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Successive radii of convex bodies
(Los radios sucesivos de un cuerpo convexo)

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AUTORIZA:

La presentación de la Tesis Doctoral titulada ”Successive radii of convex bodies (Los radios sucesivos de un cuerpo convexo)”, realizada por D. Bernardo González Merino, bajo mi inmediata dirección y supervisión, y que presenta para la obtención del grado de Doctor por la Universidad de Murcia.

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Mª Angeles H. C.

Fdo.: María de los Angeles Hernández Cifre

No podría comenzar esta memoria de otra manera que no sea recordando toda la ayuda y el apoyo recibido por la personas con las que bien he tenido el placer de trabajar, bien he convivido, sufrido y disfrutado estos últimos casi cuatro años.
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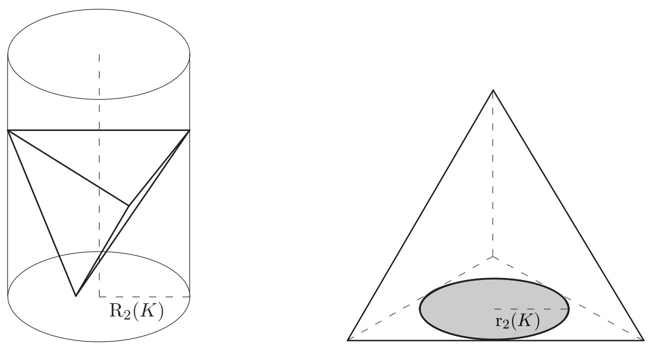
Contents

Resumen/Summary

A lo largo de este trabajo, la mayor parte de nuestros esfuerzos se han centrado en investigar el comportamiento de los llamados *radios sucesivos* de un cuerpo convexo (conjunto convexo y compacto). Nos gustaría comenzar remarcando que el *circunradio* R, el *inradio* r, el *diámetro* D y la *anchura mínima* ω son casos particulares de tales funcionales, y casi con total seguridad los más conocidos y estudiados.

Los radios sucesivos se construyen a partir de máximos o mínimos de circunradios e inradios de proyecciones o secciones del cuerpo convexo. Esto permite definir un total de siete familias de radios sucesivos, de entre las cuales nos vamos a centrar fundamentalmente en dos de ellas: los más clásicos R_i y r_i, $i = 1, \dots, n$. Estos dos funcionales particulares admiten una definición alternativa mucho más geométrica: el *i-ésimo radio sucesivo exterior* $\mathrm{R}_i(K)$ es el menor radio que puede tener un cilindro con sección esférica i-dimensional que contenga a K, mientras que el *i-ésimo radio sucesivo interior* $\mathrm{r}_i(K)$ es el radio de la mayor bola i-dimensional contenida en K.



Los radios sucesivos R_2 y r_2 del tetraedro regular

Aunque anteriormente diversos matemáticos habían obtenido resultados sobre ciertos radios particulares, el primer estudio sistemático de tales funcionales se debe a Betke y Henk [2] en 1992.

Pero estas magnitudes no han sido sólo estudiadas en Geometría Convexa: a lo largo del segundo tercio del siglo XX, los radios sucesivos aparecen en la Teoría de los Espacios de Banach y en la Teoría de la Aproximación, siendo conocidos como *números de Gelfand y de Kolmogorov*. Éstos son valores asociados a funcionales lineales acotados entre espacios de Banach, los cuales coinciden con los radios sucesivos de un cuerpo convexo K, simétrico respecto al origen, cuando se trabaja con operadores identidad de $\mathbb{R}^n$ entre los espacios normados que se obtienen al dotar uno de ellos con la norma de bola K, y el otro con la norma euclídea $|\cdot|_2$. En particular, de esta estrecha relación se pueden traducir propiedades conocidas para los números de Gelfand y Kolmogorov en propiedades para los radios sucesivos. Desafortunadamente, el nexo de unión entre estas dos teorías se rompe al entrar en juego un cuerpo convexo K que no sea simétrico respecto al origen. Esto se debe a que los únicos conjuntos convexos compactos que pueden ser la bola unidad de un espacio de Banach son, precisamente, los centralmente simétricos.

De hecho, uno de los cuerpos convexos que juegan un papel relevante en el estudio de los radios sucesivos, no presenta simetría central: es el *símplice n-dimensional* S_n. Las medidas $\mathrm{R}(S_n)$, $\mathrm{r}(S_n)$, $\mathrm{D}(S_n)$ y $\omega(S_n)$ del símplice son fáciles de obtener; sin embargo, el cálculo de sus radios sucesivos no es en absoluto sencillo y ha sido motivo de estudio a lo largo de los años por diversos matemáticos (véanse [1, 8, 38, 51, 56, 57, 58]).

Uno de los resultados más significativos en el estudio de los radios sucesivos, a la vez que uno de los problemas centrales en este campo que aún sigue abierto, es una desigualdad demostrada independientemente por Pukhov (1979, véase [50]) y Perel'man (1987, véase [45]). Ambos probaron que si K es un cuerpo convexo n-dimensional, entonces se cumple que

$$\frac{\mathrm{R}_{n-i+1}(K)}{\mathrm{r}_i(K)} < i+1, \qquad \text{para todo } \ 1 \leq i \leq n. \tag{$*$}$$

Además, Pukhov demostró que si nos restringimos a la familia de los cuerpos convexos simétricos respecto al origen, la cota puede reducirse hasta $\sqrt{e}\min\bigl\{\sqrt{i}, \sqrt{n-i+1}\bigr\}$; por su parte, Perel'man probó que, en el caso particular del espacio euclídeo 3-dimensional $\mathbb{R}^3$, la desigualdad $(*)$ se mejora a $\mathrm{R}_2(K)/\mathrm{r}_2(K) < 2.151\ldots$ Los casos particulares $i=1$ e $i=n$ corresponden, respectivamente, a los teoremas clásicos de Jung (1901, véase [38]) y Steinhagen (1921, véase [56]), en los cuales, las cotas óptimas, a saber,

$$\frac{\mathrm{R}_n(K)}{\mathrm{r}_1(K)} \leq \sqrt{\frac{2n}{n+1}} \qquad \text{y} \qquad \frac{\mathrm{R}_1(K)}{\mathrm{r}_n(K)} \leq \begin{cases} \sqrt{n} & \text{para } n \text{ impar}, \\ \frac{n+1}{\sqrt{n+2}} & \text{para } n \text{ par}, \end{cases}$$

respectivamente, distan mucho de las que se obtendrían de $(*)$. Por otro lado, en estas dos desigualdades de Jung y Steinhagen el símplice regular es un cuerpo extremal (es decir, se alcanza la igualdad). Se conjetura que, para cualquier otro valor del índice, $i = 2, \ldots, n-1$, el símplice regular daría también la cota óptima en $(*)$.

Existen muchos resultados importantes que estudian diversas propiedades de los radios sucesivos de un cuerpo convexo, o que los relacionan con otras medidas geométricas, aunque no los estudiaremos en este trabajo; mencionamos, por ejemplo, las generalizaciones de los teoremas de Jung ([32]) y Steinhagen ([3]), la relación entre los radios sucesivos y el volumen o los volúmenes intrínsecos ([2, 33]), o aspectos computacionales de estos funcionales ([28, 29]).

Grosso modo, esta Tesis Doctoral está dedicada al estudio de los radios sucesivos de cuerpos convexos, tanto de familias particulares de conjuntos, como su relación con uno de las nociones elementales para los conjuntos del espacio euclídeo: la suma vectorial o *de Minkowski*. A continuación vamos a describir el contenido específico de cada uno de los cuatro capítulos en los que se ha estructurado este trabajo.

La memoria comienza con un *primer capítulo* introductorio, en el que se establece la notación a seguir y se presentan brevemente los conceptos y resultados que serán fundamentales en el posterior desarrollo de los contenidos, tanto de convexidad general, como de los radios sucesivos en particular. De este modo, la *primera sección* está dedicada a recordar nociones básicas como la de p-bola, suma de Minkowski, politopo, función e hiperplano soportes, cuerpo polar, métrica de Hausdorff, etc., así como algunas de sus propiedades fundamentales. Un concepto especialmente importante será el de *suma de Firey* o *p-suma* de dos cuerpos convexos, que tiene la suma de Minkowski como un caso particular ($p = 1$).

En la *segunda sección* del capítulo, introducimos formalmente todas las familias existentes de radios sucesivos, estudiando sus principales propiedades. En particular, llevamos a cabo un estudio exhaustivo de la continuidad de estos funcionales (respecto a la métrica de Hausdorff), dado que no hemos podido encontrar en la literatura específica ningún resultado o referencia a esta propiedad; de hecho, todas las familias de radios sucesivos resultan ser continuas *excepto*, sorprendentemente, una de ellas: los radios interiores r_i definidos al comienzo de este resumen. Concluimos el capítulo con una *tercera sección* dedicada a recordar algunas desigualdades y relaciones entre distintos radios sucesivos de un cuerpo convexo que serán de utilidad en el desarrollo posterior del trabajo.

A continuación pasamos a describir el contenido del *segundo capítulo*, en el que nos centramos en el estudio de los radios sucesivos de familias particulares de cuerpos convexos. En la *primera sección* presentamos resultados conocidos sobre radios sucesivos de familias de conjuntos, tales como elipsoides, cajas y crosspolitopos ortogonales y símplices regulares. Como ya se ha mencionado al comienzo de la introducción, si consideramos el operador identidad entre dos espacios normados, uno de ellos dotado con la norma euclídea, entonces los números de Gelfand y Kolmogorov coinciden con los radios sucesivos de la bola unidad del otro espacio. En la *segunda sección* estudiamos en profundidad y demostramos la conexión existente entre estos números y los radios sucesivos de un cuerpo simétrico respecto al origen, lo que va a permitir obtener, en la *tercera sección* del capítulo, el valor preciso, en unos casos, y cotas, en otros, de los radios sucesivos de cualquier dilatación ortogonal de las p-bolas unidad, para $1 \leq p \leq \infty$. Incluimos también una demostración, desde el punto de vista geométrico, de estos resultados.

Es un hecho bien conocido que si K es un cuerpo convexo de anchura constante, es decir, tal que $\mathrm{D}(K) = \omega(K)$, entonces se verifica que $\mathrm{D}(K) = \mathrm{r}(K) + \mathrm{R}(K)$. En la *cuarta sección* estudiamos los radios sucesivos de este tipo de conjuntos, y nos planteamos si es posible generalizar la propiedad anterior a los demás radios sucesivos, lo que, de hecho, va a depender de los radios involucrados.

Finalmente, consideramos la familia de cuerpos denominados conjuntos *p-tangenciales* de la bola euclídea, demostrando que cumplen la propiedad de ser $\{\mathrm{r}_{p+1},\dots,\mathrm{r}_{n-1}\}$-*isoradiales.*

En el *tercer capítulo* consideramos la desigualdad $(*)$ de Pukhov y Perel'man, la cual estudiamos con detalle en la *primera sección.* A continuación, nos planteamos un problema de *tipo Pukhov-Perel'man* para otra familia de radios sucesivos interiores, es decir, acotar superiormente la razón $\mathrm{R}_{n-i+1}(K)/\widetilde{\mathrm{r}}_i(K)$, donde $\widetilde{\mathrm{r}}_i(K)$ se define como el máximo de los inradios de proyecciones i-dimensionales de K. Haciendo uso de la estimación de este cociente, en la *tercera sección* conseguimos mejorar la desigualdad de Pukhov-Perel'man en el caso de un cuerpo convexo de $\mathbb{R}^3$ simétrico respecto al origen, para lo cual estudiamos la relación existente entre los radios sucesivos interiores $\widetilde{\mathrm{r}}_2(K)$ y $\mathrm{r}_2(K)$. Al final de la sección, y haciendo uso de las técnicas desarrolladas por Perel'man, probamos que, para todo cuerpo convexo $K\subset\mathbb{R}^n$, $n\geq 3$, se tiene que

$$\frac{\mathrm{R}_{n-1}(K)}{\mathrm{r}_2(K)}\leq 2\sqrt{2}\sqrt{\frac{n-1}{n}},$$

lo que mejora la cota de Pukhov y Perel'man en el caso general cuando $i=2$. Concluimos el capítulo estableciendo una estimación de la cota superior del producto $\mathrm{r}_i(K)\mathrm{R}_i(K^*)$ para cuerpos convexos con el origen en su interior (la cota inferior ya era conocida).

Los resultados centrales de esta memoria se recogen en el *cuarto* y último capítulo. Dados dos cuerpos convexos K,K', resulta natural preguntarse cómo se relacionan los radios sucesivos de su suma de Minkowski, $K+K'$, con los de K y K'. De hecho, el comportamiento del diámetro, la anchura mínima, el circunradio y el inradio a este respecto es bien conocido, a saber,

$$\begin{aligned}\mathrm{D}(K+K')&\leq\mathrm{D}(K)+\mathrm{D}(K'),\quad &\omega(K+K')&\geq\omega(K)+\omega(K'),\\ \mathrm{R}(K+K')&\leq\mathrm{R}(K)+\mathrm{R}(K'),\quad &\mathrm{r}(K+K')&\geq\mathrm{r}(K)+\mathrm{r}(K').\end{aligned}$$

Así, en las *dos primeras secciones* hemos obtenido las cotas inferiores (óptimas) para los radios sucesivos $\mathrm{R}_i(K+K')$ (respectivamente, $\mathrm{r}_i(K+K')$) en función de $\mathrm{R}_i(K)$ y $\mathrm{R}_i(K')$ (respectivamente, $\mathrm{r}_i(K)$ y $\mathrm{r}_i(K')$). Se demuestra además que, salvo en el caso del circunradio $\mathrm{R}=\mathrm{R}_n$ (respectivamente, el diámetro $\mathrm{D}=2\mathrm{r}_1$), no existe cota superior posible. Este "problema" desaparece cuando se consideran sumas particulares de cuerpos (por ejemplo, cuando uno de los sumandos es la bola euclídea o cuando se suma un cuerpo convexo K con su opuesto $-K$), lo que estudiamos en la *tercera sección* del capítulo. La construcción llevada a cabo en el caso de la bola euclídea va a permitir además obtener como una sorprendente consecuencia que los radios sucesivos r_i no son funcionales continuos cuando $2\leq i\leq n-1$.

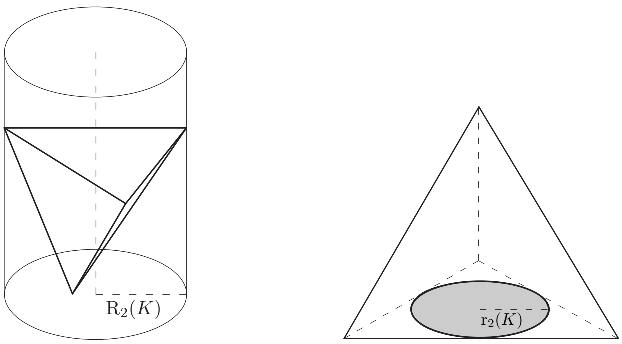
A continuación realizamos un estudio análogo al de las primeras secciones de este cuarto capítulo, considerando ahora un tipo de suma de cuerpos convexos más general: la p-suma $K+_pK'$ de dos conjuntos K y K'. Ésta se define a partir de la función soporte, y generaliza la suma de Minkowski, ya que esta última no es otra cosa que la 1-suma de cuerpos convexos. La p-suma de conjuntos fue introducida por Firey en 1962 ([19]), y ha dado lugar a una importante teoría conocida hoy en día como la Teoría de Brunn-Minkowski-Firey. Así, en la *cuarta sección* obtenemos todas las cotas posibles (óptimas) de los radios sucesivos $\mathrm{R}_i(K +_p K')$ y $\mathrm{r}_i(K +_p K')$ en términos de los correspondientes funcionales de K y K'.

Para concluir el capítulo y completar el estudio de los radios sucesivos respecto a la suma de Minkowski, en la *quinta y última sección* demostramos los resultados correspondientes para las restantes familias de radios sucesivos definidas en el primer capítulo.

Los resultados originales que aparecen recogidos en esta memoria pueden encontrarse en los trabajos [10, 22, 23, 24, 25, 26].

Throughout this dissertation, we will mainly focus our efforts in studying the behavior of the so-called *successive radii* of a convex body (compact and convex set). First, we would like to remark that the *circumradius* R, the *inradius* r, the *diameter* D and the *minimal width* ω are particular cases of these functionals, and almost certainly the best known and studied.

Successive radii are defined as the maximum or the minimum of the circumradii and the inradii of projections or sections of the convex body. This allows to define altogether seven families of successive radii, although we will mainly focus in two of them: the "classic" R_i and r_i, $i = 1, \ldots, n$. These two functionals admit an alternative geometric definition: the *i-th successive outer radii* $\mathrm{R}_i(K)$ is the smallest radius of a solid cylinder with i-dimensional spherical cross section containing K, whereas the *i-th successive inner radii* $\mathrm{r}_i(K)$ is the radius of the greatest i-dimensional ball contained in K.



The successive radii R_2 y r_2 of the regular tetrahedron

Although several mathematicians had previously obtained results on some particular radii, the first systematic study of these functionals was made by Betke and Henk [2] in 1992.

But these radii have not been only studied in Convex Geometry: during the second third of the twentieth century, successive radii appeared in Banach Space Theory and Approximation Theory, known as *Gelfand and Kolmogorov numbers*. These are values associated to bounded linear functionals between Banach spaces, which coincide with the successive radii of a convex body K, symmetric with respect to the origin, when identity operators of $\mathbb{R}^n$ are considered between finite dimensional normed spaces endowed with the norm of unit ball K and the Euclidean norm $|\cdot|_2$. In particular, from this close relation, known properties of the Gelfand and Kolmogorov numbers can be translated to the successive radii. Unfortunately, the link between these two theories comes to the end when a non-0-symmetric convex body K comes into play. It is due to the only compact convex sets that can be the unit ball of a Banach space are, precisely, the 0-symmetric ones.

In fact, one of the convex bodies playing an important role in the study of the successive radii, is not 0-symmetric: the *n-dimensional simplex* S_n. The measures $\mathrm{R}(S_n)$, $\mathrm{r}(S_n)$, $\mathrm{D}(S_n)$ and $\omega(S_n)$ of the simplex can be easily obtained; however, the computation of its successive radii is not easy at all and has been studied throughout the years by many mathematicians (see [1, 8, 38, 51, 56, 57, 58]).

One of the most relevant results in the study of the successive radii, as well as still an open problem in this field, is an inequality independently proved by Pukhov (1979, see [50]) and Perel'man (1987, see [45]). They showed that if K is an n-dimensional convex body, then it holds that

$$\frac{\mathrm{R}_{n-i+1}(K)}{\mathrm{r}_i(K)} < i+1, \qquad \text{for all } \ 1 \leq i \leq n. \tag{$\dagger$}$$

Moreover, Pukhov proved that if we restrict to the family of 0-symmetric convex bodies, the above bound can be replaced by $\sqrt{e}\min\{\sqrt{i}, \sqrt{n-i+1}\}$, whereas Perel'man showed that in the particular case of the 3-dimensional Euclidean space $\mathbb{R}^3$ (with $i=2$), inequality (†) can be improved to $\mathrm{R}_2(K)/\mathrm{r}_2(K) < 2.151\ldots$ The cases $i=1$ and $i=n$ correspond, respectively, to the classical theorems of Jung (1901, see [38]) and Steinhagen (1921, see [56]), where the optimal bounds, namely,

$$\frac{\mathrm{R}_n(K)}{\mathrm{r}_1(K)} \leq \sqrt{\frac{2n}{n+1}} \qquad \text{and} \qquad \frac{\mathrm{R}_1(K)}{\mathrm{r}_n(K)} \leq \begin{cases} \sqrt{n} & \text{for } n \text{ odd,} \\ \frac{n+1}{\sqrt{n+2}} & \text{for } n \text{ even,} \end{cases}$$

are far away from those obtained in (†). On the other hand, the regular simplex is an extremal set for Jung and Steinhagen's inequalities (i.e., equality is obtained). It is conjectured that for any other index $i = 2, \ldots, n-1$, the regular simplex also gives the optimal bound in (†).

There exist many important results studying different properties of the successive radii of a convex body or relating them with other geometric measures, although we will not deal with them in this work. We mention, for instance, the generalizations of Jung's theorem ([32]) and Steinhagen's theorem ([3]), the relation between the successive radii and the volume or the intrinsic volumes ([2, 33]), or computational aspects of these functionals ([28, 29]).

We can say, roughly speaking, that the thesis is devoted to the study of the successive radii of convex bodies regarding both, particular families of sets, and the relation with one of the elementary notions in Convexity: the vectorial or *Minkowski* addition. Next we are going to describe the specific contents of each chapter in which this dissertation has been organized.

The work starts with an introductory *first chapter* in which we establish the notation and introduce the concepts and results that will be needed further on, both about general Convexity and, in particular, about successive radii. Thus, in the *first section* the basic notions such as p-balls, Minkowski addition, polytopes, supporting hyperplane, support function, polar body, Hausdorff metric, etc., are recalled, as well as some of their fundamental properties. A particularly important concept will be the *Firey addition* or *p-sum* of two convex bodies, which includes the Minkowski addition as a particular case ($p = 1$).

In the *second section* of the chapter, we define all different families of successive radii and study their main properties. In particular, we carefully study the continuity of these functionals (with respect to the Hausdorff metric), because we have been unable to find any result or reference to this property in the specialized literature; in fact, all families of successive radii turn out to be continuous *except*, surprisingly, one of them: the inner radii r_i which were defined at the beginning of this summary. We finish the chapter with a *third section* devoted to recall some inequalities and relations between different successive radii of a convex body, that will be useful in the further development of the work.

Next we describe the contents of the *second chapter*, which we focus in the study of the successive radii of particular families of convex bodies. In the *first section* we list the known results on successive radii of families of sets, such as ellipsoids, orthogonal boxes and crosspolytopes, and regular simplices. As we already mentioned at the beginning of the introduction, if we consider the identity operator between two finite dimensional normed spaces, one of them endowed with the Euclidean norm, then the Gelfand and Kolmogorov numbers coincide with the successive radii of the unit ball of the other space. In the *second section* we deeply study and prove the connection between these numbers and the successive radii of a 0-symmetric body. This will allow to get, in the *third section* of the chapter, the precise value (in some cases) or bounds for the successive radii of any orthogonally dilated image of the unit p-balls, for $1 \leq p \leq \infty$. We will also include a geometrical proof of these results.

It is well known that if K is a constant width set, i.e., such that $\mathrm{D}(K) = \omega(K)$, then it holds $\mathrm{D}(K) = \mathrm{r}(K) + \mathrm{R}(K)$. In the *fourth section* we study the successive radii of this type of sets, wondering if there exists a generalization of the above property to the remaining successive radii. We see that this will depend on the involved radii. Finally, we consider the so-called *p-tangential bodies* of the Euclidean unit ball, and prove that they satisfy a very particular property called $\{\mathrm{r}_{p+1}, \dots, \mathrm{r}_{n-1}\}$-*isoradiality*.

In the *third chapter* we consider the Pukhov-Perel'man inequality (†), which we carefully recall in the *first section*. Next, we study a *Pukhov-Perel'man type* inequality for a different family of successive inner radii, i.e., to bound by above the quotient $\mathrm{R}_{n-i+1}(K)/\widetilde{\mathrm{r}}_i(K)$, where $\widetilde{\mathrm{r}}_i(K)$ is defined as the maximum of the inradii of all i-dimensional projections of K. Using this bound, in the *third section* we improve the Pukhov-Perel'man inequality in the case of a 0-symmetric convex body in $\mathbb{R}^3$, for which we have to study the relation between the inner radii $\widetilde{\mathrm{r}}_2(K)$ and $\mathrm{r}_2(K)$. At the end of the section, following the original idea of the proof of Perel'man for dimension 3, but slightly modifying some steps, we show that for any convex body $K \subset \mathbb{R}^n$, $n \geq 3$, it holds that

$$\frac{\mathrm{R}_{n-1}(K)}{\mathrm{r}_2(K)} \leq 2\sqrt{2}\sqrt{\frac{n-1}{n}},$$

which improves the bound in (†), in the general case, when $i = 2$. We conclude the chapter establishing an estimate for the upper bound of the product $\mathrm{r}_i(K)\mathrm{R}_i(K^*)$ for convex bodies containing the origin in the interior (the lower bound is already known).

The main results of this work are collected in the *fourth* and last chapter. Given two convex bodies K and K', it is a natural question to ask how the successive radii of their Minkowski addition, $K + K'$, are related to the radii of K and K'. Indeed, the behavior of the diameter, the minimal width, the circumradius and the inradius in this respect is well known, namely,

$$\begin{aligned} \mathrm{D}(K+K') &\leq \mathrm{D}(K) + \mathrm{D}(K'), \quad & \omega(K+K') &\geq \omega(K) + \omega(K'), \\ \mathrm{R}(K+K') &\leq \mathrm{R}(K) + \mathrm{R}(K'), \quad & \mathrm{r}(K+K') &\geq \mathrm{r}(K) + \mathrm{r}(K'). \end{aligned}$$

Thus, in the *first two sections* we obtain the (optimal) lower bounds for the outer successive radii $\mathrm{R}_i(K+K')$ (respectively, $\mathrm{r}_i(K+K')$) in terms of $\mathrm{R}_i(K)$ and $\mathrm{R}_i(K')$ (respectively, $\mathrm{r}_i(K)$ and $\mathrm{r}_i(K')$). We also prove that, except in the case of the circumradius $\mathrm{R} = \mathrm{R}_n$ (respectively, the diameter $\mathrm{D} = 2\mathrm{r}_1$), there exists no upper bound. This "problem" disappears when particular sums of convex bodies are considered (for instance, if one of the summands is the Euclidean ball or when we take the Minkowski addition of a convex body K and its opposite $-K$), which will be studied in the *third section* of the chapter. The construction carried out in the case of the Euclidean ball will allow to obtain, as a surprising consequence, that the inner radii r_i are not continuous functionals if $2 \leq i \leq n-1$.

Then we consider the analogous problem to the one studied in the first sections of this fourth chapter, but involving a more general type of addition of convex bodies: the p-sum $K +_p K'$ of two sets K and K'. It is defined by means of the support function, and generalizes the Minkowski addition since the latter is just the 1-sum of convex bodies. The p-sum of sets was introduced by Firey in 1962 ([19]), and has given rise to a theory nowadays known as the Brunn-Minkowski-Firey theory. Thus, in the *fourth section*, we obtain all possible (optimal) bounds for the successive radii $\mathrm{R}_i(K +_p K')$ and $\mathrm{r}_i(K +_p K')$ in terms of the corresponding functionals of K and K'.

In order to conclude the chapter and complete the study of the successive radii with respect to the Minkowski addition, in the *fifth and last section* we prove the corresponding results for the remaining families of successive radii defined in the first chapter.

The original results which are contained in this dissertation can be found in the papers [10, 22, 23, 24, 25, 26].

Chapter 1

Preliminaries. The successive radii

The first chapter is devoted to make a brief survey of the notions and results in convexity that we will need throughout the dissertation. More precisely, in the first section we introduce basic definitions and properties in Convexity. In the rest of the chapter, we define the main concept in this work, the successive radii, and study all properties and inequalities among them that will be needed in the other chapters.

1.1 Convex bodies. Basic properties

Throughout this dissertation, we will use the following standard notation. We write $\mathbb{R}^n$ to denote the n-dimensional Euclidean space, endowed with the standard inner product $\langle \cdot, \cdot \rangle$ and the Euclidean norm $|\cdot|_2$, and we denote by $\{\mathrm{e}_1, \dots, \mathrm{e}_n\}$ the standard basis of $\mathbb{R}^n$. If $x, y \in \mathbb{R}^n$, we write $[x, y]$ for the line segment with endpoints x, y. The *closure* of a set $C \subseteq \mathbb{R}^n$ is denoted by $\mathrm{cl}\, C$, its *boundary* by $\mathrm{bd}\, C$ and its interior by $\mathrm{int}\, C$. The *dimension* of C, i.e., the dimension of the smallest affine subspace containing C (its *affine hull*, $\mathrm{aff}\, C$) is denoted by $\dim C$, and we write $\mathrm{relbd}\, C$ to denote its relative boundary, i.e., the boundary of C relative to its affine hull.

The set of all i-dimensional subspaces in $\mathbb{R}^n$, $1 \leq i \leq n$, is represented by $\mathcal{L}_i^n$, and for $L \in \mathcal{L}_i^n$, $L^\perp$ denotes its orthogonal complement. For a set $C \subseteq \mathbb{R}^n$ and $L \in \mathcal{L}_i^n$, we denote by $C|L$ the orthogonal projection of C onto L.

The following definitions and properties are well known and can be found in any book on Convexity, for instance [5, 16, 30, 53].

Definition 1.1.1. *A (non-empty) set $C \subseteq \mathbb{R}^n$ is said to be* convex *if, whenever two points $x, y \in C$, then $\lambda x + (1-\lambda)y \in C$, for all $0 \leq \lambda \leq 1$. A* convex body *$K \subset \mathbb{R}^n$ is a compact convex set.*

From now on $\mathcal{K}^n$ will denote the set of all convex bodies in $\mathbb{R}^n$. The subset of $\mathcal{K}^n$ consisting of all convex bodies containing the origin 0 is denoted by $\mathcal{K}^n_0$. Let B_n be the (closed) unit Euclidean ball centered at the origin and $\mathbb{S}^{n-1} = \left\{u \in \mathbb{R}^n : |u|_2 = 1\right\}$ be the $(n-1)$-dimensional unit sphere. If $L \in \mathcal{L}^n_i$ then we denote by $B_{i,L}$ the unit i-dimensional Euclidean ball in L, or in other words, $B_{i,L} = B_n \cap L$. Moreover, for $p \geq 1$ let B^p_n be the unit p-ball associated to the p-norm $|\cdot|_p$, i.e.,

$$B^p_n = \left\{ x = (x_1, \dots, x_n)^\intercal \in \mathbb{R}^n : |x|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p} \leq 1 \right\},$$

with $|x|_\infty = \max\{|x_i| : i = 1, \dots, n\}$ (see Figure 1.1). In particular we have $B^2_n = B_n$ and, for $L \in \mathcal{L}^n_i$, we will write $B^p_{i,L} = B^p_n \cap L$ and $B^2_{i,L} = B_{i,L}$.

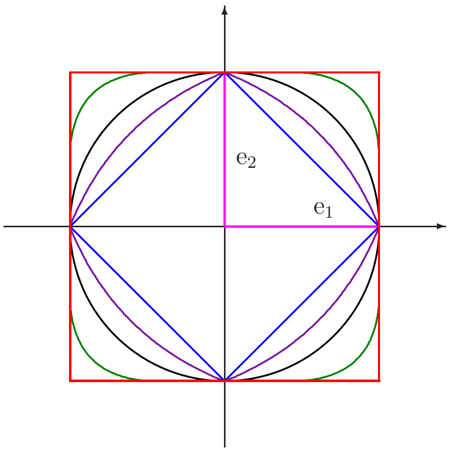


Figure 1.1: The 2-dimensional unit p-balls in the cases $1 \leq p \leq 2 \leq q \leq \infty$

Definition 1.1.2. *Let $C, C' \subseteq \mathbb{R}^n$. The* Minkowski addition *of C and C' is defined by*

$$C + C' = \{x + y : x \in C \text{ and } y \in C'\}.$$

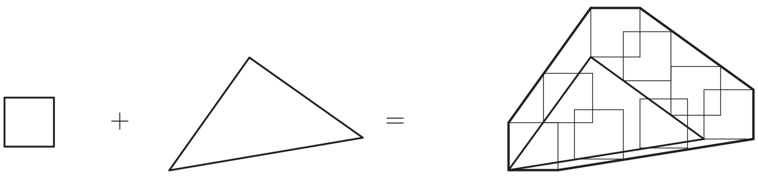


Figure 1.2: The Minkowski addition of a square and a triangle

If $K, K' \in \mathcal{K}^n$, then $K + K'$ is clearly a convex body, and we write $\lambda K = \{\lambda x : x \in K\}$, for $\lambda \in \mathbb{R}$. For the sake of brevity we will write $K - K' = K + (-K')$. The special case of the Minkowski sum $K - K$ is called the *difference body* of K, which is always a 0-symmetric convex body ($K \in \mathcal{K}^n$ is said to be 0-symmetric if $K = -K$). Moreover, the so called *central symmetral* of $K \in \mathcal{K}^n$ is defined as $K^0 = (K - K)/2$ (see [5, p. 79]).

For every $C \subseteq \mathbb{R}^n$, there exists a convex set containing it. The intersection of all convex sets containing C is the *convex hull* of C, and it will be denoted by $\operatorname{conv} C$; thus $\operatorname{conv} C$ is the smallest convex set containing C. The convex hull of a compact set is always a convex body; in particular, the convex hull of a finite number of points is so, and the family of all of them represents a very important class of convex bodies:

Definition 1.1.3. *A* polytope *is the convex hull of finitely many points in* $\mathbb{R}^n$ *(its* vertices*).*

An *n-dimensional simplex* is the convex hull of $n+1$ affinely independent points. In particular, we will denote by S_n the n-dimensional regular simplex given by

$$S_n = -\frac{1}{n+1}(1, \dots, 1)^\intercal + \operatorname{conv}\{\mathrm{e}_1, \dots, \mathrm{e}_{n+1}\},$$

embedded into the hyperplane $\left\{x = (x_1, \dots, x_{n+1})^\intercal \in \mathbb{R}^n : \sum_{j=1}^{n+1} x_j = 0\right\}$.

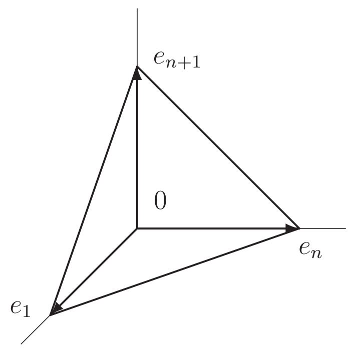


Figure 1.3: Obtaining the regular simplex S_2 embedded into $\mathbb{R}^3$

A *parallelotope* is the Minkowski sum of a finite number of linearly independent line segments. In the case that those segments are pairwise orthogonal, we call it an *orthogonal box*. In particular, we denote by C_n the n-dimensional cube of edge-length 2, that is

$$C_n = [-\mathrm{e}_1, \mathrm{e}_1] + \cdots + [-\mathrm{e}_n, \mathrm{e}_n].$$

An *n-dimensional crosspolytope* is the convex hull of n linearly independent vectors and their opposites, and it is called *orthogonal crosspolytope* when those vectors are pairwise orthogonal. In particular, we denote by C_n^* the regular crosspolytope

$$C_n^* = \operatorname{conv}\{\pm \mathrm{e}_i : i = 1, \dots, n\}.$$

Definition 1.1.4. *Let $K \in \mathcal{K}^n$. A point $z \in K$ is an extreme point of K if it cannot be written in the form $z = \lambda x + (1-\lambda)y$ with $x, y \in K$ and $\lambda \in (0,1)$.*

Thus, for instance, all vertices in a polytope are extreme points.

The space of convex bodies $\mathcal{K}^n$ is endowed with the *Hausdorff metric*, namely,

$$\delta_{\mathcal{H}}(K,K') = \min\bigl\{\lambda \geq 0 : K \subseteq K' + \lambda B_n,\, K' \subseteq K + \lambda B_n\bigr\} \quad \text{for } K, K' \in \mathcal{K}^n,$$

which allows to consider continuity of functionals defined on $\mathcal{K}^n$ as well as a distance between convex bodies. We now present the famous *Blaschke selection theorem*, which provides a very useful tool for proving the existence of convex bodies with specific properties.

Theorem 1.1.1 (Blaschke selection theorem). *Any bounded sequence of convex bodies in $\mathbb{R}^n$ contains a subsequence converging to a convex body (in the Hausdorff metric).*

Therefore, the pair $(\mathcal{K}^n, \delta_{\mathcal{H}})$ is a complete metric space.

In spite of the fact that many of the following properties and definitions are valid for closed convex sets, in order to simplify the exposition we will restrict them to compact ones, since we will always work under the hypothesis of compactness.

Definition 1.1.5. *Let $K \in \mathcal{K}^n$. A hyperplane H is called a* supporting hyperplane *of the set K if $H \cap K \neq \emptyset$ and K is contained in one of the two halfspaces determined by H, which is called the* supporting halfspace *of K.*

The following classical result concerning supporting hyperplanes will be needed in the following.

Theorem 1.1.2. *At every point of the boundary of a convex body $K \in \mathcal{K}^n$ there exists a supporting hyperplane of K. Furthermore, for every $u \in \mathbb{S}^{n-1}$ there is a supporting hyperplane of K with outer normal vector u.*

Let $K, K' \in \mathcal{K}^n$. A hyperplane H *separates* K and K' if $K \subset H^-$ and $K' \subset H^+$ or viceversa. Here H^- and H^+ denote the two closed halfspaces bounded by H. The sets K, K' are said to be *strictly separated* by H if $K \subset \operatorname{int} H^-$ and $K' \subset \operatorname{int} H^+$ or viceversa.

Definition 1.1.6. *The* support function *of a convex body $K \in \mathcal{K}^n$ in the direction $u \in \mathbb{R}^n$, denoted by $h(K,u)$ is the real valued function defined by*

$$h(K,u) = \max\bigl\{\langle x, u\rangle : x \in K\bigr\}.$$

The intuitive geometrical meaning of the support function is simple. For a unit vector $u \in \mathbb{S}^{n-1}$, $h(K,u)$ is the signed distance of the supporting hyperplane to K with exterior normal vector u from the origin; moreover, the distance is negative if and only if u points into the open halfspace containing the origin.

Some other properties of the support function are collected in the next proposition.

Proposition 1.1.1. *Let $K, K' \in \mathcal{K}^n$ and $u, v \in \mathbb{S}^{n-1}$.*

1. $h(K+K',u) = h(K,u) + h(K',u)$ *and* $h(\lambda K, u) = \lambda h(K,u)$ *for all* $\lambda \geq 0$.
2. $K \subseteq K'$ *if and only if* $h(K,u) \leq h(K',u)$.
3. $h(K, u+v) \leq h(K,u) + h(K,v)$ *and* $h(K,\lambda u) = \lambda h(K,u)$ *for all* $\lambda \geq 0$. *Hence,* $h(K,\cdot)$ *is a convex function.*
4. $h(K,-u) = h(-K,u)$.
5. *The value of the support function of* K *at any vector is attained in an extreme point.*

An important consequence of the first above property is that $(\mathcal{K}^n, +)$ has the cancellation law, i.e., the equality $K + M = K' + M$ for convex bodies $K, K', M \in \mathcal{K}^n$ implies $K = K'$.

Using the support function, in [19] Firey introduced the following binary operation between two convex bodies, generalizing the usual Minkowski addition.

Definition 1.1.7. *Let $p \geq 1$ and $K, K' \in \mathcal{K}^n_0$. The* p-sum *(or* Firey addition*) of K and K' is the unique convex body $K +_p K'$ for which the support function*

$$h(K +_p K', \cdot)^p = h(K,\cdot)^p + h(K',\cdot)^p. \tag{1.1}$$

If $p = 1$, formula (1.1) defines the usual Minkowski sum $K + K'$, and for $p = \infty$ it holds that $h(K +_\infty K', u) = \max\{h(K,u), h(K',u)\}$, i.e.,

$$K +_\infty K' = \mathrm{conv}(K \cup K').$$

Moreover, in [19, Theorem 1] it is shown that for all $1 \leq p \leq q \leq \infty$,

$$K +_q K' \subseteq K +_p K'. \tag{1.2}$$

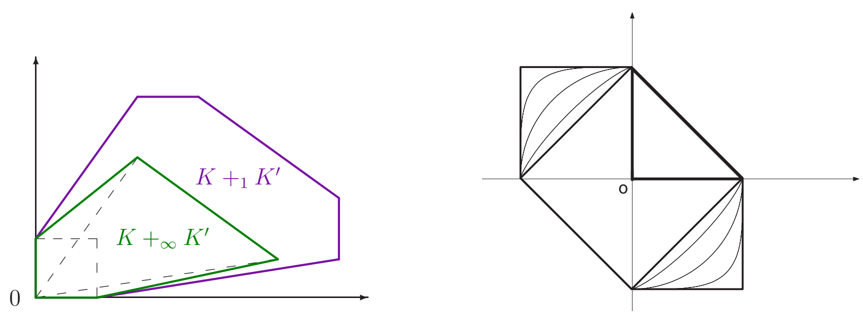


Figure 1.4: 1-sum and ∞-sum of a square and a triangle (left); p-sum of a triangle and its opposite (right)

In [19, Theorem 1] it was also shown that for any $1 \leq p \leq \infty$,

$$\frac{1}{2^{(p-1)/p}}(K + K') \subseteq K +_p K' \subseteq K + K'. \tag{1.3}$$

We observe that for the p-sums of sets, except in the case $p = 1$, the translation invariance is lost.

In [42, 43] Lutwak studied p-sums of convex bodies systematically, and developed a theory nowadays known as Brunn-Minkowski-Firey theory. In the last years many important developments of this theory have come out; we mention e.g. [4, 13, 44] and the references inside.

In this point we would like to notice that usually p-sums are defined for convex bodies containing the origin as a relative interior point, since this condition is needed in some aspects of the Brunn-Minkowski-Firey theory; however, regarding the functionals we are working on, this condition can be withdrawn, and thus we allow the origin to lie on the boundary of the convex bodies.

Definition 1.1.8. *Let $K \in \mathcal{K}^n$ with $0 \in \operatorname{int} K$. The* polar *body of K is defined as the set*

$$K^* = \left\{x \in \mathbb{R}^n : \langle x, y\rangle \leq 1 \ \textit{for all} \ y \in K\right\}.$$

It can be easily proved that the polar of a convex body K containing the origin in its interior is also a convex and compact set with $0 \in \operatorname{int} K^*$. We enumerate some properties that we will use in the following sections. Let $K, K' \in \mathcal{K}^n$ with $0 \in \operatorname{int} K, \operatorname{int} K'$. Then:

1. Let M be an $(n \times n)$-regular matrix. Then $(MK)^* = M^{-\intercal}K^*$. In particular, $(\lambda K)^* = \lambda^{-1}K^*$ for all $\lambda \neq 0$.
2. If $K \subseteq K'$ then $(K')^* \subseteq K^*$.
3. $B_n = B_n^*$ and $K = K^*$ if and only if $K = B_n$.

We observe that the cube C_n and the crosspolytope C_n^* are dual to each other.

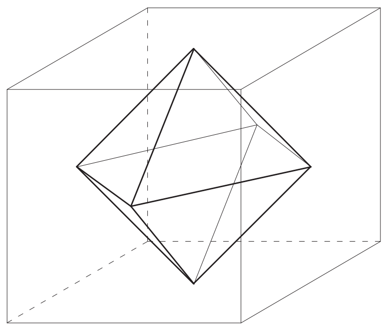


Figure 1.5: The cube C_n and its polar, the crosspolytope C_n^*

1.2 The successive radii. Definitions and basic properties

The smallest Euclidean ball that contains a convex body $K \in \mathcal{K}^n$ (respectively, one of the biggest Euclidean balls contained in K) is called the *circumball* (respectively, the *inball*) and its radius is the *circumradius* $\mathrm{R}(K)$ (respectively, the *inradius* $\mathrm{r}(K)$) of K. The greatest distance between two points of K is called the *diameter* of K, and is denoted by $\mathrm{D}(K)$. The value $h(K,u)+h(K,-u)$, $u\in\mathbb{S}^{n-1}$, is the distance between the two parallel supporting hyperplanes of K with outer normal vector u (the *width in the direction* u) and the smallest of all these values is called the *minimal width* of K, $\omega(K)$ (see Figure 1.6).

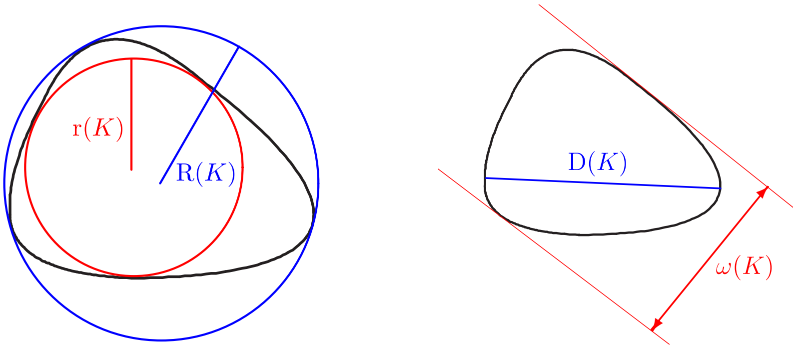


Figure 1.6: A convex body and its circumradius, inradius, diameter and (minimal) width

The first result we present is a characterization of the circumball and the inball of a convex body K. It makes easier to handle with the in- and circumradius and shows that the calculation of them is, in some sense, a discrete computation.

Theorem 1.2.1. *Let $K\in\mathcal{K}^n$, $\mathrm{r},\mathrm{R}>0$ and $z_1,z_2\in\mathbb{R}^n$ be such that $z_1+\mathrm{r}B_n\subseteq K\subseteq z_2+\mathrm{R}B_n$. The following conditions are equivalent:*

- *$z_1+\mathrm{r}B_n$ is the inball of K (respectively, $z_2+\mathrm{R}B_n$ is the circumball of K) and therefore $\mathrm{r}=\mathrm{r}(K)$ (respectively, $\mathrm{R}=\mathrm{R}(K)$);*
- *there exist contact points $x_1,\dots,x_j\in\mathrm{bd}\,K\cap\mathrm{bd}(z_1+\mathrm{r}B_n)$, $2\le j\le n+1$ (respectively, $y_1,\dots,y_k\in\mathrm{bd}\,K\cap\mathrm{bd}(z_2+\mathrm{R}B_n)$, $2\le k\le n+1$) such that $z_1\in\mathrm{conv}\{x_1,\dots,x_j\}$ (respectively, $z_2\in\mathrm{conv}\{y_1,\dots,y_k\}$).*

With all the notation above, eight families of *successive outer and inner radii* can be defined in terms of the circumradius and the inradius of sections and projections of a convex body. For it, if f is a functional on $\mathcal{K}^n$ depending on the dimension in which a convex body K is embedded, and if K is contained in an affine space A, then we write $f(K;A)$ to stress that f has to be evaluated with respect to the space A.

Definition 1.2.1. *Let $K\in\mathcal{K}^n$ and $i\in\{1,\dots,n\}$. Then*

$$\begin{aligned}
\mathrm{R}_i(K)&=\min_{L\in\mathcal{L}^n_i}\mathrm{R}(K|L), &\quad \mathrm{r}_i(K)&=\max_{L\in\mathcal{L}^n_i}\max_{x\in L^\perp}\mathrm{r}\bigl(K\cap(x+L);x+L\bigr),\\
\overline{\mathrm{R}}_i(K)&=\max_{L\in\mathcal{L}^n_i}\mathrm{R}(K|L), &\quad \overline{\mathrm{r}}_i(K)&=\min_{L\in\mathcal{L}^n_i}\max_{x\in L^\perp}\mathrm{r}\bigl(K\cap(x+L);x+L\bigr).
\end{aligned}\tag{1.4}$$

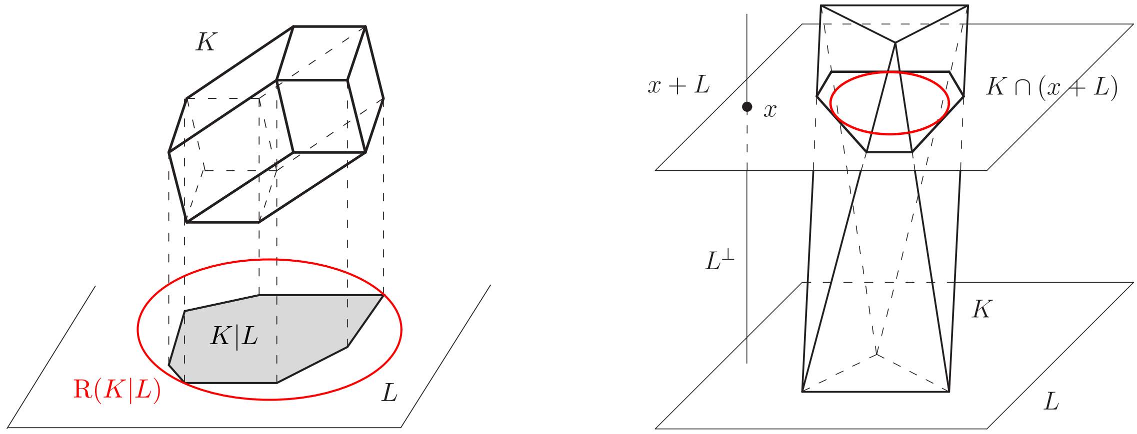


Figure 1.7: Successive radii defined via sections and projections

We observe that $\mathrm{R}_i(K)$ is the smallest radius of a solid cylinder with i-dimensional spherical cross section containing K, whereas $\mathrm{r}_i(K)$ is the radius of the greatest i-dimensional ball contained in K (see Figure 1.8).

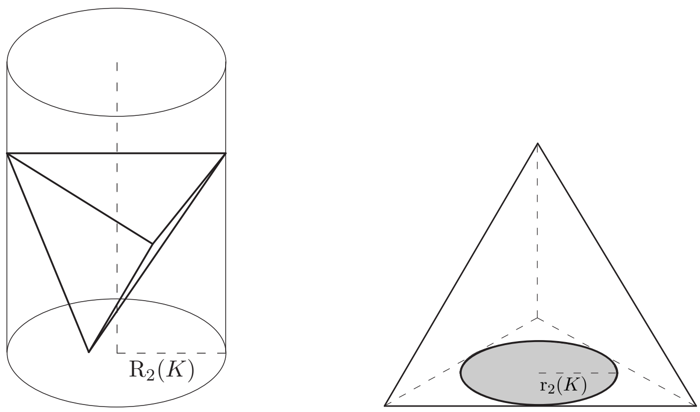


Figure 1.8: The successive radii R_2 and r_2 of a tetrahedron

If we replace projections by sections in Definition 1.2.1 (and viceversa) we get four other series of successive radii.

Definition 1.2.2. *Let $K \in \mathcal{K}^n$ and $i \in \{1, \ldots, n\}$.*

$$\begin{aligned} \widetilde{\mathrm{R}}_i(K) &= \min_{L \in \mathcal{L}_i^n} \max_{x \in L^\perp} \mathrm{R}\big(K \cap (x+L)\big), & \widetilde{\mathrm{r}}_i(K) &= \max_{L \in \mathcal{L}_i^n} \mathrm{r}(K|L;L), \\ \widehat{\mathrm{R}}_i(K) &= \max_{L \in \mathcal{L}_i^n} \max_{x \in L^\perp} \mathrm{R}\big(K \cap (x+L)\big), & \widehat{\mathrm{r}}_i(K) &= \min_{L \in \mathcal{L}_i^n} \mathrm{r}(K|L;L). \end{aligned} \tag{1.5}$$

The first systematic study of these successive outer and inner radii was made in [2].

It is already known (see [11, Theorem 3.3]) that $\overline{\mathrm{R}}_i(K)=\widehat{\mathrm{R}}_i(K)$ for all $i=1,\dots,n$. The first and last radius of each family coincides either with the circumradius, the inradius, the minimal width or the diameter, namely,

$$\begin{aligned} &\mathrm{R}(K)=\mathrm{R}_n(K)=\overline{\mathrm{R}}_n(K)=\widetilde{\mathrm{R}}_n(K), \quad \mathrm{r}(K)=\mathrm{r}_n(K)=\bar{\mathrm{r}}_n(K)=\tilde{\mathrm{r}}_n(K)=\widehat{\mathrm{r}}_n(K),\\ &\frac{\mathrm{D}(K)}{2}=\overline{\mathrm{R}}_1(K)=\mathrm{r}_1(K)=\tilde{\mathrm{r}}_1(K), \quad \frac{\omega(K)}{2}=\mathrm{R}_1(K)=\widetilde{\mathrm{R}}_1(K)=\bar{\mathrm{r}}_1(K)=\widehat{\mathrm{r}}_1(K). \end{aligned} \tag{1.6}$$

It is clear that all outer successive radii form an increasing sequence in i, i.e.,

$$\mathrm{R}_i(K)\leq\mathrm{R}_{i+1}(K), \quad \overline{\mathrm{R}}_i(K)\leq\overline{\mathrm{R}}_{i+1}(K) \quad \text{and} \quad \widetilde{\mathrm{R}}_i(K)\leq\widetilde{\mathrm{R}}_{i+1}(K),$$

$1\leq i\leq n-1$, whereas inner successive radii are decreasing in i. Moreover, they are all monotone and homogeneous functions of degree 1.

Next we deal with the continuity of the successive radii. Since we have been not able to find any remark/argument regarding this property, we include here the proofs for completeness. Surprisingly, not all of them will by continuous functionals on $(K^n,\delta_{\mathcal{H}})$.

1.2.1 On the continuity of the successive radii

We write

$$f:\big(\{K\in\mathcal{K}^n:\dim K=n\},\delta_{\mathcal{H}}\big)\longrightarrow\mathbb{R}_{\geq0}$$

to denote any successive radii of Definitions 1.2.1 and 1.2.2 restricted to the class of convex bodies with non-empty interior.

Proposition 1.2.1. *f is a continuous functional.*

Proof. Let $K\in\mathcal{K}^n$ with $\dim K=n$, and let $K_m\in\mathcal{K}^n$, $m\in\mathbb{N}$, be such that $\lim_{m\to\infty}K_m=K$ (in the Hausdorff metric). Then, $\delta_m=\delta_{\mathcal{H}}(K_m,K)\longrightarrow0$ when m goes to ∞. Since $\operatorname{int}K\neq\emptyset$ and both, successive radii and Hausdorff distance, are invariant by translations, we can assume that $\mathrm{r}B_n\subseteq K$ for some $\mathrm{r}>0$. Choosing m so large that $\delta_m<\mathrm{r}$, by the definition of $\delta_{\mathcal{H}}$ we have that

$$K_m\subseteq K+\delta_mB_n\subseteq\left(1+\frac{\delta_m}{\mathrm{r}}\right)K$$

and

$$\left(1-\frac{\delta_m}{\mathrm{r}}\right)K+\frac{\delta_m}{\mathrm{r}}K=K\subseteq K_m+\delta_mB_n\subseteq K_m+\frac{\delta_m}{\mathrm{r}}K,$$

i.e., $(1-\delta_m/\mathrm{r})K\subseteq K_m$. Therefore, the homogeneity and monotonicity of the radii imply that

$$\left(1-\frac{\delta_m}{\mathrm{r}}\right)f(K)\leq f(K_m)\leq\left(1+\frac{\delta_m}{\mathrm{r}}\right)f(K),$$

and thus, $\lim_{m\to\infty}f(K_m)=f(K)$, as required. □

However, not all radii are continuous functionals on the full domain $\mathcal{K}^n$. Indeed, in Chapter 4 it will be proved that the inner radius $\mathrm{r}_i : \mathcal{K}^n \longrightarrow \mathbb{R}_{\geq 0}$, for all $2 \leq i \leq n-1$, is not continuous (see Remark 4.3.3).

Next proposition shows that all other radii are continuous defined on $\mathcal{K}^n$.

Proposition 1.2.2. *Successive radii R_i, $\overline{\mathrm{R}}_i$, $\widetilde{\mathrm{R}}_i$, $\bar{\mathrm{r}}_i$, $\widetilde{\mathrm{r}}_i$ and $\widehat{\mathrm{r}}_i$ are continuous functionals on $\mathcal{K}^n$.*

Proof. Let $i \in \{1, \dots, n\}$ be fixed and let $K, K_m \in \mathcal{K}^n$, $m \in \mathbb{N}$, be such that $\lim_{m\to\infty} K_m = K$. Then it holds that

$$\begin{aligned} &\delta_m = \delta_{\mathcal{H}}(K_m, K) \longrightarrow 0 \quad \text{when } \ m \to \infty \ \text{ and} \\ &K \subseteq K_m + \delta_m B_n \quad \text{and} \quad K_m \subseteq K + \delta_m B_n, \quad \text{for all } \ m \in \mathbb{N}. \end{aligned} \tag{1.7}$$

First we show that the successive radii R_i, $\overline{\mathrm{R}}_i$, $\widetilde{\mathrm{r}}_i$ and $\widehat{\mathrm{r}}_i$ are continuous. Using (1.7) we get that, for every $L \in \mathcal{L}_i^n$, the circumradius verifies

$$\begin{aligned} \mathrm{R}(K_m|L) &\leq \mathrm{R}\big((K+\delta_m B_n)|L\big) = \mathrm{R}(K|L+\delta_m B_n|L) = \mathrm{R}(K|L)+\delta_m, \\ \mathrm{R}(K|L) &\leq \mathrm{R}\big((K_m+\delta_m B_n)|L\big) = \mathrm{R}(K_m|L+\delta_m B_n|L) = \mathrm{R}(K_m|L)+\delta_m \end{aligned}$$

(respectively, the analogous inequalities for the inradius r). Then, taking the minimum and the maximum over all $L \in \mathcal{L}_i^n$ in both sides of the above inequalities we get that

$$\begin{aligned} &\mathrm{R}_i(K_m) \leq \mathrm{R}_i(K)+\delta_m \quad \text{and} \quad \mathrm{R}_i(K) \leq \mathrm{R}_i(K_m)+\delta_m, \quad \text{and} \\ &\overline{\mathrm{R}}_i(K_m) \leq \overline{\mathrm{R}}_i(K)+\delta_m \quad \text{and} \quad \overline{\mathrm{R}}_i(K) \leq \overline{\mathrm{R}}_i(K_m)+\delta_m \end{aligned}$$

(respectively, the corresponding inequalities for $\widehat{\mathrm{r}}_i$ and $\widetilde{\mathrm{r}}_i$). Therefore, $\lim_{m\to\infty} \mathrm{R}_i(K_m) = \mathrm{R}_i(K)$ and $\lim_{m\to\infty} \overline{\mathrm{R}}_i(K_m) = \overline{\mathrm{R}}_i(K)$ (respectively, $\lim_{m\to\infty} \widehat{\mathrm{r}}_i(K_m) = \widehat{\mathrm{r}}_i(K)$ and $\lim_{m\to\infty} \widetilde{\mathrm{r}}_i(K_m) = \widetilde{\mathrm{r}}_i(K)$).

Next we prove the continuity of $\widetilde{\mathrm{R}}_i$, which needs more refined arguments. First we show that for any $\varepsilon > 0$ and $m \in \mathbb{N}$ large enough it holds

$$\widetilde{\mathrm{R}}_i(K_m) \leq \widetilde{\mathrm{R}}_i(K)+\varepsilon. \tag{1.8}$$

Since $\lim_{m\to\infty} K_m = K$, for every $k \in \mathbb{N}$ there exists $m_k \in \mathbb{N}$ such that $K_m \subseteq K + (1/k)B_n$ for all $m \geq m_k$, and therefore

$$\widetilde{\mathrm{R}}_i(K_m) \leq \widetilde{\mathrm{R}}_i\left(K+\frac{1}{k}B_n\right).$$

So, in order to prove (1.8) it suffices to show that

$$\widetilde{\mathrm{R}}_i\left(K+\frac{1}{k}B_n\right) \leq \widetilde{\mathrm{R}}_i(K)+\varepsilon \quad \text{for some } \ k \in \mathbb{N}. \tag{1.9}$$

We suppose the contrary and we will get a contradiction. Thus, we assume that

$$\widetilde{\mathrm{R}}_i\left(K+\frac{1}{k}B_n\right) > \widetilde{\mathrm{R}}_i(K)+\varepsilon \quad \text{for all } \ k \in \mathbb{N}. \tag{1.10}$$

On the one hand, let $L_0 \in \mathcal{L}_i^n$ be such that

$$\widetilde{\mathrm{R}}_i(K) = \max_{x \in L_0^\perp} \mathrm{R}\bigl(K \cap (x + L_0)\bigr), \tag{1.11}$$

for which it clearly holds

$$\max_{x \in L_0^\perp} \mathrm{R}\left(\left(K + \frac{1}{k}B_n\right) \cap (x + L_0)\right) \geq \widetilde{\mathrm{R}}_i\left(K + \frac{1}{k}B_n\right). \tag{1.12}$$

Moreover, for each $k \in \mathbb{N}$, there exists $x_k \in L_0^\perp$ such that

$$\max_{x \in L_0^\perp} \mathrm{R}\left(\left(K + \frac{1}{k}B_n\right) \cap (x + L_0)\right) = \mathrm{R}\left(\left(K + \frac{1}{k}B_n\right) \cap (x_k + L_0)\right). \tag{1.13}$$

Since the sequence of convex bodies $\bigl(\bigl(K + (1/k)B_n\bigr) \cap (x_k + L_0)\bigr)_k$ is bounded, there exists a subsequence converging to a convex body (see Theorem 1.1.1). Without loss of generality we can suppose it is the same sequence. So, let

$$\overline{K} = \lim_{k \to \infty} \left[\left(K + \frac{1}{k}B_n\right) \cap (x_k + L_0)\right].$$

We notice that every set in the sequence is an i-dimensional convex body which is contained in the affine subspace $x_k + L_0$, $k \in \mathbb{N}$. Therefore there exists $x_0 \in L_0^\perp$ such that $\overline{K} \subset x_0 + L_0$. Then, the continuity of the circumradius, (1.13), (1.12), (1.10) and (1.11) imply that

$$\mathrm{R}(\overline{K}) = \lim_{k \to \infty} \mathrm{R}\left(\left(K + \frac{1}{k}B_n\right) \cap (x_k + L_0)\right) \geq \widetilde{\mathrm{R}}_i(K) + \varepsilon = \max_{x \in L_0^\perp} \mathrm{R}\bigl(K \cap (x + L_0)\bigr) + \varepsilon. \tag{1.14}$$

On the other hand, there exist points $p_1, \ldots, p_j \in \overline{K}$ such that $\mathrm{R}(\overline{K}) = \mathrm{R}\bigl(\mathrm{conv}\{p_1, \ldots, p_j\}\bigr)$, $2 \leq j \leq i+1$ (see Theorem 1.2.1). Since $p_\ell \in \overline{K} = \lim_{k \to \infty}\Bigl[\bigl(K + (1/k)B_n\bigr) \cap (x_k + L_0)\Bigr]$, $1 \leq \ell \leq j$, there exist sequences $(y_k^\ell)_k \subset K$ and $(u_k^\ell)_k \subset B_n$ such that

$$p_\ell = \lim_{k \to \infty}\left(y_k^\ell + \frac{1}{k}u_k^\ell\right).$$

But since $\lim_{k \to \infty}(1/k)u_k^\ell = 0$, then the sequence $(y_k^\ell)_k$ converges for all $\ell = 1, \ldots, j$ and

$$p_\ell = \lim_{k \to \infty}\left(y_k^\ell + \frac{1}{k}u_k^\ell\right) = \lim_{k \to \infty} y_k^\ell \in K.$$

Therefore, $p_1, \ldots, p_j \in K \cap (x_0 + L_0)$ because $\overline{K} \subset x_0 + L_0$, and thus

$$\mathrm{R}(\overline{K}) = \mathrm{R}\bigl(\mathrm{conv}\{p_1, \ldots, p_j\}\bigr) \leq \mathrm{R}\bigl(K \cap (x_0 + L_0)\bigr),$$

which contradicts (1.14). It shows (1.9) and hence, for that value of k, there exists $m_k \in \mathbb{N}$ such that for all $m \geq m_k$ (1.8) holds.

Analogously it can be proved that for all $\varepsilon > 0$, there exist $m_{k'} \in \mathbb{N}$ such that if $m \geq m_{k'}$ then

$$\widetilde{\mathrm{R}}_i(K) \leq \widetilde{\mathrm{R}}_i(K_m) + \varepsilon. \tag{1.15}$$

Thus, taking $m_0 = \max\{m_k, m_{k'}\}$, for all $m \geq m_0$ it holds

$$\left|\widetilde{\mathrm{R}}_i(K_m) - \widetilde{\mathrm{R}}_i(K)\right| \leq \varepsilon,$$

which shows that $\lim_{m\to\infty} \widetilde{\mathrm{R}}_i(K_m) = \widetilde{\mathrm{R}}_i(K)$.

Finally we prove the continuity of the functional $\bar{\mathrm{r}}_i$, for which we can restrict to the class of convex bodies $K \in \mathcal{K}^n$ with $\operatorname{int} K = \emptyset$. Let $u \in \mathbb{R}^n$, $u \neq 0$, be such that $K \subset u^\perp$, and let $L \in \mathcal{L}_i^n$ with $u \in L$. Then, since $\dim\bigl(K \cap (x+L)\bigr) \leq i-1$, we have that

$$\mathrm{r}\bigl(K \cap (x+L); x+L\bigr) = 0$$

for every $x \in L^\perp$, and hence $\bar{\mathrm{r}}_i(K) = 0$. It remains to be shown that

$$\lim_{m\to\infty} \bar{\mathrm{r}}_i(K_m) = 0.$$

Again, using (1.7) we get that, for all $x \in L^\perp$,

$$\mathrm{r}\bigl(K_m \cap (x+L); x+L\bigr) \leq \mathrm{r}\bigl((K + \delta_m B_n) \cap (x+L); x+L\bigr) \leq \delta_m,$$

and hence $\max_{x\in L^\perp} \mathrm{r}\bigl(K_m \cap (x+L); x+L\bigr) \leq \delta_m$. Thus we conclude that $\lim_{m\to\infty} \bar{\mathrm{r}}_i(K_m) = 0$. $\square$

1.3 Inequalities for successive radii

Successive radii satisfy many inequalities. Here we collect some of the most relevant ones, which will be needed throughout the dissertation. Just from the definitions it can be seen that for $K \in \mathcal{K}^n$ and all $i = 1, \dots, n$ it holds

$$\widetilde{\mathrm{R}}_i(K) \leq \mathrm{R}_i(K) \leq \overline{\mathrm{R}}_i(K), \quad \bar{\mathrm{r}}_i(K) \leq \hat{\mathrm{r}}_i(K) \leq \tilde{\mathrm{r}}_i(K) \quad \text{and} \quad \mathrm{r}_i(K) \leq \tilde{\mathrm{r}}_i(K). \tag{1.16}$$

In this work we are mainly interested in the “classical” radii R_i and r_i. So, although many of the properties we are going to present are also known for the remaining successive radii we will just state them for these two particular functionals. The following relation holds between the inner radii r_i and the outer radii R_{n-i+1}.

Proposition 1.3.1 (Betke & Henk, [2]). *Let $K \in \mathcal{K}^n$ and $i \in \{1, \dots, n\}$. Then*

$$\mathrm{r}_i(K) \leq \mathrm{R}_{n-i+1}(K). \tag{1.17}$$

The inequality is best possible, as B_n shows. A famous open problem in this context is the one of determining the best upper bound for the ratio $\mathrm{R}_{n-i+1}(K)/\mathrm{r}_i(K)$. Up to now, only the bound $i+1$ is known (see Theorem 3.1.1 in Chapter 3).

Gritzmann and Klee (see [28, (1.2)]) studied the relation between the successive radii R_i and r_i of a convex body K and its polar body K^*.

Proposition 1.3.2 (Gritzmann & Klee, [28]). *Let $K \in \mathcal{K}^n$ be a 0-symmetric convex body, $0 \in \operatorname{int} K$, and $i \in \{1, \dots, n\}$. Then*

$$\mathrm{r}_i(K)\mathrm{R}_i(K^*) = 1 \quad \textit{and} \quad \mathrm{R}_i(K)\mathrm{r}_i(K^*) = 1. \tag{1.18}$$

If we remove the 0-symmetry condition from this property, Henk proved a generalization of the previous result (see [31, Proposition 2.3]).

Proposition 1.3.3 (Henk, [31]). *Let $K \in \mathcal{K}^n$ with $0 \in \operatorname{int} K$ and $i \in \{1, \dots, n\}$. Then*

$$\mathrm{r}_i(K)\mathrm{R}_i(K^*) \geq 1 \quad \textit{and} \quad \mathrm{R}_i(K)\mathrm{r}_i(K^*) \geq 1.$$

Regarding relations between successive radii and other measures (intrinsic volumes, roots of Steiner polynomials and successive minima), we refer for instance, to [2, 33, 34].

If we restrict to the particular cases $i = 1$ and $i = n$ (i.e., inequalities among r, D, ω and R, see (1.6)) many other inequalities are known. For instance, from (1.16) the trivial relations

$$2\mathrm{r}(K) \leq \omega(K) \leq \mathrm{D}(K) \leq 2\mathrm{R}(K) \tag{1.19}$$

are obtained for any $K \in \mathcal{K}^n$, which are best possible. For example, equality holds in the first and the third inequalities for any 0-symmetric convex body, and in the second one for constant width sets. The reverse inequalities are given by the theorems of *Steinhagen* and *Jung*.

Theorem 1.3.1 (Steinhagen, [56]). *Let $K \in \mathcal{K}^n$. Then*

$$\mathrm{r}(K) \geq \begin{cases} \frac{1}{2\sqrt{n}}\,\omega(K) & \textit{for } n \textit{ odd}, \\ \frac{\sqrt{n+2}}{2(n+1)}\,\omega(K) & \textit{for } n \textit{ even}. \end{cases} \tag{1.20}$$

Theorem 1.3.2 (Jung, [38]). *Let $K \in \mathcal{K}^n$. Then*

$$\sqrt{\frac{2(n+1)}{n}}\,\mathrm{R}(K) \leq \mathrm{D}(K). \tag{1.21}$$

These results have been extended to successive radii in [3] and [32], respectively. In particular, we will need the following generalization of Jung's theorem:

Theorem 1.3.3 (Henk, [32]). *Let $K \in \mathcal{K}^n$ and $1 \leq j \leq i \leq n$. Then*

$$\overline{\mathrm{R}}_i(K) \leq \sqrt{\frac{i(j+1)}{j(i+1)}}\overline{\mathrm{R}}_j(K),$$

and equality holds for $i > j$ if and only if K contains a regular i-dimensional simplex with edge-length $\sqrt{2(j+1)/j}\,\overline{\mathrm{R}}_j(K)$.

Chapter 2

Successive radii for special families of convex bodies

Frequently, the study of some general properties of a geometric functional entails the necessity of a better understanding on particular families of convex bodies. For example, when we obtain a sharp inequality, the study of the equality case amounts to this particular question. But this problem has interest by itself; for instance, regarding the successive radii, several mathematicians have worked on this topic, although so far, only orthogonal boxes, orthogonal crosspolytopes [6, 8, 17], simplices [1, 8, 12, 51, 57, 58] and ellipsoids [31] have been studied and their radii explicitly given. In the case of other families as unit p-balls, the problem is far away from a complete solution.

We start this chapter recalling the known results on successive radii of the above mentioned families of convex bodies. The classical successive radii R_i and r_i are closely related to some notions in approximation theory, namely, they are particular cases of the so-called Gelfand and Kolmogorov numbers of identity operators between finite dimensional normed spaces. Next we introduce this numbers, stating the necessary notation from Banach Space Theory and Approximation Theory, and we explain their close relation to successive radii. As a consequence, we obtain the values of the successive radii of unit p-balls. Finally we consider other families of convex bodies, as constant width sets and p-tangential bodies, and we prove properties of some of their successive radii.

All new results collected in this chapter appear in [26].

2.1 Known successive radii of certain families of convex bodies

It is well known (see [31, p. 18]) that if E is the ellipsoid

$$E=\left\{x=(x_1,\dots,x_n)^\intercal\in\mathbb{R}^n:\left(\frac{x_1}{t_1}\right)^2+\cdots+\left(\frac{x_n}{t_n}\right)^2\leq 1\right\}$$

with $t_1\geq\cdots\geq t_n>0$, then

$$\mathrm{R}_i(E)=t_{n-i+1}\quad\text{and}\quad \mathrm{r}_i(E)=t_i \tag{2.1}$$

for $i=1,\dots,n$.

We observe that the ellipsoid E can be represented as $E=D_tB_n$, where $D_t=\mathrm{diag}\{t_1,\dots,t_n\}$ is the diagonal matrix with diagonal $t=(t_1,\dots,t_n)$.

We consider now the regular crosspolytope C_n^* and the cube C_n. If we apply on them the linear transformation D_t, we get orthogonal boxes $D_tC_n=\sum_{j=1}^n[-t_i,t_i]$ and orthogonal crosspolytopes $D_tC_n^*=\mathrm{conv}\{\pm t_1\mathrm{e}_1,\dots,\pm t_n\mathrm{e}_n\}$. Next proposition was proved in [17]. We state it here following the above notation.

Proposition 2.1.1 (Everett et al., [17]). *Let $1\leq i\leq n$. Then*

$$\mathrm{r}_i(D_tC_n)=\sqrt{\frac{t_{k+1}^2+\cdots+t_n^2}{i-k}},$$

where $k\in\{0,\dots,i-1\}$ is the smallest integer satisfying

$$t_{k+1}\leq\sqrt{\frac{t_{k+2}^2+\cdots+t_n^2}{i-k-1}},$$

and

$$\mathrm{R}_i(D_tC_n^*)=\sqrt{\frac{i-k}{\frac{1}{t_1^2}+\cdots+\frac{1}{t_{n-k+1}^2}}},$$

where $k\in\{0,\dots,i-1\}$ is the smallest integer satisfying

$$t_{n-k+1}\geq\sqrt{\frac{i-k-1}{\frac{1}{t_1^2}+\cdots+\frac{1}{t_{n-k}^2}}}.$$

We remark here that the second formula is obtained by duality (see (1.18)), since

$$D_tC_n^*=(D_t^{-1}C_n)^*.$$

With respect to the outer radii of D_tC_n (respectively the inner radii of $D_tC_n^*$) the following proposition was shown in [8].

Proposition 2.1.2 (Brandenberg, [8]). *Let $i \in \{1, \dots, n\}$. Then*

$$\mathrm{R}_i(D_t C_n) = \sqrt{t_{n-i+1}^2 + \cdots + t_n^2}$$

and

$$\mathrm{r}_i(D_t C_n^*) = \frac{1}{\sqrt{\frac{1}{t_1^2} + \cdots + \frac{1}{t_i^2}}}.$$

Again, the second identity holds by duality. If $t_1 = \cdots = t_n = 1$ then $D_t = \mathrm{I}_n$ and the following corollary holds.

Corollary 2.1.1. *Let $i \in \{1, \dots, n\}$. Then*

$$\mathrm{R}_i(C_n) = \sqrt{i}, \qquad \mathrm{r}_i(C_n) = \sqrt{\frac{n}{i}}$$

and

$$\mathrm{R}_i(C_n^*) = \sqrt{\frac{i}{n}}, \qquad \mathrm{r}_i(C_n^*) = \sqrt{\frac{1}{i}}.$$

We mention here that the outer (inner) radii of the regular crosspolytope (cube) were previously known. In [51] Pukhov gives the references for the papers in which these values are computed.

The inner radii r_i of regular simplices were studied in [1], where Ball used a well known result of John [37], which also plays an important role in the computation of some outer radii R_i of S_n.

The first studied outer radii were R_1 by Steinhagen in [56] and R_n by Jung in [38]. The case $i = n-1$ with i odd was studied by Weißbach in [57, 58] whereas the case $i = n-1$ with i even was studied by Brandenberg in [8]. The remaining cases were proved by Pukhov in [51].

We collect all radii of the regular simplex in the next proposition.

Proposition 2.1.3. *Let $\overline{S}_n$ be the n-dimensional regular simplex of circumradius* 1. *Then*

$$\mathrm{R}_i(\overline{S}_n) = \begin{cases} \sqrt{\frac{i}{n}}, & i \notin \{1, n-1\} \text{ or } n \text{ odd}, \\ \frac{n+1}{n}\sqrt{\frac{1}{n+2}}, & i = 1 \text{ and } n \text{ even}, \\ \frac{2n-3}{2n}, & i = n-1 \text{ and } n \text{ even} \end{cases} \qquad \text{and} \qquad \mathrm{r}_i(\overline{S}_n) = \sqrt{\frac{n+1}{i(i+1)n}}.$$

2.2 Gelfand numbers, Kolmogorov numbers and successive radii of symmetric convex bodies

The authors of [28] already mentioned the close relation of successive radii to notions of width studied in approximation theory, see e.g. [14, 48, 49]. Nevertheless, it seems that up to now this intimate connection has not been so far highlighted in its full generality. So some results proved for successive radii in recent years can be translated from corresponding results about Gelfand numbers and Kolmogorov numbers of identity operators between finite dimensional normed spaces. Our aim in this section is to point out the formal connection between successive radii and Gelfand and Kolmogorov numbers and to translate results from approximation theory to the geometric setting of successive radii.

We start introducing the necessary notation from Banach space theory and approximation theory. The letters X, Y always stand for Banach spaces. The dual space of all bounded linear functionals on X will be denoted by X'. In this particular setting, we will also represent the action of $a \in X'$ on $x \in X$ by $\langle x, a\rangle$. The Banach space $\mathcal{L}(X,Y)$ is the space of all linear bounded operators from X to Y with the usual operator norm, denoted by $\|\cdot\|$. Then, the dual operator $T' \in \mathcal{L}(Y', X')$ of $T \in \mathcal{L}(X,Y)$ is given by $\langle x, T'b\rangle = \langle Tx, b\rangle$ for $x \in X$ and $b \in Y'$. It satisfies $\|T'\| = \|T\|$. We denote by $\|\cdot\|_X$ the norm on the Banach space X.

Definition 2.2.1. *Let $T \in \mathcal{L}(X,Y)$. The k-th* approximation number *is defined as*

$$a_k(T) := \inf\big\{\|T - R\| : R \in \mathcal{L}(X,Y),\, \operatorname{rank} R < k\big\},$$

the k-th Gelfand number *as*

$$c_k(T) := \inf\left\{\|T_{|_M}\| : M \text{ linear subspace of } X,\, \operatorname{codim} M < k\right\},$$

and the k-th Kolmogorov number *as*

$$d_k(T) := \inf\big\{\|q_N T\| : N \text{ linear subspace of } Y,\, \dim N < k\big\};$$

here $T_{|_M}$ is the restriction of T to the subspace M and q_N denotes the quotient mapping $Y \longrightarrow Y/N$.

More explicit descriptions of the Gelfand and Kolmogorov numbers are

$$\begin{aligned} c_k(T) &= \inf_{\substack{M \subset X \\ \operatorname{codim} M < k}} \sup_{x \in M, \|x\|_X \leq 1} \|Tx\|_Y, \\ d_k(T) &= \inf_{\substack{N \subset Y \\ \dim N < k}} \sup_{x \in X, \|x\|_X \leq 1} \inf_{y \in N} \|Tx - y\|_Y. \end{aligned}$$

In the following lemma we collect some basic known facts about these quantities. For this and more information on s-numbers of operators in the normed case we refer to [39, 48].

Lemma 2.2.1. *Let $s \in \{a, c, d\}$, $k \in \{1, \ldots, n\}$ and $T \in \mathcal{L}(X,Y)$. Then:*

i) $\|T\| \geq s_1(T) \geq s_2(T) \geq s_3(T) \geq \cdots \geq 0$.

ii) $s_k(STR) \leq \|S\| s_k(T) \|R\|$, *for all operators R, S for which the product STR is defined.*

iii) $c_k(T) \leq a_k(T)$ *and* $d_k(T) \leq a_k(T)$.

iv) $c_k(T) = a_k(T)$ *if* X *is a Hilbert space and* $d_k(T) = a_k(T)$ *whenever* Y *is a Hilbert space.*

v) $a_k(T') = a_k(T)$, *and* $d_k(T') = c_k(T)$ *if* T *is a compact operator between Banach spaces.*

In order to state the connection of the above numbers with the successive radii, we need the well-known correspondence between a 0-symmetric convex body $K \in \mathcal{K}^n$ and the n-dimensional normed space $X_K = \big(\mathbb{R}^n, |\cdot|_K\big)$ with unit ball K. For two such bodies K and E, let I_K^E denote the identity operator of $\mathbb{R}^n$ considered as an operator between the corresponding normed spaces, $X_K \longrightarrow X_E$. If $K = B_n^p$, then we abbreviate I_p^E for I_K^E. Similarly, if $E = B_n^q$, we write I_K^q for I_K^E. Now the notation I_p^q is self-explaining.

We recall that for a 0-symmetric convex body $K \in \mathcal{K}^n$, K^* is the unit ball of the dual space of X_K, i.e., $X_K' = X_{K^*}$. Moreover, it holds

$$\big(I_K^E\big)' = I_{E^*}^{K^*}. \tag{2.2}$$

The following theorem gives the formal connection between the Gelfand and Kolmogorov numbers, and the successive radii.

Theorem 2.2.1 ([26]). *Let* $K \in \mathcal{K}^n$ *be* 0*-symmetric. For all* $i = 1, \ldots, n$ *it holds*

$$\mathrm{r}_i(K) = c_{n-i+1}\big(I_2^K\big)^{-1} = d_{n-i+1}\big(I_{K^*}^2\big)^{-1} = a_{n-i+1}\big(I_2^K\big)^{-1}$$

and

$$\mathrm{R}_i(K) = d_{n-i+1}\big(I_K^2\big) = c_{n-i+1}\big(I_2^{K^*}\big) = a_{n-i+1}\big(I_K^2\big).$$

Proof. The last two equalities between the Gelfand, Kolmogorov and approximation numbers follow immediately from the properties of these numbers stated above (see Lemma 2.2.1 and (2.2)).

For a 0-symmetric convex body K, the definition of $\mathrm{r}_i(K)$ reduces to

$$\mathrm{r}_i(K) = \max_{L \in \mathcal{L}_i^n} \mathrm{r}(K \cap L; L).$$

Let $L \in \mathcal{L}_i^n$ be any i-dimensional linear subspace of $\mathbb{R}^n$. Observe that

$$\big\|I_2^K{}_{|L}\big\| = \min\big\{R > 0 : |x|_K \leq R|x|_2 \text{ for all } x \in L\big\}$$

and

$$\mathrm{r}(K \cap L; L) = \max\big\{r > 0 : rB_{i,L} \subset K \cap L\big\} = \max\left\{r > 0 : |x|_K \leq \frac{1}{r}|x|_2 \text{ for all } x \in L\right\}.$$

Thus it follows that

$$\mathrm{r}(K \cap L; L) = \big\|I_2^K{}_{|L}\big\|^{-1},$$

and taking the maximum over $L \in \mathcal{L}_i^n$, which is the same as taking the maximum over all L with $\operatorname{codim} L < n - i + 1$, we get $\mathrm{r}_i(K) = c_{n-i+1}\big(I_2^K\big)^{-1}$.

The equality for the successive outer radii is now deduced from the above stated duality relation $d_{n-i+1}\big(I_K^2\big) = c_{n-i+1}\big(I_2^{K^*}\big)$, the previously proved identity $\mathrm{r}_i(K^*) = c_{n-i+1}\big(I_2^{K^*}\big)^{-1}$, and the known relation $\mathrm{R}_i(K)\mathrm{r}_i(K^*) = 1$ (see (1.18)). We would like to emphasize that this identity can be also seen as a special case of the duality relation $c_k(T') = d_k(T)$ between Gelfand and Kolmogorov numbers. For completeness we give a self-contained short argument. To this end, we observe that

$$\mathrm{R}(K|L) = \min\{R > 0 : K|L \subset RB_n\} = \min\big\{R > 0 : |P_L x|_2 \le R|x|_K \text{ for all } x \in \mathbb{R}^n\big\} = \big\|P_L I_K^2\big\|,$$

where P_L denotes the map giving the orthogonal projection onto L in the Euclidean space ℓ_2^n. Then, since $\mathrm{R}(K|L) = \|P_L I_K^2\|$ (see [47, Proposition 11.6.2]), it follows that

$$\mathrm{R}_i(K) = \min_{L\in\mathcal{L}_i^n} \mathrm{R}(K|L) = \min_{L\in\mathcal{L}_i^n} \big\|P_L I_K^2\big\| = d_{n-i+1}\big(I_K^2\big) = c_{n-i+1}\big(I_2^{K^*}\big) = \frac{1}{\mathrm{r}_i(K^*)}. \qquad \square$$

2.3 Successive radii of p-balls

In this section we use the general characterization of inner and outer successive radii by approximation quantities given in Theorem 2.2.1 to deduce exact values and sharp asymptotic estimates for successive radii of p-balls. This also shows that the results for $p = 1$ and $p = \infty$ referred to in Corollary 2.1.1 can be also derived from known results about Gelfand and Kolmogorov numbers.

We start collecting the known results for Gelfand and Kolmogorov numbers $c_k(I_2^p)$ and $d_k(I_p^2)$ for $1 \le p \le \infty$. It was proved by Steckin [55] and Pietsch [46] that for all $k = 1, \dots, n$,

$$d_k(I_1^2) = c_k(I_2^\infty) = \sqrt{\frac{n-k+1}{n}} \quad \text{and} \quad c_k(I_1^2) = d_k(I_2^\infty) = \sqrt{n-k+1}.$$

By Theorem 2.2.1 this immediately implies Corollary 2.1.1. Pietsch actually computed all the s-numbers

$$a_k(I_p^q) = c_k(I_p^q) = d_k(I_p^q) = (n-k+1)^{1/q-1/p}$$

when $1 \le q \le p \le \infty$. In particular, it holds

$$d_k(I_p^2) = (n-k+1)^{1/2-1/p} \quad \text{and} \quad c_k(I_2^q) = (n-k+1)^{1/q-1/2}$$

for $2 \le p \le \infty$ and $1 \le q \le 2$. Then, using Theorem 2.2.1, we get as a direct consequence for successive radii the next theorem.

Theorem 2.3.1 ([26]). *Let $p \ge 2$ and $1 \le q \le 2$. For all $i = 1, \dots, n$ it holds*

$$\mathrm{R}_i(B_n^p) = i^{1/2-1/p} \quad \textit{and} \quad \mathrm{r}_i(B_n^q) = i^{1/2-1/q}.$$

Here we also sketch a geometrical proof of this theorem. We point out that it partly follows the idea of the proof of [47, Theorem 11.11.4], from a geometric point of view. In order to prove it, we start by computing the circumradius of the p-balls B_n^p, $p \ge 1$. We include the proof for completeness, since we have not been able to find it in the literature.

Lemma 2.3.1. *If* $1 \leq p \leq 2$ *then* $\mathrm{R}(B_n^p) = 1$. *If* $p \geq 2$ *then* $\mathrm{R}(B_n^p) = n^{1/2-1/p}$.

Proof. The case $1 \leq p \leq 2$ is trivial, because $B_n^p \subseteq B_n$ and the points $\pm \mathrm{e}_1 \in \operatorname{bd} B_n^p$. So, we assume $p \geq 2$. Notice that it suffices to show that

$$\max_{x \in \operatorname{bd} B_n^p} |x|_2 = \left| \frac{1}{n^{1/p}} (1, \dots, 1)^\intercal \right|_2 = n^{1/2-1/p}.$$

It is an easy computation to check that the maximum of the function

$$f(x_1, \dots, x_n) := \sqrt{x_1^2 + \cdots + x_n^2}$$

under the condition $|x_1|^p + \cdots + |x_n|^p = 1$ is attained precisely when $|x_1| = \cdots = |x_n|$, i.e., for $|x_j| = (1/n)^{1/p}$, $j = 1, \dots, n$. □

We also observe the following property. If $P \subset \mathbb{R}^n$ is a polytope with $0 \in \operatorname{int} P$ then, for any $L \in \mathcal{L}_i^n$, $P_L = P \cap L$ is an i-dimensional polytope. Let v be a vertex of P_L and we denote by F the smallest (in the sense of dimension) face of P containing v, which gives $F \cap L = \{v\}$. If we assume that $\dim F > n - i$, then it would be $\dim(F + L) = i + \dim F > n$, which is not possible. Therefore $\dim F \leq n - i$, i.e., we have proved the following fact:

If $P \subset \mathbb{R}^n$ is a polytope with $0 \in \operatorname{int} P$, then any $L \in \mathcal{L}_i^n$ intersects P in one of its $(n-i)$-faces. (2.3)

Proof of Theorem 2.3.1. In order to prove that $\mathrm{R}_i(B_n^p) = i^{1/2-1/p}$, $p \geq 2$, it suffices to show

$$\mathrm{R}(B_n^p \cap L) \geq i^{1/2-1/p} \quad \text{for all } \ L \in \mathcal{L}_i^n; \tag{2.4}$$

then, using Lemma 2.3.1, since $\mathrm{R}(B_n^p | L) \geq \mathrm{R}(B_n^p \cap L)$ and

$$\mathrm{R}\big(B_n^p | \operatorname{lin}\{\mathrm{e}_1, \dots, \mathrm{e}_i\}\big) = \mathrm{R}\big(B_n^p \cap \operatorname{lin}\{\mathrm{e}_1, \dots, \mathrm{e}_i\}\big) = i^{1/2-1/p},$$

we get that $\mathrm{R}_i(B_n^p) = i^{1/2-1/p}$, as required.

Let $L \in \mathcal{L}_i^n$. By (2.3) there exists an $(n-i)$-face F_{n-i} of the cube B_n^∞ such that $L \cap F_{n-i} \neq \emptyset$. Let $x \in L \cap F_{n-i}$. Without loss of generality we may assume that

$$F_{n-i} = \big\{(t_1, \dots, t_{n-i}, 1, \dots, 1)^\intercal \in \mathbb{R}^n : |t_j| \leq 1, j = 1, \dots, n-i\big\},$$

i.e., $x = (x_1, \dots, x_{n-i}, 1, \dots, 1)^\intercal$ with $|x_j| \leq 1$, $j = 1, \dots, n-i$. Moreover, let

$$\lambda = \left(i + \sum_{j=1}^{n-i} |x_j|^p \right)^{-1/p} \in (0, 1].$$

Then $z = \lambda x \in L \cap \operatorname{bd} B_n^p$, and since $p \geq 2$ and $|x_j| \leq 1$, we clearly get

$$|z|_2 = \frac{\left(i + \sum_{j=1}^{n-i} |x_j|^2\right)^{1/2}}{\left(i + \sum_{j=1}^{n-i} |x_j|^p\right)^{1/p}} \geq \frac{\left(i + \sum_{j=1}^{n-i} |x_j|^p\right)^{1/2}}{\left(i + \sum_{j=1}^{n-i} |x_j|^p\right)^{1/p}} = \left(i + \sum_{j=1}^{n-i} |x_j|^p\right)^{1/2-1/p} \geq i^{1/2-1/p}.$$

It shows (2.4). Finally, the value for the inner radii comes from (1.18) and the fact that $(B_n^p)^* = B_n^q$ with $1/p + 1/q = 1$. $\square$

The computation of the remaining Kolmogorov and Gelfand numbers of identity operators I_p^q turned out to be more complicated. In the relevant cases for us, the exact values seem to be very difficult to determine. Nevertheless, matching lower and upper bounds up to multiplicative constants are known. Here we use the notation $a_{n,i} \asymp b_{n,i}$ for some double sequences $a_{n,i}, b_{n,i}$ of non-negative real numbers to mean that there exist absolute constants $c, C > 0$ such that $c\, a_{n,i} \leq b_{n,i} \leq C\, a_{n,i}$. The result we need is due to Gluskin [21], who proved that, for $q \geq 2$,

$$c_k(I_2^q) \asymp \begin{cases} \left(\frac{n-k+1}{n}\right)^{1/2} & \text{for } 1 \leq k \leq n+1-n^{2/q}, \\ n^{1/q-1/2} & \text{for } n+1-n^{2/q} \leq k \leq n, \end{cases}$$

and, by duality, for $1 < p \leq 2$,

$$d_k(I_p^2) \asymp \begin{cases} \left(\frac{n-k+1}{n}\right)^{1/2} & \text{for } 1 \leq k \leq n+1-n^{2(1-1/p)}, \\ n^{1/2-1/p} & \text{for } n+1-n^{2(1-1/p)} \leq k \leq n. \end{cases}$$

By Theorem 2.2.1, the direct consequence for successive radii is the next theorem.

Theorem 2.3.2 ([26]). *Let $1 \leq p \leq 2$ and $q \geq 2$. For all $i = 1, \ldots, n$ it holds*

$$\mathrm{R}_i(B_n^p) \asymp \begin{cases} \left(\frac{i}{n}\right)^{1/2} & \text{for } i \geq n^{2(1-1/p)}, \\ n^{1/2-1/p} & \text{for } i \leq n^{2(1-1/p)}, \end{cases}$$

and

$$\mathrm{r}_i(B_n^q) \asymp \begin{cases} \left(\frac{n}{i}\right)^{1/2} & \text{for } i \geq n^{2/q}, \\ n^{1/2-1/q} & \text{for } i \leq n^{2/q}. \end{cases}$$

We also have been able to state a different lower (respectively, upper) bound for the outer (inner) radii in the case $1 \leq p \leq 2$ ($q \geq 2$). We observe that both results are not comparable since, in many cases, the previous bounds are better, but not always.

Theorem 2.3.3. *Let $1 \leq p \leq 2$ and $q \geq 2$. For all $i = 1, \ldots, n$ it holds*

$$\mathrm{R}_i(B_n^p) \geq (n-i+1)^{1/2-1/p} \quad \text{and} \quad \mathrm{r}_i(B_n^q) \leq (n-i+1)^{1/2-1/q}.$$

Proof. Again, by (1.18), the lower bound for R_i would immediately imply the upper bound for r_i. The proof is similar to the one of Theorem 2.3.1 but now considering the crosspolytope B_n^1. Let $L \in \mathcal{L}_i^n$. By (2.3) there exists an $(n-i)$-face F_{n-i} of B_n^1 such that $L \cap F_{n-i} \neq \emptyset$, and we may assume that

$$F_{n-i} = \left\{ (t_1, \dots, t_{n-i+1}, 0, \dots, 0)^\intercal \in \mathbb{R}^n : t_j \geq 0, \sum_{j=1}^{n-i+1} t_j = 1 \right\}.$$

For $x = (x_1, \dots, x_{n-i+1}, 0, \dots, 0)^\intercal \in L \cap F_{n-i}$, it clearly holds that

$$z = \frac{1}{\left(\sum_{j=1}^{n-i+1} x_j^p\right)^{1/p}} x \in L \cap \mathrm{bd}\, B_n^p,$$

and since $1 \leq p \leq 2$, Hölder's inequality for $2/p$ (see e.g. [30, p. 15]) implies that

$$|z|_2 = \frac{\left(\sum_{j=1}^{n-i+1} x_j^2\right)^{1/2}}{\left(\sum_{j=1}^{n-i+1} x_j^p\right)^{1/p}} \geq \frac{\left(\sum_{j=1}^{n-i+1} x_j^2\right)^{1/2}}{\left[\left(\sum_{j=1}^{n-i+1} x_j^2\right)^{p/2} \left(\sum_{j=1}^{n-i+1} 1\right)^{(2-p)/2}\right]^{1/p}} = \frac{1}{(n-i+1)^{1/p-1/2}}.$$

Therefore,

$$\mathrm{R}(B_n^p|L) \geq \mathrm{R}(B_n^p \cap L) \geq (n-i+1)^{1/2-1/p}$$

for all $L \in \mathcal{L}_i^n$, which shows that $\mathrm{R}_i(B_n^p) \geq (n-i+1)^{1/2-1/p}$. □

In connection with the approximation of embeddings between function spaces, considerable work has been done to compute the Gelfand and Kolmogorov numbers of diagonal operators. We will now translate some of this work into results for successive radii. Let D_t be the diagonal matrix with diagonal $t = (t_1, \dots, t_n)$ with $t_1 \geq t_2 \geq \dots \geq t_n > 0$. The following result is a special case of [47, Theorem 11.11.4].

Proposition 2.3.1 (Pietsch, [47]). *Let $1 \leq q \leq 2$ and $p \geq 2$ and define positive numbers r, s by $1/r = 1/q - 1/2$ and $1/s = 1/2 - 1/p$. Then*

$$c_k\big(D_t : \ell_2^n \longrightarrow \ell_q^n\big) = \left(\sum_{j=k}^n t_j^r\right)^{1/r} \quad \textit{and} \quad d_k\big(D_t : \ell_p^n \longrightarrow \ell_2^n\big) = \left(\sum_{j=k}^n t_j^s\right)^{1/s}.$$

Let $K_p = D_t B_n^p$, $p \geq 2$, and $K^q = D_t^{-1} B_n^q$, $1 \leq q \leq 2$. This is, K_p and K^q are orthogonally dilated images of the balls B_n^p and B_n^q, t_i and t_i^{-1} being the respective lengths of the half-axes in the direction e_i. Thus from the properties of the Gelfand and Kolmogorov numbers, we directly obtain from Proposition 2.3.1 that

$$c_k\big(I_2^{K^q}\big) = \left(\sum_{j=k}^n t_j^r\right)^{1/r} \quad \text{and} \quad d_k\big(I_{K_p}^2\big) = \left(\sum_{j=k}^n t_j^s\right)^{1/s}.$$

Finally, Theorem 2.2.1 leads to the following result.

Theorem 2.3.4 ([26]). *Let $1 \leq q \leq 2$ and $p \geq 2$ and define positive numbers r, s by $1/r = 1/q - 1/2$ and $1/s = 1/2 - 1/p$. Let $t = (t_1, \dots, t_n)$ be such that $t_1 \geq t_2 \geq \dots \geq t_n > 0$ and let $K_p = D_t B_n^p$ and $K^q = D_t^{-1} B_n^q$. Then*

$$\mathrm{r}_i(K^q) = \left(\sum_{j=n-i+1}^{n} t_j^r \right)^{-1/r} \quad \textit{and} \quad \mathrm{R}_i(K_p) = \left(\sum_{j=n-i+1}^{n} t_j^s \right)^{1/s}.$$

For $q = 1$ and $p = \infty$, the values of the inner radii of orthogonal crosspolytopes and the outer radii of orthogonal boxes are obtained (see Proposition 2.1.2); for $p = q = 2$ the successive radii of the ellipsoids can be deduced (see (2.1)).

We also remark that the values of the outer radii of orthogonal crosspolytopes (and so the inner radii of orthogonal boxes) can be derived from [47, Theorem 11.11.7] via Theorem 2.2.1 (see Proposition 2.1.1). Finally, we mention that the results from [35, 36] can be used to compute (estimate, up to multiplicative constants) the successive radii of unit balls of symmetric n-dimensional normed spaces; in particular this applies to unit balls of Lorentz and Orlicz sequence spaces.

2.4 Successive radii of constant width sets

A convex body $K \in \mathcal{K}^n$ is said to have constant width b if it has the same width b in all directions, i.e., if

$$\omega(K) = \mathrm{D}(K) = \mathrm{b}.$$

The set of convex bodies of constant width will be denoted by $\mathcal{W}^n$.

Constant width sets have been intensively studied along the last century. In the plane they are well known (the unit disc and the Reuleaux triangle are the best known examples, see Figure 2.1), whereas the situation becomes much more complicated in dimension $n \geq 3$ (see e.g. [5, §15], [16, Ch. 7] and [15] for detailed surveys).

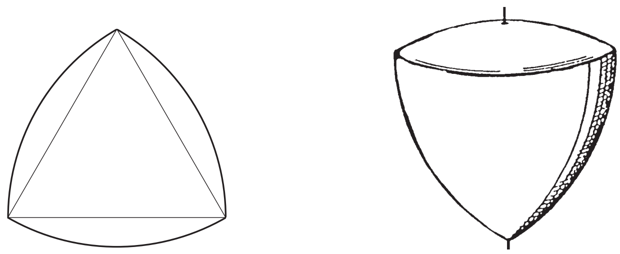


Figure 2.1: The Reuleaux triangle and its revolution in dimension 3

The best known 3-dimensional constant width sets are the revolution of planar convex bodies with constant width (see Figure 2.1), and the so-called Meissner bodies, which are constructed, roughly speaking, in the following way. Let T_3 be a 3-dimensional regular tetrahedron with edge length b, and consider the intersection K of four balls of radius b having the vertices of T_3 as centers. Then K is bounded by four pieces of sphere which meet in six circular arcs. However, K is not a constant width set, because the distance between two of those opposite circular arcs is strictly greater than b. The Meissner bodies are then obtained rounding suitably three of those arcs (see Figure 2.2). Notice that two Meissner bodies can be constructed, depending on the three rounded arcs either converge to a vertex or form a triangle. For a more detailed construction of the Meissner bodies we refer to [5, p. 144].

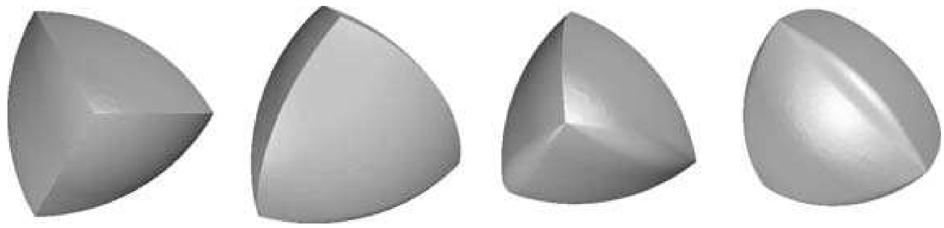


Figure 2.2: A Meissner body. Figure obtained from [40]

Let $K \in \mathcal{W}^n$. It is well known (see e.g. [16, p. 125]) that the inball and the circumball of K are concentric and both,

$$\mathrm{R}(K) + \mathrm{r}(K) = \mathrm{b} \quad \text{and} \quad \mathrm{D}(K) + \omega(K) = 2\mathrm{b}. \tag{2.5}$$

So the natural question arises if an analogous relation holds for the more general in- and outer radii, namely, if

$$\mathrm{R}_i(K) + \mathrm{r}_i(K) = \mathrm{b}, \quad i = 1, \dots, n. \tag{2.6}$$

The next theorem shows that this relation is, in general, not true except, of course, when $i = 1, n$.

Theorem 2.4.1 ([26]). *Let $K \in \mathcal{W}^n$ with width* b*. Then $\mathrm{R}_i(K) + \mathrm{r}_i(K) \leq \mathrm{b}$, and the inequality can be strict, as the Meissner body shows.*

Proof. For the proof of this theorem the inner radii $\widetilde{\mathrm{r}}_i$ will play an important role. For $K \in \mathcal{W}^n$ with width b, let $L' \in \mathcal{L}_i^n$ be such that

$$\mathrm{r}(K|L'; L') = \widetilde{\mathrm{r}}_i(K). \tag{2.7}$$

It is well-known (see e.g. [5, p. 135]) that every orthogonal projection of a constant width set is also a body of constant width having the same width. Then, using (2.5) one can easily obtain that

$$\mathrm{b} = \mathrm{R}(K|L') + \mathrm{r}(K|L'; L') \geq \mathrm{R}_i(K) + \widetilde{\mathrm{r}}_i(K) \geq \mathrm{R}_i(K) + \mathrm{r}_i(K). \tag{2.8}$$

So it remains to prove that the inequality can be strict. Let $K_M \in \mathcal{W}^3$ be a Meissner body with width b. It is known (see e.g. [7, p. 37]) that the orthogonal projection of K_M onto the plane Π

determined by two opposite edges of the generating tetrahedron is a 2-dimensional ball with radius $\mathrm{b}/2$. Then, since $\mathrm{R}_2(K_M) \geq \mathrm{R}_1(K_M) = \mathrm{b}/2$ and $\mathrm{R}(K_M|\Pi) = \mathrm{b}/2$, we get $\mathrm{R}_2(K_M) = \mathrm{b}/2$. So, we have to prove that $\mathrm{r}_2(K_M) < \mathrm{b}/2$. In order to show it, we assume $\mathrm{r}_2(K_M) = \mathrm{b}/2$, and we will get a contradiction.

From the definition of $\mathrm{r}_2(K_M)$, there exist $L \in \mathcal{L}_2^3$ and $x \in L^\perp$ such that

$$\frac{\mathrm{b}}{2} = \mathrm{r}_2(K_M) = \mathrm{r}\big(K_M \cap (x+L); x+L\big),$$

and thus there exists a circle C of radius $\mathrm{b}/2$ contained in $K_M \cap (x+L)$. Moreover, we observe that $C = K_M \cap (x+L)$, otherwise there would exist a point $p \in \big(K_M \cap (x+L)\big)\backslash C$, and then the diameter $\mathrm{D}(K_M) \geq \mathrm{D}\big(K_M \cap (x+L)\big) > \mathrm{b}$, which is not possible. Let $y \in \operatorname{int} K_M$ be such that $C = y + (\mathrm{b}/2)B_{i,L}$, and let $v \in \operatorname{relbd} B_{i,L}$. Then the point $y + (\mathrm{b}/2)v \in y + (\mathrm{b}/2)\operatorname{relbd} B_{i,L} = \operatorname{relbd} C$ and thus, $y + (\mathrm{b}/2)v$ cannot be a vertex of K_M.

On the one hand, if the point $y + (\mathrm{b}/2)v$ lies on one of the four pieces of sphere bounding the set K_M, by the construction of the Meissner body and taking into account that the line segment $\big[y - (\mathrm{b}/2)v, y + (\mathrm{b}/2)v\big] \subset C$ and that it has length b, then $y - (\mathrm{b}/2)v$ should be the opposite vertex, which is not possible. On the other hand, if $y + (\mathrm{b}/2)v$ lies on one of the (rounded) arcs, then C should touch one of the opposite sphere pieces of K_M, which leads to the previous case and again to a contradiction. □

It can be easily seen (see Proposition 2.4.1) that if the radii $\widetilde{\mathrm{r}}_i$ are involved, then it is possible to get an equality relation of the type (2.6). Moreover, it is well-known that for any constant width set $K \in \mathcal{W}^n$ of width b it holds

$$\mathrm{b}\left(1 - \sqrt{\frac{n}{2(n+1)}}\right) \leq \mathrm{r}(K) \leq \mathrm{R}(K) \leq \mathrm{b}\sqrt{\frac{n}{2(n+1)}} \tag{2.9}$$

(see e.g. [15, p. 68] or [16, p. 125]); the analogous result for these inner and the outer radii can be easily obtained.

Proposition 2.4.1 ([26]). *For any $K \in \mathcal{W}^n$ of width* b *and all $i = 1, \ldots, n$ it holds*

$$\mathrm{R}_i(K) + \widetilde{\mathrm{r}}_i(K) = \mathrm{b}$$

and

$$\mathrm{b}\left(1 - \sqrt{\frac{i}{2(i+1)}}\right) < \widetilde{\mathrm{r}}_i(K) \leq \mathrm{R}_i(K) < \mathrm{b}\sqrt{\frac{i}{2(i+1)}}. \tag{2.10}$$

Proof. Notice that for any $K \in \mathcal{W}^n$, say of width b, and for any $i = 1, \ldots, n$, the i-plane $L' \in \mathcal{L}_i^n$ giving the value for $\mathrm{R}_i(K)$ gives also $\widetilde{\mathrm{r}}_i(K)$: indeed, if $\mathrm{R}_i(K) = \mathrm{R}(K|L')$, since $K|L$ is also a constant width set of width b satisfying $\mathrm{R}(K|L) + \mathrm{r}(K|L;L) = \mathrm{b}$ for all $L \in \mathcal{L}_i^n$ (see (2.5)), then

$$\mathrm{r}(K|L';L') = \mathrm{b} - \mathrm{R}(K|L') \geq \mathrm{b} - \mathrm{R}(K|L) = \mathrm{r}(K|L;L)$$

for all $L \in \mathcal{L}_i^n$, and so $\widetilde{\mathrm{r}}_i(K) = \mathrm{r}(K|L'; L')$. Therefore,

$$\mathrm{R}_i(K) + \widetilde{\mathrm{r}}_i(K) = \mathrm{R}(K|L') + \mathrm{r}(K|L'; L') = \mathrm{b},$$

and moreover, applying (2.9) to the i-dimensional set $K|L'$ gives the left and right inequalities in (2.10). In order to conclude the proof of (2.10) we notice that, since K is a constant width set and $\widetilde{\mathrm{r}}_1(K) = \mathrm{D}(K)/2$ and $\mathrm{R}_1(K) = \omega(K)/2$, then

$$\widetilde{\mathrm{r}}_i(K) \le \widetilde{\mathrm{r}}_1(K) = \frac{\mathrm{D}(K)}{2} = \frac{\omega(K)}{2} = \mathrm{R}_1(K) \le \mathrm{R}_i(K).$$

We observe that the equality $\widetilde{\mathrm{r}}_i(K) = \mathrm{R}_i(K)$ holds for any constant width set $K \in \mathcal{W}^n$ such that $K|L' = (\mathrm{b}/2)B_{i,L'}$. □

Now we prove an analogous result in the case of the successive radii $\widehat{\mathrm{r}}_i(K)$ and $\overline{\mathrm{R}}_i(K)$, for which the involved inequalities are sharp. We recall here the equality $\overline{\mathrm{R}}_i(K) = \widehat{\mathrm{R}}_i(K)$, which would allow to write next proposition in a more "tasteful" way.

Proposition 2.4.2. *For any $K \in \mathcal{W}^n$ of width b and all $i = 1, \dots, n$ it holds*

$$\overline{\mathrm{R}}_i(K) + \widehat{\mathrm{r}}_i(K) = \mathrm{b} \tag{2.11}$$

and

$$\mathrm{b}\left(1 - \sqrt{\frac{i}{2(i+1)}}\right) \le \widehat{\mathrm{r}}_i(K) \le \overline{\mathrm{R}}_i(K) \le \mathrm{b}\sqrt{\frac{i}{2(i+1)}}. \tag{2.12}$$

All inequalities are best possible.

Proof. Since $K|L$ is a convex body of constant width b for all $L \in \mathcal{L}_i^n$, we obtain using (2.5) that

$$\overline{\mathrm{R}}_i(K) = \max_{L \in \mathcal{L}_i^n} \mathrm{R}(K|L) = \max_{L \in \mathcal{L}_i^n}(\mathrm{b} - \mathrm{r}(K|L; L)) = \mathrm{b} - \min_{L \in \mathcal{L}_i^n} \mathrm{r}(K|L; L) = \mathrm{b} - \widehat{\mathrm{r}}_i(K).$$

Let $L' \in \mathcal{L}_i^n$ be such that $\overline{\mathrm{R}}_i(K) = \mathrm{R}(K|L')$. The right inequality in (2.9) applied to the i-dimensional set $K|L'$ and (2.11) imply the left and right inequalities in (2.12).

Finally, since K is a constant width set and $\widehat{\mathrm{r}}_1(K) = \omega(K)/2$ and $\overline{\mathrm{R}}_1(K) = \mathrm{D}(K)/2$, then $\widehat{\mathrm{r}}_i(K) \le \widehat{\mathrm{r}}_1(K) = \omega(K)/2 = \mathrm{D}(K)/2 = \overline{\mathrm{R}}_1(K) \le \overline{\mathrm{R}}_i(K)$.

Now we show that all inequalities are sharp. Let $i \in \{2, \dots, n\}$ be fixed and let $K \in \mathcal{W}^n$ be of constant width b such that $\widehat{\mathrm{r}}_i(K) = \overline{\mathrm{R}}_i(K)$. This implies that $\mathrm{r}(K|L; L) = \mathrm{R}(K|L)$ for all $L \in \mathcal{L}_i^n$, and therefore, $K|L$ is an i-dimensional ball of radius $\mathrm{b}/2$. Then K is the ball $(\mathrm{b}/2)B_n$ (see [20, Corollary 3.1.6]).

Let $i \in \{1, \dots, n\}$ and let $K \in \mathcal{W}^n$ be of constant width b such that equality in the last inequality in (2.12) is attained. Let $L' \in \mathcal{L}_i^n$ be such that $\overline{\mathrm{R}}_i(K) = \mathrm{R}(K|L')$. The identity

$$\mathrm{R}(K|L') = \mathrm{b}\sqrt{\frac{i}{2(i+1)}}$$

is Jung's equality condition, and holds (see [32, 38]) if and only if $K|L'$ contains an i-dimensional regular simplex of edge-length $\sqrt{2(i+1)/i}\,\mathrm{R}(K|L') = \mathrm{b}$. We would like to remark that there exists at least a constant width set satisfying the above condition. Indeed, if we take the i-dimensional regular simplex $\overline{S}_i$ of edge-length b, by [54, Theorem 1], there exists $K \in \mathcal{W}^n$ such that $\overline{S}_i \subset K$, $\mathrm{R}(K) = \mathrm{R}(\overline{S}_i)$ and $\mathrm{D}(K) = \mathrm{D}(\overline{S}_i) = \mathrm{b}$. Then

$$\mathrm{R}(K) = \mathrm{R}(\overline{S}_i) = \overline{\mathrm{R}}_i(\overline{S}_i) \le \overline{\mathrm{R}}_i(K) \le \mathrm{R}(K),$$

and thus

$$\overline{\mathrm{R}}_i(K) = \mathrm{R}(\overline{S}_i) = \mathrm{D}(\overline{S}_i)\sqrt{\frac{i}{2(i+1)}} = \mathrm{D}(K)\sqrt{\frac{i}{2(i+1)}} = \mathrm{b}\sqrt{\frac{i}{2(i+1)}}.$$

The above identity together with (2.11) also imply equality in the left inequality of (2.12):

$$\widehat{\mathrm{r}}_i(K) = \mathrm{b}\left(1 - \sqrt{\frac{i}{2(i+1)}}\right). \qquad \square$$

2.5 A property on p-tangential bodies

We conclude the chapter stating a property for the so-called p-tangential bodies. A convex body $K \in \mathcal{K}^n$ containing the Euclidean ball B_n is called a p-tangential body of B_n, $0 \le p \le n-1$, if each support plane of K that is not a support plane of B_n contains only $(p-1)$-singular points of K [53, p. 76]. Here $x \in \mathrm{bd}\,K$ is said to be an r-singular point of K if the dimension of the normal cone at x is at least $n-r$. We recall that the normal cone of K at x consists of all outward normal vectors of K at x (together with the zero vector). For further characterizations and properties of p-tangential bodies we refer to [53, Section 2.2].

So a 0-tangential body of B_n is B_n itself, and each p-tangential body of B_n is also a q-tangential body for $p \le q \le n-1$. We observe that a 1-tangential body can be seen as the convex hull of B_n and countably many points such that the line segment joining any pair of those points intersects the ball (see Figure 2.3).

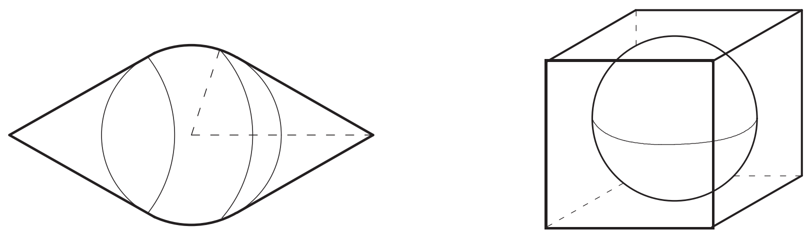


Figure 2.3: A 1-tangential body (cap-body) and a 2-tangential body

A celebrated result of Favard [18] states a nice characterization of n-dimensional p-tangential bodies in terms of the so-called *quermassintegrals* of K, namely, that the $n-p+1$ first ones coincide.

We are not going to enter in the definition and study of these measures; for the interested reader we refer to [53, p. 367]. Here we show a result in the spirit of the above mentioned Favard theorem, in the sense that now, for a p-tangential body, many inner radii also coincide.

Proposition 2.5.1 ([26]). *Let $K \in \mathcal{K}^n$ be a p-tangential body of B_n, $0 \leq p \leq n-1$. Then*

$$\mathrm{r}_n(K) = \mathrm{r}_{n-1}(K) = \cdots = \mathrm{r}_{p+1}(K) = 1.$$

Proof. It is a direct consequence from the definition that any p-tangential body of B_n has inradius 1. So, if $p = n-1$ then $\mathrm{r}_n(K) = 1$ and the result follows. Thus, we assume $1 \leq p \leq n-2$.

Since the inner radii form a decreasing sequence then

$$1 = \mathrm{r}_n(K) \leq \mathrm{r}_{n-1}(K) \leq \cdots \leq \mathrm{r}_{p+1}(K),$$

and it suffices to show that $\mathrm{r}_{p+1}(K) \leq 1$. So we assume $\mathrm{r}_{p+1}(K) > 1$ and we will get a contradiction. On the one hand, by definition of inner radii, there exist $x \in \mathbb{R}^n$ and $L \in \mathcal{L}^n_{p+1}$ such that

$$x + \mathrm{r}_{p+1}(K) B_{p+1,L} \subseteq K. \tag{2.13}$$

On the other hand, in [52, Lemma 2.5] it is shown, in particular, that K is a p-tangential body of B_n, $1 \leq p \leq n-2$, if and only if $K|u^\perp$ is a p-tangential body of $B_{n-1,u^\perp}$ for any unit vector $u \in \mathbb{R}^n$. From this result it can be easily obtained that the orthogonal projection $K|L$ is again a p-tangential body of the ball $B_n|L = B_{p+1,L}$, and then

$$\mathrm{r}(K|L; L) = \mathrm{r}(B_{p+1,L}; L) = 1. \tag{2.14}$$

Moreover, from (2.13) we get that $x|L + \mathrm{r}_{p+1}(K) B_{p+1,L} \subseteq K|L$, and then, together with (2.14), we obtain the desired contradiction:

$$1 = \mathrm{r}(K|L; L) \geq \mathrm{r}_{p+1}(K) > 1. \qquad \square$$

This results shows (see [9, Lemma 3.2]) that p-tangential bodies of the Euclidean ball B_n are $\{\mathrm{r}_{p+1}, \ldots, \mathrm{r}_{n-1}\}$-*isoradial*. We recall that a convex body K is called r_j-isoradial if for every $L \in \mathcal{L}^n_j$ there exist $x \in \mathbb{R}^n$ such that $\bigl(x + \mathrm{r}_j(K) B_n\bigr) \cap (x + L) \subset K$, and is said to be $\{\mathrm{r}_j : j \in I\}$-isoradial, for a subset $I \subset \{1, \ldots, n-1\}$, if it is r_j-isoradial for all $j \in I$.

Chapter 3

On the ratio between successive radii

The well known relations between diameter and circumradius, and minimal width and inradius, were obtained by Jung and Steinhagen (see Theorems 1.3.2 and 1.3.1, respectively):

$$\frac{\mathrm{R}_n(K)}{\mathrm{r}_1(K)} \leq \sqrt{\frac{2n}{n+1}} \quad \text{and} \quad \frac{\mathrm{R}_1(K)}{\mathrm{r}_n(K)} \leq \begin{cases} \sqrt{n} & \text{for } n \text{ odd,} \\ \frac{n+1}{\sqrt{n+2}} & \text{for } n \text{ even.} \end{cases} \tag{3.1}$$

The regular n-simplex gives equality in both inequalities (see Proposition 2.1.3). These relations can be seen as particular cases of a more general question: to determine the optimal upper bound for the quotient $\mathrm{R}_{n-i+1}/\mathrm{r}_i$. In 1979 and 1987 Pukhov and Perel'man proved that $i+1$ bounds from above this ratio, but it is far away from the optimal value. In this chapter we first study this type of inequality for 0-symmetric convex bodies for the same outer radii but involving a different inner radii, namely, $\widetilde{\mathrm{r}}_i$. Next we prove upper bounds for the quotient between these two inner radii, r_i and $\widetilde{\mathrm{r}}_i$, which will allow to get an improvement of the Pukhov-Perel'man inequality for $n = 3$ in the 0-symmetric case. We also improve the Pukhov-Perel'man's bound for general convex bodies in arbitrary dimension when $i = 2$.

All new results collected in this chapter appear in [10, 22].

3.1 On the Pukhov-Perel'man inequality

Pukhov [50] and Perel'man [45] showed independently the following result:

Theorem 3.1.1 (Perel'man, [45] & Pukhov, [50]). *Let $K \in \mathcal{K}^n$ and $1 \leq i \leq n$. Then it holds*

$$\frac{\mathrm{R}_{n-i+1}(K)}{\mathrm{r}_i(K)} < i + 1. \tag{3.2}$$

But the optimal bound is still not known. It is conjectured that the regular n-simplex provides the optimal upper bound: if $i = 1, n$, then $\mathrm{R}_{n-i+1}(S_n)/\mathrm{r}_i(S_n)$ takes the values of (3.1); for $i = 2$ and n even,

$$\frac{\mathrm{R}_{n-1}(S_n)}{\mathrm{r}_2(S_n)} = \frac{(2n-1)\sqrt{3}}{\sqrt{2n(n+1)}};$$

in the remaining cases

$$\frac{\mathrm{R}_{n-i+1}(S_n)}{\mathrm{r}_i(S_n)} = \sqrt{1-\frac{i}{n+1}}\sqrt{i(i+1)} \sim (i+1)\sqrt{1-\frac{i}{n+1}}$$

(see Proposition 2.1.3).

In [2] the best possible lower bound for the above ratio was obtained (see Proposition 1.3.1): for any $K \in \mathcal{K}^n$ and $i \in \{1, \dots, n\}$, it holds $\mathrm{R}_{n-i+1}(K)/\mathrm{r}_i(K) \geq 1$, with equality for the ball.

Moreover, in the particular case $n = 3$ (and $i = 2$), Perel'man [45] improved the result reducing the bound in (3.2) from 3 to $2.151\dots$:

$$\frac{\mathrm{R}_2(K)}{\mathrm{r}_2(K)} < 2.151\dots. \tag{3.3}$$

On the other hand, in the case of a 0-symmetric convex body $K \in \mathcal{K}^n$, Pukhov proved in [50] that

$$\frac{\mathrm{R}_{n-i+1}(K)}{\mathrm{r}_i(K)} < \sqrt{e}\,\min\left\{\sqrt{i}, \sqrt{n-i+1}\right\}. \tag{3.4}$$

The optimal bound is also not known. It is conjectured that both, the regular cube and the regular crosspolytope provide the optimal upper bound:

$$\frac{\mathrm{R}_{n-i+1}(K)}{\mathrm{r}_i(K)} \leq \sqrt{\frac{(n-i+1)i}{n}}.$$

The precise values of the successive radii of the regular cube and crosspolytope can be found in Corollary 2.1.1.

In [50] Pukhov also improved the bound given in (3.4) in the particular case $i = 2$, obtaining that the ratio

$$\frac{\mathrm{R}_{n-1}(K)}{\mathrm{r}_2(K)} < 2.$$

In [22] we gave an alternative geometrical proof of this Pukhov result for $n = 3$. In this dissertation we have slightly improved this last bound (see Section 3.3).

We notice that the problem on bounding the ratio $\mathrm{R}_j/\mathrm{r}_i$, $1 \leq i, j \leq n$, has only interest when $j = n - i + 1$:

Proposition 3.1.1 ([22]). *If $j > n - i + 1$ there is no upper bound for $\mathrm{R}_j(K)/\mathrm{r}_i(K)$.*

Proof. Notice that since we assume $j > n-i+1$, then $i > 1$. We are going to find a convex body K such that for $j > n-i+1$, the above ratio is arbitrarily large. It suffices to consider the $(i-1)$-dimensional ball $B_{i-1,L}$, with $L \in \mathcal{L}^n_{i-1}$. On the one hand, since $\dim B_{i-1,L} = i-1$, then $\mathrm{r}_i(B_{i-1,L}) = 0$; on the other hand, we can assume that $B_{i-1,L} \subset (RB_{j,L'}) \times (L')^{\perp}$, for suitable $R > 0$, where $L' \in \mathcal{L}^n_j$. Since

$$\dim L + \dim L' = i - 1 + j > i - 1 + n - i + 1 = n,$$

then L and L' have, at least, a common straight line ℓ. Hence

$$B_{i-1,L} \cap \ell = [-u, u] \subset RB_{j,L'}, \qquad \text{with} \ \ |u|_2 = 1,$$

and thus $R \geq 1$. Therefore $\mathrm{R}_j(B_{i-1,L}) \geq R \geq 1$, and then the quotient $\mathrm{R}_j/\mathrm{r}_i$ is not bounded by above. It suffices to consider the convex hull of $B_{i-1,L}$ and suitable sufficiently close points in order to get a convex body in $\mathbb{R}^n$ with non-empty interior and verifying the same property. □

As it was already noticed in [45], we observe that if $j < n-i+1$, since the outer radii form an increasing sequence, knowing the optimal bound for the ratio $\mathrm{R}_{n-i+1}/\mathrm{r}_i$ would give immediately the required upper bound for $\mathrm{R}_j/\mathrm{r}_i$. Therefore, $\mathrm{R}_{n-i+1}/\mathrm{r}_i$ is the only ratio needed to be considered.

We can also state the same problem but when other successive radii are considered.

3.2 A Pukhov-Perel'man type inequality for inner radii defined via projections

We consider here the Pukhov-Perel'man problem for the ratio $\mathrm{R}_{n-i+1}/\widetilde{\mathrm{r}}_i$.

Theorem 3.2.1 ([22]). *Let $K \in \mathcal{K}^n$ be a 0-symmetric convex body and $1 \leq i \leq n$. Then*

$$\frac{\mathrm{R}_{n-i+1}(K)}{\widetilde{\mathrm{r}}_i(K)} \leq \sqrt{n-i+1}. \tag{3.5}$$

We observe that if $i = n$, equality holds for all 0-symmetric convex bodies.

Proof. Let $L_1 \in \mathcal{L}^n_i$ be an arbitrary linear subspace and we consider $K|L_1$, which is also a 0-symmetric convex body. For the sake of brevity we write $\widetilde{\mathrm{r}}_i = \widetilde{\mathrm{r}}_i(K)$. Then it holds

$$\rho_1 = \mathrm{r}(K|L_1; L_1) \leq \widetilde{\mathrm{r}}_i.$$

Let $u_1 \in L_1$ be the unit vector such that $\omega(K|L_1; L_1)$ is the width in the direction u_1, i.e., such that $\omega(K|L_1; L_1) = h(K|L_1, u_1) + h(K|L_1, -u_1)$. Then,

$$K|L_1 \subset \left\{ y \in L_1 : |\langle y, u_1 \rangle| \leq \frac{\omega(K|L_1; L_1)}{2} \right\} = \{ y \in L_1 : |\langle y, u_1 \rangle| \leq \rho_1 \}$$

because $K|L_1$ is 0-symmetric, which implies that $\omega(K|L_1;L_1) = 2\rho_1$. Moreover, since we are working with the orthogonal projection onto L_1, it holds

$$K \subset \{x \in \mathbb{R}^n : |\langle x, u_1\rangle| \leq \rho_1\}.$$

Notice that we can assume $i \leq n-1$, because for $i = n$ it is trivial that for any 0-symmetric convex set, $\omega(K) = 2\mathrm{r}(K)$. Now we consider $u_1^\perp \in \mathcal{L}^n_{n-1}$ and let $L_2 \subset u_1^\perp$ be an i-dimensional linear subspace. With an analogous argument to the above one we know that there exists a suitable $u_2 \in L_2$ such that

$$K|L_2 \subset \{y \in L_2 : |\langle y, u_2\rangle| \leq \rho_2\},$$

with $\rho_2 = \mathrm{r}(K|L_2;L_2) \leq \widetilde{\mathrm{r}}_i$. Again we can conclude that

$$K \subset \{x \in \mathbb{R}^n : |\langle x, u_2\rangle| \leq \rho_2\}.$$

Next, if $i \leq n-2$, we consider $\mathrm{lin}\{u_1,u_2\}^\perp \in \mathcal{L}^n_{n-2}$, and we take L_3 to be an i-dimensional subspace of $\mathrm{lin}\{u_1,u_2\}^\perp$.

Using an iterative argument, in the $(n-i+1)$-step we obtain $n-i+1$ pairwise orthogonal unit vectors $u_1,\dots,u_{n-i+1}$ (by the construction) and positive real numbers $\rho_j \leq \widetilde{\mathrm{r}}_i$, for $j = 1,\dots,n-i+1$, such that

$$K \subset \bigcap_{j=1}^{n-i+1} \{x \in \mathbb{R}^n : |\langle x, u_j\rangle| \leq \rho_j\} = \Big\{x \in \mathbb{R}^n : |\langle x, u_j\rangle| \leq \rho_j \text{ for } j = 1,\dots,n-i+1\Big\}. \tag{3.6}$$

Thus writing $H_{n-i+1} = \mathrm{lin}\{u_1,\dots,u_{n-i+1}\}$ and denoting by $C_{l_1,\dots,l_{n-i+1}}$ the 0-symmetric orthogonal box contained in H_{n-i+1} with edge-lengths $l_1,\dots,l_{n-i+1}$, we get as a consequence of (3.6) that

$$K|H_{n-i+1} \subseteq C_{2\rho_1,\dots,2\rho_{n-i+1}} \subseteq C_{2\widetilde{\mathrm{r}}_i,\dots,2\widetilde{\mathrm{r}}_i}, \tag{3.7}$$

i.e., it is contained in the $(n-i+1)$-cube of H_{n-i+1} with edge-length $2\widetilde{\mathrm{r}}_i$. Hence

$$\mathrm{R}(K|H_{n-i+1}) \leq \mathrm{R}(C_{2\widetilde{\mathrm{r}}_i,\dots,2\widetilde{\mathrm{r}}_i}) = \sqrt{n-i+1}\,\widetilde{\mathrm{r}}_i,$$

and therefore

$$\mathrm{R}_{n-i+1}(K) \leq \mathrm{R}(K|H_{n-i+1}) \leq \sqrt{n-i+1}\,\widetilde{\mathrm{r}}_i = \sqrt{n-i+1}\,\widetilde{\mathrm{r}}_i(K). \qquad \square$$

If $K \in \mathcal{K}^n$ is an arbitrary convex body (not necessarily 0-symmetric) then, a similar argument to the above one allows to show that a suitable projection of K onto an $(n-i+1)$-dimensional linear subspace H is contained in an orthogonal box $C_{\omega_1,\dots,\omega_{n-i+1}}$ (see (3.7)) with edge-lengths $\omega_j := \omega(K|L_j;L_j)$, where $L_j \in \mathcal{L}^n_i$, $j = 1,\dots,n-i+1$, are suitably chosen.

Using Steinhagen's theorem (see (1.20)) in the subspace L_j, namely,

$$\omega(K|L_j;L_j) \leq \begin{cases} 2\sqrt{i}\ \mathrm{r}(K|L_j;L_j) & \text{for } i \text{ odd},\\ 2\frac{i+1}{\sqrt{i+2}}\,\mathrm{r}(K|L_j;L_j) & \text{for } i \text{ even},\end{cases}$$

and since $\mathrm{r}(K|L_j;L_j) \leq \widetilde{\mathrm{r}}_i(K)$ for all $1 \leq j \leq n-i+1$, we finally obtain the following result.

Proposition 3.2.1 ([22]). *Let $K \in \mathcal{K}^n$ and $1 \le i \le n$. Then*

$$\frac{\mathrm{R}_{n-i+1}(K)}{\widetilde{\mathrm{r}}_i(K)} \le \begin{cases} \sqrt{i}\sqrt{n-i+1} & \textit{for } i \textit{ odd,} \\ \frac{i+1}{\sqrt{i+2}}\sqrt{n-i+1} & \textit{for } i \textit{ even.} \end{cases}$$

We observe that in order to avoid the parity distinction for i, both bounds above should be replaced by $\sqrt{i+1/3}\sqrt{n-i+1}$. We also notice that these bounds for the ratio $\mathrm{R}_{n-i+1}(K)/\widetilde{\mathrm{r}}_i(K)$, depending on the values of n and i, can improve Pukhov-Perel'man's bound $i+1$.

3.3 On the ratio between two successive inner radii. Improving the Pukhov-Perel'man inequality

We prove now a relation between $\mathrm{r}_i(K)$ and $\widetilde{\mathrm{r}}_i(K)$ for 0-symmetric convex bodies. From the definition of inner radii we trivially have $\mathrm{r}_i(K) \le \widetilde{\mathrm{r}}_i(K)$ for all $i = 1, \dots, n$ and any $K \in \mathcal{K}^n$ (see (1.16)). We would like to point out the existence of a reverse relation: the following proposition provides a (non-sharp) lower bound for $\mathrm{r}_i(K)$ in terms of $\widetilde{\mathrm{r}}_i(K)$ when K is 0-symmetric. We remark that in the cases $i = 1, n$ both inner radii coincide.

Proposition 3.3.1 ([22]). *Let $K \in \mathcal{K}^n$ be a 0-symmetric convex body and $1 \le i \le n$. Then $\widetilde{\mathrm{r}}_i(K) \le \sqrt{i}\,\mathrm{r}_i(K)$.*

Proof. Without loss of generality we assume that $L = \mathrm{lin}\{\mathrm{e}_1, \dots, \mathrm{e}_i\} \in \mathcal{L}_i^n$ is the i-dimensional linear subspace such that $\widetilde{\mathrm{r}}_i(K) = \mathrm{r}(K|L; L)$. The central symmetry of K ensures that $\widetilde{\mathrm{r}}_i(K) B_{i,L} \subseteq K|L$. Now let

$$u_j = \widetilde{\mathrm{r}}_i(K)\mathrm{e}_j \in \big(\widetilde{\mathrm{r}}_i(K)\,\mathrm{relbd}\, B_{i,L}\big) \subseteq K|L, \quad j = 1, \dots, i.$$

These points u_j are projections of points of the original body K, i.e., there exist numbers $a_k^l \in \mathbb{R}$ for $k = i+1, \dots, n$, $l = 1, \dots, i$, such that

$$v_j := u_j + \big(0, \dots, 0, a_{i+1}^j, \dots, a_n^j\big)^\intercal \in K, \quad j = 1, \dots, i,$$

and since K is a 0-symmetric convex body, $C = \mathrm{conv}\{\pm v_1, \dots, \pm v_i\} \subseteq K$. Next we show that

$$\mathrm{r}(C; \mathrm{lin}\, C) \ge \mathrm{r}\big(\mathrm{conv}\{\pm u_1, \dots, \pm u_i\}; L\big).$$

Since C is 0-symmetric, then

$$\mathrm{r}(C; \mathrm{lin}\, C) = \min_{x \in \mathrm{relbd}\, C} |x|_2$$

and so we may choose $x \in \mathrm{relbd}\, C$ such that $\mathrm{r}(C; \mathrm{lin}\, C) = |x|_2$. Let

$$x = \sum_{j=1}^{i} (\lambda_j - \mu_j) v_j,$$

with $\lambda_j, \mu_j \geq 0$ for $j = 1, \dots, i$ and $\sum_{j=1}^{i}(\lambda_j + \mu_j) = 1$. Then

$$\begin{aligned}|x|_2^2 &= \left|\sum_{j=1}^{i}(\lambda_j - \mu_j)u_j\right|_2^2 + \left|\sum_{j=1}^{i}(\lambda_j - \mu_j)\bigl(0,\dots,0,a_{i+1}^j,\dots,a_n^j\bigr)^{\intercal}\right|_2^2 \\ &\geq \left|\sum_{j=1}^{i}(\lambda_j - \mu_j)u_j\right|_2^2 = \bigl|x|L\bigr|_2^2.\end{aligned}$$

Since $x|L \in \operatorname{relbd}\operatorname{conv}\{\pm u_1, \dots, \pm u_i\}$, we get that

$$\mathrm{r}(C; \operatorname{lin} C) = |x|_2 \geq \bigl|x|L\bigr|_2 \geq \mathrm{r}\bigl(\operatorname{conv}\{\pm u_1, \dots, \pm u_i\}; L\bigr).$$

Thus we can conclude that

$$\mathrm{r}_i(K) \geq \mathrm{r}(C; \operatorname{lin} C) \geq \mathrm{r}\bigl(\operatorname{conv}\{\pm u_1, \dots, \pm u_i\}; L\bigr) = \widetilde{\mathrm{r}}_i(K)\,\mathrm{r}\bigl(\operatorname{conv}\{\pm \mathrm{e}_1, \dots, \pm \mathrm{e}_i\}; L\bigr) = \widetilde{\mathrm{r}}_i(K)\frac{1}{\sqrt{i}},$$

and we get the required inequality, $\widetilde{\mathrm{r}}_i(K) \leq \sqrt{i}\,\mathrm{r}_i(K)$. □

If K is an arbitrary convex body (not necessarily 0-symmetric) then, a similar argument to the above one allows to show the next proposition.

Proposition 3.3.2. *Let $K \in \mathcal{K}^n$ and $1 \leq i \leq n$. Then $\widetilde{\mathrm{r}}_i(K) \leq i\mathrm{r}_i(K)$.*

Proof. After a suitable rigid motion we can suppose that there exists $L \in \mathcal{L}_i^n$ such that

$$\widetilde{\mathrm{r}}_i(K) B_{i,L} \subseteq K|L.$$

We take points $p_1, \dots, p_{i+1} \in \operatorname{relbd}(\widetilde{\mathrm{r}}_i(K) B_{i,L})$ being the vertices of an i-dimensional regular simplex of L, $S = \operatorname{conv}\{p_j : j = 1, \dots, i+1\}$. There exist points $q_1, \dots, q_{i+1} \in K$ such that $q_j|L = p_j$, $j = 1, \dots, i+1$, and we call $S' = \operatorname{conv}\{q_j : j = 1, \dots, i+1\} \subset K$. Using a similar argument to the one of Proposition 3.3.1, it can be shown that

$$\mathrm{r}_i(K) \geq \mathrm{r}(K \cap \operatorname{aff} S'; \operatorname{aff} S') \geq \mathrm{r}(S' \cap \operatorname{aff} S'; \operatorname{aff} S') \geq \mathrm{r}(S \cap L; L) = \frac{1}{i}\widetilde{\mathrm{r}}_i(K). \qquad \square$$

In the case $n = 3$ and $i = 2$, we have been able to improve the bound given in Proposition 3.3.1.

Theorem 3.3.1. *Let $K \in \mathcal{K}^3$ be a 0-symmetric convex body. Then*

$$\widetilde{\mathrm{r}}_2(K) \leq \frac{2}{\sqrt{3}}\,\mathrm{r}_2(K).$$

The inequality is best possible.

We first show a lemma that will be needed in the proof of Theorem 3.3.1.

Lemma 3.3.1. *Let $K \in \mathcal{K}^3$ be a 0-symmetric convex body, $L = \mathrm{lin}\{\mathrm{e}_1, \mathrm{e}_2\}$ and $\mathrm{r} > 0$ be such that $\mathrm{r}B_{2,L} \subseteq K|L$. Then, there exist a regular hexagon $\mathrm{conv}\{\pm p_i : i = 1,2,3\}$ inscribed in $\mathrm{r}B_{2,L}$ and points $\pm q_i \in K$, $i = 1,2,3$, such that $\pm q_i|L = \pm p_i$, $i = 1,2,3$, and $\dim \mathrm{conv}\{\pm q_i : i = 1,2,3\} = 2$.*

Proof. For a fixed $u_1 \in \mathrm{relbd}(\mathrm{r}B_{2,L})$, we consider the regular hexagon inscribed in $\mathrm{r}B_{2,L}$ and having u_1 as a vertex, and call $\overline{u}_1, \widetilde{u}_1$ the closest vertices to u_1 (see Figure 3.1).

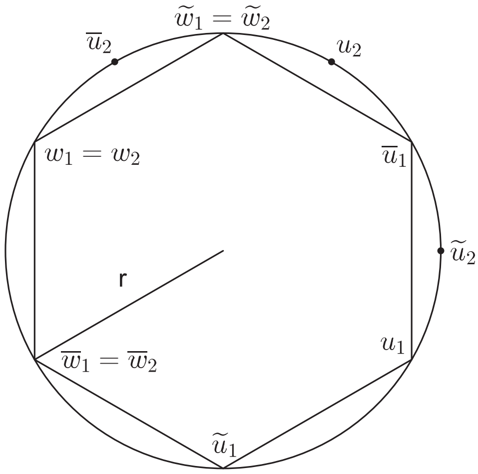


Figure 3.1: The starting regular hexagon and the construction of the sequences

Since $u_1, \overline{u}_1, \widetilde{u}_1 \in K|L$, there exist points $x_1^u, \overline{x}_1^u, \widetilde{x}_1^u \in K$ such that

$$x_1^u|L = u_1, \quad \overline{x}_1^u|L = \overline{u}_1 \quad \text{and} \quad \widetilde{x}_1^u|L = \widetilde{u}_1.$$

If $x_1^u \in \mathrm{lin}\{\overline{x}_1^u, \widetilde{x}_1^u\}$, then $\mathrm{conv}\{\pm x_1^u, \pm\overline{x}_1^u, \pm\widetilde{x}_1^u\}$ is a 2-dimensional convex body whose projection onto L is the regular hexagon $\mathrm{conv}\{\pm u_1, \pm\overline{u}_1, \pm\widetilde{u}_1\}$. In this case, $p_1 := u_1$, $p_2 := \overline{u}_1$, $p_3 := \widetilde{u}_1$, and $q_1 := x_1^u$, $q_2 := \overline{x}_1^u$, $q_3 := \widetilde{x}_1^u$ show the lemma. So, we assume $x_1^u \notin \mathrm{lin}\{\overline{x}_1^u, \widetilde{x}_1^u\}$.

We observe that $x_1^u \in \mathrm{lin}\{\overline{x}_1^u, \widetilde{x}_1^u\}$ if and only if there exist $t, s \in \mathbb{R}$ such that

$$t\big(\overline{u}_1, \overline{x}_{13}^u\big)^\intercal + s\big(\widetilde{u}_1, \widetilde{x}_{13}^u\big)^\intercal = t\overline{x}_1^u + s\widetilde{x}_1^u = x_1^u = \big(u_1, x_{13}^u\big)^\intercal,$$

which holds if and only if $t\overline{u}_1 + s\widetilde{u}_1 = u_1$ and $t\overline{x}_{13}^u + s\widetilde{x}_{13}^u = x_{13}^u$. Since $u_1, \overline{u}_1, \widetilde{u}_1$ are consecutive vertices of a regular hexagon, the unique solution of $t\overline{u}_1 + s\widetilde{u}_1 = u_1$ is $t = s = 1$. Therefore, $x_1^u \notin \mathrm{lin}\{\overline{x}_1^u, \widetilde{x}_1^u\}$ if and only if $\overline{x}_{13}^u + \widetilde{x}_{13}^u \neq x_{13}^u$. We suppose without loss of generality that $\overline{x}_{13}^u + \widetilde{x}_{13}^u > x_{13}^u$. For the rest of the proof we will use the same notation in the construction of the points, namely: from any point $v \in \mathrm{relbd}(\mathrm{r}B_{2,L})$, we derive $\overline{v}$, $\widetilde{v}$, x^v, etc.

We write $w_1 := -u_1$. Then $\overline{w}_1 = -\overline{u}_1$, $\widetilde{w}_1 = -\widetilde{u}_1$ and the symmetry of K imply that $x_1^w = -x_1^u$, $\overline{x}_1^w = -\overline{x}_1^u$, $\widetilde{x}_1^w = -\widetilde{x}_1^u$, and thus

$$\overline{x}_{13}^w + \widetilde{x}_{13}^w = -\overline{x}_{13}^u - \widetilde{x}_{13}^u < -x_{13}^u = x_{13}^w.$$

Let $u_2 \in \mathrm{relbd}(\mathrm{r}B_{2,L})$ be the "midpoint" on the circumference $\mathrm{relbd}(\mathrm{r}B_{2,L})$ between u_1 and w_1. If $x^u_{23} = \overline{x}^u_{23} + \widetilde{x}^u_{23}$ then $p_1 := u_2$, $p_2 := \overline{u}_2$, $p_3 := \widetilde{u}_2$, and $q_1 := x^u_2$, $q_2 := \overline{x}^u_2$, $q_3 := \widetilde{x}^u_2$ show the lemma. If that is not the case, then we can assume that $\overline{x}^u_{23} + \widetilde{x}^u_{23} > x^u_{23}$ and define $w_2 := w_1$ (see Figure 3.1); otherwise we just take w_2 to be the midpoint and define $u_2 := u_1$. In the next step we take again the midpoint $u_3 = (u_2 + w_2)/\left|u_2 + w_2\right|_2 \in \mathrm{relbd}(\mathrm{r}B_{2,L})$ and do the same construction.

Iterating the process, either we find three points p_i, $i = 1, 2, 3$, verifying the required condition in some step, or we get two sequences $(u_n)_n, (w_n)_n \subset \mathrm{relbd}(\mathrm{r}B_{2,L})$, satisfying the following properties:

- $d(u_n, w_n) = (1/2)d(u_{n-1}, w_{n-1})$, where $d(a, b)$ is the length of the shortest arc in $\mathrm{relbd}(\mathrm{r}B_{2,L})$ joining the points $a, b \in \mathrm{relbd}(\mathrm{r}B_{2,L})$.
- $\lim_{n\to\infty} u_n = \lim_{n\to\infty} w_n \in \mathrm{relbd}(\mathrm{r}B_{2,L})$. Let $p_1 := \lim_{n\to\infty} u_n$.
- The vertices of the two corresponding hexagons sequences tend to the appropriate limit, say $\lim_{n\to\infty} \overline{u}_n = \lim_{n\to\infty} \overline{w}_n =: p_2$ and $\lim_{n\to\infty} \widetilde{u}_n = \lim_{n\to\infty} \widetilde{w}_n =: p_3$.
- $\overline{x}^u_{n3} + \widetilde{x}^u_{n3} > x^u_{n3}$ and $\overline{x}^w_{n3} + \widetilde{x}^w_{n3} < x^w_{n3}$, for all $n \in \mathbb{N}$.

With this process, we also get sequences of points in K, namely $(x^u_n)_n$, $(\overline{x}^u_n)_n$, $(\widetilde{x}^u_n)_n$, $(x^w_n)_n$, $(\overline{x}^w_n)_n$ and $(\widetilde{x}^w_n)_n$. Since they are bounded sequences (because they are contained in K), there exist convergent subsequences in K and we can suppose without loss of generality that they are the same sequences. Thus

$$\lim_{n\to\infty} x^u_n = x^u_0 \in K, \quad \lim_{n\to\infty} \overline{x}^u_n = \overline{x}^u_0 \in K, \quad \lim_{n\to\infty} \widetilde{x}^u_n = \widetilde{x}^u_0 \in K,$$
$$\lim_{n\to\infty} x^w_n = x^w_0 \in K, \quad \lim_{n\to\infty} \overline{x}^w_n = \overline{x}^w_0 \in K, \quad \lim_{n\to\infty} \widetilde{x}^w_n = \widetilde{x}^w_0 \in K.$$

We observe that

$$x^u_0|L = \left(\lim_{n\to\infty} x^u_n\right)|L = \lim_{n\to\infty} (x^u_n|L) = \lim_{n\to\infty} u_n = p_1,$$

and analogously,

$$x^w_0|L = p_1, \quad \overline{x}^u_0|L = \overline{x}^w_0|L = p_2 \quad \text{and} \quad \widetilde{x}^u_0|L = \widetilde{x}^w_0|L = p_3.$$

We notice also that

$$\overline{x}^u_{03} + \widetilde{x}^u_{03} = \left(\lim_{n\to\infty} \overline{x}^u_n\right)_3 + \left(\lim_{n\to\infty} \widetilde{x}^u_n\right)_3 = \lim_{n\to\infty} \overline{x}^u_{n3} + \lim_{n\to\infty} \widetilde{x}^u_{n3} = \lim_{n\to\infty} \left(\overline{x}^u_{n3} + \widetilde{x}^u_{n3}\right) \geq \lim_{n\to\infty} x^u_{n3} = x^u_{03},$$

and analogously, $\overline{x}^w_{03} + \widetilde{x}^w_{03} \leq x^w_{03}$.

If $\overline{x}^u_{03} + \widetilde{x}^u_{03} = x^u_{03}$ then the set of points $q_1 := x^u_0$, $q_2 := \overline{x}^u_0$, $q_3 := \widetilde{x}^u_0$ together with p_1, p_2, p_3 show the lemma. Otherwise, $\overline{x}^u_{03} + \widetilde{x}^u_{03} > x^u_{03}$. We observe that if $\overline{x}^w_{03} + \widetilde{x}^u_{03} \leq x^u_{03}$ then the lemma is proved: in fact, if this is the case, there exists $\lambda \in [0, 1)$ such that

$$\left(\lambda\overline{x}^u_0 + (1-\lambda)\overline{x}^w_0\right)_3 + \widetilde{x}^u_{03} = \lambda\overline{x}^u_{03} + (1-\lambda)\overline{x}^w_{03} + \widetilde{x}^u_{03} = x^u_{03},$$

with

$$\lambda\overline{x}_0^u + (1-\lambda)\overline{x}_0^w \in K, \quad \big(\lambda\overline{x}_0^u + (1-\lambda)\overline{x}_0^w\big)|L = \lambda p_1 + (1-\lambda)p_1 = p_1,$$

and thus the set of points $q_1 := x_0^u$, $q_2 := \lambda\overline{x}_0^u + (1-\lambda)\overline{x}_0^w$, $q_3 := \widetilde{x}_0^u$ shows the lemma.

So we assume that $\overline{x}_{03}^w + \widetilde{x}_{03}^u > x_{03}^u$. Similarly, we now have that if $\overline{x}_{03}^w + \widetilde{x}_{03}^w \leq x_{03}^u$, then there exists $\lambda \in [0,1)$ such that

$$\overline{x}_{03}^w + \big(\lambda\widetilde{x}_0^u + (1-\lambda)\widetilde{x}_0^w\big)_3 = \overline{x}_{03}^w + \lambda\widetilde{x}_{03}^u + (1-\lambda)\widetilde{x}_{03}^w = x_{03}^u,$$

and hence the set of points $q_1 := x_0^u$, $q_2 := \overline{x}_0^w$, $q_3 := \lambda\widetilde{x}_0^u + (1-\lambda)\widetilde{x}_0^w$ shows the lemma.

So we assume once more that this is not the case, i.e., that $\overline{x}_{03}^w + \widetilde{x}_{03}^w > x_{03}^u$. But then, since $\overline{x}_{03}^w + \widetilde{x}_{03}^w \leq x_{03}^w$ there exists $\lambda \in [0,1)$ such that

$$\overline{x}_{03}^w + \widetilde{x}_{03}^w = \lambda x_{03}^u + (1-\lambda)x_{03}^w = \big(\lambda x_0^u + (1-\lambda)x_0^w\big)_3,$$

and thus the points $q_1 := \lambda x_0^u + (1-\lambda)x_0^w$, $q_2 := \overline{x}_0^w$, $q_3 := \widetilde{x}_0^w$ show the lemma. $\square$

We now prove Theorem 3.3.1.

Proof of Theorem 3.3.1. By definition of $\widetilde{\mathrm{r}}_2(K)$, there exists $L \in \mathcal{L}_2^3$ such that $\widetilde{\mathrm{r}}_2(K) = \mathrm{r}(K|L;L)$. After a suitable rigid motion, we can assume without loss of generality that $L = \mathrm{lin}\{\mathrm{e}_1, \mathrm{e}_2\}$ and that $\mathrm{r}(K|L;L)B_{2,L} \subseteq K|L$. We now apply Lemma 3.3.1 and find an inscribed regular hexagon

$$H = \mathrm{conv}\{\pm p_i : i = 1,2,3\} \subset \mathrm{r}(K|L;L)B_{2,L}$$

and points $\pm q_i \in K$, $i = 1,2,3$, such that

$$\pm q_i|L = \pm p_i,\ i = 1,2,3, \quad \text{and} \quad \dim\mathrm{conv}\{\pm q_i : i = 1,2,3\} = 2.$$

We call $C = \mathrm{conv}\{\pm q_i : i = 1,2,3\}$ and $L' = \mathrm{lin}\, C$. Then,

$$\mathrm{r}_2(K) \geq \mathrm{r}(K \cap L'; L') \geq \mathrm{r}(C; L').$$

We now show that $\mathrm{r}(C;L') \geq \mathrm{r}(H;L)$. Clearly,

$$\mathrm{r}(C;L') = \min_{x \in \mathrm{relbd}\, C} |x|_2 = |x_0|_2$$

for some $x_0 \in \mathrm{relbd}\, C$. We can suppose that the points q_1 and q_2 are consecutive vertices and that $x_0 = \lambda q_1 + (1-\lambda)q_2$, for some $\lambda \in (0,1)$. Since $q_j|L = p_j$, we have $q_j = (p_j, q_{j3})^\intercal$, $j = 1,2$, and then

$$|x_0|_2^2 = \big|\lambda q_1 + (1-\lambda)q_2\big|_2^2 = \big|\lambda p_1 + (1-\lambda)p_2\big|_2^2 + \big|\lambda q_{13} + (1-\lambda)q_{23}\big|^2 \geq \big|\lambda p_1 + (1-\lambda)p_2\big|_2^2.$$

The point $\lambda p_1 + (1-\lambda)p_2 \in \mathrm{relbd}\, H$, and therefore

$$\big|\lambda p_1 + (1-\lambda)p_2\big|_2 \geq \min_{y \in \mathrm{relbd}\, H} |y|_2 = \mathrm{r}(H;L).$$

From that, we get $\mathrm{r}(C;L') = |x_0|_2 \geq \mathrm{r}(H;L)$ and then

$$\mathrm{r}_2(K) \geq \mathrm{r}(C;L') \geq \mathrm{r}(H;L) = \frac{\sqrt{3}}{2}\,\widetilde{\mathrm{r}}_2(K).$$

It remains to be shown that the inequality is best possible. Let $P_\varepsilon = \operatorname{conv}\{\pm v_1, \pm v_2, \pm v_3\}$ be the non-regular triangular antiprism in $\mathbb{R}^3$ with vertices

$$v_1 = \left(\frac{1}{\sqrt{3}}, 1, \varepsilon\right)^{\intercal}, \quad v_2 = \left(\frac{1}{\sqrt{3}}, -1, \varepsilon\right)^{\intercal}, \quad v_3 = \left(-\frac{2}{\sqrt{3}}, 0, \varepsilon\right)^{\intercal},$$

$\varepsilon > 0$ (see Figure 3.2). First we prove that $\mathrm{r}_2(P_\varepsilon) = \sqrt{3}/2$ for ε small enough.

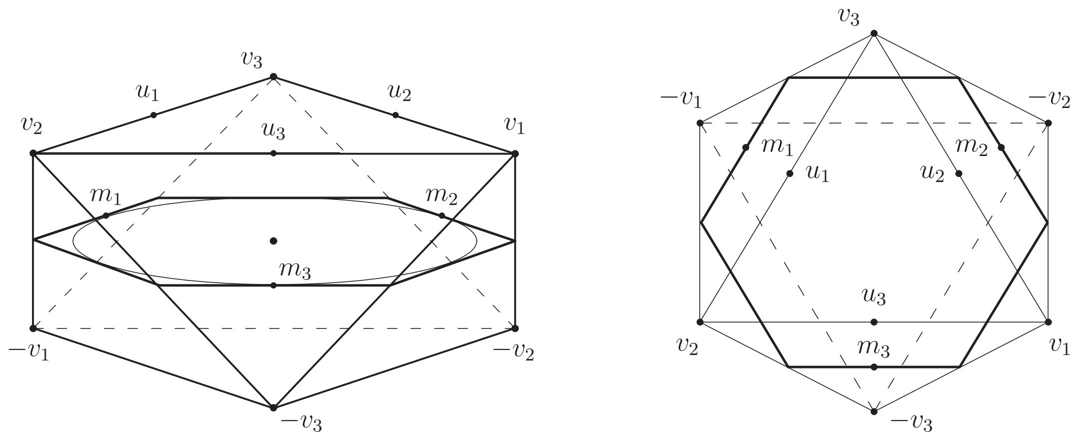


Figure 3.2: Triangular antiprism with $\mathrm{r}_2(P_\varepsilon + rB_3) > \mathrm{r}_2(P_\varepsilon) + r$

Let $u_1 = (1/2)(v_2 + v_3)$, $u_2 = (1/2)(v_1 + v_3)$, $u_3 = (1/2)(v_1 + v_2)$ be the middle points of the edges of the triangle contained in the plane $z = \varepsilon$, and let $m_j = (1/2)(-v_j + u_j)$, $j = 1, 2, 3$ (see Figure 3.2). It is easy to check that $|m_j|_2 = \sqrt{3}/2$ and $|u_j|_2 = \sqrt{1/3 + \varepsilon^2}$, for all $j = 1, 2, 3$. Then $|u_j|_2 \leq \sqrt{3}/2$ if and only if $\varepsilon \leq \sqrt{5/12}$ and hence, for any $\varepsilon \leq \sqrt{5/12}$, all segments

$$\Big\{\big[(0,0,\varepsilon)^{\intercal}, u_j\big], [u_j, m_j] : j = 1,2,3\Big\} \subset \operatorname{bd} P_\varepsilon \cap \frac{\sqrt{3}}{2} B_3.$$

Now we can prove that $\mathrm{r}_2(P_\varepsilon) = \sqrt{3}/2$ for $\varepsilon \leq \sqrt{5/12}$. Notice that since P_ε is 0-symmetric, then $\mathrm{r}_2(P_\varepsilon) = \max_{L \in \mathcal{L}_2^3} \mathrm{r}\big(P_\varepsilon \cap L; L\big)$. If $L = \operatorname{lin}\{\mathrm{e}_1, \mathrm{e}_2\}$, then $P_\varepsilon \cap \operatorname{lin}\{\mathrm{e}_1, \mathrm{e}_2\}$ is the regular hexagon with apothem $|m_j|_2$, and so with incircle $(\sqrt{3}/2) B_{2, \operatorname{lin}\{\mathrm{e}_1, \mathrm{e}_2\}}$ (see Figure 3.2). Therefore,

$$\mathrm{r}(P_\varepsilon \cap \operatorname{lin}\{\mathrm{e}_1, \mathrm{e}_2\}; \operatorname{lin}\{\mathrm{e}_1, \mathrm{e}_2\}) = \frac{\sqrt{3}}{2}.$$

Now let $L \in \mathcal{L}_2^3$, $L \neq \operatorname{lin}\{\mathrm{e}_1, \mathrm{e}_2\}$. Clearly $L \cap \operatorname{lin}\{\mathrm{e}_1, \mathrm{e}_2\}$ is a 1-dimensional subspace which intersects the relative interior of, at least, one of the segments with end-points m_j, $j = 1, 2, 3$, say $[m_1, m_2]$. Then there exists a point

$$u \in L \cap \operatorname{bd} P_\varepsilon \cap \Big\{[m_1, u_1], \big[u_1, (0,0,\varepsilon)^{\intercal}\big], \big[(0,0,\varepsilon)^{\intercal}, u_2\big], [u_2, m_2]\Big\}$$

with $|u|_2 \leq \sqrt{3}/2$, which ensures that $\mathrm{r}(P_\varepsilon \cap L; L) \leq \sqrt{3}/2$ for all $L \in \mathcal{L}_2^3$, $L \neq \mathrm{lin}\{\mathrm{e}_1, \mathrm{e}_2\}$. Thus we can conclude that

$$\mathrm{r}_2(P_\varepsilon) = \frac{\sqrt{3}}{2} \quad \text{if } \varepsilon \leq \sqrt{\frac{5}{12}}.$$

Since the set $P_\varepsilon | \mathrm{lin}\{\mathrm{e}_1, \mathrm{e}_2\}$ is a regular hexagon with 2-dimensional inradius 1, then $\widetilde{\mathrm{r}}_2(P_\varepsilon) \geq 1$. Therefore

$$1 \leq \widetilde{\mathrm{r}}_2(P_\varepsilon) \leq \frac{2}{\sqrt{3}} \mathrm{r}_2(P_\varepsilon) = \frac{2}{\sqrt{3}} \frac{\sqrt{3}}{2} = 1,$$

and thus $\widetilde{\mathrm{r}}_2(P_\varepsilon) = (2/\sqrt{3})\, \mathrm{r}_2(P_\varepsilon)$. □

We observe that for $n = 3$ and $i = 2$, Theorem 3.2.1 and Theorem 3.3.1 together improve the upper bound proved by Pukhov, namely, $\mathrm{R}_2(K)/\mathrm{r}_2(K) < 2$.

Corollary 3.3.1. *Let $K \in \mathcal{K}^3$ be a 0-symmetric convex body. Then*

$$\frac{\mathrm{R}_2(K)}{\mathrm{r}_2(K)} \leq \sqrt{2}\frac{2}{\sqrt{3}} = 1.632\ldots$$

Proof. From Theorem 3.2.1 and Theorem 3.3.1 we get

$$\frac{\mathrm{R}_2(K)}{\mathrm{r}_2(K)} = \frac{\mathrm{R}_2(K)}{\widetilde{\mathrm{r}}_2(K)} \frac{\widetilde{\mathrm{r}}_2(K)}{\mathrm{r}_2(K)} \leq \sqrt{2}\frac{2}{\sqrt{3}} = 1.632\ldots$$ □

We observe that for $n = 3$ and $i = 2$ the optimal bound is still far away, since it is supposed to be $2/\sqrt{3} \approx 1.154\ldots$.

In the particular case $i = 2$, we also have been able to improve the Pukhov-Perel'man's bound for general convex bodies in arbitrary dimension. We follow the idea of the proof of Perel'man in [45] for dimension 3, slightly modifying some steps. Unfortunately, this technique cannot be extended to all indices $i \leq n - 1$.

Theorem 3.3.2. *Let $K \in \mathcal{K}^n$ with $n \geq 3$. Then*

$$\frac{\mathrm{R}_{n-1}(K)}{\mathrm{r}_2(K)} \leq 2\sqrt{2}\sqrt{\frac{n-1}{n}}.$$

Proof. After a suitable translation of K, we can suppose that the diameter of K is given by $\mathrm{D}(K) = 2\,|p|_2$ for $p, -p \in K$. Let $p_1, p_2 \in K|p^\perp$ be such that $|p_1 - p_2|_2 = \mathrm{D}(K|p^\perp)$. We are going to prove that

$$\mathrm{D}(K|p^\perp) \leq 4\mathrm{r}_2(K). \tag{3.8}$$

So, we assume the contrary, $\mathrm{D}(K|p^\perp) > 4\mathrm{r}_2(K)$, and we will get a contradiction. Let $q_1, q_2 \in K$ be such that $q_j|p^\perp = p_j$, for $j = 1, 2$, and we write (see Figure 3.3)

$$P = \mathrm{conv}\left\{\frac{1}{2}(p + q_j), \frac{1}{2}(-p + q_j) : j = 1, 2\right\} \subset K.$$

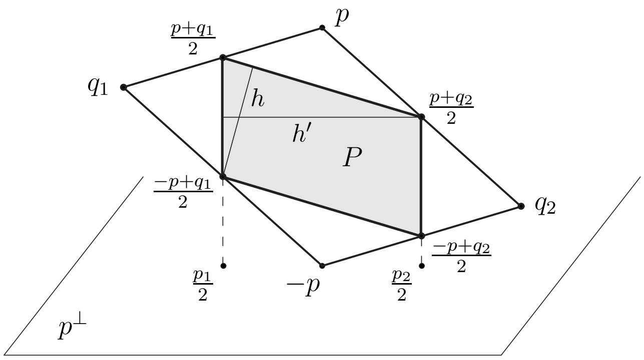


Figure 3.3: The parallelogram P (for the sake of clearness, the hyperplane $p^\perp$ has been translated to $-p$)

We first observe that P is a (2-dimensional) parallelogram, because

$$\begin{aligned}&\frac{1}{2}(p+q_1)-\frac{1}{2}(p+q_2)=\frac{1}{2}(q_1-q_2)=\frac{1}{2}(-p+q_1)-\frac{1}{2}(-p+q_2) \quad \text{and}\\ &\frac{1}{2}(p+q_1)-\frac{1}{2}(-p+q_1)=p=\frac{1}{2}(p+q_2)-\frac{1}{2}(-p+q_2),\end{aligned} \tag{3.9}$$

and since P is a 0-symmetric convex body, $\mathrm{r}(P;\operatorname{aff}P)=\omega(P;\operatorname{aff}P)/2$.

Next we compute the width $\omega(P;\operatorname{aff}P)$. Let h,h' denote the heights of the parallelogram P corresponding to the edges $\big[(p+q_1)/2,(p+q_2)/2\big]$ and $\big[(p+q_1)/2,(-p+q_1)/2\big]$, respectively (see Figure 3.3). From (3.9) we get, on the one hand, that h is just the distance between the orthogonal projections onto $p^\perp$ of the points $(p+q_1)/2$ and $(p+q_2)/2$, i.e., the distance between $p_1/2$ and $p_2/2$. Thus, $h'=|p_1-p_2|_2/2=\mathrm{D}(K|p^\perp)/2$. On the other hand, since

$$\frac{\left|\frac{p+q_1}{2}-\frac{-p+q_1}{2}\right|_2}{h}=\frac{\left|\frac{p+q_1}{2}-\frac{p+q_2}{2}\right|_2}{h'},$$

then we have

$$h=\frac{2h'\,|p|_2}{|q_1-q_2|_2}=\frac{h'\,\mathrm{D}(K)}{|q_1-q_2|_2}\geq h',$$

where the inequality comes from the fact that $q_1,q_2\in K$ and then $|q_1-q_2|_2\leq\mathrm{D}(K)$. Therefore

$$\omega(P;\operatorname{aff}P)=\min\{h,h'\}=h'=\frac{\mathrm{D}(K|p^\perp)}{2},$$

and hence

$$\mathrm{r}(K\cap\operatorname{aff}P;\operatorname{aff}P)\geq\mathrm{r}(P;\operatorname{aff}P)=\frac{\omega(P;\operatorname{aff}P)}{2}=\frac{\mathrm{D}(K|p^\perp)}{4}>\mathrm{r}_2(K),$$

a contradiction.

This shows (3.8), and then, applying Jung's Theorem 1.3.2 to the $(n-1)$-dimensional convex body $K|p^\perp$, we finally get that

$$\mathrm{R}_{n-1}(K)\leq\mathrm{R}(K|p^\perp)\leq\sqrt{\frac{n-1}{2n}}\mathrm{D}(K|p^\perp)\leq2\sqrt{2}\sqrt{\frac{n-1}{n}}\mathrm{r}_2(K). \qquad \square$$

3.4 Additional properties for successive radii

We recall that for any convex body $K \in \mathcal{K}^n$ with $0 \in \operatorname{int} K$ and all $i = 1, \dots, n$, it holds

$$\mathrm{r}_i(K)\,\mathrm{R}_i(K^*) \geq 1 \quad \text{and} \quad \mathrm{R}_i(K)\,\mathrm{r}_i(K^*) \geq 1.$$

Moreover, the equality (in both inequalities) is attained for all 0-symmetric convex bodies (see Proposition 1.3.3 and (1.18)).

One might think about upper bounds for those quotients. Unfortunately they are, in general, not bounded by above. A simple example is given by the sequence of 1-dimensional convex bodies $K = [-\mathrm{e}_1, \varepsilon \mathrm{e}_1] \subset \mathbb{R}$ with $0 < \varepsilon < 1$. Then $K^* = \left[-\mathrm{e}_1, (1/\varepsilon)\mathrm{e}_1\right]$ and therefore

$$\mathrm{R}(K)\mathrm{r}(K^*) = \frac{1+\varepsilon}{2}\,\frac{1+\frac{1}{\varepsilon}}{2} = \frac{(1+\varepsilon)^2}{4\varepsilon}.$$

If ε goes to 0, then the product becomes arbitrarily large. This shows the necessity of imposing more restrictions on K. We have proved the following result.

Proposition 3.4.1. *Let $K \in \mathcal{K}^n$ with $0 \in \operatorname{int} K$, $1 \leq i \leq n$ and $\lambda, \mu > 0$ be such that*

$$\lambda B_n \subseteq K \subseteq \mu B_n.$$

Then

$$\mathrm{R}_i(K)\mathrm{r}_i(K^*) \leq \min\left\{\frac{1+\mu/\lambda}{\sqrt{2}}, \frac{\mu}{\lambda}\right\}.$$

Proof. Since $K \subseteq \mu B_n$ then $\mathrm{R}_i(K) \leq \mu$, and since $\lambda B_n \subseteq K$ we get $K^* \subseteq (1/\lambda)B_n$, and hence $\mathrm{r}_i(K^*) \leq 1/\lambda$. Thus we get the bound $\mathrm{R}_i(K)\mathrm{r}_i(K^*) \leq \mu/\lambda$.

So we have to prove the inequality $\mathrm{R}_i(K)\mathrm{r}_i(K^*) \leq (1+\mu/\lambda)/\sqrt{2}$. We first show the inclusion

$$\frac{1}{2(1+\mu/\lambda)}(K^* - K^*) \subseteq (K-K)^*. \tag{3.10}$$

Let $x_1, x_2 \in K^*$ and $y_1, y_2 \in K$. From the definition of polar body we have $\langle x_1, y_1\rangle, \langle x_2, y_2\rangle \leq 1$ and it also holds that $x_j \in K^* \subseteq (\lambda B_n)^* = (1/\lambda)B_n$ and $y_j \in \mu B_n$ for $j = 1, 2$. Therefore

$$\begin{aligned}\left\langle \frac{1}{2(1+\mu/\lambda)}(x_1 - x_2), y_1 - y_2 \right\rangle &\leq \frac{1}{2(1+\mu/\lambda)}(1 + 1 + |x_1|_2\,|y_2|_2 + |x_2|_2\,|y_1|_2) \\ &\leq \frac{1}{2(1+\mu/\lambda)}\left(2 + 2\,\frac{\mu}{\lambda}\right) = 1,\end{aligned}$$

and then

$$\frac{1}{2(1+\mu/\lambda)}(x_1 - x_2) \in (K-K)^*.$$

It shows (3.10). In the next chapter we will prove that

$$\mathrm{R}_i(K) \leq \frac{1}{\sqrt{2}}\sqrt{\frac{i}{i+1}}\mathrm{R}_i(K-K) \tag{3.11}$$

(see Proposition 4.3.2). Then, the known inequality $\mathrm{r}_i(K^*) \leq \mathrm{r}_i(K^* - K^*)$ (see [33, Lemma 2.1, Remark 2.1] and also Proposition 4.3.2) together with (3.10) and (3.11) imply that

$$\begin{aligned}\mathrm{R}_i(K)\mathrm{r}_i(K^*) &\leq \frac{1}{2\sqrt{2}}\sqrt{\frac{i}{i+1}}\,\mathrm{R}_i(K-K)\,\mathrm{r}_i(K^*-K^*)\\ &\leq \frac{1}{2\sqrt{2}}\sqrt{\frac{i}{i+1}}\,2\left(1+\frac{\mu}{\lambda}\right)\mathrm{R}_i(K-K)\,\mathrm{r}_i\big((K-K)^*\big) = \frac{1+\mu/\lambda}{\sqrt{2}}\sqrt{\frac{i}{i+1}} < \frac{1+\mu/\lambda}{\sqrt{2}},\end{aligned}$$

where in the above equality we have used (1.18) because $K-K$ is a 0-symmetric convex body. $\square$

In this dissertation we have also considered the following natural question. We know that

$$\mathrm{r}_1(K) \geq \cdots \geq \mathrm{r}_n(K) \geq 0 \quad \text{and} \quad 0 \leq \mathrm{R}_1(K) \leq \cdots \leq \mathrm{R}_n(K) \tag{3.12}$$

for any convex body $K \in \mathcal{K}^n$ (see Section 1.2). So, we can ask whether any given set of non-negative numbers satisfying the above left (respectively, right) inequalities can arise as the inner (respectively, outer) successive radii of some convex body. The following proposition answers this question in the positive.

Proposition 3.4.2. *Let $a_1 \geq \cdots \geq a_n > 0$, $0 < b_1 \leq \cdots \leq b_n$ be two sequences of positive real numbers. Then there exist convex bodies $K, K' \in \mathcal{K}^n$ such that $\mathrm{r}_i(K) = a_i$ and $\mathrm{R}_i(K') = b_i$, for all $i = 1, \ldots, n$.*

Proof. Let

$$K = \mathrm{conv}\left\{a_j B_{j,\mathrm{lin}\{\mathrm{e}_1,\ldots,\mathrm{e}_j\}} : j = 1, \ldots, n\right\}.$$

Clearly $\mathrm{r}_j(K) \geq a_j$ for all $j = 1, \ldots, n$, and thus we have to show that $\mathrm{r}_j(K) \leq a_j$. Since K is a 0-symmetric convex body, let $L \in \mathcal{L}^n_j$ be such that

$$\mathrm{r}_j(K) = \mathrm{r}(K \cap L; L) = \min_{x \in \mathrm{relbd}(K\cap L)} |x|_2 .$$

We observe that $\dim\big(L \cap \mathrm{lin}\{\mathrm{e}_j, \ldots, \mathrm{e}_n\}\big) \geq j + (n-j+1) - n = 1$, and therefore, there exists a point $x_0 \in \mathrm{relbd}\,K \cap \big(L \cap \mathrm{lin}\{\mathrm{e}_j, \ldots, \mathrm{e}_n\}\big)$. Then, on the one hand, we can write $x_0 = \sum_{i=1}^n \lambda_i z_i$, with $\lambda_i \geq 0$, $i = 1, \ldots, n$, $\sum_{i=1}^n \lambda_i = 1$ and $z_i \in a_i\,\mathrm{relbd}\,B_{i,\mathrm{lin}\{\mathrm{e}_i,\ldots,\mathrm{e}_n\}}$ for $i = 1, \ldots, n$. Moreover, since $z_i \in a_i\,\mathrm{relbd}\,B_{i,\mathrm{lin}\{\mathrm{e}_i,\ldots,\mathrm{e}_n\}}$, there exist $\mu^i_k \geq 0$, $k = 1, \ldots, i$, with $\sum_{k=1}^i (\mu^i_k)^2 = 1$ such that $z_i = a_i\big(\sum_{k=1}^i \mu^i_k \mathrm{e}_k\big)$. Therefore,

$$x_0 = \sum_{i=1}^n \lambda_i z_i = \sum_{i=1}^n \lambda_i a_i \left(\sum_{k=1}^i \mu^i_k \mathrm{e}_k\right) = \sum_{k=1}^n \left(\sum_{i=k}^n \lambda_i a_i \mu^i_k\right) \mathrm{e}_k.$$

On the other hand, since $x_0 \in \mathrm{lin}\{\mathrm{e}_j, \ldots, \mathrm{e}_n\}$, the above expression reduces to

$$x_0 = \sum_{k=j}^n \left(\sum_{i=k}^n \lambda_i a_i \mu^i_k\right) \mathrm{e}_k = \sum_{i=j}^n \lambda_i a_i \left(\sum_{k=j}^i \mu^i_k \mathrm{e}_k\right),$$

and hence,

$$|x_0|_2 = \left|\sum_{i=j}^{n}\lambda_i a_i\left(\sum_{k=j}^{i}\mu_k^i \mathrm{e}_k\right)\right|_2 \leq \sum_{i=j}^{n}\lambda_i a_i\left|\sum_{k=j}^{i}\mu_k^i \mathrm{e}_k\right|_2 = \sum_{i=j}^{n}\lambda_i a_i\left(\sum_{k=j}^{i}(\mu_k^i)^2\right)^{1/2}$$
$$\leq \sum_{i=j}^{n}\lambda_i a_i \leq a_j\sum_{i=j}^{n}\lambda_i \leq a_j.$$

Thus we can conclude that

$$\mathrm{r}_j(K) = \min_{x\in\mathrm{relbd}(K\cap L)}|x|_2 \leq |x_0|_2 \leq a_j,$$

as required. It shows the first part of the proposition.

Now we assume $0 \leq b_1 \leq \cdots \leq b_n$, and we consider the set of numbers $\{1/b_1,\ldots,1/b_n\}$, which satisfies that

$$\frac{1}{b_1} \geq \cdots \geq \frac{1}{b_n} \geq 0.$$

Then, we have shown that there exists a convex body $K \in \mathcal{K}^n$, which is in addition 0-symmetric, such that $\mathrm{r}_i(K) = 1/b_i$, for all $i = 1,\ldots,n$. Thus, using (1.18) we get that

$$\mathrm{R}_i(K^*) = \frac{1}{\mathrm{r}_i(K)} = b_i.$$

The set K^* gives the result. □

A collection of inequalities relating m geometric measures, which assure that any set of m non-negative real numbers satisfying the inequalities arise as the corresponding measures of a convex body, is called a *full* (or *complete*) *system* of inequalities. Thus, Proposition 3.4.2 shows that the sets of inequalities $0 \leq \mathrm{R}_1 \leq \cdots \leq \mathrm{R}_1$ and $\mathrm{r}_1 \geq \cdots \geq \mathrm{r}_n \geq 0$ are full systems. In [10] we have determined the full system of inequalities corresponding to the successive radii $\mathrm{R}_1, \mathrm{R}_n, \mathrm{r}_1, \mathrm{r}_n$ of planar convex bodies.

Chapter 4

Successive radii and (Firey-)Minkowski addition

It is well known (see [53, p. 42]) that in the case of the diameter, the minimal width, the inradius and the circumradius, for two convex bodies $K, K' \in \mathcal{K}^n$ it holds

$$\begin{aligned} \mathrm{D}(K+K') &\leq \mathrm{D}(K)+\mathrm{D}(K'), \quad \omega(K+K') \geq \omega(K)+\omega(K'), \\ \mathrm{R}(K+K') &\leq \mathrm{R}(K)+\mathrm{R}(K'), \quad \mathrm{r}(K+K') \geq \mathrm{r}(K)+\mathrm{r}(K'). \end{aligned} \tag{4.1}$$

Equality holds in all inequalities, for example, when $K = K'$.

We observe that inequalities (4.1) can be translated as inequalities for the first and the last of all possible outer and inner radii, since

$$\begin{aligned} \mathrm{R}(K) = \mathrm{R}_n(K) = \overline{\mathrm{R}}_n(K) = \widetilde{\mathrm{R}}_n(K), &\qquad \mathrm{r}(K) = \mathrm{r}_n(K) = \bar{\mathrm{r}}_n(K) = \widetilde{\mathrm{r}}_n(K) = \widehat{\mathrm{r}}_n(K), \\ \frac{\mathrm{D}(K)}{2} = \overline{\mathrm{R}}_1(K) = \mathrm{r}_1(K) = \widetilde{\mathrm{r}}_1(K), &\qquad \frac{\omega(K)}{2} = \mathrm{R}_1(K) = \widetilde{\mathrm{R}}_1(K) = \bar{\mathrm{r}}_1(K) = \widehat{\mathrm{r}}_1(K). \end{aligned}$$

Hence the question arises to study the relation between Minkowski addition and all remaining successive inner and outer radii.

In this chapter we first study all possible upper and lower bounds for the classical outer and inner radii R_i and r_i, with respect to the Minkowski addition. We also show that these bounds can be improved when special sums of convex bodies are considered. Next we relate these radii with the more general p-sum of two convex bodies. These results extend the previous ones, since the p-sum coincides with the Minkowski addition in the case $p = 1$. Finally, we study the remaining families of successive outer and inner radii.

All new results collected in this chapter appear in [23, 24, 25].

4.1 Bounds for the outer successive radii and Minkowski addition

We start this section showing the relations between the outer radii R_i and Minkowski addition.

Theorem 4.1.1 ([25]). *Let $K, K' \in \mathcal{K}^n$. Then*

$$\begin{aligned} \mathrm{R}_1(K+K') &\geq \mathrm{R}_1(K)+\mathrm{R}_1(K'), \\ \sqrt{2}\mathrm{R}_i(K+K') &\geq \mathrm{R}_i(K)+\mathrm{R}_i(K'), \quad i=2,\dots,n. \end{aligned} \tag{4.2}$$

All inequalities are best possible.

Proof. The lower bound for $\mathrm{R}_1(K+K')$ (minimal width) is well-known (cf. (4.1)), and equality holds for instance when $K=K'$. So we prove (4.2) for $i=2,\dots,n$.

Let $L \in \mathcal{L}_i^n$. Without loss of generality we may assume that $\mathrm{R}(K|L)B_{i,L}$ and $\mathrm{R}(K'|L)B_{i,L}$ are the circumballs of $K|L$ and $K'|L$ respectively. Then it is well-known (see Theorem 1.2.1) that there exist contact points

$$\begin{aligned} \{u_1,\dots,u_k\} &\subseteq \mathrm{relbd}(K|L)\cap\mathrm{relbd}\big(\mathrm{R}(K|L)B_{i,L}\big), \\ \{v_1,\dots,v_l\} &\subseteq \mathrm{relbd}(K'|L)\cap\mathrm{relbd}\big(\mathrm{R}(K'|L)B_{i,L}\big), \end{aligned}$$

with $2 \leq k,l \leq i+1$, such that $0 \in \mathrm{conv}\{u_1,\dots,u_k\}\cap\mathrm{conv}\{v_1,\dots,v_l\}$.

Now we assume that there exist $t \in L$ and a positive number $\rho < \big(\mathrm{R}(K|L)^2+\mathrm{R}(K'|L)^2\big)^{1/2}$ such that $(K+K')|L \subseteq t+\rho B_{i,L}$, and we will get a contradiction.

Notice first that since $0 \in \mathrm{conv}\{u_1,\dots,u_k\}$, there exists a point, say u_1, such that $\langle u_1,t\rangle \leq 0$: indeed, if for all $i=1,\dots,k$ it holds $\langle u_i,t\rangle > 0$, then $\mathrm{conv}\{u_1,\dots,u_k\}$ and the origin 0 can be strictly separated by a hyperplane with (outer) normal vector t (see [53, p. 12]), which contradicts the fact that $0 \in \mathrm{conv}\{u_1,\dots,u_k\}$. Then we get

$$|u_1-t|_2^2 = \mathrm{R}(K|L)^2 - 2\langle u_1,t\rangle + |t|_2^2 \geq \mathrm{R}(K|L)^2.$$

Next we take the vector u_1-t. Notice that $u_1-t \neq 0$ because $\langle u_1,t\rangle \leq 0$ and $u_1 \neq 0$. Since $0 \in \mathrm{conv}\{v_1,\dots,v_l\}$, an analogous argument to the previous one shows that there exists a point, say v_1, such that $\langle v_1,u_1-t\rangle \geq 0$. Finally we consider the point

$$u_1+v_1 \in K|L+K'|L=(K+K')|L \subseteq t+\rho B_{i,L},$$

for which, using the above conditions, we get

$$|u_1+v_1-t|_2^2 = |u_1-t|_2^2 + 2\langle u_1-t,v_1\rangle + |v_1|_2^2 \geq \mathrm{R}(K|L)^2+\mathrm{R}(K'|L)^2 > \rho^2,$$

a contradiction. Therefore $\rho \geq \big(\mathrm{R}(K|L)^2+\mathrm{R}(K'|L)^2\big)^{1/2}$ and, in particular, the same holds for the circumradius of $(K+K')|L$. Hence we finally get

$$\mathrm{R}\big((K+K')|L\big) \geq \big(\mathrm{R}(K|L)^2+\mathrm{R}(K'|L)^2\big)^{1/2} \geq \frac{\sqrt{2}}{2}\big(\mathrm{R}(K|L)+\mathrm{R}(K'|L)\big)$$

for all $L \in \mathcal{L}_i^n$. Now let $L_i \in \mathcal{L}_i^n$ be such that $\mathrm{R}_i(K+K') = \mathrm{R}\big((K+K')|L_i\big)$. Then, together with the above property, we can conclude that

$$\mathrm{R}_i(K+K') = \mathrm{R}\big((K+K')|L_i\big) \geq \frac{1}{\sqrt{2}}\big(\mathrm{R}(K|L_i) + \mathrm{R}(K'|L_i)\big) \geq \frac{1}{\sqrt{2}}\big(\mathrm{R}_i(K) + \mathrm{R}_i(K')\big),$$

which proves (4.2) for $i = 2, \dots, n$.

It remains to be shown that these inequalities are best possible. We fix $i \in \{2, \dots, n\}$ and consider the convex bodies

$$K = [-\mathrm{e}_1, \mathrm{e}_1] + \sum_{k=i+1}^{n} [-\mathrm{e}_k, \mathrm{e}_k], \quad K' = [-\mathrm{e}_2, \mathrm{e}_2] + \sum_{k=i+1}^{n} [-\mathrm{e}_k, \mathrm{e}_k]. \tag{4.3}$$

Here for $i = n$ we are just taking $K = [-\mathrm{e}_1, \mathrm{e}_1]$, $K' = [-\mathrm{e}_2, \mathrm{e}_2]$. Since K and K' are both $(n-i+1)$-cubes with edges parallel to the coordinate axes and length 2, it is clear that $\mathrm{R}(K|L), \mathrm{R}(K'|L) \geq 1$ for all $L \in \mathcal{L}_i^n$. Moreover, if $L = \mathrm{lin}\{\mathrm{e}_1, \dots, \mathrm{e}_i\}$ then $\mathrm{R}(K|L) = \mathrm{R}(K'|L) = 1$. This shows that $\mathrm{R}_i(K) = \mathrm{R}_i(K') = 1$. Now we take the sum

$$K + K' = [-\mathrm{e}_1, \mathrm{e}_1] + [-\mathrm{e}_2, \mathrm{e}_2] + 2\sum_{k=i+1}^{n} [-\mathrm{e}_k, \mathrm{e}_k],$$

an $(n-i+2)$-dimensional parallelepiped with edges again parallel to the coordinate axes and lengths 2 and 4. Then it is easy to see that

$$\mathrm{R}_i(K+K') = \mathrm{R}\big((K+K')|\,\mathrm{lin}\{\mathrm{e}_1, \dots, \mathrm{e}_i\}\big) = \sqrt{2} = \frac{1}{\sqrt{2}}\big(\mathrm{R}_i(K) + \mathrm{R}_i(K')\big),$$

which concludes the proof of the theorem. □

We already know that there exists an upper bound for $\mathrm{R}_n(K+K')$ in terms of the sum of the circumradii, namely,

$$\mathrm{R}_n(K+K') \leq \mathrm{R}_n(K) + \mathrm{R}_n(K')$$

(cf. (4.1)). So, the natural question arises whether there exists an upper bound for the remaining outer radii. Next proposition answers this question.

Proposition 4.1.1 ([25]). *Let $K, K' \in \mathcal{K}^n$. For all $i = 1, \dots, n-1$, there exists no constant $c > 0$ such that $c\mathrm{R}_i(K+K') \leq \mathrm{R}_i(K) + \mathrm{R}_i(K')$.*

Proof. For any $i \in \{1, \dots, n-1\}$ fixed, we define the convex bodies

$$K = [-\mathrm{e}_{n-i+1}, \mathrm{e}_{n-i+1}] \quad \text{and} \quad K' = \sum_{k=1}^{n-i} [-\mathrm{e}_k, \mathrm{e}_k].$$

Notice that the projections $K|\,\mathrm{lin}\{\mathrm{e}_{n-i}, \mathrm{e}_{n-i+2}, \dots, \mathrm{e}_n\} = K'|\,\mathrm{lin}\{\mathrm{e}_{n-i+1}, \dots, \mathrm{e}_n\} = \{0\}$, and hence both $\mathrm{R}_i(K) = \mathrm{R}_i(K') = 0$, i.e., $\mathrm{R}_i(K) + \mathrm{R}_i(K') = 0$. However, $K + K' = \sum_{k=1}^{n-i+1} [-\mathrm{e}_k, \mathrm{e}_k]$ is an $(n-i+1)$-dimensional convex body, which implies that the dimension $\dim\big((K+K')|L\big) \geq 1$ for all $L \in \mathcal{L}_i^n$, and thus $\mathrm{R}(K+K') > 0$. Hence we conclude that there exists no constant $c > 0$ such that $c\,\mathrm{R}_i(K+K') \leq \mathrm{R}_i(K) + \mathrm{R}_i(K')$ for any $i = 1, \dots, n-1$. □

4.2 Bounds for the inner successive radii and Minkowski addition

First, we state some preliminary results in Linear Algebra which will be needed in the proof of the main theorem.

Lemma 4.2.1 ([25]). *For $1 \leq i \leq n/2$, let $L, L' \in \mathcal{L}_i^n$ be such that $L \cap L' = \{0\}$. Then there exist orthonormal bases $\{u_1, \ldots, u_i\}$ and $\{v_1, \ldots, v_i\}$ of L and L' respectively, such that the 2-dimensional subspaces $\operatorname{lin}\{u_1, v_1\}, \ldots, \operatorname{lin}\{u_i, v_i\}$ are pairwise orthogonal.*

Proof. Throughout the proof we will always work with $L + L'$ as the main vector space instead of $\mathbb{R}^n$ when considering subspaces, orthogonal complements, projections... Moreover we will identify $L + L' \equiv \mathbb{R}^{2i}$ for the sake of brevity. We distinguish two cases.

Case (i): First we suppose that $L \cap L'^{\perp} = \{0\}$. Denoting by π' the orthogonal projection onto L', it clearly holds that $\pi'(L) = L'$. We assume, without loss of generality, that $L' = \operatorname{lin}\{\mathrm{e}_{i+1}, \ldots, \mathrm{e}_{2i}\}$, and let $w_j \in L$ be such that $\pi'(w_j) = \mathrm{e}_{i+j}$ for $j = 1, \ldots, i$. Let $W = (w_1 \cdots w_i) \in \mathbb{R}^{2i \times i}$ be the $(2i \times i)$-matrix with column vectors w_j, which takes the form

$$W = \begin{pmatrix} M \\ \mathrm{I}_i \end{pmatrix}, \quad M \in \mathbb{R}^{i \times i}.$$

Here I_i denotes the $(i \times i)$-identity matrix. Then the singular value decomposition of a real matrix (see e.g. [41, p. 80]) ensures the existence of orthogonal matrices $U, V \in \mathbb{R}^{i \times i}$ and a diagonal matrix $D = \operatorname{diag}\{d_1, \ldots, d_i\}$ such that $U^{\intercal} M V = D$. We write $U = (u'_1 \cdots u'_i)$ and $V = (v'_1 \cdots v'_i)$, with $u'_j = (u'_{j1}, \ldots, u'_{ji})^{\intercal}$ and $v'_j = (v'_{j1}, \ldots, v'_{ji})^{\intercal}$. Notice that, on the one hand,

$$WV = \left(\sum_{k=1}^{i} v'_{1k} w_k \cdots \sum_{k=1}^{i} v'_{ik} w_k \right),$$

i.e., the column vectors of WV are linear combinations of $\{w_1, \ldots, w_i\}$. So they lie in L. On the other hand,

$$WV = \begin{pmatrix} M \\ \mathrm{I}_i \end{pmatrix} V = \begin{pmatrix} MV \\ V \end{pmatrix} = \begin{pmatrix} UD \\ V \end{pmatrix} = \begin{pmatrix} d_1 u'_1 & \cdots & d_i u'_i \\ v'_1 & \cdots & v'_i \end{pmatrix}.$$

Therefore, the column vectors $(d_j u'_j, v'_j)^{\intercal} \in L$ for all $j = 1, \ldots, i$. Notice that $d_j \neq 0$, $j = 1, \ldots, i$, otherwise we would get $(0, v'_j)^{\intercal} \in L \cap L' = \{0\}$, which is a contradiction. Moreover, it holds that the set $\left\{(d_j u'_j, v'_j)^{\intercal} : j = 1, \ldots, i\right\}$ consists of non-zero pairwise orthogonal vectors, since

$$\left\langle (d_j u'_j, v'_j)^{\intercal}, (d_k u'_k, v'_k)^{\intercal} \right\rangle = \left\langle d_j u'_j, d_k u'_k \right\rangle + \left\langle v'_j, v'_k \right\rangle = 0$$

for all $j \neq k$, $j, k \in \{1, \ldots, i\}$, because U, V are orthogonal matrices. Then, we define the vectors

$$u_j = \frac{1}{\left|(d_j u'_j, v'_j)^{\intercal}\right|_2} (d_j u'_j, v'_j)^{\intercal} \in L, \quad v_j = \frac{1}{\left|(0, v'_j)^{\intercal}\right|_2} (0, v'_j)^{\intercal} \in L',$$

for $j = 1, \dots, i$. By construction, $\{u_1, \dots, u_i\}$ and $\{v_1, \dots, v_i\}$ are orthonormal bases of L and L' respectively. Moreover, for $au_j + bv_j \in \operatorname{lin}\{u_j, v_j\}$ and $cu_k + dv_k \in \operatorname{lin}\{u_k, v_k\}$ with $j \neq k$, $j, k \in \{1, \dots, i\}$, we get

$$\langle au_j + bv_j, cu_k + dv_k \rangle = ad \langle u_j, v_k \rangle + bc \langle v_j, u_k \rangle = 0,$$

i.e., the 2-dimensional subspaces $\operatorname{lin}\{u_1, v_1\}, \dots, \operatorname{lin}\{u_i, v_i\}$ are pairwise orthogonal, as required.

Case (ii): Now we assume $L \cap L'^{\perp} \neq \{0\}$. Since $L^{\perp} \cap L' = (L + L'^{\perp})^{\perp}$, we have

$$\dim(L^{\perp} \cap L') = \dim(L + L'^{\perp})^{\perp} = 2i - \dim L - \dim L'^{\perp} + \dim(L \cap L'^{\perp}) = \dim(L \cap L'^{\perp}).$$

So, let $m = \dim(L^{\perp} \cap L') = \dim(L \cap L'^{\perp})$, $0 < m \leq i$, and let $\{u_1, \dots, u_m\}$ and $\{v_1, \dots, v_m\}$ be orthonormal bases of $L \cap L'^{\perp}$ and $L^{\perp} \cap L'$, respectively. We define $\bar{L} = (L \cap L'^{\perp}) + (L^{\perp} \cap L')$. Then

$$\bar{L}^{\perp} \cap L = \left[(L^{\perp} + L') \cap (L + L'^{\perp})\right] \cap L = (L^{\perp} + L') \cap L$$

and hence

$$\begin{aligned}
\dim(\bar{L}^{\perp} \cap L) &= \dim\big((L^{\perp} + L') \cap L\big) \\
&= \left[\dim L^{\perp} + \dim L' - \dim(L^{\perp} \cap L')\right] + \dim L - \dim(L^{\perp} + L' + L) \\
&= i + i - m + i - 2i = i - m.
\end{aligned}$$

Analogously we get $\dim(\bar{L}^{\perp} \cap L') = i - m$. Moreover it is clear that the intersection

$$\big(\bar{L}^{\perp} \cap L\big) \cap \big(\bar{L}^{\perp} \cap L'\big)^{\perp} = \{0\},$$

and thus we can apply the previous case (i) to the subspaces $\bar{L}^{\perp} \cap L, \bar{L}^{\perp} \cap L' \subset \bar{L}^{\perp}$ to get orthonormal bases $\{u_{m+1}, \dots, u_i\}$ and $\{v_{m+1}, \dots, v_i\}$ of $\bar{L}^{\perp} \cap L$ and $\bar{L}^{\perp} \cap L'$ respectively, such that the 2-dimensional subspaces $\operatorname{lin}\{u_{m+1}, v_{m+1}\}, \dots, \operatorname{lin}\{u_i, v_i\}$ are pairwise orthogonal. Embedding these vectors in the canonical way in $\mathbb{R}^{2i}$ we get orthonormal bases of L and L' verifying the required property. □

Lemma 4.2.2 ([25]). *Let $L, L' \in \mathcal{L}_i^n$. There exist orthonormal bases $\{u_1, \dots, u_i\}$ and $\{v_1, \dots, v_i\}$ of L and L' respectively, such that $\langle u_j, v_j \rangle \geq 0$ for all $j = 1, \dots, i$ and such that the vectors $\{u_1 + v_1, \dots, u_i + v_i\}$ are pairwise orthogonal.*

Proof. Let $k = \dim L \cap L' \leq i$ and let $w_1, \dots, w_k$ be an orthonormal basis of $L \cap L'$. Then we define $u_j = w_j \in L$ and $v_j = w_j \in L'$, for all $1 \leq j \leq k$. The vectors $\{u_1 + v_1, \dots, u_k + v_k\}$ are trivially pairwise orthogonal since $u_j + v_j = 2w_j$, and moreover, $\langle u_j, v_j \rangle = 1$, $j = 1, \dots, k$. So, they verify the required properties, and we have to complete them to bases of L and L'.

Let $\bar{L} = L \cap L'$ and consider $L \cap \bar{L}^{\perp}$ and $L' \cap \bar{L}^{\perp}$. Notice that

$$\dim L \cap \bar{L}^{\perp} = \dim L + \dim \bar{L}^{\perp} - \dim(L + \bar{L}^{\perp}) = i + (n - k) - n = i - k,$$

since $L+\bar{L}^{\perp}=\mathbb{R}^n$. Analogously $\dim L'\cap\bar{L}^{\perp}=i-k$. Moreover,

$$\left(L\cap\bar{L}^{\perp}\right)\cap\left(L'\cap\bar{L}^{\perp}\right)=L\cap L'\cap\bar{L}^{\perp}=\bar{L}\cap\bar{L}^{\perp}=\{0\},$$

and thus we can apply Lemma 4.2.1 to the subspaces $L\cap\bar{L}^{\perp}, L'\cap\bar{L}^{\perp}\in\mathcal{L}^n_{i-k}$ to get the existence of orthonormal bases

$$\{u_{k+1},\dots,u_i\}\subset L\cap\bar{L}^{\perp}\quad\text{ and }\quad\{v_{k+1},\dots,v_i\}\subset L'\cap\bar{L}^{\perp}$$

such that the subspaces $\operatorname{lin}\{u_{k+1},v_{k+1}\},\dots,\operatorname{lin}\{u_i,v_i\}$ are pairwise orthogonal. Notice that the vectors v_j can be chosen such that $\langle u_j,v_j\rangle\geq 0$ for all $j=k+1,\dots,i$, otherwise we just have to replace v_j by $-v_j$. Since $u_j,v_j\in\bar{L}^{\perp}$ for all $j=k+1,\dots,i$, together with the previously selected vectors, we obtain orthonormal bases $\{u_1,\dots,u_i\}$ and $\{v_1,\dots,v_i\}$ of L and L' respectively, verifying also that $\langle u_j,v_j\rangle\geq 0$ for all $j=1,\dots,i$. Moreover, since $u_j+v_j\in\operatorname{lin}\{u_j,v_j\}$ for $j=k+1,\dots,i$ and these 2-dimensional subspaces are pairwise orthogonal, we also get the required orthogonality property for the vectors u_j+v_j, $j=1,\dots,i$. □

Now we state the bounds for the inner radii r_i with respect to Minkowski addition.

Theorem 4.2.1 ([25]). *Let $K,K'\in\mathcal{K}^n$. Then*

$$\begin{aligned}\sqrt{2}\mathrm{r}_i(K+K')&\geq\mathrm{r}_i(K)+\mathrm{r}_i(K'),\quad i=1,\dots,n-1,\\ \mathrm{r}_n(K+K')&\geq\mathrm{r}_n(K)+\mathrm{r}_n(K').\end{aligned}\tag{4.4}$$

All inequalities are best possible.

Proof. The lower bound for $\mathrm{r}_n(K+K')$ (inradius) is well-known (cf. (4.1)), and equality holds for instance when $K=K'$. So we prove (4.4) for $i=1,\dots,n-1$.

Without loss of generality we may assume that $\mathrm{r}_i(K)=\mathrm{r}(K\cap L;L)$ and $\mathrm{r}_i(K')=\mathrm{r}(K'\cap L';L')$ for $L,L'\in\mathcal{L}^n_i$, i.e., that the greatest i-dimensional balls contained in K and K' are $\mathrm{r}(K\cap L;L)B_{i,L}$ and $\mathrm{r}(K'\cap L';L')B_{i,L'}$, respectively. For the sake of brevity we write $\mathrm{r}=\mathrm{r}(K\cap L;L)=\mathrm{r}_i(K)$ and $\mathrm{r}'=\mathrm{r}(K'\cap L';L')=\mathrm{r}_i(K')$. Thus it suffices to show that inequality (4.4) holds for i-dimensional balls, i.e., that

$$\sqrt{2}\mathrm{r}_i(\mathrm{r}B_{i,L}+\mathrm{r}'B_{i,L'})\geq\mathrm{r}+\mathrm{r}',\tag{4.5}$$

since, taking into account that $\mathrm{r}B_{i,L}+\mathrm{r}'B_{i,L'}\subseteq K+K'$, we have

$$\sqrt{2}\mathrm{r}_i(K+K')\geq\sqrt{2}\mathrm{r}_i(\mathrm{r}B_{i,L}+\mathrm{r}'B_{i,L'})\geq\mathrm{r}+\mathrm{r}'=\mathrm{r}_i(K)+\mathrm{r}_i(K').$$

So we have to prove (4.5). By Lemma 4.2.2 we can assure the existence of two subsets of pairwise orthogonal vectors

$$\{u_1,\dots,u_i\}\in\operatorname{relbd}(\mathrm{r}B_{i,L})\quad\text{ and }\quad\{v_1,\dots,v_i\}\in\operatorname{relbd}(\mathrm{r}'B_{i,L'}),$$

such that $\{u_1+v_1,\dots,u_i+v_i\}$ are also pairwise orthogonal with $\langle u_j,v_j\rangle\geq 0$, $j=1,\dots,i$. Let $\bar{L}=\operatorname{lin}\{u_1+v_1,\dots,u_i+v_i\}\in\mathcal{L}_i^n$. Next we show that the i-dimensional ball

$$\left[\mathrm{r}^2+(\mathrm{r}')^2\right]^{1/2}B_{i,\bar{L}}\subset \mathrm{r}B_{i,L}+\mathrm{r}'B_{i,L'}. \tag{4.6}$$

Notice first that

$$|u_j+v_j|_2^2=|u_j|_2^2+|v_j|_2^2+2\langle u_j,v_j\rangle\geq \mathrm{r}^2+(\mathrm{r}')^2.$$

Then, denoting by

$$\mathcal{E}=\left\{\sum_{j=1}^i\lambda_j(u_j+v_j):\lambda_j\in[-1,1],\sum_{j=1}^i\lambda_j^2\leq 1\right\}$$

the 0-symmetric ellipsoid with semi-axes $\{u_j+v_j, j=1,\dots,i\}$, it trivially holds that

$$\left[\mathrm{r}^2+(\mathrm{r}')^2\right]^{1/2}B_{i,\bar{L}}\subseteq\mathcal{E}.$$

Thus, in order to show (4.6) it suffices to prove the inclusion $\mathcal{E}\subset \mathrm{r}B_{i,L}+\mathrm{r}'B_{i,L'}$, i.e., that

$$\sum_{j=1}^i\lambda_j(u_j+v_j)\in \mathrm{r}B_{i,L}+\mathrm{r}'B_{i,L'}\quad\text{for}\quad\sum_{j=1}^i\lambda_j^2=1.$$

Clearly, $\sum_{j=1}^i\lambda_ju_j\in L$, and moreover, since $\{u_1,\dots,u_i\}$ are pairwise orthogonal vectors with $|u_j|_2=\mathrm{r}$, we have

$$\left|\sum_{j=1}^i\lambda_ju_j\right|_2^2=\sum_{j=1}^i\lambda_j^2\,|u_j|_2^2=\mathrm{r}^2\sum_{j=1}^i\lambda_j^2=\mathrm{r}^2.$$

Therefore, $\sum_{j=1}^i\lambda_ju_j\in \mathrm{r}B_{i,L}$. Analogously we get $\sum_{j=1}^i\lambda_jv_j\in \mathrm{r}'B_{i,L'}$ and thus

$$\sum_{j=1}^i\lambda_j(u_j+v_j)=\sum_{j=1}^i\lambda_ju_j+\sum_{j=1}^i\lambda_jv_j\in \mathrm{r}B_{i,L}+\mathrm{r}'B_{i,L'}.$$

This shows (4.6) and we can conclude that

$$\mathrm{r}_i(\mathrm{r}B_{i,L}+\mathrm{r}'B_{i,L'})\geq \mathrm{r}_i\left(\left[\mathrm{r}^2+(\mathrm{r}')^2\right]^{1/2}B_{i,\bar{L}}\right)=\left[\mathrm{r}^2+(\mathrm{r}')^2\right]^{1/2}\geq\frac{1}{\sqrt{2}}(\mathrm{r}+\mathrm{r}'),$$

which gives the required inequality (4.5).

It remains to be shown that these inequalities are best possible. We fix $i\in\{1,\dots,n-1\}$. Let $j=2i-n$ if $2i\geq n$, and $j=0$ otherwise, and we consider the i-dimensional linear subspaces

$$L=\operatorname{lin}\{\mathrm{e}_1,\dots,\mathrm{e}_j,\mathrm{e}_{j+1},\dots,\mathrm{e}_i\},\quad L'=\operatorname{lin}\{\mathrm{e}_1,\dots,\mathrm{e}_j,\mathrm{e}_{i+1},\dots,\mathrm{e}_{2i-j}\}. \tag{4.7}$$

We are going to show that equality in (4.4) is attained for the i-dimensional unit balls $B_{i,L}$ and $B_{i,L'}$. Notice that if we prove the inequality

$$\mathrm{r}_i(B_{i,L}+B_{i,L'})\leq\sqrt{2} \tag{4.8}$$

then by (4.4) we can conclude that

$$\sqrt{2} \geq \mathrm{r}_i(B_{i,L} + B_{i,L'}) \geq \frac{1}{\sqrt{2}} \left[\mathrm{r}_i(B_{i,L}) + \mathrm{r}_i(B_{i,L'})\right] = \frac{1}{\sqrt{2}} (1+1) = \sqrt{2},$$

which gives the required result. Observe first that since $B_{i,L} + B_{i,L'}$ is a 0-symmetric convex body, for any $\bar{L} \in \mathcal{L}_i^n$ we have

$$\max_{x \in \bar{L}^\perp} \mathrm{r}\big((B_{i,L} + B_{i,L'}) \cap (x + \bar{L}); x + \bar{L}\big) = \mathrm{r}\big((B_{i,L} + B_{i,L'}) \cap \bar{L}; \bar{L}\big).$$

Therefore in order to show (4.8) it suffices to prove that

$$\mathrm{r}\big((B_{i,L} + B_{i,L'}) \cap \bar{L}; \bar{L}\big) \leq \sqrt{2} \quad \text{for all } \ \bar{L} \in \mathcal{L}_i^n. \tag{4.9}$$

If $\dim\big((B_{i,L} + B_{i,L'}) \cap \bar{L}\big) < i$ for $\bar{L} \in \mathcal{L}_i^n$ then $\mathrm{r}\big((B_{i,L} + B_{i,L'}) \cap \bar{L}; \bar{L}\big) = 0$. So we take $\bar{L} \in \mathcal{L}_i^n$ with $\dim\big((B_{i,L} + B_{i,L'}) \cap \bar{L}\big) = i$. Notice that if we find $x \in \mathrm{relbd}\big((B_{i,L} + B_{i,L'}) \cap \bar{L}\big)$ with $|x|_2 \leq \sqrt{2}$, then we immediately get (4.9). In order to find such an x, let $L'' = \mathrm{lin}\{\mathrm{e}_{j+1}, \dots, \mathrm{e}_n\}$. If $j = 2i - n$ (i.e., if $2i \geq n$) then

$$\begin{aligned}\dim(\bar{L} \cap L'') &= \dim \bar{L} + \dim L'' - \dim(\bar{L} + L'') = i + n - j - \dim(\bar{L} + L'') \\ &\geq i + n - j - n = i - j = i - 2i + n = n - i \geq 1,\end{aligned}$$

and moreover, $L + L' = \mathbb{R}^n$, i.e., $\dim(B_{i,L} + B_{i,L'}) = n$. On the other hand, if $j = 0$ then $L'' = \mathbb{R}^n$, and so $\bar{L} \cap L'' = \bar{L}$. Therefore, in both cases, $\dim\big((B_{i,L} + B_{i,L'}) \cap \bar{L} \cap L''\big) \geq 1$, which ensures the existence of a boundary point $x \in \mathrm{relbd}(B_{i,L} + B_{i,L'}) \cap \bar{L} \cap L''$. Since any $x \in \mathrm{relbd}(B_{i,L} + B_{i,L'}) \cap L''$ is expressed in the form

$$x = \sum_{k=j+1}^{i} \lambda_k \mathrm{e}_k + \sum_{k=i+1}^{2i-j} \mu_k \mathrm{e}_k, \quad \text{with} \ \sum_{k=j+1}^{i} \lambda_k^2 = 1, \sum_{k=i+1}^{2i-j} \mu_k^2 = 1,$$

we trivially get

$$|x|_2^2 = \sum_{k=j+1}^{i} \lambda_k^2 + \sum_{k=i+1}^{2i-j} \mu_k^2 = 2.$$

This shows (4.9) and concludes the proof. □

We already know that there exists also an upper bound for $\mathrm{r}_1(K + K')$ in terms of the sum of the diameters, namely,

$$\mathrm{r}_1(K + K') \leq \mathrm{r}_1(K) + \mathrm{r}_1(K')$$

(cf. (4.1)). So, the natural question arises whether there exists an upper bound for the remaining inner radii. Next proposition answers this question.

Proposition 4.2.1 ([25]). *Let $K, K' \in \mathcal{K}^n$. For all $i = 2, \dots, n$, there exists no constant $c > 0$ such that $c\mathrm{r}_i(K + K') \leq \mathrm{r}_i(K) + \mathrm{r}_i(K')$.*

Proof. For any $i \in \{2, \dots, n\}$ fixed, we define the convex bodies

$$K = [-\mathrm{e}_1, \mathrm{e}_1] \quad \text{and} \quad K' = \sum_{k=2}^{i} [-\mathrm{e}_k, \mathrm{e}_k]. \tag{4.10}$$

Since K and K' are, respectively, a 1-dimensional and an $(i-1)$-dimensional convex body, then $\mathrm{r}_i(K) = \mathrm{r}_i(K') = 0$. However, $K + K' = \sum_{k=1}^{i} [-\mathrm{e}_k, \mathrm{e}_k]$ and clearly $\mathrm{r}_i(K + K') = 1$. Hence we can conclude that there exists no constant $c > 0$ such that $c\,\mathrm{r}_i(K + K') \leq \mathrm{r}_i(K) + \mathrm{r}_i(K')$ for any $i = 2, \dots, n$. □

The bounds obtained in Theorems 4.1.1 and 4.2.1 can be improved when sums of special convex bodies are considered. Moreover, reverse inequalities to (4.2) and (4.4) exist for these special sums (cf. Propositions 4.1.1 and 4.2.1). We deal with this questions in the next section.

4.3 Minkowski addition of special convex bodies

Observe that equality in (4.2) and (4.4) is attained in both cases for convex bodies with empty interior. Also the non-existence of the reverse inequalities is due to this particular type of bodies (see Propositions 4.1.1 and 4.2.1). Thus the question arises whether those inequalities can be improved if convex bodies with non-empty interior are considered. So we ask, in particular, for the special case when one of the bodies involved is the Euclidean ball.

Proposition 4.3.1 ([25]). *Let $K \in \mathcal{K}^n$ and $r \geq 0$. Then for all $i = 1, \dots, n$,*

$$\mathrm{R}_i(K + rB_n) = \mathrm{R}_i(K) + r \quad \text{and} \quad \mathrm{r}_i(K + rB_n) \geq \mathrm{r}_i(K) + r.$$

All inequalities are best possible and for $i = 2, \dots, n-1$ they can be strict.

Proof. The identity for R_i is a straightforward computation:

$$\mathrm{R}_i(K + rB_n) = \min_{L \in \mathcal{L}_i^n} \mathrm{R}\big((K + rB_n)|L\big) = \min_{L \in \mathcal{L}_i^n} \mathrm{R}(K|L + rB_n|L) = \mathrm{R}_i(K) + r.$$

Now we show the lower bound for $\mathrm{r}_i(K + rB_n)$. First we notice that for any $L \in \mathcal{L}_i^n$ and $x \in \mathbb{R}^n$, we have

$$K \cap (x + L) + rB_{i,L} \subseteq (K + rB_n) \cap (x + L).$$

Indeed, if $z \in K \cap (x + L) + rB_{i,L}$ then $z = x + l + ru$, where $l \in L$, $x + l \in K$ and $u \in B_{i,L}$, and thus $z = x + l + ru \in (K + rB_n) \cap (x + L)$.

Let $L_i \in \mathcal{L}_i^n$ and $x \in L_i^{\perp}$ be such that $\mathrm{r}_i(K) = \mathrm{r}\big(K \cap (x + L_i); x + L_i\big)$. Then using the above property we get

$$\begin{aligned} \mathrm{r}_i(K + rB_n) &\geq \mathrm{r}\big((K + rB_n) \cap (x + L_i); x + L_i\big) \geq \mathrm{r}\big(K \cap (x + L_i) + rB_{i,L_i}; x + L_i\big) \\ &= \mathrm{r}\big(K \cap (x + L_i); x + L_i\big) + r = \mathrm{r}_i(K) + r. \end{aligned}$$

Equality holds, for instance, if $K = B_n$. Finally we show that, unlike the R_i case, there exist convex bodies with $\mathrm{r}_i(K + rB_n) > \mathrm{r}_i(K) + r$.

We consider the non-regular triangular antiprism $P_\varepsilon = \operatorname{conv}\{\pm v_1, \pm v_2, \pm v_3\}$ in $\mathbb{R}^3$ with vertices

$$v_1 = \left(\frac{1}{\sqrt{3}}, 1, \varepsilon\right)^\intercal, \quad v_2 = \left(\frac{1}{\sqrt{3}}, -1, \varepsilon\right)^\intercal, \quad v_3 = \left(-\frac{2}{\sqrt{3}}, 0, \varepsilon\right)^\intercal,$$

for $\varepsilon > 0$ (see Figure 3.2). In the proof of Theorem 3.3.1 it was shown that $\mathrm{r}_2(P_\varepsilon) = \sqrt{3}/2$ for $\varepsilon \le \sqrt{5/12}$. So, if we show that

$$\mathrm{r}_2(P_\varepsilon + rB_3) \ge 1 + \sqrt{r^2 - \varepsilon^2} \quad \text{for } r \ge \varepsilon, \tag{4.11}$$

then we will get that

$$\mathrm{r}_2(P_\varepsilon + rB_3) \ge 1 + \sqrt{r^2 - \varepsilon^2} > \frac{\sqrt{3}}{2} + r = \mathrm{r}_2(P_\varepsilon) + r$$

for $\varepsilon \le \sqrt{5/12}$ and $r > \left(2 + \sqrt{3}\right)\left(\varepsilon^2 - \sqrt{3} + 7/4\right) \ge \varepsilon$, as required.

Observe that in order to prove (4.11) it suffices to show that

$$\mathrm{r}\big((P_\varepsilon + rB_3) \cap \operatorname{lin}\{\mathrm{e}_1, \mathrm{e}_2\}; \operatorname{lin}\{\mathrm{e}_1, \mathrm{e}_2\}\big) \ge 1 + \sqrt{r^2 - \varepsilon^2}.$$

Denoting by $\pm\bar{v}_j = \pm v_j | \operatorname{lin}\{\mathrm{e}_1, \mathrm{e}_2\}$, it is a straightforward computation to check that

$$(\pm v_j + rB_3) \cap \operatorname{lin}\{\mathrm{e}_1, \mathrm{e}_2\} = \pm\bar{v}_j + \sqrt{r^2 - \varepsilon^2} B_{2,\operatorname{lin}\{\mathrm{e}_1,\mathrm{e}_2\}}.$$

Since $(\pm v_j + rB_3) \cap \operatorname{lin}\{\mathrm{e}_1, \mathrm{e}_2\} \subset (P_\varepsilon + rB_3) \cap \operatorname{lin}\{\mathrm{e}_1, \mathrm{e}_2\}$, then

$$\begin{aligned}(P_\varepsilon + rB_3) \cap \operatorname{lin}\{\mathrm{e}_1, \mathrm{e}_2\} &\supset \operatorname{conv}\left\{\pm\bar{v}_j + \sqrt{r^2 - \varepsilon^2} B_{2,\operatorname{lin}\{\mathrm{e}_1,\mathrm{e}_2\}} : j = 1, 2, 3\right\} \\ &= \big(P_\varepsilon | \operatorname{lin}\{\mathrm{e}_1, \mathrm{e}_2\}\big) + \sqrt{r^2 - \varepsilon^2} B_{2,\operatorname{lin}\{\mathrm{e}_1,\mathrm{e}_2\}}.\end{aligned}$$

Notice that the projected body $H = P_\varepsilon | \operatorname{lin}\{\mathrm{e}_1, \mathrm{e}_2\}$ is the regular hexagon in the plane $\operatorname{lin}\{\mathrm{e}_1, \mathrm{e}_2\}$ with vertices $\pm\bar{v}_j$, $j = 1, 2, 3$, which has 2-dimensional inradius $\mathrm{r}\big(H; \operatorname{lin}\{\mathrm{e}_1, \mathrm{e}_2\}\big) = 1$. Thus,

$$\begin{aligned}\mathrm{r}\big((P_\varepsilon + rB_3) \cap \operatorname{lin}\{\mathrm{e}_1, \mathrm{e}_2\}; \operatorname{lin}\{\mathrm{e}_1, \mathrm{e}_2\}\big) &\ge \mathrm{r}\Big(H + \sqrt{r^2 - \varepsilon^2} B_{2,\operatorname{lin}\{\mathrm{e}_1,\mathrm{e}_2\}}; \operatorname{lin}\{\mathrm{e}_1, \mathrm{e}_2\}\Big) \\ &= \mathrm{r}\big(H; \operatorname{lin}\{\mathrm{e}_1, \mathrm{e}_2\}\big) + \sqrt{r^2 - \varepsilon^2} = 1 + \sqrt{r^2 - \varepsilon^2},\end{aligned}$$

which shows (4.11) and finishes the proof. □

We observe that in this particular case it is possible to bound by above the inner radii of the sum which, in general, is not feasible (see Proposition 4.2.1).

Remark 4.3.1 ([25]). *There exist upper bounds for* $\mathrm{r}_i(K + rB_n)$ *in terms of* $\mathrm{r}_i(K)$ *and* r. *Namely, using* (3.2) *and Proposition 4.3.1, it is straightforward to get*

$$\mathrm{r}_i(K + rB_n) \le \mathrm{R}_{n-i+1}(K + rB_n) = \mathrm{R}_{n-i+1}(K) + r < (i + 1)\mathrm{r}_i(K) + r,$$

although this bound is far from being optimal.

Remark 4.3.2 ([25]). *Let $K, K' \in \mathcal{K}^n$. If K' has non-empty interior, i.e., if* $\mathrm{r}(K') > 0$*, then we have* $\mathrm{r}_i(K+K') \geq \mathrm{r}_i\bigl(K+\mathrm{r}(K')B_n\bigr) \geq \mathrm{r}_i(K)+\mathrm{r}(K')$*. Thus, in order to improve the constant* $\sqrt{2}$ *in* (4.4)*, the inradius of the body has to be involved.*

Remark 4.3.3 ([25]). *The family of triangular antiprisms P_ε considered in the proof of Proposition 4.3.1 shows that the functional* $\mathrm{r}_i : \mathcal{K}^n \longrightarrow \mathbb{R}_{\geq 0}$*, $i = 2, \dots, n-1$, is not continuous: using the previous notation and taking $\varepsilon = 1/k$, we have* $\lim_{k\to\infty} P_{1/k} = H$*, but*

$$\lim_{k\to\infty} \mathrm{r}_2(P_{1/k}) = \frac{\sqrt{3}}{2} < 1 = \mathrm{r}_2(H).$$

However, as shown in Section 1.2, $\mathrm{r}_i : \{K \in \mathcal{K}^n : \dim K = n\} \longrightarrow \mathbb{R}_{\geq 0}$ *is a continuous map.*

We are also interested in the behavior of the successive radii regarding the special case of the Minkowski sum $K - K$. In [33, Lemma 2.1, Remark 2.1] it was shown that for the central symmetral $K^0 = (K-K)/2$ it holds $\mathrm{R}_i(K^0) \leq \mathrm{R}_i(K)$ and $\mathrm{r}_i(K^0) \geq \mathrm{r}_i(K)$ for all $i = 1, \dots, n$. The next proposition completes this particular case, by showing that the bounds in (4.2) and (4.4) can be improved and that there are non-trivial reverse inequalities (cf. Propositions 4.1.1 and 4.2.1).

Proposition 4.3.2 ([25]). *Let $K \in \mathcal{K}^n$. Then for all $i = 1, \dots, n$,*

$$\begin{aligned} &a) \quad \sqrt{2}\sqrt{\frac{i+1}{i}}\mathrm{R}_i(K) \leq \mathrm{R}_i(K-K) \leq 2\mathrm{R}_i(K), \\ &b) \qquad\qquad 2\mathrm{r}_i(K) \leq \mathrm{r}_i(K-K) < 2(i+1)\mathrm{r}_i(K). \end{aligned} \tag{4.12}$$

All inequalities except for the upper bound in (b) are best possible.

Proof. The right hand side in (4.12.a) and the left hand side in (4.12.b) are known (see [33, Lemma 2.1, Remark 2.1]). In order to prove the left inequality in (4.12.a) let $L_i \in \mathcal{L}_i^n$ be such that $\mathrm{R}_i(K-K) = \mathrm{R}\bigl((K-K)|L_i\bigr)$ for any fixed $i \in \{1, \dots, n\}$. It is clear that $K^0|L_i = (K|L_i)^0$. Then, since central symmetry preserves the diameter (see e.g. [5, p. 79]) and using the well-known Jung inequality (see Theorem 1.3.2) in dimension i, we get

$$\begin{aligned} \mathrm{R}_i(K-K) &= \mathrm{R}\bigl((K-K)|L_i\bigr) = 2\mathrm{R}\bigl(K^0|L_i\bigr) = 2\mathrm{R}\bigl((K|L_i)^0\bigr) = \mathrm{D}\bigl((K|L_i)^0\bigr) \\ &= \mathrm{D}(K|L_i) \geq \sqrt{\frac{2(i+1)}{i}}\mathrm{R}(K|L_i) \geq \sqrt{\frac{2(i+1)}{i}}\mathrm{R}_i(K). \end{aligned}$$

Equality in the Jung inequality holds for the i-dimensional regular simplex S_i as well as for every convex body of diameter D containing the regular simplex of edge-length D. Hence, in our case, equality holds for any convex body K such that $\mathrm{R}_i(K) = \mathrm{R}(K|L_i)$ and such that $K|L_i$ is an extremal set in Jung's inequality. For instance, equality holds for $K = S_i + M\overline{C}_{n-i}$, where $\overline{C}_{n-i} \subset (\operatorname{aff} S_i)^\perp$ represents the $(n-i)$-dimensional unit cube and $M > 0$ is sufficiently large.

The right hand side in (4.12.b) is a direct consequence of (3.2) and the already mentioned property of the central symmetrization, $\mathrm{R}_i(K^0) \leq \mathrm{R}_i(K)$:

$$\mathrm{r}_i(K-K) = 2\mathrm{r}_i(K^0) \leq 2\mathrm{R}_{n-i+1}(K^0) \leq 2\mathrm{R}_{n-i+1}(K) < 2(i+1)\mathrm{r}_i(K). \qquad \square$$

Corollary 4.3.1. *Let $K \in \mathcal{K}^n$. Then for all $i = 1, \dots, n$,*

$$\begin{aligned} &a) && \sqrt{\frac{i+1}{2i}}\mathrm{R}_i(K) \leq \mathrm{R}_i(K^0) \leq \mathrm{R}_i(K), \\ &b) && \mathrm{r}_i(K) \leq \mathrm{r}_i(K^0) < (i+1)\mathrm{r}_i(K). \end{aligned}$$

All inequalities except for the upper bound in (b) are best possible.

4.4 Successive radii and the Firey addition

In this section we generalize Theorems 4.1.1 and 4.2.1 when the Firey addition, i.e., p-sums, of convex bodies is considered (see Definition 1.1.7).

First we observe that for any $1 \leq p \leq q$ it holds $|\cdot|_q \leq |\cdot|_p$ and it is a direct consequence of Hölder's inequality for q/p (see e.g. [30, p. 15]) that $|\cdot|_p \leq n^{1/p-1/q}|\cdot|_q$: for $x = (x_1, \dots, x_n)^\intercal \in \mathbb{R}^n$,

$$|x|_p = \left(\sum_{i=1}^n 1\, x_i^p\right)^{1/p} \leq \left(\sum_{i=1}^n 1\right)^{(q-p)/(pq)} \left(\sum_{i=1}^n x_i^q\right)^{1/q} = n^{1/p-1/q}|x|_q.$$

Clearly, they are equivalent to the inclusions

$$B_n^p \subseteq B_n^q \subseteq n^{1/p-1/q} B_n^p.$$

On the other hand, it is known (see e.g. [27]) that $[-\mathrm{e}_1, \mathrm{e}_1] +_p \cdots +_p [-\mathrm{e}_n, \mathrm{e}_n] = B_n^q$ for $q \geq 1$ such that $1/p + 1/q = 1$. Therefore we get, in particular, that

$$[-\mathrm{e}_1, \mathrm{e}_1] +_p \cdots +_p [-\mathrm{e}_n, \mathrm{e}_n] \subseteq \begin{cases} n^{1/p-1/2} B_n & \text{for } 1 \leq p \leq 2, \\ B_n & \text{for } p \geq 2. \end{cases} \tag{4.13}$$

We start by proving the lower bound for the outer radii $\mathrm{R}_i(K +_p K')$ of the p-sum of two convex bodies in terms of the corresponding radii.

Theorem 4.4.1 ([24]). *Let $K, K' \in \mathcal{K}_0^n$ and $p \geq 1$. Then*

$$\begin{aligned} 2^{\frac{p-1}{p}}\, \mathrm{R}_1(K +_p K') &\geq \mathrm{R}_1(K) + \mathrm{R}_1(K') && \text{for all } p \geq 1, \\ 2^{\frac{3p-2}{2p}}\, \mathrm{R}_i(K +_p K') &\geq \mathrm{R}_i(K) + \mathrm{R}_i(K') && \text{for } 1 \leq p \leq 2,\ i = 2, \dots, n, \\ \mathrm{R}_i(K +_p K') &\geq \max\bigl\{\mathrm{R}_i(K), \mathrm{R}_i(K')\bigr\} && \text{for } p \geq 2,\ i = 2, \dots, n. \end{aligned} \tag{4.14}$$

All inequalities are best possible.

Proof. By (1.3) we have $2^{(p-1)/p}\, \mathrm{R}_i(K +_p K') \geq \mathrm{R}_i(K + K')$ for all $i = 1, \dots, n$, and then applying Theorem 4.1.1 we get

$$\begin{aligned} 2^{(p-1)/p}\, \mathrm{R}_1(K +_p K') &\geq \mathrm{R}_1(K + K') \geq \mathrm{R}_1(K) + \mathrm{R}_1(K'), \\ 2^{(p-1)/p}\, \mathrm{R}_i(K +_p K') &\geq \mathrm{R}_i(K + K') \geq \frac{1}{\sqrt{2}}\bigl(\mathrm{R}_i(K) + \mathrm{R}_i(K')\bigr), \end{aligned}$$

for $i = 2, \dots, n$, which gives the two first inequalities in (4.14). Notice also that it always holds $K, K' \subseteq K +_p K'$, which leads to

$$\mathrm{R}_i(K +_p K') \geq \max\bigl\{\mathrm{R}_i(K), \mathrm{R}_i(K')\bigr\}.$$

Since for any real numbers $a, b \geq 0$ it holds that if $p \geq 2$ then $\max\{a, b\} \geq 1/2^{(3p-2)/(2p)}(a + b)$, the third inequality in (4.14) is obtained.

So it remains to be shown that the three inequalities are best possible. For the first one, let $K = K'$. Then $K +_p K = 2^{1/p}K$ and thus

$$2^{(p-1)/p}\,\mathrm{R}_1(K +_p K) = 2^{(p-1)/p}\,2^{1/p}\,\mathrm{R}_1(K) = \mathrm{R}_1(K) + \mathrm{R}_1(K).$$

Next we fix $i \in \{2, \dots, n\}$ and consider the convex bodies

$$K = [-\mathrm{e}_1, \mathrm{e}_1] + \sum_{k=i+1}^{n} [-\mathrm{e}_k, \mathrm{e}_k], \quad K' = [-\mathrm{e}_2, \mathrm{e}_2] + \sum_{k=i+1}^{n} [-\mathrm{e}_k, \mathrm{e}_k]$$

i.e., the 0-symmetric $(n - i + 1)$-cubes with edges parallel to the coordinate axes and length 2, of the subspaces $L_j = \{\mathrm{e}_j, \mathrm{e}_{i+1}, \dots, \mathrm{e}_n\}$, $j = 1, 2$. Here for $i = n$ we are just taking $K = [-\mathrm{e}_1, \mathrm{e}_1]$ and $K' = [-\mathrm{e}_2, \mathrm{e}_2]$ (see (4.3)). In the proof of Theorem 4.1.1 it was shown that $\mathrm{R}_i(K) = \mathrm{R}_i(K') = 1$.

First, let $1 \leq p \leq 2$, and we compute $\mathrm{R}_i(K +_p K')$. Let $L \in \mathcal{L}_i^n$. On the one hand, since $\dim(L_j \cap L) \geq (n - i + 1) + i - n = 1$, there exist points $x \in K \cap L$ and $x' \in K' \cap L$ with $|x|_2, |x'|_2 \geq 1$, and because of the central symmetry, we may assume that $\langle x, x' \rangle \geq 0$. Then

$$\left|\frac{(x + x')}{2^{(p-1)/p}}\right|_2 \geq \frac{\bigl(|x|_2^2 + |x'|_2^2\bigr)^{1/2}}{2^{(p-1)/p}} \geq \frac{2^{1/2}}{2^{(p-1)/p}} = 2^{(2-p)/(2p)},$$

and thus, since $(x + x')/2^{(p-1)/p} \in (K +_p K') \cap L$ (see (1.3)), we get

$$\mathrm{R}\bigl((K +_p K')|L\bigr) \geq \mathrm{R}\bigl((K +_p K') \cap L\bigr) \geq 2^{(2-p)/(2p)} \tag{4.15}$$

for all $L \in \mathcal{L}_i^n$. On the other hand, since the orthogonal projection of the p-sum of two convex bodies onto any lower dimensional linear subspace is the p-sum of the projections (see [19, pp. 21–22]), and using (4.13), we get

$$\begin{aligned}(K +_p K')|\operatorname{lin}\{\mathrm{e}_1, \dots, \mathrm{e}_i\} &= K|\operatorname{lin}\{e_1, \dots, e_i\} +_p K'|\operatorname{lin}\{e_1, \dots, e_i\}\\ &= K|\operatorname{lin}\{\mathrm{e}_1\} +_p K'|\operatorname{lin}\{\mathrm{e}_2\} = [-\mathrm{e}_1, \mathrm{e}_1] +_p [-\mathrm{e}_2, \mathrm{e}_2]\\ &\subseteq 2^{(2-p)/(2p)} B_{2,\operatorname{lin}\{\mathrm{e}_1,\mathrm{e}_2\}},\end{aligned}$$

which gives $\mathrm{R}\bigl((K +_p K')|\operatorname{lin}\{e_1, \dots, e_i\}\bigr) \leq 2^{(2-p)/(2p)}$. Then, together with (4.15) we get the equality $\mathrm{R}\bigl((K +_p K')|\operatorname{lin}\{e_1, \dots, e_i\}\bigr) = 2^{(2-p)/(2p)}$, and moreover,

$$\begin{aligned}\mathrm{R}_i(K +_p K') &= \min_{L \in \mathcal{L}_i^n} \mathrm{R}\bigl((K +_p K')|L\bigr) = 2^{(2-p)/(2p)} = \frac{1}{2^{(3p-2)/(2p)}}(1 + 1)\\ &= \frac{1}{2^{(3p-2)/(2p)}}\bigl(\mathrm{R}_i(K) + \mathrm{R}_i(K')\bigr).\end{aligned}$$

Let $p \geq 2$. First notice that $\mathrm{R}_i(K+_pK') \geq \mathrm{R}_i(K), \mathrm{R}_i(K') = 1$. With an analogous argument as before, but using (4.13) when $p \geq 2$, we get that $\mathrm{R}\big((K+_pK')|\operatorname{lin}\{e_1,\dots,e_i\}\big) \leq \mathrm{R}\left(B_{2,\operatorname{lin}\{\mathrm{e}_1,\mathrm{e}_2\}}\right) = 1$. Both inequalities give $\mathrm{R}_i(K+_pK') = 1 = \mathrm{R}_i(K), \mathrm{R}_i(K')$. □

The above theorem becomes Theorem 4.1.1 when $p = 1$. We also notice that in Theorem 4.4.1, the last two inequalities are valid for all $p \geq 1$. We point out that the distinction depending on the range of p is needed for the sharpness.

Regarding a reverse inequality, in the case of the circumradius we easily get, using (1.3) and (4.1), that

$$\mathrm{R}_n(K+_pK') \leq \mathrm{R}_n(K+K') \leq \mathrm{R}_n(K) + \mathrm{R}_n(K'),$$

and the inequality is tight, as shown when we take $K' = \{0\}$. However, there is no chance to get a reverse inequality for all outer radii, as next proposition shows.

Proposition 4.4.1 ([24]). *Let $K, K' \in \mathcal{K}^n_0$ and $p \geq 1$. For all $i = 1,\dots,n-1$, there exists no constant $c > 0$ such that $c\,\mathrm{R}_i(K+_pK') \leq \mathrm{R}_i(K) + \mathrm{R}_i(K')$.*

Proof. Let $i \in \{1,\dots,n-1\}$ be fixed. We take the convex bodies

$$K = [-\mathrm{e}_{n-i+1}, \mathrm{e}_{n-i+1}] \quad \text{and} \quad K' = \sum_{k=1}^{n-i}[-\mathrm{e}_k, \mathrm{e}_k].$$

On the one hand, we observe that

$$K|\operatorname{lin}\{\mathrm{e}_{n-i}, \mathrm{e}_{n-i+2},\dots,\mathrm{e}_n\} = K'|\operatorname{lin}\{\mathrm{e}_{n-i+1},\dots,\mathrm{e}_n\} = \{0\},$$

and hence both $\mathrm{R}_i(K) = \mathrm{R}_i(K') = 0$, i.e., $\mathrm{R}_i(K) + \mathrm{R}_i(K') = 0$. On the other hand,

$$K+_pK' \supseteq \frac{1}{2^{(p-1)/p}}(K+K') = \frac{1}{2^{(p-1)/p}}\sum_{j=1}^{n-i+1}[-\mathrm{e}_j, \mathrm{e}_j],$$

i.e., the p-sum $K+_pK'$ contains an $(n-i+1)$-dimensional convex body, which implies that $\dim\big((K+_pK')|L\big) \geq 1$ for all $L \in \mathcal{L}^n_i$. Then, $\mathrm{R}_i(K+_pK') > 0$. Hence we conclude that there exists no constant $c > 0$ such that $c\,\mathrm{R}_i(K+_pK') \leq \mathrm{R}_i(K) + \mathrm{R}_i(K')$ for any $i = 1,\dots,n-1$. □

Remark 4.4.1 ([24]). *The inequality $\mathrm{R}(K+_pK') \leq \mathrm{R}(K) + \mathrm{R}(K')$, $p \geq 1$, can be strengthened in the particular case when the circumcenter of both $K, K' \in \mathcal{K}^n_0$ lies in the origin. In this case,*

$$h(K+_pK', u) = \big(h(K,u)^p + h(K',u)^p\big)^{1/p} \leq \big(\mathrm{R}(K)^p + \mathrm{R}(K')^p\big)^{1/p}$$

for all $u \in \mathbb{S}^{n-1}$, which implies, in particular, that the circumradius of the p-sum of K, K' is not greater than the p-sum of the circumradii,

$$\mathrm{R}(K+_pK') \leq \big(\mathrm{R}(K)^p + \mathrm{R}(K')^p\big)^{1/p}.$$

If $K = K'$ equality holds.

In the case of the successive inner radii $\mathrm{r}_i(K +_p K')$ we get the following theorem.

Theorem 4.4.2 ([24]). *Let $K, K' \in \mathcal{K}^n_0$ and $p \geq 1$. Then*

$$\begin{aligned}
2^{\frac{p-1}{p}} \mathrm{r}_n(K +_p K') &\geq \mathrm{r}_n(K) + \mathrm{r}_n(K') \quad \text{for all } p \geq 1,\\
2^{\frac{3p-2}{2p}} \mathrm{r}_i(K +_p K') &\geq \mathrm{r}_i(K) + \mathrm{r}_i(K') \quad \text{for } 1 \leq p \leq 2,\ i = 1, \dots, n-1,\\
\mathrm{r}_i(K +_p K') &\geq \max\bigl\{\mathrm{r}_i(K), \mathrm{r}_i(K')\bigr\} \quad \text{for } p \geq 2,\ i = 1, \dots, n-1.
\end{aligned} \tag{4.16}$$

All inequalities are best possible.

Proof. By (1.3) we have $2^{(p-1)/p} \mathrm{r}_i(K +_p K') \geq \mathrm{r}_i(K + K')$ for $i = 1, \dots, n$, and applying Theorem 4.2.1 we get

$$\begin{aligned}
2^{(p-1)/p} \mathrm{r}_n(K +_p K') &\geq \mathrm{r}_n(K + K') \geq \mathrm{r}_n(K) + \mathrm{r}_n(K'),\\
2^{(p-1)/p} \mathrm{r}_i(K +_p K') &\geq \mathrm{r}_i(K + K') \geq \frac{1}{\sqrt{2}}\bigl(\mathrm{r}_i(K) + \mathrm{r}_i(K')\bigr),
\end{aligned}$$

$i = 1, \dots, n-1$, which gives the two first inequalities in (4.16). Again, since $K, K' \subseteq K +_p K'$,

$$\mathrm{r}_i(K +_p K') \geq \max\bigl\{\mathrm{r}_i(K), \mathrm{r}_i(K')\bigr\},$$

which leads to the third inequality in (4.16).

So, we have to show that these inequalities are tight. For the first one, with $K = K'$ we get

$$2^{(p-1)/p} \mathrm{r}_n(K +_p K) = 2^{(p-1)/p} 2^{1/p} \mathrm{r}_n(K) = \mathrm{r}_n(K) + \mathrm{r}_n(K).$$

Next we fix $i \in \{1, \dots, n-1\}$, and consider the following convex bodies: for $j = 2i-n$ if $2i > n$, and $j = 0$ otherwise, we take the i-dimensional unit balls $K = B_{i,L}$ and $K' = B_{i,L'}$ of the i-dimensional linear subspaces

$$L = \mathrm{lin}\{\mathrm{e}_1, \dots, \mathrm{e}_j, \mathrm{e}_{j+1}, \dots, \mathrm{e}_i\}, \quad L' = \mathrm{lin}\{\mathrm{e}_1, \dots, \mathrm{e}_j, \mathrm{e}_{i+1}, \dots, \mathrm{e}_{2i-j}\};$$

here, for $j = 0$, we are taking the subspaces $L = \mathrm{lin}\{\mathrm{e}_1, \dots, \mathrm{e}_i\}$ and $L' = \mathrm{lin}\{\mathrm{e}_{i+1}, \dots, \mathrm{e}_{2i}\}$. Clearly $\mathrm{r}_i(B_{i,L}) = \mathrm{r}_i(B_{i,L'}) = 1$.

Notice that since $B_{i,L}, B_{i,L'}$ are 0-symmetric, then $B_{i,L} +_p B_{i,L'}$ is also 0-symmetric, and then

$$\mathrm{r}_i(B_{i,L} +_p B_{i,L'}) = \max_{\bar{L} \in \mathcal{L}^n_i} \mathrm{r}\bigl((B_{i,L} +_p B_{i,L'}) \cap \bar{L}; \bar{L}\bigr).$$

Let $1 \leq p \leq 2$. We are going to show that

$$\mathrm{r}\bigl((B_{i,L} +_p B_{i,L'}) \cap \bar{L}; \bar{L}\bigr) \leq 2^{(2-p)/(2p)} \tag{4.17}$$

for all $\bar{L} \in \mathcal{L}^n_i$, which will imply that

$$\begin{aligned}
2^{(2-p)/(2p)} = \frac{1}{2^{(3p-2)/(2p)}}(1+1) &= \frac{1}{2^{(3p-2)/(2p)}}\bigl(\mathrm{r}_i(B_{i,L}) + \mathrm{r}_i(B_{i,L'})\bigr)\\
&\leq \mathrm{r}_i(B_{i,L} +_p B_{i,L'}) \leq 2^{(2-p)/(2p)},
\end{aligned}$$

i.e., $\mathrm{r}_i(B_{i,L} +_p B_{i,L'}) = 1/2^{(3p-2)/(2p)}\big(\mathrm{r}_i(B_{i,L}) + \mathrm{r}_i(B_{i,L'})\big)$, as required. Let $L'' = \mathrm{lin}\{\mathrm{e}_{j+1}, \ldots, \mathrm{e}_n\}$. If $2i \leq n$ then $L'' = \mathbb{R}^n$ and thus, for all $\bar{L} \in \mathcal{L}_i^n$, it holds $\dim(\bar{L} \cap L'') = \dim \bar{L} = i \geq 1$; analogously, if $2i > n$, then

$$\begin{aligned}\dim(\bar{L} \cap L'') &= \dim \bar{L} + \dim L'' - \dim(\bar{L} + L'') = i + n - j - \dim(\bar{L} + L'')\\ &\geq i + n - j - n = n - i \geq 1.\end{aligned}$$

Therefore, since $\dim(B_{i,L} +_p B_{i,L'}) = n$, then, for all $\bar{L} \in \mathcal{L}_i^n$, we can always find a boundary point

$$z \in \mathrm{relbd}(B_{i,L} +_p B_{i,L'}) \cap \bar{L} \cap L'', \quad z \neq 0.$$

Moreover, notice that, in particular, $z \in \mathrm{lin}\{\mathrm{e}_{j+1}, \ldots, \mathrm{e}_{2i-j}\}$, and so it can be expressed in the form

$$z = x + x' \in \mathrm{lin}\{\mathrm{e}_{j+1}, \ldots, \mathrm{e}_i\} + \mathrm{lin}\{\mathrm{e}_{i+1}, \ldots, \mathrm{e}_{2i-j}\} = (L \cap L'') + (L' \cap L'');$$

we observe that x, x' lie in orthogonal subspaces. Writing $u = z/|z|_2$, we have

$$|z|_2 = \langle z, u\rangle \leq h\left(B_{i,L} +_p B_{i,L'}, u\right) = \Big(h\left(B_{i,L}, u\right)^p + h\left(B_{i,L'}, u\right)^p\Big)^{1/p},$$

and since

$$h(B_{i,L}, u) = \max_{y \in B_{i,L}} \langle y, u\rangle = \frac{1}{|z|_2} \max_{y \in B_{i,L}} \langle y, x\rangle = \frac{1}{|z|_2}\left\langle \frac{x}{|x|_2}, x\right\rangle = h\left(\left[-\frac{x}{|x|_2}, \frac{x}{|x|_2}\right], u\right)$$

and analogously

$$h(B_{i,L'}, u) = h\left(\left[-\frac{x'}{|x'|_2}, \frac{x'}{|x'|_2}\right], u\right),$$

we obtain that

$$\begin{aligned}|z|_2 &\leq \left(h\left(\left[-\frac{x}{|x|_2}, \frac{x}{|x|_2}\right], u\right)^p + h\left(\left[-\frac{x'}{|x'|_2}, \frac{x'}{|x'|_2}\right], u\right)^p\right)^{1/p}\\ &= h\left(\left[-\frac{x}{|x|_2}, \frac{x}{|x|_2}\right] +_p \left[-\frac{x'}{|x'|_2}, \frac{x'}{|x'|_2}\right], u\right) = h\left(B^{p/(p-1)}_{2,\mathrm{lin}\{x,x'\}}, u\right) \leq \mathrm{R}\left(B^{p/(p-1)}_{2,\mathrm{lin}\{x,x'\}}\right)\\ &\leq 2^{(2-p)/(2p)}\mathrm{R}\left(B_{2,\mathrm{lin}\{x,x'\}}\right) = 2^{(2-p)/(2p)}\end{aligned}$$

by (4.13). This implies $\mathrm{r}\big((B_{i,L} +_p B_{i,L'}) \cap \bar{L}; \bar{L}\big) \leq 2^{(2-p)/(2p)}$, showing (4.17) and concluding the proof of the case $1 \leq p \leq 2$.

Now let $p \geq 2$. Notice that $\mathrm{r}_i\left(B_{i,L} +_p B_{i,L'}\right) \geq \mathrm{r}_i\big(B_{i,L}\big), \mathrm{r}_i\big(B_{i,L'}\big) = 1$. So, it suffices to show that $\mathrm{r}\big((B_{i,L} +_p B_{i,L'}) \cap \bar{L}; \bar{L}\big) \leq 1$ for all $\bar{L} \in \mathcal{L}_i^n$. With an analogous argument as before, but using (4.13) when $p \geq 2$, we get that there exists $z \in \mathrm{relbd}(B_{i,L} +_p B_{i,L'}) \cap \bar{L} \cap L''$ such that

$$|z|_2 \leq \mathrm{R}\left(B^{p/(p-1)}_{2,\mathrm{lin}\{x,x'\}}\right) \leq \mathrm{R}\left(B_{2,\mathrm{lin}\{x,x'\}}\right) = 1.$$

It shows that $\mathrm{r}\big((B_{i,L} +_p B_{i,L'}) \cap \bar{L}; \bar{L}\big) \leq 1$ and concludes the proof. $\square$

The above result becomes Theorem 4.2.1 when $p = 1$. We notice that in Theorem 4.4.2, the last two inequalities are valid for all $p \geq 1$. We point out that the distinction depending on the range of p is needed for the sharpness.

We deal again with the possible existence of a reverse inequality. In the case of the diameter we easily get, using (1.3) and (4.1), that

$$\mathrm{r}_1(K +_p K') \leq \mathrm{r}_1(K + K') \leq \mathrm{r}_1(K) + \mathrm{r}_1(K'),$$

and the inequality is tight, as shown when we take $K' = \{0\}$. However, there is no chance to get a reverse inequality for all inner radii, as next proposition shows.

Proposition 4.4.2 ([24]). *Let $K, K' \in \mathcal{K}^n_0$ and $p \geq 1$. For all $i = 2, \dots, n$, there exists no constant $c > 0$ such that $c\,\mathrm{r}_i(K +_p K') \leq \mathrm{r}_i(K) + \mathrm{r}_i(K')$.*

Proof. Let $i \in \{2, \dots, n\}$ be fixed, and we take the convex bodies

$$K = [-\mathrm{e}_1, \mathrm{e}_1] \quad \text{and} \quad K' = \sum_{k=2}^{i} [-\mathrm{e}_k, \mathrm{e}_k],$$

which satisfy $\mathrm{r}_i(K) = \mathrm{r}_i(K') = 0$, because they have dimensions $\dim K = 1$ and $\dim K' = i - 1$. However,

$$\mathrm{r}_i(K +_p K') \geq 2^{-(p-1)/p}\mathrm{r}_i(K + K') = 2^{-(p-1)/p}\mathrm{r}_i\left(\sum_{k=1}^{i}[-\mathrm{e}_k, \mathrm{e}_k]\right) = 2^{-(p-1)/p},$$

which shows that there exists no constant $c > 0$ such that $c\,\mathrm{r}_i(K +_p K') \leq \mathrm{r}_i(K) + \mathrm{r}_i(K')$ for any $i = 2, \dots, n$. □

4.4.1 The p-difference body of a convex set

The *p-difference body* of a convex body $K \in \mathcal{K}^n_0$ is defined as the p-sum $K -_p K := K +_p (-K)$ (see Figure 1.4), which is also a 0-symmetric convex body: in fact,

$$h(K -_p K, -u)^p = h(K, -u)^p + h(-K, -u)^p = h(-K, u)^p + h(K, u)^p = h(K -_p K, u)^p.$$

Obviously, when $p = 1$, the p-difference body coincides with the usual difference body.

In [4], a sharp Rogers-Shephard inequality for the p-difference body of a planar convex body was obtained, i.e., the best (upper) bound for the volume of the set $K -_p K$ in terms of the volume of the original body K. Here we are interested in obtaining upper and lower bounds for the in- and outer radii of the p-difference body $K -_p K$ in terms of the ones of K. In Proposition 4.3.2 we have already studied the behavior of the radii regarding the usual difference body.

Next result extends (4.12) to the p-difference body, showing moreover that the bounds in (4.14) and (4.16) can be improved and that, in this particular case, there are non-trivial reverse inequalities (cf. Propositions 4.4.1 and 4.4.2).

Proposition 4.4.3 ([24]). *Let $K \in \mathcal{K}_0^n$. Then for all $i = 1, \dots, n$ and all $p \geq 1$,*

$$\left.\begin{array}{ll} \textit{If } p \leq 2, & 2^{1/p-1/2}\sqrt{\frac{i+1}{i}} \\ \textit{If } p \geq 2, & \max\left\{2^{1/p-1/2}\sqrt{\frac{i+1}{i}}, 1\right\} \end{array}\right\} \mathrm{R}_i(K) \leq \mathrm{R}_i(K -_p K) \leq 2\mathrm{R}_i(K), \tag{4.18}$$

$$2^{1/p}\mathrm{r}_i(K) \leq \mathrm{r}_i(K -_p K) < 2(i+1)\mathrm{r}_i(K). \tag{4.19}$$

The upper bound and the lower bound when $1 \leq p \leq 2$ in (4.18) are best possible. Lower bound in (4.19) is best possible.

Proof. By (1.3) we have, for all $i = 1, \dots, n$, that

$$\frac{1}{2^{(p-1)/p}} \mathrm{R}_i(K - K) \leq \mathrm{R}_i(K -_p K) \leq \mathrm{R}_i(K - K),$$

and analogously for the inner radii r_i. Then applying (4.12) we directly get (4.18) and (4.19). We note that if $p \geq 2$, then $\max\bigl\{2^{1/p-1/2}\sqrt{(i+1)/i}, 1\bigr\} = 1$ for all $i \geq 2$ and most of the values of p.

So we deal with the sharpness of the inequalities, starting with the left hand side in (4.19). In this case, just notice that if K is a 0-symmetric convex body then $K = -K$ and hence

$$\mathrm{r}_i(K -_p K) = \mathrm{r}_i(K +_p K) = 2^{1/p}\mathrm{r}_i(K).$$

Next we study the right hand side in (4.18). We fix $i \in \{1, \dots, n\}$ and consider the convex body

$$K = [0, \mathrm{e}_1] + \sum_{j=i+1}^{n} [-\mathrm{e}_j, \mathrm{e}_j],$$

for which it clearly holds

$$\mathrm{R}_i(K) = \mathrm{R}\bigl(K|\operatorname{lin}\{\mathrm{e}_1, \dots, \mathrm{e}_i\}\bigr) = \mathrm{R}\bigl([0, \mathrm{e}_1]\bigr) = \frac{1}{2};$$

here, if $i = n$ we are taking $K = [0, \mathrm{e}_1]$. Now, on the one hand, we notice that

$$(K -_p K)|\operatorname{lin}\{\mathrm{e}_1, \dots, \mathrm{e}_i\} = [0, \mathrm{e}_1] +_p [-\mathrm{e}_1, 0]$$

and that, by (1.2),

$$[-\mathrm{e}_1, \mathrm{e}_1] = \operatorname{conv}\bigl([0, \mathrm{e}_1] \cup [-\mathrm{e}_1, 0]\bigr) \subseteq [0, \mathrm{e}_1] +_p [-\mathrm{e}_1, 0] \subseteq [0, \mathrm{e}_1] + [-\mathrm{e}_1, 0] = [-\mathrm{e}_1, \mathrm{e}_1],$$

i.e., $(K -_p K)|\operatorname{lin}\{\mathrm{e}_1, \dots, \mathrm{e}_i\} = [0, \mathrm{e}_1] +_p [-\mathrm{e}_1, 0] = [-\mathrm{e}_1, \mathrm{e}_1]$.

On the other hand we observe that $\operatorname{conv}\bigl(K \cup (-K)\bigr) = C_{n-i+1}$ is the $(n-i+1)$-dimensional cube of edge-length 2 contained in $\operatorname{lin}\{\mathrm{e}_1, \mathrm{e}_{i+1}, \dots, \mathrm{e}_n\}$ and thus, by (1.2), we get that for all $L \in \mathcal{L}_i^n$

$$\begin{aligned} \mathrm{R}\bigl((K -_p K)|L\bigr) &\geq \mathrm{R}\bigl(\operatorname{conv}\bigl(K \cup (-K)\bigr)|L\bigr) = \mathrm{R}(C_{n-i+1}|L) \\ &\geq \mathrm{R}\bigl(C_{n-i+1}|\operatorname{lin}\{\mathrm{e}_1, \dots, \mathrm{e}_i\}\bigr) = \mathrm{R}\bigl([-\mathrm{e}_1, \mathrm{e}_1]\bigr) = \mathrm{R}\bigl((K -_p K)|\operatorname{lin}\{\mathrm{e}_1, \dots, \mathrm{e}_i\}\bigr). \end{aligned}$$

Therefore,

$$\mathrm{R}_i(K -_p K) = \mathrm{R}\big((K -_p K)|\operatorname{lin}\{\mathrm{e}_1, \dots, \mathrm{e}_i\}\big) = 1 = 2\mathrm{R}_i(K).$$

Finally we consider the equality case for the left hand side in (4.18) when $1 \le p \le 2$. If $i = n$, let S_n be the n-simplex, embedded in $\mathbb{R}^{n+1}$ and lying in the hyperplane

$$\left\{ x = (x_1, \dots, x_{n+1})^\intercal \in \mathbb{R}^{n+1} : \sum_{j=1}^{n+1} x_j = 0 \right\},$$

given by

$$S_n = \operatorname{conv}\left\{ p_k : p_{kk} = \frac{n}{n+1},\, p_{kj} = \frac{-1}{n+1} \text{ for } j \neq k,\, k = 1, \dots, n+1 \right\}.$$

Since $S_n -_p S_n$ is a 0-symmetric n-dimensional convex body, then

$$\mathrm{R}_n(S_n -_p S_n) = \max\left\{ h(S_n -_p S_n, u) : |u|_2 = 1 \text{ and } \sum_{j=1}^{n+1} u_j = 0 \right\}.$$

Let $u \in \mathbb{R}^{n+1}$ with $|u|_2 = 1$ and $\sum_{j=1}^{n+1} u_j = 0$. We recall that the value of the support function of a convex body at any vector is attained in an extreme point (cf. e.g. [30, Theorem 5.6]), so, in order to compute $h(S_n, u)$ it suffices to consider the vertices of S_n. Since

$$\langle p_k, u \rangle = \frac{n}{n+1} u_k - \frac{1}{n+1} \sum_{j \neq k} u_j = u_k,$$

then

$$h(S_n, u) = \max\big\{ \langle p_k, u \rangle : k = 1, \dots, n+1 \big\} = \max\{u_1, \dots, u_{n+1}\}.$$

Without loss of generality we assume $u_1 \ge \dots \ge u_{n+1}$, and notice that $u_1 \ge 0$, $u_{n+1} \le 0$. Thus

$$h(S_n -_p S_n, u)^p = h(S_n, u)^p + h(-S_n, u)^p = h(S_n, u)^p + h(S_n, -u)^p = u_1^p + (-u_{n+1})^p.$$

Then, some elementary calculations show that the maximum of the function

$$f(u_1, \dots, u_{n+1}) = u_1^p + (-u_{n+1})^p, \quad 1 \le p \le 2,$$

under the three conditions $|u|_2 = 1$, $\sum_{j=1}^{n+1} u_j = 0$ and $u_1 \ge \dots \ge u_{n+1}$, is attained in the point $\big(1/\sqrt{2}, 0, \dots, 0, -1/\sqrt{2}\big)^\intercal$; for the sake of clearness, we will sketch these computations afterwards. Therefore,

$$\mathrm{R}_n(S_n -_p S_n) = \left(\frac{1}{2^{p/2}} + \frac{1}{2^{p/2}} \right)^{1/p} = 2^{1/p - 1/2}.$$

Since $\mathrm{R}_n(S_n) = \sqrt{n/(n+1)}$, then we get the required equality:

$$\mathrm{R}_n(S_n -_p S_n) = 2^{1/p-1/2} = 2^{1/p-1/2} \sqrt{\frac{n+1}{n}}\, \mathrm{R}_n(S_n).$$

If $i < n$, we take the i-dimensional simplex S_i and consider the convex body $K = S_i + M\overline{C}_{n-i}$, where $\overline{C}_{n-i} \subset (\operatorname{aff} S_i)^\perp$ represents the $(n-i)$-dimensional unit cube and $M > 0$ is sufficiently large such that $\mathrm{R}_i(K -_p K) = \mathrm{R}(S_i -_p S_i)$ and $\mathrm{R}_i(K) = \mathrm{R}(S_i)$. The above argument gives the result. $\square$

Remark 4.4.2. *We check that the maximum of the function* $u_1^p+(-u_{n+1})^p$, $1\leq p\leq 2$, *subjects to* $u_1+\cdots+u_{n+1}=0$, $u_1^2+\cdots+u_{n+1}^2=1$ *and* $u_1\geq\cdots\geq u_{n+1}$, *is attained when* $u_1=-u_{n+1}=1/\sqrt{2}$ *and* $u_j=0$ *for* $j=2,\dots,n$. *For the sake of brevity we sketch here the case* $n=2$.

Since $u_2=-u_1-u_3$, *we have to maximize the function*

$$f(u_1,u_3)=u_1^p+(-u_3)^p,\quad 1\leq p\leq 2,$$

subjects to $u_1^2+u_3^2+(u_1+u_3)^2=1$ *and* $u_1\geq-(u_1+u_3)\geq u_3$*; notice that* $u_1\geq 0$ *and* $u_3\leq 0$. *Using the Lagrange multipliers method and taking into account that both* $2u_1+u_3\neq 0$ *and* $u_1+2u_3\neq 0$ *(otherwise we would get* $u_1=u_2=u_3=0$ *as the only solution, which is not possible), we obtain*

$$u_1^p+2u_1(-u_3)^{p-1}=(-u_3)^p-2u_3u_1^{p-1}. \tag{4.20}$$

If we assume $u_1>-u_3$, *since both* $u_1,u_3\neq 0$, *then we would get from* (4.20) *that* $u_1^{p-2}>(-u_3)^{p-2}$, *a contradiction because* $1\leq p\leq 2$. *The case* $u_1<-u_3$ *is analogous. Therefore,* (4.20) *holds if and only if* $u_1=-u_3$, *and hence* $u_2=-u_1-u_3=0$ *and* $u_1=1/\sqrt{2}=-u_3$.

4.5 Bounds for other families of successive radii

In this last section, we get upper and lower bounds for the other families of successive radii defined in Chapter 1. First, we study the outer radii.

Proposition 4.5.1 ([23]). *Let* $K,K'\in\mathcal{K}^n$. *Then*

$$\begin{aligned}\widetilde{\mathrm{R}}_1(K+K')&\geq\widetilde{\mathrm{R}}_1(K)+\widetilde{\mathrm{R}}_1(K'),\\ \sqrt{2}\,\widetilde{\mathrm{R}}_i(K+K')&\geq\widetilde{\mathrm{R}}_i(K)+\widetilde{\mathrm{R}}_i(K'),\quad i=2,\dots,n.\end{aligned}$$

All inequalities are best possible. Moreover, $\widetilde{\mathrm{R}}_n(K+K')\leq\widetilde{\mathrm{R}}_n(K)+\widetilde{\mathrm{R}}_n(K')$, *and there exists no constant* $c>0$ *such that* $c\widetilde{\mathrm{R}}_i(K+K')\leq\widetilde{\mathrm{R}}_i(K)+\widetilde{\mathrm{R}}_i(K')$ *for all* $i=1,\dots,n-1$.

Proof. The lower bound for $\widetilde{\mathrm{R}}_1(K+K')$ is well known (see (4.1)). Equality holds, for instance, when $K=K'$.

Now let $i\in\{2,\dots,n\}$ and $L\in\mathcal{L}_i^n$ be such that

$$\widetilde{\mathrm{R}}_i(K+K')=\max_{x\in L^\perp}\mathrm{R}\big((K+K')\cap(x+L)\big).$$

After suitable translations of K and K', we may suppose without loss of generality that

$$\mathrm{R}(K\cap L)=\max_{x\in L^\perp}\mathrm{R}\big(K\cap(x+L)\big)\quad\text{and}\quad\mathrm{R}(K'\cap L)=\max_{x\in L^\perp}\mathrm{R}\big(K'\cap(x+L)\big).$$

The trivial relation $(K\cap L)+(K'\cap L)\subseteq(K+K')\cap L$ and the circumradius monotonicity imply

$$\widetilde{\mathrm{R}}_i(K+K')=\max_{x\in L^\perp}\mathrm{R}\big((K+K')\cap(x+L)\big)\geq\mathrm{R}\big((K+K')\cap L\big)\geq\mathrm{R}\big((K\cap L)+(K'\cap L)\big).$$

By Theorem 4.1.1 we have

$$\sqrt{2}\,\mathrm{R}\big((K\cap L)+(K'\cap L)\big)\geq \mathrm{R}(K\cap L)+\mathrm{R}(K'\cap L),$$

and hence

$$\begin{aligned}\sqrt{2}\,\widetilde{\mathrm{R}}_i(K+K')&\geq\sqrt{2}\,\mathrm{R}\big((K\cap L)+(K'\cap L)\big)\geq \mathrm{R}(K\cap L)+\mathrm{R}(K'\cap L)\\&=\max_{x\in L^\perp}\mathrm{R}\big(K\cap(x+L)\big)+\max_{x\in L^\perp}\mathrm{R}\big(K'\cap(x+L)\big)\geq\widetilde{\mathrm{R}}_i(K)+\widetilde{\mathrm{R}}_i(K').\end{aligned}$$

It remains to be shown that these inequalities are best possible. We fix $i\in\{2,\dots,n\}$ and consider the convex bodies

$$K=[-\mathrm{e}_1,\mathrm{e}_1]+\sum_{k=i+1}^{n}[-\mathrm{e}_k,\mathrm{e}_k]\quad\text{and}\quad K'=[-\mathrm{e}_2,\mathrm{e}_2]+\sum_{k=i+1}^{n}[-\mathrm{e}_k,\mathrm{e}_k].$$

Here when $i=n$ we are just taking $K=[-\mathrm{e}_1,\mathrm{e}_1]$ and $K'=[-\mathrm{e}_2,\mathrm{e}_2]$. It can be proved in the same way as in Theorem 4.1.1 that $\widetilde{\mathrm{R}}_i(K)=\widetilde{\mathrm{R}}_i(K')=1$ and $\widetilde{\mathrm{R}}_i(K+K')=\sqrt{2}$, obtaining that

$$\sqrt{2}\,\widetilde{\mathrm{R}}_i(K+K')=2=1+1=\widetilde{\mathrm{R}}_i(K)+\widetilde{\mathrm{R}}_i(K').$$

We prove now the reverse inequalities. The upper bound of $\widetilde{\mathrm{R}}_n(K+K')$ is well known (see (4.1)). Equality holds when $K=K'$. Let $i\in\{1,\dots,n-1\}$ be fixed and we take

$$K=[-\mathrm{e}_{n-i+1},\mathrm{e}_{n-i+1}]\quad\text{and}\quad K'=\sum_{k=1}^{n-i}[-\mathrm{e}_k,\mathrm{e}_k].$$

Since $\mathrm{R}_i(K)=\mathrm{R}_i(K')=0$ (see Proposition 4.1.1), then $\widetilde{\mathrm{R}}_i(K)=\widetilde{\mathrm{R}}_i(K')=0$, and therefore $\widetilde{\mathrm{R}}_i(K)+\widetilde{\mathrm{R}}_i(K')=0$. However, $K+K'=\sum_{k=1}^{n-i+1}[-\mathrm{e}_k,\mathrm{e}_k]$ is an $(n-i+1)$-dimensional convex body, which implies that the dimension $\dim\big((K+K')\cap L\big)\geq 1$ for all $L\in\mathcal{L}^n_i$, and thus $\widetilde{\mathrm{R}}_i(K+K')>0$. So, there exists no constant $c>0$ such that $c\,\widetilde{\mathrm{R}}_i(K+K')\leq\widetilde{\mathrm{R}}_i(K)+\widetilde{\mathrm{R}}_i(K')$ for any $i=1,\dots,n-1$. □

Proposition 4.5.2 ([23]). *Let $K,K'\in\mathcal{K}^n$. Then,*

$$\left.\begin{array}{ll}\textit{if}\ \ i=1, & \frac{1}{\sqrt{2}}\\ \textit{if}\ \ i=2,\dots,n, & \sqrt{\frac{i(n+1)}{2(i+1)n}}\end{array}\right\}\big(\overline{\mathrm{R}}_i(K)+\overline{\mathrm{R}}_i(K')\big)\leq\overline{\mathrm{R}}_i(K+K')\leq\overline{\mathrm{R}}_i(K)+\overline{\mathrm{R}}_i(K'). \tag{4.21}$$

The upper bound is best possible. The lower bounds are best possible in the cases $i=1,n$.

Proof. We start proving the upper bound. For $i\in\{1,\dots,n\}$ fixed, let $L\in\mathcal{L}^n_i$ be such that

$$\overline{\mathrm{R}}_i(K+K')=\mathrm{R}\big((K+K')|L\big).$$

Since $(K+K')|L=K|L+K'|L$, by (4.1) we have

$$\overline{\mathrm{R}}_i(K+K')=\mathrm{R}\big((K+K')|L\big)=\mathrm{R}(K|L+K'|L)\leq\mathrm{R}(K|L)+\mathrm{R}(K'|L)\leq\overline{\mathrm{R}}_i(K)+\overline{\mathrm{R}}_i(K').$$

Equality holds, for instance, if $K = K'$.

Since $\overline{\mathrm{R}}_1 = \mathrm{r}_1$, the lower bound for $\overline{\mathrm{R}}_1(K+K')$ follows from Theorem 4.2.1. For $i \in \{2,\dots,n\}$ fixed, the lower bound for $\overline{\mathrm{R}}_i(K+K')$ is an easy consequence of Theorems 1.3.3 and 4.1.1, and the fact that outer radii form an increasing sequence:

$$\sqrt{\frac{n(i+1)}{i(n+1)}}\,\overline{\mathrm{R}}_i(K{+}K') \geq \overline{\mathrm{R}}_n(K{+}K') = \mathrm{R}(K{+}K') \geq \frac{1}{\sqrt{2}}\left(\mathrm{R}(K){+}\mathrm{R}(K')\right) \geq \frac{1}{\sqrt{2}}\left(\overline{\mathrm{R}}_i(K) + \overline{\mathrm{R}}_i(K')\right).$$

The sharpness in the lower bound when $i = 1, n$ follows from Theorem 4.1.1. $\square$

We observe that the left inequality in (4.21) behaves asymptotically as (4.2) when i (and hence n) goes to infinity, because

$$\lim_{i\to\infty}\sqrt{\frac{i(n+1)}{2n(i+1)}} = \frac{1}{\sqrt{2}}.$$

In the next propositions we show the possible bounds for the remaining families of inner radii.

Proposition 4.5.3 ([23]). *Let $K, K' \in \mathcal{K}^n$. Then*

$$\begin{aligned}
\sqrt{2}\,\widetilde{\mathrm{r}}_1(K+K') &\geq \widetilde{\mathrm{r}}_1(K) + \widetilde{\mathrm{r}}_1(K'),\\
2\,\widetilde{\mathrm{r}}_i(K+K') &> \widetilde{\mathrm{r}}_i(K) + \widetilde{\mathrm{r}}_i(K') \quad \textit{for} \quad i = 2,\dots,n-1,\\
\widetilde{\mathrm{r}}_n(K+K') &\geq \widetilde{\mathrm{r}}_n(K) + \widetilde{\mathrm{r}}_n(K').
\end{aligned}$$

The first and the third inequalities are best possible. Moreover, $\widetilde{\mathrm{r}}_1(K+K') \leq \widetilde{\mathrm{r}}_1(K) + \widetilde{\mathrm{r}}_1(K')$ and there exists no constant $c > 0$ such that $c\widetilde{\mathrm{r}}_i(K+K') \leq \widetilde{\mathrm{r}}_i(K) + \widetilde{\mathrm{r}}_i(K')$ for $i = 2,\dots,n$.

Proof. Theorem 4.2.1 implies the diameter inequality, namely, $\sqrt{2}\widetilde{\mathrm{r}}_1(K+K') \geq \widetilde{\mathrm{r}}_1(K) + \widetilde{\mathrm{r}}_1(K')$, whereas $\widetilde{\mathrm{r}}_n(K+K') \geq \widetilde{\mathrm{r}}_n(K) + \widetilde{\mathrm{r}}_n(K')$ is well known (see (4.1)).

Let $i \in \{2,\dots,n-1\}$ be fixed. Since $K, K' \subseteq K+K'$ (up to translations), then

$$\max\bigl\{\widetilde{\mathrm{r}}_i(K), \widetilde{\mathrm{r}}_i(K')\bigr\} \leq \widetilde{\mathrm{r}}_i(K+K'),$$

and hence $\widetilde{\mathrm{r}}_i(K) + \widetilde{\mathrm{r}}_i(K') < 2\,\widetilde{\mathrm{r}}_i(K+K')$, although this inequality is not tight. The sharpness in the cases $i = 1, n$ follows from Theorem 4.2.1.

Regarding the reverse inequalities, it holds $\widetilde{\mathrm{r}}_1(K+K') \leq \widetilde{\mathrm{r}}_1(K) + \widetilde{\mathrm{r}}_1(K')$ (see (4.1)). Now let $i \in \{2,\dots,n\}$ be fixed and we define

$$K = [-\mathrm{e}_1, \mathrm{e}_1] \quad \text{and} \quad K' = \sum_{j=2}^{i}[-\mathrm{e}_j, \mathrm{e}_j].$$

Let $L \in \mathcal{L}_i^n$. On the one hand, since $\dim K, \dim K' \leq i-1$, then $\dim K|L, \dim K|L' \leq i-1$ and therefore $\widetilde{\mathrm{r}}_i(K) = \widetilde{\mathrm{r}}_i(K') = 0$. On the other hand, $K+K' = \sum_{j=1}^{i}[-\mathrm{e}_j, \mathrm{e}_j]$ is the i-dimensional cube with edge-length 2, and it clearly holds $\widetilde{\mathrm{r}}_i(K+K') = 1$. Thus there exists no constant $c > 0$ such that $c\,\widetilde{\mathrm{r}}_i(K+K') \leq \widetilde{\mathrm{r}}_i(K) + \widetilde{\mathrm{r}}_i(K')$ for any $i = 2,\dots,n$. $\square$

Proposition 4.5.4 ([23]). *Let $K, K' \in \mathcal{K}^n$. Then*

$$\bar{\mathrm{r}}_i(K+K') \geq \bar{\mathrm{r}}_i(K) + \bar{\mathrm{r}}_i(K'), \quad i = 1, \dots, n. \tag{4.22}$$

The inequalities are best possible. Moreover, for all $i = 1, \dots, n$, there exists no constant $c > 0$ such that $c\,\bar{\mathrm{r}}_i(K+K') \leq \bar{\mathrm{r}}_i(K) + \bar{\mathrm{r}}_i(K')$.

Proof. For fixed $i \in \{1, \dots, n\}$, let $L \in \mathcal{L}_i^n$ and $x \in L^\perp$ be such that

$$\bar{\mathrm{r}}_i(K+K') = \mathrm{r}\big((K+K') \cap (x+L); x+L\big).$$

After suitable translations of K and K' we may suppose without loss of generality that

$$\mathrm{r}(K \cap L; L) = \max_{y \in L^\perp} \mathrm{r}\big(K \cap (y+L); y+L\big) \quad \text{and} \quad \mathrm{r}(K' \cap L; L) = \max_{y \in L^\perp} \mathrm{r}\big(K' \cap (y+L); y+L\big).$$

The trivial relation

$$(K \cap L) + (K' \cap L) \subseteq (K+K') \cap L,$$

together with the monotonicity of the inradius and (4.1), imply that

$$\mathrm{r}\big((K+K') \cap L; L\big) \geq \mathrm{r}(K \cap L; L) + \mathrm{r}(K' \cap L; L).$$

Therefore

$$\begin{aligned}\bar{\mathrm{r}}_i(K+K') = \mathrm{r}\big((K+K') \cap (x+L); x+L\big) &\geq \mathrm{r}\big((K+K') \cap L; L\big) \geq \mathrm{r}(K \cap L; L) + \mathrm{r}(K' \cap L; L) \\ &\geq \bar{\mathrm{r}}_i(K) + \bar{\mathrm{r}}_i(K').\end{aligned}$$

Equality holds, for instance, when $K = K'$.

In order to prove the last assertion, we consider the convex bodies

$$K = [-\mathrm{e}_1, \mathrm{e}_1] \quad \text{and} \quad K' = \sum_{j=2}^n [-\mathrm{e}_j, \mathrm{e}_j].$$

Let $L \in \mathcal{L}_i^n$ be such that $\mathrm{e}_n \in L$, and we take $x \in L^\perp$. Then $\dim\big(K \cap (x+L)\big) < i$ and therefore

$$\max_{x \in L^\perp} \mathrm{r}\big(K \cap (x+L); x+L\big) = 0,$$

which implies $\bar{\mathrm{r}}_i(K) = 0$. Let $L' \in \mathcal{L}_i^n$ be such that $\mathrm{e}_1 \in L'$ and we fix a point $x \in (L')^\perp$. Then $\dim\big(K' \cap (x+L')\big) < i$ and thus $\max_{x \in (L')^\perp} \mathrm{r}\big(K' \cap (x+L'); x+L'\big) = 0$. Hence $\bar{\mathrm{r}}_i(K') = 0$. But on the other hand $K + K' = C_n$, which implies $\bar{\mathrm{r}}_i(K+K') = 1$. We conclude that there exists no constant $c > 0$ such that $c\,\bar{\mathrm{r}}_i(K+K') \leq \bar{\mathrm{r}}_i(K) + \bar{\mathrm{r}}_i(K')$. □

Proposition 4.5.5 ([23]). *Let $K, K' \in \mathcal{K}^n$. Then*

$$\widehat{\mathrm{r}}_i(K+K') \geq \widehat{\mathrm{r}}_i(K) + \widehat{\mathrm{r}}_i(K'), \quad i = 1, \dots, n.$$

All inequalities are best possible. Moreover, for all $i = 1, \dots, n$, there exists no constant $c > 0$ such that $c\,\widehat{\mathrm{r}}_i(K+K') \leq \widehat{\mathrm{r}}_i(K) + \widehat{\mathrm{r}}_i(K')$.

Proof. Let $i \in \{1, \ldots, n\}$ and $L \in \mathcal{L}_i^n$ be such that

$$\widehat{\mathrm{r}}_i(K + K') = \mathrm{r}\big((K + K')|L; L\big).$$

By the monotonicity of the inradius and (4.1) we get

$$\widehat{\mathrm{r}}_i(K + K') = \mathrm{r}\big((K + K')|L; L\big) = \mathrm{r}(K|L + K'|L; L) \geq \mathrm{r}(K|L; L) + \mathrm{r}(K'|L; L) \geq \widehat{\mathrm{r}}_i(K) + \widehat{\mathrm{r}}_i(K').$$

Equality holds, for instance, when $K = K'$.

Now we show that there exist no reverse inequalities. Let $i \in \{1, \ldots, n\}$ be fixed, and we take

$$K = [-\mathrm{e}_1, \mathrm{e}_1] \quad \text{and} \quad K' = \sum_{j=2}^{n} [-\mathrm{e}_j, \mathrm{e}_j].$$

Let $L \in \mathcal{L}_i^n$ be such that $L \subset \mathrm{e}_1^{\perp}$ if $i \leq n - 1$. Then $K|L = \{0\}$ and hence $\widehat{\mathrm{r}}_i(K) = 0$. In the case $i = n$ it holds $\widehat{\mathrm{r}}_n(K) = 0$. We take now

$$L = \mathrm{lin}\{\mathrm{e}_1, \mathrm{e}_{n-i+2}, \ldots, \mathrm{e}_n\}$$

if $i \geq 2$ or $L = \mathrm{lin}\{\mathrm{e}_1\}$ if $i = 1$. We observe that in both cases $K'|L$ is an $(i - 1)$-dimensional convex body. Therefore $\widehat{\mathrm{r}}_i(K') = 0$. But $K + K' = C_n$, for which $\widehat{\mathrm{r}}_i(C_n) = 1$. Thus there exists no constant $c > 0$ such that $c\,\widehat{\mathrm{r}}_i(K + K') \leq \widehat{\mathrm{r}}_i(K) + \widehat{\mathrm{r}}_i(K')$ for every $i = 1, \ldots, n$. $\square$

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