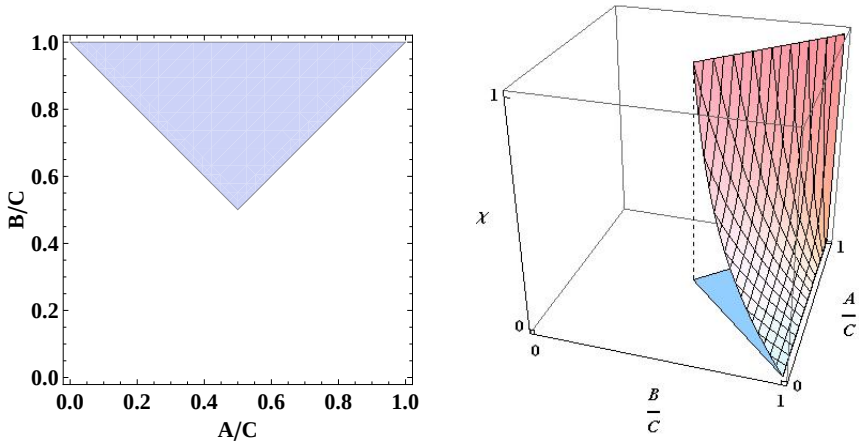
therefore, the set of possible moments of inertia of a real rigid body are given in Fig. 2.2(a).

Furthermore, at this point Andoyer defined a *triaxiality coefficient* as a function of the moments of inertial, which is given by

$$\chi = \frac{C(B-A)}{C(A+B) - 2AB} = \frac{a_1 - a_2}{a_1 + a_2 - 2a_3}. \tag{2.40}$$

where $\{a_1, a_2, a_3\} = \{1/A,\ 1/B,\ 1/C\}$.

This quantity, taking into account the previous constraints, allows us to get an idea of the different shapes of a real rigid body since the axis of maximum moment of inertia usually corresponds to the most oblate part of the body (see Fig. 2.2(b)).



(a) (b)

Figure 2.2: Constraints to be considered on the inertia metrics of a real rigid body: $A < B < C$ and $A + B > C$. (a) Region which satisfies the relations among the moments of inertia representing real rigid bodies. (b) Triaxiality surface taking into account the above constraints.

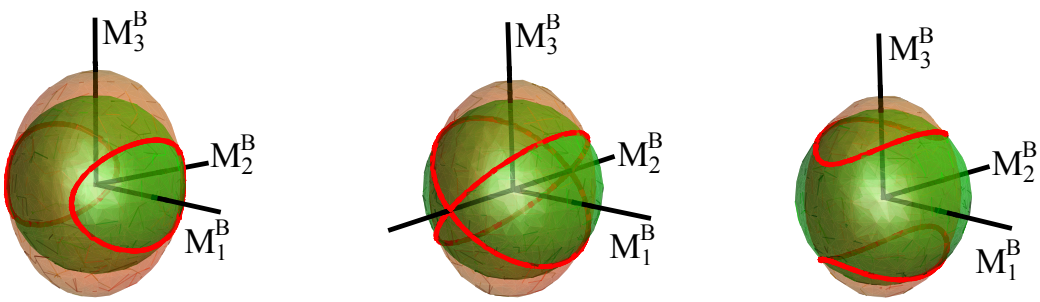
2.2.2 The geometry and the first integrals of the free rigid body

The free rigid body is characterized by two conserved quatities: the energy and the angular momentum. According to the previous description, denoting $h \equiv E$, we may write

$$\frac{(M_1^{\mathcal{B}})^2}{A} + \frac{(M_2^{\mathcal{B}})^2}{B} + \frac{(M_3^{\mathcal{B}})^2}{C} = 2h, \tag{2.41}$$

$$(M_1^{\mathcal{B}})^2 + (M_2^{\mathcal{B}})^2 + (M_3^{\mathcal{B}})^2 = M^2. \tag{2.42}$$

Note that Eqs. (2.41) and (2.42) define an ellipsoid and a sphere respectively, and the intersection between these two quadrics gives the solution curve of the problem. In particular, Fig. 2.3 shows different intersection between a sphere and an ellipsoid. Thus, the sphere can protrude from the ellipsoid or vice-versa, existing a separatrix at the middle of these two cases.



(a) $A < \frac{M^2}{2\mathcal{H}} < B < C$ (b) $A < B = \frac{M^2}{2\mathcal{H}} < C$ (c) $A < B < \frac{M^2}{2\mathcal{H}} < C$

Figure 2.3: Intersections between the ellipsoid and the sphere standing for the energy integral and the angular momentum respectively. (a) The sphere protrudes from the ellipsoid. (b) Intersection giving the separatrix of the problem. (c) The ellipsoid protrudes from the sphere.

As the components of vector $\boldsymbol{M}$ in the frame $\mathcal{N}$ ($\boldsymbol{M}_{\mathcal{N}}$) are given by

$$M_1^{\mathcal{N}} = 0, \qquad M_2^{\mathcal{N}} = 0, \qquad M_3^{\mathcal{N}} = M, \tag{2.43}$$

we may express these components in the frame $\mathcal{B}$ by means of the three following rotations

$$\boldsymbol{M}_{\mathcal{B}} = R_3(\nu)R_1(J)R_3(\mu)\,\boldsymbol{M}_{\mathcal{N}} \tag{2.44}$$

giving[2]

$$M_1^{\mathcal{B}} = M\sin\nu\sin J, \qquad M_2^{\mathcal{B}} = M\cos\nu\sin J, \qquad M_3^{\mathcal{B}} = M\cos J. \tag{2.45}$$

Furthermore, taking into account (2.43), we may express the components of $\boldsymbol{M}$ in the frame $\mathcal{B}$ by

$$\boldsymbol{M}_{\mathcal{S}} = R_3(-\lambda)R_1(-I)\,\boldsymbol{M}_{\mathcal{N}} \tag{2.46}$$

obtaining

$$M_1^{\mathcal{S}} = M\sin\lambda\sin I, \qquad M_2^{\mathcal{S}} = -M\cos\lambda\sin I, \qquad M_3^{\mathcal{S}} = M\cos I. \tag{2.47}$$

Moreover, we know that the components of the angular momentum vector $\boldsymbol{M}$ are related to the components of the angular velocity vector $\boldsymbol{\omega}$ by the expression

$$\boldsymbol{M} = \mathbb{I}\boldsymbol{\omega} \tag{2.48}$$

whence

$$M_1^{\mathcal{B}} = A\omega_1^{\mathcal{B}}, \qquad M_2^{\mathcal{B}} = B\omega_2^{\mathcal{B}}, \qquad M_3^{\mathcal{B}} = C\omega_3^{\mathcal{B}}. \tag{2.49}$$

and from Eq. (2.38) we may also write

$$E = \frac{1}{2}\left[\frac{(M_1^{\mathcal{B}})^2}{A} + \frac{(M_2^{\mathcal{B}})^2}{B} + \frac{(M_3^{\mathcal{B}})^2}{C}\right]. \tag{2.50}$$

Then, taking into account Eq. 2.45, the components of the angular velocity vector in the frame $\mathcal{B}$ are

$$\omega_1^{\mathcal{B}} = \frac{1}{A}M\sin\nu\sin J, \qquad \omega_2^{\mathcal{B}} = \frac{1}{B}M\cos\nu\sin J, \qquad \omega_3^{\mathcal{B}} = \frac{1}{C}M\cos J. \tag{2.51}$$

and hence this components can be again expressed in the frame $\mathcal{N}$ by the opposite of the aforementioned three rotations

$$\boldsymbol{\Omega} = R_3(-\mu)R_1(-J)R_3(-\nu)\,\boldsymbol{\omega} \tag{2.52}$$

which yields

$$\omega_1^{\mathcal{N}} = M\sin J\Big[(a_1-a_2)\sin\nu\cos\nu\cos\mu + \sin\mu\cos J\big[a_3 - (a_1\sin^2\nu + a_2\cos^2\nu)\big]\Big], \tag{2.53}$$

$$\omega_2^{\mathcal{N}} = M\sin J\Big[(a_1-a_2)\sin\nu\cos\nu\sin\mu + \cos\mu\cos J\big[-a_3 + (a_1\sin^2\nu + a_2\cos^2\nu)\big]\Big], \tag{2.54}$$

$$\omega_3^{\mathcal{N}} = M\big(a_3\cos^2 J + \sin^2 J(a_1\sin^2\nu + a_2\cos^2\nu)\big). \tag{2.55}$$

[2]Note that the signs of these components depend on the initial position of the vector (i.e. position when $t_0 = 0$) according to the value of the previous angles. Because of this, different signs have been used along the literature (see [Scheeres (2011), Lawden (1989), Vallejo (1995), Jacobi (1851)]).

Poinsot's Theorem. *If the angular velocity vector whose components are given in Eqs. 2.53-2.55 corresponds to the velocity of a free rigid body, then the third component Ω_3 depends only on the two integrals of the system: the energy and the angular momentum.*

Proof.

Expressing the integral of the energy (2.50) as a function of the angular velocity in the body frame, after some calculations we have

$$E = \frac{M^2}{2}\left[\left(a_1 \sin^2 \nu + a_2 \cos^2 \nu\right) \sin^2 J + a_3 \cos^2 J\right], \tag{2.56}$$

therefore, by comparing this expression with the one shown in Eq. (2.55), it inmediately follows that

$$\omega_3^{\mathcal{N}} = \frac{2E}{M}. \tag{2.57}$$

2.3 The symplectic character of the Andoyer variables

As said in the introduction, from now on we will develop our work in Hamiltonian form. At present, most researchers usually carry out their research works based on Euler variables. We make here a different choice, namely, we will use the symplectic chart built on Andoyer variables [Andoyer (1923), Deprit (1967)]. Another way to show the symplectic character can be found in [Heard (2006)].

On one hand, following Deprit-Elipe [Deprit & Elipe (1993)] we may find the expression for an infinitesimal rotation, which is given by

$$d\boldsymbol{\mathcal{R}} = d\lambda \boldsymbol{s_3} + dI \boldsymbol{\ell_s} + d\mu \boldsymbol{n} + dJ \boldsymbol{\ell_b} + d\ell \boldsymbol{b_3} \tag{2.58}$$

and on the other hand, the vector $\boldsymbol{M}$ can be written as

$$\boldsymbol{M} = M\boldsymbol{n} = M \sin I \sin \lambda \boldsymbol{s_1} - M \sin I \cos \lambda \boldsymbol{s_2} + M \cos I \boldsymbol{s_3} \tag{2.59}$$

hence the 1-form is

$$\begin{aligned} \boldsymbol{M} \cdot d\boldsymbol{\mathcal{R}} &= (\boldsymbol{M} \cdot \boldsymbol{s_3}) d\lambda + (\boldsymbol{M} \cdot \boldsymbol{\ell_s}) dI + (\boldsymbol{M} \cdot \boldsymbol{n}) d\mu + (\boldsymbol{M} \cdot \boldsymbol{\ell_b}) dJ + (\boldsymbol{M} \cdot \boldsymbol{b_3}) d\ell & (2.60) \\ &= (\boldsymbol{M} \cdot \boldsymbol{s_3}) d\lambda + (\boldsymbol{M} \cdot \boldsymbol{n}) d\mu + (\boldsymbol{M} \cdot \boldsymbol{b_3}) d\ell & (2.61) \end{aligned}$$

and it is clear that the new conjugate momenta are defined as

$$\Lambda = \boldsymbol{M} \cdot \boldsymbol{s_3} = M \cos I, \tag{2.62}$$
$$M = \boldsymbol{M} \cdot \boldsymbol{n} = M, \tag{2.63}$$
$$N = \boldsymbol{M} \cdot \boldsymbol{b_3} = M \cos J. \tag{2.64}$$

It may also checked that, according to (2.45), the components of $\boldsymbol{M}$ referred to the body frame are expressed to be

$$M_1 = C\omega_1 = \boldsymbol{M} \cdot \boldsymbol{b_1} = M\boldsymbol{n} \cdot \boldsymbol{b_1} = M \sin \nu \sin J = \sqrt{M^2 - N^2} \sin \nu, \tag{2.65}$$
$$M_2 = B\omega_2 = \boldsymbol{M} \cdot \boldsymbol{b_2} = M\boldsymbol{n} \cdot \boldsymbol{b_2} = M \cos \nu \sin J = \sqrt{M^2 - N^2} \cos \nu, \tag{2.66}$$
$$M_3 = A\omega_3 = \boldsymbol{M} \cdot \boldsymbol{b_3} = M\boldsymbol{n} \cdot \boldsymbol{b_3} = N, \tag{2.67}$$

where $\boldsymbol{\omega}= \{\omega_1, \omega_2, \omega_3\}$ are the components of the angular velocity vector in the body frame and, in what follows, we will denote $\{M_1^{\mathcal{B}},\, M_2^{\mathcal{B}},\, M_3^{\mathcal{B}}\} = \{M_1,\, M_2,\, M_3\}$ for the sake of clarity.

Then, the Eq. (2.56) may be written in symplectic form giving[3]

$$\mathcal{H}(\nu, N, M; a_1, a_2, a_3) = \frac{1}{2}(a_1 \sin^2 \nu + a_2 \cos^2 \nu)(M^2 - N^2) + \frac{a_3}{2} N^2. \tag{2.69}$$

where $a_1 = 1/A$, $a_2 = 1/B$, $a_3 = 1/C$ are the inverse of the moments of inertia.

2.4 On the integration of the problem in Andoyer variables

Despite the integration of the free rigid body can be found in several books and articles [Jacobi (1851), Andoyer (1923)], in this section we tackle again this integration due to the importance of fixing notation when manipulating elliptic functions and elliptic integrals. Thus the free rigid body in Hamiltonian formalism take the form

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial X}, \qquad \dot{X} = -\frac{\partial \mathcal{H}}{\partial x}, \tag{2.70}$$

where $\mathcal{H}(x, X)$ is called the Hamiltonian function of the system. In our case,

$$\mathcal{H}(\lambda, \mu, \nu, \Lambda, M, N) = E(-, -, \nu, -, M, N)$$

with $\mathcal{H}$ given by (2.69). Observe that the Hamiltonian does not depend on two of the variables (λ, μ) and one momenta (Λ) which means that they are cyclical variables. This fact is which makes the use of Andoyer variables so convinient for the study of the rigid body dynamics. We suggest the reader to compare the Hamiltonian function in Andoyer variables versus the same function in Euler variables to realize the benefit of these new variables.

The previous system of differential equations is given explicitly by

$$\dot{\nu} = \frac{\mathrm{d}\nu}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial N} = N(a_3 - a_1 \sin^2 \nu - a_2 \cos^2 \nu), \tag{2.71}$$

$$\dot{N} = \frac{\mathrm{d}N}{\mathrm{d}t} = -\frac{\partial \mathcal{H}}{\partial \nu} = (a_2 - a_1)(M^2 - N^2) \sin \nu \cos \nu, \tag{2.72}$$

$$\dot{\mu} = \frac{\mathrm{d}\mu}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial M} = M(a_1 \sin^2 \nu + a_2 \cos^2 \nu), \tag{2.73}$$

$$\dot{M} = \frac{\mathrm{d}M}{\mathrm{d}t} = -\frac{\partial \mathcal{H}}{\partial \mu} = 0, \tag{2.74}$$

$$\dot{\lambda} = \frac{\mathrm{d}\lambda}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial \Lambda} = 0, \tag{2.75}$$

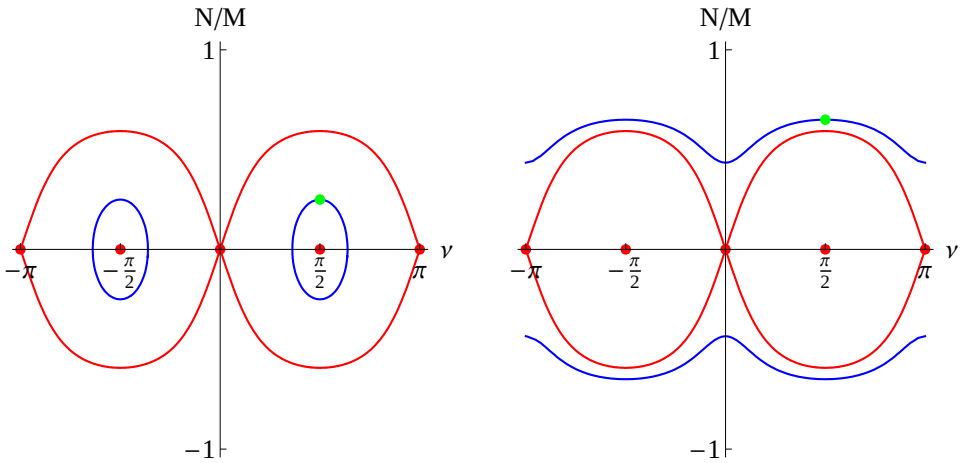
$$\dot{\Lambda} = \frac{\mathrm{d}\Lambda}{\mathrm{d}t} = -\frac{\partial \mathcal{H}}{\partial \lambda} = 0. \tag{2.76}$$

[3] According to [Ferrer & Lara (2010a)], this Hamiltonian can also be given as a function of the *triaxiality coefficient* by

$$\mathcal{H}(\nu, N, M; a_1, a_2, a_3; \chi) = \frac{1}{2} a_3 M^2 + \frac{1}{4}(a_1 + a_2 - 2a_3)(M^2 - N^2)\left(1 - \chi \cos 2\nu\right), \tag{2.68}$$

As it is well known, the solution of the problem expressed in Andoyer variables can be found by solving the system (2.71)-(2.72) plus the quadrature given by (2.73). Nevertheless, depending on initial conditions, two different solutions are possible as the reduced phase space shows in Fig. 2.4. Indeed, there is a region above and below the separatrix where the angle ν circulates (that is, $0 \leq \nu \leq 2\pi$) and another region around the equilibria (inside the separatrix) where ν does not circulate. Besides, it is also well known that solutions are given in terms of Jacobi elliptic functions and elliptic integrals, like many other dynamical systems. In fact, the three main Jacobian elliptic functions solve a number of systems of differential equations (see [Molero *et al.*(2013), Molero *et al.*(2013)]).

Then, in this section we integrate the problem and show the general solutions wherever initial conditions are taken, but excluding particular solutions concerning on relative equilibria and the separatrix, which will be analyzed in forthcoming papers.



(a) $A < \frac{M^2}{2\mathcal{H}} < B < C$ (b) $A < B < \frac{M^2}{2\mathcal{H}} < C$

Figure 2.4: Reduced phase space of the system. (a) When initial conditions are taken inside the separatrix (for instance at green point), both N and ν are bounded by the blue curve, that is, the angle ν does not circulate. (b) When initial conditions are taken outside the separatrix (see again the green point), the angle ν circulates.

2.4.1 Circulation

In order to solve the system (2.71)-(2.72), we can work out $\sin^2\nu$ and $\cos^2\nu$ from (2.69), that is

$$\sin^2\nu = \frac{2\mathcal{H} - a_3N^2 - a_2(M^2 - N^2)}{(a_1 - a_2)(M^2 - N^2)}, \qquad \cos^2\nu = \frac{a_1(M^2 - N^2) - (2\mathcal{H} - a_3N^2)}{(a_1 - a_2)(M^2 - N^2)}, \tag{2.77}$$

and replace it in Eq. (2.72), giving

$$\dot{N} = \sqrt{[(2\mathcal{H} - a_3N^2) - a_2(M^2 - N^2)][a_1(M^2 - N^2) - (2\mathcal{H} - a_3N^2)]}. \tag{2.78}$$

Thus, taking a specific value of the energy given by the initial conditions $h = \mathcal{H}(\nu_0, N_0)$ and integrating the previous equation, after several calculations we have

$$N(t) = R\,\mathrm{dn}(s\,t|m). \tag{2.79}$$

where it can be found that

$$m = \frac{(B-A)(2hC-M^2)}{(C-B)(M^2-2hA)}, \quad s^2 = \frac{(C-B)(M^2-2hA)}{ABC}, \quad R^2 = \frac{C(M^2-2hA)}{C-A}. \tag{2.80}$$

At this point one may observe that we could have used the solution of the Euler equations given in [Lawden (1989), Tantalo (1993)] to integrate the system. Thus, considering again Eqs. (2.65)-(2.67) and (2.79)-(2.80), we can equate

$$M_1(t) = A\omega_1 = P\,\mathrm{cn}(s\,t|m) = \sqrt{M^2-N^2}\sin\nu(t), \tag{2.81}$$

$$M_2(t) = B\omega_2 = Q\,\mathrm{sn}(s\,t|m) = \sqrt{M^2-N^2}\cos\nu(t), \tag{2.82}$$

$$M_3(t) = C\omega_3 = R\,\mathrm{dn}(s\,t|m) = N(t), \tag{2.83}$$

where R has been already defined in (2.80) and

$$P^2 = \frac{A(2hC-M^2)}{C-A}, \quad Q^2 = \frac{B(2hC-M^2)}{C-B}. \tag{2.84}$$

Eq. (2.83) leads inmediately to Eq. (2.79) and we can also work out $\sin\nu$ and $\cos\nu$ from (2.81) and (2.82), respectively, hence, taking into account (2.79), the angle $\nu(t)$ is defined unambiguously by means of

$$\sin\nu(t) = \frac{P}{\sqrt{M^2-N(t)^2}}\,\mathrm{cn}(s\,t|m) = \frac{\mathrm{cn}(s\,t|m)}{\sqrt{1-n\,\mathrm{sn}^2(s\,t|m)}}, \tag{2.85}$$

$$\cos\nu(t) = \frac{Q}{\sqrt{M^2-N(t)^2}}\,\mathrm{sn}(s\,t|m) = \sqrt{\frac{B(C-A)}{A(C-B)}}\frac{\mathrm{sn}(s\,t|m)}{\sqrt{1-n^2\,\mathrm{sn}(s\,t|m)}}, \tag{2.86}$$

where from (2.80), it follows inmediately that

$$n = \frac{m\,R^2}{R^2-M^2} = \frac{C(B-A)}{A(C-B)}, \tag{2.87}$$

which is related to the elliptic integral of the third kind as we will see below.

We could also obtain the own angle by working out $\sin^2\nu$ from Eq. (2.69) and, by replacing it into Eq. (2.71) together with Eq. (2.79), after several calculations it yields

$$\nu(t) = -\arctan\left(\sqrt{\frac{B(C-A)}{A(C-B)}}\,\mathrm{sc}(s\,t|m)\right). \tag{2.88}$$

Finally, as indicated previously, the system is completely integrated by solving the quadrature caming from (2.73), that is

$$\mu(t) = M\int_0^t (a_1\sin^2\nu(t) + a_2\cos^2\nu(t))\,\mathrm{d}t. \tag{2.89}$$

Then, from (2.69) we can work out

$$a_1\sin^2\nu + a_2\cos^2\nu = \frac{2h-a_3N^2}{M^2-N^2} \tag{2.90}$$

hence, taking into account (2.79) and the relation $\mathrm{dn}^2(s\,t|m) = 1 - m\,\mathrm{sn}^2(s\,t|m)$, we may write Eq. (2.89)

$$\begin{aligned}
\mu(t) &= M\int_0^t \frac{2h - a_3N^2}{M^2 - N^2}\,\mathrm{d}t \\
&= M\int_0^t \frac{2h - a_3R^2\,\mathrm{dn}^2(s\,t|m)}{M^2 - R^2\,\mathrm{dn}^2(s\,t|m)}\,\mathrm{d}t \\
&= M\int_0^t \left(a_3 + \frac{2h - a_3M^2}{M^2 - R^2\,\mathrm{dn}^2(s\,t|m)}\right)\mathrm{d}t \\
&= M\int_0^t \left(a_3 + \frac{2h - a_3M^2}{M^2 - R^2}\frac{1}{1 + n\,\mathrm{sn}^2(s\,t|m)}\right)\mathrm{d}t.
\end{aligned}$$

where the elliptic parameter n has been defined in (2.87) [4] and

$$\frac{2h - a_3M^2}{M^2 - R^2} = \frac{C - A}{AC}, \tag{2.91}$$

hence

$$\mu(t) = M\left(\int_0^t a_3\,\mathrm{d}t + \frac{C - A}{AC}\int_0^t \frac{1}{1 - n\,\mathrm{sn}^2(s\,t|m)}\,\mathrm{d}t\right) \tag{2.92}$$

and finally it leads to

$$\mu(t) = M\left[\frac{1}{C}t + \frac{1}{s}\frac{C - A}{AC}\,\Pi(-n;\mathrm{am}(s\,t|m)|m)\right]. \tag{2.93}$$

where Π is the Legendre elliptic integral of the third kind. We will talk about this function again in the following chapter when the complete reduction of the problem is tackled.

2.4.2 Libration

Proceeding in the same way for the libration case (see Fig. 2.4(a)) we find that

$$N(t) = R\,\mathrm{cn}(\tilde{s}\,t|\tilde{m}), \tag{2.94}$$

where R is the same as the previous case and

$$\tilde{m} = \frac{(C - B)(M^2 - 2hA)}{(B - A)(2hC - M^2)}, \quad \tilde{s}^2 = \frac{(B - A)(2hC - M^2)}{ABC}. \tag{2.95}$$

In order to obtain the new equation for μ, we start again from Eq. (2.89), and taking into account Eqs. (2.87) and (2.95) we have

$$\mu(t) = M\left[\frac{1}{C}t + \frac{1}{\tilde{s}}\frac{C - A}{AC}\,\Pi(-\tilde{n};\mathrm{am}(\tilde{s}\,t|\tilde{m})|\tilde{m})\right]. \tag{2.96}$$

[4] Note that Tantalo in [Tantalo (1993)] used a different definition of the elliptic integral of the third kind where $\eta = -n$.

where

$$\tilde{n} = \frac{C(M^2 - 2hA)}{A(2hC - M^2)}. \tag{2.97}$$

Note that this elliptic parameter depends not only on the moments of inertia but also on the firts integrals. Meanwhile it depends only on the moments of inertia for the circulation case.

Finally, the solution for the angle ν is found by replacing (2.94) into (2.77).

2.5 On the integration of the free rigid body by regularization

2.5.1 The Euler equations with a "new time"

Every text in classical mechanics devotes a chapter to the free rigid body. The reason is that it is one of the classical integrable systems in dynamics. The Euler fundamental equations are

$$A\frac{\mathrm{d}\omega_1}{\mathrm{d}t} + (C - B)\,\omega_2\,\omega_3 = 0, \tag{2.98}$$

$$B\frac{\mathrm{d}\omega_2}{\mathrm{d}t} + (A - C)\,\omega_1\,\omega_3 = 0, \tag{2.99}$$

$$C\frac{\mathrm{d}\omega_3}{\mathrm{d}t} + (B - A)\,\omega_1\,\omega_2 = 0. \tag{2.100}$$

which have two well known first integrals given by

$$A\omega_1^2 + B\omega_2^2 + C\omega_3^2 = 2h, \tag{2.101}$$

$$A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2 = M^2 \tag{2.102}$$

where h is the kinetic energy and M is the angular momentum.

The treatment of these equations has been already done in the aforementioned classical texts [Whittaker (1937), Landau & Lifshitz (1976), Goldstein (2002)]. Nevertheless, an alternative way to deal with Euler equations has been recently proposed by [Molero *et al.*(2013)]. Indeed, Eq. (2.98)-(2.100) may be put into the form

$$\frac{\mathrm{d}\omega_1}{\mathrm{d}t} = c_1\,\omega_2\omega_3, \qquad \frac{\mathrm{d}\omega_2}{\mathrm{d}t} = c_2\,\omega_1\omega_3, \qquad \frac{\mathrm{d}\omega_3}{\mathrm{d}t} = c_3\,\omega_1\omega_2, \tag{2.103}$$

where

$$c_1 = -\frac{C - B}{A} < 0, \qquad c_2 = \frac{C - A}{B} > 0, \qquad c_3 = -\frac{B - A}{C} < 0. \tag{2.104}$$

Then, considering the system (2.103) and the regularization $t \to \tau$ given by the differential relation

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \omega_3, \tag{2.105}$$

in the domain where $\omega_3 \neq 0$, the previous system (2.103) takes the form

$$\frac{\mathrm{d}\omega_1}{\mathrm{d}\tau} = c_1\,\omega_2, \qquad \frac{\mathrm{d}\omega_2}{\mathrm{d}\tau} = c_2\,\omega_1, \qquad \omega_3\frac{\mathrm{d}\omega_3}{\mathrm{d}\tau} = c_3\,\omega_1\omega_2, \tag{2.106}$$

showing that the first two equations decouple from the third. The reduced system is no more than a linear differential system with constant coefficients whose solution is given by a combination of exponentials. Depending on the sign of c_1 we will find the classic harmonic oscillator or a hyperbolic motion. In the case of the rigid body where $c_1 < 0$ and $c_2 > 0$ the solution is expressed by means of trigonometric functions. Then, the third equation is trivially solved by quadrature. Note also that the linear differential system has a new integral given by

$$c_2\omega_1^2 + |c_2|\omega_2^2 = k \tag{2.107}$$

where from (2.101) and (2.102), it can be checked that

$$k = \frac{2hC - M^2}{A\,B}. \tag{2.108}$$

which is no more than a linear combination of the two first integrals of the problem.

The solution of the above linear differential system is given by

$$\omega_1(\tau) = d_1\cos(\eta\tau), \qquad \omega_2(\tau) = d_2\sin(\eta\tau), \tag{2.109}$$

and taking into account (2.104) and (2.108), by replacing (2.109) in the first two equations of (2.106), we have three three equations to solve three unknowns d_1, d_2 and η, which are found to be

$$d_1^2 = \frac{2hC - M^2}{A(C-A)}, \tag{2.110}$$

$$d_2^2 = \frac{2hC - M^2}{B(C-B)}, \tag{2.111}$$

$$\eta^2 = \frac{(C-B)(C-A)}{A\,B}. \tag{2.112}$$

where we will take the positive roots of the previous quantities.

Once solved the linear system, taking into account (2.109), the third equation of (2.106) may be solved and put into the form

$$\omega_3^2 = \frac{d_1 d_2|c_3|}{2\eta}(1 - 2\eta\, c_i)\left[1 - \frac{2}{1-2\eta c_i}\sin^2(\eta\tau)\right] \tag{2.113}$$

where c_i is an integration constant. To find it in terms of the moments of inertia and the first integrals, we can express (2.101) as

$$\omega_3^2 = \frac{2h - Ad_1^2}{C}\left[1 + \frac{Ad_1^2 - Bd_2^2}{2h - Ad_1^2}\sin^2(\eta\tau)\right] \tag{2.114}$$

and by identifying terms we have that

$$c_i = \frac{1}{2\eta} - \frac{2h - Ad_1^2}{C}\frac{1}{d_1 d_2|c_3|}, \tag{2.115}$$

which we replace in (2.113), giving

$$\omega_3(\tau) = d_3\sqrt{1 - m\sin^2(\eta\tau)} \tag{2.116}$$

where

$$d_3^2 = \frac{M^2 - 2hA}{C(C-A)}, \qquad m = \frac{(B-A)(2hC - M^2)}{(C-B)(M^2 - 2hA)}. \tag{2.117}$$

being m again the elliptic modulus and $d_3 = R$, both already given in (2.80).

Finally, the time is recovered by the quadrature (2.105) to get

$$t = \int \frac{\mathrm{d}\tau}{\omega_3(\tau)} = \frac{1}{d_3} \int \frac{\mathrm{d}\tau}{\sqrt{1 - m\sin^2(\eta\tau)}} = \frac{1}{d_3\eta}\,\mathrm{F}(\eta\tau|m), \tag{2.118}$$

and taking into account that $d_3\eta = s$ as given in (2.80), we may express

$$t = \frac{1}{s}\,\mathrm{F}(\eta\tau|m) = \sqrt{\frac{A\,B\,C}{(C-B)(M^2 - 2hA)}}\,\mathrm{F}(\eta\tau|m). \tag{2.119}$$

where F is the incomplete elliptic integral of the first kind.

2.5.2 Completing the integration

Once obtained $\omega_1 = \omega_1(\tau)$, $\omega_2 = \omega_2(\tau)$ and $\omega_3 = \omega_3(\tau)$ given respectively by (2.109) and (2.116), we have also solved Andoyer variables ν and N according to Eqs. (2.81)-(2.83). However μ remains to be found under this new scenario. Therefore, applying (2.105) to Eq. (2.73), it can be put into the form

$$\omega_3 \frac{\mathrm{d}\mu}{\mathrm{d}\tau} = M(a_1 \sin^2\nu + a_2\cos^2\nu). \tag{2.120}$$

Considering (2.90), (2.83) and (2.116) we may write

$$\begin{aligned}
\mu(\tau) &= M \int \frac{1}{\omega_3} \frac{2h - a_3 N^2}{M^2 - N^2}\,\mathrm{d}\tau \\
&= M \int \frac{1}{\omega_3} \frac{2h - a_3 d_3^2\,[1 - m\sin^2(\eta\tau)]}{M^2 - R^2[1 - m\sin^2(\eta\tau)]}\,\mathrm{d}\tau \\
&= M \int \frac{1}{\omega_3} \left(a_3 + \frac{2h - a_3 M^2}{M^2 - d_3^2} \frac{1}{1 - n\sin^2(\eta\tau)} \right) \mathrm{d}\tau \\
&= \frac{M}{d_3} \left[a_3 \int \frac{\mathrm{d}\tau}{\sqrt{1 - m\sin^2(\eta\tau)}} + \frac{C-A}{A\,C} \int \frac{\mathrm{d}\tau}{[1 - n\sin^2(\eta\tau)]\sqrt{1 - m\sin^2(\eta\tau)}} \right] \\
&= \frac{M}{s} \left[\frac{1}{C}\,\mathrm{F}(\eta\tau|m) + \frac{C-A}{A\,C}\Pi(n;\eta\tau|m) \right]
\end{aligned} \tag{2.121}$$

where n is the elliptic parameter given in (2.87).

Note finally that (2.121) can be directly derived from (2.93) by replacing (2.119).

2.6 On the phases of the free rigid body

For the last two decades there has been a renewal of interest in geometric aspects of the rigid body dynamics [Marsden & Ratiu (1999)]. Among them, one of the aspects recently revisited has to do with the phases of the free rigid body, that is, to compute how much one of the angles of the problem has rotated when the other one has made a rotation. Within this vein Montgomery germinal paper [Montgomery (1991)] connecting Berry phase with classical mechanics has spurred several publications of which we just mention a few: [Levi (1993), Levi (2002)], [Zhuravlev (1996)], [Bates (2005)], [Natario (2010)]. In particular, as we will prove below, Montgomery gave a formula to compute how much the angle μ has rotated when ν has rotated 2π. Additionally, we will give new expressions to compute the other phase, that is, how much the angle ν has rotated when μ has rotated 2π [Ferrer & Molero (2013)]. However, in order to connect with the aforementioned literature, we will use $A > B > C$ for this section.

2.6.1 Spherical area defined by the integral surfaces

To reckon these phases, a fundamental quantity is the spherical area given by the intersection of the surfaces defined by the first integrals of the system. Although its explicit expression is not needed in the previous proofs of Montgomery formula, we will base our proof on it. Thus, we bring here that area whose determination may be found in vectorial calculus handbooks (or see also [Tantalo (1993)]). Indeed, the surface area $\mathcal{A}$ on the sphere limited by the intersection of the sphere (2.42) with the ellipsoid (2.41) is given by:

(i) If $A > B > \frac{M^2}{2h} > C$ then

$$\mathcal{A} = 2\pi M^2 + 4Ms^{-1}\left(\frac{M^2 - 2hC}{C}\,\mathrm{K}(m) - \frac{M^2(A-C)}{AC}\,\Pi(-n|m)\right) \tag{2.122}$$

where n, m and s have been defined above.

(ii) If $A > B = \frac{M^2}{2h} > C$ then

$$\mathcal{A} = 8M^2 \arctan(\sqrt{1+n} - \sqrt{n}), \tag{2.123}$$

and of course, if $A > B > \frac{M^2}{2h} = C$ then $\mathcal{A} = 0$.

2.6.2 The trajectory in configuration space

We quote from [Zhuravlev (1996)]: "Poinsot's *geometrical interpretation* is not complete if we mean by a complete geometrical interpretation a clear representation of the sequence of positions of a rigid body in a space without indicating the instants of time at which these positions are reached. In other words, a complete geometrical interpretation is the interpretation of the trajectory in a configuration manifold (group $SO(3)$) without indicating the schedule of motion in it." With the previous differential equations (2.73) and (2.71) we may eliminate the time and to obtain the expression $\mu = \mu(\nu)$ asked by Zhuravlev for the complete geometric interpretation. Indeed, we divide first Eq. (2.71) by Eq. (2.73). Then, we take into account (2.69) and fix a

value of the energy, which gives $N = N(\nu; M, \mathcal{H}_0, a_i)$, we obtain

$$\frac{\mathrm{d}\nu}{\mathrm{d}\mu} = \frac{N(a_3 - a_1 \sin^2 \nu - a_2 \cos^2 \nu)}{M(a_1 \sin^2 \nu + a_2 \cos^2 \nu)} \tag{2.124}$$

from where

$$\mu - \mu_0 = \int_{\nu_0}^{\nu} \frac{a_1 \sin^2 \nu + a_2 \cos^2 \nu}{\sqrt{(d - a_1 \sin^2 \nu - a_2 \cos^2 \nu)(a_3 - a_1 \sin^2 \nu - a_2 \cos^2 \nu)}} \mathrm{d}\nu \tag{2.125}$$

with $d = 2\mathcal{H}/M^2$.

The previous quadrature may be computed by introducing an auxiliary variable $\nu \to \phi$ given by

$$\sin \nu = \frac{\sqrt{1 - \alpha_3} \sin \phi}{\sqrt{1 - \alpha_3 \sin^2 \phi}}, \quad \cos \nu = \frac{\cos \phi}{\sqrt{1 - \alpha_3 \sin^2 \phi}}, \quad \mathrm{d}\nu = \frac{\sqrt{1 - \alpha_3}}{1 - \alpha_3 \sin^2 \phi} \mathrm{d}\phi, \tag{2.126}$$

with

$$\alpha_3 = \frac{M^2(A - B)}{B(2hA - M^2)}. \tag{2.127}$$

Indeed, taking $\mu_0 = 0$ at the instant when $\nu_0 = 0$ (and therefore $\phi_0 = 0$) and taking into account [Fukushima (2012)] (formula 6), after some computations we can express $\mu = \mu(\nu)$ as

$$\mu[\nu(\phi)] = \frac{1}{\sqrt{(a_3 - a_2)(d - a_1)}} \left[d\,\mathrm{F}(\phi|m) - (d - a_2)\Pi(\alpha_3; \phi|m) \right]. \tag{2.128}$$

This expression will play a key role when looking for the Montgomery formula without requiring the explicit involvement of time.

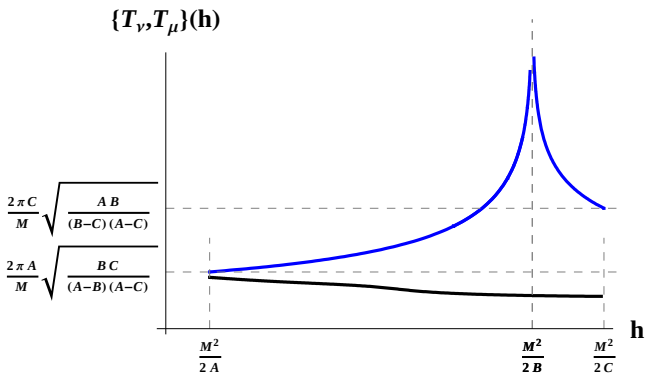
2.6.3 The periods of the Andoyer variables

With the solution of the free rigid body in explicit form given in Section 2.4 (but for the case $A > B > C$), we ask first for the period of the two angles involved (T_ν and T_μ). Fig. 2.5 gives two examples of the evolution of the period of the angles as functions of the energy for fixed values of the moments of inertia. It is important to remark the biparametric character of the system given by the ratios B/A and C/A (see Fig. 2.2). As it is pointed out in [Borisov (2008)], the periods of the two angles will show a different behaviour according to these ratios. Indeed, when ν circulates, there are conditions leading us to $T_\nu = T_\mu$ as observed in Fig. 2.5(b), which means that, depending on the shape of the body, when one of the angles has rotated 2π, the other angle could not have reached this value or even have exceeded it depending on the energy.

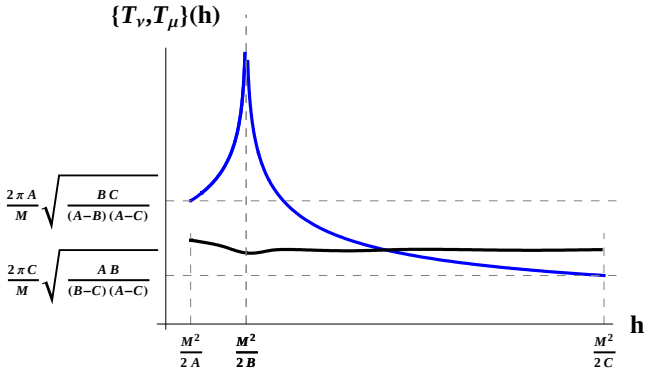
We compute first that period of the angle ν. From (2.81) we obtain

$$T_\nu = 4s^{-1}\,\mathrm{K}(m) = 4\sqrt{\frac{ABC}{(B - C)(2hA - M^2)}}\,\mathrm{K}(m). \tag{2.129}$$

as a function of the energy; we should keep in mind that apart from the explicit presence of h, we have also that $m = m(h)$. We have illustrated that functional relation by the two upper



(a) $(A, B, C) = (1, 0.55, 0.5)$



(b) $(A, B, C) = (1, 0.8, 0.35)$

Figure 2.5: Example of the evolution of the periods T_ν and T_μ for two different triaxial bodies. The blue curve corresponding to the evolution of T_ν has been plotted from (2.129); the black curve corresponding to T_μ has been plotted with *Mathematica* [Wolfram Mathematica 9.0] by solving implicitly (2.130) for a list of energy values within the range where ν circulates.

curves in Fig. 2.5. Doing the same with μ from (2.93) for the case $A > B > C$, replacing t by the period, we obtain

$$2\pi = M\left[\frac{1}{C}T_\mu - \frac{1}{s}\frac{A-C}{AC}\,\Pi(-n;\mathrm{am}(s\,T_\mu|m)|m)\right] \tag{2.130}$$

whose inversion lead us to $T_\mu = T_\mu(h)$ as shown by the lower curve in Fig. 2.5.

Now, once found the Andoyer variables as a function of time, we ask ourselves the question put by Montgomery [Montgomery (1991)]: how much the angle μ has rotated when ν has revolved 2π radians, i.e. for the time $t = T_\nu$?

2.6.4 Recovering Montgomery formula via equations of integration

Now, in answering Montgomery question we consider the period of ν given by (2.129). Then, if this period is replaced into Eq. (2.93), knowing that

$$\Pi(-n;\mathrm{am}(s\,T_\nu|m)|m) = \Pi(-n;\mathrm{am}(4\,\mathrm{K}(m)|m)|m) = \Pi(-n;2\pi|m) = 4\Pi(-n|m) \tag{2.131}$$

we have that

$$\mu(T_\nu) = 4Ms^{-1}\left(\frac{1}{C}\,\mathrm{K}(m) - \frac{A-C}{AC}\,\Pi(-n|m)\right). \tag{2.132}$$

Note that this expression relates to the spherical area defined by the intersection of the energy ellipsoid $\mathcal{E}_h$ and the momentum sphere $\mathcal{S}_M$, whose expression was already given (2.122). In our case, examination of the solution shows that the previous oriented area is $\mathcal{A}_O = -\mathcal{A}$. Thus, the corresponding solid angle is equal to

$$\Omega = \frac{\mathcal{A}_O}{M^2} = -2\pi - 4M^{-1}s^{-1}\left(\frac{M^2-2hC}{C}\,\mathrm{K}(m) - \frac{M^2(A-C)}{AC}\,\Pi(-n|m)\right). \tag{2.133}$$

which can be written

$$\begin{aligned}
\Omega &= -2\pi - 4Ms^{-1}\left(\frac{1-2hC/M^2}{C}\,\mathrm{K}(m) - \frac{A-C}{AC}\,\Pi(-n|m)\right)\\
&= -2\pi - \underbrace{4Ms^{-1}\left(\frac{1}{C}\,\mathrm{K}(m) - \frac{A-C}{AC}\,\Pi(-n|m)\right)}_{\mu(T_\nu)} + \frac{2h}{M}\underbrace{4s^{-1}\mathrm{K}(m)}_{T_\nu}\\
&= -2\pi - \mu(T_\nu) + \frac{2h}{M}\,T_\nu,
\end{aligned} \tag{2.134}$$

that we may write

$$2\pi + \mu(T_\nu) = \frac{2h}{M}T_\nu - \Omega = \Delta\theta. \tag{2.135}$$

Finally, considering that μ could have given more than one rotation when $\nu = 2\pi$ we will have

$$\mu(T_\nu) = 2k\pi + \Delta\theta(\mathrm{mod}\,2\pi), \qquad 0 \le k. \tag{2.136}$$

2.6.5 Recovering Montgomery formula via the trajectory $\mu(\nu)$

Montgomery's question can be answered by obtaining the value of the Andoyer angle μ when ν has rotated 2π. To do this, we can make use of the relation given by (2.128). Thus, after $\Delta t = T_\nu$ we have that

$$\mu(\Delta\nu = 2\pi) = 4M^{-1}s^{-1}\left[2h\,\mathrm{K}(m) - \frac{2hB-M^2}{B}\Pi(\alpha_3|m)\right]. \tag{2.137}$$

On the other side, note that this expression apparently does not relate to the spherical area (2.122) defined by the intersection of the energy ellipsoid and the momentum. Nevertheless, by using the relation between the Legendre first and third complete elliptic integrals given by

$$(\alpha_3 - m)\,\Pi(\alpha_3|m) = (n+m)\left[\Pi(-n|m) - \frac{m}{n+m}\,\mathrm{K}(m)\right], \tag{2.138}$$

the aforementioned spherical area (2.122) may be written in the form

$$\mathcal{A} = 2\pi M^2 - 4Ms^{-1}\frac{2hB-M^2}{B}\Pi(\alpha_3|m). \tag{2.139}$$

In our case, the oriented area is again $\mathcal{A}_O = -\mathcal{A}$. Thus,

$$\mathcal{A}_O = -2\pi M^2 + 4Ms^{-1}\frac{2hB - M^2}{B}\Pi(\alpha_3|m). \tag{2.140}$$

The corresponding solid angle is equal to

$$\Omega = \frac{\mathcal{A}_O}{M^2} = -2\pi + 4M^{-1}s^{-1}\frac{2hB - M^2}{B}\Pi(\alpha_3|m), \tag{2.141}$$

which, taking into account (2.129) and (2.137), can be written

$$\begin{aligned}\Omega &= -2\pi - \mu(\nu = 2\pi) + \frac{2h}{M}\underbrace{4s^{-1}\mathrm{K}(m)}_{T_\nu} \\ &= -2\pi - \mu(\nu = 2\pi) + \frac{2h\,T_\nu}{M},\end{aligned} \tag{2.142}$$

that we may write

$$2\pi + \mu(\nu = 2\pi) = \frac{2h}{M}T_\nu - \Omega = \Delta\theta. \tag{2.143}$$

Like in the previous proof, as μ could have given more than one rotation when $\nu = 2\pi$ we will have

$$\mu(\nu = 2\pi) = 2\pi k + \Delta\theta(\mathrm{mod}\,2\pi), \qquad 0 \leq k. \tag{2.144}$$

2.6.6 A new phase for the free rigid body

As we have just shown, depending on the shape of the body, Montgomery formula could require the use of a counter allowing us to keep control of the number of turns given by μ, if necessary. This circumstance forces us to resort to $\mathrm{mod}\,2\pi$ which leads us finally to the $\Delta\mu$ associated to $t = T_\nu$.

During the preparation of this note we came into the consideration that in our problem there is another phase, that is, to reckon how much has rotated ν when μ has rotated 2π. Indeed, taking $\mu = 2\pi$ and replacing $d = 2h/M^2$ in (2.128), we may write

$$2\pi = \frac{M}{s}\left[\frac{2h}{M^2}\,\mathrm{F}(\Delta\phi|m) - \frac{2hB - M^2}{B}\Pi(\alpha_3;\Delta\phi|m)\right], \tag{2.145}$$

where $\Delta\phi$ represents the fraction of the auxiliary angle we are looking for. Then, taking into account (2.141), this last equation can be put into the form

$$2\pi = \frac{2h}{s\,M}\,\mathrm{F}(\Delta\phi|m) - \frac{1}{4}(2\pi + \Omega)\frac{\Pi(\alpha_3;\Delta\phi|m)}{\Pi(\alpha_3|m)}, \tag{2.146}$$

which is an implicit relation from where one can obtain $\Delta\phi$ when known the first integrals and the solid angle. The corresponding $\Delta\nu = \Delta\nu(\Delta\phi)$ can then be found by using (3.86).

If one wishes to identify dynamic and geometric phases, they are contained in the previous formulas (2.130) and (2.146) both being considered as parametrized by the energy h in the form

$$T_\mu = T_\mu(h), \qquad \Delta\nu = \Delta\nu(h). \tag{2.147}$$

Chapter 3

On the complete reduction of the free rigid body

The complete reduction of the Euler-Poinsot problem may be performed in different sets of variables. Action-angle variables are usually preferred because of their suitability to approach perturbed rigid-body motion, but they are just one among the variety of sets of canonical variables that integrate the problem. Thus, in this chapter we will revisit them and present an alternate set of variables proposed by Ferrer and Lara [Lara & Ferrer (2010b)] which shows an important advantage over the action-angle one: the transformation from and to Andoyer variables is given in an explicit form while implicit equations must be solved when dealing with action-angle variables.

Meanwhile many authors still rely on Eulerian variables when dealing with rotational dynamics (see for instance [Marsden & Ratiu (1999)]), the formulation of H-J-P equation in Andoyer variables fully take into account the symmetries of this system [Andoyer (1923), Deprit (1967)]. Thus, we use Andoyer variables to find a family of transformations allowing the complete reduction of the Euler-Poinsot problem. However, in the first stages of this procedure the explicit expression of the Hamiltonian in the new variables is still to be chosen, hence symmetries and other geometric considerations should be addressed in the search for new variables. These symmetries relate to the symplectic charts used to formulate the H-J equation, and the corresponding transformation. It is worth to notice that we only deal with canonical transformations derived from integrable Hamiltonians, which are cyclic in all but one variable, allowing the analysis of different choices and avoiding to deal with rather cumbersome expressions.

Indeed, although a general expression for the new variables is here proposed within the frame given by Ferrer and Lara in [Ferrer & Lara (2010b)], where different choices produce different sets of variables, in this chapter we will deal only with the set proposed by Sadov [Sadov (1970a)] and the aforementioned set proposed by Ferrer and Lara. It is worth to remark that both sets of variables are obtained by using the same angle as an intermediate variable when solving the quadratures involved in the H-J-P equation. Furthermore, as the Ferrer-Lara's set is non-action-angle, their variables are not considered to be angles.

Due to the existence of a relatively wide literature on complete reductions of the free rigid body problem, we show a table containing the notation used by several authors.

	Authors				
Param.	Ferrer-Lara	Deprit-Elipe	Sadov	Kinoshita	Hitzl-Breakwell
Old var.	$(\nu, \mu, \lambda, N, M, \Lambda)$	(ℓ, g, h, L, G, H)	$(\varphi, \psi, h, G_\zeta, G, L)$	$(\ell_1, g_1, h_1, L_1, G_1, H_1)$	$(q_1, q_2, q_3, p_1, p_2, p_3)$
New var.	(ℓ, g, h, L, G, H)	$(\delta, \gamma, ?, \Delta, \Gamma, ?)$	(f, ν, h, I, G, L)	$(\beta_1, \beta_2, \beta_3, \alpha_1, \alpha_2, \alpha_3)$	$(\beta_1, \beta_2, \beta_3, \alpha_1, \alpha_2, \alpha_3)$
M. inertia	$A < B < \Delta < C$	$I_1 < I_2 < J < I_3$	$A < B < C$	$A < B < C$	$A < B < C$
Int. var.	ψ	$\{\phi, \psi\}$	$\xi = -\psi$	δ	$\delta; (\pi/2 - \delta)?$
Module	m	$k_3(\alpha_3, \alpha_3^0)$	$\lambda^2 = m$	k	k
Charac.	$-f$	α_3	$\kappa^2 = f$	$e_1; 1 + \kappa^2 = \frac{1+e_1}{1-e_1}$	$\nu = \frac{f}{2+f}$

Table 3.1: Table containing the different variables used by several authors in the complete reductions of the free rigid body.

3.1 A General Form of the Hamilton-Jacobi Equation

We borrow from [Ferrer & Lara (2010b)] the explanation of the basic concepts related to the Hamilton-Jacobi-Poincaré equation.

We only deal with Hamiltonians of the type

$$\mathcal{K}(x_0, x, X_0, X; \mu) \equiv (X_0 + \mathcal{H})\, \chi, \tag{3.1}$$

where $x = (x_1, \ldots, x_n)$ are coordinates and $X = (X_1, \ldots, X_n)$ conjugate momenta; x_0 is the independent variable and X_0 its conjugate momentum in the extended phase space formulation, in which we restrict to the manifold $\mathcal{K} = 0$; μ is a vector of parameters, and the Hamiltonian $\mathcal{H}$ as well as the "regularizing factor" χ may depend on all or some of the parameters defining μ:

$$\chi = \chi(x_0, x, X_0, X; \mu), \qquad \mathcal{H} = \mathcal{H}(x_0, x, -, X; \mu). \tag{3.2}$$

A dash in the place of a variable is used to remark that the corresponding variable is not present.

Hamilton equations are

$$\begin{aligned} \frac{\mathrm{d}x_0}{\mathrm{d}\tau} &= \frac{\partial \mathcal{K}}{\partial X_0}, & \frac{\mathrm{d}X_0}{\mathrm{d}\tau} &= -\frac{\partial \mathcal{K}}{\partial x_0}, \\ \frac{\mathrm{d}x_i}{\mathrm{d}\tau} &= \frac{\partial \mathcal{K}}{\partial X_i}, & \frac{\mathrm{d}X_i}{\mathrm{d}\tau} &= -\frac{\partial \mathcal{K}}{\partial x_i}, \qquad (i = 1, \ldots, n) \end{aligned} \tag{3.3}$$

where τ is the evolution parameter of the flow. Note that the first of the previous equations reads

$$\frac{\mathrm{d}x_0}{\mathrm{d}\tau} = \chi \tag{3.4}$$

which tells the function χ ought to verify that $\chi > 0$ in its domain. Moreover, in the case of conservative systems $\mathcal{H} = \mathcal{H}(-, x, -, X)$, X_0 is an integral, and the manifold $\mathcal{K} = 0$ may be also seen as $X_0 = -\mathcal{H} =$ constant.

We are interested in canonical transformations

$$(x_0, x, X_0, X) \xrightarrow{\mathcal{T}_\Phi} (y_0, y, Y_0, Y) \tag{3.5}$$

in the sense of Poincaré. More precisely, we look for transformations such that they simplify Hamiltonian systems defined by functions $\mathcal{H}$ which can be written as

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 \tag{3.6}$$

where $\mathcal{H}_0 = \mathcal{H}_0(-, x, -, X)$ defines an integrable system, and $\mathcal{H}_1 = \mathcal{H}_1(x_0, x, X)$ is a perturbation. Specifically, we focus on canonical transformations such that the Hamiltonian in the new variables $K = K_0 + K_1$ satisfies

$$K_0 = (X_0 + \mathcal{H}_0)\,\chi = \Phi(-, -, Y_0, Y), \tag{3.7}$$

i.e. the full reduction of the unperturbed part is carried out.

The transformations are defined by

$$X_i = \frac{\partial \mathcal{W}}{\partial x_i}, \qquad y_i = \frac{\partial \mathcal{W}}{\partial Y_i}, \qquad i = 0, \ldots, n \tag{3.8}$$

derived from a generator $\mathcal{W} = \mathcal{W}(x_0, x, Y_0, Y, \mu)$ that is a complete solution of the generalized H-J equation

$$\left[\frac{\partial \mathcal{W}}{\partial x_0} + \mathcal{H}_0\left(x, \frac{\partial \mathcal{W}}{\partial x}\right)\right] \chi\left(x, x_0, \frac{\partial \mathcal{W}}{\partial x_0}, \frac{\partial \mathcal{W}}{\partial x}\right) = \Phi(Y_0, Y). \tag{3.9}$$

This Eq. (3.9) is what we refers as the variant of Poincaré to the H-J equation; the classical case chooses $\Phi(-, -, Y_0, Y) \equiv 0$.

Thus, the Hamiltonian $\mathcal{K}$ in the new variables will take the form

$$\mathcal{K} = \mathcal{K}_0 + \mathcal{K}_1 = \Phi \, + \, \mathcal{H}_1\,\chi \tag{3.10}$$

where $\mathcal{H}_1$ and χ are expressed in the new variables. Note that in what follows we take $\mathcal{H}_1 = 0$.

In this paper we limit to generators of the form

$$\mathcal{W} = \sum_{0 \le i < n-1} x_i\, Y_i + \mathcal{R}(x_n, Y_0, Y), \tag{3.11}$$

and regularizing function $\chi = \chi(x_n, X_0, X_1, \ldots, X_{n-1})$. Hence, from Eq. (3.9) we may write

$$\mathcal{H}_0\left(x, Y_1, \ldots, Y_{n-1}, \frac{\partial \mathcal{R}}{\partial x_n}\right) = \frac{\Phi(Y_0, Y_1, \ldots, Y_n)}{\chi(x_n, Y_0, Y_1, \ldots, Y_{n-1})} - Y_0. \tag{3.12}$$

Depending on the form of $\mathcal{H}_0$, Eq. (3.12) may be solved for $\partial\mathcal{R}/\partial x_n$ and, therefore, $\mathcal{R}$ is computed from a quadrature, whose solution will depend on the choices made for Φ and χ. Note that, in fact, there is no reason why we should impose on $\mathcal{H}_0$ to be cyclic in x_0. What we have presented above, properly adapted, remains valid if we lift that constraint. This is referred in the literature as nonautonomuos systems; the driven oscillator, the relativistic particle, etc are just simple examples within that category (for other systems of interest see for instance recent papers in Journal of Geometric Mechanics). In the families we will study below the possible presence of x_0 occurs in the perturbing part.

Apart from the general case of Eq. (3.12), transformations non-based on the homogeneous formalism ($\chi \equiv 1$, $\Phi = Y_0 + \Psi$), adopt the simpler formulation

$$\mathcal{H}_0\left(x_n, Y_1, \ldots, Y_{n-1}, \frac{\partial \mathcal{R}}{\partial x_n}\right) = \Psi(Y_1, \ldots, Y_n). \tag{3.13}$$

3.2 The Hamilton-Jacobi-Poincaré in Andoyer variables

In this section we will tackle the complete reduction of the Euler-Poinsot problem from Andoyer variables $(\nu, \mu, \lambda, N, M, \Lambda)$ to a new set of canonical variables (ℓ, g, h, L, G, H) by solving the well-known Hamilton-Jacobi-Poincaré equation. The goal is that the new reduced Hamiltonian depends only on the new momenta. Note that in the unperturbed problem given by (3.14), λ and Λ do not appear because they are constant.

The Hamiltonian of the torque-free motion in Andoyer variables is [Deprit (1967)]

$$\mathcal{H}_0(\nu, -, N, M; \upsilon) = \left(\frac{\sin^2\nu}{2A} + \frac{\cos^2\nu}{2B}\right)(M^2 - N^2) + \frac{N^2}{2C}. \tag{3.14}$$

where $\upsilon = \{A, B, C\}$ is a vector containing the three principal moments of inertia where, in our case, with no loss of generality $A < B < C$. We note that λ, Λ and μ are cyclic and therefore $\lambda = \lambda_0$, $\Lambda = \Lambda_0$, and $M = M_0$ are constant. Then the Hamiltonian of the Euler-Poinsot problem is of 1-DOF, and the integration of the Hamilton equations for μ, ν, and N can be solved by quadrature, accepting a closed form solution in elliptic integrals. In the case of zero inclination of the angular momentum plane with respect to either the inertial plane or the equatorial plane of the body, Andoyer variables are singular. These singularities are virtual and may be avoided using a different set of variables [Fukushima (1994)].

In order to carry out the complete reduction of the previous Hamiltonian we have to solve the Hamilton-Jacobi equation. To do this, we take in the transformation the two new coordinates and the two old momenta as

$$\ell = \frac{\partial \mathcal{S}}{\partial L}, \quad g = \frac{\partial \mathcal{S}}{\partial G}, \quad M = \frac{\partial \mathcal{S}}{\partial \mu} \quad \text{and} \quad N = \frac{\partial \mathcal{S}}{\partial \nu}, \tag{3.15}$$

and by means of this transformation we have to look for two new functions $\mathcal{K}$ and $\mathcal{S}$ satisfying the differential equation

$$\mathcal{K}_0(L, G) = \left(\frac{\sin^2\nu}{2A} + \frac{\cos^2\nu}{2B}\right)\left(\frac{\partial \mathcal{S}}{\partial \mu}\right)^2 + \left(\frac{1}{2C} - \frac{\sin^2\nu}{2A} - \frac{\cos^2\nu}{2B}\right)\left(\frac{\partial \mathcal{S}}{\partial \nu}\right)^2. \tag{3.16}$$

Since the angle μ is cyclic in (3.14) we may separate the coordinates ν and μ by choosing a $\mathcal{S}$-function as

$$\mathcal{S}(\nu, \mu, L, G) = G\mu + \mathcal{W}(\nu, L, G), \tag{3.17}$$

hence from (3.15) we have

$$\ell = \frac{\partial \mathcal{W}}{\partial L}, \quad g = \mu + \frac{\partial \mathcal{W}}{\partial G}, \quad M = G \quad \text{and} \quad N = \frac{\partial \mathcal{W}}{\partial \nu}, \tag{3.18}$$

then, by replacing (3.18) in (3.16) we can work out $\partial\mathcal{W}/\partial\nu$

$$\frac{\partial \mathcal{W}}{\partial \nu} = G\sqrt{Q} \quad \Longrightarrow \quad \mathcal{W} = \int_{\nu_0}^{\nu} G\sqrt{Q}\mathrm{d}\nu, \tag{3.19}$$

where

$$Q(\Delta) = \frac{\dfrac{\sin^2\nu}{A} + \dfrac{\cos^2\nu}{B} + \dfrac{1}{\Delta}}{\dfrac{\sin^2\nu}{A} + \dfrac{\cos^2\nu}{B} + \dfrac{1}{C}} \qquad \text{with} \qquad \frac{1}{\Delta} = \frac{2\mathcal{K}_0}{G^2}. \tag{3.20}$$

Note that $\Delta = \Delta(\mathcal{K}_0, G) = \Delta(L, G)$. Note also that, as can be clearly seen in Eq. (3.19), $Q \geq 0$, which indicates that all our formulas are valid as long as $A < B < \Delta < C$.

Since $\mathcal{W}$ has been obained in (3.19) as a function of Δ, we can already look for the transformation equations from (3.18). For instance, for the first equation we have

$$\ell = \frac{\partial \mathcal{W}}{\partial L} = \frac{\partial}{\partial L}\left(\int_{\nu_0}^{\nu} G\sqrt{Q}\,\mathrm{d}\nu\right) = \int_{\nu_0}^{\nu} G\,\frac{\partial Q/\partial L}{2\sqrt{Q}}\,\mathrm{d}\nu, \tag{3.21}$$

where

$$\frac{\partial Q}{\partial L} = \frac{\partial Q}{\partial(1/\Delta)}\frac{\partial(1/\Delta)}{\partial \mathcal{K}_0}\frac{\partial \mathcal{K}_0}{\partial L} = \frac{\partial Q}{\partial(1/\Delta)}\frac{2}{G^2}\frac{\partial \mathcal{K}_0}{\partial L}, \tag{3.22}$$

and by replacing (3.22) in (3.21), after some calculations we have

$$\ell = \frac{1}{G}\frac{\partial \mathcal{K}_0}{\partial L}\,I_2, \qquad I_2 = \int_{\nu_0}^{\nu} \frac{1}{\sqrt{Q}}\frac{\partial Q}{\partial(1/\Delta)}\,\mathrm{d}\nu. \tag{3.23}$$

Proceeding as before, we can find the equation of g as

$$g = \mu + I_1 - \frac{I_2}{G^2}\left(2\mathcal{K}_0 - G\,\frac{\partial \mathcal{K}_0}{\partial G}\right), \qquad I_1 = \int_{\nu_0}^{\nu} \sqrt{Q}\,\mathrm{d}\nu, \tag{3.24}$$

and, from the fouth euation of (3.18), we finally find

$$N = \frac{\partial \mathcal{W}}{\partial \nu} = \frac{\partial}{\partial \nu}\left(\int_{\nu_0}^{\nu} G\sqrt{Q}\,\mathrm{d}\nu\right) = G\sqrt{Q}, \tag{3.25}$$

hence the transformation equations are[1]

$$\ell = \frac{1}{G}\frac{\partial \mathcal{K}_0}{\partial L}\,I_2, \tag{3.26}$$

$$g = \mu + I_1 - \frac{1}{G^2}\left(2\mathcal{K}_0 - G\,\frac{\partial \mathcal{K}_0}{\partial G}\right) I_2, \tag{3.27}$$

$$M = G, \tag{3.28}$$

$$N = G\sqrt{Q}. \tag{3.29}$$

In the spirit of the Ferrer-Lara paper [Ferrer & Lara (2010b)], the Hamilton-Jacobi Poincaré scheme is the frame in the search for canonical transformations either for a qualitative analysis or perturbation procedures and so on. We show below that from the previous equations a more compact form of them can be found by imposing

$$2\,\mathcal{K}_0 - G\frac{\partial \mathcal{K}_0}{\partial G} = 0, \tag{3.30}$$

from where one can see that for any function $\mathcal{J}(L)$, a new Hamiltonian given by

$$\mathcal{K}_0(G, L) = \frac{1}{2}\frac{G^2}{\mathcal{J}(L)} \tag{3.31}$$

[1] A recent reference where these expressions can be found is [Lara & Ferrer (2013)]. Note that the definitions of the quadratures I_1 and I_2 are changed.

is a solution of the differential equation (3.30).

The consequence of choosing (3.31) is that the transformation (3.26)-(3.29) take now the more compact form

$$\ell = -\frac{G}{2}\frac{1}{\mathcal{J}^2(L)}\frac{\mathrm{d}\mathcal{J}(L)}{\mathrm{d}L}\,I_2, \tag{3.32}$$

$$g = \mu + I_1, \tag{3.33}$$

$$M = G, \tag{3.34}$$

$$N = G\sqrt{Q}. \tag{3.35}$$

Note that in the literature there is a transformation within this scenario. When we choose $\mathcal{J}(L) = L$ we have Deprit-Elipe's transformation (see [Deprit & Elipe (1993)]). The analysis of the family of transformations defined by $\mathcal{J}(L)$ is the content of a forthcoming paper by Molero and Ferrer.

3.3 Legendre elliptic integrals and the Standard Hamiltonian

Nevertheless, in order to get the transformation equations we find convenient to set

$$\beta = L/G, \tag{3.36}$$

thus we will only have to manage a variable as a momenta, not two of them. Moreover, as we will see below, this new scenario will allow us a better flexibility to make choices of new variables because we will make decisions for the value of β, not for L and G independently. Thus, we can differentitate

$$\frac{\partial\beta}{\partial L} = \frac{1}{G}, \tag{3.37}$$

$$\frac{\partial\beta}{\partial G} = -\frac{\beta}{G} \quad\Longrightarrow\quad \frac{\partial\mathcal{K}_0}{\partial G} = -\frac{\partial\mathcal{K}_0}{\partial\beta}\frac{\beta}{G}, \tag{3.38}$$

and replacing this in Eq. (3.26)-(3.29) we get [2]

$$\ell = \frac{I_2}{G^2}\frac{\partial\mathcal{K}_0}{\partial\beta}, \tag{3.39}$$

$$g = \mu + I_1 - \frac{I_2}{G^2}\left(2\mathcal{K}_0 + \beta\,\frac{\partial\mathcal{K}_0}{\partial\beta}\right), \tag{3.40}$$

$$M = G, \tag{3.41}$$

$$N = G\sqrt{Q}. \tag{3.42}$$

The following step consists of computing the quadratures I_1 and I_2. To do this we make use of the well-known change of variable (see [Sadov (1970a), Sadov (1970b)])

$$\cos\psi = \frac{\sqrt{1+f}\sin\nu}{\sqrt{1+f\sin^2\nu}}, \quad \sin\psi = \frac{\cos\nu}{\sqrt{1+f\sin^2\nu}}, \tag{3.43}$$

[2] These expression can be found in [Lara & Ferrer (2010b)].

where[3]

$$f = \frac{C(B-A)}{A(C-B)} > 0. \tag{3.44}$$

However, we have $Q(\nu)$ hence we need to express ν as a function of ψ, that is, inverting (3.43) we have

$$\sin\nu = \frac{\cos\psi}{\sqrt{1+f\sin^2\psi}}, \quad \cos\nu = \frac{\sqrt{1+f}\sin\psi}{\sqrt{1+f\sin^2\psi}}, \tag{3.45}$$

and from the first equation of (3.45) we get

$$\mathrm{d}\nu = -\frac{\sqrt{1+f}}{1+f\sin^2\psi}\,\mathrm{d}\psi. \tag{3.46}$$

In order to solve the cuadrature, we are interested in expressing Q as a function only of sines, thus, by using the well-known relation $\sin^2\nu + \cos^2\nu = 1$ and doing some calculations we have

$$Q = \frac{C(\Delta-B)}{\Delta(C-B)}\frac{1+p\sin^2\nu}{1+f\sin^2\nu} \quad \text{with} \quad p = \frac{\Delta(B-A)}{A(\Delta-B)}, \tag{3.47}$$

hence, by doing some more calculations in Eqs. (3.23) and (3.24) we obtain the standard form of the elliptic integrals of the first and third kinds, which allow us to give the quadratures I_1 and I_2 as

$$I_1 = \gamma\left[\frac{m}{f+m}\,\mathrm{F}(\psi|m) - \Pi(-f;\psi|m)\right], \tag{3.48}$$

$$I_2 = \gamma\,\frac{A\,C}{C-A}\,\mathrm{F}(\psi|m), \tag{3.49}$$

where

$$m = \frac{(C-\Delta)(B-A)}{(C-B)(\Delta-A)}, \tag{3.50}$$

$$\gamma = \sqrt{\frac{(1+f)(f+m)}{f}}, \tag{3.51}$$

being $0 < m < 1$ the *elliptic modulus*, from which it is deduced that $\Delta < C$.

Furthermore, from the second equation of (3.20) and Eq. (3.50) we get

$$\Delta(m) = \frac{m\,A(C-B)+C(B-A)}{m(C-B)+(B-A)} \quad \Longrightarrow \quad \frac{1}{\Delta} = \frac{1}{A}\left(1-\frac{C-A}{C}\frac{f}{f+m}\right) \tag{3.52}$$

and by replacing it in the second equation of (3.20) we have

$$\mathcal{K}_0(L,G) = \frac{G^2}{2A}\left(1-\frac{C-A}{C}\frac{f}{f+m}\right), \tag{3.53}$$

[3]In his treatment of the full reduction, [Kinoshita (1972)] uses a sligthly different intermediate variable $\delta = \psi \pm \pi/2$ and, indtead of f, he built his expressions on a triaxiality coefficient defined by Andoyer $e_1 = \chi$.

which appears in Sadov papers with no number attached to it and Ferrer-Lara called *The Standard Hamiltonian* in [Lara & Ferrer (2010b)]. Note that $\mathcal{K}_0$ is explicitly a function of G, hence, at least, it has to be a function of L through m, but, in general $m = m(L, G)$ which indicates that we may make choices either by means of the new Hamiltonian $\mathcal{K}_0$ or by means of the elliptic modulus itself.

As the Standard Hamiltonian is already known, we can now find its partial derivatives. However, this is not a trivial task because the result of $\partial\mathcal{K}_0/\partial\beta$ is not the same for (3.39) and (3.40) since, as above commented, we take derivatives in (3.39) according to L knowing that $\mathcal{K}_0 = \mathcal{K}_0(G, m(L, G))$ and we take derivatives in (3.40) according to G. Starting from (3.39), since $\mathcal{K}_0$ depends only on L through m, we have that

$$\frac{\partial\mathcal{K}_0}{\partial\beta} = \frac{\partial\mathcal{K}_0}{\partial m}\frac{\partial m}{\partial\beta}. \tag{3.54}$$

From Eq. (3.53) we may obtain

$$\frac{\partial\mathcal{K}_0}{\partial m} = \frac{fG^2(C-A)}{2AC(f+m)^2} \tag{3.55}$$

hence, by replacing (3.55) and (3.54) in (3.39) and taking into account (3.49) we finally get

$$\ell = \frac{1}{2\gamma}\frac{1+f}{f+m}\frac{\partial m}{\partial\beta}\,\mathrm{F}(\psi|m). \tag{3.56}$$

Nevertheless, for (3.40) we have to take into account that $\mathcal{K}_0$ depends explicitly on G and through m, therefore, making use of (3.38)

$$\begin{aligned}
\frac{\partial\mathcal{K}_0}{\partial\beta} &= -\frac{G}{\beta}\frac{\partial\mathcal{K}_0}{\partial G} \\
&= -\frac{G}{\beta}\left[\frac{G}{A}\left(1-\frac{C-A}{C}\frac{f}{f+m}\right)+\frac{G^2}{2A}\frac{1}{(f+m)^2}\frac{f}{C}(C-A)\frac{\partial m}{\partial G}\right] \\
&= -\frac{G^2}{A\beta}\left(1-\frac{C-A}{C}\frac{f}{f+m}\right)+\frac{G^2}{2A}\frac{1}{(f+m)^2}\frac{f}{C}(C-A)\frac{\partial m}{\partial\beta}.
\end{aligned} \tag{3.57}$$

Once more, by replacing (3.57) and (3.53) in (3.40) and taking into account (3.48) and (3.49) we finally get

$$g = \mu + \gamma\left[\frac{1}{f+m}\left(m - \frac{f}{f+m}\frac{\beta}{2}\frac{\partial m}{\partial\beta}\right)\mathrm{F}(\psi|m) - \Pi(-f;\psi|m)\right] \tag{3.58}$$

Since Eq. (3.41) remains unchanged, we just have to update Eq. (3.42). To do this, if we express the first equation of (3.20) as a function only of cosines, after several calculations we find

$$N = G\sqrt{\frac{f}{f+m}}\sqrt{1-m\sin^2\psi}. \tag{3.59}$$

Therefore, taking into account Eq. (3.51), the new transformation equations can finally be written as [4]

$$\ell = \frac{1}{X(m)}\,\mathrm{F}(\psi|m), \tag{3.60}$$

$$g = \mu + \sqrt{\frac{(1+f)(f+m)}{f}}\left[Y(m,\beta)\mathrm{F}(\psi|m) - \Pi(-f;\psi|m)\right], \tag{3.61}$$

$$M = G, \tag{3.62}$$

$$N = G\sqrt{\frac{f}{f+m}}\sqrt{1-m\sin^2\psi}. \tag{3.63}$$

where

$$X(m) = \frac{2}{\partial m/\partial\beta}\frac{(f+m)^{3/2}}{\sqrt{f(1+f)}}, \qquad Y(m,\beta) = \frac{1}{f+m}\left(m - \frac{f}{f+m}\frac{\beta}{2}\frac{\partial m}{\partial\beta}\right). \tag{3.64}$$

are the functions on which we will do the different choices leading to the different sets of new variables.

Nevertheless, the previous equations are still formed by a mixture of old and new variables and we need both the direct and the inverse transformation equations. In order to get the set of direct equations, Eqs. (3.62) and (3.63) have to be inverted. To do this, by working out $\partial m/\partial\beta$ from the first equation of (3.64) and replacing it in the second equation of (3.64), we find

$$\beta(X(m),Y(m)) = \frac{L}{G} = X(m)\sqrt{\frac{(1+f)(f+m)}{f}}\left[\frac{m}{f+m} - Y(m)\right], \tag{3.65}$$

from which one can work out L as a function of the old variables. Therefore, the direct transformation equations are given by

$$\ell = \frac{1}{X(m)}\,\mathrm{F}(\psi|m), \tag{3.66}$$

$$g = \mu + \sqrt{\frac{(1+f)(f+m)}{f}}\left[Y(m,\beta)\,\mathrm{F}(\psi|m) - \Pi(-f;\psi|m)\right], \tag{3.67}$$

$$G = M, \tag{3.68}$$

$$L = M\,X(m)\sqrt{\frac{(1+f)(f+m)}{f}}\left[\frac{m}{f+m} - Y(m,\beta)\right]. \tag{3.69}$$

[4] If we had not chosen to set $\beta = L/G$, the new transformation equations would be given by

$$\begin{aligned}
\ell &= \frac{G}{2\gamma}\frac{1+f}{f+m}\frac{\partial m}{\partial L}\,\mathrm{F}(\psi|m),\\
g &= \mu + \gamma\left[\frac{1}{f+m}\left(m + \frac{f}{f+m}\frac{G}{2}\frac{\partial m}{\partial G}\right)\mathrm{F}(\psi|m) - \Pi(-f;\psi|m)\right],\\
M &= G,\\
N &= G\sqrt{\frac{f}{f+m}}\sqrt{1-m\sin^2\psi}.
\end{aligned}$$

It is also possible to obtain the set of inverse transformation equations, that is, formulas expressing the old variables as functions of the new variables. Thus, by working out ψ from Eq. (3.66) we have

$$\psi = \mathrm{am}\,(X(m)\,\ell|m)\,, \tag{3.70}$$

and replacing it in Eq. (3.45), (3.67) and (3.63), the set of inverse transformation equations can be written as

$$\sin\nu = \frac{\mathrm{cn}\,(X(m)\,\ell|m)}{\sqrt{1+f\,\mathrm{sn}^2\,(X(m)\,\ell|m)}}, \quad \cos\nu = \frac{\sqrt{1+f}\,\mathrm{sn}\,(X(m)\,\ell|m)}{\sqrt{1+f\,\mathrm{sn}^2\,(X(m)\,\ell|m)}} \tag{3.71}$$

$$\mu = g - \sqrt{\frac{(1+f)(f+m)}{f}}\,[X(m)Y(m,\beta)\,\ell - \Pi(-f;\mathrm{am}(X(m)\,\ell)|m)|m]\,, \tag{3.72}$$

$$M = G, \tag{3.73}$$

$$N = G\sqrt{\frac{f}{f+m}}\mathrm{dn}\,(X(m)\,\ell|m)\,. \tag{3.74}$$

As said before, we may make choices either on the new Hamiltonian function $\mathcal{K}_0$ or on the elliptic modulus m, that is, if a new Hamiltonian function (o elliptic modulus) has been chosen for any reason, it is necessary to verify the existence of a transformation making possible such a choice. The point is that if the choice is very fussy, it may not be found a transformation satisfying it or being this too complex, therefore, to avoid this problem it is easier and safer to look for sets of new variables after obtained the transformation equations, making choices on them.

Following [Lara & Ferrer (2010a)] three properties are always desirable in every symplectic transformation:

1. to retain the topology of the Euler-Poinsot problem. In our case, this means that the problem has to be still of two degrees of freedom.

2. to adhere to KAM conditions, which is always satisfied when the Hessian of the transformation is not equal to zero.

3. to be explicit and as simple as possible. Implicit equations are more difficult to manipulate.

In what remains in this chapter, we concentrate in two choices within the previous scenario.

3.4 Action-angle variables. Sadov transformation

3.4.1 The original transformation

The proposal by Sadov (see [Sadov (1970a)]) consisted of taking the two new coordinates to be angles. To achieve this goal under our scheme, $X(m)$ and $Y(m)$ in Eq. (3.64) has to be set as

$$X(m) = -\frac{2\,\mathrm{K}(m)}{\pi}, \qquad Y(m) = \frac{\Pi(-f|m)}{\mathrm{K}(m)}, \tag{3.75}$$

which produces, for instance, that the first direct transformation equation (3.66) can be written as

$$\ell = -\frac{\pi}{2\,\mathrm{K}(m)}\,\mathrm{F}(\psi|m). \tag{3.76}$$

Observe that ℓ is actually an angle. When $\mathrm{F}(\psi|m) = 0$ then $\ell = 0$, and when $\mathrm{F}(\psi|m) = \pm 4\,\mathrm{K}(m)$ then $\ell = \pm 2\pi$ (see Appendix A for further details on elliptic functions). Proceeding as in the previous section, by replacing (3.75) in (3.65), the new expression for β is given by

$$\beta = \frac{2}{\pi}\sqrt{\frac{(1+f)(f+m)}{f}}\left[\Pi(-f|m) - \frac{m}{f+m}\,\mathrm{K}(m)\right], \tag{3.77}$$

from which can be seen that, in this case, we do not explicitly work out m as a function of β, therefore the Standard Hamiltonian given in (3.53) remains unchanged. This means that we may not express explicitly the new Hamiltonian as a function of the new momenta, but implicitly through the Eq. (3.77).

Since the action-angle variables will be use in the subsequent sections and they will be compared with Ferrer-Lara variables, it is necessary to give a new name to the action-angle variables, thus we set $\ell \equiv \varphi_\nu$; $g \equiv \varphi_\mu$; $L \equiv I_N$; $G \equiv I_M$. Taking into account this change of notation, by replacing Eq. (3.75) in Eqs. (3.66)-(3.69) the direct transformation equations are given by[5]

$$\begin{aligned}
\varphi_\nu &= -\frac{\pi}{2K(m)}F(\psi|m), && (3.78)\\
\varphi_\mu &= \mu + \sqrt{(1+f)(f+m)/f}\left[\frac{\Pi(-f|m)}{K(m)}F(\psi|m) - \Pi(-f,\psi|m)\right], && (3.79)\\
I_M &= M, && (3.80)\\
I_N &= \frac{2M}{\pi}\sqrt{(1+f)(f+m)/f}\left[\Pi(-f|m) - \frac{m}{f+m}K(m)\right]. && (3.81)
\end{aligned}$$

Proceeding as before, the set of inverse transformation equations is given by

$$\begin{aligned}
\sin\nu &= \frac{\mathrm{cn}\left(\frac{2K(m)}{\pi}\,\varphi_\nu|m\right)}{\sqrt{1+f\,\mathrm{sn}^2\left(\frac{2K(m)}{\pi}\,\varphi_\nu|m\right)}}, \quad \cos\nu = -\frac{\sqrt{1+f}\,\mathrm{sn}\left(\frac{2K(m)}{\pi}\,\varphi_\nu|m\right)}{\sqrt{1+f\,\mathrm{sn}^2\left(\frac{2K(m)}{\pi}\,\varphi_\nu|m\right)}}, && (3.82)\\
\mu &= \varphi_\mu + \gamma\left[\frac{2}{\pi}\,\Pi(-f|m)\,\varphi_\nu + \Pi\left(-f, -\mathrm{am}\left(\frac{2K(m)}{\pi}\,\varphi_\nu|m\right)|m\right)\right], && (3.83)\\
M &= I_M, && (3.84)\\
N &= I_M\sqrt{\frac{f}{f+m}}\,\mathrm{dn}\left(\frac{2K(m)}{\pi}\,\varphi_\nu|m\right). && (3.85)
\end{aligned}$$

[5]Note that, according to [Sadov (1970a)], $f \equiv \kappa^2$ and $m \equiv \lambda^2$. Note also that by setting $\psi = -\xi$, the transformation (3.78)-(3.81) become the same than Sadov's one except for the definition of the elliptic integral Π, where the characteristic has a different sign.

3.4.2 New expression for Sadov transformation

In the literature there is a well-known alternative auxiliary variable when dealing with the quadratures related to the H-J equation given by

$$\sin\nu = \frac{\sqrt{1-\alpha_3}\sin\phi}{\sqrt{1-\alpha_3\sin^2\phi}}, \quad \cos\nu = \frac{\cos\phi}{\sqrt{1-\alpha_3\sin^2\phi}}, \quad \mathrm{d}\nu = \frac{\sqrt{1-\alpha_3}}{1-\alpha_3\sin^2\phi}\mathrm{d}\phi, \tag{3.86}$$

hence the transformation (3.78)-(3.81) takes the form

$$\varphi_\nu = \frac{\pi}{2\mathrm{K}(m)}\,\mathrm{F}(\phi|m), \tag{3.87}$$

$$\varphi_\mu = \mu + \gamma\left[\Pi(\alpha_3,\phi|m) - \frac{\Pi(\alpha_3|m)}{\mathrm{K}(m)}\,\mathrm{F}(\phi|m)\right], \tag{3.88}$$

$$I_M = M, \tag{3.89}$$

$$I_N = \frac{2M}{\pi}\gamma\,\Pi(\alpha_3|m), \tag{3.90}$$

where

$$\gamma = (1-m)\sqrt{\frac{\epsilon(1-\epsilon)}{\epsilon+m(1-\epsilon)}}, \quad \epsilon = \frac{f}{1+f} = \frac{C(B-A)}{B(C-A)} < 1, \quad \alpha_3 = \epsilon + m(1-\epsilon). \tag{3.91}$$

To obtain the inverse set of transformation equations, from (3.87) we have

$$\phi = \mathrm{am}\left(\frac{2\mathrm{K}(m)}{\pi}\,\varphi_\nu\middle|m\right), \tag{3.92}$$

which transforms Eq. (3.86) into

$$\sin\nu = -\frac{\sqrt{1-\alpha_3}\,\mathrm{sn}\left(\frac{2\mathrm{K}(m)}{\pi}\,\varphi_\nu\middle|m\right)}{\sqrt{1-\alpha_3\,\mathrm{sn}^2\left(\frac{2\mathrm{K}(m)}{\pi}\,\varphi_\nu\middle|m\right)}}, \quad \cos\nu = \frac{\mathrm{cn}\left(\frac{2\mathrm{K}(m)}{\pi}\,\varphi_\nu\middle|m\right)}{\sqrt{1-\alpha_3\,\mathrm{sn}^2\left(\frac{2\mathrm{K}(m)}{\pi}\,\varphi_\nu\middle|m\right)}}, \tag{3.93}$$

and the rest of the equations are given by

$$\mu = \varphi_\mu - \gamma\left[\Pi(\alpha_3,\phi|m) - \frac{2}{\pi}\,\varphi_\nu\,\Pi(\alpha_3|m)\right], \tag{3.94}$$

$$M = I_M, \tag{3.95}$$

$$N = I_M\sqrt{\frac{\epsilon(1-m)}{\epsilon+m(1-\epsilon)}}\,\frac{1}{\mathrm{dn}\left(\frac{2\mathrm{K}(m)}{\pi}\,\varphi_\nu\middle|m\right)}. \tag{3.96}$$

3.4.3 The transformation in Jacobi Theta functions

The way in which perturbation theory will be treated in this Memoir needs to make use of the transformation expressed in Jacobi Theta functions. Since the original Sadov's set has been already given in [Sadov (1970b), Barkin (1992)] and [Vallejo (1995)] (pages 47 − 48), we will only look for the expressions of the new set as given in Eqs. (3.93)-(3.96).

As observed, according to (3.93), to express the above transformation in Jacobi Theta functions, the first obstacle one finds is how to express in these Theta functions (biparametric functions) a three-parameter expression, this is, φ_ν, m and α_3. This problem can be avoided[6] by means of a mathematical resource given by

$$m\,\mathrm{sn}^2(a|m) = \alpha_3, \qquad \alpha_3 > m \tag{3.97}$$

from where the amplitude

$$a = \mathrm{F}\left(\arcsin\sqrt{\frac{\alpha_3}{m}}\middle|m\right) = \mathrm{F}\left(\frac{\pi}{2} - i\varphi\middle|m\right). \tag{3.98}$$

According to [Byrd & Friedman (1971)] (formula 115.01) we can express

$$\mathrm{F}(\pi/2 - i\varphi) = \mathrm{F}(\beta|m) - i\,\mathrm{F}(\alpha|1-m), \tag{3.99}$$

where it can be found that

$$\sin\beta = \frac{\pi}{2}, \qquad \sin\alpha = \frac{\tanh\varphi}{\sqrt{1-m}}, \tag{3.100}$$

and therefore (3.98) can be expressed as

$$a = \mathrm{K}(m) - i\,\mathrm{F}\left[\arcsin\left(\frac{\tanh\varphi}{\sqrt{1-m}}\right)\middle|1-m\right]. \tag{3.101}$$

To find φ in our notation, taking into account [Lawden (1989)] (formula 5.7.44), after some calculations we see that

$$a = \mathrm{K}(m) - i\,\mathrm{F}\left[\arcsin\sqrt{\frac{\alpha_3 - m}{\alpha_3(1-m)}}\middle|1-m\right]. \tag{3.102}$$

However, since Theta functions deal with angles as arguments and a is an amplitude, we are interested in denoting

$$a = \frac{2\,\mathrm{K}(m)}{\pi}\,z \tag{3.103}$$

hence

$$z = \frac{\pi}{2\,\mathrm{K}(m)}\,a = \frac{\pi}{2} - i\,\frac{\pi}{2\,\mathrm{K}(m)}\,\mathrm{F}\left[\arcsin\sqrt{\frac{\alpha_3 - m}{\alpha_3(1-m)}}\middle|1-m\right]. \tag{3.104}$$

In particular, with these equations, the procedure to express the term $1 - \alpha_3\,\mathrm{sn}^2$ in Jacobi Theta functions is carried out by applying the quasi-addition formulas of the Theta functions. Specifically, from formula 1.4.19 in [Lawden (1989)] (page 9) we have

$$\vartheta_4(x+y,q)\vartheta_4(x-y,q)\vartheta_4^2(0,q) = \vartheta_4^2(x,q)\vartheta_4^2(y,q) - \vartheta_1^2(x,q)\vartheta_1^2(y,q) \tag{3.105}$$

[6] The reader may also see [Byrd & Friedman (1971), Vallejo (1995), **?**]

where x and y are two angles and

$$\vartheta_4^2(0,q) = \frac{2\mathrm{K}(m)}{\pi}\sqrt{1-m} \tag{3.106}$$

Furthermore, taking into account

$$\mathrm{sn}(u|m) = \frac{1}{\sqrt[4]{m}}\frac{\vartheta_1(\alpha_1,q)}{\vartheta_4(\alpha_1,q)}, \tag{3.107}$$

$$\mathrm{cn}(u|m) = \sqrt[4]{\frac{1-m}{m}}\frac{\vartheta_2(\alpha_1,q)}{\vartheta_4(\alpha_1,q)}, \tag{3.108}$$

$$\mathrm{dn}(u|m) = \sqrt[4]{1-m}\,\frac{\vartheta_3(\alpha_1,q)}{\vartheta_4(\alpha_1,q)}. \tag{3.109}$$

where $\alpha_1 = \{x,y\}$ the above expression can be put into the form

$$\begin{aligned}\vartheta_4(x+y,q)\vartheta_4(x-y,q)\vartheta_4^2(0,q) &= \vartheta_4^2(x,q)\vartheta_4^2(y,q)\left[1-\frac{\vartheta_1^2(x,q)}{\vartheta_4^2(x,q)}\frac{\vartheta_1^2(y,q)}{\vartheta_4^2(y,q)}\right] \\ &= \vartheta_4^2(x,q)\vartheta_4^2(y,q)\left[1-m\,\mathrm{sn}^2\left(\frac{2\mathrm{K}(m)}{\pi}x|m\right)\mathrm{sn}^2\left(\frac{2\mathrm{K}(m)}{\pi}y|m\right)\right],\end{aligned} \tag{3.110}$$

and note further that if we call $x = \varphi_\nu$ and $y = z$, by taking into account (5.10), then

$$\vartheta_4(\varphi_\nu+z,q)\vartheta_4(\varphi_\nu-z,q) = \frac{\vartheta_4^2(\varphi_\nu,q)\vartheta_4^2(z,q)}{\vartheta_4^2(0,q)}\left[1-\alpha_3\,\mathrm{sn}^2\left(\frac{2\mathrm{K}(m)}{\pi}\varphi_\nu|m\right)\right], \tag{3.111}$$

and therefore we can solve the factor

$$1-\alpha_3\,\mathrm{sn}^2\left(\frac{2\mathrm{K}(m)}{\pi}\varphi_\nu|m\right) = \frac{\vartheta_4(\varphi_\nu+z,q)\vartheta_4(\varphi_\nu-z,q)}{\vartheta_4^2(\varphi_\nu,q)\vartheta_4^2(z,q)}\,\vartheta_4^2(0,q). \tag{3.112}$$

Then, taking into account (5.16) and Eqs. (A.71)-(A.73), (3.93) is expressed

$$\sin\nu = -\frac{1}{\sqrt[4]{m(1-m)}}\sqrt{\frac{\pi(1-\alpha_3)}{2\mathrm{K}(m)}}\frac{\vartheta_1(\varphi_\nu,q)\vartheta_4(z,q)}{\sqrt{\vartheta_4(\varphi_\nu+z,q)\vartheta_1(\varphi_\nu-z,q)}} \tag{3.113}$$

$$\cos\nu = \frac{1}{\sqrt[4]{m}}\sqrt{\frac{\pi}{2\mathrm{K}(m)}}\frac{\vartheta_2(\varphi_\nu,q)\vartheta_4(z,q)}{\sqrt{\vartheta_4(\varphi_\nu+z,q)\vartheta_4(\varphi_\nu-z,q)}}, \tag{3.114}$$

On the other side, taking into account (3.95)-(3.96), we express

$$\cos J = \frac{N}{M} = \sqrt{\frac{\epsilon(1-m)}{\epsilon+m(1-\epsilon)}}\frac{1}{\mathrm{dn}\left(\frac{2\mathrm{K}(m)}{\pi}\varphi_\nu\middle|m\right)} \tag{3.115}$$

from where, knowing that $\alpha_3 = \epsilon + m(1-\epsilon)$, then

$$\sin J = \sqrt{\frac{m}{\alpha_3}}\,\frac{\sqrt{1-\alpha_3\,\mathrm{sn}^2\left(\frac{2\mathrm{K}(m)}{\pi}\varphi_\nu\middle|m\right)}}{\mathrm{dn}\left(\frac{2\mathrm{K}(m)}{\pi}\varphi_\nu\middle|m\right)}. \tag{3.116}$$

Then, both can be expressed in Theta functions as

$$\cos J = \frac{1}{\sqrt[4]{1-m}}\sqrt{\frac{\epsilon(1-m)}{\epsilon+m(1-\epsilon)}}\frac{\vartheta_4(\varphi_\nu,q)}{\vartheta_3(\varphi_\nu,q)} \tag{3.117}$$

$$\sin J = \sqrt{\frac{m}{\alpha_3}}\sqrt{\frac{2\mathrm{K}(m)}{\pi}}\frac{\sqrt{\vartheta_4(\varphi_\nu+z,q)\vartheta_4(\varphi_\nu-z,q)}}{\vartheta_3(\varphi_\nu,q)\vartheta_4(z,q)}, \tag{3.118}$$

Finally, we also need to express Eq. (3.94) in Theta functions. To do this, taking into account that

$$\Pi\left[\alpha_3;\mathrm{am}\left(\frac{2\mathrm{K(m)}}{\pi}\varphi_\nu\middle|\mathrm{m}\right)\middle|m\right] = \mathcal{M}\frac{2\mathrm{K}(m)}{\pi}\varphi_\nu + \frac{1}{2}\frac{\mathrm{sn}(a|m)}{\mathrm{cn}(a|m)\,\mathrm{dn}(a|m)}\ln\left[\frac{\vartheta_4(\varphi_\nu-z,q)}{\vartheta_4(\varphi_\nu+z,q)}\right] \tag{3.119}$$

$$\Pi(\alpha_3|m) = \mathcal{M}\,\mathrm{K}(m), \tag{3.120}$$

where

$$\mathcal{M} = 1 + \frac{\mathrm{sn}(a|m)}{\mathrm{cn}(a|m)\,\mathrm{dn}(a|m)}\frac{\pi}{2\mathrm{K}(m)}\frac{\vartheta_4'(z,q)}{\vartheta_4(z,q)} \tag{3.121}$$

and from (5.10)

$$\mathrm{sn}(a|m) = \sqrt{\frac{\alpha_3}{m}}, \qquad \mathrm{cn}(a|m) = i\sqrt{\frac{\alpha_3-m}{m}}, \qquad \mathrm{dn}(a|m) = \sqrt{1-\alpha_3} \tag{3.122}$$

then, after a few calculations, Eq. (3.94) can be put into the form

$$\omega = \varphi_\mu - \mu = -\frac{i}{2}\ln\left[\frac{\vartheta_4(\varphi_\nu-z,q)}{\vartheta_4(\varphi_\nu+z,q)}\right] \tag{3.123}$$

from where

$$e^{i\omega} = \sqrt{\frac{\vartheta_4(\varphi_\nu-z,q)}{\vartheta_4(\varphi_\nu+z,q)}}. \tag{3.124}$$

However, in the same way as ν and J, sine and cosine of μ are also looked for in this section. Thus, knowing that

$$\sin\mu = \frac{e^{i\mu}-e^{-i\mu}}{2i} = \frac{e^{i(\varphi_\mu-\omega)}-e^{-i(\varphi_\mu-\omega)}}{2i} = \frac{e^{i\varphi_\mu}e^{-i\omega}-e^{-i\varphi_\mu}e^{i\omega}}{2i} \tag{3.125}$$

$$\cos\mu = \frac{e^{i\mu}+e^{-i\mu}}{2} = \frac{e^{i(\varphi_\mu-\omega)}+e^{-i(\varphi_\mu-\omega)}}{2} = \frac{e^{i\varphi_\mu}e^{-i\omega}+e^{-i\varphi_\mu}e^{i\omega}}{2}, \tag{3.126}$$

using (3.124), we finally have

$$\sin\mu = \frac{e^{i\varphi_\mu}\vartheta_4(\varphi_\nu+z,q)-e^{-i\varphi_\mu}\vartheta_4(\varphi_\nu-z,q)}{2i\sqrt{\vartheta_4(\varphi_\nu+z,q)\vartheta_4(\varphi_\nu-z,q)}} \tag{3.127}$$

$$\cos\mu = \frac{e^{i\varphi_\mu}\vartheta_4(\varphi_\nu+z,q)+e^{-i\varphi_\mu}\vartheta_4(\varphi_\nu-z,q)}{2\sqrt{\vartheta_4(\varphi_\nu+z,q)\vartheta_4(\varphi_\nu-z,q)}} \tag{3.128}$$

3.5 Ferrer-Lara transformation

The proposal by Ferrer and Lara consists of making $X(m)$ and $Y(m)$ to be constant in (3.64). Specifically, they proposed to set[7]

$$X(m) = -1, \qquad Y(m,\beta) = 1, \tag{3.129}$$

which produces that Eq. (3.65) become

$$\beta = \sqrt{\frac{f(1+f)}{f+m}}, \tag{3.130}$$

from which one can explicitly work out m as a function of $\beta(L,G)$,

$$m(L,G) = f\left[(1+f)\frac{G^2}{L^2} - 1\right], \tag{3.131}$$

and by replacing it in Eq. (3.53) we obtain the new Hamiltonian function to be

$$\mathcal{K}_0(L,G) = \frac{1}{2A}G^2 - \frac{1}{2}\left(\frac{1}{B} - \frac{1}{C}\right)L^2. \tag{3.132}$$

Note that, since $0 < m < 1$, from Eq. (3.130) can be identified the lower and upper limit for β. That is, if $m = 0$, then $\beta = \sqrt{1+f}$, but if $m = 1$, then $\beta = \sqrt{m}$, therefore

$$\sqrt{f} < \beta = \frac{L}{G} < \sqrt{1+f}. \tag{3.133}$$

Finally, we can easily find the direct transformation equations by replacing Eq. (3.129) in the system (3.66)-(3.69), getting

$$\ell = -\mathrm{F}(\psi|m), \tag{3.134}$$

$$g = \mu + \sqrt{\frac{(1+f)(f+m)}{f}}\left[\mathrm{F}(\psi|m) - \Pi(-f;\psi|m)\right], \tag{3.135}$$

$$G = M, \tag{3.136}$$

$$L = N\sqrt{\frac{1+f}{1-m\sin^2\psi}}. \tag{3.137}$$

where to obtain the Eq. (3.137) we should take into account Eqs. (3.130) and (3.63). In the same way, proceeding as before, we may get the inverse transformation equations by replacing Eq. (3.129) in the system (3.71)-(3.74), getting

$$\sin\nu = \frac{\mathrm{cn}\,(\ell|m)}{\sqrt{1+f\,\mathrm{sn}^2\,(\ell|m)}}, \quad \cos\nu = -\frac{\sqrt{1+f}\,\mathrm{sn}\,(\ell|m)}{\sqrt{1+f\,\mathrm{sn}^2\,(\ell|m)}} \tag{3.138}$$

$$\mu = g + (1+f)\frac{G}{L}\left[\ell + \Pi(-f;-\mathrm{am}(\ell)|m)|m\right], \tag{3.139}$$

$$M = G, \tag{3.140}$$

$$N = \frac{L}{\sqrt{1+f}}\,\mathrm{dn}\,(\ell|m). \tag{3.141}$$

[7] In fact there are two papers where Ferrer-Lara have proposed two different new Hamiltonians, both quadratic in the momenta. In what follows we will refer to the one presented in [Lara & Ferrer (2010b), Lara & Ferrer (2012)].

where to obtain Eqs. (3.139) and (3.141) we should take into account Eq. (3.130) again.

Both action-angle and Ferrer-Lara will be the sets of variables we will focus in the following sections in order to study the perturbed rigid body. However, from the above equations we may already see the advantages and disadvantages of both sets. While Ferrer-Lara set has the advantage to be an explicit transformation from and to Andoyer variables, its coordinates are not angles, which is the main advantage of action-angle set permiting easily to do Fourier series expansions. Nevertheless, as commented before, the transformation in action-angle variables is not explicit, which is an important disadvantage as will be seen below.

Chapter 4

First order closed form solution of a fast rotating satellite under gravity gradient torque

In this chapter a closed form approach of the perturbed problem is tackled in the two sets of variables given in the previous chapter. It is important to remark that most of the ideas here presented rely heavily on the articles recently published by Ferrer and Lara [Lara & Ferrer (2010a), Lara & Ferrer (2010b), Lara & Ferrer (2012), Lara & Ferrer (2013)].

First of all the perturbing function has to be analyzed. As commented in Chapter 1, since the dimensions of a satellite are small when compared with its distance to the origin of the fixed reference frame, situated in a nearly spherical perturbing body, the disturbing potential is assumed to be formulated in the MacCullagh approximation [MacCullagh (1840)]. Besides, we will assume that the rigid body will move in a rotating frame consisting of a circular orbit, hence the orbital motion is already given integrated. Additionally, a further simplification can be accomplished when the body is also assumed to be in fast rotation compared with the orbital rate. In such a case, a fast rotating approximation is considered and the fast Andoyer variable μ may be averaged before applying the perturbation method, in our case the Lie-Deprit algorithm (see Section 1.3.2). Therefore, our averaged disturbing function depends on $(\nu, -, \lambda, N, M, \Lambda)$ in Andoyer variables.

As it was demonstrated by Hitzl and Breakwell [Hitzl & Breakwell (1971)], the problem of the attitude propagation of a triaxial satellite under gravity-gradient perturbations admits a closed form solution at least in the case of a fast rotating satellite. The solution is based on a perturbation approach in which the complete reduction of the torque-free rotation Hamiltonian is taken as the zero-order part. However, the formulation of the perturbing function in the new sets of canonical variables requires the use of Jacobi elliptic functions.

While the Hitzl and Breakwell's solution deals only with the secular terms of the problem, Ferrer and Lara have been recently pointed out that the periodic terms of the solution can be also provided in closed form [Lara & Ferrer (2012)]. In the new solution, the averaging is obtained as the result of a canonical transformation of the Lie type which is computed using Deprit's method [Deprit (1969)]. As we will see below, the use of a set of non-action-angle variables like

Ferrer-Lara's one is related to the introduction of mixed terms in the transformation equations of the averaging, thus constraining the time validity of the solution. This fact compels us to revisit our problem from the point of view of action-angle variables and perturbation theory. Thus, we will provide a complete closed form solution to the problem both in action-angle variables and in Ferrer-Lara variables, and both for the secular and periodic terms, in order to compare these two solutions by showing the advantages and disadvantages of each approach.

4.1 Gravity-gradient torque. Disturbing function in new variables

In this section we will show the perturbing potential over which the perturbation approaches will be accomplished both in action-angle variables and Ferrer-Lara variables. As recalled, the Hamiltonian of the torque-free motion in Andoyer variables introduced in (2.69) can be put into the form

$$\mathcal{H}_0 = \left(\frac{\sin^2\nu}{A} + \frac{\cos^2\nu}{B}\right)\frac{M^2 - N^2}{2} + \frac{N^2}{2C} = \frac{M^2}{2C}\left[1 + \left(\frac{\sin^2\nu}{A/C} + \frac{\cos^2\nu}{B/C} - 1\right)\sin^2 J\right]. \tag{4.1}$$

In order to formulate the gravity-gravient torque, the following preliminary assumptions have to be made:

1. We assume that the dimensions of the rigid body are small when compared with the distance to the perturbing body, which allow us to truncate the disturbing potential to the MacCullagh's term [MacCullagh (1840)].
2. We assume that the non-sphericity of the rigid body does not affect its orbital motion about the distant body, which is therefore Keplerian.
3. We limit to the case of circular orbital motion with constant radius r.

Thus, under the previous assumptions we may neglect the Keplerian part of MacCullagh's potential and limit our study to the disturbing potential given by

$$V = -\frac{\mathcal{G}\, M_\odot}{2r^3}\,(A + B + C - 3\,D)\,, \tag{4.2}$$

where $\mathcal{G}$ is the gravitational constant, $M_\odot$ is the mass of the disturbing body, r is the distance between the centers of mass of both bodies, and

$$D = A\,\gamma_1^2 + B\,\gamma_2^2 + C\,\gamma_3^2 \tag{4.3}$$

is the moment of inertia of the rigid body with respect to an axis in the direction of the line joining its center of mass with the perturber, of direction cosines γ_1, γ_2, and γ_3.

By replacing Eq. (4.3) in Eq. (4.2) and taking into account that $\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1$, we get

$$\mathcal{V} = -\frac{M^2}{2C^2}\left(\frac{n}{M/C}\right)^2 \left[(C - B)(1 - 3\gamma_3^2) - (B - A)(1 - 3\gamma_1^2)\right], \tag{4.4}$$

where n is the constant orbital mean motion given therefore by

$$n = \sqrt{\frac{\mathcal{G}M_{\odot}}{r^3}}. \tag{4.5}$$

If the orbital plane is chosen as the inertial reference frame, then the orbital reference frame is related to the body frame by the following composition of the rotations:

$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} = R_3(\nu)\, R_1(J)\, R_3(\mu)\, R_1(I)\, R_3(\phi) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{4.6}$$

where $\phi = \lambda - \tilde{\theta}$ and $\tilde{\theta}$ is the usual polar coordinate of the orbital motion given by $\tilde{\theta} = \tilde{\theta}_0 + n\,t$ as corresponds to a uniform circular motion.

Then, by replacing γ_1 and γ_3 as given by Eq. (4.6) in the disturbing potential (4.4), after several calculations we get that,

$$\mathcal{V} = -\frac{M^2}{2C^2}\left(\frac{n}{M/C}\right)^2 \frac{1}{16}\left[(2C - B - A)\mathcal{V}_1 + \frac{3}{2}(B - A)\mathcal{V}_2\right] \tag{4.7}$$

which separates the "axisymmetric part" of the potential

$$\begin{aligned} \mathcal{V}_1 \;=\; & (4 - 6s_J^2)\left(2 - 3s_I^2 + 3s_I^2\, C_{1,0,0}\right) \\ & -12 s_J c_J s_I \left[(1 - c_I)\, C_{-2,1,0} + 2c_I\, C_{0,1,0} - (1 + c_I)\, C_{2,1,0}\right] \\ & +3s_J^2 \left[(1 - c_I)^2\, C_{-2,2,0} + 2s_I^2\, C_{0,2,0} + (1 + c_I)^2\, C_{2,2,0}\right] \end{aligned} \tag{4.8}$$

which is independent of ν, from the "tri-axiality part"

$$\begin{aligned} \mathcal{V}_2 \;=\; & 6s_I^2 s_J^2\, (C_{2,0,-2} + C_{2,0,2}) - 4(1 - 3c_I^2)s_J^2\, C_{0,0,2} \\ & +(1 + c_J)^2 \left[(1 - c_I)^2\, C_{-2,2,2} + 2s_I^2\, C_{0,2,2} + (1 + c_I)^2\, C_{2,2,2}\right] \\ & +(1 - c_J)^2 \left[(1 - c_I)^2\, C_{-2,2,-2} + 2s_I^2\, C_{0,2,-2} + (1 + c_I)^2\, C_{2,2,-2}\right] \\ & +4s_I s_J\, (1 + c_J) \left[(1 - c_I)\, C_{-2,1,2} + 2c_I\, C_{0,1,2} - (1 + c_I)\, C_{2,1,2}\right] \\ & -4s_I s_J\, (1 - c_J) \left[(1 - c_I)\, C_{-2,1,-2} + 2c_I\, C_{0,1,-2} - (1 + c_I)\, C_{2,1,-2}\right], \end{aligned} \tag{4.9}$$

which carries the ν contribution to the perturbation. Note that $C_{i,j,k} \equiv \cos(i\phi + j\mu + k\nu)$ and the notation has been abreviated by writing $c_I \equiv \cos I$, $s_I \equiv \sin I$, $c_J \equiv \cos J$, and $s_J \equiv \sin J$.

Nevertheless, in this work we will consider that the rate of variation of μ is much faster than the rate of variation of ν and the mean orbital motion n. Then, short periodic terms related to μ may be neglected hence the previous potential contributions are now given by

$$\langle \mathcal{V}_1 \rangle_\mu \;=\; -2(1 - 3c_J^2)(2 - 3s_I^2 + 3s_I^2 \cos 2\phi), \tag{4.10}$$

$$\langle \mathcal{V}_2 \rangle_\mu \;=\; 4(1 - c_J^2)(2 - 3s_I^2 + 3s_I^2 \cos 2\phi) \cos 2\nu. \tag{4.11}$$

and therefore, recalling (4.7), the gravity-gradient torque exerted on a fast rotating satellite can be approximated by

$$\langle \mathcal{V} \rangle_\mu = -\frac{M^2}{2C^2}\left(\frac{n}{M/C}\right)^2 \frac{1}{16}\left[(2C - B - A)\, \langle \mathcal{V}_1 \rangle_\mu + \frac{3}{2}(B - A)\, \langle \mathcal{V}_2 \rangle_\mu\right]. \tag{4.12}$$

Finally, if Eqs. (4.10)-(4.11) are replaced in Eq. (4.12), we face a fast rotating satellite problem which is defined by the Hamiltonian $\mathcal{H} = \mathcal{H}_0 + \langle \mathcal{V} \rangle_\mu$, that is

$$\begin{aligned} \mathcal{H} &= \frac{M^2}{2C}\left\{\left(\frac{\sin^2\nu}{A/C}+\frac{\cos^2\nu}{B/C}\right)s_J^2 + c_J^2 + \frac{1}{8C}\left(\frac{n}{M/C}\right)^2(2-3s_I^2+3s_I^2\cos 2\phi)\right. \\ &\quad \left.\times\left[(2C-B-A)(1-3c_J^2)-3(B-A)(1-c_J^2)\cos 2\nu\right]\right\}, \end{aligned} \tag{4.13}$$

where $n^2/(M/C)^2$ will our *small parameter* since it is a small quantity and, therefore, the fast rotating satellite problem can be approached by perturbation methods.

Once obtained the Hamiltonian of the problem, it remains to express the disturbing function (4.12) in the new variables given in the previous chapter. Thus, from Eqs. (3.73) and (3.74), we find that

$$c_J = \sqrt{\frac{f}{f+m}}\,\mathrm{dn}(u|m), \qquad u = X(m)\,\ell, \tag{4.14}$$

and $c_I = I_\Lambda/I_M$ in action-angle variables or $c_I = H/G$ in Ferrer-Lara variables. Then, taking into account Eq. (3.71), the disturbing potential (4.12) is given by

$$\begin{aligned} \langle \mathcal{V} \rangle_\mu \equiv U &= \frac{n^2}{16}(2-3s_I^2+3s_I^2\cos 2\phi)\left\{(2C-B-A)\left[1-3\frac{f}{f+m}\mathrm{dn}^2(u|m)\right]\right. \\ &\quad \left.+3(B-A)\left[1-\frac{f}{f+m}\mathrm{dn}^2(u|m)\right]\left[1-2\frac{(1+f)\,\mathrm{sn}^2(u|m)}{1+f\,\mathrm{sn}^2(u|m)}\right]\right\}, \end{aligned} \tag{4.15}$$

where $\phi = h - \tilde{\theta}$. Note that m, and therefore u, are still remain to be defined depending on the specific set of variables to be used. Note further that the elliptic functions involved in (4.15) are squared and therefore, in what follows we can neglect in the argument u the negative sign from (3.129) and (3.75) leading to the two different sets of variables here analyzed.

4.2 Rotating frame

As the reader noted in the previous section, the existence of an orbital motion shows that we are actually immersed in a roto-translatory problem. In fact, since the goal of this work is to show how the rotational motion evolves, we will use the easiest frame given by a uniform circular motion trivially integrated as a lineal function of time.

The explicit appearance of the time can be avoided by moving to a rotating frame at the same rotation rate as the orbital motion (see [San-Juan (2012)], page 4). Because of that, the new variable $\phi = \lambda - nt$ (with $\tilde{\theta}_0 = 0$) has been introduced, which is the argument of the ascending node of the invariant plane with respect to the inertial plane, in a rotating frame with orbital rate $d\tilde{\theta}/dt = n$. Indeed, note that as

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{\mathrm{d}h}{\mathrm{d}t} - n = \frac{\partial \mathcal{H}}{\partial H} - n = \frac{\partial}{\partial H}(\mathcal{H} - nH), \tag{4.16}$$

the change of reference frame requires the introduction of the Coriolis term $-nH$ in the Hamiltonian. The result is a new conservative Hamiltonian $\mathcal{K}$ with $\Phi \equiv \Lambda = H$ is now the conjugate momenta of ϕ, and then we finally have that

$$\mathcal{K} = \mathcal{K}_0 - n\,\Phi + U \tag{4.17}$$

where we consider the Coriolis term to be of first order and the gravity-gradient potential U to be a second order quantity.

4.3 Perturbation approach

As an alternative to recent efforts in giving a solution to the attitude propagation of a fast rotating triaxial satellite under gravity-gradient torque, the new solution will be computed by the Lie-Deprit approach and given in closed form, either for the secular or periodic terms, therefore being valid for any triaxial satellite. In our case, as commented above, we assume that the gravity-gradient torque is of higher order than the Coriolis term, which in turn is of higher order than torque-free rotation. This allows us to split the averaging procedure into two parts. As the first stages of the perturbation approach can be obtained in a general form, i.e. without the need of taking any particular set of variables, we use $(\ell, g, \phi, L, G, \Phi)$ to be this general variables before any choice.

Average over ℓ

First, we look for a canonical transformation $(\ell, g, \phi, L, G, \Phi) \xrightarrow{T_\ell} (\ell', g', \phi', L', G', \Phi')$ that removes the variable ℓ from the Hamiltonian.

As indicated before, we set

$$H_{0,0} = \mathcal{K}_0, \qquad H_{1,0} = -n\,\Phi, \qquad H_{2,0} = 2\,U,$$

where all the functions are assumed to be expressed in prime variables but, for the sake of brevity, we drop the prime notation in what follows when there is no risk of confusion.

The first step in the computation of the Lie triangle gives

$$\frac{G}{2A}\frac{C-A}{C}\frac{f}{(f+m)^2}\frac{\partial m}{\partial \beta}\frac{\partial W_1}{\partial \ell} = H_{1,0} - H_{0,1}, \tag{4.18}$$

However, as $H_{1,0}$ does not depend on ℓ or g, we can choose

$$H_{0,1} = H_{1,0}, \tag{4.19}$$

and the first term in the generating function is found to be $W_1 = 0$. Because the vanishing of W_1, the next step gives

$$\frac{G}{2A}\frac{C-A}{C}\frac{f}{(f+m)^2}\frac{\partial m}{\partial \beta}\frac{\partial W_2}{\partial \ell} = H_{2,0} - H_{0,2}. \tag{4.20}$$

Then, we can choose $H_{0,2} = \langle H_{2,0}\rangle_\ell$ and recalling that Jacobian elliptic functions are $4K(m)$-periodic, therefore,

$$H_{0,2} = \frac{1}{T}\int_0^T H_{2,0}\,d\ell = \frac{1}{4\mathrm{K}(m)}\int_0^{4\mathrm{K}(m)} H_{2,0}\,du. \tag{4.21}$$

from where we get

$$H_{0,2} = \frac{n^2}{4}\,\kappa\,(2 - 3s_I^2 + 3s_I^2\cos 2\phi), \tag{4.22}$$

where

$$\kappa = (B-A)\left\{\frac{C-A}{B-A} + 1 - 3\,\frac{1+f}{f+m}\left[1 + \frac{C-B}{B}\,\frac{\mathrm{E}(m)}{\mathrm{K}(m)}\right]\right\}, \tag{4.23}$$

and $\mathrm{E}(m)$ is the complete elliptic integral of the second kind.

Then, the term W_2 of the generating function can be solved from Eq. (4.20). We get

$$W_2 = -\frac{3}{2}\frac{n^2}{G}(C-B)A\sqrt{\frac{f(1+f)}{f+m}}Z(\psi|m)(1-3s_I^2\sin^2\phi), \tag{4.24}$$

where

$$Z(\psi|m) = E(\psi|m) - \frac{\mathrm{E}(m)}{\mathrm{K}(m)}\,F(\psi|m) = \mathrm{E}(\mathrm{am}(u|m)|m) - \frac{\mathrm{E}(m)}{\mathrm{K}(m)}\,u \tag{4.25}$$

is the Jacobi Zeta function, which is $2K(m)$-periodic (see A.3 for further details) and $\mathrm{E}(\psi|m)$ is the incomplete elliptic integral of the second kind. Note, therefore, that the generating function of the first averaging is intrinsically expressed in terms of Jacobi Zeta function for this perturbing function, regardless of the used variables.

4.4 Perturbation approach in action-angle variables

As indicated in Eq. (3.77) (Section 3.4), in action-angle variables we can not express m as a function of β and, therefore, as a function of the new momenta, hence Eq. (4.14) remains unchanged, only taking into account that $u = -\frac{2K(m)}{\pi}\,\varphi_\nu$. Thus, taking into account the general scheme shown in Section 4.3, next we follow it by using action-angle variables.

4.4.1 Transformation equations of the first averaging

Before computing these transformation equations, some previous partial derivatives have to be calculated. In particular, because $m = m(I_N, I_M)$, it worths to recall that

$$\frac{\partial \mathrm{E}(m)}{\partial(I_N, I_M)} = \frac{\partial \mathrm{E}(m)}{\partial m}\frac{\partial m}{\partial(I_N, I_M)} = \frac{1}{2m}[\mathrm{E}(m) - \mathrm{K}(m)]\frac{\partial m}{\partial(I_N, I_M)}, \tag{4.26}$$

$$\frac{\partial \mathrm{K}(m)}{\partial(I_N, I_M)} = \frac{\partial \mathrm{K}(m)}{\partial m}\frac{\partial m}{\partial(I_N, I_M)} = \frac{1}{2m}\left[\frac{1}{1-m}\mathrm{E}(m) - \mathrm{K}(m)\right]\frac{\partial m}{\partial(I_N, I_M)}, \tag{4.27}$$

As m is an implicit function of β, some calculations are needed to find its partial derivatives with respect to the momenta. Thus, taking into account Eq. (3.77) we find

$$\frac{\partial I_N}{\partial m} = -\frac{I_M}{\pi}\frac{\sqrt{f(1+f)}}{(f+m)^{3/2}}\mathrm{K}(m) \quad \text{and} \quad m = \phi_1(\beta = I_N/I_M), \tag{4.28}$$

where ϕ_1 is a certain function of the momenta that we cannot obtain as said before. Nevertheless, from Eq. (4.28),we can find that

$$\frac{\partial \phi_1}{\partial \beta} = \frac{1}{\partial\beta/\partial\phi_1} = I_M\frac{1}{\partial I_N/\partial\phi_1} = I_M\frac{1}{\partial I_N/\partial m} = I_M\frac{\partial m}{\partial I_N} = -\frac{\pi(f+m)^{3/2}}{\sqrt{f(1+f)}\mathrm{K}(m)}. \tag{4.29}$$

Finally, from Eq. (4.29) and Eq. (4.28), we obtain

$$\frac{\partial m}{\partial I_N} = \frac{\partial \phi_1}{\partial \beta}\frac{\partial \beta}{\partial I_N} = -\frac{\pi(f+m)^{3/2}}{I_M\sqrt{f(1+f)}\mathrm{K}(m)}, \tag{4.30}$$

$$\frac{\partial m}{\partial I_M} = \frac{\partial \phi_1}{\partial \beta}\frac{\partial \beta}{\partial I_M} = \frac{I_N}{I_M^2}\frac{\pi(f+m)^{3/2}}{\sqrt{f(1+f)}\mathrm{K}(m)} \tag{4.31}$$

$$= \frac{2(f+m)}{I_M\, f\,\mathrm{K}(m)}[(f+m)\Pi(-f,m) - m\mathrm{K}(m)], \tag{4.32}$$

from which it follows that

$$\frac{I_M^2}{I_N}\frac{\partial m}{\partial I_M} = -I_M\frac{\partial m}{\partial I_N} = \frac{\pi(f+m)^{3/2}}{\sqrt{f(1+f)}\mathrm{K}(m)}. \tag{4.33}$$

In this case, the transformation equations are

$$\xi = \xi' + \frac{1}{2}\{\xi'; W_2\}, \qquad \xi \in (\varphi_\nu, \varphi_\mu, \phi, I_N, I_M, I_\Lambda) \tag{4.34}$$

where $\{a; b\}$ stands for the Poisson bracket of the functions a and b.

Then, calling $\Delta\xi = \xi - \xi'$, we find

$$\Delta\varphi_\nu = \frac{3\pi n^2}{8I_M^2\mathrm{K}(m)}A(C-B)(1-3s_I^2\sin^2\phi)\left[2(f+m)\frac{\partial Z(\psi|m)}{\partial m} - Z(\psi|m)\right], \tag{4.35}$$

$$\Delta\varphi_\mu = -\frac{I_N}{I_M}\left[\Delta\varphi_\nu - \frac{3n^2}{4I_NI_M}A(C-B)\sqrt{\frac{f(1+f)}{f+m}}Z(\psi|m)(1-3s_I^2\sin^2\phi + 6c_I^2\sin^2\phi)\right], \tag{4.36}$$

$$\Delta\phi = -\frac{3n^2}{4\,I_M^2}A(C-B)\sqrt{\frac{f(1+f)}{f+m}}Z(\psi|m)\,6\,c_I\sin^2\phi, \tag{4.37}$$

$$\Delta I_N = \frac{3n^2}{4\,I_M}A(C-B)\sqrt{\frac{f(1+f)}{f+m}}(1-3s_I^2\sin^2\phi)\frac{2}{\pi}\left[\mathrm{E}(m) - \mathrm{dn}^2\left(\frac{2\mathrm{K}(m)}{\pi}\varphi_\nu|m\right)\mathrm{K}(m)\right], \tag{4.38}$$

$$\Delta I_M = 0, \tag{4.39}$$

$$\Delta I_\Lambda = -\frac{3n^2}{4\,I_M}A(C-B)\sqrt{\frac{f(1+f)}{f+m}}Z(\psi|m)\,3\,s_I^2\sin(2\phi). \tag{4.40}$$

The derivative of the Jacobi Zeta function with respect to the elliptic modulus required by $\Delta\varphi_\nu$ in Eq. (4.35)-(4.36) (see A.5.2) is given, after several calculations, by

$$\frac{\partial}{\partial m}Z(\psi|m) = \frac{\mathrm{cn}(u|m)}{2(1-m)}[\mathrm{sn}(u|m)\,\mathrm{dn}(u|m) - \mathrm{cn}(u|m)\,\mathrm{Z}(\psi|m)], \tag{4.41}$$

where $\psi = \mathrm{am}(u|m)$ and $u = -2\mathrm{K}(m)\,\varphi_\nu/\pi$. Moreover, Fig. 4.1 shows its periodicity.

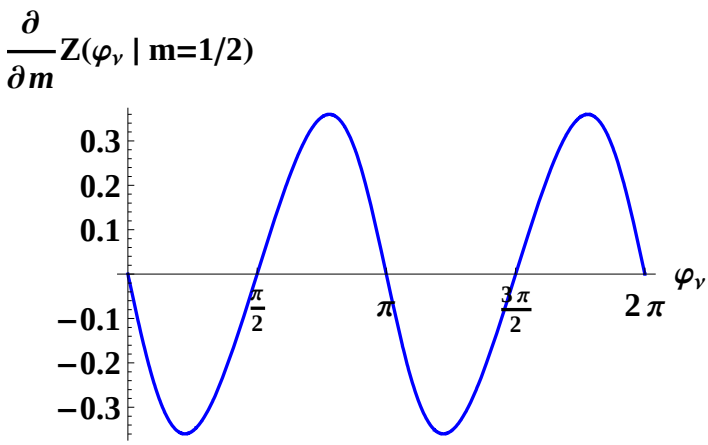


Figure 4.1: Derivative of Z(am$(-2\mathrm{K}(m)\,\varphi_\nu/\pi|m)|m)$ with respect to the modulus for the case $m = 1/2$.

4.4.2 Average over ϕ

A new Lie transform $(\varphi'_\nu, \varphi'_\mu, \phi', I'_N, I'_M, I'_\Lambda) \longrightarrow (\varphi''_\nu, \varphi''_\mu, \phi'', I''_N, I''_M, I''_\Lambda)$ such that it removes the angle ϕ is then computed assuming $\kappa \neq 0$ (details on this case is given in [Molero & Ferrer (2013)]). We start by setting $K_{0,0} = \mathcal{K}$, $K_{1,0} = H_{0,1}$, and $K_{2,0} = H_{0,2}$, all of them evaluated in the double prime variables, although we drop the primes from the notation for brevity again.

Because the only angle appearing in the Hamiltonian is ϕ'', we may assume that the new generating function $V = \sum_{i\geq 0}(\varepsilon^i/i!)\,V_{i+1}$ depends only on this angle. Then, the first step in the Lie triangle gives $0 = K_{1,0} - K_{0,1}$ and we trivially choose $K_{0,1} \equiv K_{1,0}$ while V_1 remains unknown at this step. The second step gives

$$K_{0,2} = 2\,n\,\frac{\partial V_1}{\partial \phi} + K_{2,0} \tag{4.42}$$

from where we choose $K_{0,2} = \frac{1}{2\pi}\int_0^{2\pi} K_{2,0}\,\mathrm{d}\phi$, giving

$$K_{0,2} = \frac{n^2}{4}\left(3\frac{I_\Lambda^2}{I_M^2} - 1\right)\kappa \tag{4.43}$$

Then, from Eq. (4.42) $V_1 = \frac{1}{2n}\int(K_{0,2} - K_{2,0})\,\mathrm{d}\phi$, resulting in

$$V_1 = -\frac{3n}{16}\left(1 - \frac{I_\Lambda^2}{I_M^2}\right)\kappa\sin 2\phi \tag{4.44}$$

4.4.3 Transformation equations of the second averaging

This time, the transformation equations are

$$\xi' = \xi'' + \{\xi''; V_1\}, \qquad \xi \in (\varphi''_\nu, \varphi''_\mu, \phi'', I''_N, I''_M, I''_\Lambda).$$

Then, calling $\delta\xi = \xi' - \xi''$, we find

$$\begin{aligned}
\delta\varphi_\nu &= \frac{9\pi\, n}{16\, I_M\, \mathrm{K}(m)}\left(1-\frac{I_\Lambda^2}{I_M^2}\right)(B-A)\sqrt{\frac{1+f}{f(f+m)}}\,\sin 2\phi \qquad (4.45)\\
&\quad\times\left\{1-\frac{C-B}{B}\frac{f+m}{2m}\left[2\frac{\mathrm{E}(m)}{\mathrm{K}(m)}-\frac{\mathrm{E}^2(m)}{(1-m)\mathrm{K}^2(m)}-1\right]+\frac{C-B}{B}\frac{\mathrm{E}(m)}{\mathrm{K}(m)}\right\},\\
\delta\phi &= \frac{3\,n}{8}\frac{I_\Lambda}{I_M^2}\,\kappa\sin 2\phi, \qquad (4.46)\\
\delta\varphi_\mu &= -\frac{I_\Lambda}{I_M}\,\delta\phi-\frac{I_N}{I_M}\,\delta\varphi_\nu, \qquad (4.47)\\
\delta I_N &= 0, \qquad (4.48)\\
\delta I_M &= 0, \qquad (4.49)\\
\delta I_\Lambda &= \frac{3\,n}{8}\left(1-\frac{I_\Lambda^2}{I_M^2}\right)\kappa\cos 2\phi. \qquad (4.50)
\end{aligned}$$

4.4.4 Secular terms

After the double averaging we find the secular Hamiltonian $\mathcal{S} = K_{0,0} + K_{0,1} + \frac{1}{2}K_{0,2}$ given by

$$\begin{aligned}
\mathcal{S} &= \frac{I_M^2}{2A}\left(1-\frac{C-A}{C}\frac{f}{f+m}\right)-n\,I_\Lambda \qquad (4.51)\\
&\quad-\frac{n^2}{8}\left(1-3\frac{I_\Lambda^2}{I_M^2}\right)(B-A)\left\{\frac{C-A}{B-A}+1-3\,\frac{1+f}{f+m}\left[1+\frac{C-B}{B}\,\frac{E(m)}{K(m)}\right]\right\}
\end{aligned}$$

where L, G, and therefore m, and Φ are constant. The secular frequencies of the motion are obtained from Hamilton equations

$$\begin{aligned}
\frac{\mathrm{d}\varphi_\nu}{\mathrm{d}t} &= -\frac{\pi\, I_M}{2A\mathrm{K}(m)}\frac{C-A}{C}\sqrt{\frac{f}{(1+f)(f+m)}}+\frac{3\pi\, n^2}{8I_M\mathrm{K}(m)}\left(1-3\frac{I_\Lambda^2}{I_M^2}\right)(B-A)\sqrt{\frac{1+f}{f(f+m)}}\\
&\quad\times\left\{1-\frac{C-B}{B}\frac{f+m}{2m}\left[2\frac{\mathrm{E}(m)}{\mathrm{K}(m)}-\frac{\mathrm{E}^2(m)}{(1-m)\mathrm{K}^2(m)}-1\right]+\frac{C-B}{B}\frac{\mathrm{E}(m)}{\mathrm{K}(m)}\right\}=n_{\varphi_\nu} \qquad (4.52)\\
\frac{\mathrm{d}\phi}{\mathrm{d}t} &= -n+\frac{3n^2}{4}\frac{I_\Lambda}{I_M^2}(B-A)\left\{\frac{C-A}{B-A}+1-3\,\frac{1+f}{f+m}\left[1+\frac{C-B}{B}\,\frac{E(m)}{K(m)}\right]\right\}=n_\phi \qquad (4.53)\\
\frac{\mathrm{d}\varphi_\mu}{\mathrm{d}t} &= \frac{I_M}{A}\left(1-\frac{C-A}{C}\frac{f}{f+m}\right)-\frac{I_\Lambda}{I_M}(n_\phi+n)-\frac{I_N}{I_M}\,n_{\varphi_\nu}=n_{\varphi_\mu} \qquad (4.54)
\end{aligned}$$

and $\mathrm{d}h/\mathrm{d}t = n_h = n + n_\phi$.

4.4.5 Results

In order to illustrate the application of the above theory, a sample application is here provided. We will base on the orbit and inertia parameters of a PEGASUS-A satellite taken from Ref. [Cochran (1972)] except for considering a Keplerian circular orbit, in agreement with the assumptions of our theory. Thus,

$$A = 1.03068\times 10^5\,\mathrm{kg\,m^2},\quad B = 3.33455\times 10^5\,\mathrm{kg\,m^2},\quad C = 3.94992\times 10^5\,\mathrm{kg\,m^2}$$

and

$$M = 5.842 \times 10^5\,\mathrm{kg\,m^2/min}, \quad n = 3.71^\circ/\mathrm{min}$$

Besides, we set the initial conditions $\mu = 2\,\mathrm{rad}$, $\nu = 1\,\mathrm{rad}$, $\lambda = -0.1\,\mathrm{rad}$, $J = 10^\circ$, $i = 70^\circ$.

For the integration we use internal units such that $M = C = 1$. Then, using first Eqs. (4.35)-(4.40), and later Eqs. (4.46)-(4.50) we propagate the initial conditions to the different phase spaces collected in Table 4.1, and the solutions are given in Fig. 4.2. Note that the variables denoted as x^* are given by

$$x^* = x' + \Delta x', \qquad x' = x'' + \delta x'', \tag{4.55}$$

where $x \in (\varphi_\nu, \varphi_\mu, \phi, I_N, I_M, I_\Lambda)$ and $x''(t)$ is obtained directly from the double-averaged Hamiltonian by the Eqs. (4.52)-(4.54).

object	**var.**	**non-averaged**	**prime**	**double-prime**	**sec. frequency**
PEGASUS	φ_ν	-0.1479898511	-0.1481370529	-0.1448526999	-0.6501504248
	φ_μ	1.5775303900	1.577664962	1.574852779	1.6830026275
	ϕ	-0.1	-0.09999987511	-0.1009172983	-0.0441809427
	I_N	0.9548381629	0.9548769383	0.9548769383	
	I_M	1.0	1.0	1.0	
	I_Λ	0.3420201433	0.3420169296	0.3531301948	

Table 4.1: Initial conditions in the different phase spaces for Pegasus.

Specifically, Fig. 4.2 shows the differences between the numerical solution and the complete analytical solution (after recovering periodic terms). Observe how φ_ν delays with respect to the analytical solution with a constant rate of about 0.002 units per orbital period and how the motion is hardly ever affected by short-period terms. On the contrary, note that φ_μ advances over the analytical solution with a very similar constant rate of about 0.003 units per orbital period. The evolution of ϕ is similar to φ_ν, but the variation is roughly a ten times lower per orbital period and only affected by periodic terms related to the orbital motion. We further appreciate that I_N differs only in low-amplitude periodic terms. In turn, I_Λ advances each period with a very slow rate lower than 10^{-5} units per orbital period.

4.5 Perturbation approach in Ferrer-Lara variables

In this section we will again apply perturbation theory to our system in order to find a new canonical transformation which allows us to obtain a new Hamiltonian as a fuction only of the momenta up to a certain order (secular terms), therefore moving the coordinates to high order terms. In this case, as indicated in Eq. (3.130) (Section 3.5), in Ferrer-Lara variables we can express m as a function of β and, therefore, as a function of the new momenta (L and G), hence Eq. (4.14), knowing that $u = -\ell$, can be put into the form

$$c_J = \frac{1}{\sqrt{1+f}}\,\frac{L}{G}\,\mathrm{dn}(\ell|m), \tag{4.56}$$

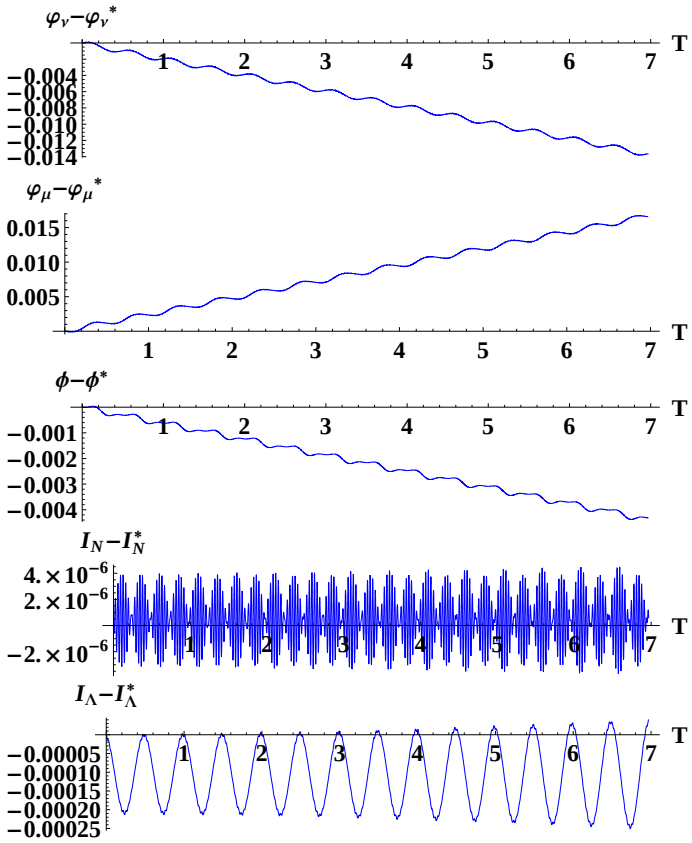


Figure 4.2: Differences between the perturbed motion in elliptic functions and the averaged motion with $x^* = x' + \Delta x'$ and $x' = x'' + \delta x''$ in internal units in action-angle variables. T stands for orbital periods.

while $\Lambda = H$, $M = G$ and $c_I = H/G$ remain unchanged. As a consequence of the explicit dependence of m with respect to the momenta, taking into account (3.130) and (4.56) or (4.15), the new disturbing function is given by

$$\begin{aligned}(T : \langle \mathcal{V} \rangle_\mu \equiv U) \;=\; & \frac{n^2}{16}\left(2 - 3s_I^2 + 3s_I^2 \cos 2\phi\right)\left\{(2C - B - A)\left[1 - 3f\,\frac{\mathrm{dn}^2(\ell|m)}{f+m}\right]\right. \\ & \left. +3\,(B - A)\left[1 - f\,\frac{\mathrm{dn}^2(\ell|m)}{f+m}\right]\left[1 - 2\,\frac{(1+f)\,\mathrm{sn}^2(\ell|m)}{1 + f\,\mathrm{sn}^2(\ell|m)}\right]\right\} \qquad (4.57)\end{aligned}$$

Thus, taking into account again the general scheme shown in Section 4.3, next we follow it by using Ferrer-Lara variables.

4.5.1 Transformation equations of the first averaging

Note that as $m \equiv m(G, L)$, it worths to recall that

$$\begin{aligned}\frac{\partial E(m)}{\partial(L,G)} &= \frac{\partial E(m)}{\partial m}\,\frac{\partial m}{\partial(L,G)} = \frac{1}{2m}\left[E(m) - K(m)\right]\frac{\partial m}{\partial(L,G)}, \\ \frac{\partial K(m)}{\partial(L,G)} &= \frac{\partial K(m)}{\partial m}\,\frac{\partial m}{\partial(L,G)} = \frac{1}{2m}\left[\frac{1}{1-m}\,E(m) - K(m)\right]\frac{\partial m}{\partial(L,G)}\end{aligned}$$

where, cf. Eq. (3.131),

$$G\,\frac{\partial m}{\partial G} = -L\,\frac{\partial m}{\partial L} = 2\,(m+f). \tag{4.58}$$

In this case, the transformation equations are

$$\xi = \xi' + \frac{1}{2}\{\xi'; W_2\}, \qquad \xi \in (\ell, g, \phi, L, G, \Phi)$$

where $\{a; b\}$ stands for the Poisson bracket of the functions a and b.

Then, calling $\Delta\xi = \xi - \xi'$, we find

$$\Delta\ell = \frac{3n^2}{4G^2}\,A\,(C-B)\left[Z(u|m) - 2(f+m)\,\frac{\partial Z(u|m)}{\partial m}\right]\left(1 - 3s_I^2\sin^2\phi\right) \tag{4.59}$$

$$\Delta g = -\frac{L}{G}\left[\Delta\ell + \frac{3n^2}{4G^2}\,A\,(C-B)\,Z(u|m)\left(1 - 3s_I^2\sin^2\phi + 6c_I^2\sin^2\phi\right)\right] \tag{4.60}$$

$$\Delta\phi = \frac{3n^2}{4G^2}\,A\,(C-B)\,\frac{L}{G}\,Z(u|m)\,6c_I\sin^2\phi \tag{4.61}$$

$$\Delta L = \frac{3n^2}{4G^2}\,A\,(C-B)\,L\left[\frac{E(m)}{K(m)} - \mathrm{dn}(\ell|m)^2\right]\left(1 - 3s_I^2\sin^2\phi\right) \tag{4.62}$$

$$\Delta G = 0 \tag{4.63}$$

$$\Delta\Phi = \frac{3n^2}{4G^2}\,A\,(C-B)\,L\,Z(u|m)\,3{s_I}^2\sin 2\phi \tag{4.64}$$

where we remind that the right member of the equations should be assumed in the prime variables.

The derivative of the equation of the center with respect to the elliptic modulus required by $\Delta\ell$ in Eqs. (4.59)-(4.60) is

$$\frac{\partial}{\partial m}\,Z\big(\mathrm{am}(\ell|m)\,\big|\,m\big) = \frac{\partial}{\partial m}\,E\big(\mathrm{am}(\ell|m)\,\big|\,m\big) - \ell\,\frac{\partial}{\partial m}\frac{E(m)}{K(m)}$$

where

$$\begin{aligned}
\frac{\partial}{\partial m}\,\frac{E(m)}{K(m)} &= \frac{1}{2m}\left[2\,\frac{E(m)}{K(m)} - \frac{E^2(m)}{(1-m)\,K^2(m)} - 1\right],\\
\frac{\partial}{\partial m}\,E\big(\mathrm{am}(\ell|m)\,\big|\,m\big) &= \frac{1}{2(1-m)}\Big[\mathrm{dn}(\ell|m)\,\mathrm{cn}(\ell|m)\,\mathrm{sn}(\ell|m)\\
&\qquad - \mathrm{cn}^2(\ell|m)\,E(u|m) - (1-m)\,\mathrm{sn}^2(\ell|m)\,\ell\Big],
\end{aligned}$$

and hence

$$\begin{aligned}
\frac{\partial}{\partial m}\,Z(u|m) &= \frac{\mathrm{sn}(\ell|m)\,\mathrm{cn}(\ell|m)\,\mathrm{dn}(\ell|m)}{2(1-m)} - \frac{\mathrm{cn}^2(\ell|m)}{2(1-m)}\,E\big(\mathrm{am}(\ell|m)\,\big|\,m\big)\\
&\quad - \frac{1}{2m}\left[2\,\frac{E(m)}{K(m)} - \frac{E^2(m)}{(1-m)\,K^2(m)} - \mathrm{dn}^2(\ell|m)\right]\ell
\end{aligned} \tag{4.65}$$

that is not periodic, as illustrated in the right plot of Fig. 4.3. This means that the transformation equations for the variables ℓ and g are affected of mixed terms.

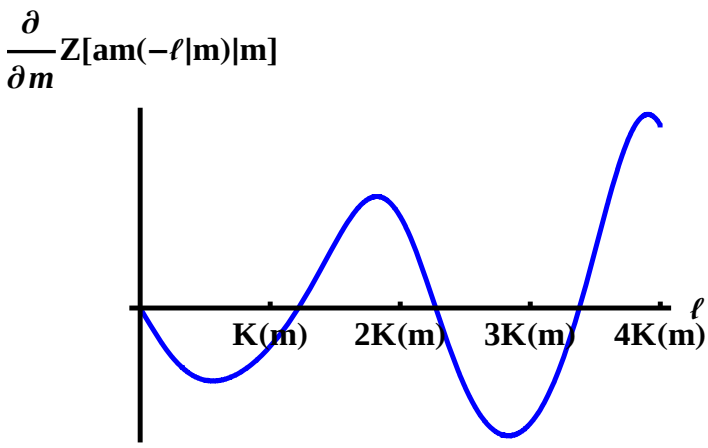


Figure 4.3: Derivative of the Jacobi Zeta function $Z\big(\mathrm{am}(-\ell\,|\,m)|m\big)$ with respect to the modulus in the case $m = 1/2$.

Note that this lack of periodicity does not contradict the properties of the Jacobi zeta function, whose derivative with respect to the modulus is guaranteed to be periodic only when the argument of the Jacobi zeta function is independent of the modulus. Nevertheless, some cases can be found in which this derivative is also a periodic function, thus avoiding the appearance of mixed terms in the transformation equations. That is the case of action-angle variables.

4.5.2 Average over ϕ

A new Lie transform $(\ell', g', \phi', L', G', \Phi') \longrightarrow (\ell'', g'', \phi'', L'', G'', \Phi'')$ such that it removes the angle ϕ is then computed. We start by setting $K_{0,0} = \mathcal{K}$, $K_{1,0} = H_{0,1}$, and $K_{2,0} = H_{0,2}$, all of then evaluated in the double prime variables, although we drop the primes from the notation for brevity.

Because the only angle that appears in the Hamiltonian is ϕ'', we may assume that the new generating function $V = \sum_{i\geq 0}(\varepsilon^i/i!)\,V_{i+1}$ only depends on this angle. Then, the first step in the Lie triangle gives $0 = K_{1,0} - K_{0,1}$ and we trivially choose $K_{0,1} \equiv K_{1,0}$ while V_1 remains unknown at this step. The second step gives

$$K_{0,2} = 2\,n\,\frac{\partial V_1}{\partial \phi} + K_{2,0} \tag{4.66}$$

We choose $K_{0,2} = \frac{1}{2\pi}\int_0^{2\pi} K_{2,0}\,\mathrm{d}\phi$, giving

$$K_{0,2} = \frac{n^2}{3}\left(3\frac{\Phi^2}{G^2} - 1\right)\kappa \tag{4.67}$$

Then, from Eq. (4.66) $V_1 = \frac{1}{2\,n}\int(K_{0,2} - K_{2,0})\,\mathrm{d}\phi$, resulting in

$$V_1 = -\frac{n}{4}\left(1 - \frac{\Phi^2}{G^2}\right)\kappa\sin 2\phi \tag{4.68}$$

4.5.3 Transformation equations of the second averaging

Now, the transformation equations are

$$\xi' = \xi'' + \{\xi''; V_1\}, \qquad \xi \in (\ell, g, \phi, L, G, \Phi).$$

Then, calling $\delta\xi = \xi' - \xi''$, we find

$$\begin{aligned}
\delta\ell &= \frac{n}{2L}\,\frac{9}{4}\,A\,\frac{C-B}{C}\left(1-\frac{\Phi^2}{G^2}\right)\frac{L^2}{G^2} \qquad (4.69)\\
&\quad\times\left\{1+\frac{C-B}{B}\,\frac{f+m}{2m}\left[1-\frac{2f}{f+m}\,\frac{E(m)}{K(m)}+\frac{1}{1-m}\,\frac{E^2(m)}{K^2(m)}\right]\right\}\sin 2\phi\\
\delta\phi &= \frac{n}{2\Phi}\,\frac{\Phi^2}{G^2}\,\kappa\sin 2\phi \qquad (4.70)\\
\delta g &= -\frac{\Phi}{G}\,\delta\phi - \frac{L}{G}\,\delta\ell \qquad (4.71)\\
\delta L &= 0 \qquad (4.72)\\
\delta G &= 0 \qquad (4.73)\\
\delta\Phi &= \frac{n}{2}\left(1-\frac{\Phi^2}{G^2}\right)\kappa\cos 2\phi \qquad (4.74)
\end{aligned}$$

where the right member of the equations must be expressed in the double-prime variables.

4.5.4 Secular terms

After the double averaging we find the secular Hamiltonian $\mathcal{S} = K_{0,0} + K_{0,1} + \frac{1}{2}K_{0,2}$ given by

$$\begin{aligned}
\mathcal{S} &= \frac{G^2}{2A} - \left(\frac{1}{B}-\frac{1}{C}\right)\frac{L^2}{2} - n\,\Phi \qquad (4.75)\\
&\quad -\frac{n^2}{8}\left(1-3\frac{\Phi^2}{G^2}\right)(B-A)\left\{\frac{C-A}{B-A}+1-3\,\frac{1+f}{f+m}\left[1+\frac{C-B}{B}\,\frac{E(m)}{K(m)}\right]\right\}
\end{aligned}$$

where L, G, and therefore m, and Φ are constant. The secular frequencies of the motion are obtained from Hamilton equations

$$\begin{aligned}
\frac{\mathrm{d}\ell}{\mathrm{d}t} &= -\left(\frac{1}{B}-\frac{1}{C}\right)L + L\,\frac{3n^2}{4G^2}\,A\,\frac{C-B}{C}\left(1-3\frac{\Phi^2}{G^2}\right) \qquad (4.76)\\
&\quad\times\left\{1-\frac{C-B}{B}\,\frac{f}{m}\,\frac{E(m)}{K(m)}+\frac{C-B}{B}\,\frac{f+m}{2m}\left[1+\frac{1}{1-m}\,\frac{E^2(m)}{K^2(m)}\right]\right\}\\
\frac{\mathrm{d}\phi}{\mathrm{d}t} &= -n + \Phi\,\frac{3n^2}{4G^2}(B-A)\left\{\frac{C-A}{B-A}+1-3\,\frac{1+f}{f+m}\left[1+\frac{C-B}{B}\,\frac{E(m)}{K(m)}\right]\right\} \qquad (4.77)\\
\frac{\mathrm{d}g}{\mathrm{d}t} &= \frac{1}{A}\,G - \frac{\Phi}{G}\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}+n\right) - \frac{L}{G}\left[\frac{\mathrm{d}\ell}{\mathrm{d}t}+\left(\frac{1}{B}-\frac{1}{C}\right)L\right] \qquad (4.78)
\end{aligned}$$

and $\mathrm{d}h/\mathrm{d}t = n + \mathrm{d}\phi/\mathrm{d}t$.

4.5.5 Results

In this section we repeat the procedure followed in action-angle variables but now in Ferrer-Lara. Thus, using first Eqs. (4.59)-(4.64), and later Eqs. (4.69)-(4.74) we propagate the initial conditions to the different phase spaces collected in Table 4.1. Note that, as in the previous section, all the variables denoted as x^* are given by

$$x^* = x' + \Delta x', \qquad x' = x'' + \delta x'', \tag{4.79}$$

where $x \in (\ell,\, g,\, \phi,\, L,\, G,\, \Phi)$ and $x''(t)$ is obtained directly from the double-averaged Hamiltonian by the Eqs. (4.78)-(**??**).

object	var.	non-averaged	prime	double-prime	sec. frequency
PEGASUS	ℓ	-0.1626833313	-0.1628298853	-0.1592197766	-0.7146350295
	g	2.0665318080	2.067093641	2.053430312	3.8310289890
	ϕ	-0.1	-0.09999987511	-0.1009172983	-0.0441809427
	L	3.8744459575	3.874481234	3.874481234	
	G	1.0	1.0	1.0	
	H	0.3420201433	0.3420169296	0.3531301948	

Table 4.2: Initial conditions in the different phase spaces for Pegasus satellite.

As before, Fig. 4.4 shows the differences between the numerical solution and complete analytical solution (after recovering periodic terms). We can see that ℓ delays with respect to the analytical solution with a constant rate of about 0.002 units per orbital period and the motion is affected by short-period terms. On the contrary, note that g advances over the analytical solution with a constant rate of about 0.01 units per orbital period and is also affected by short-period terms. The evolution of ϕ is similar to ℓ, but the variation is roughly a ten times lower per orbital period and only affected by periodic terms related to the orbital motion. As in the case of the action-angle variables, L differs only in low-amplitude periodic terms. In turn, Φ advances again each period with a very slow rate lower than 10^{-5} units per orbital period.

4.6 Concluding remarks and comparisons

The attitude evolution in the non-averaged phase space is presented in Fig. 4.5, where to better appreciate details introduced by the gravity-gradient torque, we subtract to each variable the constant rate of the torque-free motion, represented by the tilde variables. From Hamilton equations derived from $\mathcal{K}_0 - n\,\Phi$, with $\mathcal{K}_0$ given in Eq. (**??**), we find in Ferrer-Lara variables: $\tilde{L} = L_0$, $\tilde{G} = G_0$, $\tilde{\Phi} = \Phi_0$, and

$$\tilde{\ell} = \ell_0 - \left(\frac{1}{B} - \frac{1}{C}\right) L_0\, t, \tag{4.80}$$

$$\tilde{g} = g_0 + \frac{1}{A}\, G_0\, t, \tag{4.81}$$

$$\tilde{\phi} = \phi_0 - n\, t. \tag{4.82}$$

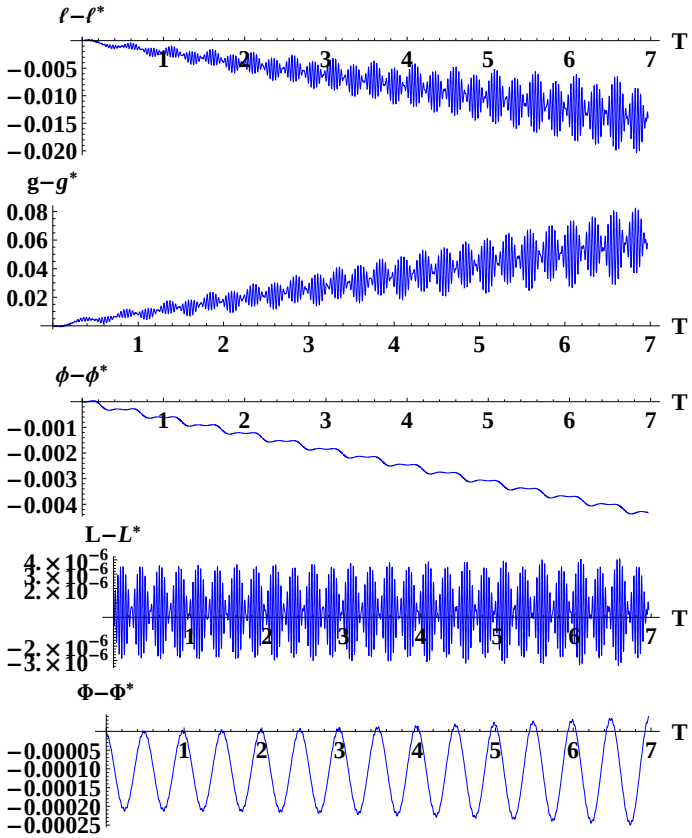


Figure 4.4: Differences between the perturbed motion in elliptic functions and the averaged motion with $x^* = x' + \Delta x'$ and $x' = x'' + \delta x''$ in internal units in Ferrer-Lara variables. T stands for orbital periods.

In turn, we find in action-angle variables: $\tilde{I}_N = I_{N_0}$, $\tilde{I}_M = I_{M_0}$, $\tilde{I}_\Lambda = I_{\Lambda_0}$, and

$$\tilde{\varphi}_\nu = \varphi_{\nu_0} - \frac{\pi}{2}\frac{C-A}{AC}\frac{f\,I_{M_0}}{\sqrt{f(1+f)(f+m)}\,\mathrm{K}(m)}\,t, \tag{4.83}$$

$$\tilde{\varphi}_\mu = \varphi_{\mu_0} + \left[\frac{I_{M_0}}{A}\left(1 - \frac{C-A}{C}\frac{f}{f+m}\right) + \frac{\pi}{2}\frac{C-A}{AC}\frac{f\,I_{N_0}}{\sqrt{f(1+f)(f+m)}\,\mathrm{K}(m)}\right]t, \tag{4.84}$$

$$\tilde{\phi} = \phi_0 - n\,t. \tag{4.85}$$

This Fig. 4.5 shows how both variables reflect the effects of the perturbation for the case of Pegasus. Note that this effect is comparable for the variables φ_ν and ℓ, but clearly unequal for the variables φ_μ and g. Observe that after seven orbital periods, the differences between the non-perturbed and the perturbed problem for the action-angle variable φ_ν is still not greater that fourteen hundredth units, while the differences for the Ferrer-Lara variable g is nearly ninety degrees. Differences of quality for the momenta are not noticeable.

Fig. 4.6 shows a comparison of the two variables for the differences between original perturbed problem and the double-averaged system. Note the reduction of the slope of the deviation observed with respect to the previous figure for the two angles, specially in Ferrer-Lara variables, which means that the new second-order secular Hamiltonian is enough to reproduce the secular trend of the perturbing function. However, the oscillations for the action-angle variables are

still about one order of magnitude lower tha in the case of Ferrer-Lara. Differences are not appreciated for the momenta.

Finally, the effects are again comparable between the two sets of variables when periodic terms are considered in Fig. 4.7. However, differences for the angles are in the same order of magnitude after seven orbital periods. Values of the momenta are also comparable.

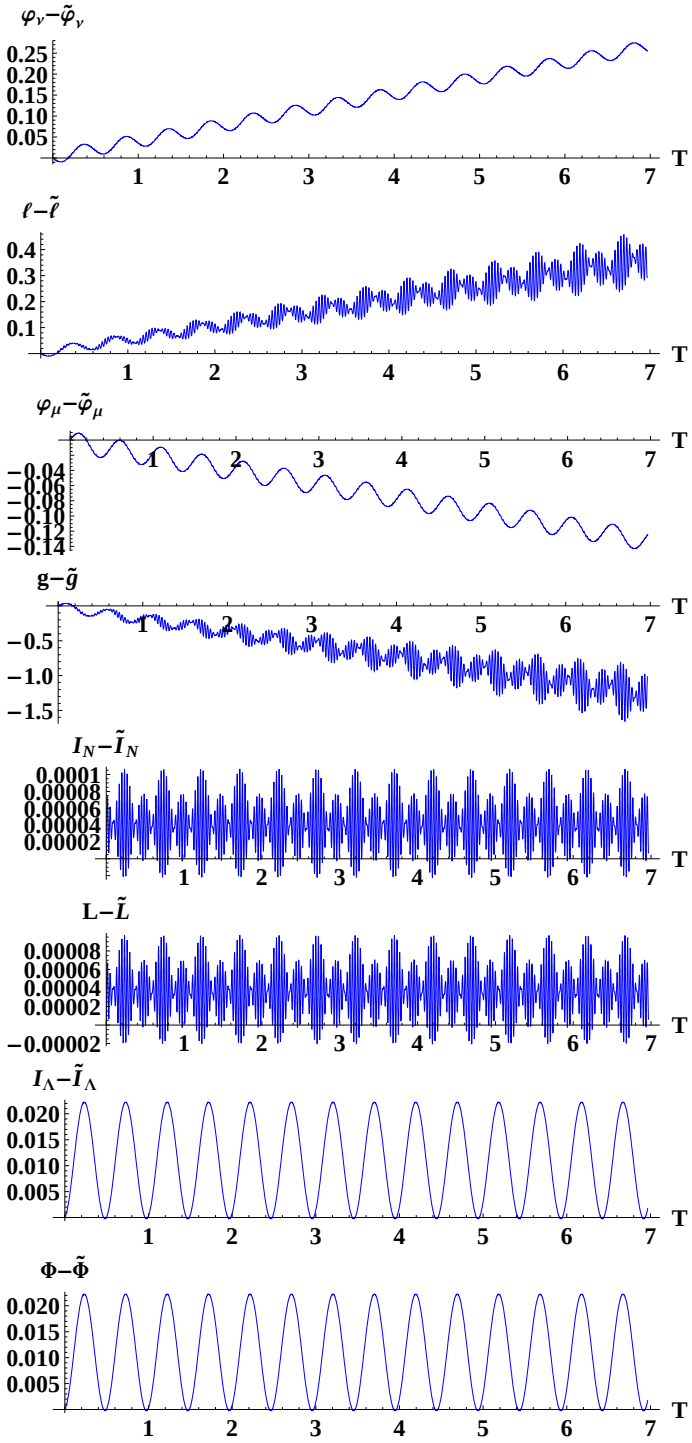


Figure 4.5: Comparisons Ferrer-Lara variables vs. action-angle variables considering the differences between the perturbed non-averaged motion and the non-perturbed motion. Only the angles are shown in internal units. T stands for orbital periods.

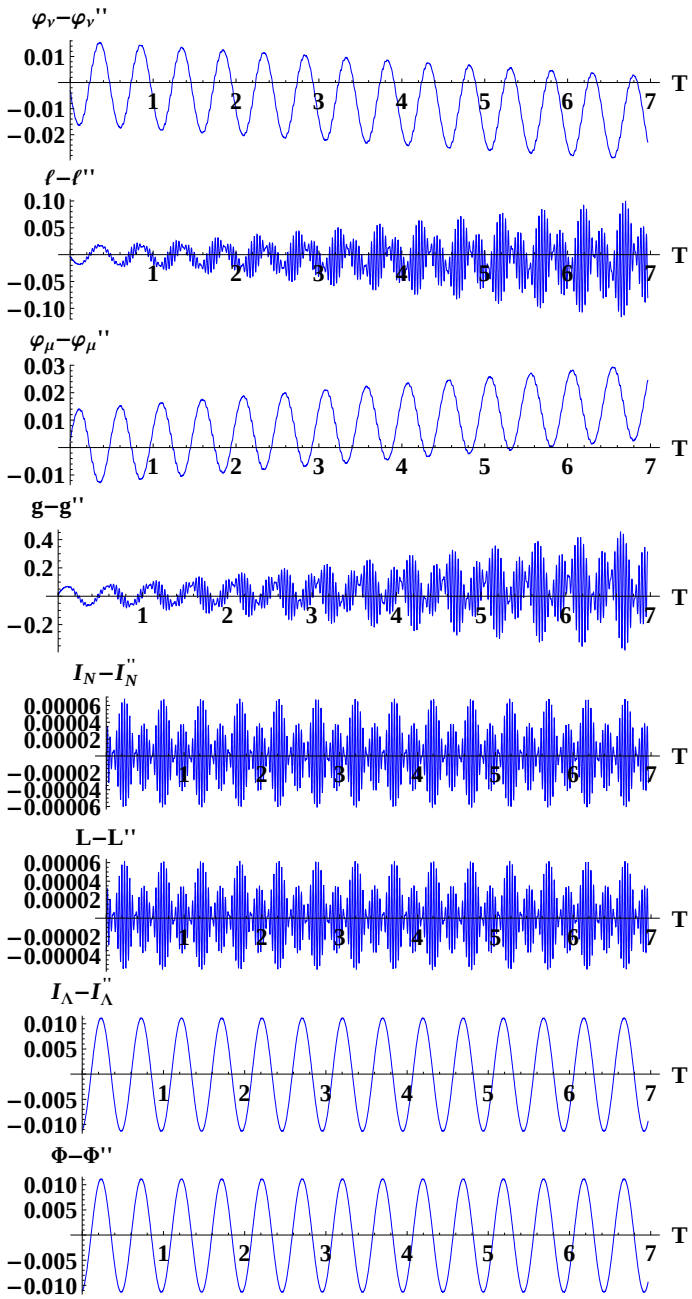


Figure 4.6: Comparisons Ferrer-Lara variables vs. action-angle variables considering the differences between the perturbed non-averaged motion and the secular (double prime) motion. Only the angles are shown in internal units. T stands for orbital periods

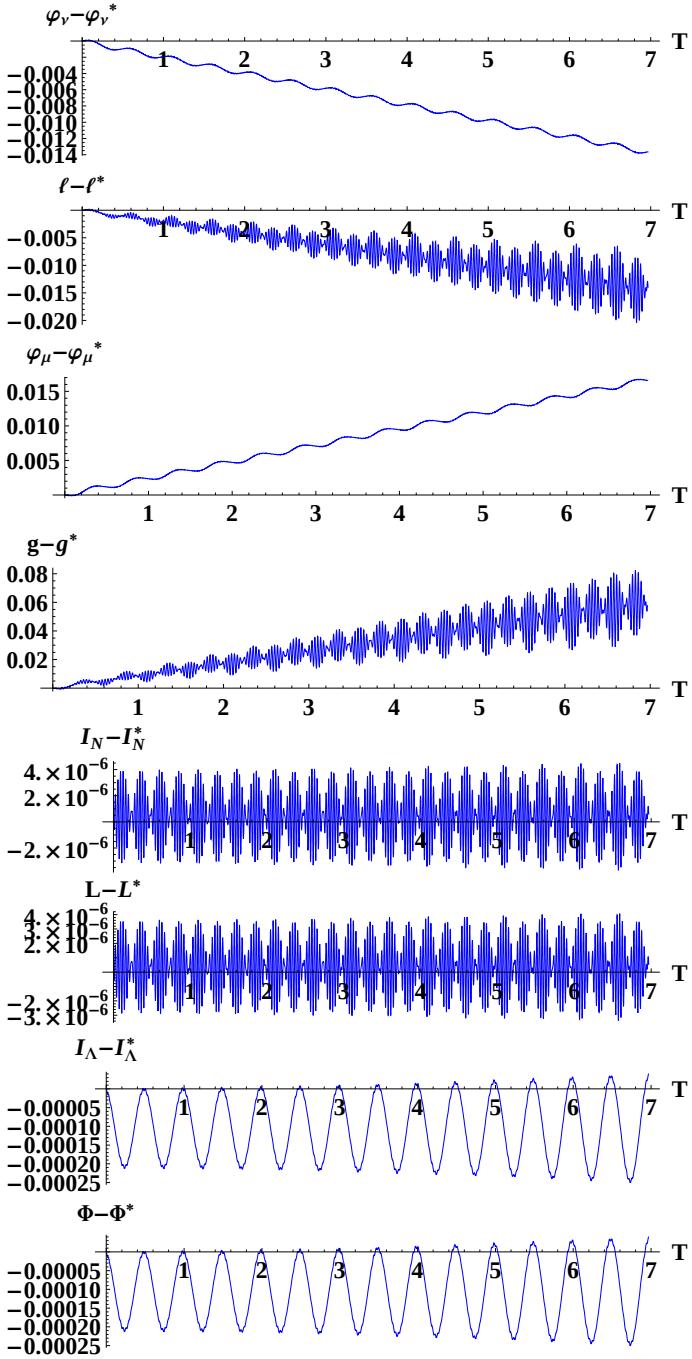


Figure 4.7: Comparisons Ferrer-Lara variables vs. action-angle variables considering the differences between the perturbed non-averaged motion and the averaged motion with $x^* = x' + \Delta x'$ and $x' = x'' + \delta x''$. Only the angles are shown in internal units. T stands for orbital periods

Chapter 5

A perturbed rigid body. Expansions approach

Due to the difficulty computing and dealing with elliptic functions and integrals in the past, some efforts were carried out to avoid its direct manipulation. Indeed, the classical method is to expand the elliptic function as a power series of the modulus or the argument, among other similar techniques rapidly convergent when the modulus is close to zero. Nevertheless, similar and much more rapidly convergent developments can be obtained when the elliptic function is expanded as a power series of the Jacobi's nome instead of the modulus (see A.6.1).

Expressing elliptic functions as a power series sometimes involves a lost of certain precision in exchange for a faster calculation time. In fact, computing elliptic functions is normally much slower than computing the trigonometric functions arising when a series expansion of the elliptic function has been accomplished. Then, in this chapter we consider the treatment of the elliptic functions involved in the perturbing function as power series. In other words, we explore whether there is a good trigonometric approach of the elliptic solution given in the previous chapter.

Indeed, this lack of precision is often acceptable since, many times, what engineers look for is to compute rapidly new solutions from new data, without the need of waiting for such a solution by numerical integrators. Therefore, these fast solutions allow us to make also fast decisions to control a satellite, for instance. Moreover, depending on the satellite, these solutions can be quite accurate when the mission is programmed to be not very long in time.

As commented above, since the elliptic nome is the best quantity to tackle the series expansions, a good way to involve the nome in the problem is achieved by first expressing the elliptic functions in terms of Jacobi Theta functions, which are in fact series expansions in the nome. However the elliptic functions seldom appear isolated in the expressions but rather together with other variables and parameters of the problem, hence one can be interested not only in the expansion of the elliptic functions themselves but also in the expansions of those factors or quotients where the elliptic functions and the other parameters are present. Since some authors (see [Sadov (1970a), Sadov (1970b), Hitzl & Breakwell (1971), Vallejo (1995)]) opted for Fourier series of such expressions revolving around the nome, this chapter focuses on elaborating a simpler analytical theory based on a Taylor expansion by Mathematica of these expressions previously given in Theta functions.

To do this, there are two elements to control the truncation order of the expansions. On one hand, the approximation of the perturbing function itself where the elliptic functions are involved. Note further that two orders of truncation have to be controlled: the first one has to do with the approximation order of the Theta functions and the second with the Taylor expansion. On the other hand, performing numerical simulations by truncating the disturbing function up to different orders is always a good help in these cases, but it is known that there are ongoing investigations about numerical methods in rotational dynamics [Celledoni & Zanna (2010)], hence analytical theories can not be neglected at all.

5.1 On series expansions of the perturbation

As shown in Chapter 4, the perturbing function analyzed in this Memoir is a simplified version of a more general model given by Eq. (4.4) after carrying out an averaging of the variable μ when it is much faster than the variable ν and the mean orbital motion n. The model revolves around the calculation of the direction cosines of the line joining the center of mass of the rigid body with the perturber, which are part of the perturbing function. This calculation depends on how the different frames of the problem have been chosen. In our model the orbital plane is chosen as the inertial reference frame but in a more general model, the orbital plane may not match with the intertial frame, hence in [Vallejo (1995)], the orbital frame $\{\boldsymbol{r_1}, \boldsymbol{r_2}, \boldsymbol{r_3}\}$ is defined by taking $\boldsymbol{r_1}$ in the same direction as the line joining the center of mass of the rigid body with the perturber; $\boldsymbol{r_3}$ in the direction of the orbital angular momentum; and $\boldsymbol{r_2} = \boldsymbol{r_3} \times \boldsymbol{r_1}$. Nevertheless, in order to simplify the calculations, Vallejo set as reference frame the plane perpendicular to the total angular momentum vector (also an integral of the problem), which is the sum of the orbital angular momentum and the rotational angular momentum.

In short, as can be noted in [Barkin (1992)], [Vallejo (1995)] (page 56) or [Barkin (1998)] (page 201), the direction cosines may be expressed as functions of the components of the rotation matrix connecting the body frame ($\mathcal{B}$) with the angular momentum frame ($\mathcal{N}$), both defined in Section 2.1. That is $[b_{i,j}] = R_3(-\mu)\, R_1(-J)\, R_3(-\nu)$, which gives[1]

$$[b_{i,j}] = \begin{pmatrix} \cos\nu\cos\mu - \sin\nu\sin\mu\cos J & -\sin\nu\cos\mu - \cos\nu\sin\mu\cos J & \sin\mu\sin J \\ \cos\nu\sin\mu + \sin\nu\cos\mu\cos J & -\sin\nu\sin\mu + \cos\nu\cos\mu\cos J & -\cos\mu\sin J \\ \sin\nu\sin J & \cos\nu\sin J & \cos J \end{pmatrix}. \quad (5.1)$$

where it can be noted that (5.1) is the same matrix as [Sadov (1970b)] (page 14, formulas (3.2), matrix S_2)[2] and its components the same, expressed in elliptic functions, as [Barkin (1998)] (page 188, formulas (27)), but not the same as [Vallejo (1995)] (page 53, formula (3.7)), which is clearly a typo. Note also that all the components of this matrix can be expressed in terms of Jacobi Theta functions with argument φ_ν (see [Vallejo (1995)] page 48) except for the cases $\sin\mu$ and $\cos\mu$ where the new Sadov variable φ_μ also appears.

As commented before, Fourier series expansions have been the classical way to carry out the series expansions of the disturbing function in rotational dynamics. To address this work, [Sadov (1970b)] pointed out the convenience of separating (5.1) as the product of two new

[1] In order to compute these rotations with *Mathematica* in a right way we may type, for instance: $[b_{i,j}]$ = Transpose[R_μ].Transpose[R_J].Transpose[R_ν] where R_μ = RotationMatrix[$-\mu, \{0,0,1\}$], R_J = RotationMatrix[$-J, \{1,0,0\}$] and R_ν = RotationMatrix[$-\nu, \{0,0,1\}$].

[2] Observe that $\varphi \equiv \nu$, $\psi \equiv \mu$ and $\vartheta \equiv J$

matrices, one of them only containing the variable φ_μ and the other one, denoted as Vallejo $[B_{i,j}]$, containing a new variable

$$\omega(\text{Vallejo}) = \Delta(\text{Sadov}) = \varphi_\mu - \mu, \tag{5.2}$$

which may be also expressed in terms of Theta functions with argument φ_μ.

Indeed, Sadov and Vallejo denoted

$$[b_{i,j}] = \begin{pmatrix} \cos\varphi_\mu & -\sin\varphi_\mu & 0 \\ \sin\varphi_\mu & \cos\varphi_\mu & 0 \\ 0 & 0 & 1 \end{pmatrix} [B^S_{i,j}] = \begin{pmatrix} -\exp(i\varphi_\mu)/(2i) & \exp(-i\varphi_\mu)/(2i) & 0 \\ \exp(i\varphi_\mu)/2 & \exp(-i\varphi_\mu)/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} [B^V_{i,j}], \tag{5.3}$$

where $[B^S_{i,j}]$ denotes the components of the matrix defined by [Sadov (1970b)] (page 21, formulas 3.21) and $[B^V_{i,j}]$ denotes the components of the matrix defined by [Vallejo (1995)] (page 24, formulas 3.10).

To obtain these components, the quasi-addition formulas of the Jacobi Theta functions have to be used (see formula (5.12) for further details), and once obtained, the Fourier series expansions may be tackled by the method depicted in [Sadov (1970b)] (pages 23-25) or [Vallejo (1995)] (page 6, Section 1.3). More information about Fourier series expansions of the elliptic functions can be found in [Whittaker & Watson (1927), Armitage & Eberlein (1972), Wang & Guo (1989), Hall (1995)].

Nevertheless, as pointed out in [Barkin (1998)] (page 190) and [Vallejo (1995)] (pages $56-57$), the expression of the perturbing function does not contain exactly the values of $[b_{i,j}]$, but rather squares and products of them. Therefore, the series expansions needed when dealing with the perturbing funcion are really these squares and products. Note that Vallejo indicated this fact on page 57 of his thesis, but he did not give the new expansions which can be found in [Barkin (1998)] (Appendix).

5.2 Series expansions in the case of a fast rotating satellite

As done in Chapter 4, before applying perturbation theory one has to express the perturbing function in Jacobi Theta functions. Recall that, in (4.15), this disturbing function was

$$\begin{aligned} U &= \frac{n^2}{16}(2 - 3s_I^2 + 3s_I^2\cos 2\phi)\left\{(2C - B - A)\left[1 - 3\frac{f}{f+m}\mathrm{dn}^2(X(m)\ell|m)\right]\right. \\ &\quad \left. +3(B-A)\left[1 - \frac{f}{f+m}\mathrm{dn}^2(X(m)\ell|m)\right]\left[1 - 2\frac{(1+f)\,\mathrm{sn}^2(X(m)\ell|m)}{1 + f\,\mathrm{sn}^2(X(m)\ell|m)}\right]\right\}. \end{aligned} \tag{5.4}$$

Nevertheless it is more convenient to start expressing this function not in new variables but in Andoyer variables. Thus, taking into account Eqs. (4.10)-(4.12) and doing a few calculations we can express the disturbing function as

$$U = \frac{n^2}{16}(2 - 3s_I^2 + 3s_I^2\cos 2\phi)\left[(2C - B - A)(3s_J^2 - 2) + 3(B-A)s_J^2(1 - 2\cos^2\nu)\right], \tag{5.5}$$

where[3]

$$\cos^2\nu = \frac{(1+f)\,\mathrm{sn}^2(X(m)\ell|m)}{1+f\,\mathrm{sn}^2(X(m)\ell|m)}, \tag{5.6}$$

$$\sin J = \sqrt{\frac{m}{f+m}}\sqrt{1+f\,\mathrm{sn}^2(X(m)\ell|m)}, \tag{5.7}$$

being

$$\cos J = \sqrt{\frac{f}{f+m}}\,\mathrm{dn}(X(m)\ell|m).$$

As we show in the previous section, we could already replace the elliptic funcions by the Jacobi Theta functions using Eqs. (A.71)-(A.73), but the goal is now to prove that both (5.6) and (5.7) can be expessed in Jacobi Theta functions including the term $1+f\,\mathrm{sn}^2$ in the developments.

5.2.1 Disturbing function in Jacobi Theta functions (action-angle variables)

In action-angle variables, Eqs. (5.6) and (5.7) are expressed as

$$\cos^2\nu = \frac{(1+f)\,\mathrm{sn}^2\left(\frac{2\mathrm{K}(m)}{\pi}\varphi_\nu|m\right)}{1+f\,\mathrm{sn}^2\left(\frac{2\mathrm{K}(m)}{\pi}\varphi_\nu|m\right)}, \tag{5.8}$$

$$\sin J = \sqrt{\frac{m}{f+m}}\sqrt{1+f\,\mathrm{sn}^2\left(\frac{2\mathrm{K}(m)}{\pi}\varphi_\nu|m\right)}, \tag{5.9}$$

As observed, the first obstacle one finds is how to deal with Jacobi Theta functions (biparametric functions) a three-parameter expression, this is, φ_ν, m and f. This problem can be avoided (see Appendix A.4.2)[4] by means of a mathematical resource given by

$$m\,\mathrm{sn}^2(a|m) = -f, \tag{5.10}$$

where

$$a = \frac{2\mathrm{K}(m)}{\pi}\,z\,i \qquad \text{and} \qquad z = \frac{\pi}{2\mathrm{K}}\,\mathrm{F}\left(\arctan\sqrt{\frac{f}{m}}\middle|1-m\right). \tag{5.11}$$

In particular, with these equations, the procedure to express the term $1+f\,\mathrm{sn}^2$ in Jacobi Theta functions is carried out by applying the quasi-addition formulas of the Theta functions. Specifically, from formula 1.4.19 in [Lawden (1989)] (page 9) we have

$$\vartheta_4(x+y,q)\vartheta_4(x-y,q)\vartheta_4^2(0,q) = \vartheta_4^2(x,q)\vartheta_4^2(y,q) - \vartheta_1^2(x,q)\vartheta_1^2(y,q) \tag{5.12}$$

where x and y are two angles and

$$\vartheta_4^2(0,q) = \frac{2\mathrm{K}(m)}{\pi}\sqrt{1-m} \tag{5.13}$$

[3]Recall that $X(m)=-1$ in Ferrer-Lara variables and $X(m)=-\frac{2\mathrm{K}(m)}{\pi}$ in action-angle variables.
[4]The reader may also see [Byrd & Friedman (1971), Vallejo (1995)]

Furthermore, taking into account (A.71) this expression can be put into the form

$$\begin{aligned}\vartheta_4(x+y,q)\vartheta_4(x-y,q)\vartheta_4^2(0,q) &= \vartheta_4^2(x,q)\vartheta_4^2(y,q)\left[1-\frac{\vartheta_1^2(x,q)}{\vartheta_4^2(x,q)}\frac{\vartheta_1^2(y,q)}{\vartheta_4^2(y,q)}\right] \\ &= \vartheta_4^2(x,q)\vartheta_4^2(y,q)\left[1-m\,\mathrm{sn}^2\left(\frac{2\mathrm{K}(m)}{\pi}x|m\right)\mathrm{sn}^2\left(\frac{2\mathrm{K}(m)}{\pi}y|m\right)\right],\end{aligned} \tag{5.14}$$

and note further that if we call $x=\varphi_\nu$ and $y=iz$, by taking into account (5.10), then

$$\vartheta_4(\varphi_\nu+iz,q)\vartheta_4(\varphi_\nu-iz,q)=\frac{\vartheta_4^2(\varphi_\nu,q)\vartheta_4^2(iz,q)}{\vartheta_4^2(0,q)}\left[1+f\,\mathrm{sn}^2\left(\frac{2\mathrm{K}(m)}{\pi}\,\varphi_\nu|m\right)\right], \tag{5.15}$$

and therefore we can solve the factor

$$1+f\,\mathrm{sn}^2\left(\frac{2\mathrm{K}(m)}{\pi}\,\varphi_\nu|m\right)=\frac{\vartheta_4(\varphi_\nu+iz,q)\vartheta_4(\varphi_\nu-iz,q)}{\vartheta_4^2(\varphi_\nu,q)\vartheta_4^2(iz,q)}\,\vartheta_4^2(0,q). \tag{5.16}$$

Then, taking into account (5.16) and (A.71), we can finally express (5.8) and (5.9) respectively as[5]

$$\cos^2\nu = \frac{1+f}{\sqrt{m(1-m)}}\,\frac{\pi}{2\mathrm{K}(m)}\,\frac{\vartheta_1^2(\varphi_\nu,q)\vartheta_4^2(iz,q)}{\vartheta_4(\varphi_\nu+iz,q)\vartheta_4(\varphi_\nu-iz,q)}, \tag{5.17}$$

$$\sin J = \sqrt[4]{1-m}\sqrt{\frac{m}{f+m}}\sqrt{\frac{2\mathrm{K}(m)}{\pi}}\,\frac{\sqrt{\vartheta_4(\varphi_\nu+iz,q)\vartheta_4(\varphi_\nu-iz,q)}}{\vartheta_4(\varphi_\nu,q)\vartheta_4(iz,q)}. \tag{5.18}$$

We can now express the disturbing function (5.5) in terms of the Theta functions. However, as we have to make some averages when applying the Lie-Deprit algorithm, we find convenient to expand this disturbing function showing all its summands. It will allow us to distinguish easier the secular terms and the periodic terms of the function. Thus, taking into account (5.17) and (5.18) we have

$$\widetilde{A} = -\frac{2n^2}{16}(2-3s_I^2+3s_I^2\cos 2\phi)(2C-B-A), \tag{5.19}$$

$$\begin{aligned}\widetilde{B} &= \frac{6n^2}{16}(2-3s_I^2+3s_I^2\cos 2\phi)(C-A)\,s_J^2 \\ &= \frac{6n^2}{16}(2-3s_I^2+3s_I^2\cos 2\phi)(C-A)\,\frac{m\sqrt{1-m}}{f+m}\,\frac{2\mathrm{K}(m)}{\pi}\,\frac{\vartheta_4(\varphi_\nu+iz,q)\vartheta_4(\varphi_\nu-iz,q)}{\vartheta_4^2(\varphi_\nu,q)\vartheta_4^2(iz,q)},\end{aligned} \tag{5.20}$$

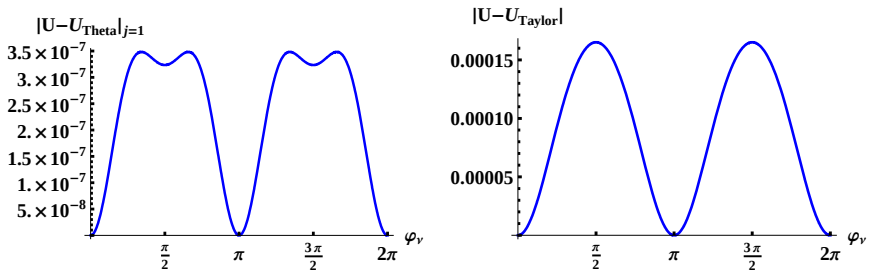
$$\begin{aligned}\widetilde{C} &= -\frac{6n^2}{16}(2-3s_I^2+3s_I^2\cos 2\phi)(B-A)\,s_J^2\cos^2\nu \\ &= -\frac{6n^2}{16}(2-3s_I^2+3s_I^2\cos 2\phi)(B-A)\,\frac{\sqrt{m}(1+f)}{f+m}\,\frac{\vartheta_1^2(\varphi_\nu,q)}{\vartheta_4^2(\varphi_\nu,q)},\end{aligned} \tag{5.21}$$

where of course $U=\widetilde{A}+\widetilde{B}+\widetilde{C}$. Note also that $\widetilde{A}$ is secular itself and the other two terms are a combination of both secular and periodic summands. We can also observe that we have not expressed the modulus in Theta functions, as well as the complete integral of the first kind. This decision has been made because the value of the modulus is one of the first known values by means of the energy and the moments of inertia. Moreover, the calculation of $\mathrm{K}(m)$ is currently very fast and efficient.

[5]For the benefit of the reader, some more expressions of the Andoyer variables as function of Theta functions are collected in [Vallejo (1995)] (page 46).

5.2.2 Taylor expansion of the disturbing function (action-angle variables)

In order to complete the treatment of the perturbing function, we will further perfom a Taylor series expansion in the nome of those quotients involving Theta functions in the previous summands; in particular $\widetilde{B}$ and $\widetilde{C}$. However, before applying Taylor we have to choose a truncation order for Theta functions. As the reader can see in Fig. 5.1(a), using the initial conditions shown in the previous chapter, when truncated as little as $j = 1$, the differences between the perturbing function evaluated with Mathematica elliptic functions and the truncated version are never greater than 10^{-7}, which is in fact a very good approximation taking into account that the small parameter is around 10^{-3} as we commented above.



(a) (b)

Figure 5.1: Different approximations of the perturbing function in action-angle variables. (a) Differences between the perturbing function evaluated with Mathematica elliptic functions according to (5.4) and the same function in Theta functions truncated up to order $j = 1$. (b) Differences between the perturbing function evaluated with Mathematica elliptic functions and the function truncated up to order $j = 1$ and later expressed as a power series expansion in the nome up to order $\mathcal{O}(q^3)$.

Since in this work we will perform a Taylor series expansion in the nome, a truncation order of such a development must be chosen. Initially, as we do not have information about the convergence speed, some numerical trials can be carried out for different truncations. Indeed, Fig. 5.2 shows two numerical approaches of PEGASUS-A satellite when truncations up to $\mathcal{O}(q^3)$ and $\mathcal{O}(q^{12})$ are developed. Note that, for this body and initial conditions, an $\mathcal{O}(q^{12})$-expansion have to be developed in order to get an approach whose differences when compared with the exact model are at the same level as the small parameter for the three angles of the problem. Note further the improvements achieved with respect to the momenta.

Nevertheless, if we repeat the above test applied to a different body (different moments of inertia) but with the same initial conditions as Pegasus, excellent results are found in Fig. 5.3 when truncated as little as $\mathcal{O}(q^3)$. Therefore, we are interested in developing a perturbation approach up to order $\mathcal{O}(q^3)$ to check the quality of such an approximation. Obviously, as we have just seen, this approximation will no be specially good for a number of objects, but it is known that, for a number of space missions, decisions in real-time prevail over great precisions for these objects or satellites, so this kind of low-precision (in some cases) algorithms are usually good for this purpose.

Therefore, in this work we will perform a power series expansion in the nome up to $\mathcal{O}(q^3)$. Once analyzed some numerical simulations, we can also see the behaviour of the perturbing function

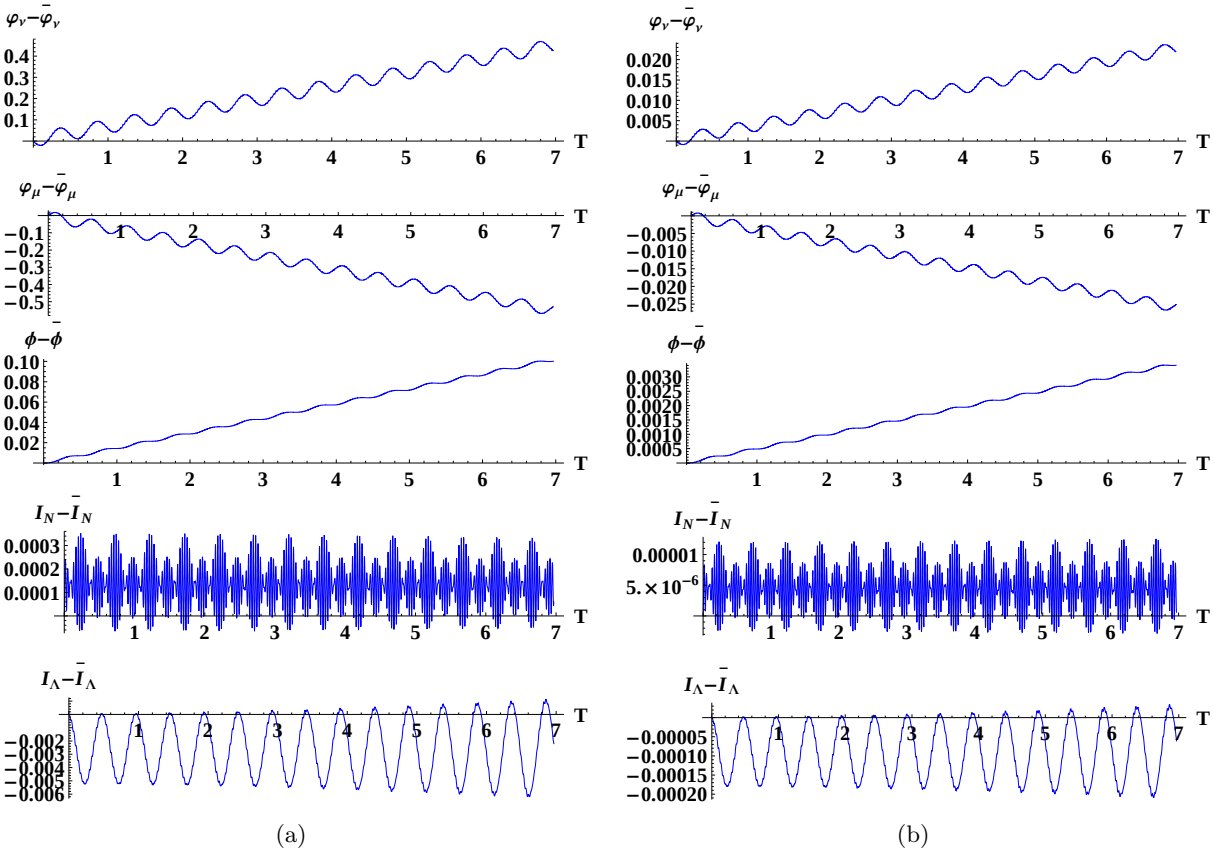


Figure 5.2: Some numerical trials to have an idea on how the convergence evolves according to the truncation order of a Taylor series expansion of the perturbing function. (a) Differences between the perturbed motion in elliptic functions (φ_ν, φ_μ, ϕ, I_N, I_M, I_Λ) and the motion when the perturbing function is expanded as a Taylor power series up to $\mathcal{O}(q^3)$ ($\bar{\varphi}_\nu$, $\bar{\varphi}_\mu$, $\bar{\phi}$, $\bar{I}_N$, $\bar{I}_M$, $\bar{I}_\Lambda$) in internal units for the initial conditions of Pegasus. T stands for orbital periods. (b) The same when the perturbing function is expanded up to $\mathcal{O}(q^{12})$.

with respect to the truncation order. Fig. 5.1(b) shows the differences between the perturbing function in elliptic functions and the same function expanded as a Taylor series up to $\mathcal{O}(q^3)$. Observe that this time the errors are never greater than 10^{-4}, the same order of magnitude as the perturbing function itself.

For the benefit of the reader, we next give the expressions of the expansions of the above

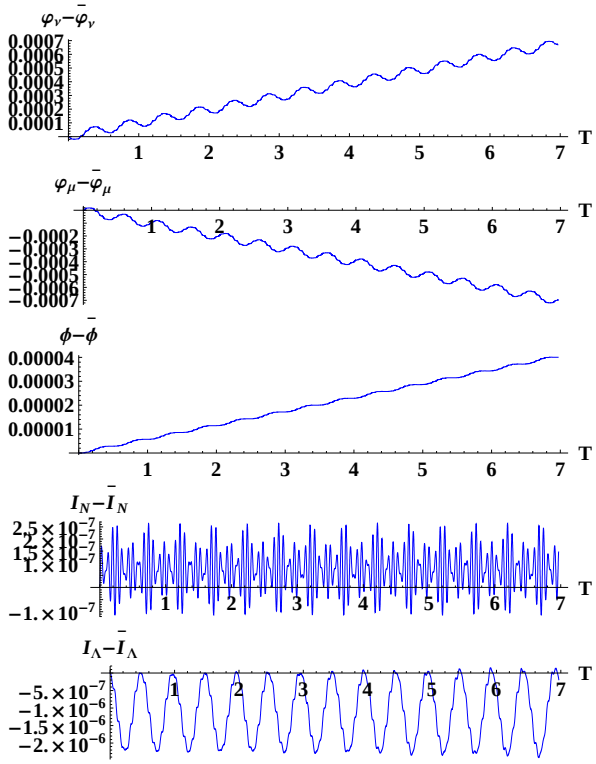


Figure 5.3: Differences between the perturbed motion in elliptic functions (φ_ν, φ_μ, ϕ, I_N, I_M, I_Λ) and the motion when the perturbing function is expanded as a Taylor power series up to $\mathcal{O}(q^3)$ ($\bar{\varphi}_\nu$, $\bar{\varphi}_\mu$, $\bar{\phi}$, $\bar{I}_N$, $\bar{I}_M$, $\bar{I}_\Lambda$) in internal units for initial conditions given by: $a = 0.7$, $b = 0.8$, $c = 1.0$, $f = 0.714$, $m = 0.019$. T stands for orbital periods.

summands:

$$\widetilde{A}_{SE} = \widetilde{A} = -\frac{2n^2}{16}(2 - 3s_I^2 + 3s_I^2\cos 2\phi)(2C - B - A), \tag{5.22}$$

$$\begin{aligned}\widetilde{B}_{SE} = &-\frac{3n^2}{8}\,\frac{m\sqrt{1-m}}{f+m}\,\frac{2\mathrm{K}(m)}{\pi}(2 - 3s_I^2 + 3s_I^2\cos 2\phi)(C - A)\\ &\Big\{-1 - 12q^2 + 16q^3 + 4q(-1 + 2q - 13q^2)\cosh(2z) + 8q^2(-1 + 2q)\cosh(4z) - 12q^3\cosh(6z)\\ &+\big[4q(-1 + 2q - 13q^2) + 4q(1 - 4q + 18q^2)\cosh(2z) + 8q^2(1 - 4q)\cosh(4z) + 12q^3\cosh(6z)\big]\cos(2\varphi_\nu)\\ &+\big[8q^2(-1 + 2q) + 8q^2(1 - 4q)\cosh(2z) + 16q^3\cosh(4z)\big]\cos(4\varphi_\nu)\\ &+\big[-12q^3 + 12q^3\cosh(2z)\big]\cos(6\varphi_\nu)\Big\}\end{aligned} \tag{5.23}$$

$$\begin{aligned}\widetilde{C}_{SE} = &-\frac{3n^2}{4}\,\frac{(1+f)\sqrt{m}}{f+m}(2 - 3s_I^2 + 3s_I^2\cos 2\phi)(B - A)\sqrt{q}\\ &\Big\{1 - 2q + 6q^2 + (-1 + 4q - 11q^2)\cos(2\varphi_\nu) + 2q(-1 + 4q)\cos(4\varphi_\nu) - 3q^2\cos(6\varphi_\nu)\Big\}.\end{aligned} \tag{5.24}$$

where the formulas

$$\cos(x\,\varphi_\nu + y\,iz) + \cos(x\,\varphi_\nu - y\,iz) = 2\cos(x\,\varphi_\nu)\cosh(y\,z)$$

$$\sin(x\,\varphi_\nu + y\,iz) + \sin(x\,\varphi_\nu - y\,iz) = 2\sin(x\,\varphi_\nu)\cosh(y\,z)$$

has been taken into account. Note further that the terms $\widetilde{B}_{SE}$ and $\widetilde{C}_{SE}$ can be expressed as a secular part plus a periodic part in the angle φ_ν. In the case of $\widetilde{B}_{SE}$, observe that the second line of the expression corresponds to the secular part and the rest of the lines correspond to the periodic part.

5.2.3 On the bounding of the Taylor expansion of the disturbing function

As noticed in Figs. 5.1(b) and 5.2, the convergence speed is not as good as it could be expected according to the value of the nome. Indeed, taking into account the initial conditions used in the previous chapter, we have $m \approx 0.32$ which involves $q(m) \approx 10^{-2}$. Since a three-order deveplopment has been carried out for the disturbing functions, it should be expected errors not greater than $q^3 \approx 10^{-6}$, however we have errors around 10^{-4}.

Observe that the larger errors are introduced by the term $\widetilde{B}_{SE}$ due to the presence of complex arguments. When Taylor approach is accomplished, cosines of complex angles become hyperbolic cosines according to

$$\cos(ikz) = \cosh(kz)$$

and therefore the presence of no bounded functions arises in the problem. Nevertheless, although slowly, this term $\widetilde{B}_{SE}$ converges, hence the powers of the nome compensate the effects of the hyperbolic cosines. Indeed, note that, for instance, the secular part of (5.23) can be expressed as

$$\sum_{k=0}^{3} q^k P_{3-k}(q) \cosh(2kz)$$

where $P_{3-k}(q)$ denotes a $(3-k)$-degree polynomial in the nome. Table 5.1 shows how the products $q^k \cosh(2kz)$ decrease slowly according to the value of k when consider Pegasus initial conditions. However, table 5.2 shows how the products $q^k \cosh(2kz)$ decrease faster when different initial conditions are considered.

	q^k	$\cosh(2kz)$	$q^k \cosh(2kz)$
$k=1$	2.42×10^{-2}	12.8738	0.3119
$k=2$	5.87×10^{-4}	330.4706	0.1940
$k=3$	1.42×10^{-5}	$8,495.9695$	0.1209
$k=4$	3.44×10^{-7}	$218,420.7645$	0.0753

Table 5.1: Convergence speed according to the value of q^k and $\cosh(2kz)$ taking into account the initial conditions for Pegasus.

5.2.4 Disturbing function in Ferrer-Lara variables

The expressions given in previous section are valid when used Ferrer-Lara variables, but we have to note that the variable ℓ is not an angle. Since Theta functions have to receive an angle as

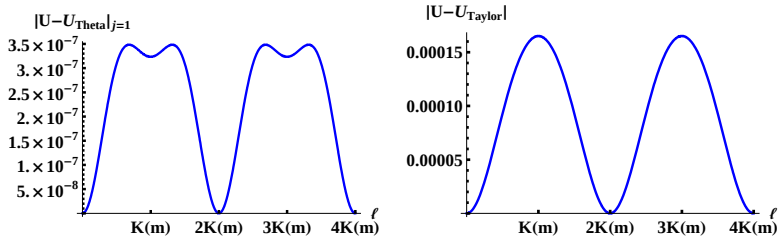
	q^k	$\cosh(2kz)$	$q^k \cosh(2kz)$
$k=1$	1.22×10^{-3}	55.5826	0.0681
$k=2$	1.50×10^{-6}	$6,177.8716$	9.29×10^{-3}
$k=3$	1.84×10^{-9}	$686,709.9097$	1.26×10^{-3}
$k=4$	2.26×10^{-12}	7.63×10^{7}	1.72×10^{-4}

Table 5.2: Convergence speed according to the value of q^k and $\cosh(2kz)$ taking into account initial conditions such that $a = 0.7$, $b = 0.8$, $c = 1.0$, $f = 0.714$, $m = 0.019$.

argument, one has to transform previously this variable into an angle by means of

$$\alpha = \frac{\pi}{2K(m)}\,\ell. \tag{5.25}$$

which was already given in (A.70). Then, φ_ν must be replaced by α in Eqs. (5.8)-(5.24). However, it is important to remark that, in Ferrer-Lara variables our angle is $\alpha = \alpha(\ell, m)$. Moreover, Fig. 5.4 (equivalent to Fig. 5.1) shows that Ferrer-Lara variables maintain the same errors as action-angle variables.



(a) (b)

Figure 5.4: Different approximations of the perturbing function in Ferrer-Lara variables. (a) Differences between the perturbing function evaluated with Mathematica elliptic functions according to (5.4) and the same function in Theta functions truncated up to order $j = 1$. (b) Differences between the perturbing function evaluated with Mathematica elliptic functions and the function truncated up to order $j = 1$ and later expressed as a power series expansion in the nome up to order $q = 3$.

5.3 Perturbation approach in action-angle variables

In this section we will repeat the procedure carried out in close form in the previous chapter, but starting from the series expansion in the nome of the disturbing function.

5.3.1 Average over φ_ν

Like in the previous chapter, we look for a canonical transformation that removes the variable φ_ν from the Hamiltonian. We set

$$H_{0,0} = \mathcal{K}_0, \qquad H_{1,0} = -n\,\Phi, \qquad H_{2,0} = 2\,U,$$

where all the functions are assumed to be expressed in prime variables. For the sake of brevity, we drop the prime notation in what follows when there is no risk of confusion.

Taking into account Eqs. (4.18) and (4.29) in the previous chapter, the first step in the computation of the Lie triangle gives

$$-\frac{I_M}{A}\frac{C-A}{C}\frac{\pi}{2\mathrm{K}(m)}\sqrt{\frac{f}{(1+f)(f+m)}}\frac{\partial W_1}{\partial \varphi_\nu} = H_{1,0} - H_{0,1}, \tag{5.26}$$

As $H_{1,0}$ does not depend on φ_ν or g, we choose

$$H_{0,1} = H_{1,0}, \tag{5.27}$$

and the first term of the generating function is $W_1 = 0$. Because the vanishing of W_1, the next step gives

$$-\frac{I_M}{A}\frac{C-A}{C}\frac{\pi}{2\mathrm{K}(m)}\sqrt{\frac{f}{(1+f)(f+m)}}\frac{\partial W_2}{\partial \varphi_\nu} = H_{2,0} - H_{0,2}. \tag{5.28}$$

where, by inspecting Eqs. (5.22)-(5.24), we can easily express $H_{2,0}$ as a secular term (S) plus a periodic term (P), that is $H_{2,0} = H_{2,0}(S) + H_{2,0}(P)$.

We then choose $H_{0,2} = H_{2,0}(S)$ where, after several calculations we have

$$\begin{aligned} H_{0,2} \;=\; & -\frac{n^2}{4}(2-3s_I^2+3s_I^2\cos 2\phi)\Big\{(2C-B-A)+6(B-A)\frac{(1+f)\sqrt{m}}{f+m}\sqrt{q}\,X_0 \\ & -3(C-A)\frac{m\sqrt{1-m}}{f+m}\frac{2\mathrm{K}(m)}{\pi}\sum_{k=0}^{3}[Y_{2k}\cosh(2kz)]\Big\}, \end{aligned} \tag{5.29}$$

and Table 5.3 shows the polynomials involved in the above average.

Table 5.3: Polynomials involved in $H_{0,2}$

X_0	$=$	$1-2q+6q^2$
Y_0	$=$	$1+12q^2-16q^3$
Y_2	$=$	$4q-8q^2+52q^3$
Y_4	$=$	$8q^2-16q^3$
Y_6	$=$	$12q^3$

Finally, according to (5.30), in order to get W_2 we have to solve the differential equation

$$-\frac{I_M}{A}\frac{C-A}{C}\frac{\pi}{2\mathrm{K}(m)}\sqrt{\frac{f}{(1+f)(f+m)}}\frac{\partial W_2}{\partial \varphi_\nu} = H_{2,0}(P). \tag{5.30}$$

whose solution, after several calculations, is given by

$$
\begin{aligned}
W_2 \;=\; & \frac{3n^2}{4}\frac{C}{C-A}\frac{A}{I_M}\sqrt{\frac{m(1+f)}{f(f+m)}}\frac{2\mathrm{K}(m)}{\pi}(2-3s_I^2+3s_I^2\cos 2\phi) \\
& \Big\{(B-A)(1+f)\sqrt{q}\sum_{k=1}^{3}[\mathcal{P}_{2k}\sin(2k\varphi_\nu)] \\
& +4(C-A)\sqrt{m(1-m)}\,\frac{2\mathrm{K}(m)}{\pi}\sinh^2 z\sum_{k=1}^{3}[\mathcal{Q}_{2k}\sin(2k\varphi_\nu)]\Big\}. \qquad (5.31)
\end{aligned}
$$

and Table 5.4 shows the polynomials involved in the above average.

Table 5.4: Expressions involved in W_2

$\mathcal{P}_2$	$=$	$-1+4q-11q^2$
$\mathcal{P}_4$	$=$	$-q+4q^2$
$\mathcal{P}_6$	$=$	$-q^2$
$\mathcal{Q}_2$	$=$	$q+11q^3+(4q^2-4q^3)\cosh(2z)+6q^3\cosh(4z)$
$\mathcal{Q}_4$	$=$	$q^2+4q^3\cosh(2z)$
$\mathcal{Q}_6$	$=$	q^3

5.3.2 Transformation equations of the first averaging

Now we have to follow the same procedure done in the previous chapter. Note that this procedure involves the calculation of the partial derivative of W_2 with respect to m, which is given by

$$
\begin{aligned}
\frac{\partial W_2}{\partial m} \;=\; & \left[\frac{\mathrm{E}(m)}{2m(1-m)\mathrm{K}(m)}-\frac{1}{2(f+m)}\right]W_2-\frac{3n^2}{16}\frac{C}{C-A}\frac{A}{I_M}\sqrt{\frac{m(1+f)}{f(f+m)}}\frac{\mathrm{K}(m)}{\pi^2} \\
\times\; & (2-3s_I^2+3s_I^2\cos 2\phi)\Bigg\{\frac{(B-A)(1+f)\pi^3\sqrt{q}}{m(1-m)\,\mathrm{K}^2(m)}\sum_{k=1}^{3}[\widetilde{P}_{2k}\sin(2k\varphi_\nu)]-\frac{16(C-A)}{\sqrt{m(1-m)}} \\
\times\; & \Bigg[2\sum_{k=1}^{3}[Q_{2k}\sin(2k\varphi_\nu)]\big[\sinh^2(z)(\mathrm{E}(m)-(1-m)\,\mathrm{K}(m))+2m(1-m)\Psi\,\mathrm{K}(m)\sinh(2z) \\
+\; & (1-2m)\,\mathrm{K}(m)\sinh^2(z)\big]+\frac{q\sinh^2(z)}{\mathrm{K}(m)}\Bigg[\pi^2\sum_{k=1}^{3}[\widetilde{Q}_{2k}\sin(2k\varphi_\nu)] \\
+\; & 32q\,m(1-m)\Psi\,\mathrm{K}^2(m)\sinh(2z)\sum_{k=1}^{2}[Z_{2k}\sin(2k\varphi_\nu)]\Bigg]\Bigg]\Bigg\}, \qquad (5.32)
\end{aligned}
$$

where

$$\begin{aligned}\Psi &= -\frac{1}{2m(1-m)\,\mathrm{K}(m)}\left\{\frac{\pi}{2}\left[\mathrm{E}\left(\arctan\sqrt{\frac{f}{m}}\middle|1-m\right)-m\,\mathrm{F}\left(\arctan\sqrt{\frac{f}{m}}\middle|1-m\right)\right]\right.\\ &+ \left. z\big[\mathrm{E}(m)-(1-m)\,\mathrm{K}(m)\big]\right\},\end{aligned} \tag{5.33}$$

and Table 5.5 shows the rest of the expressions involved in (5.32).

Table 5.5: Expressions involved in $\partial W_2/\partial m$

$\widetilde{\mathcal{P}}_2$	$=$	$1-12q+55q^2$
$\widetilde{\mathcal{P}}_4$	$=$	$3q-20q^2$
$\widetilde{\mathcal{P}}_6$	$=$	$5q^2$
$\widetilde{\mathcal{Q}}_2$	$=$	$1+33q^2+(8q-12q^2)\cosh(2z)+18q^2\cosh(4z)$
$\widetilde{\mathcal{Q}}_4$	$=$	$2q+12q^2\cosh(2z)$
$\widetilde{\mathcal{Q}}_6$	$=$	$3q^2$
$\widetilde{Z}_2$	$=$	$1-q+6q\cosh(2z)$
$\widetilde{Z}_4$	$=$	q

Taking into account the previous partial derivative and the function W_2 itself, the transformation equations for the first averaging are given by

$$\Delta\varphi_\nu = \frac{1}{2}\frac{\partial W_2}{\partial m}\frac{\partial m}{\partial I_N}, \tag{5.34}$$

$$\Delta\phi = \frac{3I_\Lambda}{I_M^2}\frac{1-\cos(2\phi)}{2-3s_I^2+3s_I^2\cos 2\phi}\,W_2, \tag{5.35}$$

$$\Delta\varphi_\mu = -\frac{1}{I_M}\left(I_N\,\Delta\varphi_\nu + I_\Lambda\,\Delta\phi + \frac{W_2}{2}\right), \tag{5.36}$$

$$\begin{aligned}\Delta I_N &= -\frac{3n^2}{8}\frac{C}{C-A}\frac{A}{I_M}\sqrt{\frac{m(1+f)}{f(f+m)}}\frac{2\mathrm{K}(m)}{\pi}(2-3s_I^2+3s_I^2\cos 2\phi)\\ &\left\{(B-A)(1+f)\sqrt{q}\sum_{k=1}^{3}[2k\,\mathcal{P}_{2k}\cos(2k\varphi_\nu)]\right.\\ &\left.+4(C-A)\sqrt{m(1-m)}\,\frac{2\mathrm{K}(m)}{\pi}\sinh^2 z\sum_{k=1}^{3}[2k\,\mathcal{Q}_{2k}\cos(2k\varphi_\nu)]\right\},\end{aligned} \tag{5.37}$$

$$\Delta I_M = 0, \tag{5.38}$$

$$\Delta I_\Lambda = \frac{3\,s_I^2\sin(2\phi)}{2-3s_I^2+3s_I^2\cos 2\phi}\,W_2, \tag{5.39}$$

where we recall that

$$\frac{\partial m}{\partial I_N} = \frac{\partial \phi_1}{\partial \beta}\frac{\partial \beta}{\partial I_N} = -\frac{\pi(f+m)^{3/2}}{I_M\sqrt{f(1+f)}\mathrm{K}(m)}, \tag{5.40}$$

$$\frac{\partial m}{\partial I_M} = \frac{\partial \phi_1}{\partial \beta}\frac{\partial \beta}{\partial I_M} = \frac{I_N}{I_M^2}\frac{\pi(f+m)^{3/2}}{\sqrt{f(1+f)}\mathrm{K}(m)} \tag{5.41}$$

5.3.3 Average over ϕ

Once obtained the transformation equations of the first averaging, we know that the new Hamiltonian function in the new prime variables is given by

$$\mathcal{H}(\varphi'_\nu, \varphi'_\mu, \phi', I'_N, I'_M, I'_\Lambda) = H_{0,0} + H_{0,1} + \frac{1}{2}H_{0,2}, \tag{5.42}$$

where $H_{0,1} = H_{1,0} = -n\,I'_\Lambda$ and $H_{0,2}$ is given in (5.29). For the second averaging, this will be the starting Hamiltonian function. In other words, the new secular Hamiltonian is set to be

$$\mathcal{S} = \mathcal{K}_{0,0} + \mathcal{K}_{0,1} + \frac{1}{2}\mathcal{K}_{0,2}, \tag{5.43}$$

where $\mathcal{K}_{0,0} = H_{0,0}$, $\mathcal{K}_{1,0} = H_{0,1}$ and $\mathcal{K}_{2,0} = H_{0,2}$.

Recalling Eq. (4.42), since we will only reach a first order in this averaging, Lie's algorithm leads to

$$\mathcal{K}_{0,2} = \underbrace{H_{0,2}}_{\mathcal{K}_{2,0}} + 2n\frac{\partial V1}{\partial \phi}. \tag{5.44}$$

Note that $H_{0,2}$ (see Eq. 5.29) can be expressed as a secular term ($H_{0,2}(S)$) and a periodic part ($H_{0,2}(P)$) with respect to ϕ. Indeed,

$$\begin{aligned} H_{0,2}(S) &= -\frac{n^2}{4}(2-3s_I^2)\Big\{(2C-B-A) + 6(B-A)\frac{(1+f)\sqrt{m}}{f+m}\sqrt{q}\,X_0 \\ &\quad -3(C-A)\frac{m\sqrt{1-m}}{f+m}\frac{2\mathrm{K}(m)}{\pi}\sum_{k=0}^{3}[Y_{2k}\cosh(2kz)]\Big\} \qquad (5.45)\\ &= \frac{2-3s_I^2}{2-3s_I^2+3s_I^2\cos 2\phi}H_{0,2}, \qquad (5.46)\\ H_{0,2}(P) &= -\frac{n^2}{4}(3s_I^2\cos 2\phi)\Big\{(2C-B-A) + 6(B-A)\frac{(1+f)\sqrt{m}}{f+m}\sqrt{q}\,X_0 \\ &\quad -3(C-A)\frac{m\sqrt{1-m}}{f+m}\frac{2\mathrm{K}(m)}{\pi}\sum_{k=0}^{3}[Y_{2k}\cosh(2kz)]\Big\} \qquad (5.47)\\ &= \frac{3s_I^2\cos 2\phi}{2-3s_I^2+3s_I^2\cos 2\phi}H_{0,2}. \qquad (5.48) \end{aligned}$$

We choose again

$$\mathcal{K}_{0,2} = H_{0,2}(S), \tag{5.49}$$

and then, from Eq. (5.44) we have to solve

$$V_1 = -\frac{1}{2n}\int H_{0,2}(P)\,\mathrm{d}\,\phi, \tag{5.50}$$

which leads to

$$\begin{aligned} V_1 &= \frac{3n}{16}\,s_I^2\sin 2\phi\Big\{(2C-B-A)+6(B-A)\frac{(1+f)\sqrt{m}}{f+m}\sqrt{q}\,X_0 \\ &\quad -3(C-A)\frac{m\sqrt{1-m}}{f+m}\frac{2\mathrm{K}(m)}{\pi}\sum_{k=0}^{3}[Y_{2k}\cosh(2kz)]\Big\} \qquad (5.51)\\ &= -\frac{3}{4n}\,\frac{s_I^2\sin 2\phi}{2-3s_I^2+3s_I^2\cos 2\phi}\,H_{0,2}. \qquad (5.52)\end{aligned}$$

5.3.4 Transformation equations of the second averaging

One more time, these transformation equations involve the calculation of the partial derivative of V_1 with respect to m. After several calculations this expression is given by

$$\begin{aligned} \frac{\partial V_1}{\partial m} &= \frac{3n}{16}\,s_I^2\sin 2\phi\Big\{ \\ &\quad \frac{3(B-A)(1+f)\sqrt{q}}{4\sqrt{m}(1-m)(f+m)^2\,\mathrm{K}^2(m)}\Big[\pi^2(f+m)\widetilde{Z}_2+4(f-m)(1-m)\,\mathrm{K}^2(m)\,X_0\Big] \\ &\quad -\frac{3(C-A)}{\pi\sqrt{1-m}(f+m)}\Big[\big[\mathrm{E}(m)+\big(1-2m\,\frac{1+f}{f+m}\big)\,\mathrm{K}(m)\big]\sum_{k=0}^{3}[Y_{2k}\cosh(2kz)] \\ &\qquad +2m(1-m)\,\mathrm{K}(m)\big[\Psi\sum_{k=1}^{3}[\mathcal{E}_{2k}\sinh(2kz)]\big] \\ &\qquad +\frac{\pi^2 q}{m(1-m)\mathrm{K}^2(m)}\sum_{k=0}^{3}[\mathcal{F}_{2k}\cosh(2kz)]\Big]\Big\}, \qquad (5.53)\end{aligned}$$

where the new involved polynomials $\widetilde{Z}_2$, $\mathcal{E}_{2k}$ and $\mathcal{F}_{2k}$ are given in Table 5.6.

Taking into account the previous partial derivative (5.53) and the function V_1 itself, the trans-

Table 5.6: Expressions involved in $\partial V_1/\partial m$

$\mathcal{E}_2$	$=$	$8q - 16q^2 + 104q^3$
$\mathcal{E}_4$	$=$	$32q^2 - 64q^3$
$\mathcal{E}_6$	$=$	$72q^3$
$\mathcal{F}_0$	$=$	$6q - 12q^2$
$\mathcal{F}_2$	$=$	$1 - 4q + 39q^2$
$\mathcal{F}_4$	$=$	$4q - 12q^2$
$\mathcal{F}_6$	$=$	$9q^2$
$\widetilde{Z}_2$	$=$	$1 - 6q + 30q^2$

formation equations for the second averaging are given by

$$\delta\varphi_\nu = \frac{\partial V_1}{\partial m}\frac{\partial m}{\partial I_N}, \tag{5.54}$$

$$\delta\phi = -\frac{2}{s_I^2}\frac{I_\Lambda}{I_M^2}\,V_1, \tag{5.55}$$

$$\delta\varphi_\mu = -\frac{1}{I_M}\left(I_N\,\delta\varphi_\nu + I_\Lambda\,\delta\phi\right), \tag{5.56}$$

$$\delta I_N = 0, \tag{5.57}$$

$$\delta I_M = 0, \tag{5.58}$$

$$\delta I_\Lambda = -2\frac{\cos 2\phi}{\sin 2\phi}\,V_1, \tag{5.59}$$

5.3.5 Secular terms

After the double averaging we find the secular Hamiltonian $\mathcal{S} = K_{0,0} + K_{0,1} + \frac{1}{2}K_{0,2}$ given by

$$\begin{aligned}\mathcal{S} &= \frac{I_M^2}{2A}\left(1 - \frac{C-A}{C}\frac{f}{f+m}\right) - n\,I_\Lambda - \frac{n^2}{8}(2-3s_I^2)\Bigg\{(2C-B-A)\\ &+ 6(B-A)\frac{(1+f)\sqrt{m}}{f+m}\sqrt{q}\,X_0 - 3(C-A)\frac{m\sqrt{1-m}}{f+m}\frac{2\mathrm{K}(m)}{\pi}\sum_{k=0}^{3}[Y_{2k}\cosh(2kz)]\Bigg\}\end{aligned} \tag{5.60}$$

where I_N, I_M, and therefore m, and I_Λ are constant. The secular frequencies of the motion are obtained from Hamilton equations

$$\frac{\mathrm{d}\varphi_\nu}{\mathrm{d}t} = \left(\frac{I_M^2}{2A}\frac{C-A}{C}\frac{f}{(f+m)^2} - \frac{2n}{3}\frac{2-3s_I^2}{s_I^2\sin 2\phi}\frac{\partial V_1}{\partial m}\right)\frac{\partial m}{\partial I_N} = n_{\varphi_\nu} \tag{5.61}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = -n + \frac{3I_\Lambda}{I_M^2}\frac{H_{0,2}}{2-3s_I^2+3s_I^2\cos 2\phi} = n_\phi \tag{5.62}$$

$$\frac{\mathrm{d}\varphi_\mu}{\mathrm{d}t} = \frac{I_M}{A}\left(1 - \frac{C-A}{C}\frac{f}{f+m}\right) - \frac{I_\Lambda}{I_M}(n_\phi + n) - \frac{I_N}{I_M}\,n_{\varphi_\nu} = n_{\varphi_\mu} \tag{5.63}$$

and $\mathrm{d}h/\mathrm{d}t = n_h = n + n_\phi$.

5.3.6 Results

Once obtained the equations allowing us to have a new double-averaged Hamiltonian of our perturbed problem, a first set of results can be offered. Table 5.7 shows the propagation of the initial conditions for the two different bodies considered by using the expressions given by Eq. (5.35)-(5.39). Fig. 5.5 shows these results. Note that all the variables denoted as x^* are given by

$$x^* = x' + \Delta x', \qquad x' = x'' + \delta x'', \tag{5.64}$$

where $x \in (\varphi_\nu, \varphi_\mu, \phi, I_N, I_M, \mathbb{I}_\Lambda)$ and $x''(t)$ is obtained directly from the double-averaged Hamiltonian by the Eqs. (5.61)-(5.63).

object	**var.**	**non-averaged**	**prime**	**double-prime**	**sec. frequency**
PEGASUS	φ_ν	-0.1479895115	-0.1476002434	-0.1510420588	-0.6506356386
	φ_μ	1.5775303900	1.577172216	1.580866560	1.6835946526
	ϕ	-0.1	-0.1000002879	-0.1011478868	-0.0442806225
	I_N	0.9548381629	0.9547486244	0.9547486244	
	I_M	1.0	1.0	1.0	
	I_Λ	0.3420201433	0.3420275517	0.3557644344	
OBJECT X	φ_ν	-0.4540080976	-0.4540617779	-0.4519849708	-0.3211640629
	φ_μ	1.8832288438	1.883280647	1.881429408	1.3269767332
	ϕ	-0.1	-0.09999998044	-0.1005426193	-0.0440180974
	I_N	0.9825021167	0.9825035675	0.9825035675	
	I_M	1.0	1.0	1.0	
	I_Λ	0.3420201433	0.3420196399	0.3487250577	

Table 5.7: Initial conditions in the different phase spaces for both Pegasus and an unknown object with initial conditions such that $a = 0.7$, $b = 0.8$, $c = 1.0$, $f = 0.714$, $m = 0.019$.

As we can see, the results verify the numerical simulations shown in Figs. 5.2 and 5.3. Furthermore, while for Pegasus φ_ν advances over the analytical solution with a constant rate of about 0.06 units per orbital period, the same variable delays with respect to the analytical solution when OBJECT X is considered with a lower constant rate of about -0.0008 units per orbital period. Both motions are only affected by long-period terms. On the contrary, it can be observed how for Pegasus φ_μ delays with respect to the analytical solution while it advances for OBJECT X with a constant rate ten times lower per orbital period. The evolution of ϕ is similar to φ_ν, but the variation for OBJECT X is roughly a hundred times lower per orbital period and both are only affected by periodic terms related to the orbital motion. We further appreciate that I_N differs in both cases only in low-amplitude periodic terms (lower-amplitude terms for OBJECT X) and is the only variable affected by short-period effects. In turn, I_Λ advances in a similar way in both cases but with a higher rate for Pegasus.

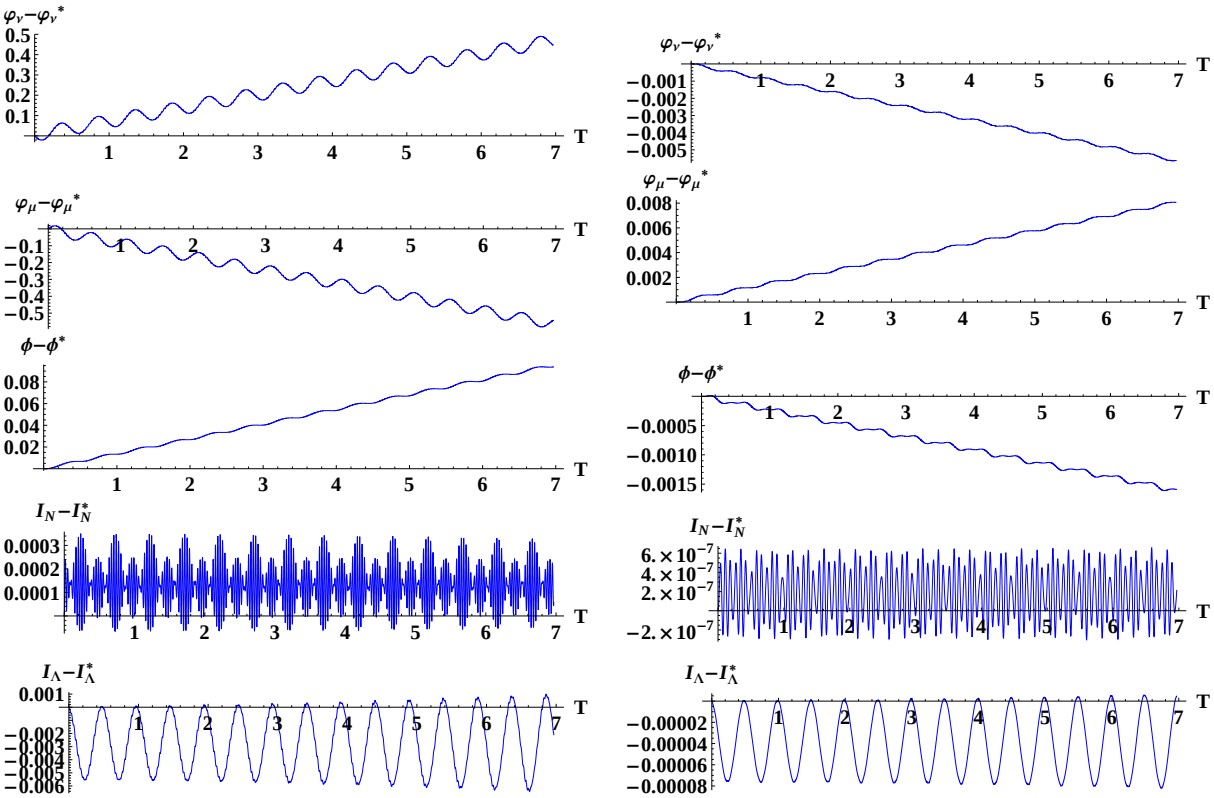


Figure 5.5: Differences between the perturbed motion in elliptic functions and the averaged motion with $x^* = x' + \Delta x'$ and $x' = x'' + \delta x''$ in internal units. T stands for orbital periods. (a) Results for Pegasus. (b) Results for an unknown object with initial conditions such that $a = 0.7$, $b = 0.8$, $c = 1.0$, $f = 0.714$, $m = 0.019$.

5.4 Perturbation approach in Ferrer-Lara variables

In this section we will repeat the procedure carried out in close form in the previous chapter, but starting from the series expansion in the nome of the disturbing function.

5.4.1 Average over ℓ

Like in the previous chapter, we look for a canonical transformation that removes the variable φ_ν from the Hamiltonian. We set

$$H_{0,0} = \mathcal{K}_0, \qquad H_{1,0} = -n\,\Phi, \qquad H_{2,0} = 2\,U,$$

where all the functions are assumed to be expressed in prime variables. For the sake of brevity, we drop the prime notation in what follows when there is no risk of confusion.

Taking into account Eq. (4.18) in the previous chapter, the first step in the computation of the Lie triangle gives

$$-\left(\frac{1}{B} - \frac{1}{C}\right) L\,\frac{\partial W_1}{\partial \ell} = H_{1,0} - H_{0,1} \tag{5.65}$$

As $H_{1,0}$ does not depend on ℓ or g, we choose

$$H_{0,1} = H_{1,0}, \tag{5.66}$$

and the first term of the generating function is $W_1 = 0$. Because the vanishing of W_1, the next step gives

$$-\left(\frac{1}{B} - \frac{1}{C}\right) L \frac{\partial W_2}{\partial \ell} = H_{2,0} - H_{0,2}. \tag{5.67}$$

where, by inspecting Eqs. (5.22)-(5.24), we can easily express $H_{2,0}$ as a secular term (S) plus a periodic term (P), that is $H_{2,0} = H_{2,0}(S) + H_{2,0}(P)$.

We then choose $H_{0,2} = H_{2,0}(S)$ where, after several calculations we have

$$\begin{aligned} H_{0,2} &= -\frac{n^2}{4}(2 - 3s_I^2 + 3s_I^2 \cos 2\phi)\Bigg\{(2C - B - A) + 6(B - A)\frac{(1+f)\sqrt{m}}{f+m}\sqrt{q}\, X_0 \\ &\quad -3(C - A)\frac{m\sqrt{1-m}}{f+m}\frac{2\mathrm{K}(m)}{\pi}\sum_{k=0}^{3}[Y_{2k}\cosh(2kz)]\Bigg\}, \end{aligned} \tag{5.68}$$

and Table 5.8 shows the polynomials involved in the above average. Note that this table is the same as Table 5.3, but we bring it here again for the benefit of the reader.

Table 5.8: Polynomials involved in $H_{0,2}$

X_0	$=$	$1 - 2q + 6q^2$
Y_0	$=$	$1 + 12q^2 - 16q^3$
Y_2	$=$	$4q - 8q^2 + 52q^3$
Y_4	$=$	$8q^2 - 16q^3$
Y_6	$=$	$12q^3$

Finally, according to (5.69), in order to get W_2 we have to solve the differential equation

$$-\left(\frac{1}{B} - \frac{1}{C}\right) L \frac{\partial W_2}{\partial \ell} = H_{2,0}(P). \tag{5.69}$$

whose solution, after several calculations, is given by

$$\begin{aligned} W_2 &= \frac{3n^2}{4L}\frac{B\,C}{C-B}\frac{\sqrt{m}}{f+m}\frac{2\mathrm{K}(m)}{\pi}(2 - 3s_I^2 + 3s_I^2\cos 2\phi) \\ &\quad \Bigg\{(B-A)(1+f)\sqrt{q}\sum_{k=1}^{3}[\mathcal{P}_{2k}\sin(2k\alpha)] \\ &\quad +4(C-A)\sqrt{m(1-m)}\,\frac{2\mathrm{K}(m)}{\pi}\sinh^2 z\sum_{k=1}^{3}[\mathcal{Q}_{2k}\sin(2k\alpha)]\Bigg\}. \end{aligned} \tag{5.70}$$

and Table 5.9 shows the polynomials involved in the above average. Note again that this table is the same as Table 5.4.

Table 5.9: Expressions involved in W_2

$\mathcal{P}_2$	$=$	$-1+4q-11q^2$
$\mathcal{P}_4$	$=$	$-q+4q^2$
$\mathcal{P}_6$	$=$	$-q^2$
$\mathcal{Q}_2$	$=$	$q+11q^3+(4q^2-4q^3)\cosh(2z)+6q^3\cosh(4z)$
$\mathcal{Q}_4$	$=$	$q^2+4q^3\cosh(2z)$
$\mathcal{Q}_6$	$=$	q^3

5.4.2 Transformation equations of the first averaging

Now we have to follow the same procedure done in the previous chapter. Note that this procedure involves the calculation of the partial derivative of W_2 with respect to m, which is given by

$$\begin{aligned}
\frac{\partial W_2}{\partial m} &= \left(\frac{\mathrm{E}(m)}{2m(1-m)\,\mathrm{K}(m)}-\frac{1}{f+m}\right)+\frac{3n^2}{4L}\frac{B\,C}{C-B}\frac{\sqrt{m}}{f+m}\frac{2\,\mathrm{K}(m)}{\pi}(2-3s_I^2+3s_I^2\cos 2\phi)\\
&\quad\left\{\frac{(B-A)(1+f)\sqrt{q}}{2m(1-m)\,\mathrm{K}^2(m)}\left[\frac{\pi^2}{4}\sum_{k=1}^{3}[\mathcal{P}_{2k}\sin(2k\alpha)]-\frac{\pi^2}{2}\sum_{k=1}^{3}[\mathcal{Z}_{2k}\sin(2k\alpha)]\right.\right.\\
&\qquad\left.-2\alpha\,\mathrm{K}(m)\big[\mathrm{E}(m)-(1-m)\,\mathrm{K}(m)\big]\sum_{k=1}^{3}[\widetilde{\mathcal{Z}}_{2k}\cos(2k\alpha)]\right]\\
&\quad+\frac{4(C-A)\sinh(z)}{\pi\sqrt{m(1-m)}}\sum_{k=1}^{3}[\mathcal{Q}_{2k}\sin(2k\alpha)]\\
&\quad\times\left[\mathrm{E}(m)\sinh(z)-m\,\mathrm{K}(m)\big[\sinh(z)-4(1-m)\Psi\cosh(z)\big]\right]\\
&\quad+\frac{8(C-A)\sinh^2(z)}{\pi\sqrt{m(1-m)}\,\mathrm{K}(m)}\left[\sum_{k=1}^{3}[\widetilde{\mathcal{X}}_{2k}\sin(2k\alpha)]-\alpha\,\mathrm{K}(m)\big[\mathrm{E}(m)-(1-m)\,\mathrm{K}(m)\big]\right.\\
&\qquad\left.\left.\times\sum_{k=1}^{3}[\mathcal{X}_{2k}\cos(2k\alpha)]\right]\right\}
\end{aligned}\tag{5.71}$$

where

$$\begin{aligned}
\Psi &= -\frac{1}{2m(1-m)\,\mathrm{K}(m)}\left\{\frac{\pi}{2}\left[\mathrm{E}\left(\arctan\sqrt{\frac{f}{m}}\middle|1-m\right)-m\,\mathrm{F}\left(\arctan\sqrt{\frac{f}{m}}\middle|1-m\right)\right]\right.\\
&\quad+\left.z\big[\mathrm{E}(m)-(1-m)\,\mathrm{K}(m)\big]\right\},
\end{aligned}\tag{5.72}$$

and Table 5.10 shows the rest of the expressions involved in (5.71).

Taking into account the previous partial derivative and the function W_2 itself, the transformation

Table 5.10: Expressions involved in $\partial W_2/\partial m$

$\mathcal{Z}_2$	$=$	$-4q+22q^2$
$\mathcal{Z}_4$	$=$	$q-8q^2$
$\mathcal{Z}_6$	$=$	$2q^2$
$\widetilde{\mathcal{Z}}_2$	$=$	$\mathcal{P}_2$
$\widetilde{\mathcal{Z}}_4$	$=$	$2\mathcal{P}_4$
$\widetilde{\mathcal{Z}}_6$	$=$	$3\mathcal{P}_6$
$\mathcal{X}_2$	$=$	$\mathcal{Q}_2$
$\mathcal{X}_4$	$=$	$2\mathcal{Q}_4$
$\mathcal{X}_6$	$=$	$3\mathcal{Q}_6$
$\widetilde{\mathcal{X}}_2$	$=$	$\frac{\pi^2}{4}q\big[1+33q^2+(8q-12q^2)\cosh(2z)+18q^2\cosh(4z)\big]$
		$+8q^2\Psi m(1-m)\mathrm{K}^2(m)\big[(1-q)\sinh(2z)+3q\sinh(4z)\big]$
$\widetilde{\mathcal{X}}_4$	$=$	$\frac{\pi^2}{2}q^2[1+6q\cosh(2z)]+8q^3\Psi m(1-m)\mathrm{K}^2(m)\sinh(2z)$
$\widetilde{\mathcal{X}}_6$	$=$	$\frac{3\pi^2}{4}q^3$

equations for the first averaging are given by

$$\Delta\ell = -\frac{1}{2L}W_2+\frac{1}{2}\frac{\partial W_2}{\partial m}\frac{\partial m}{\partial L}, \tag{5.73}$$

$$\Delta\phi = \frac{3\Phi}{G^2}\frac{1-\cos(2\phi)}{2-3s_I^2+3s_I^2\cos 2\phi}W_2, \tag{5.74}$$

$$\Delta g = -\frac{1}{G}\left[L\left(\Delta\ell+\frac{W_2}{2L}\right)+\Phi\,\Delta\phi\right], \tag{5.75}$$

$$\begin{aligned}\Delta L = & -\frac{3n^2}{8L}\frac{B\,C}{C-B}\frac{\sqrt{m}}{f+m}\frac{2\mathrm{K}(m)}{\pi}(2-3s_I^2+3s_I^2\cos 2\phi)\\ & \left\{(B-A)(1+f)\sqrt{q}\sum_{k=1}^{3}\left[k\frac{\pi}{\mathrm{K}(m)}\,\mathcal{P}_{2k}\cos(2k\alpha)\right]\right.\\ & \left.+4(C-A)\sqrt{m(1-m)}\,\frac{2\mathrm{K}(m)}{\pi}\sinh^2 z\sum_{k=1}^{3}\left[k\frac{\pi}{\mathrm{K}(m)}\,\mathcal{Q}_{2k}\cos(2k\alpha)\right]\right\},\end{aligned} \tag{5.76}$$

$$\Delta G = 0, \tag{5.77}$$

$$\Delta\Phi = \frac{3\,s_I^2\sin(2\phi)}{2-3s_I^2+3s_I^2\cos 2\phi}W_2, \tag{5.78}$$

where we recall that

$$G\frac{\partial m}{\partial G} = -L\frac{\partial m}{\partial L} = 2(f+m). \tag{5.79}$$

5.4.3 Average over ϕ

Once obtained the transformation equations of the first averaging, we know that the new Hamiltonian function in the new prime variables is given by

$$\mathcal{H}(\ell', g', \phi', L', G', \Phi') = H_{0,0} + H_{0,1} + \frac{1}{2} H_{0,2}, \tag{5.80}$$

where $H_{0,1} = H_{1,0} = -n\,\Phi$ and $H_{0,2}$ is given in (5.68). For the second averaging, this will be the starting Hamiltonian function. In other words, the new secular Hamiltonian is set to be

$$\mathcal{S} = \mathcal{K}_{0,0} + \mathcal{K}_{0,1} + \frac{1}{2} \mathcal{K}_{0,2}, \tag{5.81}$$

where $\mathcal{K}_{0,0} = H_{0,0}$, $\mathcal{K}_{1,0} = H_{0,1}$ and $\mathcal{K}_{2,0} = H_{0,2}$.

Recalling Eq. (4.66), since we will only reach a first order in this averaging, Lie's algorithm leads to

$$\mathcal{K}_{0,2} = \underbrace{H_{0,2}}_{\mathcal{K}_{2,0}} + 2n \frac{\partial V1}{\partial \phi}. \tag{5.82}$$

Note that $H_{0,2}$ (see Eq. 5.68) can be expressed as a secular term ($H_{0,2}(S)$) and a periodic part ($H_{0,2}(P)$) with respect to ϕ. Indeed,

$$\begin{aligned} H_{0,2}(S) &= -\frac{n^2}{4}(2-3s_I^2)\Big\{(2C-B-A) + 6(B-A)\frac{(1+f)\sqrt{m}}{f+m}\sqrt{q}\,X_0 \\ &\quad -3(C-A)\frac{m\sqrt{1-m}}{f+m}\frac{2\mathrm{K}(m)}{\pi}\sum_{k=0}^{3}[Y_{2k}\cosh(2kz)]\Big\} && (5.83) \\ &= \frac{2-3s_I^2}{2-3s_I^2+3s_I^2\cos 2\phi}\,H_{0,2}, && (5.84) \\ H_{0,2}(P) &= -\frac{n^2}{4}(3s_I^2\cos 2\phi)\Big\{(2C-B-A) + 6(B-A)\frac{(1+f)\sqrt{m}}{f+m}\sqrt{q}\,X_0 \\ &\quad -3(C-A)\frac{m\sqrt{1-m}}{f+m}\frac{2\mathrm{K}(m)}{\pi}\sum_{k=0}^{3}[Y_{2k}\cosh(2kz)]\Big\} && (5.85) \\ &= \frac{3s_I^2\cos 2\phi}{2-3s_I^2+3s_I^2\cos 2\phi}\,H_{0,2}. && (5.86) \end{aligned}$$

We choose again

$$\mathcal{K}_{0,2} = H_{0,2}(S), \tag{5.87}$$

and then, from Eq. (5.82) we have to solve

$$V_1 = -\frac{1}{2n}\int H_{0,2}(P)\,\mathrm{d}\,\phi, \tag{5.88}$$

which leads to

$$
\begin{aligned}
V_1 &= \frac{3n}{16}\, s_I^2 \sin 2\phi \Bigg\{ (2C-B-A) + 6(B-A)\frac{(1+f)\sqrt{m}}{f+m}\sqrt{q}\, X_0 \\
&\quad -3(C-A)\frac{m\sqrt{1-m}}{f+m}\frac{2\mathrm{K}(m)}{\pi}\sum_{k=0}^{3}[Y_{2k}\cosh(2kz)]\Bigg\} && (5.89)\\
&= -\frac{3}{4n}\,\frac{s_I^2\sin 2\phi}{2-3s_I^2+3s_I^2\cos 2\phi}\, H_{0,2}. && (5.90)
\end{aligned}
$$

5.4.4 Transformation equations of the second averaging

One more time, these transformation equations involve the calculation of the partial derivative of V_1 with respect to m. After several calculations this expression is given by

$$
\begin{aligned}
\frac{\partial V_1}{\partial m} &= \frac{3n}{16}\, s_I^2 \sin 2\phi \Bigg\{ \frac{3(B-A)(1+f)\sqrt{q}}{4\sqrt{m}(1-m)(f+m)^2\,\mathrm{K}^2(m)}\Big[\pi^2(f+m)\widetilde{Z}_2 + 4(f-m)(1-m)\,\mathrm{K}^2(m)\,X_0\Big] \\
&- \frac{3(C-A)}{\pi\sqrt{1-m}(f+m)}\Bigg[\Big[\mathrm{E}(m) + \big(1-2m\,\frac{1+f}{f+m}\big)\,\mathrm{K}(m)\Big]\sum_{k=0}^{3}[Y_{2k}\cosh(2kz)] \\
&+ 2m(1-m)\,\mathrm{K}(m)\Big[\Phi\sum_{k=1}^{3}[\mathcal{E}_{2k}\sinh(2kz)]\Big] + \frac{\pi^2 q}{m(1-m)\mathrm{K}^2(m)}\sum_{k=0}^{3}[\mathcal{F}_{2k}\cosh(2kz)]\Bigg]\Bigg\}, \quad (5.91)
\end{aligned}
$$

where the new involved polynomials $\widetilde{Z}_2$, $\mathcal{E}_{2k}$ and $\mathcal{F}_{2k}$ are given in Table 5.11.

Table 5.11: Expressions involved in $\partial V_1/\partial m$

$\mathcal{E}_2$	=	$8q - 16q^2 + 104q^3$
$\mathcal{E}_4$	=	$32q^2 - 64q^3$
$\mathcal{E}_6$	=	$72q^3$
$\mathcal{F}_0$	=	$6q - 12q^2$
$\mathcal{F}_2$	=	$1 - 4q + 39q^2$
$\mathcal{F}_4$	=	$4q - 12q^2$
$\mathcal{F}_6$	=	$9q^2$
$\widetilde{Z}_2$	=	$1 - 6q + 30q^2$

Taking into account the previous partial derivative (5.91) and the function V_1 itself, the trans-

formation equations for the second averaging are given by

$$\delta\ell = -\frac{2}{L}(f+m)\,\frac{\partial V_1}{\partial m}, \tag{5.92}$$

$$\delta\phi = -\frac{2}{s_I^2}\frac{\Phi}{G^2}\,V_1, \tag{5.93}$$

$$\delta g = -\frac{1}{G}\left(L\,\delta\ell + \Phi\,\delta\phi\right), \tag{5.94}$$

$$\delta L = 0, \tag{5.95}$$

$$\delta G = 0, \tag{5.96}$$

$$\delta\Phi = -2\frac{\cos 2\phi}{\sin 2\phi}\,V_1, \tag{5.97}$$

5.4.5 Secular terms

After the double averaging we find the secular Hamiltonian $\mathcal{S} = K_{0,0} + K_{0,1} + \frac{1}{2}K_{0,2}$ given by

$$\begin{aligned}\mathcal{S} &= \frac{G^2}{2A} - \left(\frac{1}{B}-\frac{1}{C}\right)\frac{L^2}{2} - n\,\Phi - \frac{n^2}{8}(2-3s_I^2)\Big\{(2C-B-A)\\ &+ 6(B-A)\frac{(1+f)\sqrt{m}}{f+m}\sqrt{q}\,X_0 - 3(C-A)\frac{m\sqrt{1-m}}{f+m}\frac{2\mathrm{K}(m)}{\pi}\sum_{k=0}^{3}[Y_{2k}\cosh(2kz)]\Big\}\end{aligned} \tag{5.98}$$

where I_N, I_M, and therefore m, and I_Λ are constant. The secular frequencies of the motion are obtained from Hamilton equations

$$\frac{\mathrm{d}\ell}{\mathrm{d}t} = -\left(\frac{1}{B}-\frac{1}{C}\right)L - \frac{2n}{3}\,\frac{2-3s_I^2}{s_I^2\sin 2\phi}\,\frac{\partial V_1}{\partial m}\frac{\partial m}{\partial L} = n_\ell \tag{5.99}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = -n + \frac{3\Phi}{G^2}\,\frac{H_{0,2}}{2-3s_I^2+3s_I^2\cos 2\phi} = n_\phi \tag{5.100}$$

$$\frac{\mathrm{d}g}{\mathrm{d}t} = \frac{G}{A} - \frac{\Phi}{G}(n_\phi+n) - \frac{L}{G}\left[n_\ell + \left(\frac{1}{B}-\frac{1}{C}\right)L\right] = n_g \tag{5.101}$$

and $\mathrm{d}h/\mathrm{d}t = n_h = n + n_\phi$.

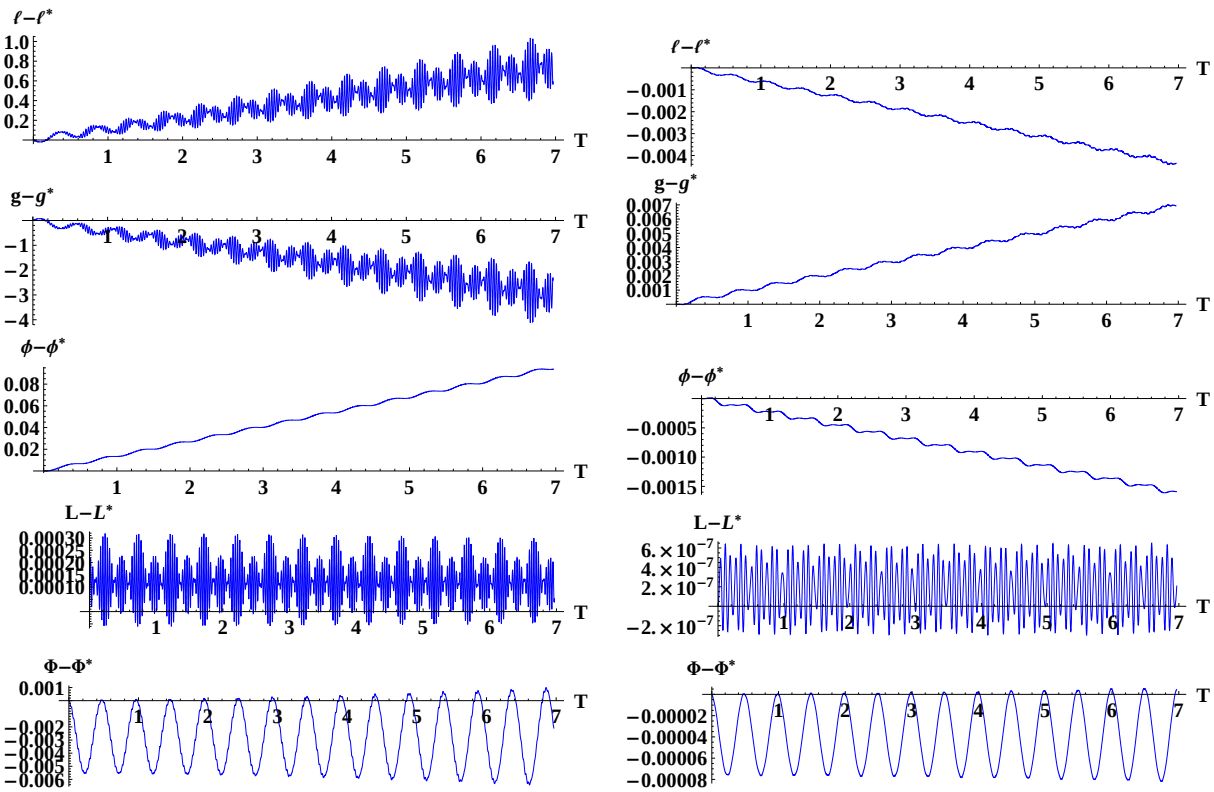
5.4.6 Results

Once obtained the new equations in Ferrer-Lara variables, table 5.12 shows the propagation of the initial conditions for the two different bodies considered by using the expressions given by the Eqs. (5.54)-(5.59). Fig. 5.6 shows the equivalent results in the new variables. Note that all the variables denoted as x^* are given again by Eq. (5.64) where $x \in (\ell,\, g,\, \phi,\, L,\, G,\, \Phi)$ and $x''(t)$ is obtained directly from the double-averaged Hamiltonian by the Eqs. (5.99)-(5.101).

As we can see, in broad outline, the trend of the variables is similar to that given in action-angle variables, this is, for Pegasus, when a variable advances or delays with respect to the analytical

object	var.	non-averaged	prime	double-prime	sec. frequency
PEGASUS	ℓ	-0.1626833313	-0.1622900734	-0.1660744347	-0.7153797544
	g	2.0665318080	2.065021987	2.080092257	3.8340342373
	ϕ	-0.1	-0.1000002879	-0.1011478868	-0.0442803523
	L	3.87444595753	3.874364524	3.874364524	
	G	1.0	1.0	1.0	
	Φ	0.3420201433	0.3420275517	0.3557644341	
OBJECT X	ℓ	-0.4562382482	-0.4562920019	-0.4542049941	-0.3227428032
	g	2.0265571677	2.026625672	2.024118795	1.4283694946
	ϕ	-0.1	-0.09999998044	-0.10054261926	-0.04401811484
	L	1.2918518846	1.2918533283	1.2918533283	
	G	1.0	1.0	1.0	
	Φ	0.3420201433	0.3420196399	0.3487250577	

Table 5.12: Initial conditions in the different phase spaces for both Pegasus and an unknown object with initial conditions such that $a = 0.7$, $b = 0.8$, $c = 1.0$, $f = 0.714$, $m = 0.019$.



(a) (b)

Figure 5.6: Differences between the perturbed motion in elliptic functions and the averaged motion with $x^* = x' + \Delta x'$ and $x' = x'' + \delta x''$ in internal units. T stands for orbital periods. (a) Results for Pegasus. (b) Results for an unknown object with initial conditions such that $a = 0.7$, $b = 0.8$, $c = 1.0$, $f = 0.714$, $m = 0.019$.

theory, the opposite situation happens with OBJECT X. It can be also appreciated that all the variables for OBJECT X are hardly ever affected by short-period effects and they advance or delay with constant rates of about hundreds times lower than the variables measuring Pegasus' motion.

5.5 Concluding remarks and comparisons

As commented at the begining of this chapter, a new and simpler approach based on a Taylor expansion in the nome of the perturbing function (5.4) is decided to carry out. Expansions of the elliptic modulus and the complete integral of the first kind have been neglected due to their steadily appearance when partial derivatives of the nome are computed. Furthermore, a better convergence of Taylor expansions is obtained when the characteristic f is added to the argument of the Theta functions through the angle z. Nevertheless, the addition of this angle involves the treatment of complex arguments in those functions, which leads to the arising of no bounded functions in our theory.

Indeed, as the disturbing function can be splitted as the sum of three terms, one of them is unavoidably expressed by a quotient of Theta functions with complex arguments (see Eq. (5.23)) while a fast convergence of the other two terms is always guaranteed. Thus, hyperbolic sines and cosines of the angle z appear when a Taylor expansion of this quotient is developed. Since these hyperbolic functions are always multiplied by powers of the nome, the convergence speed depends on how low is each power of the nome compensating for the value of the corresponding hyperbolic function, which depends in turn on the initial conditions and above all on the moments of the inertia. Note the different results shown in Tables 5.1 and 5.2.

Moreover, as expected, all the expressions given along this chapter as far as the pertubation theory concerns shows that the treatment of the perturbative scheme is nearly the same for both type of variables here studied. Indeed, a few differences between these equations are noticed because of the different nature of both variables, i.e., while in Ferrer-Lara variables the elliptic modulus is an explicit function of the momenta, one must deal with implicit functions in action-angle variables. Note, for instance, that the functions $H_{0,2}$ and V_1 are equal for both variables.

With respect to the numerical calculations, Fig. 5.7 shows how both variables reflect the effects of the perturbation for the case of Pegasus. Note that this effect is comparable for the variables φ_ν and ℓ, but clearly unequal for the variables φ_μ and g. Observe that after seven orbital periods, the differences between the non-perturbed and the perturbed problem for the action-angle variable φ_ν is still not greater that fourteen hundredth, while the differences for the Ferrer-Lara variable g is nearly ninety degrees. Differences of quality for the momenta are not noticeable.

When compared the original perturbed problem with the double-averaged system (Fig. 5.8), the same features as in the previous case are observed. For these initial conditions, the new double-averaged Hamiltonian is not sufficient to reduce the slope of the differences between the numerical simulation and the analytical theory. Unfortunately, the same situation is noticed when also periodic terms are considered (see Fig. 5.9). As a consequence, a 3-order series expansion theory does not seem to be the most proper solution to measure the attitude dynamic of an object like Pegasus for a long period of time, above all in Ferrer-Lara variables.

Nevertheless, the quality of our 3-order series expansion theory should not be judged by taking only one object or satellite. Indeed, considering a new set of moments of inertia corresponding to a different object with different geometry, Fig. 5.10 shows that the worse results are in the thousandth after seven orbital periods. Note further that Ferrer-Lara variables are even a little bit better than action-angle variables within this range of approximation. Note also that when the value of the elliptic modulus is so low, the ripple observed in Ferrer-Lara variables is strongly damped. As seen in the previous chapter, this phenomena is due to the different damping level of the partial derivative of the Jacobi Zeta function with respect to the modulus.

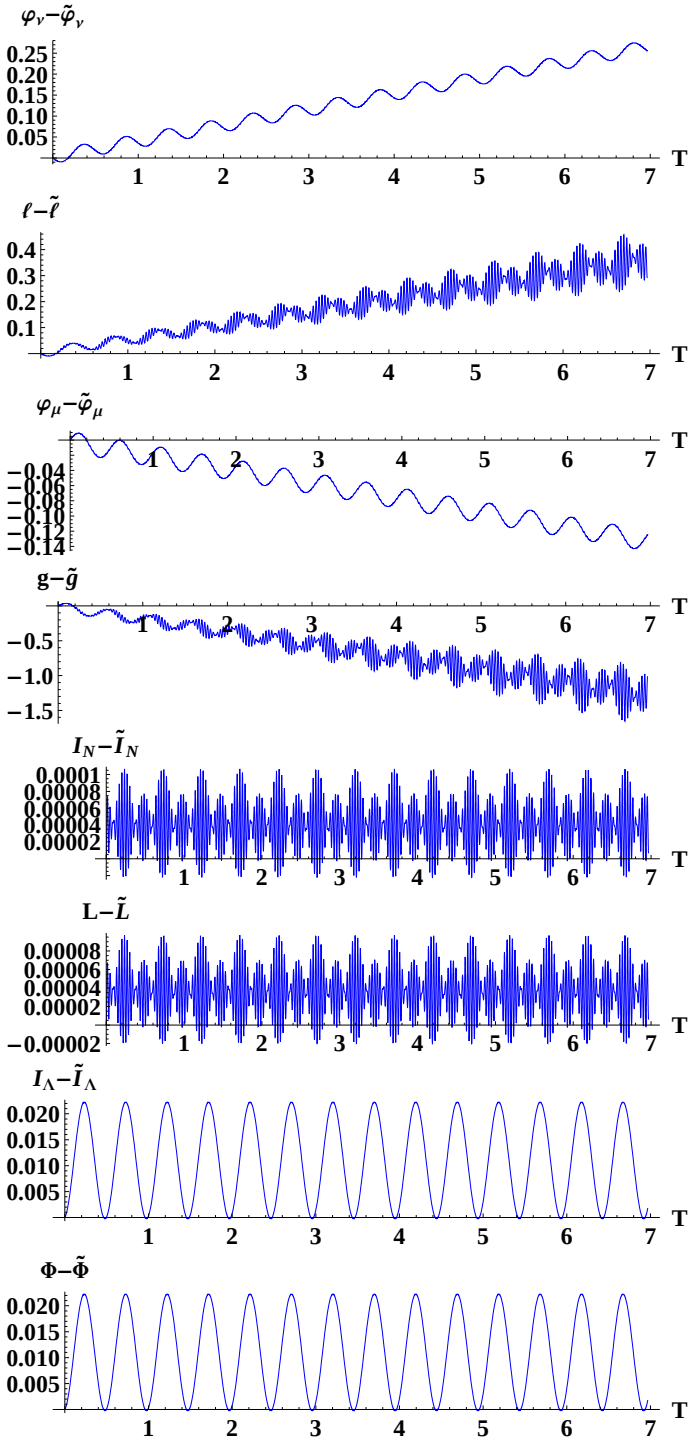


Figure 5.7: Comparisons action-angle variables vs. Ferrer-Lara variables considering the differences between the original perturbed motion and the non-perturbed motion in internal units. T stands for orbital periods.

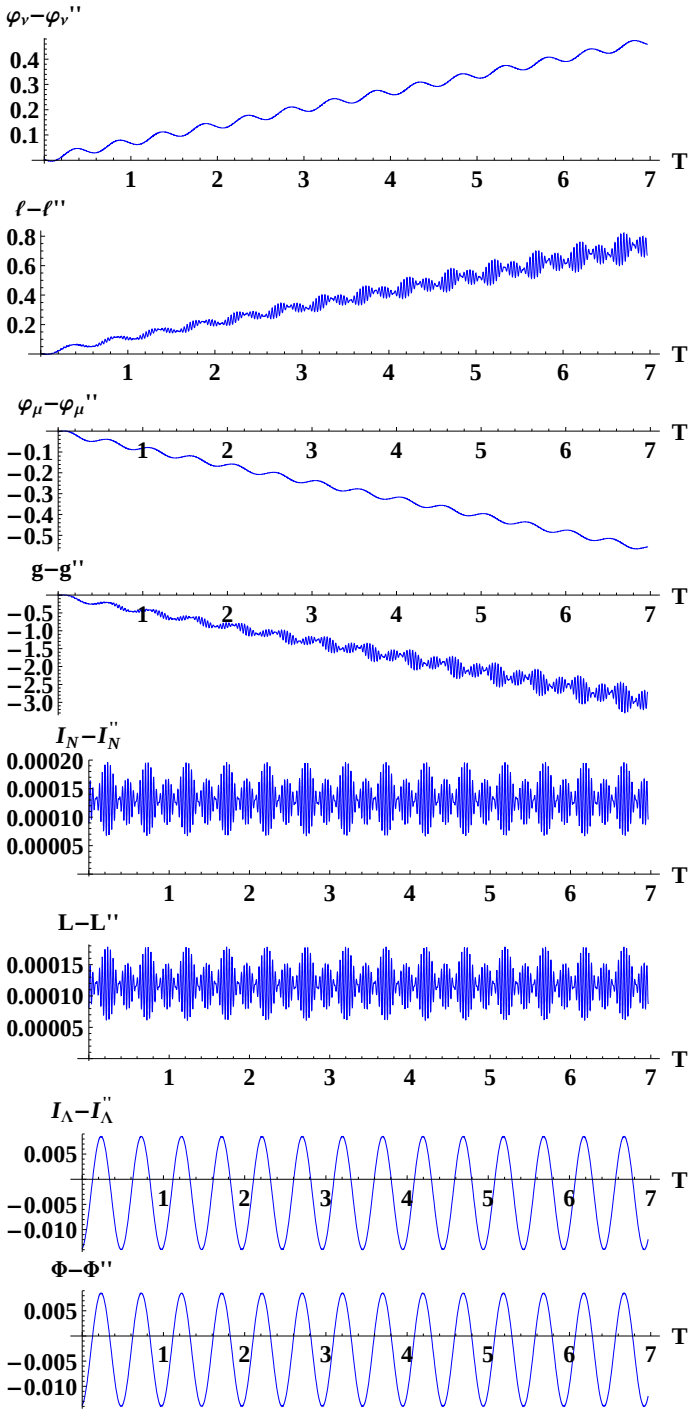


Figure 5.8: Comparisons action-angle variables vs. Ferrer-Lara variables considering the differences between the original perturbed motion and the secular (double-prime) motion in internal units. T stands for orbital periods

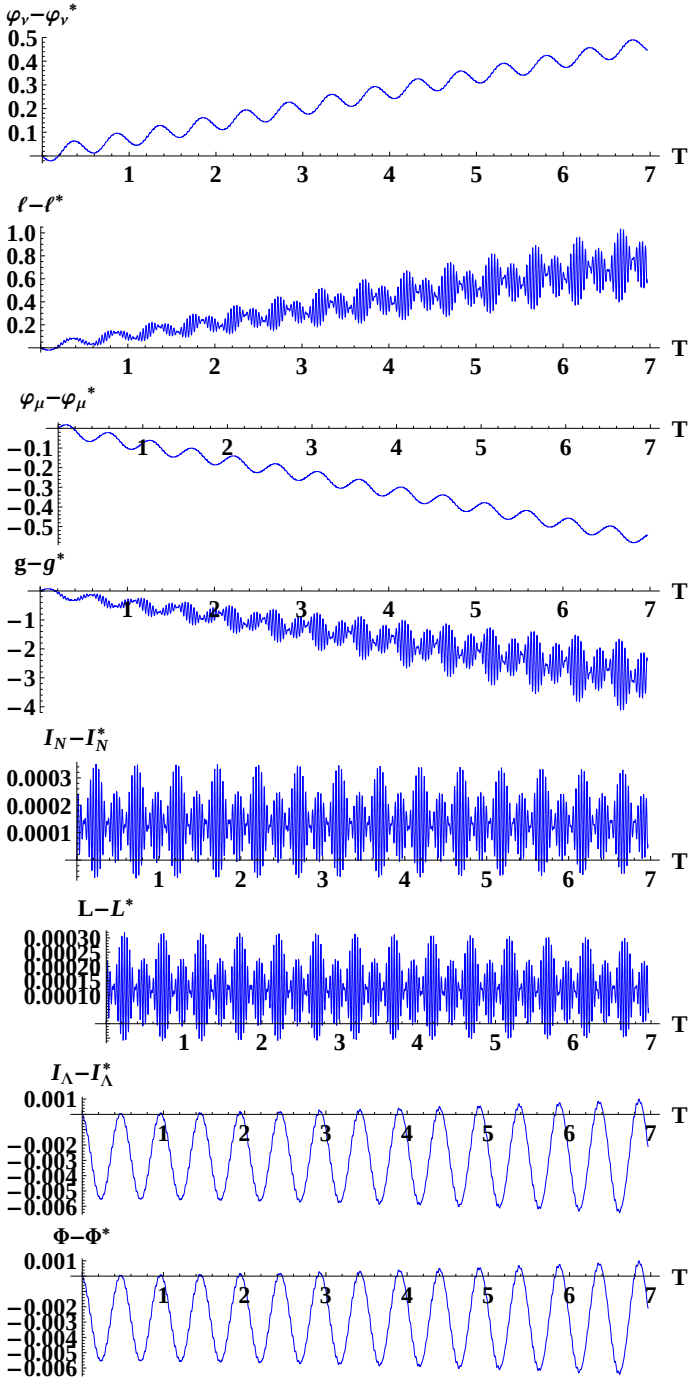


Figure 5.9: Comparisons Ferrer-Lara variables vs. action-angle variables considering the differences between the perturbed non-averaged motion and the averaged motion with $x^* = x' + \Delta x'$ and $x' = x'' + \delta x''$ in internal units. T stands for orbital periods

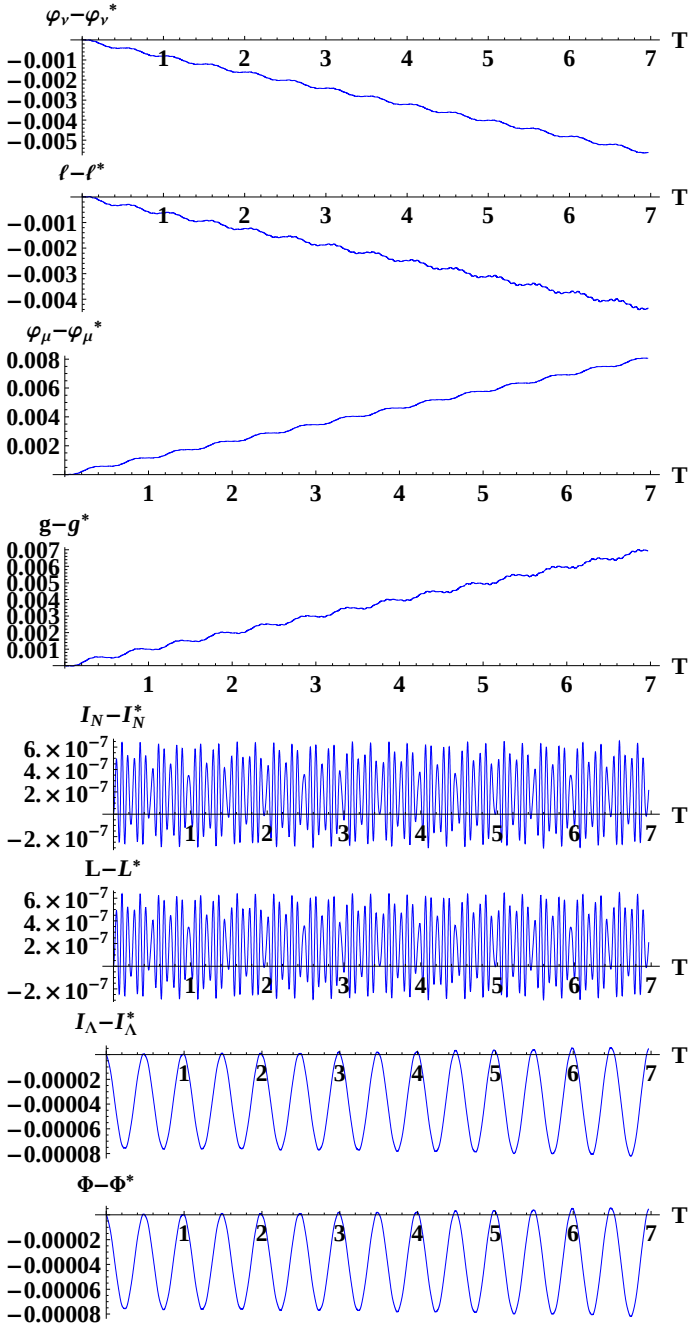


Figure 5.10: Comparisons Ferrer-Lara variables vs. action-angle variables considering the differences between the perturbed non-averaged motion and the averaged motion with $x^* = x' + \Delta x'$ and $x' = x'' + \delta x''$, taking into account new initial conditions such that $a = 0.7$, $b = 0.8$, $c = 1.0$, $f = 0.714$, $m = 0.019$ in internal units. T stands for orbital periods

Conclusions

Taking into account the contribution of each one of the chapters in this Memoir, the next paragraphs contain the conclusions we have reached.

The main contribution of Chapter 2 is twofold. On one hand, after recalling some basic facts on the rigid body dynamics, we show a different way to address the integration of the free rigid body problem by a regularization of time (Section 2.5). This procedure allows us to reach the solution of the system avoiding the explicit presence of the Jacobi elliptic functions in favour of trigonometric functions. The presence of the elliptic integrals remains in the transformation. On the other hand, due to a renewal of interest in geometric aspects of the rigid body dynamics, Section 2.6 proves the Montgomery formula from the Andoyer variables' point of view as well as proposing a new phase for the problem, that is, to reckon how much has rotated ν when μ has rotated 2π, complementing Montgomery's approach.

Chapter 3 presents in Section 3.4.2 a new version of the classical Sadov transfomation to get the complete reduction of the problem, together with the calculations involved to express such a transformation in terms of Jacobi Theta functions. This new version is based on the use of a different intermediary variable allowing us to introduce a new quantity $0 < \alpha_3 < 1$ instead of $f > 0$. Although this new approach itself does not improve the convergence of the series expansions in Chapter 5, its possible benefits should be studied by means of other perturbations.

Chapters 4 and 5 focus on the treatment of the perturbed model by the use of Lie transformations with the aim of studying the behaviour of two different sets of variables (Sadov vs. Ferrer-Lara) when both are applied to the same problem. As known, Sadov variables have the drawback of requiring the inversion of implicit equations when one has to obtain the value of the elliptic modulus as a function of the momenta. In particular, once the torque-free Hamiltonian has been reduced by using the Hamilton-Jacobi-Poincaré method, elliptic modulus can not be explicitly expressed as a function of the two momenta, which makes more difficult the later analytical manipulation of its partial derivatives, as well as the time needed to the inversion of the equations from a computational point of view. Unlike Sadov and closer to Hitzl and Breakwell approximation, to avoid the above disadvantages, Ferrer and Lara have recently proposed a new set of variables allowing us to treat the problem in an explicit way. Indeed, the new explicit Hamiltonian is quadratic in the two momenta, hence its manipulation is easier.

In spite of these differences, both sets of variables have a common analytical feature: the generating function making possible the averaging of the new variable associated to the Andoyer variable ν is expressed in terms of the Jacobi Zeta function. As well as one may anticipate non-trivial difficulties when trying to extend the closed form integration of the fast rotating satellite problem to higher orders, the presence of this function involves different consequences

depending on whether the selected variables are action-angle or Ferrer-Lara. Indeed, the partial derivative of the Zeta function with respect to the elliptic modulus is different for both sets of variables. In particular, while this derivative is shown to be periodic in action-angle variables (Fig. 4.1), in Ferrer-Lara variables is found to be non-periodic (Fig.4.3) because this derivative introduces mixed terms (see Appendix A.5 for further details), whose effects are less pronounced when the value of the triaxiality is lower and vice-versa.

This fact produces an undesired and increasing curly effect when computing the motion in Ferrer-Lara variables (see Fig. 4.4). Naturally, as it is shown in Chapter 5, in spite of a series expansion of the perturbing function can be developed in order to avoid the explicit manipulation of elliptic functions, the curly effect is still affecting the behaviour of the solution (see Fig. 5.6). Besides, it has been also proved that although this undesired effect may be not important in the time scales used in astrodynamics, a remedy is found when using action-angle variables, which are not affected by this effect.

With respect to Taylor expansions of the perturbing function, the fact of using Theta functions as an intermediate step to this expansion shows a number of pros and cons. On one side, the use of Theta functions allows us to introduce the nome as the small quantity over which the expansion can be accomplished for a faster convergence, since it is usually lower than any other quantity like the elliptic modulus. On the other side, the presence of these functions involves dealing with complex arguments whose series expansions introduce hyperbolic functions, which are non-bounded. Therefore, the convergence speed will depend mainly on the shape of each body. In particular, as it has been shown in Chapter 5, the larger is the triaxiality, a higher order Taylor expansion is needed and more troublesome expressions have to be handled.

Under this scenario, the existence of a closed form theory is clearly justified. In fact, the main advantage of using elliptic functions in a closed form is the validity of the expressions regardless of the shape of the satellite or rigid body considered.

Appendix A

Elliptic integrals and functions

Within academic world trigonometric functions have allowed us to solve a wide variety of problems in mechanics. Nevertheless, when one faces problems like getting the length of an ellipse or the solution of the motion of a simple pendulum or Duffing oscillator, one finds that the answer involves certain quadratures whose solution is not given by trigonometric functions. In fact, many dynamical systems are solved by this kind of quadratures which, reduced to their canonical forms, are known as *elliptic integrals*.

This Appendix does not pretend to be a summary of some handbooks ([Byrd & Friedman (1971), Gradshteyn & Ryzhik (2007), Abramowitz & Stegun (1972)] and its recent digitalized version [Olver *et al.* (2012)]), but rather a short guide which grew in the process of writing this Memoir. This process started by the basic definitions. An interested reader might complement the following paragraph with the reading of the paper [**?**] where a more friendly problem is analyzed in some detail.

A.1 Canonical form of the Legendre elliptic integrals and Jacobi elliptic functions

Indeed, consider the three elliptic integrals whose canonical forms are given by

$$F(\varphi|m) = \int_0^z \frac{dx}{\sqrt{(1-x^2)(1-m\,x^2)}} = \int_0^\varphi \frac{d\varphi}{\sqrt{1-m\sin^2\varphi}}, \tag{A.1}$$

$$E(\varphi|m) = \int_0^z \sqrt{\frac{1-m\,x^2}{1-x^2}}\,dx = \int_0^\varphi \sqrt{1-m\sin^2\varphi}\,d\varphi, \tag{A.2}$$

$$\Pi(n;\varphi|m) = \int_0^z \frac{dt}{(1-n\,x^2)\sqrt{(1-x^2)(1-m\,x^2)}} = \int_0^\varphi \frac{d\varphi}{(1-n\sin^2\varphi)\sqrt{1-m\sin^2\varphi}}, \tag{A.3}$$

which are the Legendre incomplete elliptic integrals of the first, second and third kind[1] respectively. Note that $x = \sin\varphi$ where $0 < \varphi < \pi/2$ is an angle and $0 < m < 1$ is a parameter known as *elliptic modulus*. It is important to remark that, depending on the problem, this parameter may be less than zero or greater than one, changing the form of these expressions as indicated in Apendix A.2.

In turn, the complete elliptic integrals of the first, second and third kind, respectively given by

$$\mathrm{K}(m) = F(\pi/2, m), \qquad \mathrm{E}(m) = \mathrm{E}(\pi/2, m) \qquad \Pi(n|m) = \Pi(\pi/2, n, m), \tag{A.5}$$

then, if we denote $u = F(\varphi, m)$, one can invert this expression by introducing the function *Jacobi Amplitude* (see Fig. A.1), that is

$$\varphi = \arcsin x = \mathrm{am}(u|m), \tag{A.6}$$

from where we get that

$$x = \sin \mathrm{am}(u|m) = \mathrm{sn}(u|m), \tag{A.7}$$

which is the Jacobi elliptic function known as *elliptic sine*. Note that the elliptic modulus plays an important role within the function; when $m = 0$ the elliptic function become trigonometric and when $m = 1$ become hyperbolic, hence the elliptic functions may be considered as a generalization of the trigonometric ones.

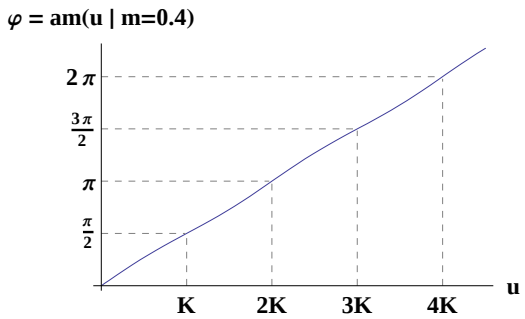


Figure A.1: Graphic representation of the relation between φ and u.

On the other hand, since $\sin^2\varphi + \cos^2\varphi = 1$, then $\sin^2 \mathrm{am}(u|m) + \cos^2 \mathrm{am}(u|m) = 1$, which gives $\mathrm{sn}^2(u|m) + \mathrm{cn}^2(u|m) = 1$ and therefore we have $\mathrm{cn}^2(u|m) = 1 - \mathrm{sn}^2(u|m)$. As well, for different purposes, it is sometimes necessary to use what is known as *complementary elliptic modulus* which is given by $m' = 1 - m$. Moreover, a number of additional formulas can be found among the elliptic functions (see [Byrd & Friedman (1971)] and [Abramowitz & Stegun (1972)]) but, in general, we finally refer to the relation

$$\mathrm{dn}^2(u|m) = 1 - m\,\mathrm{sn}^2(u|m), \tag{A.8}$$

which arises in Section 4 being part of the disturbing function of this work.

[1]Note that other authors like [Tantalo (1993)] define a slightly different elliptic integral of the third kind given by

$$\widetilde{\Pi}(\tilde{n}; \varphi|m) = \int_0^{\varphi} \frac{d\varphi}{(1 + \tilde{n}\sin^2\varphi)\sqrt{1 - m\sin^2\varphi}} = \Pi(-n; \varphi|m) \tag{A.4}$$

Like trigonometric functions, elliptic functions are periodic. However, their periods are not constant because they are function of the elliptic modulus. Specifically we have

$$T_{\mathrm{sn}} = T_{\mathrm{cn}} = 4\mathrm{K}(m) \quad \text{and} \quad T_{\mathrm{dn}} = 2\mathrm{K}(m), \tag{A.9}$$

where T_{sn} denotes the period of the elliptic sine function.

A.2 On elliptic integrals and functions with $\{m, n\} \notin [0, 1]$

It is known that, depending on the characteristics of the considered problem, the value of the elliptic modulus may change and even be out of the range $[0, 1]$. Then, there is a set of alternative formulas to treat this kind of problems which are shown next.

A.2.1 Elliptic integrals with $\{m, n\} < 0$

When our problem or application has a negative value of the modulus m and / or the characteristic n, the classical formulas used to compute elliptic integrals have to be changed by the following ones.

Complete elliptic integrals with $\{m, n\} < 0$

$$\mathrm{K}(m) = \frac{1}{\sqrt{1-m}}\,\mathrm{K}\left(\frac{m}{m-1}\right), \tag{A.10}$$

$$\mathrm{E}(m) = \sqrt{1-m}\,\mathrm{E}\left(\frac{m}{m-1}\right), \tag{A.11}$$

$$\Pi(n|m) = \frac{1}{(m-n)\sqrt{1-m}}\left[m\,\mathrm{K}\left(\frac{m}{m-1}\right) - n\,\Pi\left(\frac{m-n}{m-1}\middle|\frac{m}{m-1}\right)\right]. \tag{A.12}$$

Incomplete elliptic integrals with $\{m, n\} < 0$

$$\mathrm{F}(\phi|m) = \frac{1}{\sqrt{1-m}}\,\mathrm{F}\left(\theta\middle|\frac{m}{m-1}\right), \tag{A.13}$$

$$= \frac{1}{\sqrt{1-m}}\left[\mathrm{K}\left(\frac{m}{m-1}\right) - \mathrm{F}\left(\frac{\pi}{2}-\phi\middle|\frac{m}{m-1}\right)\right], \tag{A.14}$$

$$\mathrm{E}(\phi|m) = \frac{m\sin\theta\cos\theta}{\sqrt{1-m\cos^2\theta}} + \sqrt{1-m}\,\mathrm{E}\left(\theta\middle|\frac{m}{m-1}\right), \tag{A.15}$$

$$\Pi(n;\phi|m) = \frac{1}{(m-n)\sqrt{1-m}}\left[m\,\mathrm{F}\left(\theta\middle|\frac{m}{m-1}\right) - n\,\Pi\left(\frac{m-n}{m-1};\theta\middle|\frac{m}{m-1}\right)\right], \tag{A.16}$$

where

$$\sin\theta = \frac{\sqrt{1-m}\sin\phi}{\sqrt{1-m\sin^2\phi}}, \quad cd = cn/dn, \quad u = \mathrm{F}(\phi|m). \tag{A.17}$$

A.2.2 Elliptic integrals with $\{m, n\} > 1$

When our problem or application has a value greater than one of the modulus m and / or the characteristic n, the classical formulas used to compute elliptic integrals have to be changed by the following ones.

Incomplete elliptic integrals with $\{m, n\} > 1$

$$\mathrm{F}(\phi|m) = \frac{1}{\sqrt{m}}\,\mathrm{F}\left[\arcsin(\sqrt{m}\sin\phi)\middle|\frac{1}{m}\right], \tag{A.18}$$

$$\mathrm{E}(\phi|m) = \frac{1-m}{\sqrt{m}}\,\mathrm{F}\left[\arcsin(\sqrt{m}\sin\phi)\middle|\frac{1}{m}\right] + \sqrt{m}\,\mathrm{E}\left[\arcsin(\sqrt{m}\sin\phi)\middle|\frac{1}{m}\right], \tag{A.19}$$

$$\Pi(n, \phi|m) = \frac{1}{\sqrt{m}}\,\Pi\left[\frac{n}{m}; \arcsin(\sqrt{m}\sin\phi)\middle|\frac{1}{m}\right]. \tag{A.20}$$

Complete elliptic integrals with$\{m, n\} > 1$

$$\mathrm{K}(m) = \frac{1}{\sqrt{m}}\,\mathrm{K}\left(\frac{1}{m}\right), \tag{A.21}$$

$$\mathrm{E}(m) = \sqrt{m}\,\mathrm{E}\left(\frac{1}{m}\right) + \frac{1-m}{\sqrt{m}}\,\mathrm{K}\left(\frac{1}{m}\right), \tag{A.22}$$

$$\Pi(n|m) = \frac{1}{\sqrt{m}}\,\Pi\left[\frac{n}{m}\middle|\frac{1}{m}\right]. \tag{A.23}$$

NOTE: In order to see the expressions of the elliptic integral of the third kind (complete and incomplete), go to page 225 of Byrd & Friedman and the following, and consult the cases for values $n > 1$ and / or $m > 1$. See also Abramowitz & Stegun page 599 and the following.

A.2.3 Elliptic functions with $m < 0$

When our problem or application has a negative value of the modulus m, the classical formulas used to compute elliptic functions have to be changed by the following ones.

$$\mathrm{sn}(u|m) = \frac{1}{\sqrt{1-m}}\,\mathrm{sd}\left(\sqrt{1-m}\,u\middle|\frac{m}{m-1}\right), \tag{A.24}$$

$$\mathrm{cn}(u|m) = \mathrm{cd}\left(\sqrt{1-m}\,u\middle|\frac{m}{m-1}\right), \tag{A.25}$$

$$\mathrm{dn}(u|m) = \mathrm{nd}\left(\sqrt{1-m}\,u\middle|\frac{m}{m-1}\right), \tag{A.26}$$

where $sd = sn/dn$, $cd = cn/dn$ and $nd = 1/dn$.

A.2.4 Elliptic functions with $m > 1$

When our problem or application has a value greater than one of the modulus m, the classical formulas used to compute elliptic functions have to be changed by the following ones.

$$\mathrm{sn}(u|m) = \frac{1}{\sqrt{m}}\,\mathrm{sn}(\sqrt{m}\,u|m^{-1}), \tag{A.27}$$

$$\mathrm{cn}(u|m) = \mathrm{dn}(\sqrt{m}\,u|m^{-1}), \tag{A.28}$$

$$\mathrm{dn}(u|m) = \mathrm{cn}(\sqrt{m}\,u|m^{-1}). \tag{A.29}$$

A.3 Jacobi Zeta function

Of course, new Jacobian elliptic functions have been defined from the three ones here presented (see [Lawden (1989)] for further details), but we will deal only with the functions sn, cn and dn besides one special function whose properties are key to understand the effects of the perturbation depending on the variables used to study the problem. We are talking about the Jacobi Zeta function. As well as derivation, the elliptic functions can be integrated. In particular, in order to tackle the different averages in Section 4 we have to integrate the disturbing function containing the elliptic function $\mathrm{dn}^2(u|m)$, which leads to the appearance of the Zeta function. This function, by definition, can be expressed as

$$Z(u|m) = \int \left(\mathrm{sn}^2(u|m) - \frac{\mathrm{E}(m)}{\mathrm{K}(m)}\right) \mathrm{d}u = \int \mathrm{sn}^2(u|m)\,\mathrm{d}u - \int \frac{\mathrm{E}(m)}{\mathrm{K}(m)}\,\mathrm{d}u \tag{A.30}$$

$$= \mathrm{E}(u|m) - \frac{\mathrm{E}(m)}{\mathrm{K}(m)}\,u. \tag{A.31}$$

Nevertheless, the software (*Mathematica*) forces users to introduce an angle as the argument of the function (φ in our case), which implies that this argument has to be $\mathrm{am}(u|m)$ according to (A.6). Thus, the *Mathematica* version of the Zeta-function is given by

$$Z(\varphi|m) = \mathrm{E}(\varphi|m) - \frac{\mathrm{E}(m)}{\mathrm{K}(m)}\,\mathrm{F}(\varphi|m), \tag{A.32}$$

where we must actually type *JacobiZeta[JacobiAmplitude[u,m],m]*.

Fig. A.2 shows the graphic representation of the Zeta-function. Note that the function is $2\mathrm{K}(m)$-periodic.

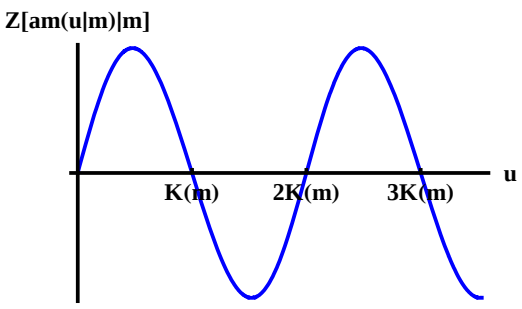


Figure A.2: Graphic representation of the Z-function.

A.3.1 On the derivative of the Jacobi Zeta function

As Byrd&Friedman (p. 284) pointed out "the Jacobi elliptic functions are not only function of $k^2 = m$ but of the argument u which may also depends on the modulus." In order to obtain the total derivative with respect to the modulus, according to the computation with Mathematica, we find convinient to rewrite the Byrd&Friedman expression as

$$\frac{\mathrm{d}\, Z(\psi|m)}{\mathrm{d}\, m} = \frac{\partial Z}{\partial m} + \frac{\partial Z}{\partial \psi}\left(\frac{\partial \psi}{\partial m} + \frac{\partial \psi}{\partial u}\frac{\partial u}{\partial m}\right), \qquad \psi = \mathrm{am}(u|m), \tag{A.33}$$

and therefore, it is clear that the analytical derivative of the Zeta-function depends on the function u, which is different according to whether the variables you are using are Ferrer-Lara or action-angle, which is a basic aspect in this work.

A.4 On the elliptic integral of the third kind

At this point we find convenient to collect here the different expressions related to the third elliptic integral. Thus, considering [Byrd & Friedman (1971), **?**] we have

Legendre's Form. Join to the previous expression of Π, we also may write

$$\begin{aligned}
\Pi_{Legendre}(n,\varphi,m) &\equiv \int_0^y \frac{dt}{(1-n\,t^2)\sqrt{(1-t^2)(1-m\,t^2)}} \\
&= \int_0^\varphi \frac{d\vartheta}{(1-n\,\sin^2\vartheta)\sqrt{(1-\sin^2\vartheta)(1-m\,\sin^2\vartheta)}} \qquad \text{(A.34)}\\
&= \int_0^{u_1} \frac{du}{1-n\,\mathrm{sn}^2 u} \equiv \Pi(u_1,n), \qquad \text{(A.35)}
\end{aligned}$$

where $y = \sin\varphi = \mathrm{sn}u_1$, $t = \sin\vartheta = \mathrm{sn}u$, $n \neq 1\,\mathrm{or}\,m$.

Jacobi's Form. Related with the addition formulas, Jacobi found convenient to introduce a new parameter a, alternative to n, by the expression

$$n = m\,\mathrm{sn}^2 a. \tag{A.36}$$

Then, he proposed another expression for Π:

$$\Pi_{Jacobi}(u_1,a) = m\,\mathrm{sn}\,a\,\mathrm{cn}\,a\,\mathrm{dn}\,a\int_0^{u_1}\frac{\mathrm{sn}^2 u\,du}{1-m\,\mathrm{sn}^2 a\,\mathrm{sn}^2 u}, \tag{A.37}$$

and the relation between the two formulas takes the form

$$\Pi_{Legendre}(u_1,n) = u_1 + \frac{\mathrm{sn}\,a}{\mathrm{cn}\,a\,\mathrm{dn}\,a}\,\Pi_{Jacobi}(u_1,a). \tag{A.38}$$

A.4.1 Computing Π: The Z and Θ Jacobi integral functions

For our purposes a question remains to be answered: how to deal with the integral function Π which, like the function $E(u)$, is a quadrature involving the elliptic function sn. As we will

see, Jacobi based his study in the use of the addition formulas and quadratures of them. After implementing a quadrature over one of the variables in the addition formulas, he introduced the periodic Z function

$$Z(u) = E(u) - \frac{E}{K}\,u, \tag{A.39}$$

combination of two of the previous basic functions. Then, another quadrature with this new function leads to the Theta function Θ[2], defined in the form

$$\int_0 Z(u)\,\mathrm{d}u = \log\frac{\Theta(u)}{\Theta(0)}. \tag{A.40}$$

Let us present some of the steps of this sequence. One way should be to rely on Lawden [Lawden (1989)] approach. But, if we proceed, as Meyer suggests, following [Greenhill (1892)] things are slightly different; although the scheme is the same, more details are given, with no need to go back and forth through different chapters. Note that in both an application to the rigid body motion is presented.

From Greenhill (p. 191) we quote: "We can now make a fresh start, and prove the Addition Theorem for the Zeta Function independently; and then proceed to Jacobi's form of the Third Elliptic Integral..." From addition formulas

$$\mathrm{dn}(u-v) + \mathrm{dn}(u+v) = \frac{2\,\mathrm{dn}u\,\mathrm{dn}v}{1 - m\,\mathrm{sn}^2u\,\mathrm{sn}^2v}, \tag{A.41}$$

$$\mathrm{dn}(u-v) - \mathrm{dn}(u+v) = \frac{2\,m\,\mathrm{sn}u\,\mathrm{cn}u\;\mathrm{sn}v\,\mathrm{cn}v}{1 - m\,\mathrm{sn}^2u\,\mathrm{sn}^2v}, \tag{A.42}$$

where u is called the parameter, and v the argument, and multiplying them we obtain

$$\mathrm{dn}^2(u+v) - \mathrm{dn}^2(u-v) = \frac{-4m\mathrm{sn}\,u\,\mathrm{cn}\,u\,\mathrm{dn}\,u\,\mathrm{sn}\,v\,\mathrm{cn}\,v\,\mathrm{dn}\,v}{(1 - m\mathrm{sn}^2u\,\mathrm{sn}^2v)^2}. \tag{A.43}$$

Then, we integrate with respect to the argument v, according to the definition (A.39), after some manipulations, we obtain

$$Z(u+v) + Z(u-v) - 2Z(u) = \frac{-2m\mathrm{sn}\,u\,\mathrm{cn}\,u\,\mathrm{dn}\,u\,\mathrm{sn}^2v}{1 - m\mathrm{sn}^2u\,\mathrm{sn}^2v} \tag{A.44}$$

as one of the forms of the addition equation of the Zeta function, which is convenient considering the expression taken by Jacobi (A.37) for the third elliptic integral.

Then, integrating (A.44) with respect to v, we obtain

$$\Pi(v,u) = v\,Z(u) - \frac{1}{2}\int Z(u+v)\,\mathrm{d}v - \frac{1}{2}\int Z(u-v)\,\mathrm{d}v. \tag{A.45}$$

At this step Jacobi introduced a new integral function given above (A.40) called Theta function. Using it we may write (A.45) as follows

$$\Pi(v,u) = \frac{1}{2}\ln\frac{\Theta(u-v)}{\Theta(u+v)} + v\,Z(u). \tag{A.46}$$

We leave at this point the exposition of Greenhill. The reader interested will find there several expressions related to Legendre and Jacobi forms of the third integral which are used later in applications. Note that we have maintained the notation of Greenhill. Nevertheless, in the context of the whole paper, the meaning of u and v should be interchanged.

[2]Note that below Byrd and Friedman use the notation ϑ_0, which is ϑ_4 for Mathematica.

A.4.2 Comparing with Vallejo expressions

As we know, the incomplete elliptic integral of the third goes with us from the very beginning in our problem. Indeed, Π is involved in the solution of the Andoyer variable $\mu(t)$ in Chapter 2 as well as in the complete reduction of the problem in Chapter 3. The treatment of the incomplete elliptic integral of the third kind in Jacobi Theta functions is not a straightforward task. According [Vallejo (1995)] (page 5), this integral can be put into the form

$$\begin{aligned}\Pi(f;\alpha|m) &= \int_0^\alpha \frac{\mathrm{sn}^2(u|m)}{1+f\,\mathrm{sn}^2(u|m)}\,\mathrm{d}u \\ &= \mathcal{M}\,\alpha + \frac{1}{2}\,\frac{\mathrm{sn}(a\,|\,m)}{\mathrm{cn}(a\,|\,m)\,\mathrm{dn}(a\,|\,m)}\,\ln\frac{\vartheta_4(\alpha-iz,q)}{\vartheta_4(\alpha+iz,q)},\end{aligned} \tag{A.47}$$

where [3]

$$m\,\mathrm{sn}^2(a|m) = -f, \tag{A.48}$$

$$a = \frac{2\mathrm{K}(m)}{\pi}\,iz, \tag{A.49}$$

$$z = \frac{\pi}{2\mathrm{K}}\,\mathrm{F}\left(\arctan\sqrt{\frac{f}{m}}\middle|1-m\right), \tag{A.50}$$

$$\mathcal{M} = 1 + \frac{\mathrm{sn}(a|m)}{\mathrm{cn}(a|m)\,\mathrm{dn}(a|m)}\,\frac{\pi}{2K(m)}\,\frac{\vartheta_4'(iz,q)}{\vartheta_4(iz,q)}. \tag{A.51}$$

It is important to remark that the parameter a can be real or complex depending on the value of the characteristic f. Indeed, handbooks like [Byrd & Friedman (1971)] show the classification of those solutions to be taken into account according to the ordinal relation between the modulus and the characteristic.

On the other hand, as it can be noted, the challenge when trying to express the elliptic Π (a three parameter function) in Jacobi Theta functions (biparametric functions) is how to add the characteristic to these functions. The answer to this question lies in Eq. (A.48). It is clear that by the inclusion of the term $\mathrm{sn}(a|m)$, the characteristic is incorporated to the solution as well as the modulus.

The connection between Eq. (A.47), (A.46) and (A.38) is not difficult. By replacing (A.51) in (A.47) we have

$$\Pi(f;\alpha|m) = \alpha + \frac{\mathrm{sn}(a|m)}{\mathrm{cn}(a|m)\,\mathrm{dn}(a|m)}\left[\underbrace{\alpha\,\frac{\pi}{2K(m)}\,\frac{\vartheta_4'(iz,q)}{\vartheta_4(iz,q)} + \frac{1}{2}\ln\frac{\vartheta_4(\alpha-iz,q)}{\vartheta_4(\alpha+iz,q)}}_{\Pi_{Jacobi}\equiv\,\mathrm{Eq.}(A.46)}\right] \tag{A.52}$$

which is actually Eq. (A.38). Note that in Eq. (A.46), our notation sets $u=\alpha$ and $v=iz$.

[3] Note that there is a typo in the equivalent formula (A.48): the exponent of the elliptic functions is missing.

A.4.3 Dealing with formulas of Byrd and Friedman.

We would like to finish coming back to the way this integral is treated at present. Apart from the literature where fast and efficient computation of these integrals is still a topic of research (see Fukushima [Fukushima (2012)] and references therein), when we confront it from the analytical point of view Byrd and Friedman (BF) handbook [Byrd & Friedman (1971)] continues to be a basic reference.

$$\Lambda_0(\vartheta,k)=\frac{2}{\pi}\left[EF(\vartheta,k')+KE(\vartheta,k')-KF(\vartheta,k')\right],\quad \vartheta=\sin^{-1}\sqrt{\frac{1-\alpha^2}{k'^2}},$$
$$v=(\pi u_1)/2K,\quad w=[\pi F(\vartheta,k')]/2K,\quad p=\pi K'/2K,\quad q=e^{-2p},$$
$$\Omega_2=\frac{iK}{\pi}\ln\frac{H_1[u_1+iF(\vartheta,k')]}{H_1[u_1-iF(\vartheta,k')]}=\frac{iK}{\pi}\ln\frac{\vartheta_2(v+iw)}{\vartheta_2(v-iw)}$$
$$=u_1-\frac{2K}{\pi}\tan^{-1}\left[\frac{2\sum_1^\infty q^{m^2}\sin(2mv)\sinh[2m(p-w)]}{1+2\sum_1^\infty q^{m^2}\cos(2mv)\cosh[2m(p-w)]}\right].$$

$$\textbf{433.01}\quad \Pi(u_1,\alpha^2)=\int_0^{u_1}\frac{du}{1-\alpha^2\,\mathrm{sn}^2\,u}=u_1+\frac{\pi\alpha[\Omega_2-u_1\Lambda_0(\vartheta,k)]}{2\sqrt{(\alpha^2-k^2)(1-\alpha^2)}\,K},$$

Figure A.3: Snapshot of one of the six boxes given in [Byrd & Friedman (1971)], corresponding to the case $0<k<\alpha^2<1\equiv 0<\sqrt{m}<f<1$, for a third integral evaluation

Just for readers not familiar with BF we offer in Fig. A.3 a snapshot presenting the form in which their computation is organized. In particular two auxiliary functions Ω and Λ are introduced, as well as the series expansions of the Theta functions used in applications due to their fast convergence. Moreover, as parameters $k^2=m$ and $\alpha^2=f$ are involved, several cases have to be treated separately. A final detail to keep in mind is the different names given in the literature to the parameters: k^2 or m for the modulus and α^2, n or square of them for the elliptic parameter.

A.5 Derivative of the Jacobi Zeta function

As shown in Section 4, we are not interested in studying generically the derivatives of the Zeta-function, but rather in analyzing the necessary derivatives to find the solution of the Lie-Deprit's algorithm. Indeed, since such an algorithm only requires the partial derivatives of every function with respect to the momenta, and both momenta are within the elliptic modulus m pertaining to the Zeta-function, we are only interested in the derivatives with respect to the modulus. Nevertheless, as Zeta-function is a biparametric function where the argument φ is also a function of m, the derivative we are looking for is not actually a partial derivative, but a total derivative. Thus, taking into account that ψ is our angle in Section 4, we can state that

$$\frac{\mathrm{d}\,Z(\psi|m)}{\mathrm{d}\,m}=\frac{\partial Z}{\partial m}+\frac{\partial Z}{\partial\psi}\left(\frac{\partial\psi}{\partial m}+\frac{\partial\psi}{\partial u}\frac{\partial u}{\partial m}\right),\qquad \psi=\mathrm{am}(u|m),\tag{A.53}$$

and therefore, it is clear that the analytical derivative of the Zeta-function depends on the function u, which is different according to whether the variables you are using are Ferrer-Lara or action-angle.

A.5.1 Derivative in Ferrer-Lara variables

Now we have to tackle the derivative given in (A.53) but taking into account that the function $\psi = \mathrm{am}(-\ell|m)$ and hence $u = -\ell$. Note therefore that the partial derivative of u with respect to m is zero, thus we have that

$$\frac{\mathrm{d}\, Z(\psi|m)}{\mathrm{d}\, m} = \frac{\partial Z}{\partial m} + \frac{\partial Z}{\partial \psi}\frac{\partial \psi}{\partial m}, \tag{A.54}$$

where

$$\frac{\partial Z}{\partial m} = \frac{1}{2m}\left[\frac{1}{1-m}\frac{E(m)}{K(m)}\left(Z\big(\mathrm{am}(\ell|m)|m\big) - \frac{m\,\mathrm{sn}(\ell|m)\,\mathrm{cn}(\ell|m)}{\mathrm{dn}(\ell|m)}\right) - Z\big(\mathrm{am}(\ell|m)|m\big)\right], \tag{A.55}$$

$$\frac{\partial Z}{\partial \psi} = \mathrm{dn}(\ell|m) - \frac{E(m)}{K(m)}\frac{1}{\mathrm{dn}(\ell|m)}, \tag{A.56}$$

$$\frac{\partial \psi}{\partial m} = \frac{1}{2m(1-m)}\left[\mathrm{dn}(\ell|m)\Big(\mathrm{E}\big(\mathrm{am}(\ell|m)|m\big) - (1-m)\,\ell\Big) - m\,\mathrm{sn}(\ell|m)\,\mathrm{cn}(\ell|m)\right]. \tag{A.57}$$

Fig. A.4 shows the shape of the above partial derivatives with respect to ℓ. It is clear that the non-periodicity of the total derivative of Z with respect to the modulus is due to the partial derivative of ψ with respect to m, which has a linear contribution according to (A.57). This circumstance produces an increasing value of the envelope of the total derivative as seen in Fig. 4.3.

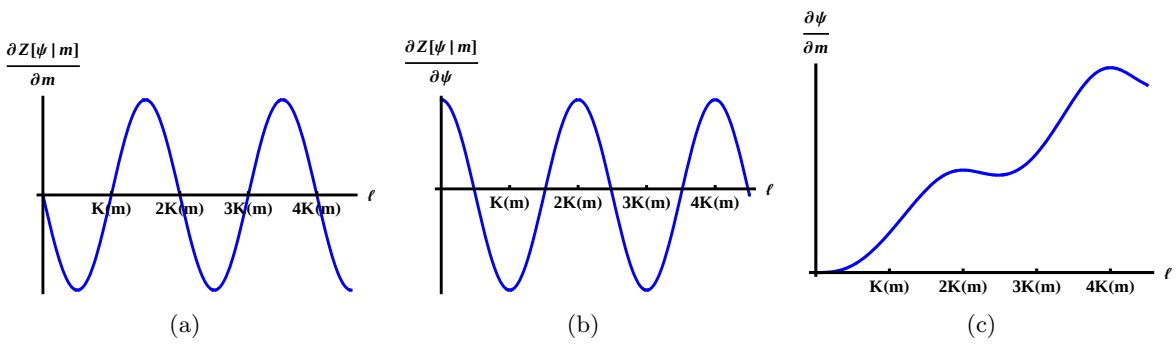


Figure A.4: Graphic representation of the three partial derivatives involved in the calculation of the total derivative of the Zeta-function with respect to the elliptic modulus in Ferrer-Lara variables.

A.5.2 Derivative in action-angle variables

We will now tackle the derivative given in (A.53) but taking into account that the function $\psi = \mathrm{am}(-2\mathrm{K}(m)\varphi_\nu/\pi|m)$ and hence $u = -2\mathrm{K}(m)\varphi_\nu/\pi$, thus we have that

$$\begin{aligned}\frac{\partial Z}{\partial m} &= \frac{1}{2m}\left\{\frac{1}{1-m}\frac{E(m)}{K(m)}\left[Z\left(\mathrm{am}\left(\frac{2\mathrm{K}(m)}{\pi}\varphi_\nu\middle|m\right)\middle|m\right)\right.\right.\\ &\left.\left.-\frac{m\,\mathrm{sn}\left(\frac{2\mathrm{K}(m)}{\pi}\varphi_\nu\middle|m\right)\mathrm{cn}\left(\frac{2\mathrm{K}(m)}{\pi}\varphi_\nu\middle|m\right)}{\mathrm{dn}\left(\frac{2\mathrm{K}(m)}{\pi}\varphi_\nu\middle|m\right)}\right]-Z\left(\mathrm{am}\left(\frac{2\mathrm{K}(m)}{\pi}\varphi_\nu\middle|m\right)|m\right)\right\}, \qquad \text{(A.58)}\end{aligned}$$

$$\frac{\partial Z}{\partial \psi} = \mathrm{dn}\left(\frac{2\mathrm{K}(m)}{\pi}\varphi_\nu\middle|m\right)-\frac{E(m)}{K(m)}\mathrm{dn}\left(\frac{2\mathrm{K}(m)}{\pi}\varphi_\nu\middle|m\right)^{-1}, \qquad \text{(A.59)}$$

$$\begin{aligned}\frac{\partial \psi}{\partial m} &= \frac{1}{2m(1-m)}\left\{\mathrm{dn}\left(\frac{2\mathrm{K}(m)}{\pi}\varphi_\nu\middle|m\right)\left[\mathrm{E}\left(\mathrm{am}\left(\frac{2\mathrm{K}(m)}{\pi}\varphi_\nu\middle|m\right)\middle|m\right)-(1-m)\frac{2\mathrm{K}(m)}{\pi}\varphi_\nu\right]-\right.\\ &\left.m\,\mathrm{sn}\left(\frac{2\mathrm{K}(m)}{\pi}\varphi_\nu\middle|m\right)\mathrm{cn}\left(\frac{2\mathrm{K}(m)}{\pi}\varphi_\nu\middle|m\right)\right\}, \qquad \text{(A.60)}\end{aligned}$$

$$\frac{\partial \psi}{\partial u} = \mathrm{dn}\left(\frac{2\mathrm{K}(m)}{\pi}\varphi_\nu\middle|m\right), \qquad \text{(A.61)}$$

$$\frac{\partial u}{\partial m} = -\frac{1}{2m(1-m)}\frac{2\mathrm{K}(m)}{\pi}\varphi_\nu\left[\frac{\mathrm{E}(m)}{\mathrm{K}(m)}-(1-m)\right]. \qquad \text{(A.62)}$$

Fig. A.5 shows the shape of the above partial derivatives with respect to φ_ν. By inspecting the figure, this time it is unclear whether the resulting total derivative is periodic or not since Figs. A.5(c) and A.5(e) are not periodic. These two figures corresponds with two partial derivatives within the brackets in (**??**), hence this part of the derivative should be analyzed more carefully. Thus, according to Eqs. (A.60)-(A.62), after a few calculations we can write

$$\frac{\partial \psi}{\partial u}\frac{\partial u}{\partial m} = -\frac{1}{2m(1-m)}\frac{2\mathrm{K}(m)}{\pi}\varphi_\nu\left[\frac{\mathrm{E}(m)}{\mathrm{K}(m)}-(1-m)\right]\mathrm{dn}\left(\frac{2\mathrm{K}(m)}{\pi}\varphi_\nu\middle|m\right), \qquad \text{(A.63)}$$

$$\begin{aligned}\frac{\partial \psi}{\partial m}+\frac{\partial \psi}{\partial u}\frac{\partial u}{\partial m} &= \frac{1}{2m(1-m)}\left[\mathrm{dn}\left(\frac{2\mathrm{K}(m)}{\pi}\varphi_\nu\middle|m\right)\right.\\ &\left.\times\mathrm{Z}\left(\mathrm{am}\left(\frac{2\mathrm{K}(m)}{\pi}\varphi_\nu\middle|m\right)\middle|m\right)-m\,\mathrm{sn}\left(\frac{2\mathrm{K}(m)}{\pi}\varphi_\nu\middle|m\right)\mathrm{cn}\left(\frac{2\mathrm{K}(m)}{\pi}\varphi_\nu\middle|m\right)\right]. \qquad \text{(A.64)}\end{aligned}$$

It is not difficult to realize that Eq. (A.63) is not periodic due to a linear contribution (see Fig. A.6(a)) but when the partial derivative of ψ with respect to m is added, Z appears in the resulting expression (A.64) making it periodic (see Fig. A.6(b)). It is therefore proved the periodicity of the total derivative of Z with respect to m in action-angle variables.

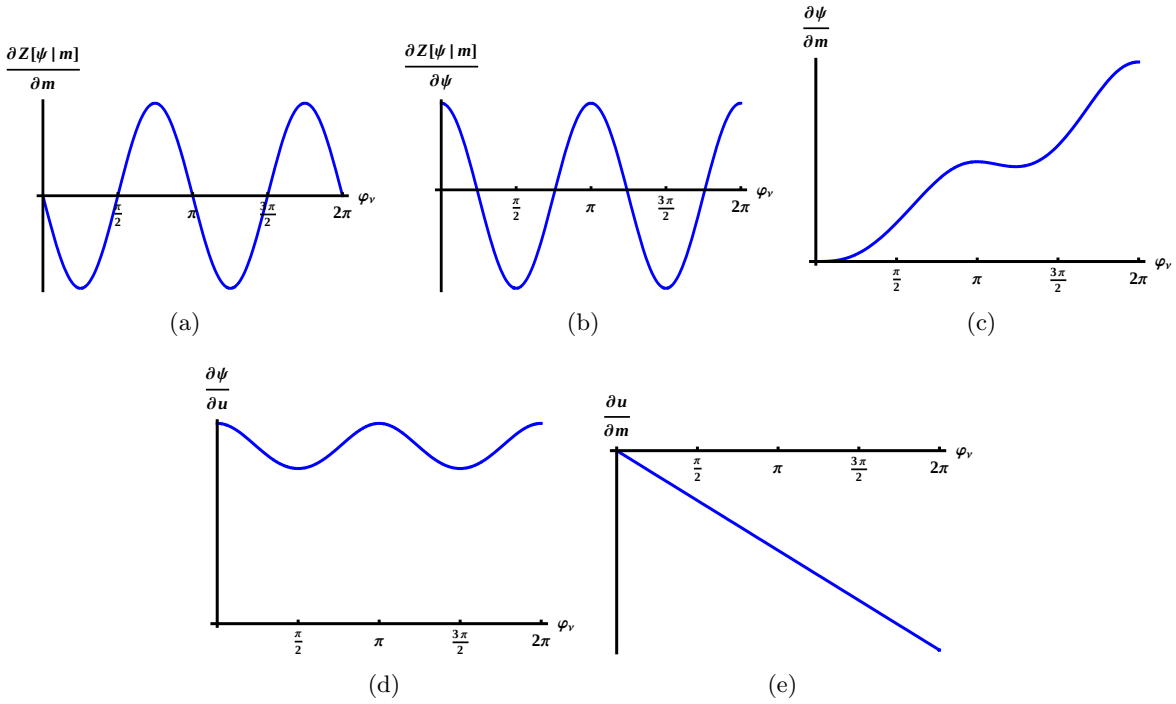


Figure A.5: Graphic representation of the five partial derivatives involved in the calculation of the total derivative of the Zeta-function with respect to the elliptic modulus in action-angle variables.

A.6 On Jacobi Theta functions

A.6.1 On Jacobi Nome

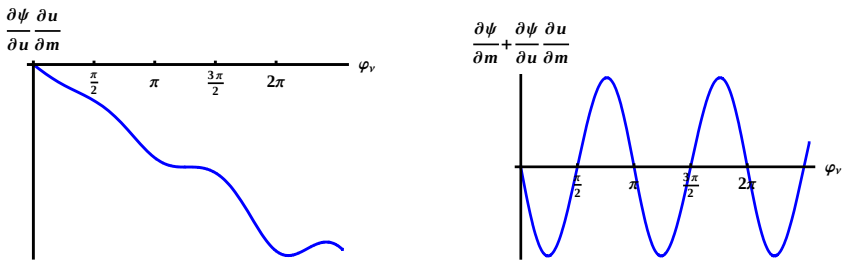
Due to the difficulty computing and dealing with elliptic functions and integrals in the past, some efforts were carried out to avoid its manipulation. The classical method is to expand the elliptic function as a power series of the modulus or the argument, among other similar techniques rapidly convergent when the modulus is close to zero. Nevertheless, similar and much more rapidly convergent developments can be obtained when the elliptic function is expanded as a power series of the Jacobi's nome instead of the modulus. The Jacobi's nome is given by

$$q(m) = \exp\left(-\pi \frac{\mathrm{K}(1-m)}{\mathrm{K}(m)}\right), \tag{A.65}$$

where it is already known that $\mathrm{K}(m)$ is the complete elliptic integral of the first kind. Fig. A.7 shows the benefits of using the nome to do the series expansions of elliptic functions.

A.6.2 On Jacobi Theta functions and their use to compute elliptic functions

The Jacobi theta functions are defined by series expansions in the nome, whose definitions may be found in classical books (see [**?**, Lawden (1989), Byrd & Friedman (1971)]). Alternatively modern computer algebra systems recognize these functions and allow us for the automatic



(a) (b)

Figure A.6: Graphic representation of some products and sums of partial derivatives showing that the total derivative of Z with respect to m is π-periodic in Action-Angle variables.

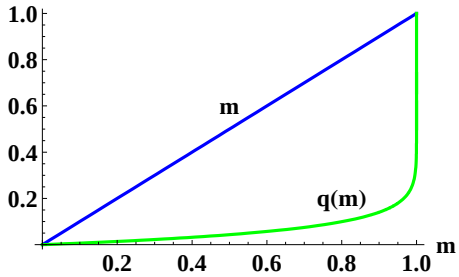


Figure A.7: Graphic representation of the Jacobi Nome (green) with respect to the modulus m (blue). Note the differences with the modulus itself.

computation of their series expansions in the nome. In our case we take and use the following definition from *Mathematica*©[4]

$$\vartheta_1(\alpha,q) = 2q^{1/4}\textstyle\sum_{j=0}^{\infty}(-1)^j q^{j(j+1)}\sin[(2j+1)\alpha], \tag{A.66}$$

$$\vartheta_2(\alpha,q) = 2q^{1/4}\textstyle\sum_{j=0}^{\infty}q^{j(j+1)}\cos[(2j+1)\alpha], \tag{A.67}$$

$$\vartheta_3(\alpha,q) = 1+2\textstyle\sum_{j=1}^{\infty}q^{j^2}\cos(2j\alpha), \tag{A.68}$$

$$\vartheta_4(\alpha,q) = 1+2\textstyle\sum_{j=1}^{\infty}(-1)^j q^{j^2}\cos(2j\alpha), \tag{A.69}$$

where the argument α is given by

$$\alpha = \frac{\pi}{2K(m)}\,u. \tag{A.70}$$

being α an angular variable coming from the Jacobi Amplitude u.

Fig. A.8 shows a graphic representation of the four Jacobi Theta functions for different values of the modulus. Note the remarkable differences when m is close to zero and when is close to one. Note further the tiny variation of θ_3 and θ_4 when m is small.

Once known the series expansions of the Jacobi Theta functions, according to [Vallejo (1995),

[4]Note that in Lawden's book [Lawden (1989)] we have $\vartheta \equiv \theta$.

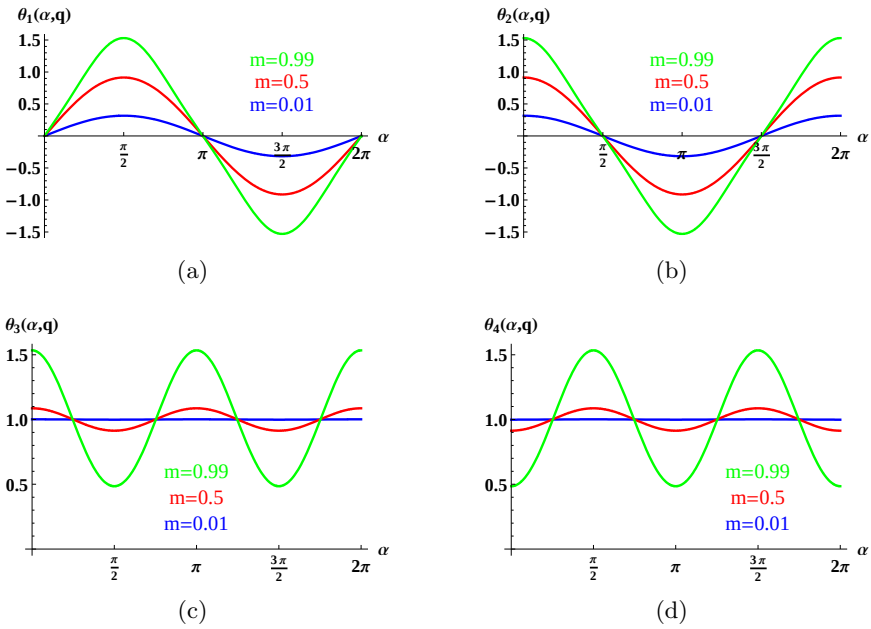


Figure A.8: Graphic representation of the Jacobi Theta functions for different values of the modulus: $m = \{0.01, 0.5, 0.99\}$.

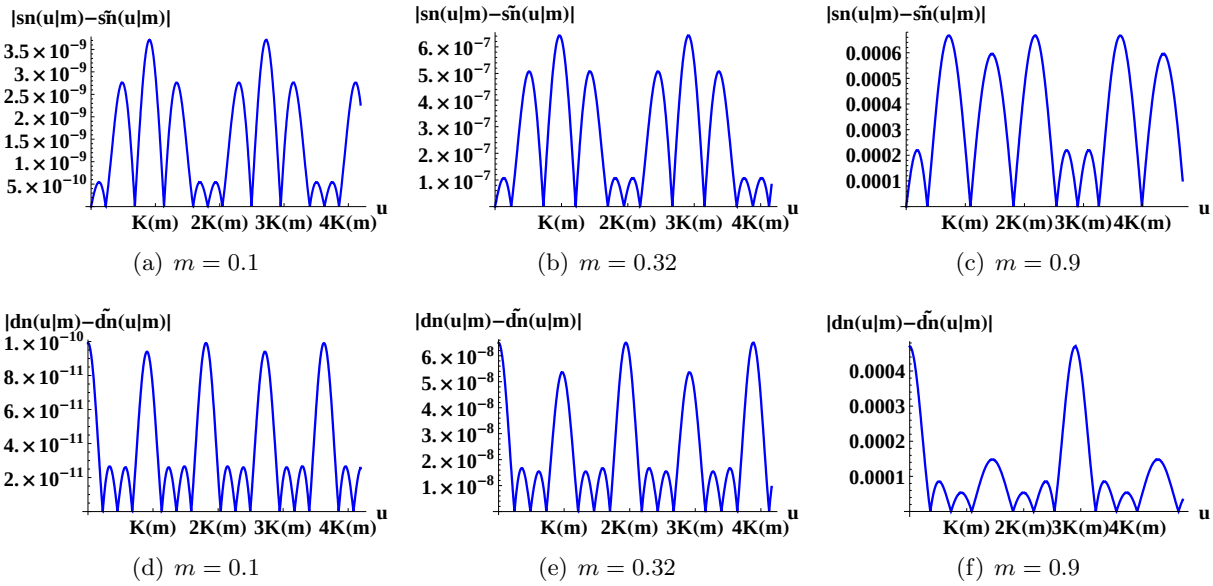


Figure A.9: (a)-(c) Differences between the $\mathrm{sn}(u|m)$ by Mathematica and its truncated version up to $j = 1$ by Jacobi theta functions ($\widetilde{\mathrm{sn}}(u|m)$) for three different values of the modulus. (d)-(f) The same as the previous set with the function $\mathrm{dn}(u|m)$.

Lawden (1989)], the three main Jacobi elliptic functions can be put into the form

$$\mathrm{sn}(u|m) = \frac{1}{\sqrt[4]{m}} \frac{\vartheta_1(\alpha, q)}{\vartheta_4(\alpha, q)}, \tag{A.71}$$

$$\mathrm{cn}(u|m) = \sqrt[4]{\frac{1-m}{m}} \frac{\vartheta_2(\alpha, q)}{\vartheta_4(\alpha, q)}, \tag{A.72}$$

$$\mathrm{dn}(u|m) = \sqrt[4]{1-m} \frac{\vartheta_3(\alpha, q)}{\vartheta_4(\alpha, q)}. \tag{A.73}$$

To illustrate the order of approximation according to Theta functions, Fig. A.9 shows the differences between the programmed elliptic functions $\mathrm{sn}(u|m)$ and $\mathrm{dn}(u|m)$ by Mathematica and the same functions expressed in Theta functions and only truncated up to $j = 1$. It can be appreciated that the error grows up with the modulus. Note further that Fig. A.9(b) and A.9(e) have been included in this set of pictures because they show the two elliptic functions involved in the disturbing function for the value of m obtained from the initial conditions given in the previous chapter. For these conditions, assuming that the Mathematica version of the functions provides the most exact value, the error for our value of m when truncated up to order $j = 1 \equiv \mathcal{O}(q^{9/4})$ is 10^{-7} for $\mathrm{sn}(u|m)$ and 10^{-8} for $\mathrm{dn}(u|m)$. At this point it is important to recall that the value of the small parameter of our perturbing function $(n\,C/M)^2$ is about 10^{-3}, hence an approximation of 10^{-7} is apparently good enough. As well, the treatment of the elliptic integral of the third kind, which is also involved in the solution of the problem, can be found in A.4.2 and [Byrd & Friedman (1971), **?**].

A.7 On Fourier series of elliptic functions

As it is well known, one of the classical ways to handle elliptic functions is to expand them as Fourier series in the nome. This was carried out, for instance, by [Sadov (1970b)] and [Vallejo (1995)]. In their works, both authors had to deal with products of elliptic functions according to the perturbing function studied, that is, products and quotients of Theta functions. Since this task can firstly be tedious, we think it is useful to show the process followed to obtain a Fourier expansion of the elliptic function $\mathrm{sn}(u|m)$. Even though this effort has already been accomplished by [Whittaker & Watson (1927)] (page 510) or [Wang & Guo (1989)] (page 567), we will show the process in more detail.

Indeed, as shown in [Whittaker & Watson (1927)], since $\mathrm{sn}(u|m)$ is an odd function, it may be put into the form

$$\mathrm{sn}(u|m) = \sum_{n=1}^{\infty} b_n \sin(nx) \tag{A.74}$$

where

$$u = \frac{2\,\mathrm{K}(m)}{\pi}\,x, \qquad 0 \le x \le 2\pi, \tag{A.75}$$

and b_n are the coefficients of the expansion. To obtain these coefficients of the serie, according to [Oppenheim *et. al.* (1927)], we multiply by e^{imx} both sides of the equation (A.74) and we take the integral within a period. Thus

$$\int_{-T/2}^{T/2} \mathrm{sn}(u|m)\mathrm{e}^{ikx} = \sum_{n=1}^{\infty} b_n \int_{-T/2}^{T/2} \mathrm{e}^{ikx} \sin(nx)\mathrm{d}x. \tag{A.76}$$

Taking into account that $\sin(nx) = (\mathrm{e}^{inx} - \mathrm{e}^{-inx})/(2i)$, the above quadrature can be expressed as

$$\int_{-T/2}^{T/2} \mathrm{e}^{ikx} \sin(nx)\mathrm{d}x = \int_{-T/2}^{T/2} \mathrm{e}^{ikx} \frac{\mathrm{e}^{inx} - \mathrm{e}^{-inx}}{2i}\mathrm{d}x = \frac{1}{2i}\left[\int_{-T/2}^{T/2} \mathrm{e}^{i(k+n)x}\mathrm{d}x - \int_{-T/2}^{T/2} \mathrm{e}^{i(k-n)x}\mathrm{d}x\right] \tag{A.77}$$

which, by the Euler relation $\mathrm{e}^{ix}\cos x + i\sin x$, can be reordered yielding

$$\begin{aligned}\int_{-T/2}^{T/2} \mathrm{e}^{ikx}\sin(nx)\mathrm{d}x &= \frac{1}{2i}\Bigg[\int_{-T/2}^{T/2}\cos[(k+n)x]\mathrm{d}x + i\int_{-T/2}^{T/2}\sin[(k+n)x]\mathrm{d}x \\ &\quad - \int_{-T/2}^{T/2}\cos[(k-n)x]\mathrm{d}x - i\int_{-T/2}^{T/2}\sin[(k-n)x]\mathrm{d}x\Bigg]. \end{aligned} \tag{A.78}$$

Observe that, if $k \neq n$, then the above four quadratures are equal to zero. However, if $k = n$, then all the quadratures are again equal to zero except for

$$\int_{-T/2}^{T/2}\cos[(k-n)x]\mathrm{d}x = \int_{-T/2}^{T/2}\mathrm{d}x = T \tag{A.79}$$

therefore, substituting it in (A.76) we get

$$\int_{-T/2}^{T/2}\mathrm{sn}(u|m)\mathrm{e}^{inx} = -\frac{T}{2i}\,b_n \tag{A.80}$$

and taking into account that $\mathrm{sn}(u|m)$ is 2π-periodic in the variable x, we may finally write

$$i\pi\, b_n = \int_{-\pi}^{\pi}\mathrm{sn}(u|m)\mathrm{e}^{inx}. \tag{A.81}$$

Now we have to compute the above quadrature. As getting the primitive function can be very complicated, we will evaluate the quadrature by the Cauchy integral theorem, that is, by computing the residues of the function at the poles inside a period parallelogram $\mathcal{C}$ as shown in Fig. A.10(b) (see [Whittaker & Watson (1927), Armitage & Eberlein (1972), Hall (1995)] for further details). Note that the corners of the parallelogram are located at the points $-\pi, \pi, \pi\tau, -2\pi+\pi\tau$ where

$$\tau = i\frac{\mathrm{K}(1-m)}{\mathrm{K}(m)} \tag{A.82}$$

and note also that $-\pi + \pi\tau/2$ and $\pi\tau/2$ are the only two poles of the integrand inside the contour.

The Laurent series expansions of the function $\mathrm{sn}(u|m)$ around these poles are, respectively[5]

$$\mathrm{sn}\left[\frac{2\mathrm{K}(m)}{\pi}\left(x-\pi+\frac{\pi\tau}{2}\right)\middle|m\right] = -\frac{\pi}{2\sqrt{m}\,\mathrm{K}(m)}\frac{1}{x} - \frac{1+m}{3\pi\sqrt{m}}\,x + \ldots \tag{A.83}$$

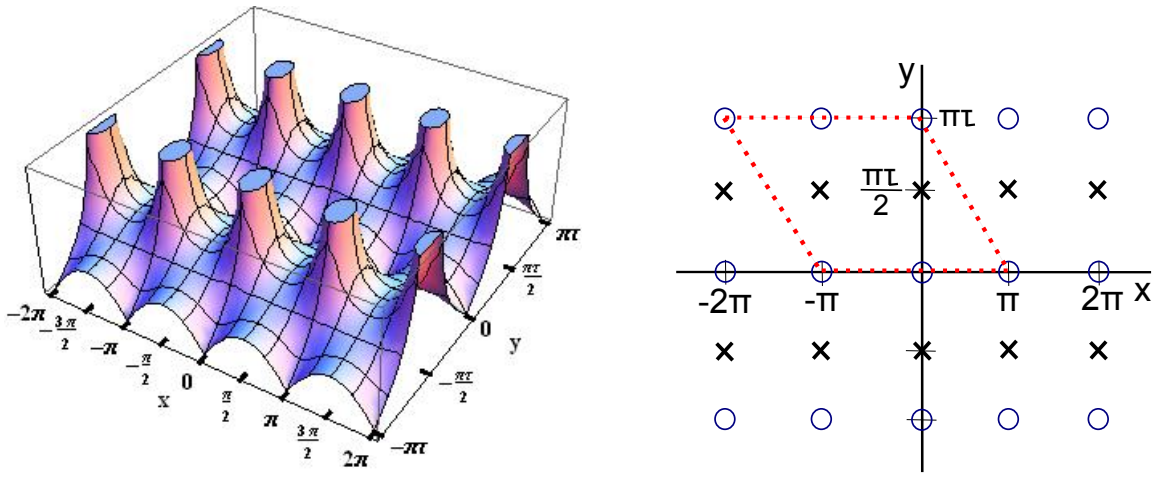
$$\mathrm{sn}\left[\frac{2\mathrm{K}(m)}{\pi}\left(x+\frac{\pi\tau}{2}\right)\middle|m\right] = \frac{\pi}{2\sqrt{m}\,\mathrm{K}(m)}\frac{1}{x} + \frac{1+m}{3\pi\sqrt{m}}\,x + \ldots \tag{A.84}$$

Since the residue of any pole is defined as the coefficient of the term x^{-1} in the corresponding Laurent expansion, then we may write

$$\mathrm{Res}\left[\mathrm{sn}(u|m)\mathrm{e}^{inx}, -\pi+\frac{\pi\tau}{2}\right] = -\frac{\pi}{2\sqrt{m}\,\mathrm{K}(m)}\mathrm{e}^{-i\pi n + in\pi\tau/2} \tag{A.85}$$

$$\mathrm{Res}\left[\mathrm{sn}(u|m)\mathrm{e}^{inx}, \frac{\pi\tau}{2}\right] = \frac{\pi}{2\sqrt{m}\,\mathrm{K}(m)}\mathrm{e}^{in\pi\tau/2}. \tag{A.86}$$

[5]Note that the Laurent expansion around the poles can be obtained with *Mathematica* as a Taylor expansion around the origin of the function shifted the value of the pole.



(a) (b)

Figure A.10: Location of poles and zeros of the function $|\mathrm{sn}[2\mathrm{K}/\pi(x+iy)|m]|$. The value of τ is given in (A.82). (a) Modular surface. (b) Period parallelogram where circles correspond to zeros and crosses represent poles of the function.

and therefore

$$\int_{\mathcal{C}} \mathrm{sn}(u|m)\mathrm{e}^{inx} = \left\{\int_{-\pi}^{\pi} + \int_{\pi}^{\pi\tau} + \int_{\pi\tau}^{-2\pi+\pi\tau} + \int_{-2\pi+\pi\tau}^{-\pi}\right\} \mathrm{sn}(u|m)\mathrm{e}^{inx} \tag{A.87}$$

$$= 2\pi i\left(\frac{\pi}{2\sqrt{m}\,\mathrm{K}(m)}\mathrm{e}^{in\pi\tau/2} - \frac{\pi}{2\sqrt{m}\,\mathrm{K}(m)}\mathrm{e}^{-i\pi n+in\pi\tau/2}\right)$$

$$= \frac{i\pi^2}{\sqrt{m}\,\mathrm{K}(m)}\, q^{n/2}\{1-(-1)^n\}. \tag{A.88}$$

Note that, from the periodic properties of $\mathrm{sn}(u|m)$ and e^{inx}, by applying the change $z = x + 2\pi$ to the fourth quadrature in (A.87) we see that

$$\int_{\pi}^{\pi\tau} + \int_{-2\pi+\pi\tau}^{-\pi} = \int_{\pi}^{\pi\tau} + \int_{\pi\tau}^{\pi} = 0 \tag{A.89}$$

and therefore, from (A.87) and (A.88) and taking into account (A.89), we may write

$$\left\{\int_{-\pi}^{\pi} - \int_{-2\pi+\pi\tau}^{\pi\tau}\right\} \mathrm{sn}(u|m)\mathrm{e}^{inx} = \frac{i\pi^2}{\sqrt{m}\,\mathrm{K}(m)}\, q^{n/2}\{1-(-1)^n\}. \tag{A.90}$$

In turn, again from the periodic properties of $\mathrm{sn}(u|m)$ and e^{inx}, by applying the change $z = x + \pi - \pi\tau$ to the second quadrature in (A.90) we see that[6]

$$\int_{-2\pi+\pi\tau}^{\pi\tau} \mathrm{sn}(u|m)\mathrm{e}^{inx} = \int_{-\pi}^{\pi} \mathrm{sn}\left[\frac{2\mathrm{K}(m)}{\pi}(z+\pi\tau-\pi)|m\right] \mathrm{e}^{in(z+\pi\tau-\pi)}$$

$$= -(-1)^n q^n \int_{-\pi}^{\pi} \mathrm{sn}(u|m)\mathrm{e}^{inx}. \tag{A.91}$$

[6]Note that $\mathrm{sn}\left[\frac{2\mathrm{K}(m)}{\pi}(x-\pi\tau+\pi)|m\right] = -\mathrm{sn}\left[\frac{2\mathrm{K}(m)}{\pi}x|m\right]$.

and substituting it in (A.90), we have

$$\{1+(-1)^n q^n\}\int_{-\pi}^{\pi}\mathrm{sn}(u|m)\mathrm{e}^{inx}=\frac{i\pi^2}{\sqrt{m}\,\mathrm{K}(m)}\,q^{n/2}\{1-(-1)^n\}. \tag{A.92}$$

Now, taking into account (A.81), the coefficients of the Fourier serie are

$$b_n=\frac{\pi}{\sqrt{m}\,\mathrm{K}(m)}\frac{q^{n/2}\{1-(-1)^n\}}{\{1+(-1)^n q^n\}}. \tag{A.93}$$

However, note that when n is even, then $b_n=0$; but when n is odd we have

$$b_n=\frac{2\pi}{\sqrt{m}\,\mathrm{K}(m)}\,\frac{q^{n/2}}{1-q^n}. \tag{A.94}$$

and therefore

$$\mathrm{sn}(u|m)=\frac{2\pi}{\sqrt{m}\,\mathrm{K}(m)}\left\{\frac{q^{1/2}\sin x}{1-q}+\frac{q^{3/2}\sin 3x}{1-q^3}+\frac{q^{5/2}\sin 5x}{1-q^5}+\dots\right\},\quad x\in\mathbb{R}. \tag{A.95}$$

Nevertheless, both sides of the last equation are analytic in the strip $|\mathrm{Im}(x)|<\pi/2\,\mathrm{Im}(\tau)$, hence by the theory of analytic continuation, we finally have

$$\mathrm{sn}(u|m)=\frac{2\pi}{\sqrt{m}\,\mathrm{K}(m)}\sum_{n=0}^{\infty}\frac{q^{n+1/2}}{1-q^{2n+1}}\sin[(2n+1)x] \tag{A.96}$$

where

$$u=\frac{2\,\mathrm{K}(m)}{\pi}\,x,\qquad 0\le x\le 2\pi. \tag{A.97}$$

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