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Weakly maxitive set functions and their possibility distributions *

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Abstract

The Shilkret integral with respect to a completely maxitive capacity is fully determined by a possibility distribution. In this paper, we introduce a weaker topological form of maxitivity and show that under this assumption the Shilkret integral is still determined by its possibility distribution for functions that are sufficiently regular. Motivated by large deviations theory, we provide a Laplace principle for maxitive integrals and characterize the possibility distribution under certain separation and convexity assumptions. Moreover, we show a maxitive integral representation result for weakly maxitive non-linear expectations. The theoretical results are illustrated by providing large deviations bounds for sequences of capacities, and by deriving a monotone analogue of Cramér's theorem.

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1. Introduction

Non-additive set functions, fuzzy measures and capacities play an important role in the theory of decision making under risk and uncertainty. They appear in possibility theory, idempotent/tropical mathematics and related fields, where maxitive measures and their maxitive integrals are studied; see, e.g., [11,12,19,30,33]. In this article we focus on the *Shilkret integral* introduced in [30]. An extension of this integral to functions taking negative values and its complete characterization was given in [6,7]. On the one hand, if Π is a possibility measure [33], then the Shilkret integral is given by

$$\int_{x}^{S} f \, \mathrm{d}\Pi = \sup_{x} f(x)\pi(x),$$

where π denotes the corresponding possibility distribution. In this case, both the set function Π and the Shilkret integral $\int^{S} f \, d\Pi$ are completely maximize. On the other hand, for the particular set function $\bar{\Pi}_{A} := \limsup_{n \to \infty} \mathbb{P}(X_n \in \mathbb{P})$

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A)^{1/n}, where $(X_n)_{n \in \mathbb{N}}$ is a sequence of random variables, the Laplace principle from the theory of large deviations ensures under reasonable assumptions that

$$\int_{x}^{S} f \,\mathrm{d}\bar{\Pi} = \sup_{x} f(x) e^{-I(x)},$$

where *I* denotes the rate function; cf. Puhalskii [28]. Although the two representations for the Shilkret integral are very similar, in general, the second one is only valid for certain continuous functions. Thus, the set function $\overline{\Pi}$ satisfies only a weaker form of maxitivity, and consequently is not a possibility measure. Nevertheless, $\overline{\pi}(x) = e^{-I(x)}$ can still be viewed as a corresponding possibility distribution. In this article, we characterize the class of set functions which admit a possibility distribution in a weaker topological sense by means of different notions of maxitivity.

In Section 2, we introduce some basics on possibility distributions and motivate our results. Section 3 includes the key bounds on maxitive integrals and their connection to weakly maxitive set functions. In Section 4, we focus on a maxitive integral representation result for weakly maxitive non-linear expectations. The results are then applied in Section 5, where we show that the basic results of large deviations theory are valid for general maxitive integrals; in particular, the equivalence between the monotone large deviation principle and the monotone Laplace principle. In Section 6, we provide conditions ensuring that the rate function is convex and study the corresponding representation. Finally, in Section 7, the theoretical results are illustrated with two examples. On the one hand, we study the asymptotic behavior of a sequence of capacities by providing some large deviations bounds, on the other hand we establish a monotone analogue of Cramér's theorem for the sample mean of i.i.d. sequences. The paper concludes with an appendix, where we provide some separation results in preordered topological groups.

2. Background and motivation

Let *E* be a non-empty set and *A* be a collection of subsets of *E* such that \emptyset , $E \in A$. A set function $\Pi: A \to \mathbb{R}$ is called a *capacity*, if $\Pi_{\emptyset} = 0$, $\Pi_E = 1$, and $\Pi_A \leq \Pi_B$ whenever $A \subset B$.¹ Among the most important examples of capacities, we find probability measures in standard probability theory [18], possibility measures in possibility theory [11], and upper/lower probabilities in the theory of imprecise probabilities [32]. A capacity Π is called a *possibility measure* [33], if there exists a *possibility distribution* $\pi: E \to [0, 1]$ such that

$$\Pi_A = \sup_{x \in A} \pi(x) \quad \text{for all } A \in \mathcal{A}.$$
(2.1)

A capacity Π which admits a possibility distribution is automatically completely maxitive, i.e.,

$$\Pi_A \le \bigvee_{i \in \mathcal{I}} \Pi_{A_i} \quad \text{whenever } A \subset \bigcup_{i \in \mathcal{I}} A_i, \tag{2.2}$$

for all $A \in \mathcal{A}$ and every family $(A_i)_{i \in \mathcal{I}} \subset \mathcal{A}$. Conversely, if \mathcal{A} is closed under arbitrary unions and complements, then any supremum preserving capacity Π admits a possibility distribution which is uniquely determined by $\pi(x) = \Pi_{\bigcap\{A \in \mathcal{A}: x \in A\}}$ for all $x \in E$; cf. [10]. Under additional continuity on the capacity and the assumption that \mathcal{A} is rich enough,² the existence of a possibility distribution is guaranteed if Π is only finitely maxitive, i.e., $\Pi_A \leq \bigvee_{i=1}^n \Pi_{B_i}$ for all $A, B_1, \ldots, B_n \in \mathcal{A}, n \in \mathbb{N}$, with $A \subset \bigcup_{i=1}^n B_i$; cf. [1,24].

The *Shilkret integral* of a function $f: E \to [0, \infty]$ with respect to a capacity Π is defined by

$$\int_{-\infty}^{S} f \, \mathrm{d}\Pi := \sup_{c \in (0,\infty)} c \Pi_{\{f > c\}}.$$
(2.3)

Originally, the Shilkret integral was introduced for supremum preserving capacities, also called maxitive probabilities or idempotent probabilities [30]. However, this definition is valid for general capacities (not necessarily maxitive) provided that f is \mathcal{A} -measurable, i.e., $\{f > c\} \in \mathcal{A}$ for all $c \in \mathbb{R}$. In case that Π admits a possibility distribution π , then the Shilkret integral takes the form

¹ Typically \mathcal{A} is endowed with some algebraic structure. Here, for the sake of generality we do not assume any structure on \mathcal{A} .

² E.g. the collection \mathcal{A} forms a σ -algebra on a separable metric space which contains all Borel sets.

$$\int_{-\infty}^{\infty} f \, \mathrm{d}\Pi = \sup_{x \in E} f(x)\pi(x) \tag{2.4}$$

for all \mathcal{A} -measurable functions $f: E \to [0, \infty]$.

In this article, we analyze to what extent a capacity is still determined by a possibility distribution if the representation (2.4) only holds for a certain class of A-measurable functions, e.g., all continuous, continuous bounded, or continuous increasing functions.³ We will see that this exactly holds when the capacity satisfies a weaker form of maxitivity.⁴ To do so, we will provide bounds for the Shilkret integral with respect to general capacities. For a capacity Π and a function $\pi: E \to [0, 1]$, a key result of this work states that

$$\int_{x \in E}^{S} f \, \mathrm{d}\Pi \ge \sup_{x \in E} f(x)\pi(x) \quad \text{if and only if} \quad \Pi_{O} \ge \sup_{x \in O} \pi(x), \tag{2.5}$$

where the first inequality holds for all (increasing) lower semicontinuous functions f, and the second inequality holds for all (upwards closed) open sets O. Likewise,

$$\int_{x \in E}^{S} f \, \mathrm{d}\Pi \le \sup_{x \in E} f(x)\pi(x) \quad \text{if and only if} \quad \Pi_{C} \le \sup_{x \in C} \pi(x), \tag{2.6}$$

where the first inequality holds for all (increasing) upper semicontinuous functions f, and the second inequality holds for all (upwards closed) closed sets C. Instead of assuming the rather restrictive assumption (2.1), the relation between the capacity and its possibility distribution is relaxed to inequalities which are required only for certain nice topological sets. As a consequence, if the capacity Π satisfies both the upper and lower bound, then the Shilkret integral has the representation (2.4) for all (increasing) continuous functions.

The Shilkret integral is only defined for non-negative functions. In order to deal with real-valued functions, we present and prove our results in terms of the maxitive integral introduced by Cattaneo [7], which is obtained as a transformation of the Shilkret integral. We say that a set function $J: \mathcal{A} \to [-\infty, 0]$ is a concentration if $J_E = 0$, $J_{\emptyset} = -\infty$, and $J_A \leq J_B$ whenever $A \subset B$. In other words, J is a concentration if and only if e^J is a capacity. The maxitive integral of an A-measurable function $f: E \to [-\infty, \infty)$ with respect to the concentration J is defined as

$$\phi_J(f) := \log \int^S e^f \, \mathrm{d}e^J = \sup_{c \in \mathbb{R}} \{c + J_{\{f > c\}}\}.$$
(2.7)

As discussed in [7], the functional ϕ_I shares the properties of a monetary risk measure [13], and satisfies in particular the translation property $\phi_J(f+c) = \phi_J(f) + c$ for all $c \in \mathbb{R}$. In contrast, the Shilkret integral (2.3) fails the translation property unless the capacity Π only assumes the values 0 and 1; cf. [9]. In particular, aside from this degenerate case, the Shilkret integral is neither a coherent prevision, nor a monetary risk measure. By defining the rate function $I: E \to [0, \infty]$ by $I(x) := -\log \pi(x)$, the bound (2.5) takes the form

$$\phi_J(f) \ge \sup_{x \in E} \{f(x) - I(x)\} \quad \text{if and only if} \quad J_O \ge -\inf_{x \in O} I(x), \tag{2.8}$$

for all (increasing) lower semicontinuous functions f, and all (upwards closed) open sets O, and the bound (2.6) translates to

$$\phi_J(f) \le \sup_{x \in E} \{f(x) - I(x)\} \quad \text{if and only if} \quad J_C \le -\inf_{x \in C} I(x), \tag{2.9}$$

for all (increasing) upper semicontinuous functions f, and all (upwards closed) closed sets C.

Similar type of bounds appear in the theory of large deviations, where the capacity has the special form $\Pi_A :=$ $\limsup_{n\to\infty} \mathbb{P}(X_n \in A)^{1/n}$ for a sequence $(X_n)_{n\in\mathbb{N}}$ of random variables with values in a completely regular topological space E. In that case, the corresponding concentration is given by $J_A = \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}(X_n \in A)$ with

³ Here, we assume that *E* is a topological preordered space. ⁴ I.e. $\Pi_A \leq \bigvee_{i=1}^n \Pi_{B_i}$ for (upwards closed) closed sets *A* and (upwards closed) open sets $B_1, \ldots, B_n, n \in \mathbb{N}$, with $A \subset \bigcup_{i=1}^n B_i$. We emphasize that no additional continuity on Π is required.

respective maxitive integral $\phi_J(f) = \lim \sup_{n\to\infty} \frac{1}{n} \log \mathbb{E}_{\mathbb{P}}[\exp(nf(X_n))]$ for all bounded continuous functions f on E. Then the equivalences (2.8) and (2.9) amount to the well-known equivalence between the large deviation principle (LDP) and the Laplace principle (LP); cf. [5,28]. In this article, we will show that the key concepts of large deviations theory can be understood and extended to the framework of weakly maxitive concentrations and their maxitive integrals; e.g., the equivalences (2.8) and (2.9) establish the equivalence between the LDP and the LP for general concentrations. This covers situations that are not captured by the standard setting of large deviations theory. For instance, in the theory of imprecise probability [32], we may be interested in large deviations bounds for upper probabilities of the form $\overline{\mathbb{P}}(A) = \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{P}(A)$ for a set \mathcal{P} of probability measures. Then we consider the concentration $J_A = \limsup_{n\to\infty} \frac{1}{n} \log \overline{\mathbb{P}}(X_n \in A)$, for which we will show in Subsection 7.1 that its maxitive integral is given by

$$\phi_J(f) = \limsup_{n \to \infty} \frac{1}{n} \log \int_0^\infty \overline{\mathbb{P}} \left(\exp(nf(X_n)) > x \right) \mathrm{d}x.$$

Standard large deviations theory provides conditions such that the usual upper bound $\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}(X_n \in C) \leq -\inf_{x\in C} I(x)$ is valid for all closed sets $C \subset E$, and the usual lower bound $\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}(X_n \in O) \geq -\inf_{x\in O} I(x)$ holds for all open sets $O \subset E$. However, one may be interested in finding bounds on certain smaller classes of sets. One of the features of the presented framework is that, by considering a preorder relation \leq on E, we may restrict ourselves to the class of upwards closed sets for which we characterize large deviation bounds. As a result, we obtain a monotone version of Cramér's theorem which provides new large deviations bounds for the sample mean of i.i.d. sequences.

3. Bounds for maxitive integral and weak maxitivity

In this section, we introduce the basic concepts and provide the key bounds for concentrations and the respective maxitive integrals. These bounds ensure a weak form of maxitivity which allows to connect concentrations and their maxitive integrals with rate functions.

3.1. Setting and notation

Let (E, \leq) be a topological preordered space.⁵ Let \mathcal{U} be a base of the topology, and define $\mathcal{U}_x := \{U \in \mathcal{U} : x \in U\}$ for all $x \in E$. Moreover, for $A \subset E$, we define the *upset* and the *downset* as

$$\uparrow A := \{y \in E : x \le y \text{ for some } x \in A\}$$
 and $\downarrow A := \{y \in E : y \le x \text{ for some } x \in A\}.$

We say that $A \subset E$ is *upwards closed* if $A = \uparrow A$, and *downwards closed* if $A = \downarrow A$. Let \mathcal{O}^{\uparrow} denote the collection of all subsets $A \subset E$ which are open and upwards closed, and \mathcal{C}^{\uparrow} be the collection of all subsets $A \subset E$ which are closed and upwards closed. In addition, let \mathcal{C}_c^{\uparrow} be the set of all $C \in \mathcal{C}^{\uparrow}$ which are compactly generated, i.e., $C = \uparrow K$ for some compact $K \subset E$. Similarly, we define the corresponding collections \mathcal{O}^{\downarrow} and \mathcal{C}^{\downarrow} of downwards closed sets, which are open and closed, respectively.

Throughout this section, we work under the following assumption.

Assumption 3.1. For every $x, y \in E$ with $x \leq y$, we assume that

- (A) for every $U_x \in \mathcal{U}_x$ there exists $U_y \in \mathcal{U}_y$ such that $U_y \subset \uparrow \mathcal{U}_x$.
- (B) for every $U_y \in \mathcal{U}_y$ there exists $U_x \in \mathcal{U}_x$ such that $U_x \subset \downarrow U_y$.

In case that the preorder is trivial, i.e., $x \le y$ if and only if x = y, then \mathcal{O}^{\uparrow} and \mathcal{C}^{\uparrow} coincide with the collections of all open and closed subsets of *E*, respectively, and the previous assumption is trivially satisfied.

⁵ Recall that a preorder is a reflexive and transitive binary relation. We do not assume any relations between the topology and the preorder.

Remark 3.2. Let *E* be a Hausdorff topological abelian group. Suppose that E_+ is a subset of *E* such that $E_+ + E_+ \subset E_+$ and $0 \in E_+$. Then the binary relation defined by

 $x \le y$ if and only if $y - x \in E_+$

is a translation invariant preorder on E. Direct verification shows that (E, \leq) satisfies Assumption 3.1.

On a functional level, the sets \mathcal{O}^{\uparrow} and \mathcal{C}^{\uparrow} correspond to the following spaces. We denote by L^{\uparrow} the set of all increasing⁶ lower semicontinuous functions $f: E \to [-\infty, \infty)$, by U^{\uparrow} the set of all increasing upper semicontinuous functions $f: E \to [-\infty, \infty)$, and by U_c^{\uparrow} the set of all $f \in U^{\uparrow}$ such that $\{f \ge c\}$ is compactly generated for all $c \in \mathbb{R}$. We first collect some basic topological properties.

Lemma 3.3. The following assertions hold.

(i) A ∈ O[↑] if and only if A^c ∈ C[↓].
(ii) A ∈ C[↑] if and only if A^c ∈ O[↓].
(iii) If f ∈ L[↑], then {f > c} ∈ O[↑] for all c ∈ ℝ.
(iv) If f ∈ U[↑], then {f ≥ c} ∈ C[↑] for all c ∈ ℝ.
(v) If U ⊂ E is open, then ↑U ∈ O[↑].
(vi) If A is upwards closed, then cl(A) is upwards closed.⁷

Proof. (i) Suppose that $A \in \mathcal{O}^{\uparrow}$. We show that A^c is downwards closed. By contradiction, assume that $A^c \neq \downarrow(A^c)$, that is, there exists $x \in \downarrow(A^c)$ such that $x \notin A^c$ (or equivalently $x \in A$). Since $x \in \downarrow(A^c)$, there exists $y \in A^c$ such that $x \leq y$. Since $x \in A$ and $x \leq y$, it follows that $y \in \uparrow A$. By assumption $\uparrow A = A$, so that $y \in A$. But this is a contradiction to $y \in A^c$.

(ii) follows along the same line of argumentation as (i).

(iii) Fix $f \in L^{\uparrow}$. We prove that $\{f > c\}$ is upwards closed. By contradiction, assume that $x \le y$ for $x \in \{f > c\}$ and $y \notin \{f > c\}$. Since $x \in \{f > c\}$, it follows that f(x) > c. Since $y \notin \{f > c\}$, we have that $f(y) \le c$. Hence f(y) < f(x), in contradiction to $f(x) \le f(y)$.

(iv) follows by similar arguments as in (iii).

(v) Let $y \in \uparrow U$, so that $x \leq y$ for some $x \in U$. Since U is open, there exists $U_x \in \mathcal{U}_x$ such that $U_x \subset U$. By Assumption 3.1, there exists $U_y \in \mathcal{U}_y$ such that $U_y \subset \uparrow U_x \subset \uparrow U$. This shows that $\uparrow U$ is open.

(vi) Suppose that $x \le y$ with $x \in cl(A)$. We have to show that $y \in cl(A)$. Fix $U_y \in U_y$. By Assumption 3.1 there exists $U_x \in U_x$ such that $U_x \subset \downarrow U_y$. Since x is in the closure of A, there exists $\tilde{x} \in U_x \cap A$. Therefore, it follows that $\tilde{x} \in \downarrow U_y$, which shows that there exists $\tilde{y} \in U_y$ with $\tilde{x} \le \tilde{y}$. Since $\tilde{y} \in \uparrow A = A$, we conclude that $U_y \cap A \ne \emptyset$. \Box

3.2. Bounds for maxitive integrals

In accordance with Section 2, we next introduce the key concepts of this article. Let J be a *concentration* on $\mathcal{OC}^{\uparrow} := \mathcal{O}^{\uparrow} \cup \mathcal{C}^{\uparrow}$, i.e., a set function $J : \mathcal{OC}^{\uparrow} \to [-\infty, 0]$ which satisfies $J_{\emptyset} = -\infty$, $J_E = 0$, and $J_A \leq J_B$ whenever $A \subset B$. The respective *maxitive integral* ϕ_J on $LU^{\uparrow} := L^{\uparrow} \cup U^{\uparrow}$ is defined by

$$\phi_J(f) := \begin{cases} \sup_{c \in \mathbb{R}} \{c + J_{\{f > c\}}\}, & \text{if } f \in L^\uparrow, \\ \sup_{c \in \mathbb{R}} \{c + J_{\{f \ge c\}}\}, & \text{otherwise.} \end{cases}$$
(3.1)

As discussed in the previous section, e^J is a capacity and ϕ_J is a transformed version of the Shilkret integral; cf. Cattaneo [7]. Notice that ϕ_J is well-defined due to Lemma 3.3. By considering the extended concentration $\overline{J}_A := \inf\{J_B : B \in \mathcal{OC}^{\uparrow}, A \subset B\}$ for all $A \subset E$, it follows by direct verification that for all $f \in LU^{\uparrow}$,

⁶ A function $f: E \to [-\infty, \infty]$ is called *increasing* if $f(x) \le f(y)$ whenever $x \le y$.

⁷ As usual, cl(A) denotes the topological closure of a subset A.

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$$\phi_J(f) = \sup_{c \in \mathbb{R}} \{c + \bar{J}_{\{f \ge c\}}\} = \sup_{c \in \mathbb{R}} \{c + \bar{J}_{\{f > c\}}\}$$

This shows that definition (3.1) is consistent with that in (2.7).⁸ The functional (3.1) shares the properties of a nonlinear expectation, i.e., it is constant preserving $\phi_J(c) = c$ for all $c \in \mathbb{R}$, and monotone $\phi_J(f) \le \phi_J(g)$ whenever $f \le g$. Also, the maxitive integral has the translation property $\phi_J(f + c) = \phi_J(f) + c$ for all $c \in \mathbb{R}$.

Next, we show that the concentration can be recovered from the maxitive integral by the evaluation at indicator functions. We always make the convention that $-\infty \cdot 0 = 0$, so that the indicator function $-\infty 1_{A^c}$ assumes the value $-\infty$ on A^c and zero on A.

Proposition 3.4. Let *J* be a concentration. Then, for every $A \in OC^{\uparrow}$,

$$J_A = \phi_J(-\infty 1_{A^c}) = \inf_{r < 0} \phi_J(r 1_{A^c}).$$

Proof. Suppose, for instance, that $A \in \mathcal{O}^{\uparrow}$. Then, $A^c \in \mathcal{C}^{\downarrow}$ due to Lemma 3.3, and therefore $-\infty \mathbf{1}_{A^c} \in L^{\uparrow}$. We obtain

$$\phi_J(-\infty 1_{A^c}) = \sup_{c \in \mathbb{R}} \{c + J_{\{-\infty 1_{A^c} > c\}}\} = \sup_{0 \le c} \{c + J_{\emptyset}\} \vee \sup_{c < 0} \{c + J_A\} = J_A.$$

Now, let r < 0, so that $r \mathbf{1}_{A^c} \in L^{\uparrow}$. Then,

$$\begin{split} \phi(r1_{A^c}) &= \sup_{c \in \mathbb{R}} \{c + J_{\{r1_{A^c} > c\}}\} \\ &= \sup_{c < r} \{c + J_{\{r1_{A^c} > c\}}\} \lor \sup_{r \le c < 0} \{c + J_{\{r1_{A^c} > c\}}\} \lor \sup_{0 \le c} \{c + J_{\{r1_{A^c} > c\}}\} \\ &= r \lor J_A \lor (-\infty) = r \lor J_A. \end{split}$$

Hence, by letting $r \to -\infty$, we conclude $\inf_{a} \phi_J(r \mathbf{1}_{A^c}) = J_A$. \Box

In accordance with Section 2, we say that a concentration J admits a rate function if there exists a function $I: E \to [0, \infty]$ such that $J_A = -\inf_{x \in A} I(x)$ for all $A \in \mathcal{OC}^{\uparrow}$. In that case, the concentration is completely maxitive in the sense that $J_A \leq \bigvee_{i \in \mathcal{I}} J_{A_i}$ for every family $(A_i)_{i \in \mathcal{I}} \subset \mathcal{OC}^{\uparrow}$ and $A \in \mathcal{OC}^{\uparrow}$ with $A \subset \bigcup_{i \in \mathcal{I}} A_i$, and the maxitive integral $\phi_J(f)$ admits the representation $\sup_{x \in E} \{f(x) - I(x)\}$ for all $f \in LU^{\uparrow}$. In the following, we relax the relation between concentrations and rate functions. As a first main result, we obtain that the maxitive integral satisfies the following upper and lower bounds.

Theorem 3.5. Let J be a concentration and $I: E \to [0, \infty]$ be a function. Then, the following equivalences hold. *First*,

$$-\inf_{x\in Q} I(x) \le J_O \quad \text{for all } O \in \mathcal{O}^{\uparrow} \tag{3.2}$$

if and only if

$$\phi_J(f) \ge \sup_{x \in E} \{f(x) - I(x)\} \quad \text{for all } f \in L^{\uparrow}.$$
(3.3)

Second,

$$J_C \le -\inf_{x \in C} I(x) \quad \text{for all } C \in \mathcal{C}^{\uparrow}$$
(3.4)

if and only if

$$\phi_J(f) \le \sup_{x \in E} \{f(x) - I(x)\} \quad \text{for all } f \in U^{\uparrow}.$$
(3.5)

Third,

⁸ In particular, for $f \in L^{\uparrow} \cap U^{\uparrow}$, it follows that the two definitions in (3.1) coincide.

$$J_C \le -\inf_{x \in C} I(x) \quad \text{for all } C \in \mathcal{C}_c^{\uparrow}$$
(3.6)

if and only if

$$\phi_J(f) \le \sup_{x \in E} \{f(x) - I(x)\} \quad \text{for all } f \in U_c^{\uparrow}.$$
(3.7)

Proof. Suppose that inequality (3.2) holds. Let $f \in L^{\uparrow}$ and $\varepsilon > 0$. Using the definition of the maximum integral ϕ_J and the inequality $J_{\{f>r\}} \ge -\inf_{x \in \{f>r\}} I(x)$ for all $r \in \mathbb{R}$,

$$\phi_J(f) \ge \sup_{r \in \mathbb{R}} \left\{ r - \inf_{x \in \{f > r\}} I(x) \right\}$$

=
$$\sup_{r \in \mathbb{R}} \sup_{x \in \{f > r\}} \left\{ r - I(x) \right\}$$

$$\ge \sup_{x \in E} \sup_{y \in \{f > f(x) - \varepsilon\}} \left\{ f(x) - \varepsilon - I(y) \right\}$$

$$\ge \sup_{x \in E} \left\{ f(x) - \varepsilon - I(x) \right\}.$$

Since $\varepsilon > 0$ was arbitrary, we obtain $\phi_J(f) \ge \sup_{x \in E} \{f(x) - I(x)\}.$

Conversely, suppose that inequality (3.3) holds. Let $O \in \mathcal{O}^{\uparrow}$, so that $-\infty 1_{O^c} \in L^{\uparrow}$. Then, by Proposition 3.4,

$$J_O = \phi_J(-\infty 1_{O^c}) \ge \sup_{x \in E} \{-\infty 1_{O^c}(x) - I(x)\} = -\inf_{x \in O} I(x).$$

Now, suppose that inequality (3.4) holds. Let $f \in U^{\uparrow}$. Using the definition of the maximum integral ϕ_J and the inequality $J_{\{f \ge r\}} \le -\inf_{x \in \{f \ge r\}} I(x)$ for all $r \in \mathbb{R}$,

$$\phi_J(f) \leq \sup_{r \in \mathbb{R}} \left\{ r - \inf_{x \in \{f \geq r\}} I(x) \right\}$$

=
$$\sup_{r \in \mathbb{R}} \sup_{x \in \{f \geq r\}} \left\{ r - I(x) \right\}$$

$$\leq \sup_{r \in \mathbb{R}} \sup_{x \in \{f \geq r\}} \left\{ f(x) - I(x) \right\}$$

$$\leq \sup_{x \in E} \left\{ f(x) - I(x) \right\}.$$

Conversely, suppose that inequality (3.5) holds. Let $C \in C^{\uparrow}$, so that $-\infty 1_{C^c} \in U^{\uparrow}$. It follows from Proposition 3.4 that

$$J_C = \phi_J(-\infty 1_{C^c}) \le \sup_{x \in E} \{-\infty 1_{C^c}(x) - I(x)\} = -\inf_{x \in C} I(x).$$

Finally, the equivalence between the inequalities (3.6) and (3.7) follows along the same line of argumentation by replacing C^{\uparrow} by C_c^{\uparrow} and U^{\uparrow} by U_c^{\uparrow} , respectively. \Box

Notice that the upper/lower bound in the previous result is specified by upper/lower semicontinuous functions. In the same spirit, in the context of the Hausdorff moment problem [22], the natural extension of the moment sequence is also determined by semicontinuous functions.

Given a concentration J, our goal now is to find a rate function $I: E \to [0, \infty]$ which satisfies the lower bound (3.2) and upper bounds (3.4) or (3.6). We first focus on the lower bound. From the previous theorem, we immediately see that

$$I_{\min}(x) := \sup_{f \in L^{\uparrow}} (f(x) - \phi_J(f)) \text{ for all } x \in E$$

satisfies $I_{\min}(x) \ge f(x) - \phi_J(f)$ for all $x \in E$ and $f \in L^{\uparrow}$, and therefore the inequalities (3.3) and (3.2). We refer to I_{\min} as the *minimal rate function*. Notice that I_{\min} is minimal in the class of functions $I : E \to [0, \infty]$ which satisfy

the inequalities (3.2) and (3.3).⁹ By definition, the function I_{min} is lower semicontinuous and increasing. Moreover, it is determined through the concentration function as follows.

Lemma 3.6. Let I_{\min} be the minimal rate function associated to a concentration J. Then, for every $x \in E$,

$$-I_{\min}(x) = \inf_{U \in \mathcal{U}_x} J_{\uparrow U}.$$

Proof. Let $x \in E$ and $U \in \mathcal{U}_x$. By Lemma 3.3, we have $(\uparrow U)^c \in \mathcal{C}^{\downarrow}$, and therefore $-\infty \mathbb{1}_{(\uparrow U)^c} \in L^{\uparrow}$. Using the definition of the minimal rate function I_{\min} and Proposition 3.4, we obtain $I_{\min}(x) \ge -\phi_J(-\infty \mathbb{1}_{(\uparrow U)^c}) = -J_{\uparrow U}$. Hence, by taking the supremum over all $U \in \mathcal{U}_x$,

$$I_{\min}(x) \geq -\inf_{U \in \mathcal{U}_x} J_{\uparrow U}.$$

As for the other inequality, let $f \in L^{\uparrow}$ and $\varepsilon > 0$. Since f is lower semicontinuous, there exists $U \in U_x$ such that $f(z) \ge f(x) - \varepsilon$ for all $z \in U$. In particular, $f(z) \ge f(x) - \varepsilon$ for all $z \in \uparrow U$ because f is increasing. Then, by monotonicity and the translation property of ϕ_J and Proposition 3.4,

$$\begin{split} \phi_J(f) &\geq \phi_J((f(x) - \varepsilon) \mathbf{1}_{\uparrow U} - \infty \mathbf{1}_{(\uparrow U)^c}) \\ &= f(x) - \varepsilon + \phi_J(-\infty \mathbf{1}_{(\uparrow U)^c}) \\ &= f(x) - \varepsilon + J_{\uparrow U}. \end{split}$$

Since $\varepsilon > 0$ was arbitrary, we get $\phi_J(f) \ge f(x) + J_{\uparrow U} \ge f(x) + \inf_{U \in \mathcal{U}_x} J_{\uparrow U}$. This shows that for all $f \in L^{\uparrow}$,

 $\inf_{U \in \mathcal{U}_x} J_{\uparrow U} \le -f(x) + \phi_J(f).$

By taking the infimum over all $f \in L^{\uparrow}$, we conclude $\inf_{U \in \mathcal{U}_{x}} J_{\uparrow U} \leq -I_{\min}(x)$. \Box

For the minimal rate function to satisfy the upper bound (3.4) or (3.6), additional assumptions on the concentration are required. We will see that the concentration must necessarily fulfill a certain form of maximizing the lower and upper bounds simultaneously hold. This is the context of the next subsection.

3.3. Weak maxitivity

As discussed in Section 2, a capacity which admits a possibility distribution in the sense of (2.4) (respectively a concentration which admits rate function) is completely maxitive. In case that inequality (2.2) is only assumed for sequences or finite sets, then the capacity is called countably maxitive or finitely maxitive, respectively. For a detailed discussion on different maxitivity concepts, we refer to [7]. In case that a concentration satisfies only upper and lower bounds, we obtain a weaker form of maxitivity.

Definition 3.7. A concentration function *J* is called *weakly maxitive* if

 $J_A \leq \bigvee_{i=1}^n J_{B_i}$ for all $A \in \mathcal{C}^{\uparrow}$ and $B_1, \ldots, B_n \in \mathcal{O}^{\uparrow}$, $n \in \mathbb{N}$, such that $A \subset \bigcup_{i=1}^n B_i$.

Likewise, a function $\phi: LU^{\uparrow} \to [-\infty, \infty]$ is called *weakly maxitive* if

 $\phi(f) \leq \bigvee_{i=1}^{n} \phi(g_i)$ for all $f \in U^{\uparrow}$ and $g_1, \ldots, g_n \in L^{\uparrow}$, $n \in \mathbb{N}$, such that $f \leq \bigvee_{i=1}^{n} g_i$.

As a direct consequence of Theorem 3.5, we obtain the following result.

⁹ In fact, if $I: E \to [0, \infty]$ is another function for which inequality (3.3) holds, then $I(x) \ge f(x) - \phi_J(f)$ for all $x \in E$ and $f \in L^{\uparrow}$, and therefore $I(x) \ge \sup_{f \in L^{\uparrow}} (f(x) - \phi_J(f)) = I_{\min}(x)$ for all $x \in E$.

Proposition 3.8. Suppose that a concentration J satisfies the inequalities (3.2) and (3.4) for a function $I: E \rightarrow [0, \infty]$. Then, J and ϕ_J are weakly maximum.

Proof. Let $A \in C^{\uparrow}$ and $B_1, \ldots, B_n \in O^{\uparrow}$, $n \in \mathbb{N}$, with $A \subset \bigcup_{i=1}^n B_i$. Then, by monotonicity of J and the inequalities (3.4) and (3.2),

$$J_A \leq -\inf_{x \in A} I(x) \leq -\inf_{x \in \bigcup_{i=1}^n B_i} I(x) = -\bigwedge_{i=1}^n \inf_{x \in B_i} I(x) \leq \bigvee_{i=1}^n J_{B_i}.$$

Furthermore, let $f \in U^{\uparrow}$ and $g_1, \ldots, g_n \in L^{\uparrow}, n \in \mathbb{N}$, with $f \leq \bigvee_{i=1}^n g_i$. Then, it follows from Theorem 3.5 that

$$\phi_J(f) \le \sup_{x \in E} \{f(x) - I(x)\}$$

$$\le \sup_{x \in E} \{\bigvee_{i=1}^n g_i(x) - I(x)\}$$

$$\le \bigvee_{i=1}^n \sup_{x \in E} \{g_i(x) - I(x)\}$$

$$\le \bigvee_{i=1}^n \phi_J(g_i). \quad \Box$$

We now turn to the converse question and provide bounds for weakly maxitive concentrations. Thereby, we focus on the upper bound.

Proposition 3.9. Suppose that J is weakly maxitive. Then, J satisfies the upper bound (3.6) for the minimal rate function I_{\min} .

Proof. Let $C \in C_c^{\uparrow}$. By definition, $C \in C^{\uparrow}$ and $C = \uparrow K$ for some compact $K \subset E$. Fix $\varepsilon > 0$. Then, by Lemma 3.6 and compactness, there exist $x_i \in K$ and $U_i \in U_{x_i}$ for i = 1, ..., n such that $K \subset \bigcup_{i=1}^n U_i$ and

$$-J_{\uparrow U_i} \ge (I_{\min}(x_i) - \varepsilon) \wedge \varepsilon^{-1}$$
 for all $i = 1, \dots, n$.

Consequently, since $\uparrow U_i \in \mathcal{O}^{\uparrow}$ for all i = 1, ..., n due to Lemma 3.3, $C \subset \bigcup_{i=1}^n \uparrow U_i$, and J is weakly maximized.

$$-J_C \ge \wedge_{i=1}^n - J_{\uparrow U_i} \ge \wedge_{i=1}^n (I_{\min}(x_i) - \varepsilon) \wedge \varepsilon^{-1} \ge (\inf_{x \in K} I_{\min}(x) - \varepsilon) \wedge \varepsilon^{-1}.$$

Since $\varepsilon > 0$ was arbitrary, we obtain that

$$-J_C \ge \inf_{x \in K} I_{\min}(x) = \inf_{x \in C} I_{\min}(x),$$

where the last equality holds because I_{\min} is increasing. \Box

For the minimal rate function to satisfy the stronger upper bound (3.4), we need an additional tightness assumption.

Definition 3.10. A concentration *J* is called *tight*, if for all $C \in C^{\uparrow}$ and $\varepsilon > 0$, there exists $K \subset E$ compact with $\uparrow (C \cap K) \in C_c^{\uparrow}$ such that

$$J_C \le \left(J_{\uparrow(C\cap K)} + \varepsilon\right) \lor (-\varepsilon^{-1}). \tag{3.8}$$

Notice that $C \cap K$ is compact. In typical situations, $\uparrow (C \cap K)$ is closed, see Lemma A.2 in Appendix A.

Corollary 3.11. Let J be an weakly maximize concentration which is tight. Then, J satisfies the upper bound (3.4) with minimal rate function I_{min} .

Proof. Let $C \in C^{\uparrow}$ and $\varepsilon > 0$. Since *J* is tight, there exists $K \subset E$ compact with $\uparrow (C \cap K) \in C_c^{\uparrow}$ such that inequality (3.8) holds. Hence, we can apply Proposition 3.9 to obtain

$$J_C \leq \left(J_{\uparrow(C\cap K)} + \varepsilon\right) \lor (-\varepsilon^{-1}) \leq \left(-\inf_{x \in \uparrow(C\cap K)} I_{\min}(x) + \varepsilon\right) \lor (-\varepsilon^{-1})$$
$$\leq \left(-\inf_{x \in C} I_{\min}(x) + \varepsilon\right) \lor (-\varepsilon^{-1}).$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $J_C \leq -\inf_{x \in C} I_{\min}(x)$. \Box

The results achieved so far can be summarized as follows. For a weakly maximized concentration J there exists a function $I: E \to [0, \infty]$ such that

$$-\inf_{x \in O} I(x) \le J_O \quad \text{and} \quad J_C \le -\inf_{x \in C} I(x)$$
(3.9)

for all $O \in O^{\uparrow}$ and $C \in C_c^{\uparrow}$, respectively $C \in C^{\uparrow}$ if the concentration is in addition tight. In this case, the respective maxitive integral is given by $\phi_J(f) = \sup_{x \in E} (f(x) - I(x))$ for all $f \in L^{\uparrow} \cap U_c^{\uparrow}$ and $f \in L^{\uparrow} \cap U^{\uparrow}$, respectively. Moreover, the function *I* can be replaced by the minimal rate function I_{\min} . In other words, the class of weakly maxitive concentrations (capacities) can be connected with a rate function (possibility distribution), which fully determines the associated maxitive integral for sufficiently regular functions. To what extent the rate function is unique will be discussed in Section 5, and how it can be determined by means of convex duality arguments in Section 6.

4. Representation of weakly maxitive non-linear expectations

As remarked in the previous section, the maxitive integral (3.1) has the same properties as a non-linear expectation which satisfies the translation property. In the following, we focus on the converse direction and investigate when a non-linear expectation with the translation property can be represented as a maxitive integral. To do so, the following continuity condition is necessary.

Lemma 4.1. For every $f \in LU^{\uparrow}$,

$$\lim_{N \to \infty} \phi_J(f \land N) = \phi_J(f) \quad and \quad \lim_{N \to \infty} \phi_J(f \lor (-N)) = \phi_J(f).$$

Proof. Let $f \in L^{\uparrow}$, the other case follows with similar arguments. By definition of the maximizer integral, for each $N \in \mathbb{N}$,

$$\phi_J(f \wedge N) = \sup_{c \in \mathbb{R}} \{c + J_{\{(f \wedge N) > c\}}\} = \sup_{c < N} \{c + J_{\{f > c\}}\} \lor \sup_{N \le c} \{c + J_{\emptyset}\} = \sup_{c < N} \{c + J_{\{f > c\}}\},$$

and therefore,

$$\lim_{N \to \infty} \phi_J(f \land N) = \sup_{N \in \mathbb{N}} \phi_J(f \land N) = \sup_{N \in \mathbb{N}} \sup_{c < N} \{c + J_{\{f > c\}}\} = \sup_{c \in \mathbb{R}} \{c + J_{\{f > c\}}\} = \phi_J(f).$$

As for the second statement, for each $N \in \mathbb{N}$,

$$\begin{split} \phi(f \lor (-N)) &= \sup_{c \in \mathbb{R}} \{c + J_{\{f \lor (-N) > c\}} \} \\ &= \sup_{c < -N} \{c + J_{\{f \lor (-N) > c\}} \} \lor \sup_{-N \leq c} \{c + J_{\{f \lor (-N) > c\}} \} \\ &= \sup_{c < -N} \{c + J_E\} \lor \sup_{-N \leq c} \{c + J_{\{f > c\}} \} \\ &\leq (-N) \lor \sup_{c \in \mathbb{R}} \{c + J_{\{f > c\}} \} \\ &= (-N) \lor \phi_J(f). \end{split}$$

Hence, by monotonicity of ϕ_J ,

$$\phi_J(f) \le \lim_{N \to \infty} \phi_J(f \lor (-N)) \le \lim_{N \to \infty} (-N) \lor \phi_J(f) = \phi_J(f). \quad \Box$$

As shown in [7, Corollary 6], every finitely maxitive non-linear expectation with the translation property admits a representation in terms of a maxitive integral; see also [7, Corollary 7] and [20, Proposition 2.2]. We next provide a related representation result for weakly maxitive non-linear expectations. Let \overline{L}^{\uparrow} , \overline{U}^{\uparrow} , and \overline{LU}^{\uparrow} denote the sets of all functions in L^{\uparrow} , U^{\uparrow} , and LU^{\uparrow} , respectively, which are bounded from above. Moreover, let C_b^{\uparrow} be the space of all increasing bounded continuous functions $f: E \to \mathbb{R}$.

Theorem 4.2. Suppose that $\psi : \overline{LU}^{\uparrow} \to [-\infty, \infty)$ satisfies

- (i) $\psi(0) = 0$,
- (ii) $\psi(f) \leq \psi(g)$ whenever $f \leq g$,
- (iii) $\psi(f+c) = \psi(f) + c$ for all $c \in \mathbb{R}$,

with concentration $J_A^{\psi} := \psi(-\infty 1_{A^c})$ for all $A \in \mathcal{OC}^{\uparrow}$. If ψ is weakly maximize, ¹⁰ then

$$\psi(f) = \phi_{J^{\psi}}(f) \quad \text{for all } f \in \overline{L}^{\uparrow} \cap \overline{U}^{\uparrow}.$$

Proof. First, suppose that $f \in C_b^{\uparrow}$ and let $a, b \in \mathbb{R}$ such that a < f(x) < b for all $x \in E$. Fix $N \in \mathbb{N}$, and define for $0 \le j \le N - 1$,

$$a_{N,j} := a + j \frac{b-a}{N}, \quad O_{N,j} = \{f > a_{N,j}\}, \text{ and } C_{N,j} = \{f \ge a_{N,j}\}.$$

We consider the simple functions

$$l_N := \bigvee_{j=0}^{N-1} \left(-\infty \mathbf{1}_{O_{N,j}^c} + a_{N,j} \mathbf{1}_{O_{N,j}} \right) \quad \text{and} \quad u_N := \bigvee_{j=0}^{N-1} \left(-\infty \mathbf{1}_{C_{N,j}^c} + a_{N,j} \mathbf{1}_{C_{N,j}} \right).$$

By construction, it holds $l_N \in \overline{L}^{\uparrow}$, $u_N \in \overline{U}^{\uparrow}$, and $f - \frac{b-a}{N} \leq l_N \leq u_N \leq f$. Using the definition of the maximizer integral and the translation property of ψ ,

$$\begin{split} \phi_{J^{\psi}}(l_N) &= \bigvee_{j=0}^{N-1} \{a_{N,j} + J_{O_{N,j}}^{\psi} \} \\ &= \bigvee_{j=0}^{N-1} \psi \left(- \infty \mathbf{1}_{O_{N,j}^c} + a_{N,j} \mathbf{1}_{O_{N,j}} \right) \\ &\leq \psi(l_N). \end{split}$$

Moreover, it follows from

$$u_N \le l_N + \frac{b-a}{N} = \bigvee_{j=0}^{N-1} \left(-\infty \mathbb{1}_{O_{N,j}^c} + \left(a_{N,j} + \frac{b-a}{N} \right) \mathbb{1}_{O_{N,j}} \right),$$

and the weak maxitivity of ψ that

$$\begin{split} \psi(u_N) &\leq \bigvee_{j=0}^{N-1} \psi\Big(-\infty \mathbf{1}_{O_{N,j}^c} + \left(a_{N,j} + \frac{b-a}{N}\right) \mathbf{1}_{O_{N,j}}\Big) \\ &= \frac{b-a}{N} + \bigvee_{j=0}^{N-1} \psi\big(-\infty \mathbf{1}_{O_{N,j}^c} + a_{N,j} \mathbf{1}_{O_{N,j}}\big) \\ &= \frac{b-a}{N} + \phi_{J^{\psi}}(l_N). \end{split}$$

In combination with $\phi_{J^{\psi}}(l_N) \leq \psi(l_N) \leq \psi(u_N)$, we obtain

$$|\psi(u_N) - \phi_{J^{\psi}}(l_N)| \le \frac{b-a}{N}.$$

Hence, as a consequence of the monotonicity and translation property of ψ and $\phi_{J^{\psi}}$,

$$\begin{aligned} |\psi(f) - \phi_{J^{\psi}}(f)| &\leq |\psi(f) - \psi(u_N)| + |\psi(u_N) - \phi_{J^{\psi}}(l_N)| + |\phi_{J^{\psi}}(l_N) - \phi_{J^{\psi}}(f)| \\ &\leq \frac{b-a}{N} + \frac{b-a}{N} + \frac{b-a}{N}. \end{aligned}$$

¹⁰ I.e., $\phi(f) \leq \bigvee_{i=1}^{n} \phi(g_i)$ for all $f \in \overline{U}^{\uparrow}$ and $g_1, \dots, g_n \in \overline{L}^{\uparrow}$, $n \in \mathbb{N}$, such that $f \leq \bigvee_{i=1}^{n} g_i$.

Letting $N \to \infty$ results in $\psi(f) = \phi_{J^{\psi}}(f)$ as desired.

Second, suppose that $f \in \overline{L}^{\uparrow} \cap \overline{U}^{\uparrow}$. Let $N \in \mathbb{N}$, so that $f \vee (-N) \in C_b^{\uparrow}$, and therefore $\psi(f \vee (-N)) = \phi_{J^{\psi}}(f \vee (-N))$ due to the previous step. Moreover, since ψ is weakly maxitive,

$$\psi(f) \le \psi(f \lor (-N)) \le \psi(f) \lor (-N).$$

Hence, it follows from Lemma 4.1 that

$$\phi_{J^{\psi}}(f) = \lim_{N \to \infty} \phi_{J^{\psi}}(f \lor (-N)) = \lim_{N \to \infty} \psi(f \lor (-N)) = \psi(f). \quad \Box$$

Remark 4.3. A weakly maxitive non-linear expectation with the translation property is fully determined on increasing continuous bounded functions by its restriction to the indicators $-\infty 1_{A^c}$ for $A \in \mathcal{OC}^{\uparrow}$. Indeed, let $\psi_1, \psi_2: \overline{LU}^{\uparrow} \rightarrow [-\infty, \infty)$ be weakly maxitive non-linear expectations with the translation property. If $\psi_1(-\infty 1_{A^c}) = \psi_2(-\infty 1_{A^c})$ for all $A \in \mathcal{OC}^{\uparrow}$, then Theorem 4.2 implies $\psi_1(f) = \psi_2(f)$ for all $f \in \overline{L}^{\uparrow} \cap \overline{U}^{\uparrow}$.

5. A Laplace principle for maxitive integrals

Theorem 4.2 allows to represent weakly maxitive non-linear expectations with the translation property in terms of a maxitive integral ϕ_J . In order to find a computable representation of the minimal rate function I_{\min} , we aim to find conditions which guarantee that I_{\min} is attained on certain spaces of bounded functions. More specifically, under an additional separation property, we focus on representations of the form

$$I_{\min}(x) = \sup_{f \in C_b^{\uparrow}} \{f(x) - \phi_J(f)\}$$

We also investigate the relation between the bounds (3.9) and a Laplace principle for general concentrations. Throughout this section, let (E, \leq) be a topological preordered space. We fix a base \mathcal{U} for the topology of E, and set $\mathcal{U}_x := \{U \in \mathcal{U} : x \in U\}$ for all $x \in E$. Additionally to Assumption 3.1, we require the following separation property.

Assumption 5.1. For every $A \in C^{\uparrow}$ and $x \notin A$, there exists an increasing continuous function $f: E \to [0, 1]$ which satisfies

$$f(x) = 0$$
 and $A \subset f^{-1}(1)$.

Similarly, for every $A \in C^{\downarrow}$ and $x \notin A$, there exists an increasing continuous function $f: E \to [0, 1]$ such that $A \subset f^{-1}(0)$ and f(x) = 1.

Remark 5.2. If the preorder \leq is trivial, then the previous assumption corresponds to that of complete regularity. Both metric spaces, and Hausdorff topological abelian groups are completely regular topological spaces and thus satisfy Assumption 5.1 for the trivial preorder. Moreover, we show in Theorem A.1 in Appendix A that every preordered Hausdorff topological abelian group (E, \leq) as in Remark 3.2 satisfies Assumption 3.1 and Assumption 5.1.

Under Assumption 5.1, the following holds.

Lemma 5.3. For any $O \in \mathcal{O}^{\uparrow}$ and $x \in O$, there exists $U \in \mathcal{U}_x$ such that $cl(\uparrow U) \subset O$.

Proof. Fix $O \in \mathcal{O}^{\uparrow}$ and $x \in O$. Due to Lemma 3.3, we have that $O^c \in \mathcal{C}^{\downarrow}$. By Assumption 5.1, there exists an increasing continuous function $f: E \to [0, 1]$ such that $O^c \subset f^{-1}(0)$ and f(x) = 1. Since $x \in \{f > 1/2\}$, there exists $U \in \mathcal{U}_x$, which satisfies $U \subset \{f > 1/2\}$. Then, since $\{f > 1/2\}$ is upwards closed, we have $\uparrow U \subset \{f > 1/2\}$. Hence, $x \in \uparrow U \subset cl(\uparrow U) \subset \{f \ge 1/2\} \subset O$. The proof is complete. \Box

The previous result allows for a better description of the minimal rate function.

Lemma 5.4. Let J be a concentration function with minimal rate function I_{\min} . Then, for every $x \in E$,

$$I_{\min}(x) = \sup_{f \in C_b^{\uparrow}} \{f(x) - \phi_J(f)\} = -\inf_{U \in \mathcal{U}_x} J_{\uparrow U} = -\inf_{U \in \mathcal{U}_x} J_{\operatorname{cl}(\uparrow U)}.$$

Proof. Let $x \in E$. By Lemma 3.6 and Lemma 5.3,

$$\sup_{f \in C_b^{\uparrow}} \{f(x) - \phi_J(f)\} \le I_{\min}(x) = -\inf_{U \in \mathcal{U}_x} J_{\uparrow U} = -\inf_{U \in \mathcal{U}_x} J_{\mathrm{cl}(\uparrow U)}.$$

Recall that $cl(\uparrow U) \in C^{\uparrow}$ due to Lemma 3.3. Let $V \in U_x$ and r < 0. It follows from Lemma 3.3 that $(\uparrow V)^c \in C^{\downarrow}$. Hence, due to Assumption 5.1, there exists an increasing continuous function $f_{U,r}: E \to [r, 0]$ with $(\uparrow U)^c \subset f_{U,r}^{-1}(r)$ and $f_{U,r}(x) = 0$. Then, we have

$$\sup_{f \in C_b^{\uparrow}} \{f(x) - \phi_J(f)\} \ge -\phi_J(f_{U,r}) \ge -\phi_J(r \mathbf{1}_{(\uparrow U)^c}).$$

Letting $r \to -\infty$ and using Proposition 3.4, we see that $\sup_{f \in C_b^{\uparrow}} \{f(x) - \phi_J(f)\} \ge -J_{\uparrow U}$. If we take the supremum over all $U \in \mathcal{U}_x$, we conclude

$$-\inf_{U\in\mathcal{U}_x}J_{\uparrow U}\leq \sup_{f\in C_h^{\uparrow}}\{f(x)-\phi_J(f)\}. \quad \Box$$

Motivated by the theory of large deviations and in accordance with the inequalities (3.9), we introduce the following concepts.

Definition 5.5. Let $I: E \to [0, \infty]$ be a function. A concentration J is said to satisfy the *monotone large deviation principle* (mLDP) with rate function I if

$$-\inf_{x \in O} I(x) \le J_O \quad \text{and} \quad J_C \le -\inf_{x \in C} I(x)$$
(5.1)

for every $O \in O^{\uparrow}$ and all $C \in C^{\uparrow}$. Moreover, a concentration *J* is said to satisfy the *monotone Laplace principle* (mLP) with rate function *I* if the maxitive integral ϕ_J has the representation

$$\phi_J(f) = \sup_{x \in E} \{f(x) - I(x)\} \quad \text{for all } f \in C_b^{\uparrow}.$$
(5.2)

The mLDP is equivalent to the bounds (3.2) and (3.4). On the other hand, while the bounds (3.3) and (3.5) imply the mLP, the converse assertion is not necessarily true. For a concentration J which satisfies either the mLDP or mLP with rate function I, it follows from $J_E = 0$ or equivalently $\phi_J(0) = 0$, that I is proper, i.e., $I(x) \in [0, \infty)$ for some $x \in E$. We do not require that rate functions are lower semicontinuous.¹¹ However, we show that the mLDP uniquely determines the rate function within the class of increasing lower semicontinuous functions. For a function $f: E \to [-\infty, \infty]$, we define its *increasing lower semicontinuous envelope* $f^{\uparrow}: E \to [-\infty, \infty]$ by

$$f^{\uparrow}(x) := \sup \left\{ g(x) \colon g \in L^{\uparrow}, \ g \le f \right\}.$$

Directly from the definition, we see that f^{\uparrow} is the greatest increasing lower semicontinuous function $g: E \to [-\infty, \infty]$ which satisfies that $g \leq f$.

Proposition 5.6. Let *J* be a concentration and $I: E \to [0, \infty]$ be a function.

- (i) If $J_C \leq -\inf_{x \in C} I(x)$ for every $C \in C^{\uparrow}$, then $I^{\uparrow} \leq I_{\min}$.
- (ii) If $-\inf_{x \in O} I(x) \le J_O$ for every $O \in \mathcal{O}^{\uparrow}$, then $I_{\min} \le I^{\uparrow}$.

¹¹ In the theory of large deviations, it is typically assumed that a rate function is lower semicontinuous and proper; see e.g. [5].

In particular, if J satisfies the mLDP with rate function I being increasing and lower semicontinuous, then $I = I_{min}$.

Proof. First, we assume that $J_C \leq -\inf_{x \in C} I(x)$ for all $C \in C^{\uparrow}$. Since $I^{\uparrow} \leq I$,

$$J_C \leq -\inf_{x \in C} I(x) \leq -\inf_{x \in C} I^{\uparrow}(x) \quad \text{for all } C \in \mathcal{C}^{\uparrow}.$$

Let $x \in E$. We apply the previous inequality and the fact that I^{\uparrow} is lower semicontinuous and increasing to show that

$$I^{\uparrow}(x) = \sup_{U \in \mathcal{U}_x} \inf_{y \in U} I^{\uparrow}(y) = \sup_{U \in \mathcal{U}_x} \inf_{y \in \uparrow U} I^{\uparrow}(y) = \sup_{U \in \mathcal{U}_x} \inf_{y \in cl(\uparrow U)} I^{\uparrow}(y) \le -\inf_{U \in \mathcal{U}_x} J_{cl(\uparrow U)} = I_{\min}(x),$$

where in the first equality we used that I^{\uparrow} is lower semicontinuous, in the second equality that I^{\uparrow} is increasing, the third equality is a consequence of Lemma 5.3, and the last equality follows from Lemma 5.4.

Second, we assume that $-\inf_{x \in O} I(x) \le J_O$ for every $O \in \mathcal{O}^{\uparrow}$. Then, it follows from Lemma 5.4 that for every $x \in E$,

$$I_{\min}(x) = -\inf_{U \in \mathcal{U}_x} J_{\uparrow U} = \sup_{U \in \mathcal{U}_x} (-J_{\uparrow U}) \le \sup_{U \in \mathcal{U}_x} \inf_{y \in \uparrow U} I(y) \le I(x),$$

where we have used that $\uparrow U \in \mathcal{O}^{\uparrow}$. This shows $I_{\min} \leq I$. Moreover, since I_{\min} is increasing and lower semicontinuous, it follows that $I_{\min} \leq I^{\uparrow}$.

In particular, if *J* satisfies the mLDP with an increasing lower semicontinuous rate function *I*, then $I = I^{\uparrow}$, so that conditions (i) and (ii) imply that $I_{\min} \le I \le I_{\min}$. \Box

It follows from Theorem 3.5 that the mLDP implies the mLP. The converse implication is more involved. As shown in the following main result, if the space is monotonically normal or the rate function has downwards compactly generated sublevel sets, then the mLP implies the mLDP.

Theorem 5.7. Let $I: E \to [0, \infty]$ be an increasing lower semicontinuous function, and suppose that one of the following conditions is satisfied:

- (i) *E* is monotonically normal,¹² if for every $A \in C^{\downarrow}$ and $B \in C^{\uparrow}$ with $A \cap B = \emptyset$ there exist $U \in O^{\downarrow}$ and $V \in O^{\uparrow}$ with $U \cap V = \emptyset$ such that $A \subset U$ and $B \subset V$.
- (ii) I has downwards compactly generated sublevel sets, i.e., if for every $\alpha \in \mathbb{R}$ there exists a compact set K_{α} such that $\{I \leq \alpha\} = \downarrow K_{\alpha}$.

Then, J satisfies the mLP with rate function I if and only if J satisfies the mLDP with rate function I. In that case, $I = I_{\min}$.

Proof. If *J* satisfies the mLDP with increasing rate function *I*, then it follows from Theorem 3.5 that *J* satisfies the mLP with rate function *I*. Moreover, Proposition 5.6 ensures that $I = I_{min}$.

Conversely, suppose that the mLP with rate function I holds. Fix $A \in C^{\uparrow}$. We first show that $J_A \leq -\inf_{x \in A} I(x)$. If $\inf_{x \in A} I(x) = 0$, then the assertion trivially holds as $J_A \leq 0$. Thus, assume that $\inf_{x \in A} I(x) > \delta$ for some $\delta > 0$ small enough. Define

$$I^{\delta}(x) := (I(x) - \delta) \wedge \delta^{-1} \text{ for } x \in E.$$

Next, we show that

 $J_A < -\alpha$,

where $\alpha := \inf_{x \in A} I^{\delta}(x) \in (0, \infty)$. Since $-\inf_{x \in E} I(x) = \phi_J(0) = 0$, it follows that $\{I \le \alpha\}$ is non-empty. Moreover, $\{I \le \alpha\} \in C^{\downarrow}$ as *I* is increasing and lower semicontinuous. In addition, it holds $\{I \le \alpha\} \cap A = \emptyset$.

¹² This notion is due to Nachbin [25] where the term 'normally ordered space' is used in the context of an ordered set. The term 'monotonically normal' is also used in the literature for preordered sets; cf. e.g., [3,29].

Suppose first that *E* is monotonically normal. It follows from [25, Theorem 2] that for each $m \in \mathbb{N}$, there exists an increasing continuous function $h_m: E \to [-m, 0]$ such that $\{I \le \alpha\} \subset h_m^{-1}(-m)$ and $A \subset h_m^{-1}(0)$.¹³ For each $m \in \mathbb{N}$,

$$J_A = \inf_{r \in \mathbb{R}} \phi_J(r \mathbf{1}_{A^c}) \le \phi_J(h_m) = -\inf_{x \in S} \{-h_m(x) + I(x)\}$$

Since $-h_m(x) + I(x) \ge m$ if $x \in \{I \le \alpha\}$, and $-h_m(x) + I(x) \ge \alpha$ if $x \notin \{I \le \alpha\}$, by choosing $m \ge \alpha$, we obtain inequality (5.3).

Suppose now that *I* has downwards compactly generated sublevel sets. In that case, $\{I \le \alpha\} = \downarrow K_{\alpha}$ for a compact set $K_{\alpha} \subset E$. Since $A \cap K_{\alpha} = \emptyset$, by Assumption 5.1, for every $y \in K_{\alpha}$ there exists an increasing continuous function $f_y: E \to [-1, 0]$ such that $f_y(y) = -1$ and $A \subset f_y^{-1}(0)$. The neighborhoods $V_y := \{f_y < -\frac{1}{2}\}$ cover the compact set K_{α} . Hence, we can find $y_1, \ldots, y_N \in K_{\alpha}$ such that $K_{\alpha} \subset \bigcup_{1 \le i \le N} V_{y_i}$. For each $m \in \mathbb{N}$, we define $h_m := \wedge_{1 \le i \le N} 2m f_{y_i}$. Then, h_m is increasing, continuous and bounded, $A \subset h_m^{-1}(0)$ and $h_m(x) \le -m$ for all $x \in K_{\alpha}$. Moreover, since h_m is increasing, it follows that $h_m(x) \le -m$ for all $x \in \{I \le \alpha\} = \downarrow K_{\alpha}$. Finally, with the same arguments as in the previous case, we obtain inequality (5.3).

In both cases, inequality (5.3) holds for all $\delta > 0$ small enough, so that

$$J_A \leq -\liminf_{\delta \downarrow 0} \inf_{x \in A} I^{\delta}(x) = -\lim_{\delta \downarrow 0} \left((\inf_{x \in A} I(x) - \delta) \wedge \delta^{-1} \right) = -\inf_{x \in A} I(x).$$

Hence, *J* satisfies the upper bound of the mLDP with increasing rate function *I*. By Proposition 5.6, we obtain $I_{\min} \ge I$. On the other hand, it follows from the representation (5.2) that $I_{\min} \le I$, and therefore $I = I_{\min}$. Finally, it follows from Theorem 3.5 that *J* satisfies the lower bound of the mLDP with rate function $I = I_{\min}$. \Box

Remark 5.8. If the preorder \leq is trivial, then the notion of monotonical normality corresponds to normality. Examples of normal spaces include metrizable spaces, and regular Lindelöf spaces; see e.g. [17].

In case that \leq is a closed preorder,¹⁴ it is shown in [21] that *E* is monotonically normal whenever *E* is second countable and locally compact. This is the case, for example, when *E* is a (second countable) topological manifold equipped with a closed preorder. Further examples of monotonically normal spaces can be found in [4,21].

In case that $E = \mathbb{R}$ is the real line, then every increasing lower semicontinuous function $I: E \to [0, \infty)$ satisfies that $\{I \le \alpha\} = (-\infty, I(\alpha)] = \bigcup \{I(\alpha)\}$, and has therefore compactly generated sublevel sets.

6. Convex rate functions

The explicit determination of the rate function is generally a difficult task. However, in the particular case where the rate function is convex, one can rely on convex duality arguments. Throughout this section, let *E* be a locally convex Hausdorff topological real vector space. Moreover, let $E_+ \subset E$ be a convex cone, i.e., $E_+ + E_+ \subset E_+$, $\lambda E_+ \subset E_+$ for all $\lambda > 0$ and $0 \in E_+$. We endow *E* with the preorder induced by E_+ , i.e., $x \le y$ if and only if $y - x \in E_+$. Then, the assumptions of Section 5 are satisfied as outlined in Remark 5.2. Let \mathcal{U} be a base for the topology of *E*, and define $\mathcal{U}_x := \{U \in \mathcal{U} : x \in U\}$ for all $x \in E$. We first provide a condition which ensures that the minimal rate function is convex.

Lemma 6.1. Let J be a concentration such that

$$J_{\frac{1}{2}\uparrow U+\frac{1}{2}\uparrow V} \geq \frac{1}{2}J_{\uparrow U} + \frac{1}{2}J_{\uparrow V} \quad \text{for all } U, V \in \mathcal{U}.$$

Then, the minimal rate function I_{\min} : $E \to [0, \infty]$ *is convex.*

Proof. Fix $x, y \in E$ and $\varepsilon > 0$. Set $z := \frac{1}{2}x + \frac{1}{2}y$. Due to Lemma 5.4, there exists $W \in \mathcal{U}_z$ such that

$$-J_{\uparrow W} \ge (I_{\min}(z) - \varepsilon) \wedge \varepsilon^{-1}.$$

¹⁴ I.e., $\{(x, y) : x \le y\}$ is closed.

¹³ [25, Theorem 2] is formulated for ordered topological spaces rather than preordered topological spaces. However, since in the proof the antisymmetry is not used, the result applies also to preordered topological spaces.

Since $O := \{(\tilde{x}, \tilde{y}) \in E \times E : \frac{1}{2}\tilde{x} + \frac{1}{2}\tilde{y} \in W\}$ is open, there exist $U \in \mathcal{U}_x$ and $V \in \mathcal{U}_y$ such that $\frac{1}{2}U + \frac{1}{2}V \subset W$. Then,

$$-[(I_{\min}(z)-\varepsilon)\wedge\varepsilon^{-1}] \ge J_{\uparrow W} \ge J_{\frac{1}{2}\uparrow U+\frac{1}{2}\uparrow V} \ge \frac{1}{2}J_{\uparrow U}+\frac{1}{2}J_{\uparrow V}.$$

Taking the infimum over all $U \in \mathcal{U}_x$ and then over all $V \in \mathcal{U}_y$,

$$(I_{\min}(z) - \varepsilon) \wedge \varepsilon^{-1} \leq \frac{1}{2}I_{\min}(x) + \frac{1}{2}I_{\min}(y).$$

Therefore, since $\varepsilon > 0$ was arbitrary, we obtain

$$I_{\min}(z) \le \frac{1}{2}I_{\min}(x) + \frac{1}{2}I_{\min}(y).$$

Now, consider the set $\mathcal{D} := \{k2^{-n} : n, k \in \mathbb{N}, k \leq 2^n\}$ of all dyadic rational numbers in the interval [0, 1]. By recursion, it can be shown that

$$I_{\min}(z) \le \lambda I_{\min}(x) + (1-\lambda)I_{\min}(y),$$

for every $\lambda \in \mathcal{D}$. As a consequence, since the map $[0, 1] \rightarrow [0, \infty]$, $\lambda \mapsto I_{\min}(\lambda x + (1 - \lambda)y)$ is lower semicontinuous and the set \mathcal{D} is dense in [0, 1], the previous inequality is valid for all $\lambda \in [0, 1]$. \Box

Let E^* be the topological dual space of E. We denote by E^*_+ the set of all $\mu \in E^*$ which are positive, i.e., $\mu(x) \ge 0$ for all $x \in E_+$. The convex conjugate of the minimal rate function I_{\min} is defined by $I^*_{\min}(\mu) := \sup_{x \in E} \{\mu(x) - I_{\min}(x)\}$ for all $\mu \in E^*$.

Proposition 6.2. Let J be a concentration. Suppose that the minimal rate function I_{\min} is convex and $I_{\min}^*(\mu) \ge \phi_J(\mu)$ for all $\mu \in E_+^*$. Then,

$$I_{\min}(x) = \sup_{\mu \in E_+^*} \{\mu(x) - \phi_J(\mu)\} \quad \text{for all } x \in E.$$
(6.1)

Proof. The minimal rate function I_{min} is lower semicontinuous and increasing. By the Fenchel-Moreau theorem [13, Theorem A.62],

$$I_{\min}(x) = \sup_{\mu \in E^*} \{\mu(x) - I^*_{\min}(\mu)\}$$

= $\sup_{\mu \in E^*_+} \{\mu(x) - I^*_{\min}(\mu)\}$ for all $x \in E$.

The second equality holds because $I_{\min}^*(\mu) = \infty$ whenever $\mu \in E^* \setminus E_+^*$. In fact, for $\mu \in E^* \setminus E_+^*$ there exists $y \le 0$ with $\mu(y) > 0$, and therefore

$$I_{\min}^{*}(\mu) \geq \sup_{\lambda>0} \{\mu(x+\lambda y) - I_{\min}(x+\lambda y)\}$$

$$\geq \sup_{\lambda>0} \{\mu(x) + \lambda\mu(y) - I_{\min}(x)\} = \infty.$$

Finally, using the definition of I_{\min} and the inequality $I^*_{\min}(\mu) \ge \phi_J(\mu)$ for all $\mu \in E^*_+$,

$$I_{\min}(x) = \sup_{f \in L^{\uparrow}} \{f(x) - \phi_J(f)\} \ge \sup_{\mu \in E^*_+} \{\mu(x) - \phi_J(\mu)\} \ge I_{\min}(x). \quad \Box$$

The hypothesis of Proposition 6.2 can be verified in some important situations; e.g., for the asymptotic concentration of sample means of i.i.d. sequences, see Subsection 7.2. In particular, it is satisfied under the mLDP, which leads to the following result. **Corollary 6.3.** Let J be a concentration which satisfies the mLDP with a rate function $I: E \to [0, \infty]$ which is convex, increasing and lower semicontinuous. Then,

$$I(x) = \sup_{\mu \in E_+^*} \{\mu(x) - \phi_J(\mu)\} \quad for \ all \ x \in E.$$

Proof. By Proposition 5.6, it holds $I = I_{\min}$. Moreover, it follows from Theorem 3.5 that $\phi_J(\mu) = I^*_{\min}(\mu)$ for all $\mu \in E^*_+$. Consequently, since $I = I_{\min}$ is convex, the claim follows from Proposition 6.2. \Box

7. Examples

We illustrate the theoretical results with two examples. First, we study the asymptotic behavior of a sequence $(\mu_n)_{n \in \mathbb{N}}$ of capacities by considering the weakly maxitive concentration $J_A := \limsup_{n \to \infty} \frac{1}{n} \log \mu_n(A)$. Large deviations bounds for sequences of capacities on \mathbb{R}^d were recently considered in [8,31]. Second, we focus on the role of the partial order by elaborating a monotone version of Cramér's theorem for which the rate function can be determined explicitly.

7.1. Asymptotic concentration of capacities

Let (E, \leq) be a topological preordered space which satisfies Assumption 3.1. We denote by \overline{B} the set of all Borel measurable functions $f: E \to [-\infty, \infty)$ which are bounded from above. In the following, let $(\mathcal{E}_n)_{n \in \mathbb{N}}$ be a sequence of sublinear expectations on \overline{B} , i.e., for each $n \in \mathbb{N}$, the functional $\mathcal{E}_n: \overline{B} \to [-\infty, \infty)$ satisfies

(i) $\mathcal{E}_n(f) \leq \mathcal{E}_n(g)$ whenever $f \leq g$, (ii) $\mathcal{E}_n(f+g) \leq \mathcal{E}_n(f) + \mathcal{E}_n(g)$, (iii) $\mathcal{E}_n(f+c) = \mathcal{E}_n(f) + c$ for all $c \in \mathbb{R}$, (iv) $\mathcal{E}_n(\lambda f) = \lambda \mathcal{E}_n(f)$ for all $\lambda \in [0, \infty)$.

A functional which satisfies the properties (i)-(iv) is also called upper expectation in robust statistics [15], coherent risk measure in mathematical finance [2], or upper coherent prevision in the theory of imprecise probabilities [32]. We remark that the theory of upper previsions does not make any measurability assumption, and in that context it is known that the conditions (i) and (iii) follow from the conditions (ii) and (iv) together with $\mathcal{E}_n(f) \leq \sup f$; see, e.g., [23,32].

Example 7.1. Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of *E*-valued random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Consider a non-empty set \mathcal{P} of probability measures on \mathcal{F} . Then, for each *n*, the upper prevision $\mathcal{E}_n(f) = \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}[f(X_n)]$ satisfies the properties (i)–(iv) above.

We consider the set function

 $J: \mathcal{OC}^{\uparrow} \to [-\infty, 0], \quad J_A := \limsup_{n \to \infty} \frac{1}{n} \log \mu_n(A),$

where μ_n denotes the corresponding capacity of \mathcal{E}_n , which is defined by $\mu_n(A) := \mathcal{E}_n(1_A)$ for all Borel sets $A \subset E$. Straightforward verification shows that each μ_n satisfies $\mu_n(\emptyset) = 0$, $\mu_n(E) = 1$, $\mu_n(A) \le \mu_n(B)$ whenever $A \subset B$, and $\mu_n(A) \le \mu_n(B) + \mu_n(C)$ whenever $A \subset B \cup C$. Moreover, *J* is a concentration which turns out to be weakly maxitive according to the principle of the largest term.¹⁵

Lemma 7.2. The concentration J is weakly maxitive.

¹⁵ The principle of the largest term is a result which is often used in the theory of large deviations; see, e.g., [27, Proposition 12.3] and [5, Lemma 1.2.15]. Namely, if $(a_n^1)_{n \in \mathbb{N}}, \ldots, (a_n^N)_{n \in \mathbb{N}}$ are $[0, \infty]$ -valued sequences, then $\limsup_{n \to \infty} \frac{1}{n} \log \sum_{i=1}^{N} a_n^i = \bigvee_{i=1}^{N} \limsup_{n \to \infty} \frac{1}{n} \log a_n^i$.

Proof. Let $C \in C^{\uparrow}$ and $O_1, O_2, \ldots, O_N \in O^{\uparrow}$ such that $C \subset \bigcup_{i=1}^N O_i$, and therefore $\mu_n(C) \leq \sum_{i=1}^N \mu_n(O_i)$. Applying the principle of the largest term,

$$J_C \leq \limsup_{n \to \infty} \frac{1}{n} \log \sum_{i=1}^N \mu_n(O_i) \leq \bigvee_{i=1}^N \limsup_{n \to \infty} \frac{1}{n} \log \mu_n(O_i) = \bigvee_{i=1}^N J_{O_i}. \quad \Box$$

In particular, the results of Section 3 are applicable. Due to Lemma 3.6, the minimal rate function is given by

$$I_{\min}(x) = -\sup_{U \in \mathcal{U}_x} \limsup_{n \to \infty} \frac{1}{n} \log \mu_n \left(\uparrow U\right) \quad \text{for all } x \in E$$

As an application of Theorem 3.5, the discussion thereafter, and Proposition 3.9, we obtain

$$-\inf_{x \in O} I_{\min}(x) \le J_O \quad \text{and} \quad J_C \le -\inf_{x \in C} I_{\min}(x)$$
(7.1)

for all $O \in O^{\uparrow}$ and $C \in C_c^{\uparrow}$, and consequently the respective bounds in Theorem 3.5 for the maxitive integral ϕ_J . Building on Theorem 4.2 and Remark 4.3, the maxitive integral ϕ_J has the following representation in terms of a sequence of Choquet integrals.

Proposition 7.3. For every $f \in \overline{L}^{\uparrow} \cap \overline{U}^{\uparrow}$,

$$\phi_J(f) = \limsup_{n \to \infty} \frac{1}{n} \log \mathcal{E}_n(\exp(nf)) = \limsup_{n \to \infty} \frac{1}{n} \log \int_0^\infty \mu_n \left(\exp(nf) > x\right) dx.$$
(7.2)

Proof. Define $\psi : \overline{LU}^{\uparrow} \to [-\infty, \infty)$ as the right hand side of (7.2). Inspection shows that ψ satisfies properties (i)-(iii) in Theorem 4.2. Next, we show that ψ is weakly maxitive. To that end, let $f \in \overline{U}^{\uparrow}$, $g_1, g_2, \ldots, g_N \in \overline{L}^{\uparrow}$, and $f \leq \bigvee_{i=1}^N g_i$. Then, by the principle of the largest term,

$$\begin{split} \psi(f) &\leq \limsup_{n \to \infty} \frac{1}{n} \log \int_{0}^{\infty} \mu_n \left(\exp\left(n(\vee_{i=1}^N g_i)\right) > x \right) \mathrm{d}x \\ &= \limsup_{n \to \infty} \frac{1}{n} \log \int_{0}^{\infty} \mu_n \left(\cup_{i=1}^N \{\exp(ng_i) > x\} \right) \mathrm{d}x \\ &\leq \limsup_{n \to \infty} \frac{1}{n} \log \sum_{i=1}^N \int_{0}^{\infty} \mu_n \left(\exp(ng_i) > x\right) \mathrm{d}x \\ &\leq \vee_{i=1}^N \limsup_{n \to \infty} \frac{1}{n} \log \left(\int_{0}^{\infty} \mu_n \left(\exp(ng_i) > x\right) \mathrm{d}x \right) \\ &= \vee_{i=1}^N \psi(g_i), \end{split}$$

which shows that ψ is weakly maximize. Moreover, for $A \in \mathcal{OC}^{\uparrow}$,

$$\psi(-\infty \mathbf{1}_{A^c}) = \limsup_{n \to \infty} \frac{1}{n} \log \int_{0}^{\infty} \mu_n \left(\exp(n(-\infty) \mathbf{1}_{A^c}) > x \right) dx$$
$$= \limsup_{n \to \infty} \frac{1}{n} \log \int_{0}^{\infty} \mu_n \left(\mathbf{1}_A > x \right) dx$$

$$= \limsup_{n \to \infty} \frac{1}{n} \log \int_{0}^{1} \mu_n(A) \, dx$$
$$= \limsup_{n \to \infty} \frac{1}{n} \log \mu_n(A)$$
$$= J_A.$$

Hence, $J^{\psi} = J$ and we can apply Theorem 4.2 to conclude $\psi = \phi_J$. In a second step, we define

$$\tilde{\psi}: \overline{LU}^{\uparrow} \to [-\infty, \infty), \quad \tilde{\psi}(f):=\limsup_{n \to \infty} \frac{1}{n} \log \mathcal{E}_n(\exp(nf)).$$

Direct verification shows that $\tilde{\psi}$ satisfies properties (i)-(iii) in Theorem 4.2. Moreover, it is weakly maxitive. Indeed, let $f \in \overline{U}^{\uparrow}$ and $g_1, g_2, \ldots, g_N \in \overline{L}^{\uparrow}$ such that $f \leq \bigvee_{i=1}^N g_i$. Again, by the principle of the largest term,

$$\begin{split} \tilde{\psi}(f) &\leq \limsup_{n \to \infty} \frac{1}{n} \log \mathcal{E}_n \left(e^{n(\vee_{i=1}^N g_i)} \right) \\ &\leq \limsup_{n \to \infty} \frac{1}{n} \log \sum_{i=1}^N \mathcal{E}_n \left(e^{ng_i} \right) \\ &\leq \vee_{i=1}^N \limsup_{n \to \infty} \frac{1}{n} \log \mathcal{E}_n \left(e^{ng_i} \right) \\ &= \vee_{i=1}^N \tilde{\psi}(g_i). \end{split}$$

Since $J_A = \tilde{\psi}(-\infty 1_{A^c})$ for all $A \in \mathcal{OC}^{\uparrow}$, it follows from Theorem 4.2 that $\tilde{\psi} = \phi_J$. Together with the first part, we obtain equality (7.3). \Box

In contrast to linear expectations in standard probability theory, a sublinear expectation \mathcal{E} is in general not determined by its associated capacity $\mu(A) = \mathcal{E}(1_A)$; see, e.g., [26]. In other words, the sublinear expectation \mathcal{E} may contain more information than the capacity μ . However, Proposition 7.3 implies that the asymptotic entropic version $\limsup_{n\to\infty} \frac{1}{n} \log \mathcal{E}_n(\exp(nf))$ is fully specified through the sequence of capacities $(\mu_n)_{n\in\mathbb{N}}$ by means of the right hand side of equation (7.2).

Under the assumptions of Theorem 5.7, it follows that ϕ_J satisfies the mLP with rate function *I* if and only if *J* satisfies the mLDP with rate function *I*. Under a slightly stronger version of the mLDP, the limit superior in (7.2) can even be replaced by a limit. More precisely, we obtain the equivalence between the following versions of the classical large deviation principle and the Laplace principle.

Corollary 7.4. Suppose that (E, \leq) satisfies Assumption 3.1 and Assumption 5.1. Let $I: E \to [0, \infty]$ be an increasing lower semicontinuous function. If $(\mu_n)_{n \in \mathbb{N}}$ satisfies

$$-\inf_{x \in O} I(x) \le \liminf_{n \to \infty} \frac{1}{n} \log \mu_n(O) \quad and \quad \limsup_{n \to \infty} \frac{1}{n} \log \mu_n(C) \le -\inf_{x \in C} I(x)$$
(7.3)

for all $O \in \mathcal{O}^{\uparrow}$ and $C \in \mathcal{C}^{\uparrow}$, then $(\mathcal{E}_n)_{n \in \mathbb{N}}$ satisfies

$$\lim_{n \to \infty} \frac{1}{n} \log \mathcal{E}_n(\exp(nf)) = \sup_{x \in E} \{f(x) - I(x)\}$$
(7.4)

for all $f \in C_b^{\uparrow}$. The converse assertion holds true if *E* is monotonically normal or *I* has compactly generated sublevel sets.

Proof. Let $f \in C_b^{\uparrow}$. It follows from Theorem 3.5 that

$$\phi_J(f) = \limsup_{n \to \infty} \frac{1}{n} \log \mathcal{E}_n(\exp(nf)) = \sup_{x \in E} \{f(x) - I(x)\}.$$

To show that the previous limit superior is a limit, we define

$$\underline{\phi}(f) := \liminf_{n \to \infty} \frac{1}{n} \log \mathcal{E}_n(\exp(nf)).$$

Fix $\varepsilon > 0$ and $x \in E$. Since f is upper semicontinuous and increasing, there exists $U \in \mathcal{O}^{\uparrow}$ with $x \in U$ such that $\inf_{y \in U} f(y) \ge f(x) - \varepsilon$. Then,

$$\underline{\phi}(f) \ge \underline{\phi}(f1_U - \infty 1_{U^c})$$

$$= f(x) - \varepsilon + \underline{\phi}(-\infty 1_{U^c})$$

$$= f(x) - \varepsilon + \liminf_{n \to \infty} \frac{1}{n} \log \mu_n(U)$$

$$\ge f(x) - \varepsilon - \inf_{y \in U} I(y)$$

$$\ge f(x) - \varepsilon - I(x).$$

Letting $\varepsilon \downarrow 0$ and taking the supremum over all $x \in E$ yields

$$\sup_{x\in E} \{f(x)-I(x)\} \le \underline{\phi}(f) \le \phi_J(f) = \sup_{x\in E} \{f(x)-I(x)\}.$$

As for the converse assertion, suppose that E is monotonically normal or I has compactly generated sublevel sets. By Theorem 5.7,

$$\limsup_{n \to \infty} \frac{1}{n} \log \mu_n(C) = J_C \le -\inf_{x \in C} I(x) \quad \text{for all } C \in \mathcal{C}^{\uparrow}.$$

To show the lower bound, fix $O \in \mathcal{O}^{\uparrow}$. Let $x \in O$ and $m \in \mathbb{N}$. By Lemma 3.3, it holds $O^c \in \mathcal{C}^{\downarrow}$. Hence, due to Assumption 5.1, there exists $f_m \in C_b^{\uparrow}$ such that $f_m(x) = 0$, $O^c \subset f_m^{-1}(-m)$, and $-m \leq f(y) \leq 0$ for all $y \in E$. We obtain

$$\begin{split} \liminf_{n \to \infty} \frac{1}{n} \log \mu_n(O) \lor (-m) &= \liminf_{n \to \infty} \frac{1}{n} \log \mathcal{E} \big(\exp(n(-\infty \mathbf{1}_{O^c})) \big) \lor (-m) \\ &\geq \liminf_{n \to \infty} \frac{1}{n} \log \mathcal{E} \big(\exp(n((-\infty \mathbf{1}_{O^c}) \lor (-m)) \big) \\ &\geq \underline{\phi}(f_m) = \phi_J(f_m) \ge f_m(x) - I(x) = -I(x). \end{split}$$

By letting $m \to \infty$ and taking the supremum over all $x \in O$,

$$\liminf_{n \to \infty} \frac{1}{n} \log \mu_n(O) \ge -\inf_{x \in O} I(x). \quad \Box$$

7.2. Sample means of i.i.d. sequences

Let *E* be a locally convex Hausdorff topological real vector space endowed with a preorder induced by a closed convex cone $E_+ \subset E$. Then, the assumptions of Section 5 are satisfied as outlined in Remark 5.2.

For each $n \in \mathbb{N}$, let $X_n := \frac{1}{n} \sum_{i=1}^{n} \xi_i$ be the sample mean of an i.i.d. sequence $(\xi_n)_{n \in \mathbb{N}}$ of *E*-valued random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For $A \in \mathcal{OC}^{\uparrow}$, we define the concentration $J_A = \lim \sup_{n \to \infty} \frac{1}{n} \log \mathbb{P}(X_n \in A)$ with minimal rate function I_{\min} . Due to Proposition 7.3, the concentration *J* is weakly maxitive and its maxitive integral admits the representation

$$\phi_J(f) = \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{E}_{\mathbb{P}}[\exp(nf(X_n))] \quad \text{for all } f \in \overline{L}^{\uparrow} \cap \overline{U}^{\uparrow}.$$
(7.5)

Lemma 7.5. For every convex set $A \in OC^{\uparrow}$,

$$J_A = \sup_{n \in \mathbb{N}} \frac{1}{n} \log \mathbb{P}(X_n \in A).$$
(7.6)

Moreover, for every convex set $O \in \mathcal{O}^{\uparrow}$ *,*

$$J_O = \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(X_n \in O) = \sup_{n \in \mathbb{N}} \frac{1}{n} \log \mathbb{P}(X_n \in O).$$
(7.7)

Proof. The equalities (7.6) and (7.7) follow from [27, Proposition 12.5] and [27, Proposition 12.2], respectively. Although [27] assumes that the state space is a separable Banach space, these particular results are also valid for general topological vector spaces. \Box

Lemma 7.6. I_{\min} : $E \rightarrow [0, \infty]$ is convex.

Proof. We adapt the proof of [27, Proposition 12.9]. Let U, V be open and convex sets. For fixed $n \in \mathbb{N}$, define $X_{n+1,2n} = \frac{1}{n} \sum_{i=n+1}^{2n} \hat{\xi}_i$. Since $X_{2n} = \frac{1}{2} (X_n + X_{n+1,2n})$, as well as X_n and $X_{n+1,2n}$ are independent,

$$\mathbb{P}(X_n \in \uparrow U)\mathbb{P}(X_{n+1,2n} \in \uparrow V) = \mathbb{P}(\{X_n \in \uparrow U\} \cap \{X_{n+1,2n} \in \uparrow V\}) \le \mathbb{P}(X_{2n} \in \frac{1}{2} \uparrow U + \frac{1}{2} \uparrow V)$$

Hence.

$$\frac{1}{2}\frac{1}{n}\log\mathbb{P}(X_n\in\uparrow U)+\frac{1}{2}\frac{1}{n}\log\mathbb{P}(X_n\in\uparrow V)\leq\frac{1}{2n}\log\mathbb{P}(X_{2n}\in\frac{1}{2}\uparrow U+\frac{1}{2}\uparrow V).$$

It follows from Lemma 7.5 that

$$J_{\frac{1}{2}\uparrow U+\frac{1}{2}\uparrow V} = \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P} \left(X_n \in \frac{1}{2}\uparrow U + \frac{1}{2}\uparrow V \right)$$

$$= \lim_{n \to \infty} \frac{1}{2n} \log \mathbb{P} \left(X_{2n} \in \frac{1}{2}\uparrow U + \frac{1}{2}\uparrow V \right)$$

$$\geq \frac{1}{2} \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P} \left(X_n \in \uparrow U \right) + \frac{1}{2} \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P} \left(X_n \in \uparrow V \right)$$

$$= \frac{1}{2} J_{\uparrow U} + \frac{1}{2} J_{\uparrow V}.$$

.

Since the topology of E is generated by the collection of all convex open subsets of E, we can apply Lemma 6.1 to conclude that I_{\min} is convex. \Box

Definition 7.7. An *E*-valued random variable ξ is called *convex tight* if for every $\varepsilon > 0$ there exists a convex compact $K \subset E$ such that $\mathbb{P}(\xi \in K) \ge 1 - \varepsilon$. Moreover, ξ is said to be *convex inner regular* if for every $\varepsilon > 0$ and each convex open $O \subset E$, there exists a convex compact $K \subset O$ such that $\mathbb{P}(\xi \in K) \geq \mathbb{P}(\xi \in O) - \varepsilon$.

By adapting [27, Proposition 12.7] to the present setting, we remark that every E-valued random variable that is convex tight is also convex inner regular. Moreover, if E is a separable Banach space, then every random variable is automatically convex tight and convex inner regular; see [27, Proposition 12.4] and [27, Proposition 12.7].

From now on, let ξ be an *E*-valued random variable distributed as ξ_1 . We need the following result from [27]. For the sake of completeness, we provide a proof.

Lemma 7.8. Let $f: E \to (0, \infty)$ be a Borel measurable function, and suppose that ξ is convex tight. Then, for every $\varepsilon > 0$, there exists a convex compact $K \subset E$ such that

$$\varepsilon^{-1} \wedge \left(\log \mathbb{E}_{\mathbb{P}}[f(\xi)] - \varepsilon \right) \leq \log \mathbb{E}_{\mathbb{P}}[f(\xi) \mathbf{1}_{K}(\xi)].$$

Proof. Suppose first that $f: E \to \mathbb{R}$ is bounded. Fix $\varepsilon > 0$. Since ξ is convex tight, there exists a compact set $K \subset E$ such that

$$\mathbb{P}(\xi \in K^c) \le (1 - e^{-\varepsilon}) \frac{\mathbb{E}_{\mathbb{P}}[f(\xi)]}{M},$$

where $|f(x)| \leq M$ for all $x \in E$. Then,

$$\mathbb{E}_{\mathbb{P}}[f(\xi)] = \mathbb{E}_{\mathbb{P}}[f(\xi)1_{K}(\xi)] + \mathbb{E}_{\mathbb{P}}[f(\xi)1_{K^{c}}(\xi)]$$

$$\leq \mathbb{E}_{\mathbb{P}}[f(\xi)1_{K}(\xi)] + M\mathbb{P}(\xi \in K^{c})$$

$$\leq \mathbb{E}_{\mathbb{P}}[f(\xi)1_{K}(\xi)] + (1 - e^{-\varepsilon})\mathbb{E}_{\mathbb{P}}[f(\xi)],$$

which shows that $\log \mathbb{E}_{\mathbb{P}}[f(\xi)] - \varepsilon \leq \log \mathbb{E}_{\mathbb{P}}[f(\xi)1_K(\xi)].$

In case that f is not bounded, due to the monotone convergence theorem, there exists $N \in \mathbb{N}$ such that

$$\varepsilon^{-1} \wedge \left(\log \mathbb{E}_{\mathbb{P}}[f(\xi)] - \varepsilon \right) \leq \log \mathbb{E}_{\mathbb{P}}[f(\xi) \wedge N] - \varepsilon/2.$$

Since $x \mapsto f(x) \wedge N$ is bounded, it follows from the first part that there exists a convex compact $K \subset E$ such that

$$\log \mathbb{E}_{\mathbb{P}}[f(\xi) \wedge N] - \varepsilon/2 \le \log \mathbb{E}_{\mathbb{P}}[f(\xi) \mathbf{1}_{K}(\xi)]. \quad \Box$$

The *logarithmic moment generating function* of ξ is defined by

$$\Lambda \colon E_+^* \to [0, \infty], \quad \Lambda(\mu) := \log \mathbb{E}_{\mathbb{P}}[\exp(\mu(\xi))].$$

In addition, we define its positive convex conjugate $\Lambda_+^* \colon E \to [0, \infty]$ by

$$\Lambda_{+}^{*}(x) = \sup_{\mu \in E_{+}^{*}} \{\mu(x) - \Lambda(\mu)\}.$$

Then the following monotone version of Cramér's theorem holds.

Theorem 7.9. Suppose that ξ is convex tight. Then,

$$I_{\min}(x) = \Lambda_{+}^{*}(x) \quad \text{for all } x \in E.$$
(7.8)

Moreover,

$$\limsup_{n \to \infty} \frac{1}{n} \mathbb{P}(X_n \in C) \le -\inf_{x \in C} \Lambda_+^*(x) \quad \text{for all } C \in \mathcal{C}_c^{\uparrow},$$
(7.9)

$$\liminf_{n \to \infty} \frac{1}{n} \mathbb{P}(X_n \in O) \ge -\inf_{x \in O} \Lambda_+^*(x) \quad \text{for all } O \in \mathcal{O}^\uparrow.$$
(7.10)

If additionally X_n is convex tight for all $n \in \mathbb{N}$, then

$$\lim_{n \to \infty} \frac{1}{n} \mathbb{P}(X_n \in O) = -\inf_{x \in O} \Lambda_+^*(x) \quad \text{for all convex } O \in \mathcal{O}^{\uparrow}.$$

Proof. Fix $\mu \in E_+^*$. Since $(\xi_n)_{n \in \mathbb{N}}$ is i.i.d., we have for all $N \in \mathbb{N}$,

$$\Lambda(\mu) = \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{E}_{\mathbb{P}}[\exp(n\mu(X_n))]$$

$$\geq \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{E}_{\mathbb{P}}[\exp(n(N \land \mu)(X_n))]$$

$$= \phi_J(N \land \mu),$$

where the last equality follows from (7.5). Letting $N \to \infty$, it follows from Lemma 4.1 that

$$\Lambda(\mu) \ge \phi_J(\mu).$$

Fix $\varepsilon > 0$. By Lemma 7.8, there exists a convex compact $K \subset E$ such that

$$\varepsilon^{-1} \wedge (\Lambda(\mu) - \varepsilon) \leq \log \mathbb{E}_{\mathbb{P}}[e^{\mu(\xi)} \mathbf{1}_K(\xi)].$$

Since $(\xi_n)_{n \in \mathbb{N}}$ is i.i.d. and *K* is convex,

$$\mathbb{E}_{\mathbb{P}}[e^{n\mu(X_n)}1_K(X_n)] \ge \mathbb{E}_{\mathbb{P}}[e^{\mu(\xi_1)}\cdots e^{\mu(\xi_n)}1_K(\xi_1)\cdots 1_K(\xi_n)] = \mathbb{E}_{\mathbb{P}}[e^{\mu(\xi)}1_K(\xi)]^n.$$

This shows that

$$\varepsilon^{-1} \wedge \left(\Lambda(\mu) - \varepsilon \right) \leq \log \mathbb{E}_{\mathbb{P}} [e^{\mu(\xi)} \mathbf{1}_{K}(\xi)]$$

$$\leq \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{E}_{\mathbb{P}} [e^{n\mu(X_{n})} \mathbf{1}_{K}(X_{n})]$$

$$\leq \sup_{x \in E} \left\{ \mu(x) - I_{\min}(x) \right\}$$

$$= I_{\min}^{*}(\mu),$$

where the last inequality follows from Lemma B.1 in the Appendix B. Since $\varepsilon > 0$ was arbitrary, we obtain $\phi_J(\mu) \le \Lambda(\mu) \le I_{\min}^*(\mu)$. Hence, since the minimal rater function I_{\min} is convex, lower semicontinuous and increasing, in line with Proposition 6.2, we obtain for all $x \in E$,

$$I_{\min}(x) \ge \sup_{\mu \in E_{+}^{*}} \{\mu(x) - \phi_{J}(\mu)\}$$

$$\ge \sup_{\mu \in E_{+}^{*}} \{\mu(x) - \Lambda(\mu)\}$$

$$\ge \sup_{\mu \in E_{+}^{*}} \{\mu(x) - I_{\min}^{*}(\mu)\}$$

$$= I_{\min}(x).$$

This shows equation (7.8). Moreover, since the concentration J is weakly maxitive, the upper bound (7.9) follows directly from Proposition 3.9. To show the lower bound (7.10), we consider the concentration $\underline{J}_A :=$ $\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}(X_n \in A)$ with minimal rate function \underline{I}_{\min} . Let \mathcal{U} be a topological base consisting of open convex sets. It follows from Lemma 3.6 that for every $x \in E$,

$$\underline{I}_{\min}(x) = -\inf_{U \in \mathcal{U}_x} \underline{J}_U = -\inf_{U \in \mathcal{U}_x} J_U = I_{\min}(x) = \Lambda_+^*(x)$$

where $J_U = \underline{J}_U$ is valid due to (7.7) in Lemma 7.5. Then the lower bound (7.10) follows from the lower bound (3.2), which is satisfied for the minimal rate function $\underline{I}_{\min} = \Lambda_+^*$.

Finally, suppose that for each $n \in \mathbb{N}$ the random variable X_n is convex tight and therefore convex inner regular. Fix $\varepsilon > 0$, a convex $O \in \mathcal{O}^{\uparrow}$ and $N \in \mathbb{N}$. Since X_N is convex inner regular, there exists a convex compact $K \subset O$ such that

$$\frac{1}{N}\log \mathbb{P}(X_N \in O) \le \frac{1}{N}\log \mathbb{P}(X_N \in K) + \varepsilon \le \frac{1}{N}\log \mathbb{P}(X_N \in \uparrow K) + \varepsilon.$$

It follows from Lemma A.2 that $\uparrow K \in C_c^{\uparrow}$ and $\uparrow K$ is a Borel set. Since $\uparrow K$ is convex, we can apply Lemma 7.5 to conclude

$$J_{\uparrow K} = \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}(X_n \in \uparrow K) = \sup_{n \in \mathbb{N}} \frac{1}{n} \log \mathbb{P}(X_n \in \uparrow K),$$

and therefore

$$\frac{1}{N}\log\mathbb{P}(X_N\in O) \leq \frac{1}{N}\log\mathbb{P}(X_N\in\uparrow K) + \varepsilon \leq \sup_{n\in\mathbb{N}}\frac{1}{n}\log\mathbb{P}(X_n\in\uparrow K) + \varepsilon = J_{\uparrow K} + \varepsilon.$$

By Proposition 3.9, it holds $J_{\uparrow K} \leq -\inf_{x \in \uparrow K} I_{\min}(x) \leq -\inf_{x \in O} I_{\min}(x)$. Hence,

$$\frac{1}{N}\log \mathbb{P}(X_N \in O) \le -\inf_{x \in O} I_{\min}(x) + \varepsilon.$$

Letting $N \to \infty$ and then $\varepsilon \downarrow 0$,

$$J_O = \lim_{N \to \infty} \frac{1}{N} \log \mathbb{P}(X_N \in O) \le -\inf_{x \in O} I_{\min}(x),$$

where the limit above exists due to Proposition 7.5. The other inequality follows from the lower bound (3.2), which is satisfied for the minimal rate function I_{\min} .

The sample mean of an i.i.d. sequence of random variables with values in \mathbb{R}^d satisfies the usual large deviation principle with rate function Λ^* ; cf. [5, Theorem 2.2.30]. In infinite dimensional spaces, the upper bound in the large deviation principle

$$\limsup_{n \to \infty} \frac{1}{n} \mathbb{P}(X_n \in C) \le -\inf_{x \in C} \Lambda^*(x)$$
(7.11)

is only known for certain sets $C \subset E$. For instance, bounds for compact sets or convex open sets are shown in [5, Theorem 6.1.3], [14] and [27]. We obtain in (7.9) a new upper bound for compactly generated sets. Moreover, each

choice of the cone E_+ yields a class of upwards closed sets for which we obtain an upper bound in terms of Λ_+^* . By Corollary 3.11, the upper bound (7.9) is valid for all $C \in C^{\uparrow}$ if the concentration $J_A = \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}(X_n \in A)$ is tight. Moreover, the lower bound in the large deviation principle

$$\liminf_{n \to \infty} \frac{1}{n} \log \mathbb{P}(X_n \in O) \ge -\inf_{x \in O} \Lambda^*(x)$$
(7.12)

is valid for all open set $O \subset E$; see e.g. [27]. Since $-\inf_{x \in O} \Lambda^*_+ \ge -\inf_{x \in O} \Lambda^*$, we obtain in (7.10) a sharper lower bound on upwards closed open sets.

8. Conclusions

It is well-known that the maxitive integral with respect to a completely maxitive concentration (capacity) is uniquely determined by a rate function (possibility distribution). In this paper, we have introduced the notion of weak maxitivity, and we have shown that, if a concentration J is weakly maxitive and tight, then the corresponding maxitive integral ϕ_J is determined by the minimal rate function I_{\min} on the space of all increasing continuous functions.

Furthermore, every maxitive integral ϕ_J is a non-linear expectation with the translation property, and we have argued that ϕ_J is weakly maxitive if J is weakly maxitive. Conversely, we have shown that every weakly maxitive non-linear expection ψ with the translation property has a maxitive integral representation $\psi = \phi_J$ on the space of all increasing continuous functions.

Motivated by the theory of large deviations, we have provided different representation results for the minimal rate function I_{\min} . First, under Assumption 5.1 (which is satisfied if the state space *E* is a preordered topological group), we have seen that the rate function is determined by the space of all bounded increasing continuous functions, i.e.,

$$I_{\min}(x) = \sup_{f \in C_h^{\uparrow}} \{f(x) - \phi_J(f)\}.$$

In addition, we have formulated monotone analogues of the large deviation principle and the Laplace principle, which have been shown to be equivalent under suitable conditions and which uniquely determine the rate function within the class of increasing lower semicontinuous functions. Second, we have focused on the case where the rate function is a convex function on a locally convex topological real vector space E. In that case, under an additional assumption which is implied by the monotone Laplace principle, the minimal rate function is specified by the dual space, i.e.,

$$I_{\min}(x) = \sup_{\mu \in E^*} \{\mu(x) - \phi_J(\mu)\}.$$

Finally, we have shown that standard large deviations theory can be understood within the framework of weakly maxitive concentrations and their maxitive integrals. In turn, the present framework enlarges the scope of large deviations theory to non-standard situations, which we have illustrated with two examples. On the one hand, we have covered the asymptotic concentration of capacities on preordered topological spaces. On the other hand, we have established new large deviation bounds for the sample mean of i.i.d. sequences on upwards closed sets by showing a monotone analogue of Cramér's theorem on locally convex topological vector spaces.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Appendix A. Auxiliary results for preordered topological groups

Let *G* be a Hausdorff topological abelian group. Given a subset $G_+ \subset G$ such that

 $G_+ + G_+ \subset G_+, \quad 0 \in G_+,$

we endow G with the preorder induced by G_+ , i.e., $x \le y$ if and only if $y - x \in G_+$. Next, we show that G satisfies Assumption 5.1.

Theorem A.1. For every $A \in C^{\uparrow}$ and $x \notin A$ there exists an increasing continuous function $f: G \to [0, 1]$ such that

$$f(x) = 0 \quad and \quad A \subset f^{-1}(1).$$

Similarly, for every $A \in C^{\downarrow}$ and $x \notin A$, there exists an increasing continuous function $f: G \to [0, 1]$ such that

 $A \subset f^{-1}(0)$ and f(x) = 1.

Proof. Notice that it suffices to prove the first statement as the second one can be obtained from the first one by applying the transformation $z \mapsto -z$. The argumentation is an adaptation of [16, Theorem 5], where it is shown that every topological group is completely regular.

Suppose that $A \in C^{\uparrow}$ and $x \notin A$. Without loss of generality, we assume that x = 0. Denote by D the set of all dyadic rational numbers of the interval [0, 1], i.e., $D := \{k2^{-n} : n, k \in \mathbb{N}, k \leq 2^n\}$. As in the proof of [16, Theorem 5], it is possible to construct a family $(V_r)_{r \in D}$ of open neighborhoods of $0 \in G$ which satisfy the following properties:

(i) $V_1 = A^c$,

- (ii) $V_r = -V_r$ for r < 1,
- (iii) $V_r \subset V_s$ if r < s,
- (iv) $V_{k2^{-n}} + V_{2^{-n}} \subset V_{(k+1)2^{-n}}$ for $k = 1, ..., 2^n 1$.

Then, we define

$$f: G \to [0, 1], \quad f(x) := 1 \land \inf\{r \in \mathcal{D} : x \in \downarrow V_r\}.$$

As usual, we set $\inf \emptyset := \infty$. Since $0 \in \bigcup V_r$ for all r, we have that f(0) = 0. Suppose that $x \in A$. Due to (i), (iii), and Lemma 3.3, we have that $\bigcup V_r \subset A^c$ for all r. Hence, if $x \in A$, we have that $x \notin \bigcup V_r$ for r, and consequently f(x) = 1. Finally, since $\bigcup V_r$ is downwards closed, it follows that f is increasing. It remains to show that f is continuous. To that end, we fix $y \in G$.

First, suppose that f(y) = 1. Let $\varepsilon > 0$ and fix $n \in \mathbb{N}$ with $2^{-(n-1)} < \varepsilon$. We show that $z \in y + V_{2^{-n}}$ implies that $z \in (\downarrow V_{k2^{-n}})^c$ for all $k < 2^n - 2$. If otherwise $z \in \downarrow V_{k2^{-n}}$, then due to (ii) and (iv),

$$y \in z - V_{2^{-n}} = z + V_{2^{-n}} \subset \downarrow V_{k2^{-n}} + V_{2^{-n}} \subset \downarrow V_{k2^{-n}} + \downarrow V_{2^{-n}} \subset \downarrow V_{(k+1)2^{-n}},$$

and therefore, $f(y) \le (k+1)2^{-n} < (2^n-1)2^{-n} < 1$, which is a contradiction to f(y) = 1. Thus, for $z \in x + V_{2^{-n}}$, we have $f(z) \ge (2^n - 2)2^{-n}$, and therefore

$$|f(y) - f(z)| = 1 - f(z) \le 1 - (2^n - 2)2^{-n} = 2^{-(n-1)} < \varepsilon.$$

We conclude that f is continuous at y.

Second, suppose that 0 < f(y) < 1. Let $\varepsilon > 0$ and $n \in \mathbb{N}$ such that $2^{-(n-1)} < \varepsilon$, $2^{-(n-1)} < 1 - f(y)$, and $2^{-(n-1)} < f(y)$. Let $k \in \mathbb{N}$ be the smallest $i \in \{1, \dots, 2^n\}$ such that $y \in \bigvee_{i \ge n}$. Then, $(k-1)2^{-n} \le f(y) \le k2^{-n}$ and

$$y \in \bigcup V_{k2^{-n}} \cap (\bigcup V_{(k-1)2^{-n}})^c$$

Since $1 - f(x) > 2^{-(n-1)}$, we have $1 - (k-1)2^{-n} \ge 1 - f(x) > 2^{-(n-1)}$, so that $k < 2^n$. Since $f(x) > 2^{-(n-1)}$, it holds $k2^{-n} \ge f(x) > 2^{-(n-1)}$, and therefore k > 2. We have $z \in (\downarrow V_{(k-2)2^{-n}})^c$ whenever $z \in y + V_{2^{-n}}$. Otherwise, due to (ii) and (iv), we obtain

$$y \in z - V_{2^{-n}} = z + V_{2^{-n}} \subset \downarrow V_{(k-2)2^{-n}} + V_{2^{-n}} \subset \downarrow V_{(k-2)2^{-n}} + \downarrow V_{2^{-n}} \subset \downarrow V_{(k-1)2^{-n}},$$

which is a contradiction. Moreover, if $z \in y + V_{2^{-m}}$, since $y \in \bigcup V_{k2^{-n}}$, it follows from (iv) that¹⁶

¹⁶ We can apply (iv). Indeed, since $f(x) \ge (k-1)2^{-n}$, we have $1 - (k-1)2^{-n} \ge 1 - f(x) > 2^{-(n-1)}$, and therefore $k < 2^n$.

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$$z \in y + V_{2^{-n}} \subset \downarrow V_{k2^{-n}} + \downarrow V_{2^{-n}} \subset \downarrow V_{(k+1)2^{-n}}.$$

Hence, for $z \in y + V_{2^{-n}}$, we have $(k-2)2^{-n} \leq f(z) \leq (k+1)2^{-n}$, and therefore

 $|f(y) - f(z)| \le 2^{-(n-1)} < \varepsilon.$

We conclude that f is continuous at y.

Third, suppose that f(y) = 0. Let $\varepsilon > 0$ and fix $n \in \mathbb{N}$ with $2^{-(n-1)} < \varepsilon$. Since f(y) = 0, we have $y \in \downarrow V_{2^{-n}}$. For $z \in y + V_{2^{-n}}$, it holds

$$z \in y + V_{2^{-n}} \subset \downarrow V_{2^{-n}} + \downarrow V_{2^{-n}} \subset \downarrow V_{2^{-(n-1)}},$$

and therefore

 $|f(y) - f(z)| = f(z) \le 2^{-(n-1)} < \varepsilon.$

We conclude that f is continuous at y. The proof is complete. \Box

Lemma A.2. Suppose that G_+ is closed. If $K \subset G$ is compact, then $\uparrow K$ is closed.

Proof. Suppose that (x_{α}) is a net in $\uparrow K$ such that $x_{\alpha} \to x$. We show that $x \in \uparrow K$. For each α , we have $x_{\alpha} = y_{\alpha} + z_{\alpha}$ where $y_{\alpha} \in K$ and $z_{\alpha} \in G_+$. Since *K* is compact there exists a subnet (y_{β}) such that $y_{\beta} \to y \in K$. Then $z_{\beta} = x_{\beta} - y_{\beta}$ converges to x - y. Since G_+ is closed, we have $x - y \in G_+$, and therefore $x \in K + G_+ = \uparrow K$. \Box

Appendix B. Auxiliary result for asymptotic concentration of capacities

Let (E, \leq) be a topological preordered space which satisfies Assumption 3.1. In line with Section 7, we consider the concentration $J_A := \limsup_{n \to \infty} \frac{1}{n} \log \mu_n(A)$ with minimal rate function I_{\min} , where $\mu_n(A) := \mathcal{E}_n(1_A)$ for all $A \in \mathcal{OC}^{\uparrow}$, and $(\mathcal{E}_n)_{n \in \mathbb{N}}$ is a sequence of sublinear expectations on \overline{B} .

Lemma B.1. Let $K \subset E$ be a compact set. Then, for every $f \in L^{\uparrow} \cap U^{\uparrow}$,

$$\limsup_{n \to \infty} \frac{1}{n} \log \mathcal{E}_n \left(\exp(nf) \mathbf{1}_K \right) = \phi_J (-\infty \mathbf{1}_{K^c} + f \mathbf{1}_K)$$
$$\leq \sup_{x \in E} \left\{ f(x) - I_{\min}(x) \right\}.$$

Proof. If $J_K = -\infty$, then the first two terms are equal to $-\infty$ as f is bounded on K, hence the inequality trivially holds. Otherwise, if $J_K > -\infty$, we consider the concentration $J_A^K := J_{A\cap K} - J_K$, $A \in \mathcal{OC}^{\uparrow}$, and the trivial preorder $\leq .^{17}$ By similar arguments as in Proposition 7.3, for every $f \in \overline{L}^{\uparrow} \cap \overline{U}^{\uparrow}$,

$$\phi_{J^{K}}(f) = \phi_{J}(-\infty \mathbf{1}_{K^{c}} + f\mathbf{1}_{K}) - J_{K} = \limsup_{n \to \infty} \frac{1}{n} \log \mathcal{E}_{n}(\exp(nf)\mathbf{1}_{K}) - J_{K}.$$
(B.1)

Moreover, since $-\infty \mathbf{1}_{K^c} + f \mathbf{1}_K \in U_c^{\uparrow}$, it follows from Theorem 3.5 that

$$\phi_J(-\infty \mathbf{1}_{K^c} + f\mathbf{1}_K) \le \sup_{x \in K} \left\{ f(x) - I_{\min}^0(x) \right\} \le \sup_{x \in E} \left\{ f(x) - I_{\min}^0(x) \right\},\tag{B.2}$$

where I_{\min}^0 denotes the minimal rate function for the concentration J w.r.t. the trivial order. By definition of the minimal rate function, it holds $I_{\min}^0 \ge I_{\min}$. Hence, we obtain the claimed assertion as a direct consequence of equation (B.1) and inequality (B.2). \Box

¹⁷ I.e., $x \le y$ if and only if x = y.

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