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Representation of weakly maxitive monetary risk measures and their rate functions

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ABSTRACT

This article provides a representation result for monetary risk measures (i.e., monotone translation-invariant functionals) satisfying a weak maxitivity property. This result can be understood as a functional analytic generalization of the Gärtner-Ellis large deviations theorem. In contrast to the classical Gärtner-Ellis theorem, the rate function is computed on an arbitrary set of continuous real-valued functions rather than the dual space. As an application of the main result, we establish a large deviations result for sequences of sublinear expectations on regular Hausdorff topological spaces.

1. Introduction

The theory of large deviations studies the asymptotic tail behavior of sequences of random variables. The earliest developments of this theory arose in the context of ruin theory in actuarial science [10,11], while Varadhan and Donsker [13,27] systematically developed the modern framework of this field. First, we recall the basic principles of large deviations theory; we refer to the excellent monograph by Dembo and Zeitouini [12] for further details and the historical background.

Let E be a regular Hausdorff topological space and let $(X_n)_{n\in\mathbb{N}}$ be a sequence of E-valued random variables defined in a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The sequence $(X_n)_{n\in\mathbb{N}}$ is said to satisfy the *large deviation principle* (LDP) with rate function $I\colon E \to [0,\infty]$ if

$$-\inf_{x\in\operatorname{int}(A)} I(x) \le \liminf_{n\to\infty} \frac{1}{n}\log \mathbb{P}(X_n \in A) \le \limsup_{n\to\infty} \frac{1}{n}\log \mathbb{P}(X_n \in A) \le -\inf_{x\in\operatorname{cl}(A)} I(x)$$

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for all Borel sets $A \subset E$.[2] Varadhan's integral lemma asserts that a sequence $(X_n)_{n\in\mathbb{N}}$ that satisfies the LDP with rate function $I(\cdot)$ also satisfies the *Laplace principle* (LP) with rate function $I(\cdot)$; that is,

$$\psi(f) = \sup_{x\in E}\{f(x) - I(x)\}$$

for all $f \in C_b(E)$.[3] We denote by $\psi(f) = \lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}_{\mathbb{P}}[e^{nf(X_n)}]$ the asymptotic entropy of f.[4] The converse of Varadhan's integral lemma also holds true under additional regularity conditions; Bryc [4] established this statement assuming that $I(\cdot)$ has compact sublevel sets, and Comman [6] proved the same assuming that E is normal. In addition, Bryc's theorem states that both the LDP and the LP hold with the rate function $I(x) = \sup_{f\in C_b(E)}\{f(x) - \psi(f)\}$ if the sequence $(X_n)_{n\in\mathbb{N}}$ is exponentially tight.[5] The functional $\psi\colon C_b(E) \to \mathbb{R}$ has some properties that are crucial in this theory. First, ψ is a monetary risk measure; that is, it is monotone (i.e., $\psi(f) \le \psi(g)$ whenever $f \le g$) and translation invariant (i.e., $\psi(f + c) = \psi(f) + c$). Second, the asymptotic entropy ψ has the remarkable property of being maxitive (i.e., $\psi(f \vee g) \le \psi(f) \vee \psi(g)$). The properties of a maxitive monetary risk measure are sufficient to prove generalized versions of all the basic results mentioned above covering this type of functional, striping away any probabilistic aspect of the theory. Bell and Bryc [3] introduced and studied a general LP for monetary risk measures[6] on $C_b(E)$ and, more recently, Kupper and Zapata [20] formulated a general LDP for this kind of functional and extended to this general setting the Varadhan-Bryc equivalence between the LDP and the LP, and Bryc's theorem.

As a continuation of the research in [20], in the present article we aim to identify new situations where a monetary risk measure satisfies the LP and the LDP and, in particular, how to compute the rate function $I(\cdot)$. While our main result applies to general monetary risk measures satisfying a weak form of maxitivity, we explain it now for easier readability for the case of the asymptotic entropy $\psi(f) = \lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}_{\mathbb{P}}[e^{nf(X_n)}]$, where for simplicity in the exposition we assume that the latter limit exists for all real-valued continuous functions f. In the following we fix an arbitrarily given non-empty set $\mathcal{H}$ of continuous real-valued functions on E, and consider the corresponding conjugate $\psi^*_{\mathcal{H}}$, which is defined by $\psi^*_{\mathcal{H}}(x) = \sup_{f\in\mathcal{H}}\{f(x) - \psi(f)\}$. We wish to establish sufficient conditions so that $(X_n)_{n\in\mathbb{N}}$ satisfies the LDP with rate function $\psi^*_{\mathcal{H}}$. We say that a point $x \in E$ is $\mathcal{H}$-exposed for $\psi^*_{\mathcal{H}}$ if there exists a function $f \in \mathcal{H}$ such that

$$\psi^*_{\mathcal{H}}(y) - f(y) > \psi^*_{\mathcal{H}}(x) - f(x) \quad \text{for all } y \ne x. \tag{1.1}$$

The interpretation of the exposing condition (1.1) is that, for a certain constant c, the curve $y \mapsto f(y) + c$ lies strictly below the curve $\psi^*_{\mathcal{H}}$ on $E \setminus \{x\}$, and agrees with it at x. Denote by $\mathscr{E}$ the set of all $\mathcal{H}$-exposed points of E. In the special case of the asymptotic entropy, the main result of this article reads as follows.

Theorem 1.1. *Suppose that the sequence $(X_n)_{n\in\mathbb{N}}$ is exponentially tight. Then*

(i) *for every closed set $C \subset E$, we have the upper bound*

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}(X_n \in C) \le -\inf_{y\in C} \psi^*_{\mathcal{H}}(x);$$

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[2] We denote by $\mathrm{int}(A)$ and $\mathrm{cl}(A)$ the topological interior and closure of $A \subset E$, respectively.
[3] We denote by $C_b(E)$ the set of all bounded continuous real-valued functions on E.
[4] The limit in $\psi(f)$ exists for all $f \in C_b(E)$ if (X_n) satisfies the LDP.
[5] That is, for all positive numbers M, there exists $K \subset E$ compact such that $\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}(X_n \in K^c) \le -M$.
[6] Bell and Bryc [3] use the term *Varadhan functional* rather than *monetary risk measure*. Here, we use the term *monetary risk measure* to build a bridge toward risk analysis.

(ii) *for every open set $O \subset E$, we have the lower bound*

$$-\inf_{y \in O \cap \mathscr{E}} \psi^*_{\mathcal{H}}(x) \leq \liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}(X_n \in O);$$

(iii) *if, moreover,*

$$\inf_{x \in O} \psi^*_{\mathcal{H}}(x) = \inf_{x \in O \cap \mathscr{E}} \psi^*_{\mathcal{H}}(x) \quad \textit{for all } O \subset E \textit{ open,}$$

then $(X_n)_{n\in\mathbb{N}}$ satisfies the LDP and the LP with rate function $\psi^*_{\mathcal{H}}$.

Of great importance, the Gärtner-Ellis theorem provides the LDP for sequences of random variables with values on a topological vector space by testing the rate function on the dual space. This well-known result turns out to be a particular instance of Theorem 1.1 for the special case when E is a topological vector space and the particular choice $\mathcal{H} = E^*$; see [12, Theorem 4.5.20]. One of the novelties of Theorem 1.1 is that it allows for different options for the testing set $\mathcal{H}$, instead of limiting us to the dual space. Moreover, it is well known that the Gärtner-Ellis theorem does not cover all cases in which an LDP exists; there are examples for which the LDP holds but it does not follow from this basic result; see [12, Remarks(d), p. 45] and [8]. In contrast, Theorem 1.1 allows for arbitrary choices of $\mathcal{H}$, covering situations where the Gärtner-Ellis theorem fails. This is illustrated in Example 5.5, where we have a situation where an LDP is not covered by the Gärtner-Ellis theorem but is captured by our choosing a family $\mathcal{H}$ of inverted V-shaped functions.

Our main result for the particular functional $\psi(f) = \lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}_{\mathbb{P}}[e^{nf(X_n)}]$ as stated above applies to very general functionals, allowing us to cover some nonstandard setups as those in [2,9,14,16,17,22,26,29]. For instance, in situations with model uncertainty, one may be interested in considering a set $\mathcal{P}$ of probability measures rather than a single probability measure $\mathbb{P}$; see [9,26]. This situation is covered by our main result by considering the robust asymptotic entropy $\psi_{\mathcal{P}}(f) = \limsup_{n\to\infty} \frac{1}{n} \log \sup_{Q\in\mathcal{P}} \mathbb{E}_Q[e^{nf(X_n)}]$. In particular, we extend to infinite-dimensional spaces the version of the Gärtner-Ellis theorem for sequences of sublinear expectations on $\mathbb{R}^d$ proven in [26] with the advantage that the rate function is now tested on arbitrary sets of continuous functions.

The present approach fully relies on topological and order properties and, in particular, an underlying probability space is not needed. We emphasize that the existing proofs of the Gärtner-Ellis theorem are based on probability concepts such as the Radon-Nykodym derivative that are not needed here; see [12, Theorem 4.5.20]. In line with [21,23], the machinery is taken from maxitive integration. More specifically, as in [21] we use the convex integral introduced by Cattaneo [5], which is conceptually related to the idempotent integral in tropical mathematics [19] and can be obtained as a transformation of the Shilkret integral [25]. In particular, we rely on the duality bounds for convex integrals and the convex integral representation of weakly maxitive monetary risk measures provided in [21].

This article is organized as follows. In Section 2 we give some preliminaries on maxitive integration. In Section 3 we focus on the integral representation of weakly maxitive monetary risk measures. In Section 4 we state and prove the main result of this article. Finally, in Section 5 we apply the main result to obtain a large deviations result for sequences of sublinear expectations on regular Hausdorff topological spaces.

2. Preliminaries on maxitive integration

Throughout this article, let E be a regular Hausdorff topological space with Borel σ-algebra $\mathcal{B}(E)$. We always use the convention that $-\infty \cdot 0 = 0$. Then, given a function $f : E \to \mathbb{R} \cup \{-\infty\}$, the function $f 1_A - \infty 1_{A^c}$ takes the same values as f on $A \subset E$ and the value $-\infty$ on A^c.

A set function $J : \mathcal{B}(E) \to [-\infty, 0]$ is said to be a *concentration* if

(a1) $J_\emptyset = -\infty$, $J_E = 0$;
(a2) $J_A \leq J_B$ whenever $A \subset B$.

We say that J is *maxitive* if

(a3) $J_{A\cup B} \leq J_A \vee J_B$.

Denote by $B(E)$ the set of all Borel measurable functions $f\colon E \to \mathbb{R} \cup \{-\infty\}$. We define the *convex integral* of $f \in B(E)$ with respect to the concentration J as[7]

$$\phi_J(f) = \sup_{c\in\mathbb{R}} \left\{c + J_{\{f>c\}}\right\}. \tag{2.1}$$

Properties (b1)–(b4) below are provided in [5], and their proofs easily adapt to the present setting. The proof of (b5) can be found in [21, Lemma 4.1].

(b1) $\phi_J(-\infty 1_{A^c}) = J_A$ for all $A \in \mathcal{B}(E)$.
(b2) $\phi_J(0) = 0$.
(b3) $\phi_J(f + c) = \phi_J(f) + c$ for all constant $c \in \mathbb{R}$.
(b4) $\phi_J(f) \leq \phi_J(g)$ whenever $f \leq g$.
(b5) $\lim_{n\to\infty} \phi_J(f \wedge n) = \phi_J(f)$ and $\lim_{n\to\infty} \phi_J(f \vee -n) = \phi_J(f)$.

If J is maxitive, then the following hold:

(b6) ϕ_J is maxitive; that is, $\phi_J(f \vee g) \leq \phi_J(f) \vee \phi_J(g)$.
(b7) ϕ_J is convex; that is, $\phi_J(\lambda f + (1-\lambda)g) \leq \lambda\phi_J(f) + (1-\lambda)\phi_J(g)$ for all $0 \leq \lambda \leq 1$.

The proofs of (b6) and (b7) can be found in [5] in Corollary 5 and Theorem 7, respectively. Denote by $L(E)$ the set of all lower semicontinuous functions $f\colon E \to \mathbb{R} \cup \{-\infty\}$ and by $U(E)$ the set of all upper semicontinuous functions $f\colon E \to \mathbb{R} \cup \{-\infty\}$. The following duality bounds were proved in [21, Theorem 3.4].[8]

Theorem 2.1. *Let J be a concentration and let $I\colon E \to [0,\infty]$ be a function. Then the following equivalences hold. First,*

$$-\inf_{x\in O} I(x) \leq J_O \quad \textit{for all open sets } O \subset E \tag{2.2}$$

if and only if

$$\phi_J(f) \geq \sup_{x\in E}\{f(x) - I(x)\} \quad \textit{for all } f \in L(E). \tag{2.3}$$

Second,

$$J_C \leq -\inf_{x\in C} I(x) \quad \textit{for all closed sets } C \subset E \tag{2.4}$$

[7] It is not difficult to show that $\phi_J(f) = \sup_{c\in\mathbb{R}} \left\{c + J_{\{f>c\}}\right\} = \sup_{c\in\mathbb{R}} \left\{c + J_{\{f\geq c\}}\right\}$; that is, we obtain an equivalent definition of $\phi_J(f)$ if the strict inequality in (2.1) is replaced by a nonstrict inequality.

[8] Theorem 3.4 in [21] deals with functions that are increasing with respect to a given preorder. To apply Theorem 3.4 in [21] here, we consider the trivial preorder (i.e., $x \leq y$ whenever $x = y$).

if and only if

$$\phi_J(f) \leq \sup_{x\in E}\{f(x) - I(x)\} \quad \textit{for all } f \in U(E). \tag{2.5}$$

The minimal rate function $I_{\min}\colon E \to [0,\infty]$ associated with a concentration J is defined as

$$I_{\min}(x) := \sup_{f\in L(E)}\{f(x) - \phi_J(f)\}. \tag{2.6}$$

Proposition 2.2. *Let J be a concentration and suppose that $I_{\min}$ is defined as in (2.6). Then $I_{\min}$ is the smallest mapping $I\colon E \to [0,\infty]$ that satisfies the equivalent inequalities (2.2) and (2.3).*

The minimal rate function has the following representation; see [21, Lemma 3.5].

Proposition 2.3. *Let J be a concentration and suppose that $I_{\min}$ is defined as in (2.6). For all $x \in E$ it holds that*

$$-I_{\min}(x) = \inf_{U\in\mathcal{U}_x} J_U,$$

where $\mathcal{U}_x$ is a base of open neighborhoods of $x \in E$.

The following notion was introduced in [21].

Definition 2.4. A concentration J is said to be *weakly maxitive* if

$$J_C \leq \vee_{i=1}^N J_{O_i} \text{ for all } C \subset E \text{ closed, } O_1, O_2, \ldots, O_N \subset E \text{ open, } N \in \mathbb{N}, \text{ such that } C \subset \cup_{i=1}^N O_i.$$

Notice that every maxitive concentration J is also weakly maxitive.

Definition 2.5. We say that a concentration J is *tight* if for every $n \in \mathbb{N}$ there exists a compact set $K \subset E$ such that $J_{K^c} < -n$.

The following result was provided in [20, Theorem 4.1] (see [20, Remark 4.2]) and [21, Corollary 3.10] under slightly different assumptions. For completeness, we provide a short proof.

Proposition 2.6. *Let J be a concentration and suppose that $I_{\min}$ is defined as in (2.6). If J is tight and weakly maxitive, then $I_{\min}$ satisfies (2.2), (2.3), (2.4), and (2.5).*

Proof. In view of Theorem 2.1 and Proposition 2.2, it is enough to show (2.4). Suppose that $C \subset E$ is closed. Fix $\varepsilon > 0$. Since J is tight, there exists a compact set such that $-J_{K^c} \geq \varepsilon^{-1}$. From Proposition 2.3 and by compactness, there exists $x_1, \ldots, x_N \in K \cap C$ and open sets $U_1, \ldots, U_N$ such that $x_i \in U_i \subset E$ for all $i \in \{1, 2, \ldots, N\}$ and

$$-J_{U_i} \geq (I(x_i) - \varepsilon) \wedge \varepsilon^{-1} \quad \text{for all } i = 1, 2, \ldots, N.$$

Set in addition $U_0 := K^c$. We have $C \subset \cup_{i=0}^N U_i$. Since J is weakly maxitive, we have

$$-J_C \geq \wedge_{i=0}^N(-J_{U_i}) \geq \wedge_{i=0}^N(I(x_i) - \varepsilon) \wedge \varepsilon^{-1} \geq \left(\inf_{x\in C} I(x) - \varepsilon\right) \wedge \varepsilon^{-1}.$$

Letting $\varepsilon \downarrow 0$, we obtain the result. $\square$

3. Integral representation of maxitive monetary risk measures

A *monetary risk measure*[9] is a function $\phi\colon B(E) \to [-\infty, \infty]$ satisfying the following:

- Normalization: $\phi(0) = 0$.
- Monotonicity: $\phi(f) \leq \phi(g)$ whenever $f \leq g$.
- Translation invariance: $\phi(f + c) = \phi(f) + c$ for all $c \in \mathbb{R}$.

We say that a monetary risk measure ϕ is *maxitive* if it satisfies

$$\phi(f \vee g) \leq \phi(f) \vee \phi(g) \quad \text{for all } f, g \in B(E).$$

From (b2)–(b4) in Section 2, the convex integral (2.1) is a monetary risk measure that is maxitive if the corresponding concentration is maxitive. In the following, we focus on the converse direction and analyze when a monetary risk measure can be represented as a convex integral. Denote by $\bar{B}(E)$ the set of all $f \in B(E)$ such that f is bounded from above. The following result was obtained in a slightly different setting in [5, Corollary 6]. We provide a short proof in the present setting.

Theorem 3.1. *Let $\phi\colon B(E) \to [-\infty, \infty]$ be a maxitive monetary risk measure and let $J_A = \phi(-\infty 1_{A^c})$ for all $A \in \mathcal{B}(A)$. Then J is a maxitive concentration, and*

$$\phi(f) = \phi_J(f)$$

for all $f \in \bar{B}(E)$.

Proof. For every function $f\colon E \to \mathbb{R} \cup \{-\infty\}$ that is bounded from above and every set $A \subset E$, we define

$$\bar{\phi}(f) := \inf_{g \in B(E)\colon f \leq g} \phi(g) \quad \text{and} \quad \bar{J}_A := \bar{\phi}(-\infty 1_{A^c}).$$

Inspection shows that $\bar{\phi}$ is finitely maxitive and additively homogeneous in the sense of [5]. By Corollary 6 in [5], we have

$$\bar{\phi}(f) = \sup_{c \in \mathbb{R}} \{c + \bar{J}_{\{f \geq c\}}\}$$

for all f that are bounded from above. In particular, for $f \in B(E)$, we have

$$\phi(f) = \bar{\phi}(f) = \sup_{c \in \mathbb{R}} \{c + \bar{J}_{\{f \geq c\}}\} = \sup_{c \in \mathbb{R}} \{c + J_{\{f \geq c\}}\} = \phi_J(f).$$

This completes the proof. $\square$

By relaxing the maxitivity condition, we may still represent a monetary risk measure on continuous functions. The following notion was introduced in [21].

Definition 3.2. A monetary risk measure $\phi\colon B(E) \to [-\infty, \infty]$ is said to be weakly maxitive if

$$\phi(f) \leq \vee_{i=1}^{N} \phi(g_i) \text{ for all } f \in U(E),\ g_1, g_2, \dots, g_N \in L(E),\ N \in \mathbb{N}, \text{ such that } f \leq \vee_{i=1}^{N} g_i.$$

[9] Here we use the terminology of [15] up to a sign change.

Define $C(E) = U(E) \cap L(E)$ and $\bar{C}(E) = C(E) \cap \bar{B}(E)$. The following result was shown in [21, Theorem 4.2].[10]

Theorem 3.3. *Let $\phi\colon B(E) \to [-\infty, \infty]$ be a weakly maxitive monetary risk measure and let $J_A = \phi(-\infty 1_{A^c})$ for all $A \in \mathcal{B}(A)$. Then J is a weakly maxitive concentration, and*

$$\phi(f) = \phi_J(f)$$

for all $f \in \bar{C}(E)$.

Given a function $\phi\colon B(E) \to [-\infty, \infty]$, we define the following sets:

$$B_\phi(E) := \{f \in B(E) \colon \text{there exists } t > 1 \text{ such that } \phi(tf) < \infty\}.$$

$$C_\phi(E) := \{f \in C(E) \colon \text{there exists } t > 1 \text{ such that } \phi(tf) < \infty\}.$$

Lemma 3.4. *Let $\phi\colon B(E) \to [-\infty, \infty]$ be a monetary risk measure. If $f \in B_\phi(E)$, then*

$$\lim_{m\to\infty} \phi\left(f 1_{\{f \geq m\}} - \infty 1_{\{f < m\}}\right) = \lim_{m\to\infty} \phi\left(f 1_{\{f > m\}} - \infty 1_{\{f \leq m\}}\right) = -\infty.$$

Proof. Suppose that $f \in B_\phi(E)$, and take $t > 1$ such that $\phi(tf) < \infty$. Fix $m \in \mathbb{N}$ and define $g = \exp(f - m)$. By translation invariance and monotonicity, we have

$$\begin{aligned}
-m + \phi\left(f 1_{\{f \geq m\}} - \infty 1_{\{f < m\}}\right) &= \phi\left(-m + f 1_{\{f \geq m\}} - \infty 1_{\{f < m\}}\right) \\
&= \phi\left(\log(g) 1_{\{g \geq 1\}} - \infty 1_{\{g < 1\}}\right) \\
&\leq \phi\left(\log(g^t)\right) \\
&= \phi\left(t(f - m)\right) \\
&= -mt + \phi\left(tf\right).
\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}
\lim_{m\to\infty} \phi\left(f 1_{\{f > m\}} - \infty 1_{\{f \leq m\}}\right) &\leq \lim_{m\to\infty} \phi\left(f 1_{\{f \geq m\}} - \infty 1_{\{f < m\}}\right) \\
&\leq \lim_{m\to\infty} \Big(m(1 - t) + \phi\left(tf\right)\Big) = -\infty,
\end{aligned}$$

where the latter limit is $-\infty$ since $t > 1$ and $\phi\left(tf\right) < -\infty$. This completes the proof. $\square$

We next extend Theorems 3.1 and 3.3 to unbounded functions as follows.

Theorem 3.5. *Let $\phi\colon B(E) \to [-\infty, \infty]$ be a monetary risk measure and let the concentration $J_A = \phi(-\infty 1_{A^c})$ for all $A \in \mathcal{B}(E)$. Then*

(1) *if ϕ is maxitive, then $\phi(f) = \phi_J(f)$ for all $f \in B_\phi(E)$;*
(2) *if ϕ is weakly maxitive, then $\phi(f) = \phi_J(f)$ for all $f \in C_\phi(E)$.*

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[10] To apply Theorem 4.2 in [21] here, we consider the trivial preorder (i.e., $x \leq y$ whenever $x = y$).

Proof. We prove (2). Suppose that ϕ is weakly maxitive. Fix $f \in C_\phi(E)$ and $n \in \mathbb{N}$. We have

$$f \leq (f \wedge n) \vee \left(f 1_{\{f>n\}} - \infty 1_{\{f \leq n\}}\right).$$

Since f is upper semicontinuous, and the functions in the maximum on the right-hand side are lower semicontinuous, it follows from the weak maxitivity of ϕ that

$$\begin{aligned} \phi(f) &\leq \phi\left(f \wedge n\right) \vee \phi\left(f 1_{\{f>n\}} - \infty 1_{\{f \leq n\}}\right) \\ &= \phi_J\left(f \wedge n\right) \vee \phi\left(f 1_{\{f>n\}} - \infty 1_{\{f \leq n\}}\right), \end{aligned} \tag{3.1}$$

where we have applied $\phi\left(f \wedge n\right) = \phi_J\left(f \wedge n\right)$ by Theorem 3.1. On the other hand,

$$\phi(f) \geq \phi(f \wedge n) = \phi_J(f \wedge n). \tag{3.2}$$

Then, (3.1) and (3.2) yield

$$\phi_J(f \wedge n) \leq \phi(f) \leq \phi_J\left(f \wedge n\right) \vee \phi\left(f 1_{\{f>n\}} - \infty 1_{\{f \leq n\}}\right).$$

We have $\lim_{n\to\infty} \phi\left(f 1_{\{f>n\}} - \infty 1_{\{f \leq n\}}\right) = -\infty$ by Lemma 3.4. By letting $n \to \infty$, we get $\phi(f) = \phi_J(f)$. □

4. Main result

Throughout this section we consider two monetary risk measures $\underline{\phi}, \overline{\phi} \colon B(E) \to [-\infty, \infty]$ that satisfy the following.

Assumption 4.1.

(1) For every $f \in B(E)$, $\underline{\phi}(f) \leq \overline{\phi}(f)$.
(2) $\overline{\phi}$ is weakly maxitive.
(3) $\underline{\phi}(f) \leq \underline{\phi}(g_1) \vee \left(\vee_{i=2}^N \overline{\phi}(g_i)\right)$ for all $f \in U(E)$, and $g_1, g_2, \ldots, g_N \in L(E)$, $N \in \mathbb{N}$, such that $f \leq \vee_{i=1}^N g_i$.

Remark 4.2. Assumption 4.1 covers the case of a single (weakly) maxitive monetary risk measure ϕ by our taking $\underline{\phi} := \overline{\phi} := \phi$. In that case, (1)–(3) are automatically satisfied. In the application in the next section, we deal with lower/upper large deviations bounds, which is the reason why we consider a pair of monetary risk measures rather than a single monetary risk measure.

We consider the concentrations $\underline{J}, \overline{J} \colon \mathcal{B}(E) \to [-\infty, 0]$ given by $\underline{J}_A = \underline{\phi}(-\infty 1_{A^c})$ and $\overline{J}_A = \overline{\phi}(-\infty 1_{A^c})$. In addition, we denote by $\underline{I}$ and $\overline{I}$ the respective minimal rate functions defined as in (2.6). As in [20], we introduce the LDP and the LP for monetary risk measures.

Definition 4.3. Suppose that $I \colon E \to [0, \infty]$ is a rate function.

- We say that the pair $\underline{\phi}, \overline{\phi}$ satisfies the LDP with rate function $I(\cdot)$ if

$$-\inf_{x \in \mathrm{int}(A)} I(x) \leq \underline{J}_A \leq \overline{J}_A \leq -\inf_{x \in \mathrm{cl}(A)} I(x) \quad \text{for all } A \in \mathcal{B}(E).$$

- We say that the pair $\underline{\phi}, \overline{\phi}$ satisfies the LP with rate function $I(\cdot)$ if

$$\underline{\phi}(f) = \overline{\phi}(f) = \sup_{x\in E}\{f(x) - I(x)\} \quad \text{for all } f \in C_{\overline{\phi}}(E).$$

Remark 4.4. In [20, Proposition 5.2] the equivalence between the LDP and the LP is proven under the hypothesis that E is normal (the normality is needed only to prove that the LP implies the LDP). Notice that in [20], the LP is defined on $C_b(E)$. Here we consider the larger space $C_{\overline{\phi}}(E)$. That the LDP implies the LP as defined above is a simple consequence of the duality bounds proven in [21] (see Theorem 2.1). We give the argument in the Appendix.

In the following, let $\mathcal{H}$ be a distinguished nonempty set of continuous real-valued functions on E. We define the *conjugate* $\overline{\phi}^*_{\mathcal{H}} : E \to [-\infty, \infty]$ of $\overline{\phi}$ with respect to $\mathcal{H}$ as

$$\overline{\phi}^*_{\mathcal{H}}(x) := \sup_{f\in\mathcal{H}}\{f(x) - \overline{\phi}(f)\}.$$

Definition 4.5. We say that $x \in E$ is an $\mathcal{H}$-*exposed point* of $\overline{\phi}^*_{\mathcal{H}}$ if there exists $f \in \mathcal{H}$ such that

$$f(y) - \overline{\phi}^*_{\mathcal{H}}(y) < f(x) - \overline{\phi}^*_{\mathcal{H}}(x) \quad \text{for all } y \neq x.$$

In that case, we say that f is an *exposing function* for x. We denote by $\mathscr{E}$ the set of all $\mathcal{H}$-exposed points $x \in E$ of $\overline{\phi}^*_{\mathcal{H}}$ that admit an exposing function $f \in \mathcal{H}$ such that

$$\overline{\phi}(f) = \underline{\phi}(f) \quad \text{and} \quad f \in B_{\overline{\phi}}(E). \tag{4.1}$$

Remark 4.6. If x is an $\mathcal{H}$-exposed point, then $\overline{\phi}^*_{\mathcal{H}}(x) < \infty$ and $f(x) > -\infty$.

We now present the main result of this section.

Theorem 4.7. *Suppose that $\overline{J}$ is tight. Then*

(i) *for all $x \in \mathscr{E}$, $\underline{I}(x) = \overline{I}(x) = \overline{\phi}^*_{\mathcal{H}}(x)$;*
(ii) *for every closed set $C \subset E$, we have the upper bound*

$$\overline{J}_C \leq -\inf_{y\in C} \overline{\phi}^*_{\mathcal{H}}(x);$$

(iii) *for every open set $O \subset E$, we have the lower bound*

$$-\inf_{y\in O\cap\mathscr{E}} \overline{\phi}^*_{\mathcal{H}}(x) \leq \underline{J}_O;$$

(iv) *if, moreover,*

$$\inf_{x\in O} \overline{\phi}^*_{\mathcal{H}}(x) = \inf_{x\in O\cap\mathscr{E}} \overline{\phi}^*_{\mathcal{H}}(x) \quad \textit{for every } O \subset E \textit{ open}, \tag{4.2}$$

then the pair $\underline{\phi}, \overline{\phi}$ verifies the LDP and the LP with rate function $\overline{\phi}^*_{\mathcal{H}}$.

Remark 4.8. Given a sequence of E-valued random variables $(X_n)_{n\in\mathbb{N}}$ defined in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we define the lower and upper asymptotic entropies by

$$\underline{\psi}(f) := \liminf_{n\to\infty} \frac{1}{n} \log \mathbb{E}_n[e^{nf(X_n)}], \quad \overline{\psi}(f) := \limsup_{n\to\infty} \frac{1}{n} \log \mathbb{E}_n[e^{nf(X_n)}].$$

We prove in Section 5 (see also [21]) that $\underline{\psi}$ and $\overline{\psi}$ satisfy Assumption 4.1. In addition, we also have that the respective concentrations $\underline{J}$ and $\overline{J}$ are given by

$$\underline{J}_A = \liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}(X_n \in A), \quad \overline{J}_A = \limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}(X_n \in A).$$

Moreover, the tightness of $\overline{J}$ is exactly the exponential tightness of the sequence $(X_n)_{n\in\mathbb{N}}$. Then we have that Theorem 1.1 is a direct consequence of Theorem 4.7.

To prove Theorem 4.7, we need some preliminary results.

Lemma 4.9. *$\overline{\phi}^*_{\mathcal{H}}(x) \le \overline{I}(x) \le \underline{I}(x)$ for all $x \in E$.*

Proof. Given $x \in E$, we know from their respective definitions that $\overline{I}(x) \le \underline{I}(x)$. Fix now $f \in \mathcal{H}$ and $n \in \mathbb{N}$. Since $\overline{\phi}$ is weakly maxitive and $f \wedge n \in \bar{C}(E)$, by Theorem 3.3 we have

$$\phi_{\overline{J}}(f \wedge n) = \overline{\phi}(f \wedge n).$$

By monotonicity, we have

$$\phi_{\overline{J}}(f \wedge n) = \overline{\phi}(f \wedge n) \le \overline{\phi}(f).$$

From (b5), letting $n \to \infty$ results in

$$\phi_{\overline{J}}(f) \le \overline{\phi}(f).$$

Thus, we have

$$f(x) - \overline{\phi}(f) \le f(x) - \phi_{\overline{J}}(f) \le \overline{I}(x).$$

Since $f \in \mathcal{H}$ was arbitrary, it follows that

$$\overline{\phi}^*_{\mathcal{H}}(x) = \sup_{f\in\mathcal{H}} \{f(x) - \overline{\phi}(f)\} \le \overline{I}(x).$$

The proof is complete. $\square$

Lemma 4.10. *Let $K \subset E$ be compact and let $x \in E$ be an $\mathcal{H}$-exposed point of $\overline{\phi}^*_{\mathcal{H}}$ with exposing function $f \in \mathcal{H}$. Then for every open set $U \subset E$ such that $x \in U$, there exists an open set $W \subset E$ such that*

(1) $K \cap U^c \subset W$,
(2) $\sup_{y\in \mathrm{cl}(W)} \left\{ f(y) - \overline{\phi}^*_{\mathcal{H}}(y) \right\} < f(x) - \overline{\phi}^*_{\mathcal{H}}(x)$.

Proof. For each $\varepsilon > 0$, define

$$V_\varepsilon := \left\{ y \in E \colon f(y) - \overline{\phi}^*_{\mathcal{H}}(y) + \varepsilon < f(x) - \overline{\phi}^*_{\mathcal{H}}(x) \right\}.$$

Since the mapping $y \mapsto f(y) - \overline{\phi}^*_{\mathcal{H}}(y)$ is upper semicontinuous, we have that V_ε is open. We claim that there exists $\varepsilon > 0$ such that $K \cap U^c \subset V_\varepsilon$. Indeed, by contradiction, assume that for every $\varepsilon > 0$ we can pick up $y_\varepsilon \in K \cap U^c$ such that

$$f(y_\varepsilon) - \overline{\phi}^*_{\mathcal{H}}(y_\varepsilon) + \varepsilon \geq f(x) - \overline{\phi}^*_{\mathcal{H}}(x).$$

Then $(y_\varepsilon)_{\varepsilon>0}$ is a net in the compact set K.[11] We can take a subnet (y_{ε_α}) such that $y_{\varepsilon_\alpha} \to y \in K \cap U^c$. Taking the limit superior on α, and using the property that $y \mapsto f(y) - \overline{\phi}^*_{\mathcal{H}}(y)$ is upper semicontinuous, we get

$$f(y) - \overline{\phi}^*_{\mathcal{H}}(y) \geq f(x) - \overline{\phi}^*_{\mathcal{H}}(x).$$

Besides, we have $y \neq x$ as $y \in U^c$ and $x \in U$. This contradicts x being an $\mathcal{H}$-exposed point. We have that $K \cap U^c$ is a compact set contained in the open set V_ε. Since E is regular, we can find an open set $W \subset E$ such that $K \cap U^c \subset \mathrm{cl}(W) \subset V_\varepsilon$. Finally, the set W meets the required conditions. □

Proposition 4.11. *Suppose that $x \in \mathscr{E}$ and that $f \in \mathcal{H}$ is an exposing function for x satisfying (4.1). If $\overline{J}$ is tight, then for every open set $U \subset E$ with $x \in U$ it holds that*

$$\overline{\phi}(f1_U - \infty 1_{U^c}) = \overline{\phi}(f) = \underline{\phi}(f) = \underline{\phi}(f1_U - \infty 1_{U^c}).$$

Proof. Let $f \in \mathcal{H}$ be an exposing function for x satisfying (4.1). Fix $n \in \mathbb{N}$. Since $\overline{J}$ is tight, there exists a compact set $K \subset E$ such that

$$\overline{J}_{K^c} < -2n. \tag{4.3}$$

From Lemma 4.10, we can find an open set $W \subset E$ such that

$$K \cap U^c \subset W, \quad \sup_{y \in \mathrm{cl}(W)} \left\{ f(y) - \overline{\phi}^*_{\mathcal{H}}(y) \right\} < f(x) - \overline{\phi}^*_{\mathcal{H}}(x). \tag{4.4}$$

Since

$$\begin{aligned} E &= K \cup K^c \\ &\subset (K \cap U) \cup (K \cap U^c) \cup \Big(K^c \cap \{f < n+1\} \Big) \cup \Big(K^c \cap \{f > n\} \Big) \\ &\subset U \cup W \cup \Big(K^c \cap \{f < n+1\} \Big) \cup \{f > n\}, \end{aligned}$$

from Assumption 4.1 we have

$$\begin{aligned} &\overline{\phi}(f) = \underline{\phi}(f) \\ &\quad \leq \underline{\phi}\left(f1_U - \infty 1_{U^c}\right) \vee \overline{\phi}\left(f1_W - \infty 1_{W^c}\right) \vee \overline{\phi}\left((n+1)1_{K^c} - \infty 1_K\right) \vee \overline{\phi}\left(f1_{\{f>n\}} - \infty 1_{\{f \leq n\}}\right). \end{aligned} \tag{4.5}$$

In addition, by the definition of $\overline{\phi}^*_{\mathcal{H}}$, we have

$$\overline{\phi}(f) \geq f(x) - \overline{\phi}^*_{\mathcal{H}}(x). \tag{4.6}$$

On the other hand, $\overline{J}$ is weakly maxitive and tight. From Theorem 3.5 and Proposition 2.6, we have

[11] Here $\{\varepsilon \colon \varepsilon > 0\}$ is regarded as a downward-directed set.

$$\begin{aligned}\overline{\phi}\left(f1_W - \infty 1_{W^c}\right) &\le \overline{\phi}\left(f1_{\mathrm{cl}(W)} - \infty 1_{\mathrm{cl}(W)^c}\right)\\ &= \phi_{\overline{J}}\left(f1_{\mathrm{cl}(W)} - \infty 1_{\mathrm{cl}(W)^c}\right)\\ &\le \sup_{y\in\mathrm{cl}(W)}\{f(y) - \overline{I}(y)\}\\ &\le \sup_{y\in\mathrm{cl}(W)}\{f(y) - \overline{\phi}^*_{\mathcal{H}}(y)\}\\ &< f(x) - \overline{\phi}^*_{\mathcal{H}}(x),\end{aligned}$$

where we have used $\overline{\phi}^*_{\mathcal{H}}(y) \le \overline{I}(y)$ from Lemma 4.9 in the second inequality and (4.4) in the third inequality. The last inequality is strict. Then in view of (4.6), we can drop the second member of the maximum in (4.5), obtaining

$$\overline{\phi}(f) \le \underline{\phi}\left(f1_U - \infty 1_{U^c}\right) \vee \overline{\phi}\left((n+1)1_{K^c} - \infty 1_K\right) \vee \overline{\phi}\left(f1_{\{f>n\}} - \infty 1_{\{f\le n\}}\right). \tag{4.7}$$

By monotonicity and translation invariance, we get

$$\begin{aligned}\overline{\phi}\left((n+1)1_{K^c} - \infty 1_K\right) &= \overline{\phi}\left(-\infty 1_K\right) + n + 1\\ &= \overline{J}_{K^c} + n + 1\\ &\le -2n + n + 1 = -n + 1,\end{aligned}$$

where we have used (4.3) in the last inequality. Therefore,

$$\lim_{n\to\infty}\overline{\phi}\left((n+1)1_{K^c} - \infty 1_K\right) = -\infty. \tag{4.8}$$

On the other hand, $f \in B_{\overline{\phi}}(E)$ by (4.1). Applying Lemma 3.4, we have

$$\lim_{n\to\infty}\overline{\phi}\left(f1_{\{f>n\}} - \infty 1_{\{f\le n\}}\right) = -\infty. \tag{4.9}$$

Consequently, letting $n \to \infty$ in (4.7) results in

$$\overline{\phi}(f) \le \underline{\phi}\left(f1_U - \infty 1_{U^c}\right).$$

Finally, by monotonicity we have

$$\underline{\phi}\left(f1_U - \infty 1_{U^c}\right) \le \overline{\phi}\left(f1_U - \infty 1_{U^c}\right) \le \overline{\phi}(f) \le \underline{\phi}\left(f1_U - \infty 1_{U^c}\right),$$

and the desired equalities follow. □

We now turn to the proof of Theorem 4.7.

Proof. Let $x \in \mathscr{E}$, and take an exposing function $f \in \mathcal{H}$ for x satisfying (4.1). Fix an open neighborhood U of x and $\varepsilon > 0$. Since f is upper semicontinuous, we can find an open neighborhood $V \subset U$ of x such that

$$f(y) < f(x) + \varepsilon \quad \text{for all } y \in V. \tag{4.10}$$

From Proposition 4.11,

$$\begin{aligned}\overline{\phi}(f) &= \underline{\phi}(f1_V - \infty 1_{V^c})\\ &\leq \underline{\phi}\big((f(x)+\varepsilon)1_V - \infty 1_{V^c}\big) = \underline{J}_V + f(x) + \varepsilon \leq \underline{J}_U + f(x) + \varepsilon.\end{aligned} \tag{4.11}$$

On the other hand, from the definition of $\overline{\phi}^*_{\mathcal{H}}$ we have

$$\overline{\phi}(f) \geq f(x) - \overline{\phi}^*_{\mathcal{H}}(x). \tag{4.12}$$

Combining (4.11) and (4.12), we get

$$-\overline{\phi}^*_{\mathcal{H}}(x) \leq \underline{J}_U + \varepsilon.$$

Letting $\varepsilon \downarrow 0$, we obtain

$$-\overline{\phi}^*_{\mathcal{H}}(x) \leq \underline{J}_U.$$

Since U was arbitrary, it follows from Proposition 2.3 that

$$\overline{\phi}^*_{\mathcal{H}}(x) \geq \underline{I}(x).$$

Finally, by Lemma 4.9 we have $\overline{\phi}^*_{\mathcal{H}}(x) \leq \overline{I}(x) \leq \underline{I}(x)$, obtaining (i).

Suppose that $C \subset E$ is closed. Since $\overline{J}$ is tight and weakly maxitive, by Proposition 2.6 we have

$$\overline{J}_C \leq -\inf_{x\in C} \overline{I}(x) \leq -\inf_{x\in C} \overline{\phi}^*_{\mathcal{H}}(x),$$

where we have used $\overline{\phi}^*_{\mathcal{H}}(x) \leq \overline{I}(x)$. This proves (ii).

Suppose now that $O \subset E$ is open. It follows from Proposition 2.2 that

$$\begin{aligned}\underline{J}_O &\geq -\inf_{x\in O} \underline{I}(x)\\ &\geq -\inf_{x\in O\cap\mathscr{E}} \underline{I}(x)\\ &= -\inf_{x\in O\cap\mathscr{E}} \overline{\phi}^*_{\mathcal{H}}(x),\end{aligned}$$

where we have used $\underline{I}(x) = \overline{\phi}^*_{\mathcal{H}}(x)$ for all $x \in \mathscr{E}$ by (i). Then (iii) follows.

Finally, we obtain that the pair $\underline{\phi}, \overline{\phi}$ satisfies the LDP with rate function $\overline{\phi}^*_{\mathcal{H}}$ as a consequence of (ii) and (iii) taking into account (4.2). In turn, from Proposition A.1 the pair $\underline{\phi}, \overline{\phi}$ also satisfies the LP with rate function $\overline{\phi}^*_{\mathcal{H}}$. □

Example 4.12.

(1) **Topological vector spaces**: Every topological vector space is regular. Then Theorem 4.7 applies to (Hausdorff) topological vector spaces. In that case, we can consider $\mathcal{H} := E^*$, the set of all linear continuous real-valued functions on E. Define $\Lambda := \overline{\phi}|_{E^*}$. Since $\overline{\phi}$ is weakly maxitive because of Assumption 4.1, we have that $\overline{\phi}$ is maxitive on the set $C(E)$. It follows from [20, Proposition 2.1] that $\overline{\phi}$ is convex on $C(E)$ and, in particular, Λ is a convex function. Then we have that $\Lambda^* := \phi^*_{E^*}$ is the convex conjugate of Λ.

(2) **Finite dimension**: Suppose now that $E = \mathbb{R}^d$ and $\mathcal{H} = (\mathbb{R}^d)^* = \mathbb{R}^d$. Consider the following conditions:
   (a) $\Lambda(y) = \underline{\phi}(y) = \overline{\phi}(y)$ for $y \in \mathbb{R}^d$.
   (b) 0 belongs to the topological interior of $\{y \in \mathbb{R}^d \colon \overline{\phi}(y) < \infty\}$.

(c) Λ is lower semicontinuous.
(d) Λ is *essentially smooth* in the sense of [12, Definition 2.3.5].
Under conditions (a)–(c) above, the condition (4.2) in Theorem 4.7 holds. This is proved by following word by word the argumentation in the proof of (c) in [12, Theorem 2.3.6]. We conclude that, in the present situation, we can replace (4.2) in Theorem 4.7 by conditions (a)–(c) above.

(3) **Exposing families of functions**: A family $(f_a)_{a\in E}$ of functions in $C_{\overline{\phi}}$ is said to be an *exposing family* for $\underline{\phi}, \overline{\phi}$ if for every $x \in E$ we have $\overline{\phi}(f_x) = \underline{\phi}(f_x) = 0$ and

$$f_x(y) < \sup_{a\in E} f_a(y) \quad \text{for all } y \neq x.$$

If $\overline{J}$ is tight and $\mathcal{H} = \{f_a : a \in E\}$ for an exposing family $(f_a)_{a\in E}$, then $\mathscr{E} = E$,

$$\overline{I}(x) = \underline{I}(x) = \overline{\phi}^*_{\mathcal{H}}(x) = \sup_{a\in E} f_a(x) \quad \text{for all } x \in E,$$

and the pair $\underline{\phi}, \overline{\phi}$ satisfies the LDP and the LP with rate function $\overline{\phi}^*_{\mathcal{H}}$. This can be proven as follows. First, since $\overline{\phi}(f_a) = 0$ for all $a \in E$, it follows that

$$\overline{\phi}^*_{\mathcal{H}}(x) = \sup_{a\in E} f_a(x) \quad \text{for all } x \in E.$$

Then, given $x \in E$, for each $y \neq x$ we have

$$\overline{\phi}^*_{\mathcal{H}}(y) - f_x(y) = \sup_{a\in E} f_a(y) - f_x(y) > 0.$$

On the other hand,

$$\overline{\phi}^*_{\mathcal{H}}(x) - f_x(x) = \sup_{a\in E} f_a(x) - f_x(x) \leq 0.$$

It follows that x is $\mathcal{H}$-exposed. Since x was arbitrary, we conclude that $\mathscr{E} = E$. Thus, the condition (4.2) is trivially satisfied, and we get the conclusions as a consequence of Theorem 4.7.

5. Large deviation principle for sequences of sublinear expectations

We finally apply Theorem 4.7 to study large deviations for sequences of sublinear expectations. Denote by $B_+(E)$ the set of all Borel measurable functions $f : E \to [0, \infty)$. A function $\mathcal{E} : B_+(E) \to [0, \infty]$ is called a *sublinear* expectation if

(1) $\mathcal{E}(c) = c$ for all constant $c \geq 0$,
(2) $\mathcal{E}(f) \leq \mathcal{E}(g)$ whenever $f \leq g$,
(3) $\mathcal{E}(f + g) \leq \mathcal{E}(f) + \mathcal{E}(g)$,
(4) $\mathcal{E}(af) = a\mathcal{E}(f)$ for all constant $a \geq 0$.

A functional that satisfies properties (1)–(4) is also called an *upper expectation* in robust statistics [18], an u*pper coherent prevision* in the theory of imprecise probabilities [28], or (up to a sign change) a *coherent risk measure* in mathematical finance [1].

Example 5.1. Suppose that X is an E-valued random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Consider a nonempty set $\mathcal{P}$ of probability measures on $\mathcal{F}$. Then the mapping $\mathcal{E} : B_+(E) \to [0, \infty]$ given by

$$\mathcal{E}(f) := \sup_{Q\in\mathcal{P}} \mathbb{E}_Q[f(X)]$$

is a sublinear expectation.

In the following, we consider a sequence $(\mathcal{E}_n)_{n\in\mathbb{N}}$ of sublinear expectations. We define the lower and upper *asymptotic entropies* $\underline{\psi}, \overline{\psi} \colon B(E) \to \overline{\mathbb{R}}$ as

$$\underline{\psi}(f) := \liminf_{n\to\infty} \frac{1}{n} \log \mathcal{E}_n(e^{nf}), \quad \overline{\psi}(f) := \limsup_{n\to\infty} \frac{1}{n} \log \mathcal{E}_n(e^{nf}).$$

Straightforward inspection shows that $\underline{\psi}$ and $\overline{\psi}$ are monetary risk measures. The following lemma is well known in large deviations theory and is often referred to as the *principle of the largest term*; see, for example, [12, Lemma 1.2.15] for (5.1) and [24, Exercise 14.8] for (5.2).

Lemma 5.2. *Suppose that $(a_n^1)_{n\in\mathbb{N}}, (a_n^2)_{n\in\mathbb{N}}, \ldots, (a_n^N)_{n\in\mathbb{N}}$ are $[0,\infty]$-valued sequences. Then*

$$\limsup_{n\to\infty} \frac{1}{n} \log \sum_{i=1}^N a_n^i \le \vee_{i=1}^N \limsup_{n\to\infty} \frac{1}{n} \log a_n^i, \tag{5.1}$$

$$\liminf_{n\to\infty} \frac{1}{n} \log \sum_{i=1}^N a_n^i \le \left(\liminf_{n\to\infty} \frac{1}{n} \log a_n^1 \right) \vee \left(\vee_{i=2}^N \limsup_{n\to\infty} \frac{1}{n} \log a_n^i \right). \tag{5.2}$$

As a consequence, we have the following lemma.

Lemma 5.3. *$\underline{\psi}$ and $\overline{\psi}$ satisfy Assumption 4.1.*

Proof. We prove that

$$\overline{\psi}(f) \le \vee_{i=1}^N \overline{\psi}(g_i)$$

for $f \in U(E)$ and $g_1, g_2, \ldots, g_n \in L(E)$ with $f \le \vee_{i=1}^n g_i$. Indeed, as a consequence of Lemma 5.2, we have

$$\begin{aligned}
\overline{\psi}(f) &\le \limsup_{n\to\infty} \frac{1}{n} \log \mathcal{E}_n \left(e^{n(\vee_{i=1}^N g_i)} \right) \\
&\le \limsup_{n\to\infty} \frac{1}{n} \log \mathcal{E}_n \left(\sum_{i=1}^N e^{n g_i} \right) \\
&\le \limsup_{n\to\infty} \frac{1}{n} \log \sum_{i=1}^N \mathcal{E}_n \left(e^{n g_i} \right) \\
&\le \vee_{i=1}^N \limsup_{n\to\infty} \frac{1}{n} \log \mathcal{E}_n \left(e^{n g_i} \right) \\
&= \vee_{i=1}^N \overline{\psi}(g_i). \quad \square
\end{aligned}$$

Adopting the usual terminology of standard large deviations theory [12], we introduce the following:

- We say that the sequence $(\mathcal{E}_n)_{n\in\mathbb{N}}$ is *exponentially tight* if for every $n \in \mathbb{N}$ there exists a compact set $K \subset E$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \log \mu_n(K^c) < -n.$$

Here $\mu_n: \mathcal{B}(E) \to [0,1]$ is the capacity associated with $\mathcal{E}_n$, which is given by $\mu_n(A) = \mathcal{E}_n(1_A)$.

- We say that $(\mathcal{E}_n)_{n\in\mathbb{N}}$ satisfies the LDP with rate function $I: E \to [0,\infty]$ if

$$
\begin{aligned}
-\inf_{x\in\operatorname{int}(A)} I(x) &\le \liminf_{n\to\infty} \frac{1}{n}\log\mu_n(A) \\
&\le \limsup_{n\to\infty}\frac{1}{n}\log\mu_n(A) \le -\inf_{x\in\operatorname{cl}(A)} I(x)
\end{aligned}
$$

  for all $A \in \mathcal{B}(E)$.

- We say that $(\mathcal{E}_n)_{n\in\mathbb{N}}$ satisfies the LP with rate function $I: E \to [0,\infty]$ if

$$
\lim_{n\to\infty}\frac{1}{n}\log\mathcal{E}_n(e^{nf}) = \sup_{x\in E}\{f(x) - I(x)\} \tag{5.3}
$$

  for all $f \in C_{\overline{\psi}}(E)$.

Notice that the exponential tightness of $(\mathcal{E}_n)_{n\in\mathbb{N}}$ is equivalent to the tightness of $\overline{J}$ in the sense of Definition 2.5.

We turn next to the main result of this section. Let $\mathcal{H}$ be a distinguished nonempty set of continuous real-valued functions on E. Denote by $\mathscr{E}$ the set of all $\mathcal{H}$-exposed points of $\overline{\psi}^*_{\mathcal{H}}$ that admit an exposing function $f \in \mathcal{H}$ such that

$$
\underline{\psi}(f) = \overline{\psi}(f) = \lim_{n\to\infty}\frac{1}{n}\log\mathcal{E}_n(e^{nf}) \quad \text{and} \quad f \in B_{\overline{\psi}}(E).
$$

In view of Lemma 5.3, we have the following result as a direct consequence of Theorem 4.7.

Theorem 5.4. *Suppose that the sequence $(\mathcal{E}_n)_{n\in\mathbb{N}}$ of sublinear expectations is exponentially tight. Then*

(i) *for every closed set $C \subset E$, we have the upper bound*

$$
\limsup_{n\to\infty}\frac{1}{n}\log\mu_n(C) \le -\inf_{y\in C}\overline{\psi}^*_{\mathcal{H}}(x);
$$

(ii) *for every open set $O \subset E$, we have the lower bound*

$$
-\inf_{y\in O\cap\mathscr{E}}\overline{\psi}^*_{\mathcal{H}}(x) \le \liminf_{n\to\infty}\frac{1}{n}\log\mu_n(O);
$$

(iii) *if, moreover,*

$$
\inf_{x\in O}\overline{\psi}^*_{\mathcal{H}}(x) = \inf_{x\in O\cap\mathscr{E}}\overline{\psi}^*_{\mathcal{H}}(x) \quad \textit{for all } O \subset E \textit{ open}, \tag{5.4}
$$

then $(\mathcal{E}_n)_{n\in\mathbb{N}}$ satisfies the LDP and the LP with rate function $\overline{\psi}^*_{\mathcal{H}}$.

In the special case when $\mathcal{H} = E^*$ is the dual space of a topological vector space E, Theorem 5.4 amounts to the well-known Gärtner-Ellis theorem; see [12, Theorem 4.5.20], and [26, Theorem 3.1] for a version for sublinear expectations. We have that Theorem 5.4 is more flexible as it allows for arbitrary choices for $\mathcal{H}$. In the following example we show a case where the LDP and the LP follow from Theorem 5.4 but it is not covered by the Gärtner-Ellis theorem.

Example 5.5. For each $n \in \mathbb{N}$, let X_n be a real-valued random variable with a (centered) Laplace distribution of parameter $1/n$; that is, X_n has density $h_n(x) = \frac{n}{2} e^{-n|x|}$ for all $x \in \mathbb{R}$.[12] Consider the lower and upper asymptotic entropies

$$\underline{\psi}(f) = \liminf_{n\to\infty} \frac{1}{n} \log \mathbb{E}_{\mathbb{P}}[e^{nf(X_n)}], \quad \overline{\psi}(f) = \limsup_{n\to\infty} \frac{1}{n} \log \mathbb{E}_{\mathbb{P}}[e^{nf(X_n)}].$$

For every $m \in \mathbb{N}$, define the compact set $K_m := [-m, m]$. Then $\overline{J}_{K_m^c} = -m$. Hence, $(X_n)_{n\in\mathbb{N}}$ is exponentially tight. Consider $\mathcal{H}_1 = \mathbb{R}^* = \mathbb{R}$ as in the Gärtner-Ellis theorem [12, Theorem 4.5.20]. In that case, for every $y \in \mathcal{H}_1 = \mathbb{R}$,

$$\underline{\psi}(y) = \overline{\psi}(y) = \lim_{n\to\infty} \frac{1}{n} \log \int_{-\infty}^{\infty} \tfrac{n}{2} e^{nyx - n|x|} dx = \begin{cases} 0 & \text{if } |y| < 1, \\ \infty & \text{if } |y| \geq 1. \end{cases}$$

Hence,

$$\overline{\psi}^*_{\mathcal{H}_1}(x) = \sup_{y\in\mathbb{R}} \{yx - \overline{\psi}(y)\} = |x|.$$

The only exposed point of $\overline{\psi}^*_{\mathcal{H}_1}$ is 0. Then for every open set $O \subset \mathbb{R}$ that does not contain the origin, the classical Gärtner-Ellis theorem gives only a trivial lower bound $-\inf_{y\in O\cap\{0\}} \overline{\psi}^*_{\mathcal{H}_1}(y) = -\infty$, and the condition (5.4) is not satisfied.

Now consider the set of continuous functions

$$\mathcal{H}_2 := \{f_a : a \in \mathbb{R}\},$$

where

$$f_a(x) := |a| - 2|x - a|.$$

Direct verification yields $\underline{\psi}(f_a) = \overline{\psi}(f_a) = 0$ for every $a \in \mathbb{R}$. Consequently, we have

$$\overline{\psi}^*_{\mathcal{H}_2}(x) = \sup_{a\in\mathbb{R}} f_a(x) = |x|.$$

This shows that $\mathcal{H}_2$ is an exposing family as in Example 4.12(3). Therefore, the set of all $\mathcal{H}_2$-exposed points is $\mathbb{R}$, and $\overline{\psi}^*_{\mathcal{H}_2}$ trivially verifies the condition (5.4) (Fig. 1).

Then (iii) in Theorem 5.4 gives the LDP

$$-\inf_{y\in\mathrm{int}(A)} |y| \leq \liminf_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}(X_n \in A) \leq \limsup_{n\to\infty} \tfrac{1}{n} \log \mathbb{P}(X_n \in A) \leq -\sup_{y\in\mathrm{cl}(A)} |y|,$$

where the lower bound is not trivial whenever $\mathrm{int}(A) \neq \emptyset$. Moreover, we obtain the LP

$$\underline{\psi}(f) = \overline{\psi}(f) = \lim_{n\to\infty} \frac{1}{n} \log \int_{-\infty}^{\infty} \tfrac{n}{2} e^{n(f(x)-|x|)} dx = \sup_{x\in\mathbb{R}} \{f(x) - |x|\}$$

for all $f \in C_{\overline{\psi}}(E)$. The present example illustrates that the Gärtner-Ellis theorem does not capture the lower bound in the LDP when the rate function has large parts that are not exposed by hyperplanes. In

[12] Equivalently, X_n is the difference of two independent random variables with exponential distribution parameter n.

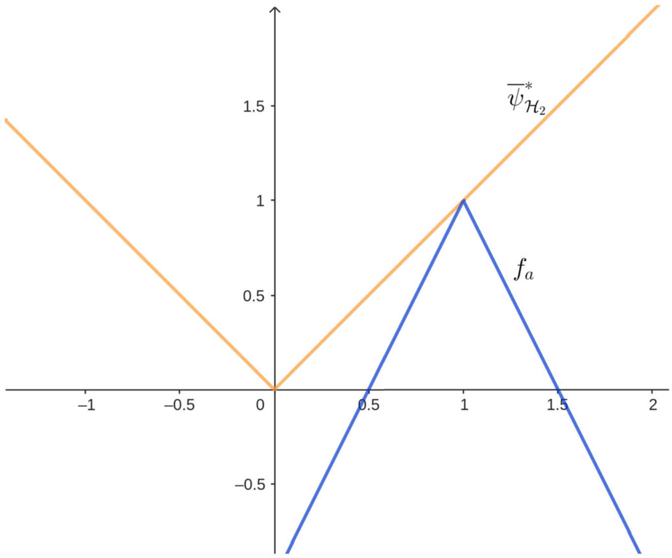

Fig. 1. Every point a is exposed with exposing function f_a. In this case, we can see the exposing function for $a = 1$.

contrast, by considering other types of exposing functions (e.g., inverted V-shaped functions as in this case), we can produce richer classes of exposing points and prove an LDP by means of Theorem 5.4.

Although the Gärtner-Ellis theorem does not capture the lower bound in the LDP, we may try other known methods to prove it. Next we briefly discuss some of these methods. We note that in this simple one-dimensional case the limit representation [12, Theorem 4.1.18] yields the rate function $I(x) = |x|$, but we would still need to prove the existence of an LDP to apply this result. Since $(X_n)_{n\in\mathbb{N}}$ is exponentially tight, we may apply Bryc's theorem [12, Theorem 4.4.2]. However, we need to verify that $\overline{\psi}(f) = \underline{\psi}(f)$ for all $f \in C_b(\mathbb{R})$ to derive the lower LDP bound, which is not straightforward. Alternatively, as proven in [7], one can replace $C_b(\mathbb{R})$ in Bryc's theorem by an algebra $\mathcal{A}$ of continuous functions separating the points of $\mathbb{R}$, or any well-separating class $\mathcal{A}$ (i.e., $\mathcal{A}$ contains the constant functions, is closed under finite infima, and separates points of $\mathbb{R}$). However, this method leads again to rather intricate classes of functions where we need to verify $\overline{\psi}(f) = \underline{\psi}(f)$. In contrast, our method leads to a family of inverted V-shaped functions for which it is directly checked that $\overline{\psi}(f) = \underline{\psi}(f) = 0$.

Remark 5.6. The main result in [26] is a version of the Gärtner-Ellis theorem for sequences of sublinear expectations in a finite-dimensional setting, which is proven by adapting the proof of the standard case. This result is a particular instance of Theorem 5.4 for the special choices $E = \mathbb{R}^d$ and $\mathcal{H} = (\mathbb{R}^d)^* = \mathbb{R}^d$, taking into account (2) in Example 4.12. As illustrated in Example 5.5, this setting does not cover all $\mathbb{R}^d$ cases in which an LDP exists. In addition, Tan and Zong [26] consider sublinear expectations of the particular form $\mathcal{E}_n(f) = \sup_{Q\in\mathcal{P}} \mathbb{E}_Q[f(X_n)]$ for some set $\mathcal{P}$ of probability measures and a sequence of random variables $(X_n)_{n\in\mathbb{N}}$. In particular, this implies that the sublinear $\mathcal{E}_n$ is continuous from below because of the monotone convergence theorem. Such a continuity condition is not needed in the present approach. Also, among other restrictions, it is assumed in [26, Assumption 3.1]) that the limit $\lim_{n\to\infty} \frac{1}{n} \log \mathcal{E}_n(\exp(nf))$ exists for all $f \in E^*$, which was not needed here. Furthermore, in [26] the LP is not derived (5.3).

Appendix A. Pairs of monetary risk measures

As a consequence of the duality bounds provided in [21] (see Theorem 2.1) and in line with [20, Proposition 5.2] we have the following.

Proposition A.1. *Suppose that $\underline{\phi}, \overline{\phi}$ is a pair of monetary risk measures such that $\overline{\phi}$ is weakly maxitive and $\underline{\phi}(f) \leq \overline{\phi}(f)$ for all $f \in B(E)$. If the pair $\underline{\phi}, \overline{\phi}$ satisfies the LDP with rate function $I(\cdot)$, then the pair $\underline{\phi}, \overline{\phi}$ satisfies the LP with rate function $I(\cdot)$.*

Proof. Suppose that the pair $\underline{\phi}, \overline{\phi}$ satisfies the LDP with rate function $I(\cdot)$, and fix $f \in C_{\overline{\phi}}(E)$. Since $\overline{\phi}$ is weakly maxitive, from Theorem 3.5 we have $\phi_{\overline{J}}(f) = \overline{\phi}(f)$. By Theorem 2.1, we have

$$\overline{\phi}(f) = \phi_{\overline{J}}(f) = \sup_{x \in E}\{f(x) - I(x)\}. \tag{A.1}$$

Given $x \in E$ and $\delta > 0$, since f is upper semicontinuous, there exists $U \in \mathcal{U}_x$ such that $\inf_{y \in U} f(y) \geq f(x) - \delta$. By monotonicity and translation invariance

$$\begin{aligned}\underline{\phi}(f) &\geq \underline{\phi}(f 1_U - \infty 1_{U^c}) \geq f(x) - \delta + \underline{J}_U \\ &\geq f(x) - \delta - \inf_{y \in U} I(y) \geq f(x) - \delta - I(x).\end{aligned}$$

Letting $\delta \downarrow 0$ and taking the supremum over all $x \in E$ yields

$$\underline{\phi}(f) \geq \sup_{x \in E}\{f(x) - I(x)\}.$$

This along with (A.1) shows that the pair $\underline{\phi}, \overline{\phi}$ satisfies the LP with rate function $I(\cdot)$. □

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