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Objective Bayesian model selection approach to the two way analysis of variance

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Abstract An objective Bayesian procedure for testing in the two way analysis of variance is proposed. In the classical methodology the main effects of the two factors and the interaction effect are formulated as linear contrasts between means of normal populations, and hypotheses of the existence of such effects are tested. In this paper, for the first time these hypotheses have been formulated as objective Bayesian model selection problems. Our development is under homoscedasticity and heteroscedasticity, providing exact solutions in both cases. Bayes factors are the key tool to choose between the models under comparison but for the usual default prior distributions they are not well defined. To avoid this difficulty Bayes factors for intrinsic priors are proposed and they are applied in this setting to test the existence of the main effects and the interaction effect. The method has been illustrated with an example and compared with the classical method. For this example, both approaches went in the same direction although the large P value for interaction (0.79) only prevents us against to reject the null, and the posterior probability of the null (0.95) was conclusive.

Keywords Bayes factors · Intrinsic priors · Linear contrasts

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1 Introduction

Along this paper the classical two way analysis of variance and the usual notation of the frequentist methodology are considered. For it, let $x_{ij1}, x_{ij2}, \ldots, x_{ijq}$, $i = 1, \ldots, k$; $j = 1, \ldots, n$ be nk random samples, each one of size q, from normal populations with means μ_{ij} and a common variance σ^2 that are unknown. The two way analysis of variance assumes that

$$x_{ijl} = \mu_{ij} + \varepsilon_{ijl}, \; i = 1, \ldots, k; \; j = 1, \ldots, n; \; l = 1, \ldots, q, \tag{1}$$

where x_{ijl} is the *l-th* observation corresponding to the *i-th* treatment and the *j-th* block, and ε_{ijl} is the *l-th* value of the error random variable corresponding to the *i-th* treatment and the *j-th* block. The error random variables are independent and identically distributed as normal variables with mean 0 and variance σ^2.

Model (1) can be rewritten as

$$x_{ijl} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \varepsilon_{ijl},$$

with

$$\sum_{i=1}^{k} \alpha_i = \sum_{j=1}^{n} \beta_j = \sum_{i=1}^{k} \gamma_{ij} = \sum_{j=1}^{n} \gamma_{ij} = 0,$$

where $\mu = \sum_{i=1}^{k} \sum_{j=1}^{n} \mu_{ij}/nk$ is the grand mean, $\alpha_i = \mu_{i.} - \mu$ with $\mu_{i.} = \sum_{j=1}^{n} \mu_{ij}/n$ is the effect of the *i-th* treatment, $\beta_j = \mu_{.j} - \mu$ with $\mu_{.j} = \sum_{i=1}^{k} \mu_{ij}/k$ is the effect of the *j-th* block and $\gamma_{ij} = \mu_{ij} - (\mu + \alpha_i + \beta_j)$ is the interaction effect of the *i-th* treatment and the *j-th* block. The objective of the two way analysis of variance is to test the following hypotheses:

$$\begin{aligned} H_T &: \alpha_1 = \alpha_2 = \cdots = \alpha_k = 0 \quad \text{(no treatments effect)}, \\ H_B &: \beta_1 = \beta_2 = \cdots = \beta_n = 0 \quad \text{(no blocks effect)}, \\ H_{TB} &: \gamma_{ij} = 0, \; i = 1, \ldots, k; \; j = 1, \ldots, n \quad \text{(no interaction effect)}. \end{aligned}$$

Our proposal here is to develop a testing procedure to treat the two way analysis of variance from an objective Bayesian perspective. The key idea is testing the null hypotheses H_T, H_B and H_{TB} formulating them as Bayesian model selection problems. This Bayesian analysis allows us to design the two way analysis of variance under the assumptions of homoscedasticity and heteroscedasticity.

The main tool to solve an objective Bayesian model selection problem is the Bayes factor for default priors. For the comparison of two Bayesian models, $M_i : \{f_i(\mathbf{z} \mid \theta_i), \; \pi_i(\theta_i)\}$, where $f_i(\mathbf{z} \mid \theta_i)$ is the parametric model to explain the data $\mathbf{z}$, and $\pi_i(\theta_i)$ is the prior distribution, $i = 1, 2$, the Bayes factor of M_2 to M_1 is

$$B_{21}(\mathbf{z}) = \frac{\int f_2(\mathbf{z} \mid \theta_2)\pi_2(\theta_2)d\theta_2}{\int f_1(\mathbf{z} \mid \theta_1)\pi_1(\theta_1)d\theta_1},$$

which is interpreted as the odds provided by the data for M_2 to M_1, see Berger and Pericchi (2015). Usually the priors $\pi_i(\theta_i)$, $i = 1, 2$, are chosen as default estimation priors $\pi_i^N(\theta_i)$, $i = 1, 2$, like the Jeffreys priors (Jeffreys 1961).

Unfortunately, these priors are usually improper and lead to a Bayes factor defined up to a positive multiplicative constant. To avoid this difficulty we have computed the Bayes factors for intrinsic priors introduced in Berger and Pericchi (1996), that is, $\pi_1^I(\theta_1) = \pi_1^N(\theta_1)$ and

$$\pi_2^I(\theta_2) = \pi_2^N(\theta_2) E_{f_2(x|\theta_2)} \left(\frac{m_1^N(x)}{m_2^N(x)} \right),$$

where x is an imaginary sample of minimal size for which the predictive distribution $m_i^N(x) = \int f_i(x \mid \theta_i)\pi_i^N(\theta_i)d\theta_i$ is finite, $i = 1, 2$. Note, that here we are comparing nested models. In each situation the null hypothesis is the simpler model that will be denoted as M_1 while M_2 is the complex model, that is the one with the larger number of parameters. A more detailed explanation can be found in Bertolino et al. (2000), and Cano et al. (2004).

To compute the intrinsic priors we have considered as default priors the conventional Jeffreys priors assuming that the location and the scale parameters are a priori independent, see Jeffreys (1961, page 138). Furthermore, the posterior probability of the simple and complex models has been computed assigning the same prior probability to both models. Finally, we will use the following notation, $\mathbf{z} = (\mathbf{x}_{11}, \ldots, \mathbf{x}_{kn})$ is our set of data, where $\mathbf{x}_{ij} = (x_{ij1}, \ldots, x_{ijq})$, $\overline{x}_{ij} = \sum_{h=1}^{q} x_{ijh}/q$, $\overline{x} = \sum_{i=1}^{k}\sum_{j=1}^{n}\sum_{h=1}^{q} x_{ijh}/N$ with $N = knq$, $s_{ij}^2 = \sum_{h=1}^{q}(x_{ijh} - \overline{x}_{ij})^2$ and $S^2 = \sum_{i=1}^{k}\sum_{j=1}^{n} s_{ij}^2$, $N_q(x \mid \mu, \Sigma)$ denotes the density of the q-dimensional normal distribution with mean μ and covariance Σ.

In summary, we present the problem in the context of Bayesian model selection. The usual default estimation priors cannot be used for this model selection problem and to solve this drawback we develop the intrinsic priors. The paper is organized as follows. For the sake of Occam's Razor principle, in Sect. 2 a Bayes factor for testing homoscedasticity versus heteroscedasticity is computed, which will enables us to use the homoscedastic approach when it can be assumed. Furthermore, in Sect. 2.1 the good behavior of this Bayes factor is illustrated with an example. In Sect. 3 the two way analysis of variance is considered in both, the homoscedastic and the heteroscedastic settings. In Sect. 3.1 a Bayes factor for intrinsic priors to test the global effect of treatments and blocks on the dependent variable is computed. In Sect. 3.2 Bayes factors for testing the effects of treatments and blocks, and the interaction effect are computed when $k = n = 2$. In this particular case, each one of the null hypotheses are formulated as a simple linear contrast between means and Bayes factors have been computed following the Bayesian methodology developed in Cano et al. (2016). However, when $k > 2$ or $n > 2$ at least one of these null hypotheses is formulated as a set of several linear contrasts. In Sect. 3.3, we develop a Bayesian solution for simultaneously testing two linear contrasts between means of heteroscedastic populations. In Sect. 3.4 an example illustrates the reasonable behavior of the Bayes factors computed in Sect. 3.2 for the homoscedastic case; furthermore, the results that are obtained from the frequentist and Bayesian methodologies are compared. Finally, in Sect. 4, we briefly summarize the results we have obtained.

2 Bayes factor for testing homoscedasticity

When homoscedasticity is present the dimension of the parameter spaces of the models under comparison diminishes. Therefore, it is of interest testing whether the homoscedastic assumption can be done. In a frequentist setting testing homoscedasticity versus heteroscedasticity can be done using Bartlett's test, see Snedecor and Cochran (1989). Under the Bayesian approach the nested models to be compared are:

$$M_1 : \left\{ f_1(\mathbf{z}|\theta_1) = \prod_{i=1}^{k}\prod_{j=1}^{n} N_q(\mathbf{x}_{ij}|\mu_{ij}1_q, \tau^2 I_q), \pi_1^N(\theta_1) = \frac{c_1}{\tau} \right\}, \tag{2}$$

and

$$M_2 : \left\{ f_2(\mathbf{z}|\theta_2) = \prod_{i=1}^{k}\prod_{j=1}^{n} N_q(\mathbf{x}_{ij}|\beta_{ij}1_q, \sigma_{ij}^2 I_q), \pi_2^N(\theta_2) = \frac{c_2}{\prod_{i=1}^{k}\prod_{j=1}^{n}\sigma_{ij}} \right\}, \tag{3}$$

where $\theta_1 = (\mu_{11}, \ldots, \mu_{kn}, \tau)$, $\theta_2 = (\beta_{11}, \ldots, \beta_{kn}, \sigma_{11}, \ldots, \sigma_{kn})$, c_1 and c_2 are arbitrary positive constants, and $\pi_1^N(\theta_1)$ and $\pi_2^N(\theta_2)$ are the default prior distribution obtained as the independent Jeffreys priors.

For sample $\mathbf{z}$, the Bayes factor for the intrinsic priors corresponding to models (2) and (3) turns out to be

$$B_{21}^I(\mathbf{z}) = \frac{1}{2^{\frac{N-3kn}{2}-1}\pi^{kn}\Gamma(\frac{N-kn}{2})} S^{N-kn} I_1, \tag{4}$$

where

$$I_1 = \int_0^\infty \tau^{kn-1} \left\{ \prod_{i=1}^{k}\prod_{j=1}^{n} \int_0^\infty \frac{exp\{-s_{ij}^2/2\sigma_{ij}^2\}}{(\sigma_{ij}^2+\tau^2)\sigma_{ij}^{q-1}} d\sigma_{ij} \right\} d\tau.$$

The integral I_1 can be computed by numerical integration using standard software, e.g. Mathematica. The proof has been omitted since Bayes factor (4) is a particular case of the Bayes factor developed in Section 5 of Bertolino et al. (2000).

2.1 Example

Next we illustrate the reasonable behavior of Bayes factor (4). We also compare the values of the posterior probabilities of M_1 with the P values of Bartlett's test.

Table 1 P values, Bayes factors and posterior probabilities of M_1 for different values of (s_{11}^2, s_{12}^2)

(s_{11}^2, s_{12}^2)	P values	B_{21}^I	$P(M_1)$
(12.83, 11.02)	0.99	0.0008	0.99
(44.32, 60.05)	0.13	0.05	0.95
(84.32, 90.05)	0.01	1.02	0.50
(230.06, 320.05)	7.27×10^{-6}	5579.77	0.0001

Table 2 P values, Bayes factors and posterior probabilities of M_1 for different values of s_{11}^2 and $s_{12}^2 = 10.34$

s_{11}^2	P values	B_{21}^I	$P(M_1)$
12.10	0.99	0.0008	0.99
75.08	0.07	0.13	0.87
110.06	0.01	1.67	0.37
210.51	0.0001	318.85	0.002

Example 1 (Testing homoscedasticity versus heteroscedasticity) Data refer to a two way analysis of variance with three treatments and two blocks. The data are the following:

$$\left\{s_{ij}^2, i = 1, 2, 3, j = 1, 2\right\} = \{(84.32, 90.05), (10.33, 10), (12.13, 11.33)\}.$$

Tables 1 and 2 show the P values and the posterior probabilities of M_1 along with their corresponding Bayes factors for different values of (s_{11}^2, s_{12}^2) while the remaining sampling values are kept fixed.

Second and third rows of Table 1 indicate that the value of the posterior probability of M_1 is large for values of (s_{11}^2, s_{12}^2) close to the remaining sampling values; therefore the homoscedastic model is favored in these cases. However, for values of (s_{11}^2, s_{12}^2) far from the remaining data the posterior probability of M_1 clearly favors the heteroscedastic model as the fifth row indicates. Furthermore, the P values for these three rows convey the same reasonable message. On the other hand, for the fourth row the messages from the P value and the posterior probability of M_1 are different. The P value suggests rejecting the null hypothesis while the posterior probability of M_1 is not concluding.

Numbers in Table 2 show similar conclusions to those in Table 1. Second and third rows indicate that for values of s_{11}^2 close to $s_{12}^2 = 10.34$ and to the remaining sampling values, the P value and the posterior probability of M_1 reasonability favor the homoscedastic model; however, for values of s_{11}^2 far from the remaining sampling values, both evidence measures favor the heteroscedastic model, as it is shown in the fourth and the fifth rows.

3 Bayes factors for the two way analysis of variance

In this Section the basic proposals of the classic two way analysis of variance are considered from an objective Bayesian perspective; that is, Bayes factors for testing the null hypotheses H_T, H_B and H_{TB} are computed. Nevertheless, firstly we test whether or not there exists a global effect of treatments and blocks on the dependent variable since if there is not such a global effect a two way analysis of variance would not be needed.

3.1 Bayes factors for testing the global effect of treatments and blocks

Let $\mathbf{z}$ be our set of data from normal populations with means μ_{ij}, if the μ_{ij}, $i = 1, \ldots, k$ and $j = 1, \ldots, n$, are all equal it will not exist a global effect of the k treatments and the n blocks on the dependent variable. Therefore, to deal with this issue a one way analysis of variance is needed. Since the Bayesian model selection approach to the one way analysis of variance under homoscedasticity and heteroscedasticity have been developed in Cano et al. (2013) and Bertolino et al. (2000), respectively, proofs will be omitted.

The nested models to be compared are:

(a) under homoscedasticity,

$$M_1 : \left\{ f_1(\mathbf{z}|\theta_1) = \prod_{i=1}^{k} \prod_{j=1}^{n} N_q(\mathbf{x}_{ij}|\mu 1_q, \tau^2 I_q), \pi_1^N(\theta_1) = c_1/\tau \right\},$$

$$M_2 : \left\{ f_2(\mathbf{z}|\theta_2) = \prod_{i=1}^{k} \prod_{j=1}^{n} N_q(\mathbf{x}_{ij}|\mu_{ij} 1_q, \sigma^2 I_q), \pi_2^N(\theta_2) = c_2/\sigma \right\},$$

where, $\theta_1 = (\mu, \tau)$ and $\theta_2 = (\mu_{11}, \ldots, \mu_{kn}, \sigma)$, and

(b) under heteroscedasticity,

$$M_1 : \left\{ f_1(\mathbf{z}|\theta_1) = \prod_{i=1}^{k} \prod_{j=1}^{n} N_q(\mathbf{x}_{ij}|\mu 1_q, \tau_{ij}^2 I_q), \pi_1^N(\theta_1) = \frac{c_1}{\prod_{i=1}^{k} \prod_{j=1}^{n} \tau_{ij}} \right\},$$

$$M_2 : \left\{ f_2(\mathbf{z}|\theta_2) = \prod_{i=1}^{k} \prod_{j=1}^{n} N_q(\mathbf{x}_{ij}|\mu_{ij} 1_q, \sigma_{ij}^2 I_q), \pi_2^N(\theta_2) = \frac{c_2}{\prod_{i=1}^{k} \prod_{j=1}^{n} \sigma_{ij}} \right\},$$

where $\theta_1 = (\mu, \tau_{11}, \ldots, \tau_{kn})$ and $\theta_2 = (\mu_{11}, \ldots, \mu_{kn}, \sigma_{11}, \ldots, \sigma_{kn})$.

The Bayes factors for intrinsic priors corresponding to these comparisons are given by:

(a) under homoscedasticity,

$$B^I_{21}(\mathbf{z}) = \frac{2N^{\frac{1}{2}}\Gamma(\frac{N}{2})\left(\sum_{i=1}^k\sum_{j=1}^n\left(s^2_{ij}+q(\overline{x}_{ij}-\overline{\overline{x}})^2\right)\right)^{\frac{N-1}{2}}}{\pi^{\frac{3}{2}}\Gamma(\frac{N-1}{2})} I_2,$$

where $I_2 = \int\limits_{-\infty}^{\infty} I_1 d\mu$ and

$$I_1 = \int\limits_0^{\pi/2} \frac{(g(q,\theta,n,k))^{-kn/2}}{(sin\theta)^{N-kn}\left(\frac{S^2}{sin^2\theta}+\frac{q\sum\limits_{i=1}^{k}\sum\limits_{j=1}^{n}(\overline{x}_{ij}-\mu)^2}{g(q,\theta,n,k)}\right)^{N/2}} d\theta,$$

with $g(q,\theta,n,k) = q(1-\frac{1}{2kn}) + sin^2\theta$, and

(b) under heteroscedasticity,

$$B^I_{21}(\mathbf{z}) = \frac{2^{3kn/2}}{\pi^{kn}} \frac{\int_{-\infty}^{\infty}\left\{\prod_{i=1}^k\prod_{j=1}^n I_{ij}\left(\mu,\overline{x}_{ij},s^2_{ij},q\right)\right\}d\mu}{\int_{-\infty}^{\infty}\left\{\prod_{i=1}^k\prod_{j=1}^n (s^2_{ij}+q(\overline{x}_{ij}-\mu)^2)^{-q/2}\right\}d\mu},$$

where

$$I_{ij}\left(\mu,\overline{x}_{ij},s^2_{ij},q\right) = \int_0^{\pi/2} \frac{d\theta}{(sin\theta)^{q-1}g\,(\theta,q)^{1/2}\,h\left(\theta,\overline{x}_{ij},s^2_{ij},q\right)},$$

with

$$h\left(\theta,\overline{x}_{ij},s^2_{ij},q\right) = \left(\frac{s^2_{ij}}{sin^2\theta}+\frac{2q(\overline{x}_{ij}-\mu)^2}{g\,(\theta,q)}\right)^{q/2},$$

and $g\,(\theta,q) = q + 2sin^2\theta$. Note that the above integrals cannot be expressed in a closed form but can be solved by numerical integration.

3.2 Bayes factors for testing the effects of treatments, blocks and the interaction

The hypotheses to be tested are:

$$\begin{aligned} H_T &: \alpha_1 = \alpha_2 = \cdots = \alpha_{k-1} = 0, \\ H_B &: \beta_1 = \beta_2 = \cdots = \beta_{n-1} = 0, \\ H_{TB} &: \gamma_{ij} = 0,\ i = 1,\ldots,k-1;\ j = 1,\ldots,n-1, \end{aligned}$$

which are formulated as $(k-1)$,$(n-1)$ and $(k-1)\times(n-1)$ linear contrasts, respectively. In particular, these null hypotheses in a two way analysis of variance with two treatments and two blocks are written in the form $\sum_{i=1}^{2}\sum_{j=1}^{2} a_{ij}\mu_{ij} = 0$ as:

$$\begin{aligned} H_T &: \frac{1}{4}\mu_{11} + \frac{1}{4}\mu_{12} - \frac{1}{4}\mu_{21} - \frac{1}{4}\mu_{22} = 0, \\ H_B &: \frac{1}{4}\mu_{11} - \frac{1}{4}\mu_{12} + \frac{1}{4}\mu_{21} - \frac{1}{4}\mu_{22} = 0, \\ H_{TB} &: \frac{1}{4}\mu_{11} - \frac{1}{4}\mu_{12} - \frac{1}{4}\mu_{21} + \frac{1}{4}\mu_{22} = 0. \end{aligned}$$

Each one of these hypotheses can be formulated as nested model comparisons in the following way:

(a) under homoscedasticity, the simple model, M_1, is

$$f_1(\mathbf{z}|\theta_1) = N_q\left(\mathbf{x}_{11}|\left(\sum_{\substack{i=1\\(i,j)\neq(1,1)}}^{2}\sum_{j=1}^{2} d_{ij}\mu_{ij}\right)1_q, \tau^2 I_q\right)\prod_{\substack{i=1\\(i,j)\neq(1,1)}}^{2}\prod_{j=1}^{2} N_q(\mathbf{x}_{ij}|\mu_{ij}1_q, \tau^2 I_q),$$

with prior distribution

$$\pi_1^N(\theta_1) = \frac{c_1}{\tau},$$

and the complex model, M_2, is

$$f_2(\mathbf{z}|\theta_2) = \prod_{i=1}^{2}\prod_{j=2}^{2} N_q(\mathbf{x}_{ij}|\beta_{ij}1_q, \sigma^2 I_q),$$

with prior distribution

$$\pi_2^N(\theta_2) = \frac{c_2}{\sigma},$$

where $\mathbf{z} = (\mathbf{x}_{11}, \mathbf{x}_{12}, \mathbf{x}_{21}, \mathbf{x}_{22})$, $\theta_1 = (\mu_{12}, \mu_{21}, \mu_{22}, \tau)$, $\theta_2 = (\beta_{11}, \beta_{12}, \beta_{21}, \beta_{22}, \sigma)$ and $d_{ij} = -a_{ij}/a_{11}$ for $i = 1, 2;\ j = 1, 2$ with $(i, j) \neq (1, 1)$, and

(b) under heteroscedasticity, the simple model, M_1, is

$$f_1(\mathbf{z}|\theta_1) = N_q\left(\mathbf{x}_{11}|\left(\sum_{\substack{i=1\\(i,j)\neq(1,1)}}^{2}\sum_{j=1}^{2} d_{ij}\mu_{ij}\right)1_q, \tau_{ij}^2 I_q\right)\prod_{\substack{i=1\\(i,j)\neq(1,1)}}^{2}\prod_{j=1}^{2} N(\mathbf{x}_{ij}|\mu_{ij}1_q, \tau_{ij}^2 I_q),$$

with prior distribution

$$\pi_1^N(\theta_1) = c_1/\prod_{i=1}^{2}\prod_{j=1}^{2}\tau_{ij},$$

and the complex model, M_2, is

$$f_2(\mathbf{z}|\theta_2) = \prod_{i=1}^{2}\prod_{j=2}^{2} N_q(\mathbf{x}_{ij}|\beta_{ij}1_q, \sigma_{ij}^2 I_q),$$

with prior distribution

$$\pi_2^N(\theta_2) = c_2/\prod_{i=1}^{2}\prod_{j=1}^{2}\sigma_{ij},$$

where $\mathbf{z} = (\mathbf{x}_{11}, \mathbf{x}_{12}, \mathbf{x}_{21}, \mathbf{x}_{22})$, $\theta_1 = (\mu_{12}, \mu_{21}, \mu_{22}, \tau_{11}, \tau_{12}, \tau_{21}, \tau_{22})$, $\theta_2 = (\beta_{11}, \beta_{12}, \beta_{21}, \beta_{22}, \sigma_{11}, \sigma_{12}, \sigma_{21}, \sigma_{22})$ and $d_{ij} = -a_{ij}/a_{11}$ for $i = 1, 2;\ j = 1, 2$ with $(i, j) \neq (1, 1)$.

Using the theory of the Bayesian model selection approach to simple linear contrasts developed in Cano et al. (2016) it is obtained that, for the sample $\mathbf{z}$, the Bayes factors for the corresponding intrinsic priors are given by:

(a) under homoscedasticity,

$$B_{21}^I(\mathbf{z}) = \frac{2\pi^{-1}}{\int \left(s^2 + qh\left(d_{ij}, \mu_{ij}, \overline{x}_{ij}\right)\right)^{-2q} d\mu_{12}d\mu_{21}d\mu_{22}} I_2,$$

where $I_2 = \int\limits_{-\infty}^{\infty} \cdots \int\limits_{-\infty}^{\infty} I_1\, d\mu_{12}d\mu_{21}d\mu_{22}$, with

$$I_1 = \int_0^{\pi/2} \frac{(sin\theta)^{-4(q-1)}\,(g\,(\theta, q))^{-2}}{\left(\frac{s^2}{sin^2\theta} + \frac{qh\left(d_{ij},\mu_{ij},\overline{x}_{ij}\right)}{g(\theta,q)}\right)^{2q}} d\theta,$$

where

$$h\left(d_{ij}, \mu_{ij}, \overline{x}_{ij}\right) = \left(\overline{x}_{11} - \sum_{\substack{i=1\\(i,j)\neq(1,1)}}^{2}\sum_{j=1}^{2} d_{ij}\mu_{ij}\right)^2 + \sum_{\substack{i=1\\(i,j)\neq(1,1)}}^{2}\sum_{j=1}^{2}\left(\overline{x}_{ij} - \mu_{ij}\right)^2,$$

and $g\,(\theta, q) = (1 - \frac{1}{8})q + sin^2\theta$, and

(b) under heteroscedasticity,

$$B_{21}^I(\mathbf{z}) = \frac{2^6 I_3}{\pi^4 I_1},$$

where

$$I_1 = \int \left(s_{11}^2 + q \left(\overline{x}_{11} - \sum_{\substack{i=1 \\ (i,j)\neq(1,1)}}^{2} \sum_{j=1}^{2} d_{ij}\mu_{ij} \right)^2 \right)^{-q/2}$$
$$\times \prod_{\substack{i=1 \\ (i,j)\neq(1,1)}}^{2} \prod_{j=1}^{2} (s_{ij}^2 + q(\overline{x}_{ij} - \mu_{ij})^2)^{-q/2} d\mu_{12} d\mu_{21} d\mu_{22},$$

with $I_3 = \int\limits_{-\infty}^{\infty} \cdots \int\limits_{-\infty}^{\infty} I_2 \, d\mu_{12} d\mu_{21} d\mu_{22}$, where

$$I_2 = \int_0^{\pi/2} \frac{d\theta}{(sin\theta)^{q-1}(g(\theta,q))^{1/2} \left(\frac{s_{11}^2}{sin^2\theta} + \frac{2q}{g(\theta,q)} \left(\overline{x}_{11} - \sum_{\substack{i=1 \\ (i,j)\neq(1,1)}}^{2} \sum_{j=1}^{2} d_{ij}\mu_{ij} \right)^2 \right)^{\frac{q}{2}}}$$
$$\times \prod_{\substack{i=1 \\ (i,j)\neq(1,1)}}^{2} \prod_{j=1}^{2} \left(\int_0^{\pi/2} \frac{d\theta}{(sin\theta)^{q-1}(g(\theta,q))^{1/2} \left(\frac{s_{ij}^2}{sin^2\theta} + \frac{2q}{g(\theta,q)} (\overline{x}_{ij} - \mu_{ij})^2 \right)^{\frac{q}{2}}} \right),$$

and $g(\theta, q) = q + 2sin^2\theta$.

Nevertheless, when $k > 2$ or $n > 2$ some of the null hypotheses H_T, H_B and H_{TB} are formulated as a set of several linear contrasts. In these cases, the Bayesian procedure developed in Cano et al. (2016) cannot be used and a method for simultaneously testing several linear contrasts is needed. For the sake of brevity we have developed a solution for simultaneously testing two linear contrasts in the general heteroscedastic case. The results that have been obtained are the following.

3.3 Simultaneous testing of two linear contrasts for heteroscedastic populations

Let $N\left(x_1|\mu_1, \sigma_1^2\right), \ldots, N\left(x_k|\mu_k, \sigma_k^2\right)$ be k normal distributions with means $\mu_1, \ldots, \mu_k$ and variances $\sigma_1^2, \ldots, \sigma_k^2$ unknown. The key idea is to find a reparameterization allowing us to simultaneously test two independent linear contrasts. For it, we consider the two following independent linear contrasts:

$$0 = a_{11}\mu_1 + a_{12}\mu_2 + \cdots + a_{1k}\mu_k, \tag{5}$$
$$0 = a_{21}\mu_1 + a_{22}\mu_2 + \cdots + a_{2k}\mu_k, \tag{6}$$

with $k > 2$. Without lost of generality, this system of two equations can be written as

$$\mu_1 = \sum_{i=3}^{k} b_{1i}\mu_i, \quad \mu_2 = \sum_{i=3}^{k} c_{2i}\mu_i.$$

Now, simultaneously testing the two linear contrasts (5) and (6) above can be formulated as a Bayesian model selection problem, where the density function $f_1(\mathbf{z}|\theta_1)$ for the simple model, M_1, is

$$N_{n_1}\left(\mathbf{x}_1|\left(\sum_{i=3}^{k} b_{1i}\mu_i\right)1_{n_1}, \tau_1^2 I_{n_1}\right) N_{n_2}\left(\mathbf{x}_2|\left(\sum_{i=3}^{k} c_{2i}\mu_i\right)1_{n_2}, \tau_2^2 I_{n_2}\right)$$
$$\times \prod_{i=3}^{k} N_{n_i}(\mathbf{x}_i|\mu_i 1_{n_i}, \tau_i^2 I_{n_i}),$$

with $\pi_1^N(\theta_1) = c_1/\prod_{i=1}^k \tau_i$, and the complex model, M_2, is

$$f_2(\mathbf{z}|\theta_2) = \prod_{i=1}^{k} N_{n_i}\left(\mathbf{x}_i|\beta_i 1_{n_i}, \sigma_i^2 I_{n_i}\right),$$

with $\pi_2^N(\theta_2) = c_2/\prod_{i=1}^k \sigma_i$, where $\mathbf{z} = \{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$, $\theta_1 = (\mu_3, \ldots, \mu_k, \tau_1, \ldots, \tau_k)$ and $\theta_2 = (\beta_1, \ldots, \beta_k, \sigma_1, \ldots, \sigma_k)$.

The sampling model M_1 is nested in M_2 and for the priors above the minimal training sample is a $2k-$dimensional random vector that consists of two observations from each population, see Berger and Pericchi (1996). Applying the standard intrinsic priors methodology we have obtained the following result.

Theorem 1 *The intrinsic priors for comparing model M_1 versus model M_2 are $\{\pi_1^N(\theta_1), \pi_2^I(\theta_2)\}$ with $\pi_2^I(\theta_2) = \int \pi_2^I(\theta_2|\theta_1)\pi_1^N(\theta_1)d\theta_1$, where*

$$\pi_2^I(\theta_2|\theta_1) = N\left(\beta_1|\sum_{i=3}^{k} b_{1i}\mu_i, \frac{\tau_1^2+\sigma_1^2}{2}\right) N\left(\beta_2|\sum_{i=3}^{k} c_{2i}\mu_i, \frac{\tau_2^2+\sigma_2^2}{2}\right)$$
$$\times \prod_{i=3}^{k} N\left(\beta_i|\mu_i, \frac{\tau_i^2+\sigma_i^2}{2}\right) \prod_{i=1}^{k} HC^+(\sigma_i|0, \tau_i),$$

and $HC^+(\sigma|0, \tau)$ denotes the half Cauchy density.

Proof It is obtained adapting the proof of Theorem 1 in Bertolino et al. (2000).

For the sample $\mathbf{z}$, the Bayes factor for these intrinsic priors turns out to be

$$B_{21}^I(\mathbf{z}) = \frac{2^{\frac{3k}{2}} I_3}{\pi^k I_1}, \tag{7}$$

where

$$I_1 = \int \left(s_1^2 + n_1 \left(\overline{x}_1 - \sum_{i=3}^{k} b_{1i}\mu_i \right)^2 \right)^{-n_1/2}$$
$$\times \left(s_2^2 + n_2 \left(\overline{x}_2 - \sum_{i=3}^{k} c_{2i}\mu_i \right)^2 \right)^{-n_2/2}$$
$$\times \prod_{i=3}^{k} (s_i^2 + n_i(\overline{x}_i - \mu_i)^2)^{-n_i/2} d\mu_3 \ldots d\mu_k,$$

and $I_3 = \int\limits_{-\infty}^{\infty} \cdots \int\limits_{-\infty}^{\infty} I_2 \, d\mu_3 \ldots d\mu_k$, with

$$I_2 = \int_0^{\pi/2} \frac{d\theta}{(sin\theta)^{n_1-1}(g(\theta,n_1))^{1/2}\left(\frac{s_1^2}{sin^2\theta} + \frac{2n_1}{g(\theta,n_1)}\left(\overline{x}_1 - \sum_{i=3}^{k} b_{1i}\mu_i\right)^2\right)^{\frac{n_1}{2}}}$$
$$\times \int_0^{\pi/2} \frac{d\theta}{(sin\theta)^{n_2-1}(g(\theta,n_2))^{1/2}\left(\frac{s_2^2}{sin^2\theta} + \frac{2n_2}{g(\theta,n_2)}\left(\overline{x}_2 - \sum_{i=3}^{k} c_{2i}\mu_i\right)^2\right)^{\frac{n_2}{2}}}$$
$$\times \prod_{i=3}^{k} \left(\int_0^{\pi/2} \frac{d\theta}{(sin\theta)^{n_i-1}(g(\theta,n_i))^{1/2}\left(\frac{s_i^2}{sin^2\theta} + \frac{2n_i}{g(\theta,n_i)}(\overline{x}_i - \mu_i)^2\right)^{\frac{n_i}{2}}} \right),$$

where $g(\theta, n) = n + 2sin^2\theta$.

Note that expression (7) is computed by first integrating out the $\beta_i\,'s$ and then transforming (τ_i, σ_i) to polar coordinates. It is not possible to compute integrals I_1 and I_3 analytically but this is not a serious inconvenience, as they can be approximated numerically with an appropriate software such as Mathematica or some common simulation techniques, see Robert and Casella (2001).

3.4 Example

To illustrate the Bayesian methodology developed in this Section we have applied it to Example 1 in Rohatgi (1984, page 822). Furthermore, we have compared our results with those obtained from the frequentist approach.

Example 2 (Comparing Two Workers in Two Machines) Suppose we wish to test whether there are any differences in two machines that are used by two workers.

Table 3 Number of units produced per day for every worker and machine, respectively

Source	Machine 1	Machine 2
Worker 1	69, 68, 72, 74, 75	95, 98, 100, 96, 97
Worker 2	81, 88, 84, 87, 88	105, 110, 107, 112, 118

Table 4 Frequentist analysis of variance for data in Table 3

Source	DF	SS	MS	P values
Workers and machines	3	4100.8	1366.93	4.91×10^{-11}
Workers	1	924.8	924.8	1.54×10^{-7}
Machines	1	3175.2	3175.2	2.12×10^{-11}
Interaction	1	0.8	0.8	0.79
Error	16	190.4	11.9	–
Total	19	4291.2		

Table 5 Bayes factors and posterior probabilities of M_1 for the two way analysis of variance for data in Table 3

Source	B_{21}^{I}	$P(M_1)$
Workers and machines	1.69×10^{8}	5.90×10^{-9}
Workers	67.47	0.01
Machines	2.96×10^{10}	3.37×10^{-11}
Interaction	0.05	0.95

Suppose, further, that four weeks are selected for this study, each week consisting of five working days. During the first and the second week worker 1 is used and during the third and the fourth week worker 2 is used. Machine 1 is used during the first and the third week and machine 2 is used during the second and the fourth week. Suppose that other conditions are identical. Data in Table 3 are the number of units produced per day.

There are two factors to be considered here. The firs one is the effect of the workers and the other one is the effect of the machines. Furthermore, the effect of the interaction between the workers and the machines is to be considered too. The results of the frequentist analysis of these effects are summarized in Table 4.

Second row of Table 4 indicates that there is a joint effect of the two factors. This effect is decomposed as follows, third and fourth rows indicate that there is an effect on the production that can be attributed to differences in the workers and another one attributed to differences in the machines. On the other hand, there is no interaction between workers and machines as it can be seen from the fifth row.

From our objective Bayesian approach the results obtained are displayed in Table 5, where it is shown that all the rows convey similar conclusions to those in Table 4.

4 Summary

We have found a satisfactory objective Bayesian model selection approach to the classic two way analysis of variance. We have shown how objective model selection priors like intrinsic priors can be applied in the two way ANOVA model. This model is nowadays very used, and therefore we will continue our research using other type of priors.

Our development is in both the homoscedastic and the heteroscedastic settings and hence deserves special attention since the exact frequentist analysis of variance is based on the homoscedastic assumption. Furthermore, a well behaved Bayes factor for testing homoscedasticity versus heteroscedasticity have been computed too.

A detailed two way analysis of variance is only needed if a joint effect of treatments and blocks on the dependent variable is previously detected; therefore, we have firstly focussed on this question. On the other hand, the effects of the two factors and the interaction effect have been formulated as linear contrasts between means of normal populations and the Bayesian methodology approach to simple linear contrasts developed in Cano et al. (2016) have been used to study the existence of these effects in the particular case when $k = n = 2$. An example illustrates the good behavior of the Bayes factor for this particular analysis of variance. However, when $k > 2$ or $n > 2$ at least some of the classic null hypotheses is formulated as a set of several linear contrasts. For the sake of brevity, we have developed a solution to simultaneously testing two linear contrasts between means in the general heteroscedastic case. Note that when we have more than two simultaneous linear contrasts the complexity of the involved computation is not increased. Furthermore, the integrals needed to compute the Bayes factors can be solved by numerical integration with current software, as Mathematica, or some common simulation techniques.

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References

Berger JO, Pericchi LR (1996) The intrinsic Bayes factor for model selection and prediction. J Am Stat Assoc 91:109–122

Berger JO, Pericchi LR (2015) Bayes factors. Wiley StatsRef: statistics reference online. Wiley, London

Bertolino F, Moreno E, Racugno W (2000) Bayesian model selection approach to analysis of variance under heteroscedasticity. Statistician 49(4):503–517

Cano JA, Kessler M, Moreno E (2004) On intrinsic priors for nonnested models. Test 13:445–463

Cano JA, Carazo C, Salmerón D (2013) Bayesian model selection approach to the one way analysis of variance under homoscedasticity. Comput Stat 28:919–931

Cano JA, Carazo C, Salmerón D (2016) Objective Bayesian model selection approach to linear contrasts for the one way analysis of variance. Stat Prob Lett 109:54–62

Jeffreys H (1961) Theory of probability. Oxford University Press, Oxford

Robert CP, Casella G (2001) Monte Carlo statistical methods. Springer, Berlin

Rohatgi VK (1984) Statistical inference. Wiley, London

Snedecor GW, Cochran WG (1989) Statistical methods, 8th edn. Iowa State University Press, Iowa City