UNIVERSIDAD DE MURCIA

ESCUELA INTERNACIONAL DE DOCTORADO

TESIS DOCTORAL

Dynamical Aspects of k-order Difference Equations

Aspectos Dinámicos de Ecuaciones en Diferencias Finitas de Orden k

D. Daniel Nieves Roldán

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Autor: D. Daniel Nieves Roldán

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MURCIA

DECLARACIÓN DE AUTORÍA Y ORIGINALIDAD
DE LA TESIS PRESENTADA PARA OBTENER EL TÍTULO DE DOCTOR
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doctorando del Programa de Doctorado en

Matemáticas

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Finally, I would like to make mine the following verses from *Hamlet* that have reminded me in many moments of my life why I am who I am.

What is a man,
If his chief good and market of his time
Be but to sleep and feed? A beast, no more.
Hamlet, W. Shakespeare.

In gratitude to all those people who, throughout my life, have encouraged me to not live like a beast.

Contents

Resumen

Las ecuaciones en diferencias son una herramienta muy útil para modelar una amplia variedad de fenómenos donde el tiempo evoluciona de manera discreta. En este sentido, es fácil encontrar infinidad de modelos en diversos campos, como la biología, la física, la economía o la ingeniería, donde este tipo de ecuaciones describen el comportamiento de dichos fenómenos. Por ello, el estudio de sus propiedades dinámicas es crucial para conocer cómo se van a comportar a largo plazo las soluciones de dichos modelos, que pueden presentar una variedad de propiedades muy rica, desde comportamientos más sencillos como la periodicidad de las soluciones o la convergencia, hasta características más complejas como la existencia de conjuntos atractores extraños o de caos.

El principal objetivo de esta disertación es analizar diferentes propiedades dinámicas asociadas a las ecuaciones en diferencias autónomas y aportar algunos avances en determinadas cuestiones abiertas sobre dichas propiedades. En esta dirección, en los capítulos de esta tesis abordamos las siguientes propiedades dinámicas: convergencia, periodicidad global, periodicidad, conjuntos de puntos de acumulación, invarianza, atracción, conjugaciones topológicas, estabilidad, bifurcaciones y permanencia. Cabe destacar que hemos realizado una amplia revisión de la literatura existente y hemos analizado y ejecutado distintas técnicas que van desde cuestiones más sencillas de análisis matemático, hasta otras más complejas propias de los sistemas dinámicos discretos.

A continuación, presentamos de forma somera el estudio desarrollado en los diferentes capítulos relativo a las distintas propiedades dinámicas analizadas e indicamos nuestras principales aportaciones en cada caso.

El Capítulo 1 recoge los conceptos y resultados básicos relativos a las diferentes cuestiones que se abordarán a lo largo de la tesis.

El Capítulo 2 parte de la conocida propiedad de que toda sucesión de números reales monótona y acotada es convergente. Este resultado fue generalizado por E.T. Copson al intercambiar la condición de monotonía por una desigualdad convexa entre los términos de la sucesión. Más adelante, la propiedad de Copson se extendió a otros ámbitos más allá de las sucesiones de números reales (véase [63] para una visión general sobre el resultado de Copson y sus distintas generalizaciones y extensiones). Nuestra principal aportación, recogida en [64], consiste en una generalización del Teorema de la Convergencia Monótona

de Lebesgue donde hemos aplicado las técnicas desarrolladas en el trabajo de Copson.

Teorema A. *Sean* $(\mathbb{X}, \Sigma, \mu)$ *un espacio de medida y* (f_n) *una sucesión de funciones medibles no negativas,* $f_n : \mathbb{X} \to [0, \infty]$*. Si la sucesión verifica la desigualdad*

$$f_{n+k}(x) \geq \sum_{j=1}^{k} \alpha_j f_{n+k-j}(x), \quad \textit{para todo } x \in \mathbb{X},$$

donde los coeficientes α_j *son números reales estrictamente positivos tales que* $\sum_{j=1}^{k} \alpha_j = 1$*, entonces existe una función medible* f *de manera que*

$$\lim_{n\to\infty} f_n(x) = f(x), \quad \textit{para todo } x \in \mathbb{X},$$

y

$$\lim_{n\to\infty} \int_{\mathbb{X}} f_n \; d\mu = \int_{\mathbb{X}} f \; d\mu.$$

Además, si suponemos la desigualdad contraria, $f_{n+k}(x) \leq \sum_{j=1}^{k} \alpha_j f_{n+k-j}(x)$*, para todo* $x \in \mathbb{X}$ *y la hipótesis adicional de ser* $f_1, \ldots, f_k \in L^1(\mu)$*, el resultado sigue siendo cierto.*

Además, ilustramos con un ejemplo la utilidad de dicha generalización a través de una sucesión de funciones no monótona que, en cambio, sí verifica la correspondiente desigualdad tipo Copson.

Seguidamente, dejamos el escenario de las sucesiones de funciones para centrarnos en nuestro principal objeto de estudio, las ecuaciones en diferencias autónomas. El Capítulo 3 versa sobre el estudio de la periodicidad global, una propiedad dinámica que, a grandes rasgos, consiste en que todas las soluciones de la ecuación son periódicas. Además, si p es el mínimo común múltiplo de todos los periodos, diremos que la ecuación es un p-ciclo. Para el estudio de dicha propiedad, nos centramos en la ecuación de tercer orden

$$x_{n+3} = x_i f(x_j, x_k),$$

donde $i, j, k \in \{n, n+1, n+2\}$ son distintos dos a dos, $f : (0, \infty)^2 \to (0, \infty)$ es continua y las condiciones iniciales son números reales positivos.

En la literatura podemos encontrar la caracterización de los p-ciclos de dicha forma cuando $p \leq 5$, véase [7]. En este sentido, nosotros hemos profundizado en el estudio de la existencia de 6-ciclos y hemos aportado algunos avances al respecto, donde las técnicas empleadas se basan en la computación de las órbitas de las soluciones, argumentos de análisis real y la resolución de ecuaciones funcionales. Por un lado, en [68] probamos la no existencia de 6-ciclos al asumir la condición adicional de simetría para f, esto es, $f(x, y) = f(y, x)$ para todo $x, y > 0$.

Teorema B. *No existen* 6*-ciclos de tercer orden de la forma* $x_{n+3} = x_i f(x_j, x_k)$*, cuando* $f : (0, \infty)^2 \to (0, \infty)$ *es una función continua simétrica y los índices* $i, j, k \in \{n, n+1, n+2\}$ *son distintos dos a dos.*

Por otro lado, en el caso de separación de variables, $f(x,y) = g(x)h(y)$, para todo $x, y > 0$, donde $g, h : (0,\infty) \to (0,\infty)$ son funciones continuas, en [70] presentamos la prueba de que $x_{n+3} = x_n \left(\frac{x_{n+2}}{x_{n+1}}\right)^2$ es el único 6-ciclo que existe.

Teorema C. *Considérese la ecuación en diferencias*

$$x_{n+3} = x_i g(x_j) h(x_k),$$

con $i, j, k \in \{n, n+1, n+2\}$ *distintos dos a dos, y* $g, h : (0,\infty) \to (0,\infty)$ *funciones continuas. El único* 6*-ciclo que muestra tal forma viene dado por*

$$x_{n+3} = x_n \left(\frac{x_{n+2}}{x_{n+1}}\right)^2.$$

A continuación, en el Capítulo 4 nos dedicamos a estudiar la periodicidad de las soluciones de ecuaciones en diferencias autónomas. Para ello, nos centramos en la familia de ecuaciones en diferencias tipo-max (esto es, ecuaciones en diferencias en cuya ley de formación aparece el operador máximo), ya que se trata de una clase de ecuaciones idónea para hacer la transición del estudio de la periodicidad global a la periodicidad: la familia presenta diversos ejemplos de ecuaciones globalmente periódicas y múltiples modelos cuyas soluciones son periódicas o finalmente periódicas. En este sentido, en el estudio [69] hacemos acopio de diferentes resultados de la literatura sobre esta clase de ecuaciones enfocándonos en las distintas técnicas y en la riqueza de su dinámica.

En esta dirección, analizamos la ecuación tipo-max de orden 4

$$x_{n+4} = \max\{x_{n+3}, x_{n+2}, x_{n+1}, 0\} - x_n, \tag{1}$$

con condiciones iniciales reales arbitrarias. En primer lugar, en [65] proporcionamos una descripción completa de su conjunto de periodos, $\mathrm{Per}(F_4)$, empleando técnicas de dinámica combinatoria donde hemos podido describir la evolución de las órbitas de las soluciones a través de un grafo.

Teorema D. *Considérese la Ecuación* (1) *y denotemos por* $\mathrm{Per}(F_4)$ *a su conjunto de periodos. Entonces*

$$\mathrm{Per}(F_4) = \{1, 8, 11\} \bigcup \left\{10 \cdot a + 11 \cdot b \mid \gcd(a,b) = 1, a \geq 1, b \geq 2a+1\right\}.$$

Además, no solo probamos que el conjunto de periodos es no acotado, sino que, a través de técnicas básicas de congruencias y álgebra general, vemos que 1674 es el mayor número natural que no pertenece a tal conjunto, véase [66].

Respecto a las soluciones no periódicas, en [62] determinamos por completo sus conjuntos de puntos de acumulación que, de hecho, son intervalos compactos de la recta real.

Teorema E. *Sean (x_1, x_2, x_3, x_4) condiciones iniciales reales arbitrarias que generan una órbita no periódica (x_n) por la Ecuación (1). Entonces, el conjunto de puntos de acumulación de (x_n) es un intervalo compacto. Además, la tupla (x_1, x_2, x_3, x_4) es equivalente a alguna tupla de condiciones iniciales (x, y, z, w), con $x = \max\{x_n : n \geq 1\}$, $x \geq w \geq y \geq z \geq 0$, y $\frac{w-z}{x} \in \mathbb{R} \setminus \mathbb{Q}$, y la órbita se acumula en el intervalo compacto $\big[\min\{w - x, -z\}, x\big]$.*

De esta forma, el capítulo no solo versa sobre el análisis de la propiedad dinámica de periodicidad, sino que también aborda otras cuestiones como los conjuntos de puntos de acumulación y la invarianza de la ecuación, para la que proporcionamos una nueva integral primera para el sistema dinámico asociado a la Ecuación (1).

Tras estudiar ecuaciones en diferencias tipo-max, en el Capítulo 5 establecemos una relación entre esta clase de ecuaciones y la conocida *Ecuación de Lozi*,

$$x_{n+1} = 1 - a|x_n| + bx_{n-1},$$

donde los parámetros a, b y las condiciones iniciales son números reales.

En concreto, basándonos en el trabajo [67], proponemos una generalización de dicha ecuación

$$y_{n+1} = \alpha|y_n| + \beta y_n + \gamma y_{n-1} + \delta,$$

donde α, β, γ y δ son números reales con $\alpha \neq 0$, a la que denominamos *Ecuación de Lozi generalizada*, y demostramos que dicha generalización es topológicamente conjugada a toda una familia de ecuaciones tipo-max.

Teorema F. *Considérese la ecuación de Lozi generalizada con $\delta = 0$,*

$$y_{n+1} = \alpha|y_n| + \beta y_n + \gamma y_{n-1}.$$

Entonces, es topológicamente conjugada a la ecuación tipo-max

$$z_{n+1} = \frac{\max\{z_n^{2\alpha}, B\}}{z_n^{\alpha-\beta} \cdot z_{n-1}^{-\gamma}} \cdot B^{\frac{\alpha+\beta+\gamma-1}{-2\alpha}},$$

para todo $B > 0$.

En particular, si $\alpha + \beta + \gamma - 1 = 0$, entonces la ecuación generalizada de Lozi es topológicamente conjugada a la ecuación tipo-max

$$z_{n+1} = \frac{\max\{z_n^{2\alpha}, B\}}{z_n^{\alpha-\beta} \cdot z_{n-1}^{-\gamma}},$$

para todo $B > 0$.

Teorema G. *Considérese la ecuación de Lozi generalizada con $\delta \neq 0$,*

$$y_{n+1} = \alpha|y_n| + \beta y_n + \gamma y_{n-1} + \delta.$$

Entonces, o bien para $A > 1$ y $\frac{\alpha}{q} > 0$, o bien para $0 < A < 1$ y $\frac{\alpha}{q} < 0$, con $q \in \mathbb{R} \setminus \{0\}$, la ecuación es topológicamente conjugada a

$$z_{n+1} = \frac{\max\{z_n^{2\alpha}, A^{\frac{-2\alpha p}{q}}\}}{z_n^{\alpha-\beta} \cdot z_{n-1}^{-\gamma}} \cdot A^{\frac{p(\alpha+\beta+\gamma-1)+\delta}{q}}, \quad \text{para todo} \ \ p \in \mathbb{R}.$$

En particular, suponiendo que $\alpha + \beta + \gamma - 1 \neq 0$, la ecuación es topológicamente conjugada a:

- $z_{n+1} = \frac{\max\{z_n^{2\alpha}, B\}}{z_n^{\alpha-\beta} \cdot z_{n-1}^{-\gamma}}$, *para todo $B > 1$, si $\frac{\delta}{\alpha+\beta+\gamma-1} > 0$.*
- $z_{n+1} = \frac{\max\{z_n^{2\alpha}, C\}}{z_n^{\alpha-\beta} \cdot z_{n-1}^{-\gamma}}$, *para todo $0 < C < 1$, si $\frac{\delta}{\alpha+\beta+\gamma-1} < 0$.*

Además, si $\alpha+\beta+\gamma-1 = 0$, adicionalmente obtenemos que la ecuación es topológicamente conjugada a:

- $z_{n+1} = \frac{\max\{z_n^{2\alpha}, 1\}}{z_n^{\alpha-\beta} \cdot z_{n-1}^{-\gamma}} \cdot B$, *para todo $B > 1$, si $\frac{\delta}{\alpha} > 0$.*
- $z_{n+1} = \frac{\max\{z_n^{2\alpha}, 1\}}{z_n^{\alpha-\beta} \cdot z_{n-1}^{-\gamma}} \cdot C$, *para todo $0 < C < 1$, si $\frac{\delta}{\alpha} < 0$.*

La potencia de estos resultados radica en que, a grandes rasgos, hemos establecido una equivalencia entre una única ecuación generalizada de Lozi, en principio más sencilla de analizar por ser una ecuación en diferencias lineal a trozos, con toda una familia de ecuaciones tipo-max, por lo que el conocimiento de la dinámica de una única ecuación generalizada de Lozi será suficiente para conocer el comportamiento de las soluciones de toda una familia de ecuaciones tipo-max.

En particular, tomando $\alpha = -a$, $\beta = 0$, $\gamma = b$ y $\delta = 1$, podemos recuperar la Ecuación de Lozi, por lo que la conjugación topológica también es aplicable a esta ecuación. En esta dirección, hemos considerado el caso $a = b$, del que ya se conocía la dinámica de la ecuación para $a \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, y hemos determinado su dinámica en los casos extremos $a = \frac{1}{2}$ y $a = -\frac{1}{2}$.

Teorema H. *Dada la ecuación en diferencias*

$$x_{n+1} = 1 - \frac{1}{2}|x_n| + \frac{1}{2}x_{n-1},$$

su dinámica viene dada por:

(a) *Un punto de equilibrio, $\overline{x} = 1$.*

(b) *Un continuo de sucesiones 2-periódicas $(\ldots, x, y, x, y, \ldots)$ con $0 \leq x, y \leq 2$, $x + y = 2$.*

(c) *El resto de soluciones convergen a una de las soluciones 2-periódicas dadas en la Parte (b).*

Teorema I. *Dada la ecuación en diferencias*

$$x_{n+1} = 1 + \frac{1}{2}|x_n| - \frac{1}{2}x_{n-1},$$

su único punto de equilibrio $\bar{x} = 1$ *es un atractor global.*

Finalmente, el Capítulo 6 se centra en la aplicación de las ecuaciones en diferencias autónomas a un campo concreto, específicamente al de la dinámica de poblaciones. En este área se utilizan dichas ecuaciones para modelar sistemas poblacionales con generaciones no superpuestas. Concretamente, abordamos un modelo de Beddington generalizado de tipo huésped-parásito con una función de escape de parasitismo arbitraria, profundizando así en los resultados ya existentes en la literatura. El modelo que analizamos es el siguiente:

$$\begin{cases} H_{n+1} &= aH_n e^{r(1-H_n)} f(bP_n) \\ P_{n+1} &= cH_n\big(1 - f(P_n)\big) \end{cases}, \tag{2}$$

donde los parámetros a, b, c, r son números reales positivos; H_n y P_n representan el tamaño de las poblaciones de huésped y de parásito en la generación n, respectivamente; los tamaños iniciales de ambas poblaciones son $H_0, P_0 \geq 0$. Además, la función de probabilidad f debe satisfacer una serie de condiciones que surgen de manera natural de la relación entre huéspedes y parásitos, en concreto, debe cumplir

$$f \in C[0,\infty) \cap C^4(0,\infty),\ f(y) > 0,\ f'(y) < 0,\ f''(y) \geq 0 \text{ para } y > 0,$$
$$\lim_{y \to 0^+} yf'(y) = 0, f(0) = 1, \text{y } f(\infty) = 0.$$

Nuestra aportación consiste en estudiar la dinámica de dicho modelo centrándonos en diversas propiedades como la acotación, la estabilidad de los puntos de equilibrio, la ocurrencia de bifurcaciones o la permanencia del sistema, [14]. Cabe mencionar que hemos considerado una función de probabilidad general f en lugar de una específica. Además, no solo comprobamos la existencia de bifurcaciones numéricamente, sino que se prueban varios escenarios de bifurcación analíticamente, avanzando así en los resultados ya existentes en la literatura.

En primer lugar, vemos que ambas poblaciones son uniformemente acotadas. Además, establecemos las condiciones para la existencia de hasta tres puntos de equilibrio: de extinción, $E_0 = (0, 0)$, que siempre existe; de exclusión, $E_0^* = \left(1 + \frac{\ln(a)}{r}, 0\right)$; y de coexistencia, $E^* = (H^*, P^*)$, con $H^*, P^* > 0$. Para dichos puntos analizamos su estabilidad local e incluso proporcionamos resultados globales para los puntos de equilibrio de extinción y exclusión, Teoremas 22 y 26, respectivamente. En el caso del punto de equilibrio interior solo establecemos resultados locales, pues el no poder determinar una expresión para dicho punto dificulta enormemente la obtención de resultados globales. El resultado que establece la casuística sobre la estabilidad local de E^* es el siguiente:

Teorema J. *Supongamos que el punto de equilibrio de coexistencia $E^* = (H^*, P^*)$ existe. Entonces,*

(a) *E^* es localmente asintóticamente estable si y solo si*

$$rH^* + cH^*f'(P^*) - 2 < -cH^*f'(P^*)(1 - rH^*) - \frac{c\big(1 - f(P^*)\big)bH^*f'(bP^*)}{f(bP^*)} < 1.$$

(b) *E^* es un repulsor si y solo si*

$$rH^* + cH^*f'(P^*) - 2 < -cH^*f'(P^*)(1 - rH^*) - \frac{c\big(1 - f(P^*)\big)bH^*f'(bP^*)}{f(bP^*)}$$

y

$$1 < -cH^*f'(P^*)(1 - rH^*) - \frac{c\big(1 - f(P^*)\big)bH^*f'(bP^*)}{f(bP^*)}.$$

(c) *E^* es un punto de silla si y solo si*

$$-cH^*f'(P^*)(1 - rH^*) - \frac{c\big(1 - f(P^*)\big)bH^*f'(bP^*)}{f(bP^*)} < rH^* + cH^*f'(P^*) - 2.$$

(d) *E^* es no hiperbólico si y solo si*

$$2 - cH^*f'(P^*)(2 - rH^*) - rH^* - \frac{c\big(1 - f(P^*)\big)bH^*f'(bP^*)}{f(bP^*)} = 0,$$

o

$$rH^* + cH^*f'(P^*) \leq 3 \quad \text{and} \quad - cH^*f'(P^*)(1 - rH^*) - \frac{c\big(1 - f(P^*)\big)bH^*f'(bP^*)}{f(bP^*)} = 1.$$

Seguidamente, probamos la aparición de bifurcaciones de tipo *duplicación de periodo* y *transcrítica* para el punto de equilibrio de exclusión; y para el de coexistencia, la ocurrencia de bifurcaciones de tipo *duplicación de periodo* y *Neimark-Sacker*. Este último caso conlleva un cambio de estabilidad en el punto de equilibrio que origina la creación de una curva cerrada invariante que puede ser estable o inestable.

Teorema K. *Supongamos que se verifican todas las hipótesis del Lema 69. Sean $E^* = (H^*, P^*)$, $d(r_0)$ y $\alpha(r_0)$ dados por (6.6), (6.40) y (6.44), respectivamente. Entonces, el punto de equilibrio interior o de coexistencia $E^* = (H^*, P^*)$ experimenta una bifurcación de Neimark–Sacker en $r = r_0$.*

- *Si $d(r_0) > 0$ y $\alpha(r_0) > 0$ ($\alpha(r_0) < 0$) entonces el Sistema (2) tiene una curva invariante cerrada atractora (inestable) cuando $r > r_0$ ($r < r_0$) y $r \approx r_0$.*

- *Si $d(r_0) < 0$ y $\alpha(r_0) > 0$ ($\alpha(r_0) < 0$) entonces el Sistema (2) tiene una curva invariante cerrada atractora (inestable) cuando $r < r_0$ ($r > r_0$) y $r \approx r_0$.*

La última propiedad dinámica estudiada en este modelo es la de permanencia, donde hemos sido capaces de establecer las condiciones necesarias para que el sistema sea permanente. Estas las hemos proporcionado en dos teoremas distintos, pues difieren en función de si $1 < ae^r < e^2$ o si $ae^r \geq e^2$.

Teorema L. *Si* $1 < ae^r < e^2$ *y* $-c\left(1+\frac{\ln(a)}{r}\right)f'_+(0) > 1$, *entonces el Sistema* (2) *es permanente.*

Teorema M. *Supongamos que* $ae^r \geq e^2$ *y* $-cf'_+(0) > e^{-A\left(\frac{\ln(a)}{r}+1\right)-B}$, *donde* A *y* B *vienen dados por* (6.46) *con* $\rho_0 = h\left(\frac{ae^{r-1}}{r}\right)$, *siendo* h *la aplicación de Ricker,* $h(x) = axe^{r(1-x)}$, $x \geq 0$. *Entonces, el Sistema* (2) *es permanente.*

Por útlimo, desarrollamos simulaciones numéricas para funciones de probabilidad concretas que ilustran todos los resultados obtenidos. En concreto, las simulaciones realizadas han sido para las funciones $f(y) = \left(1+\frac{y}{m}\right)^{-m}$, con $m > 0$; y $f(y) = e^{-y^m}$, con $0 < m \leq 1$.

En definitiva, la presente disertación recoge un amplio estudio de diferentes propiedades dinámicas de las ecuaciones en diferencias autónomas a través de diversas técnicas matemáticas, ilustrando así la riqueza de su dinámica, el alcance de sus aplicaciones y avanzando en diversos problemas y cuestiones a través de los nuevos resultados que hemos proporcionado.

Abstract

Difference equations are a very useful tool for modeling a wide variety of phenomena where time evolves discretely. In this regard, it is easy to find countless models in various fields such as biology, physics, economics, or engineering, where these types of equations describe the behaviour of such phenomena. Therefore, the study of their dynamical properties is crucial to understand how the solutions of these models will behave in the long term, which can exhibit a rich variety of properties, from simpler behaviours such as the periodic character of the solutions or their convergence, to more complex characteristics such as the existence of strange attractors or chaos.

The main objective of this dissertation is to analyze different dynamical properties associated with autonomous difference equations and to contribute with some advances in certain open questions on the topic. In this direction, in the chapters of this thesis, we address the following dynamical properties: convergence, global periodicity, periodicity, accumulation point sets, invariance, attraction, topological conjugacies, stability, bifurcations and permanence. It should be highlighted that we have conducted an extensive review of the existing literature and have analyzed and implemented various techniques ranging from simpler issues of mathematical analysis to more complex ones inherent to discrete dynamical systems.

Next, we briefly summarize the study developed in the different chapters regarding the various dynamical properties analyzed and indicate our main contributions in each case.

Chapter 1 covers the basic concepts and results related to the diverse issues that will be addressed throughout the thesis.

Chapter 2 starts from the well-known property that every bounded monotonic sequence of real numbers is convergent. This result was generalized by E.T. Copson by exchanging the monotonic condition with a convex inequality between the terms of the sequence. Later on, Copson's property was extended to other fields beyond sequences of real numbers (see [63] for a general overview about Copson's result and its different generalizations and extensions). Our main contribution, included in [64], consists of a generalization of the Monotone Convergence Theorem of Lebesgue where we have applied the techniques developed in Copson's work.

Theorem A. *Let $(\mathbb{X}, \Sigma, \mu)$ be a measure space and let (f_n) be a sequence of measurable non-negative functions, $f_n : \mathbb{X} \to [0, \infty]$. If the sequence verifies*

$$f_{n+k}(x) \geq \sum_{j=1}^{k} \alpha_j f_{n+k-j}(x), \quad \text{for all } x \in \mathbb{X}, \tag{3}$$

where the coefficients α_j are strictly positive real numbers satisfying that $\sum_{j=1}^{k} \alpha_j = 1$, then there exists a measurable function f such that

$$\lim_{n\to\infty} f_n(x) = f(x), \quad \text{for all } x \in \mathbb{X},$$

and

$$\lim_{n\to\infty} \int_{\mathbb{X}} f_n \; d\mu = \int_{\mathbb{X}} f \; d\mu.$$

Furthermore, if we suppose the reverse inequality, $f_{n+k}(x) \leq \sum_{j=1}^{k} \alpha_j f_{n+k-j}(x)$, for all $x \in \mathbb{X}$ and the additional hypothesis of being $f_1, \ldots, f_k \in L^1(\mu)$, the result remains true.

Furthermore, we illustrate with an example the usefulness of such generalization through a non-monotonic sequence of functions that, whereas, verifies the corresponding inequality of Copson's type.

Straightaway, we leave the scenario of sequences of functions to focus on our main target of study, autonomous difference equations. Chapter 3 deals with the study of global periodicity, a dynamical property that, roughly speaking, consists in that every solution of the equation is periodic. Even more, if p is the minimum common multiple of every period, we will say that the equation is a p-cycle. For the study of such property, we focus on the third order equation

$$x_{n+3} = x_i f(x_j, x_k),$$

where $i, j, k \in \{n, n+1, n+2\}$ are pairwise distinct, $f : (0,\infty)^2 \to (0, \infty)$ is continuous and the initial conditions are positive real numbers.

In the literature we can find a characterization of the p-cycles exhibiting such form when $p \leq 5$, see [7]. In this direction, we have deepen in the study of the existence of 6-cycles and we have provided some advances, where the techniques employed are based on the computation of the orbits of the solutions, arguments of real analysis and the resolution of functional equations. On the one hand, in [68] we show the non-existence of 6-cycles when we assume the additional condition of symmetry for f, that is, $f(x, y) = f(y, x)$ for every $x, y > 0$.

Theorem B. *There are no 6-cycles of third order having the form $x_{n+3} = x_i f(x_j, x_k)$, whenever $f : (0,\infty)^2 \to (0, \infty)$ is a symmetric continuous map, and $i, j, k \in \{n, n+1, n+2\}$ are pairwise distinct.*

On the other hand, in the case of separation of variables, $f(x,y) = g(x)h(y)$, for every $x, y > 0$, where $g, h : (0,\infty) \to (0,\infty)$ are continuous maps, in [70] we present the proof of that $x_{n+3} = x_n \left(\frac{x_{n+2}}{x_{n+1}}\right)^2$ is the unique 6-cycle that exists.

Theorem C. *Consider the difference equation*

$$x_{n+3} = x_i g(x_j) h(x_k),$$

where $i, j, k \in \{n, n+1, n+2\}$ *are pairwise distinct, and* $g, h : (0,\infty) \to (0,\infty)$ *are continuous maps. The unique* 6*-cycle displaying such form is given by*

$$x_{n+3} = x_n \left(\frac{x_{n+2}}{x_{n+1}}\right)^2.$$

As a next step, in Chapter 4 we deal with the study of periodicity for the solutions of autonomous difference equations. To achieve that, we focus on the family of max-type difference equations (that is, difference equations in which recurrence law appears the maximum operator), since it is a suitable class of equations to do the transition from global periodicity to periodicity: the family presents various examples of globally periodic equations and multiple models whose solutions are either periodic or eventually periodic. In this sense, in the survey [69] we gather various results from the literature concerning this class of equations focusing on the diverse techniques and the richness of their dynamics.

In this direction, we analyze the 4-order max-type equation

$$x_{n+4} = \max\{x_{n+3}, x_{n+2}, x_{n+1}, 0\} - x_n, \tag{4}$$

with arbitrary real initial conditions. Firstly, in [65] we give a complete description of its set of periods, $\text{Per}(F_4)$, employing techniques from combinatorial dynamics where we have been able to describe the evolution of the orbits of the solutions through a graph.

Theorem D. *Consider Equation* (4) *and let* $\text{Per}(F_4)$ *be its set of periods. Then*

$$\text{Per}(F_4) = \{1, 8, 11\} \bigcup \big\{10 \cdot a + 11 \cdot b \mid \gcd(a,b) = 1, a \geq 1, b \geq 2a+1\big\}.$$

Furthermore, we do not only prove that the set of periods is unbounded, but, throughout basic congruence's techniques and general algebra, we see that 1674 is the greatest natural number that does not belong to this set, see [66].

Regarding non-periodic solutions, in [62] we fully determine their accumulation point sets that, indeed, are compact intervals of the real line.

Theorem E. *Let* (x_1, x_2, x_3, x_4) *be arbitrary real initial conditions that generate a non-periodic orbit* (x_n) *under Equation* (4)*. Then, the set of accumulation points of* (x_n) *is a compact interval. Even more, the tuple* (x_1, x_2, x_3, x_4) *is equivalent to some tuple of initial conditions* (x, y, z, w)*, with* $x = \max\{x_n : n \geq 1\}$*,* $x \geq w \geq y \geq z \geq 0$*, and* $\frac{w-z}{x} \in \mathbb{R} \setminus \mathbb{Q}$*, and the orbit accumulates in the compact interval* $\big[\min\{w-x, -z\}, x\big]$*.*

In this way, the chapter not only focuses on the analysis of the dynamical property of periodicity, but also deals with other issues like accumulation point sets and the invariance of the equation, for which we give a new first integral for the associate dynamical system of Equation (4).

After studying max-type difference equations, in Chapter 5 we establish a new relation between this class of equations and the well-known *Lozi map*,

$$x_{n+1} = 1 - a|x_n| + bx_{n-1},$$

where the parameters a, b and the initial conditions are real numbers.

In concrete, based on the work [67], we propose a generalization of such equation

$$y_{n+1} = \alpha|y_n| + \beta y_n + \gamma y_{n-1} + \delta,$$

where α, β, γ and δ are real numbers with $\alpha \neq 0$, which we call *generalized Lozi map*, and we show that such generalization is topologically conjugate to a whole family of max-type difference equations.

Theorem F. *Consider the generalized Lozi map with $\delta = 0$,*

$$y_{n+1} = \alpha|y_n| + \beta y_n + \gamma y_{n-1}.$$

Then, it is topologically conjugate to the max-type equation

$$z_{n+1} = \frac{\max\{z_n^{2\alpha}, B\}}{z_n^{\alpha-\beta} \cdot z_{n-1}^{-\gamma}} \cdot B^{\frac{\alpha+\beta+\gamma-1}{-2\alpha}},$$

for all $B > 0$.

In particular, if $\alpha+\beta+\gamma-1 = 0$, then the generalized Lozi map is topologically conjugate to the max-type equation

$$z_{n+1} = \frac{\max\{z_n^{2\alpha}, B\}}{z_n^{\alpha-\beta} \cdot z_{n-1}^{-\gamma}},$$

for all $B > 0$.

Theorem G. *Consider the generalized Lozi map with $\delta \neq 0$,*

$$y_{n+1} = \alpha|y_n| + \beta y_n + \gamma y_{n-1} + \delta.$$

Then, either for $A > 1$ and $\frac{\alpha}{q} > 0$, or for $0 < A < 1$ and $\frac{\alpha}{q} < 0$, with $q \in \mathbb{R} \setminus \{0\}$, the equation is topologically conjugate to

$$z_{n+1} = \frac{\max\{z_n^{2\alpha}, A^{\frac{-2\alpha p}{q}}\}}{z_n^{\alpha-\beta} \cdot z_{n-1}^{-\gamma}} \cdot A^{\frac{p(\alpha+\beta+\gamma-1)+\delta}{q}}, \quad \text{for all} \ \ p \in \mathbb{R}.$$

In particular, assuming that $\alpha + \beta + \gamma - 1 \neq 0$, the equation is topologically conjugate to:

- $z_{n+1} = \frac{\max\{z_n^{2\alpha}, B\}}{z_n^{\alpha-\beta} \cdot z_{n-1}^{-\gamma}}$, *for all* $B > 1$, *if* $\frac{\delta}{\alpha+\beta+\gamma-1} > 0$.
- $z_{n+1} = \frac{\max\{z_n^{2\alpha}, C\}}{z_n^{\alpha-\beta} \cdot z_{n-1}^{-\gamma}}$, *for all* $0 < C < 1$, *if* $\frac{\delta}{\alpha+\beta+\gamma-1} < 0$.

Moreover, if $\alpha+\beta+\gamma-1 = 0$, *additionally we get that the equation is topologically conjugate to:*

- $z_{n+1} = \frac{\max\{z_n^{2\alpha}, 1\}}{z_n^{\alpha-\beta} \cdot z_{n-1}^{-\gamma}} \cdot B$, *for all* $B > 1$, *if* $\frac{\delta}{\alpha} > 0$.
- $z_{n+1} = \frac{\max\{z_n^{2\alpha}, 1\}}{z_n^{\alpha-\beta} \cdot z_{n-1}^{-\gamma}} \cdot C$, *for all* $0 < C < 1$, *if* $\frac{\delta}{\alpha} < 0$.

The power of this results lie in that, widely speaking, we have established an equivalence between a unique generalized Lozi map, initially simpler to analyze since it is a piecewise linear difference equation, with a whole family of max-type equations, so the knowledge of the dynamics of a unique generalized Lozi map is enough to know the behaviour of the solutions of a whole family of max-type difference equations.

In particular, taking $\alpha = -a$, $\beta = 0$, $\gamma = b$ and $\delta = 1$, we recover the Lozi map, so the topological conjugation is applicable for this equation too. In this direction, we have considered the case $a = b$, for which the dynamics of the equation for $a \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ was already known, and we have determined its dynamics in the boundary cases $a = \frac{1}{2}$ and $a = -\frac{1}{2}$.

Theorem H. *Given the difference equation*

$$x_{n+1} = 1 - \frac{1}{2}|x_n| + \frac{1}{2}x_{n-1},$$

its dynamics is given by:

(a) *An equilibrium point,* $\overline{x} = 1$.

(b) *A continuum of* 2*-periodic sequences* $(\ldots, x, y, x, y, \ldots)$ *with* $0 \leq x, y \leq 2$, $x + y = 2$.

(c) *The rest of solutions converge to one of the* 2*-periodic solutions given in Part (b).*

Theorem I. *Given the difference equation*

$$x_{n+1} = 1 + \frac{1}{2}|x_n| - \frac{1}{2}x_{n-1},$$

its unique equilibrium point $\bar{x} = 1$ *is a global attractor.*

Finally, Chapter 6 delves into the application of autonomous difference equations to a concrete field, specifically to population dynamics. In this area such equations are employed to model population systems with non-overlapping generations. In concrete, we tackle with a generalized Beddington host-parasitoid model with an arbitrary parasitism escape function, thus delving deeper into the results already existing in the literature. The analyzed model is the following:

$$\begin{cases} H_{n+1} &= aH_n e^{r(1-H_n)} f(bP_n) \\ P_{n+1} &= cH_n\big(1-f(P_n)\big) \end{cases}, \tag{5}$$

where the parameters a, b, c, r are positive real numbers; H_n and P_n represent the size of the host and parasitoid's populations at generation n, respectively; the initial sized of both populations are $H_0, P_0 \geq 0$. Moreover, the probability function f must satisfy some conditions that arise naturally from the relation of hosts and parasitoids, in concrete, it must satisfy

$$f \in C[0,\infty) \cap C^4(0,\infty),\ f(y) > 0,\ f'(y) < 0,\ f''(y) \geq 0 \text{ for } y > 0,$$
$$\lim_{y \to 0^+} yf'(y) = 0, f(0) = 1, \text{and } f(\infty) = 0.$$

Our contribution consists in studying the dynamics of this model focusing on various properties such as boundedness, the stability of the equilibrium points, the occurrence of bifurcations or the permanence of the system, [14]. It should be mentioned that we have considered a general probability function f instead of a specific one. Moreover, we do not only show numerically the existence of bifurcations, but we proof analytically various bifurcation scenarios, thus advancing on the existing results in the literature.

Firstly, we see that both populations are uniformly bounded. Furthermore, we establish the conditions for the existence of up to three equilibrium points: extinction, $E_0 = (0,0)$, which always exists; exclusion, $E_0^* = \left(1 + \frac{\ln(a)}{r}, 0\right)$; and coexistence, $E^* = (H^*, P^*)$, with $H^*, P^* > 0$. For such points we analyze their local stability and we even give global results for the extinction and exclusion equilibrium points, Theorems 22 and 26, respectively. For the interior equilibrium we only establish local results, since the impossibility of determining an expression for such point complicates enormously the achievement of global results. The result that establishes the casuistic concerning the local stability of E^* is the following:

Theorem J. *Assume that the coexistence equilibrium $E^* = (H^*, P^*)$ exists. Then,*

(a) E^* is locally asymptotically stable if and only if

$$rH^* + cH^*f'(P^*) - 2 < -cH^*f'(P^*)(1 - rH^*) - \frac{c\big(1 - f(P^*)\big)bH^*f'(bP^*)}{f(bP^*)} < 1.$$

(b) E^* is a repeller if and only if

$$rH^* + cH^*f'(P^*) - 2 < -cH^*f'(P^*)(1 - rH^*) - \frac{c\big(1 - f(P^*)\big)bH^*f'(bP^*)}{f(bP^*)}$$

and

$$1 < -cH^*f'(P^*)(1 - rH^*) - \frac{c\big(1 - f(P^*)\big)bH^*f'(bP^*)}{f(bP^*)}.$$

(c) E^* *is a saddle if and only if*

$$-cH^*f'(P^*)(1-rH^*) - \frac{c\big(1-f(P^*)\big)bH^*f'(bP^*)}{f(bP^*)} < rH^* + cH^*f'(P^*) - 2.$$

(d) E^* *is non-hyperbolic if and only if*

$$2 - cH^*f'(P^*)(2-rH^*) - rH^* - \frac{c\big(1-f(P^*)\big)bH^*f'(bP^*)}{f(bP^*)} = 0,$$

or

$$rH^* + cH^*f'(P^*) \leq 3 \quad \textit{and} \quad -cH^*f'(P^*)(1-rH^*) - \frac{c\big(1-f(P^*)\big)bH^*f'(bP^*)}{f(bP^*)} = 1.$$

Next, we prove the occurrence of bifurcations of the type *period-doubling* and *transcritical* for the exclusion equilibrium point; and for the coexistence equilibrium, the occurrence of bifurcations of the type *period-doubling* and *Neimark-Sacker*. This last case implies a change on the stability of the equilibrium point creating a closed invariant curve that can be stable or unstable.

Theorem K. *Assume that every assumption of Lemma 69 holds. Let* $E^* = (H^*, P^*)$, $d(r_0)$ *and* $\alpha(r_0)$ *be given by* (6.6), (6.40) *and* (6.44)*, respectively. Then, the interior or coexistence equilibrium point* $E^* = (H^*, P^*)$ *undergoes a Neimark–Sacker bifurcation at* $r = r_0$.

- *Assume that* $d(r_0) > 0$*. If* $\alpha(r_0) > 0$ ($\alpha(r_0) < 0$) *then System* (5) *has an attracting (unstable) closed invariant curve when* $r > r_0$ ($r < r_0$) *and* $r \approx r_0$.
- *Assume that* $d(r_0) < 0$*. If* $\alpha(r_0) > 0$ ($\alpha(r_0) < 0$) *then System* (5) *has an attracting (unstable) closed invariant curve when* $r < r_0$ ($r > r_0$) *and* $r \approx r_0$.

The last studied dynamical property in this model is permanence, where we have been able to established necessary conditions for the system to be permanent. We have given such conditions in two different theorems, since they differ depending on whether $1 < ae^r < e^2$ or $ae^r \geq e^2$.

Theorem L. *If* $1 < ae^r < e^2$ *and* $-c\left(1 + \frac{\ln(a)}{r}\right) f'_+(0) > 1$*, then System* (5) *is permanent.*

Theorem M. *Assume that* $ae^r \geq e^2$ *and* $-cf'_+(0) > e^{-A\left(\frac{\ln(a)}{r}+1\right)-B}$*, where* A *and* B *are given by* (6.46) *with* $\rho_0 = h\left(\frac{ae^{r-1}}{r}\right)$*,* h *being the Ricker map,* $h(x) = axe^{r(1-x)}$*,* $x \geq 0$*. Then, System* (5) *is permanent.*

Finally, we develop numerical simulations with concrete probability functions that show the obtained results. Specifically, the simulations have been made for the functions $f(y) = \left(1 + \frac{y}{m}\right)^{-m}$, with $m > 0$; and $f(y) = e^{-y^m}$, with $0 < m \leq 1$.

In summary, the present dissertation gathers a comprehensive study on different dynamical properties of autonomous difference equations through diverse mathematical techniques, thus illustrating the richness of their dynamics, the scope of their applications and advancing on various problems and questions through the new results that we have obtained.

Chapter 1

Basic results and notions

The main target of this chapter is to establish the basic notions related to the field of autonomous difference equations and systems of autonomous difference equations that will be used along the dissertation. The definitions, properties and results gathered here are based on [34, 40, 41, 43, 59, 93].

Firstly, we focus on the basic notions related to difference equations, namely, equilibrium points, stability, periodicity, topological conjugacies and invariants. We develop them in Sections 1.1-1.3. Then, we deal with systems of first-order difference equations. In Section 1.4, we collect the definitions concerning equilibrium points and their local stability. Next, Section 1.5 establishes the main concepts related to invariant manifolds. Section 1.6 presents the principal properties of the well-known Ricker map, $h(x) = axe^{r(1-x)}$; and finally, Section 1.7 is devoted to the occurrence of bifurcations.

1.1 Equilibrium points and stability for autonomous difference equations

An autonomous difference equation of k-order is an equation of the form

$$x_{n+k} = f(x_{n+k-1}, \dots, x_{n+1}, x_n), \quad n \geq 1, \tag{1.1}$$

where $f : \Omega \subseteq X^k \to X$ is defined on some subset Ω of a finite Cartesian product of a set X. A sequence (x_n) which satisfies Equation (1.1) for every $n \geq 1$ is called a solution of the autonomous difference equation. Furthermore, if we specify a set of initial conditions $(x_1, x_2, \dots, x_k) \in X^k$, they generate a unique solution by Equation (1.1). In what follows, every autonomous difference equation studied is determined by a map f with $X = \mathbb{R}$. For this reason, every notion and property collected in this chapter will be defined in such space.

A constant solution of (1.1), that is,

$$x_n = x^*, \quad \text{for every } n \geq 1,$$

is called an equilibrium solution of (1.1); in this case x^* is called an equilibrium point for the equation. Equivalently, an equilibrium point satisfies $x^* = f(x^*, \dots, x^*)$.

Concerning the notions of stability of equilibrium points we have the following definitions:

Definition 1. *Let x^* be an equilibrium point of Equation (1.1) and (x_n) denotes a solution of (1.1). Then, we say that*

- *x^* is locally stable if for every $\varepsilon > 0$, there exists $\delta > 0$ such that if*

$$|x_j - x^*| < \delta, \;\; j = 1, \dots, k,$$

then

$$|x_n - x^*| < \varepsilon \;\; \textit{for all} \;\; n \geq 1.$$

- *x^* is locally asymptotically stable if it is stable and, if in addition, there exists $\gamma > 0$ such that*

$$|x_j - x^*| < \gamma, \;\; j = 1, \dots, k,$$

then

$$\lim_{n \to \infty} x_n = x^*.$$

- *x^* is a global attractor if for every solution (x_n), we have*

$$\lim_{n \to \infty} x_n = x^*.$$

- *x^* is globally asymptotically stable if it is locally asymptotically stable and a global attractor.*

- *x^* is unstable if it is not locally stable.*

To study the local stability character of an equilibrium point x^*, if the map f that defines Equation (1.1) is continuously differentiable in some open neighbourhood of x^*, we can apply the so-called *Linearized Stability Theorem*. To do so, we need to introduce the linearized equation and its corresponding characteristic equation linked to Equation (1.1). In this sense, let

$$p_i = \frac{\partial f}{\partial x_i}(x^*, \dots, x^*), \quad \text{for } i = 1, \dots, k,$$

denote the partial derivative of $f(x_1, \dots, x_k)$ with respect to x_i evaluated at x^*. Thus, the linearized equation of Equation (1.1) about the equilibrium point x^* is given by

$$y_{n+k} = p_1 y_{n+k-1} + p_2 y_{n+k-2} + \dots + p_k y_n, \tag{1.2}$$

and its corresponding characteristic equation about x^* is given by

$$\lambda^k - p_1 \lambda^{k-1} - \dots - p_{k-1}\lambda - p_k = 0. \tag{1.3}$$

Theorem 1. *[33, **Theorem 5.15**] **Linearized Stability Theorem:** Consider the autonomous difference equation (1.1). Let x^* be an equilibrium point of the equation and assume that f is continuously differentiable in some open neighbourhood of x^*. Then the following statements hold:*

- *If every root of Equation (1.3) lies inside the unit disk in the complex plane, then x^* is locally asymptotically stable.*
- *If at least one root of Equation (1.3) is outside the unit disk in the complex plane, then x^* is unstable.*
- *If one root of Equation (1.3) is on the unit disk and all the other roots are either inside or on the unit disk, then x^* can be either stable, unstable or locally asymptotically stable.*

If every root of the characteristic equation (1.3) does not lie on the unit disk, we say that x^* is hyperbolic; on the contrary, we say that the equilibrium is non-hyperbolic. Moreover, for an hyperbolic equilibrium, if there exists a root of Equation (1.3) lying inside the unit disk and another root lying outside, we say that x^* is a saddle point. Notice that a saddle point is unstable.

In the particular case of autonomous difference equations of first order, $x_{n+1} = f(x_n)$, there are results that allow us to determine the stability of the equilibrium points when they are non-hyperbolic, $f'(x^*) = 1$ or $f'(x^*) = -1$. For the case $f'(x^*) = 1$ we have the following stability criteria.

Theorem 2. *[34, **Theorem 1.5**] Let x^* be a fixed point of a map $f : U \subset \mathbb{R} \to \mathbb{R}$ such that $f'(x^*) = 1$. If $f'(x)$, $f''(x)$ and $f'''(x)$ are continuous at x^*, then the following statements hold:*

- *If $f''(x^*) \neq 0$, then x^* is unstable.*
- *If $f''(x^*) = 0$ and $f'''(x^*) > 0$, then x^* is unstable.*
- *If $f''(x^*) = 0$ and $f'''(x^*) < 0$, then x^* is asymptotically stable.*

For the remaining case, $f'(x^*) = -1$, we have to introduce the notion of Schwarzian derivative of a function f, Sf, which is defined as

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left[\frac{f''(x)}{f'(x)}\right]^2 .$$

Theorem 3. *[34, **Theorem 1.6**] Let x^* be a fixed point of a map $f : U \subset \mathbb{R} \to \mathbb{R}$ such that $f'(x^*) = -1$. If $f'(x)$, $f''(x)$ and $f'''(x)$ are continuous at x^*, then the following statements hold:*

- *If $Sf(x^*) < 0$, then x^* is asymptotically stable.*
- *If $Sf(x^*) > 0$, then x^* is unstable.*

1.2 Periodicity

After dealing with the constant solutions of autonomous difference equations, we focus on periodic ones. We say that a solution (x_n) of Equation (1.1) is periodic with period p if there exists an integer $p \geq 1$ such that

$$x_{n+p} = x_n \quad \text{for all } n \geq 1. \tag{1.4}$$

If such p is the smallest positive integer for which (1.4) holds, we call it prime period.

Additionally, if every solution generated by the difference equation is periodic, we say that Equation (1.1) is globally periodic. In this scenario, since the difference equation is autonomous, it is known that the set of prime periods has to be bounded, see [84]. Therefore, if p is the least common multiple of every period of the periodic solutions, we will call the equation a p-cycle.

Also, a solution (x_n) of Equation (1.1) is called eventually periodic with period p if there exists an integer $N \geq 1$ such that $(x_n)_{n\geq N}$ is periodic with period p; that is, $x_{n+p} = x_n$ for every $n \geq N$. Moreover, if a solution of the equation converges to a periodic point, we say that the solution is asymptotically periodic.

1.3 Topological conjugacies and invariance

Difference equations can be associated to discrete dynamical systems. In concrete, given Equation (1.1), we can link it to the map $F : \Omega \subseteq \mathbb{R}^k \to \Omega$ given by

$$F(x_1, x_2, \ldots, x_k) = \big(x_2, \ldots, x_k, f(x_k, \ldots, x_2, x_1)\big).$$

According to this equivalence between difference equations and discrete dynamical systems, we can establish whether two difference equations are conjugate in terms of their correspondent dynamical systems. In this sense, given two metric spaces X_1, X_2 and two continuous maps $f_1 : X_1 \to X_1$, $f_2 : X_2 \to X_2$, we say that (X_1, f_1) is topologically conjugate to (X_2, f_2) if there exists a homeomorphism $h : X_1 \to X_2$ such that $h \circ f_1 = f_2 \circ h$. This property is of huge relevance since if we know the dynamics of one iterative map, we can directly determine the dynamics of the other one by the topological conjugacy. For instance, a p-periodic solution (x_n) of Equation (1.1) provides us with p-periodic points of the map F, that is, any tuple of k consecutive terms of (x_n) verifies

$$F^p(x_n, x_{n+1}, \ldots, x_{n+k-1}) = (x_n, x_{n+1}, \ldots, x_{n+k-1}) \quad \text{for every } n \geq 1.$$

In addition, we denote the set of periods of F by $\mathrm{Per}(F)$

$$\mathrm{Per}(F) = \{n \in \mathbb{N} : F^n(\boldsymbol{x}) = \boldsymbol{x} \text{ for some } \boldsymbol{x} \in \Omega \text{ and } F^j(\boldsymbol{x}) \neq \boldsymbol{x} \text{ for } 0 < j < n\}.$$

To illustrate the above notions, let us consider the 4-cycle of third order

$$x_{n+3} = \frac{x_n x_{n+2}}{x_{n+1}},$$

whose solutions are of the form $\left(\alpha, \beta, \delta, \frac{\alpha \cdot \delta}{\beta}\right)$, and take the homeomorphism $h(x) = e^x$. Then, the linear difference equation $y_{n+3} = y_{n+2} - y_{n+1} + y_n$ is also a 4-cycle whose solutions are of the form $(\tilde{\alpha}, \tilde{\beta}, \tilde{\delta}, \tilde{\delta} - \tilde{\beta} + \tilde{\alpha})$.

We say that Equation (1.1) possesses an invariant if there exists a non-trivial function $G : \Omega \subseteq \mathbb{R}^k \to \mathbb{R}$ such that every solution (x_n) of the equation verifies

$$G(x_n, x_{n+1}, \dots, x_{n+k-1}) = G(x_1, \dots, x_k) \text{ for all } n \geq 1.$$

For instance, $G(x, y) = (1+x+y)\left(1 + \frac{1}{x}\right)\left(1 + \frac{1}{y}\right)$ is an invariant for the well-known Lyness' Equation $x_{n+2} = \frac{1+x_{n+1}}{x_n}$. Analogously, for a discrete dynamical system, we call a first integral to a non-constant function in a nonempty open set $\mathcal{U} \subseteq \mathbb{R}^k$, $V : \mathcal{U} \to \mathbb{R}^k$, which is constant on the orbits, i.e.,

$$V\big(F(\boldsymbol{x})\big) = F(\boldsymbol{x}) \text{ for all } \boldsymbol{x} \in \mathcal{U}.$$

A set $V_1, \dots, V_k$ of first integrals of G defined in an open set $\mathcal{U}$ are *functionally dependent* if there exists a real-valued function $R : \mathcal{U} \to \mathbb{R}$ not identically zero such that $R\big(V_1(\boldsymbol{x}), \dots, V_k(\boldsymbol{x})\big) = 0$ for all $\boldsymbol{x} \in \mathcal{U}$. Otherwise, we say that they are *functionally independent*, [88, pages 84–85]. Also, we will say that G is *completely integrable* if it has n functionally independent first integrals.

Following the above example, for the discrete dynamical system associated to Lyness' Equation, $F(x, y) = \left(y, \frac{1+y}{x}\right)$, we have the following first integral:

$$V(x, y) = \frac{(1+x)(1+y)(1+x+y)}{xy}.$$

1.4 Equilibrium points and stability for systems of first-order difference equations

Beyond autonomous difference equations, we will deal with systems of first-order difference equations in some chapters of this dissertation. In this section, we establish the basic notions and properties concerning the stability of the equilibrium points of a system.

A k-dimensional system of first-order difference equations is a system of the form

$$\begin{cases} x^1_{n+1} &= f_1\big(x^1_n, x^2_n, \dots, x^k_n\big) \\ x^2_{n+1} &= f_2\big(x^1_n, x^2_n, \dots, x^k_n\big) \\ \dots & \\ x^k_{n+1} &= f_k\big(x^1_n, x^2_n, \dots, x^k_n\big) \end{cases}, \quad n \geq 1, \tag{1.5}$$

where $f_i : \Omega \subseteq \mathbb{R}^k \to \mathbb{R}$. For each $\boldsymbol{x}_1 \in \mathbb{R}^k$ there exists a unique solution $\boldsymbol{x}_n \in \mathbb{R}^k$ of System (1.5) with initial conditions $\boldsymbol{x}_1$. Such system can be associated to a map $T : \Omega \subseteq \mathbb{R}^k \to \mathbb{R}^k$ defined by

$$T\begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_k \end{pmatrix} = \begin{pmatrix} f_1(x_1, \ldots, x_k) \\ f_2(x_1, \ldots, x_k) \\ \cdots \\ f_k(x_1, \ldots, x_k) \end{pmatrix}. \tag{1.6}$$

An equilibrium point for the System (1.5), or equivalently, a fixed point for the map T is a point $\boldsymbol{x}^* = \big(x_1^*, x_2^*, \ldots, x_k^*\big)$ which verifies

$$T\big(x_1^*, x_2^*, \ldots, x_k^*\big) = \big(x_1^*, x_2^*, \ldots, x_k^*\big).$$

To establish the basic notions of stability concerning the equilibrium points of a system of difference equations, we need to introduce the notions of norms.

Definition 2. *A real-valued function on a vector space V is called a norm, $||\cdot||$, if the following properties hold:*

- $||\boldsymbol{x}|| \geq 0$ *for every* $\boldsymbol{x} \in V$ *and* $||\boldsymbol{x}|| = 0$ *if and only if* $\boldsymbol{x} = 0$*;*
- $||\alpha \cdot \boldsymbol{x}|| = |\alpha| \cdot ||\boldsymbol{x}||$ *for every* $\boldsymbol{x} \in V$ *and scalars* α*;*
- $||\boldsymbol{x} + \boldsymbol{y}|| \leq ||\boldsymbol{x}|| + ||\boldsymbol{y}||$ *for every* $\boldsymbol{x}, \boldsymbol{y} \in V$*.*

In $\mathbb{R}^k$ the most commonly used norms are:

(i) the l_1 norm: $||\boldsymbol{x}||_1 = \sum_{i=1}^{k} |x_i|$;

(ii) the l_∞ norm: $||\boldsymbol{x}||_\infty = \max_{1\leq i\leq k} |x_i|$;

(iii) the Euclidean norm l_2: $||\boldsymbol{x}||_2 = \left(\sum_{i=1}^{k} x_i^2\right)^{1/2}$.

It is relevant to emphasize that every norm on $\mathbb{R}^k$ is equivalent in the sense that given two norms, $||\cdot||$ and $||\cdot||'$, there exist constants $\alpha, \beta > 0$ such that

$$\alpha||\boldsymbol{x}|| \leq ||\boldsymbol{x}||' \leq \beta||\boldsymbol{x}||.$$

Hence, if $(\boldsymbol{x}_n)$ is a sequence in $\mathbb{R}^k$, then $||\boldsymbol{x}_n|| \to 0$ as n tends to infinity if and only if $||\boldsymbol{x}_n||' \to 0$ as n tends to infinity.

For a $k \times k$ matrix $A = (a_{ij})$, we write $\rho(A) := \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}$ to denote the spectral radius of the matrix.

Now, we present the basic notions related to stability.

Definition 3. *Let $\boldsymbol{x}^*$ be an equilibrium point of System (1.5), $\boldsymbol{x}_1 \in \mathbb{R}^k$ denote the vector of initial conditions and $(\boldsymbol{x}_n)$ a solution of the system with $\boldsymbol{x}_n \in \mathbb{R}^k$ for every $n \geq 1$.*

(i) $\boldsymbol{x}^*$ *is locally stable if given* $\varepsilon > 0$ *there exists* $\delta = \delta(\varepsilon)$ *such that* $||\boldsymbol{x}_1 - \boldsymbol{x}^*|| < \delta$ *implies* $||\boldsymbol{x}_n - \boldsymbol{x}^*|| < \varepsilon$ *for all* $n \geq 1$*. If the equilibrium is not stable, we say it is unstable.*

(ii) $\boldsymbol{x}^*$ *is attracting if there exists* μ *such that* $||\boldsymbol{x}_1 - \boldsymbol{x}^*|| < \mu$ *implies* $\lim_{n\to\infty} x_n = \boldsymbol{x}^*$.

(iii) $\boldsymbol{x}^*$ *is locally asymptotically stable if it is locally stable and attracting.*

If in parts, (ii) and (iii), $\mu = \infty$, the corresponding stability properties are said to be global.

A useful technique for the study of stability is the so-called linearization. Roughly speaking, such technique is based on the fact that the behaviour of a linear map associated to the nonlinear one determines locally its behaviour. In this direction, consider the map $T : \Omega \subseteq \mathbb{R}^k \to \mathbb{R}^k$ defined by (1.6). Then, T is continuously differentiable, or a C^1 map, if its partial derivatives exist and are continuous. We define the Jacobian matrix of T as

$$DT = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_k} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_k} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_k}{\partial x_1} & \frac{\partial f_k}{\partial x_2} & \cdots & \frac{\partial f_k}{\partial x_k} \end{pmatrix}. \tag{1.7}$$

The relevance of the Jacobian matrix of a map resides on the fact that the stability of T can be determined by the eigenvalues of the Jacobian matrix evaluated at the equilibrium point. Indeed, the linear map that maps a vector $\boldsymbol{y} \in \mathbb{R}^k$ into $DT(\boldsymbol{x}^*)(\boldsymbol{y})$ is the linear one mentioned above.

Theorem 4. *Let* $T : \Omega \subseteq \mathbb{R}^k \to \mathbb{R}^k$ *be a* C^1 *map, where* Ω *is an open subset of* $\mathbb{R}^k$*,* $\boldsymbol{x}^*$ *is a fixed point of* T *and* $DT(\boldsymbol{x}^*)$ *denotes the Jacobian matrix of* T *evaluated at the equilibrium. Then the following hold:*

- *If* $\rho\big(DT(\boldsymbol{x}^*)\big) < 1$*, then* $\boldsymbol{x}^*$ *is asymptotically stable.*
- *If* $\rho\big(DT(\boldsymbol{x}^*)\big) > 1$*, then* $\boldsymbol{x}^*$ *is unstable.*
- *If* $\rho\big(DT(\boldsymbol{x}^*)\big) = 1$*, then* $\boldsymbol{x}^*$ *may or may not be stable.*

In addition, a solution $(\boldsymbol{x}_n) \in \mathbb{R}^k$ for every $n \geq 1$ is periodic of period p, if $T^p(\boldsymbol{x}_n) = \boldsymbol{x}_n$ for every $n \geq 1$, where T^p represents the composition of the map T with itself p times. To study the stability of a p-periodic point corresponds with the study of the stability of the fixed points of the map T^p. Moreover, the Jacobian of T^p, DT^p, can be computed by multiplying the Jacobian of T p times as follows:

$$DT^p(\boldsymbol{x}) = DT\big(T^{p-1}(\boldsymbol{x})\big) \cdot DT\big(T^{p-2}(\boldsymbol{x})\big) \cdot \ldots \cdot DT\big(T(\boldsymbol{x})\big) \cdot DT(\boldsymbol{x}).$$

1.5 Invariant manifolds

In this section we focus on the notion of invariant manifold for a fixed point of a map. We establish the main concepts related to the topic and we finish with the statement of the Local Manifold Theorem, which will be used later in the dissertation to determine the local stable and unstable manifolds of the equilibrium points of a particular model whenever they are saddle points. It should be mentioned that the definitions presented here are based on the textbook [93]. However, the author deals with the matter in the framework of Banach spaces, while we have adapted it to the particular case of $\mathbb{R}^n$, our framework in this study.

Let us consider a C^k differentiable map defined on an open subset U of $\mathbb{R}^n$, $T : U \subset \mathbb{R}^n \to \mathbb{R}^n$, assume x^* is a fixed point of T and let DT_{x^*} denote the Jacobian matrix of T evaluated at x^*. For the matrix DT_{x^*}, we can divide its spectrum, i.e., its set of eigenvalues, into three sets σ_s, σ_u and σ_c, for which $\lambda \in \sigma_s$ if $|\lambda| = 1$, $\lambda \in \sigma_u$ if $|\lambda| > 1$ and $\lambda \in \sigma_c$ if $|\lambda| = 1$. For each one of these sets, we have a linear subspace that is the generalized eigenspace of σ_s, σ_u and σ_c. We denote such subspaces as E^s, E^u and E^c, respectively. Corresponding to each one of those linear subspaces, we can extend the notion to the corresponding local invariant manifolds. Concretely, given a hyperbolic fixed point x^* of the map T (recall that hyperbolic means that every eigenvalue of the Jacobian matrix lies inside the unit disk on the complex plane), and given a neighbourhood $U' \subset U$ of x^*, we define the local stable manifold for x^* in U' as

$$W^s_{loc}(x^*, U', T) = \big\{ q \in U' : T^j(q) \in U' \text{ for } j > 0 \text{ and } \big|T^j(q) - x^*\big| \to 0 \text{ as } j \to \infty \big\}.$$

For the local unstable manifold, we need to examine the past history of x^* under T. If the map T is invertible, we consider the backwards iterates of T; otherwise we need to find a replacement. In this direction, we define a *past history of a point* q to be a sequence of points $\big(q_{-j}\big)_{j=0}^{\infty}$ such that $q_0 = q$ and $T\big(q_{-j-1}\big) = q_{-j}$ for $j \geq 0$. Bearing this definition in mind, the local unstable manifold for x^* in U' is defined as

$$\begin{aligned} W^u_{loc}(x^*, U', T) \;\; = \;\; & \big\{ q \in U' : \text{ there exists some choice of the past history of } q, \\ & \big(q_{-j}\big)_{j=0}^{\infty} \subset U' \text{ such that } \big|q_{-j} - x^*\big| \to 0 \text{ as } j \to \infty \big\}. \end{aligned}$$

In the scenario where the map T is invertible, we can use the backwards iterations of the map to define the local unstable manifold. In this sense, we can define it as follows

$$W^u(x^*, U,', T) = \{ q \in U' : T^j(q) \in U' \text{ for } j > 0 \text{ and } \big|T^j(q) - x^*\big| \to 0 \text{ as } j \to -\infty \}.$$

Whenever the context makes clear that we are talking about the local stable manifolds, we will write W^s and W^u instead of W^s_{loc} and W^u_{loc}, respectively.

The following theorem states that these local stable and unstable manifolds are C^k embedded manifolds which can be represented as the graph of a map from a disk in one of the subspaces to the other subspace. To represent a closed disk in one of the subspaces, we use the following notation: for any $\delta > 0$, the closed disk about the origin of radius δ is represented by $E(\delta) = \{x \in \mathbb{R}^n : |x| \leq \delta\}$. Furthermore, in the statement of the following result, μ and λ are eigenvalues belonging to σ_s and σ_u, respectively.

Theorem 5. *[93, Theorem 10.1]* ***Local Stable Manifold Theorem:*** *Let x^* be a hyperbolic fixed point for a C^k map $T : U \subset \mathbb{R}^k \to \mathbb{R}^k$ with $k \geq 1$. We assume that the derivatives are uniformly continuous in terms of the point at which the derivative is taken. Then there is some neighbourhood of x^*, $U' \subset U$, such that $W^s(x^*, U', T)$ and $W^u(x^*, U', T)$ are each C^k embedded disks which are tangent to E^s and E^u, respectively. In fact, considering $\mathbb{R}^n = E^u \times E^s$, there is a small $r > 0$ such that taking $U' \equiv x^* + \big(E^u(r) \times E^s(r)\big)$, $W^s(x^*, U', T)$ is the graph of a C^k function $\rho_s : E^s(r) \to E^u(r)$ with $\rho_s(0) = 0$ and $D\rho_s(0) = 0$:*

$$W^s(x^*, U', T) = \big\{x^* + \big(\rho_s(y), y\big) : y \in E^s(r)\big\}.$$

Similarly, there is C^k function $\rho_u : E^u(r) \to E^s(r)$ with $\rho_u(0) = 0$ and $D\rho_u(0) = 0$ such that

$$W^u(x^*, U', T) = \big\{x^* + \big(x, \rho_u(x)\big) : x \in E^u(r)\big\}.$$

Moreover, for $r > 0$ small enough and $U' = x^* + \big(E^u(r) \times E^s(r)\big)$,

$$\begin{aligned} W^s(x^*, U', T) &= \big\{q \in U' : T^j(q) \in U' \text{ for } j \geq 0\big\} \\ &= \big\{q \in U' : T^j(q) \in U' \text{ for } j \geq 0 \text{ and } \big|T^j(q) - x^*\big| \leq \mu^j |q - x^*| \text{ for all } j \geq 0\big\}. \end{aligned}$$

This means that every point that is not on $W^s(x^*, U', T)$ leaves U' under forward iteration, and that points on $W^s(x^*, U', T)$ converge to x^* at an exponential rate given by the bound on the stable spectrum. Similarly,

$$\begin{aligned} W^u(x^*, U', T) &= \big\{q \in U' : \text{there exists some choice of the past history of } q \\ &\qquad \text{with } (q_{-j})_{j=0}^{\infty} \subset U'\big\} \\ &= \big\{q \in U' : \text{there exists some choice of the past history of } q \\ &\qquad \text{with } (q_{-j})_{j=0}^{\infty} \subset U' \text{ and } |q_{-j} - x^*| \leq \lambda^{-j} |x^* - q| \text{ for all } j \geq 0\big\}. \end{aligned}$$

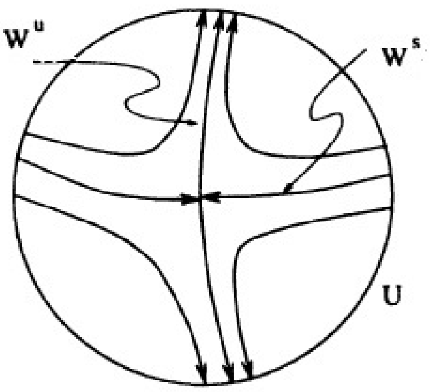


Figure 1.1: Stable and unstable manifolds in a neighbourhood of a fixed point. Image extracted from [93].

Once we have defined the local stable and unstable manifolds, we are able to obtained the global unstable manifold as

$$W^u(x^*, T) = \bigcup_{j\geq 0} T^j W^u\big(x^*, U', T\big).$$

Furthermore, if T is an invertible map, then the global stable manifold is given by

$$W^s(x^*, T) = \bigcup_{j\geq 0} T^{-j} W^s\big(x^*, U', T\big).$$

Finally, concerning the local center manifold, let $E^s \subset \mathbb{R}^s$, $E^u \subset \mathbb{R}^u$ and $E^c \subset \mathbb{R}^c$, with $s + u + c = n$. We define it as a manifold of dimension c whose graph is tangent to E^c at the origin. Here, we understand by invariant manifold a manifold embedded in its phase space verifying that it is invariant under the dynamical system generated by T, that is, for a subspace $M \subseteq \mathbb{R}^n$, whenever $x \in M$, then $T^j(x) \in M$ for every $n \geq 1$.

1.6 Ricker map

In Chapter 6, we will study in detail a host-parasitoid model, which reduces to the well-known Ricker map for some invariant regions. Such map was introduced in [92] by W.E. Ricker and is given by

$$h(x) = axe^{r(1-x)},$$

where a and r are positive real numbers and $x \geq 0$. In the sequel, we denote $\lambda := ae^r$ and consider

$$h(x) = h_{\lambda,r}(x) = \lambda xe^{-rx},$$

where we will only write $h(x)$ when no confusion can arise. Here, we gather its main properties that will be useful in the development of the mentioned chapter.

Firstly, observe that $h(x)$ attains its maximum in $x = \frac{1}{r}$ with value $h\left(\frac{1}{r}\right) = \frac{\lambda}{r}e^{-1}$; notice that $x_1^* = 0$ is always an equilibrium point for every $\lambda > 0$ and another equilibrium appears when $\lambda > 1$, namely, $x_2^* = \frac{\ln\lambda}{r}$.

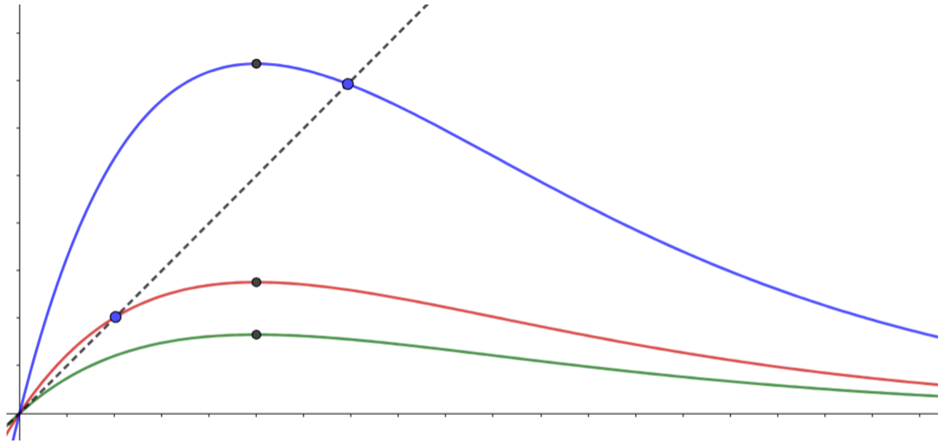


Figure 1.2: Graphs of the function $h(x) = \lambda x e^{-rx}$. The green curve corresponds to the case $\lambda \leq 1$; the red curve represents the case $1 < \lambda \leq e$; and the blue curve shows the case $\lambda > e$.

It is well-known that in the particular case $1 < \lambda < e^2$, the Ricker map has no 2-periodic points. Nevertheless, for the sake of completeness, we give a brief proof.

Proposition 1. *Let us consider the Ricker map, $h(x) = \lambda x e^{-rx}$, with $x \geq 0$, $\lambda > 0$. If $1 < \lambda < e^2$, then $h(x)$ does not have 2-periodic points.*

Proof. Let $h(x) = \lambda x e^{-rx}$, with $x \geq 0$, $\lambda > 0$. Firstly, observe that $h(x)$ has two fixed points, namely, $x_1 = 0$ and $x_2 = \frac{\ln\lambda}{r}$. The map will have 2-periodic points if the equation $h^2(x) = x$ has any solution different from the fixed points. Such equation reads as

$$x + \lambda x e^{-rx} - \frac{2\ln\lambda}{r} = 0.$$

Notice that the fixed point $x_2 = \frac{\ln\lambda}{r}$ satisfies the previous equation. Our target is to prove that x_2 is indeed the unique solution. To do so, consider $H(x) := x + \lambda x e^{-rx} - \frac{2\ln\lambda}{r}$. It is direct to check that $H'(x) = 1 + \lambda e^{-rx} - r\lambda x e^{-rx}$ attaining its minimum in $x = \frac{2}{r}$. Moreover, $H'(0) = 1 + \lambda > 0$, $\lim_{x\to+\infty} H'(x) = 1$ and $H'\left(\frac{2}{r}\right) = 1 - \lambda e^{-2} > 0$, since $1 < \lambda < e^2$. Therefore, $H'(x) > 0$ for every $x \in (0, \infty)$ and the equation $H'(x) = 0$ has no solutions in such interval. From here, as a consequence of Rolle's Theorem, the equation $H(x) = 0$ can only have, at most, one solution, namely, the fixed point $x_2 = \frac{\ln\lambda}{r}$. This completes the proof. □

As a consequence of the above result, when $1 < \lambda < e^2$, the equilibrium point $x^* = 1 + \frac{\ln(a)}{r}$ is a global attractor relative to $(0, \infty)$, since in this case there are no 2-periodic points and Coppel's Theorem (see [25, Main Theorem]) establishes that in this scenario every orbit is convergent to a fixed point. In addition, the origin is unstable for $1 < \lambda < e^2$, so x^* must be the attractor.

Finally, related to the local invariant unstable manifold, if

$$W_h^u = \left\{x \in [0, \infty) : h^{-n}(x) \xrightarrow{n\to\infty} 0\right\}$$

is meant the points whose orbits go backward to the origin, then it is straightforward to check that:

- If $\lambda \leq 1$, then $W_h^u = \{0\}$.
- If $1 < \lambda \leq e$, then $W_h^u = \left[0, \frac{1}{r} \ln \lambda\right)$.
- If $\lambda > e$, then $W_h^u = \left[0, \frac{\lambda}{r} e^{-1}\right]$.

1.7 Bifurcations

A bifurcation of a dynamical system is a qualitative change in its dynamics produced by the variation of their parameters. In this dissertation, we only deal with the possible occurrence of bifurcations of the fixed points of a discrete dynamical system. In this sense, suppose that we have a hyperbolic equilibrium point; then there are only three possible ways in which the hyperbolicity condition can be violated whenever we vary the values of the parameters: an eigenvalue λ of the corresponding Jacobian matrix evaluated at the equilibrium approaches the unit circle and we have $\lambda = 1$; the eigenvalue approaches the unit circle with $\lambda = -1$; or we have a pair of simple conjugate complex eigenvalues approaching the unit circle, $\lambda_{1,2} = e^{\pm i\delta_0}$, $0 < \delta_0 < \pi$.

The bifurcation associated with the appearance of $\lambda = 1$ is called transcritical and it occurs when two equilibrium points exchange their stability as a parameter is modified. For $\lambda = -1$, we have a period-doubling bifurcation which corresponds to the creation of a period two cycle. Finally, for a pair of complex eigenvalues with unit modulus, we have the occurrence of a Neimark-Sacker bifurcation, which is the discrete analogue to the Hopf bifurcation in continuous dynamical systems. This bifurcation happens when an equilibrium point changes its stability giving rise to the emergence of a closed invariant curve. If such curve is stable, we say that the bifurcation is supercritical, while we call it subcritical when it is unstable.

In the scenario of period-doubling bifurcation, the following result allows us to establish the stability of the period-two cycle that appears near the equilibrium point.

Theorem 6. *[41, Theorem 3.5.1] Let $f_\mu : \mathbb{R} \to \mathbb{R}$ be a one-parameter family of mappings such that f_{μ_0} has a fixed point x^* with eigenvalue -1. Assume*

(F1) $\left(\frac{\partial f}{\partial \mu}\frac{\partial^2 f}{\partial x^2} + 2\frac{\partial^2 f}{\partial x \partial \mu}\right) = \frac{\partial f}{\partial \mu}\frac{\partial^2 f}{\partial x^2} - \left(\frac{\partial f}{\partial x} - 1\right)\frac{\partial^2 f}{\partial x \partial \mu} \neq 0$ *at* (x^*, μ_0)*;*

(F2) $a := \left(\frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}\right)^2 + \frac{1}{3}\left(\frac{\partial^3 f}{\partial x^3}\right)\right) \neq 0$ *at* (x^*, μ_0)*.*

Then there is a smooth curve of fixed points of f_μ *passing through* (x^*, μ_0)*, the stability of which changes at* (x^*, μ_0)*. There is also a smooth curve* γ *passing through* (x^*, μ_0) *so that* $\gamma - \{(x^*, \mu_0)\}$ *is a union of hyperbolic period 2 orbits. The curve* γ *has quadratic tangency with the line* $\mathbb{R} \times \{\mu_0\}$ *at* (x^*, μ_0)*.*

Additionally, the sign of a *in* $(F2)$ *determines the stability and direction of bifurcation of the 2-periodic orbits. If* $a > 0$*, the orbits are stable; if* $a < 0$*, they are unstable.*

Finally, we state a result that will be useful in the determination of the occurrence of Neimark-Sacker bifurcations. Due to the fact that we will apply it in a model defined in a subset of $\mathbb{R}^2$, we state it for a map $F : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$.

Theorem 7. *[43,* ***Theorem 15.31****] Let* $F : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ *be a* C^4 *map depending on a real parameter* μ *satisfying the following conditions:*

(i) $F(\mu, 0) = 0$ *for* μ *near some fixed* μ_0*;*

(ii) $DF(\mu, 0)$ *has two non-real eigenvalues* $\lambda(\mu)$ *and* $\bar{\lambda}(\mu)$ *for* μ *near* μ_0 *with* $|\lambda(\mu_0)| = 1$*;*

(iii) $\frac{d}{d\mu}|\lambda(\mu)| > 0$ *at* $\mu = \mu_0$*;*

(iv) $\lambda^k(\mu_0) \neq 1$ *for* $k = 1, 2, 3, 4$*.*

Then there is a smooth μ*-dependent change of coordinates bringing* F *into the form*

$$F(\mu, \boldsymbol{x}) = \mathcal{F}(\mu, \boldsymbol{x}) + O\big(||\boldsymbol{x}||^5\big)$$

and there are smooth functions $a(\mu)$*,* $b(\mu)$ *and* $\omega(\mu)$ *so that in polar coordinates the function* $\mathcal{F}(\mu, \boldsymbol{x})$ *is given by*

$$\begin{pmatrix} r \\ \theta \end{pmatrix} \mapsto \begin{pmatrix} |\lambda(\mu)|r - a(\mu)r^3 \\ \theta + \omega(\mu) + b(\mu)r^2 \end{pmatrix}.$$

- *If* $a(\mu_0) > 0$*, then there is a neighbourhood* U *of the origin and a* $\delta > 0$ *such that, for* $|\mu - \mu_0| < \delta$ *and* $\boldsymbol{x}^0 \in U$*, the* ω*-limit set of* $\boldsymbol{x}^0$*:*

  ⋆ *it is the origin if* $\mu < \mu_0$*,*

  ⋆ *it belongs to a closed invariant* C^1 *curve* $\Gamma(\mu)$ *encircling the origin if* $\mu > \mu_0$*. Furthermore,* $\Gamma(\mu_0) = 0$*.*

- *If* $a(\mu_0) < 0$*, then there is a neighbourhood* U *of the origin and a* $\delta > 0$ *such that, for* $|\mu - \mu_0| < \delta$ *and* $\boldsymbol{x}^0 \in U$*, the* α*-limit set of* $\boldsymbol{x}^0$*:*

$\star$ *it is the origin if* $\mu > \mu_0$,

$\star$ *it belongs to a closed invariant* C^1 *curve* $\Gamma(\mu)$ *encircling the origin if* $\mu < \mu_0$. *Furthermore,* $\Gamma(\mu_0) = 0$.

Chapter 2

Convergence

One basic result in the field of real mathematical analysis is the widely known fact that every bounded monotonic sequence of real numbers is convergent. In the 60's of the last century, two suggestions of Professors J.M. Whittaker and J.B. Tatchell inspired E.T. Copson to generalized such result. His idea resided on the exchange of the monotonic condition by a convex inequality involving some consecutive terms of the sequence of real numbers. In concrete, in [26] he proved the following:

Theorem 8. ***Copson's Theorem:*** *Let (x_n) be a sequence of real numbers satisfying that*

$$x_{n+k} \leq \sum_{j=1}^{k} \alpha_j x_{n+k-j}, \quad n \geq 1, \tag{2.1}$$

where the coefficients α_j are strictly positive real numbers verifying that $\sum_{j=1}^{k} \alpha_j = 1$. Then, if (x_n) is bounded, it is convergent; otherwise, it diverges to $-\infty$.

Notice that if we take the reverse inequality in (2.1), the result remains true, since it is sufficient to replace x_n by $-x_n$ in order to prove it. Moreover, it should be emphasized that the condition of the coefficients α_j being strictly positive is necessary. For instance, take a bounded sequence satisfying the linear inequality $x_{n+4} \leq \frac{1}{2}(x_{n+2} + x_n)$. In this case, the subsequences (x_{2n}) and (x_{2n+1}) converge, but (x_n) is not necessarily convergent as it shows the sequence $x_n = (-1)^{n+1}$ for all $n \geq 1$.

Furthermore, in Copson's paper, besides his own proof, he included an alternative one due to R.A. Rankin. This proof is based on the definition of an auxiliary sequence, namely, $A_n = \max\{x_{n-1}, \ldots, x_{n-k}\}$, which is monotonic under the conditions of Copson's Theorem. This idea gave rise to numerous generalizations and extensions to other mathematical objects like sequences of complex numbers, sequences of functions, double sequences or Hermitian operators. The reader interested in those extensions is referred to the survey [63].

The aim of this chapter is to apply the techniques developed by Copson in [26] to generalize the well-known Monotone Convergence Theorem. Additionally, we give a further generalization by assuming an extra condition concerning the first terms of the sequence of functions (f_n).

This chapter, which is based on [64], is organized as follows: firstly, in Section 2.1, we establish the basic notions concerning Measure Theory that will be used in the sequel; Section 2.2 is devoted to prove the generalization of the Monotone Convergence Theorem.

Theorem A. *Let $(\mathbb{X}, \Sigma, \mu)$ be a measure space and let (f_n) be a sequence of measurable non-negative functions, $f_n : \mathbb{X} \to [0, \infty]$. If the sequence verifies*

$$f_{n+k}(x) \geq \sum_{j=1}^{k} \alpha_j f_{n+k-j}(x), \quad \text{for all } x \in \mathbb{X}, \tag{2.2}$$

where the coefficients α_j are strictly positive real numbers satisfying that $\sum_{j=1}^{k} \alpha_j = 1$, then there exists a measurable function f such that

$$\lim_{n\to\infty} f_n(x) = f(x), \quad \text{for all } x \in \mathbb{X},$$

and

$$\lim_{n\to\infty} \int_{\mathbb{X}} f_n \, d\mu = \int_{\mathbb{X}} f \, d\mu.$$

Furthermore, if we suppose the reverse inequality, $f_{n+k}(x) \leq \sum_{j=1}^{k} \alpha_j f_{n+k-j}(x)$, for all $x \in \mathbb{X}$ and the additional hypothesis of being $f_1, \ldots, f_k \in L^1(\mu)$, the result remains true.

Then, Section 2.3 presents an example that shows the usefulness of such generalization by considering a case where the Monotone Convergence Theorem cannot be applied. Finally, we establish some conclusions and propose some open questions related to the topic in Section 2.4.

Finally, it should be mentioned that the generalization that we present, Theorem A, motivated further research concerning varying measures. For instance, in [77], the authors developed a convergence theorem for varying measures when the sequence of functions (f_n) verifies a Copson's inequality and the sequence of measures is setwisely or weakly convergent (consult [48, Section 2.1] for the corresponding definitions). This kind of results have important applications to different fields of pure and applied sciences such as transportation problems, stochastic processes or neural networks among others. The interested reader in those applications is referred to [77] and the references therein.

2.1 Preliminary notions

In the present section, we gather the basic notions and results concerning Measure Theory that will be used in the rest of the chapter. As main reference, we have used [96].

Given an order triplet $(\mathbb{X}, \Sigma, \mu)$, where $\mathbb{X}$ is a set, Σ is a σ-algebra over such set $\mathbb{X}$, and $\mu : \Sigma \to [0, \infty]$ is a positive measure over that σ-algebra verifying that the measure of every countable collection of pairwise disjoint measurable sets is equal to the sum of their measure, that is, the so-called countable additivity property, and such that $\mu(\emptyset) = 0$, we will call it measure space and the elements of Σ will be called measurable sets.

Furthermore, let $f : \mathbb{X} \to Y$ be a function where $\mathbb{X}$ is a measurable space and Y is a topological space. If for every open set $V \subseteq Y$, we have that $f^{-1}(V) = \{x \in \mathbb{X} : f(x) \in V\}$ is a measurable set of $\mathbb{X}$, we will say that f is a measurable function. In particular, for the cases $Y = [-\infty, \infty]$ or $Y = [0, \infty]$, if we have a sequence of measurable functions (f_n), $f_n : \mathbb{X} \to Y$, then $g = \sup_{n \geq 1} f_n$ and $h = \limsup_{n\to\infty} f_n$ are measurable functions too. Additionally, for every measurable function $f : \mathbb{X} \to [-\infty, \infty]$, if $\int_X |f| \, d\mu < \infty$, we will write $f \in L^1(\mu)$. For the notion of the integral of a measurable function f, the reader is also referred to [96].

On the other hand, for a set $A \subset \mathbb{X}$, χ_A denotes the characteristic function on A defined as $\chi_A(x) = 1$ if $x \in A$, and $\chi_A(x) = 0$, otherwise.

Finally, we state two well-known results of Measure Theory, namely, the Monotone Convergence Theorem and the Dominated Convergence Theorem.

Theorem 9. ***[96, page 21] Monotone Convergence Theorem:*** *Let (f_n) be a sequence of measurable functions on $\mathbb{X}$, and suppose that*

- $0 \leq f_1(x) \leq f_2(x) \leq \cdots \leq f_n(x) \leq \cdots \leq \infty$ *for every* $x \in \mathbb{X}$.
- $\lim_{n\to\infty} f_n(x) = f(x)$ *for every* $x \in \mathbb{X}$.

Then f is measurable and

$$\lim_{n\to\infty} \int_{\mathbb{X}} f_n \, d\mu = \int_{\mathbb{X}} f \, d\mu.$$

We should emphasize that if we consider a decreasing sequence of measurable functions instead of an increasing one, the result fails to be true. For instance, if we take $\mathbb{X} = [0, +\infty]$; μ the Lebesgue measure over $\mathbb{X}$; and $f_n = \frac{1}{n}\chi_{[n,\infty)}$, then we have $\lim_{n\to\infty} f_n = 0$ and $\int_{\mathbb{X}} f \, d\mu = 0$. However,

$$\lim_{n\to\infty} \int_{\mathbb{X}} f_n \, d\mu = \lim_{n\to\infty} \int_n^{\infty} \frac{1}{n} \, d\mu = \lim_{n\to\infty} \frac{1}{n}\mu\big([n,\infty)\big) = \infty.$$

Nevertheless, in the case of decreasing sequences of measurable functions, we can give an extension of the Monotone Convergence Theorem by assuming the extra condition of $f_1 \in L^1(\mu)$, where f_1 denotes the first function of the sequence. The following result is proposed in [96] as an exercise, but its proof can be found in [4, page 49].

Theorem 10. ***Extended Monotone Convergence Theorem:*** *Let $f_n : \mathbb{X} \to [0, \infty]$ be a sequence of measurable functions with $f_1 \in L^1(\mu)$. Suppose that*

- $f_1(x) \geq f_2(x) \geq \cdots \geq f_n(x) \geq \cdots \geq 0$, *for every* $x \in \mathbb{X}$.

- $\lim_{n\to\infty} f_n(x) = f(x)$ *for every* $x \in \mathbb{X}$

Then

$$\lim_{n\to\infty} \int_{\mathbb{X}} f_n \; d\mu = \int_{\mathbb{X}} f \; d\mu.$$

Lastly, we state the Dominated Convergence Theorem.

Theorem 11. ***[96, page 26] Dominated Convergence Theorem:*** *Suppose* (f_n) *is a sequence of complex measurable functions such that* $\lim_{n\to\infty} f_n(x) = f(x)$ *exists for every* $x \in \mathbb{X}$*. If there is a function* $g \in L^1(\mu)$ *such that* $|f_n(x)| \leq g(x)$ *for all* $x \in \mathbb{X}$*, then* $f \in L^1(\mu)$ *and*

$$\lim_{n\to\infty} \int_{\mathbb{X}} f_n \; d\mu = \int_{\mathbb{X}} f \; d\mu.$$

2.2 A generalization of the Monotone Convergence Theorem

The aim of this section is to prove a generalization of the Monotone Convergence Theorem by considering a convex linear inequality involving certain terms of the sequence of measurable functions instead of the monotonicity condition. To do so, we begin by proving two auxiliary results that will be useful in the sequel.

Lemma 1. *Let* (x_n) *be a sequence of real numbers satisfying the inequality*

$$x_{n+k} \geq \sum_{j=1}^{k} \alpha_j x_{n+k-j}, \quad n \geq 1, \tag{2.3}$$

where the coefficients α_j *are strictly positive real numbers verifying* $\sum_{j=1}^{k} \alpha_j = 1$*. Then, the sequence* $A_n = \min\{x_{n-1}, \ldots, x_{n-k}\}$*, for* $n \geq k+1$*, is increasing. Moreover, if we consider the reverse inequality* $x_{n+k} \leq \sum_{j=1}^{k} \alpha_j x_{n+k-j}$*, for* $n \geq 1$*, then the sequence* $A_n = \max\{x_{n-1}, \ldots, x_{n-k}\}$*, for* $n \geq k+1$*, is decreasing.*

Proof. Assume that the sequence of real numbers (x_n) satisfies (2.3). The case where we consider the reverse inequality is analogous and we omit it.

Take the sequence $A_n = \min\{x_{n-1}, \ldots, x_{n-k}\}$ for every $n \geq k+1$. It can be easily seen that $A_n \leq x_n$. Indeed,

$$A_n = \sum_{j=1}^{k} \alpha_j A_n \leq \sum_{j=1}^{k} \alpha_j x_{n-j} \leq x_n.$$

Bearing this fact in mind, it follows that

$$\begin{aligned} A_n &= \min\{x_n, A_n\} \\ &= \min\left\{x_n, \min\{x_{n-1}, x_{n-2}, \ldots, x_{n-(k-1)}, x_{n-k}\}\right\} \\ &= \min\left\{x_{n-k}, \min\{x_n, x_{n-1}, \ldots, x_{n-(k-1)}\}\right\} \\ &\leq \min\{x_n, x_{n-1}, \ldots, x_{n-(k-1)}\} = A_{n+1}, \end{aligned}$$

and we get that (A_n) is an increasing sequence. □

Lemma 2. *Let (x_n) be a sequence of non-negative real numbers satisfying*

$$x_{n+k} \geq \sum_{j=1}^{k} \alpha_j x_{n+k-j}, \quad n \geq 1, \tag{2.4}$$

where the coefficients α_j are strictly positive real numbers verifying $\sum_{j=1}^{k} \alpha_j = 1$. Then, there exists $\lambda \in \mathbb{R} \cup \{\infty\}$ such that $x_n \leq k \cdot \lambda$ for all $n \geq 1$.

Proof. Take $B_n = x_{n-1} + \cdots + x_{n-k}$ for every $n \geq k+1$. Applying that each x_{n-j}, with $j \in \{1, \ldots, k\}$, satisfies (2.4), it is direct to see that every B_n verifies such inequality too for every $n \geq k+1$:

$$\begin{aligned}
B_n = x_{n-1} + \cdots + x_{n-k} &\geq \sum_{j=1}^{k} \alpha_j x_{n-1-j} + \sum_{j=1}^{k} \alpha_j x_{n-2-j} + \cdots + \sum_{j=1}^{k} \alpha_j x_{n-k-j} \\
&= \alpha_1 x_{n-2} + \cdots + \alpha_1 x_{n-k-1} + \cdots + \alpha_k x_{n-k-1} + \cdots + \alpha_k x_{n-2k} \\
&= \alpha_1 (x_{n-2} + \cdots + x_{n-k-1}) + \cdots + \alpha_k (x_{n-k-1} + \cdots + x_{n-2k}) \\
&= \alpha_1 B_{n-1} + \alpha_2 B_{n-2} + \cdots + \alpha_k B_{n-k}.
\end{aligned}$$

Next, let $C_n = \min\left\{B_{n-1}, \ldots, B_{n-k}\right\}$. Observe that (C_n) is an increasing sequence due to Lemma 1.

On the other hand, by Copson's Theorem, (x_n) is a convergent sequence or it diverges to infinity. Let $\lambda := \lim_{n\to\infty} x_n$. Hence, $\lim_{n\to\infty} C_n = k \cdot \lambda$ and $C_n \leq k \cdot \lambda$ for all n.

Now, since $x_j \geq 0$ for all $j \geq 1$ and

$$\begin{aligned}
C_n &= \min\left\{B_{n-1}, \ldots, B_{n-k}\right\} \\
&= \min\{x_{n-2} + \cdots + x_{n-k-1}, \ldots, x_{n-k-1} + \cdots + x_{n-2k}\},
\end{aligned}$$

we deduce that $C_n \geq x_{n-k-1}$. In conclusion, $x_n \leq C_{n+k+1} \leq k \cdot \lambda$. □

2.2.1 Proof of Theorem A

Now we proceed to prove the main result of this chapter concerning the convergence of sequences of measurable functions.

Theorem A. *Let $(\mathbb{X}, \Sigma, \mu)$ be a measure space and let (f_n) be a sequence of measurable non-negative functions, $f_n : \mathbb{X} \to [0, \infty]$. If the sequence verifies*

$$f_{n+k}(x) \geq \sum_{j=1}^{k} \alpha_j f_{n+k-j}(x), \quad \textit{for all } x \in \mathbb{X}, \tag{2.5}$$

where the coefficients α_j *are strictly positive real numbers satisfying that* $\sum_{j=1}^{k} \alpha_j = 1$*, then there exists a measurable function* f *such that*

$$\lim_{n\to\infty} f_n(x) = f(x), \quad \textit{for all } x \in \mathbb{X},$$

and

$$\lim_{n\to\infty} \int_{\mathbb{X}} f_n \, d\mu = \int_{\mathbb{X}} f \, d\mu.$$

Furthermore, if we suppose the reverse inequality, $f_{n+k}(x) \leq \sum_{j=1}^{k} \alpha_j f_{n+k-j}(x)$*, for all* $x \in \mathbb{X}$ *and the additional hypothesis of being* $f_1, \ldots, f_k \in L^1(\mu)$*, the result remains true.*

Proof. Let (f_n) be a sequence of measurable non-negative functions. Firstly, we assume the increasing case, that is, inequality (2.5). By Copson's theorem, there exists a function $f : \mathbb{X} \to [0, \infty]$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in \mathbb{X}$. Notice that f is a measurable function since it is the limit of measurable functions.

Now, set the sequence $A_n(x) = \min\{f_{n-1}(x), \ldots, f_{n-k}(x)\}$. By Lemma 1, given an $x \in \mathbb{X}$, it holds that $A_n(x) \leq A_{n+1}(x) \leq f_{n+1}(x)$ with $\lim_{n\to\infty} A_n(x) = f(x)$. Therefore, by the Monotone Convergence Theorem, we get

$$\lim_{n\to\infty} \int_{\mathbb{X}} A_n \, d\mu = \int_{\mathbb{X}} f \, d\mu.$$

Additionally, since the sequence $\left(\int_{\mathbb{X}} A_n \, d\mu\right)$ is increasing, we have $\int_{\mathbb{X}} f \, d\mu \in [0, \infty]$.

Next, we distinguish two cases. First, if $\int_{\mathbb{X}} f \, d\mu < \infty$, by Lemma 2, the sequence $C_n(x) = \min\{B_{n-1}(x), \ldots, B_{n-k}(x)\}$, where $B_n(x) = \sum_{j=1}^{k} f_{n-j}(x)$, is increasing with $\lim_{n\to\infty} C_n(x) = k \cdot f(x)$. Moreover, $f_{n-k-1}(x) \leq C_n(x) \leq k \cdot f(x)$ for all $x \in \mathbb{X}$ and $n \geq k+2$, so, by the Dominated Convergence Theorem,

$$\lim_{n\to\infty} \int_{\mathbb{X}} f_n \, d\mu = \int_{\mathbb{X}} f \, d\mu.$$

Secondly, if $\int_{\mathbb{X}} f \, d\mu = +\infty$, it is direct to achieve that $\lim_{n\to\infty} \int_{\mathbb{X}} f_n(x) \, d\mu = +\infty = \int_{\mathbb{X}} f \, d\mu$. Indeed, due to the fact that $A_n \leq f_n$, the result follows by taking limits in the inequality.

Finally, we consider the decreasing case

$$f_{n+k}(x) \leq \sum_{j=1}^{k} \alpha_j f_{n+k-j}(x), \quad \text{for all } x \in \mathbb{X}.$$

Again, by Copson's theorem, we can take a non-negative measurable function $f : \mathbb{X} \to [0, \infty]$ such that $\lim_{n\to\infty} f_n(x) = f(x)$.

Now, consider the sequence $A_n(x) = \max\{f_{n-1}(x), \ldots, f_{n-k}(x)\}$ for $n \geq k+1$, which is decreasing and verifies $f_n(x) \leq A_n(x)$ with $\lim_{n\to\infty} A_n(x) = \lim_{n\to\infty} f_n(x) = f(x)$ by Lemma 1.

Observe that $A_{k+1} \in L^1(\mu)$. Indeed, if $k = 2$, $\max\{f_1, f_2\} = \frac{|f_1+f_2|+|f_1-f_2|}{2}$, and we can extend it analogously for $k \geq 1$. Then, since $f_n(x) \leq A_n(x) \leq A_{k+1}(x)$, by the Dominated Convergence Theorem, we get

$$\lim_{n\to\infty} \int_{\mathbb{X}} f_n \, d\mu = \int_{\mathbb{X}} f \, d\mu.$$

□

It should be highlighted the necessity of the extra assumption of $f_j \in L^1(\mu)$ for $j = 1, \ldots, k$ in the decreasing case. For instance, take the sequence $f_n(x) = \frac{1}{n}\chi_{[n,\infty)}$ for $n \geq 1$, and set $k = 2$, $\alpha_1 = \alpha_2 = \frac{1}{2}$. Since $\frac{1}{n+2} \leq \frac{1}{2(n+1)} + \frac{1}{2n}$ for all $n \geq 1$, it follows that $f_{n+2}(x) \leq \frac{1}{2}f_{n+1}(x) + \frac{1}{2}f_n(x)$ for $n \geq 1$. Thus, we have that $\lim_{n\to\infty} f_n(x) = f(x) = 0$. Nevertheless, $\lim_{n\to\infty} \int_{\mathbb{X}} f_n \, d\mu = +\infty$, although $\int_{\mathbb{X}} f \, d\mu = 0$.

2.2.2 A further generalization

The generalization of the Monotone Convergence Theorem that it has been proved deals with sequences of measurable non-negative functions. In order to find a more general result involving measurable functions without the restriction of non-negativity, we must impose an extra assumption, that is, the functions $f_1, \ldots, f_k$ belong to $L^1(\mu)$. Bearing this condition in mind, we are able to prove the following:

Corollary 1. *Let $(\mathbb{X}, \Sigma, \mu)$ be a measure space and (f_n) a sequence of measurable functions satisfying inequality (2.5). If additionally, $f_1, \ldots, f_k \in L^1(\mu)$, then there exists a measurable function f such that*

$$\lim_{n\to\infty} f_n(x) = f(x), \quad \textit{for all } x \in \mathbb{X}$$

and

$$\lim_{n\to\infty} \int_{\mathbb{X}} f_n \, d\mu = \int_{\mathbb{X}} f \, d\mu.$$

Moreover, if we consider the reverse inequality in (2.5), the result remains true.

Proof. We focus on the prove of the case where (f_n) satisfies (2.5). Here, our aim is to define an appropriate sequence of non-negative measurable functions in order to apply Theorem A. For the reverse inequality, the result follows directly by considering the sequence $(-f_n)$ and applying the other case.

Take $\varphi = |f_1| + \cdots + |f_k|$. Observe that, since $f_1, \ldots, f_k$ are in $L^1(\mu)$, we deduce that $\varphi \in L^1(\mu)$ too. Next, define $g_j(x) = \varphi(x) + f_j(x), x \in \mathbb{X}, j \geq 1$, where $g_j(x) \geq 0$ for every $j = 1, \ldots, k$ and $x \in \mathbb{X}$. Furthermore, applying (2.5),

$$g_{k+1} = \varphi + f_{k+1} \geq \varphi + \sum_{j=1}^{k} \alpha_j f_{k+1-j} = \sum_{j=1}^{k} \alpha_j \big(\varphi + f_{k+1-j}\big) = \sum_{j=1}^{k} \alpha_j g_{k+1-j} \geq 0.$$

Thus, it is direct to check by induction that $g_{n+k}(x) \geq 0$ with $g_{n+k}(x) \geq \sum_{j=1}^{k} \alpha_j g_{n+k-j}(x)$ for every $x \in \mathbb{X}$ and for all $n \geq 1$. Observe that we have proved two properties: $g_j(x) \geq 0$ for every $j \geq 1$, and the sequence of non-negative measurable functions (g_n) verifies an inequality of Copson's type.

Next, we apply Theorem A to (g_n) and we deduce the existence of a measurable function g such that $\lim_{n\to\infty} g_n(x) = g(x)$ for all $x \in \mathbb{X}$, and $\lim_{n\to\infty} \int_{\mathbb{X}} g_n \, d\mu = \int_{\mathbb{X}} g \, d\mu$.

Hence, by taking $f(x) := g(x) - \varphi(x)$, we conclude that $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in \mathbb{X}$ and

$$\begin{aligned}
\int_{\mathbb{X}} f \, d\mu &= \int_{\mathbb{X}} \lim_{n\to\infty} f_n \, d\mu = \int_{\mathbb{X}} \lim_{n\to\infty} (g_n - \varphi) \, d\mu \\
&= \int_{\mathbb{X}} \lim_{n\to\infty} g_n \, d\mu - \int_{\mathbb{X}} \lim_{n\to\infty} \varphi \, d\mu \\
&= \lim_{n\to\infty} \left(\int_{\mathbb{X}} g_n \, d\mu - \int_{\mathbb{X}} \varphi \, d\mu \right) \\
&= \lim_{n\to\infty} \int_{\mathbb{X}} f_n \, d\mu,
\end{aligned}$$

since g and φ are in $L^1(\mu)$. $\square$

As it happened in the decreasing case of Theorem A, the extra assumption of $f_1, \ldots, f_k$ belonging to $L^1(\mu)$ is a necessary condition. For example, take $f_1(x) = \chi_{[1,\infty)}$ and $f_2(x) = -\frac{1}{2}\chi_{[1,\infty)}$, where $\chi_{[1,\infty)}$ is the characteristic function in the interval $[1, \infty)$, and consider the sequence of measurable functions

$$f_n(x) = \frac{(-1)^{n+1}}{2^{n-1}} \cdot f_1(x) = \frac{(-1)^{n+1}}{2^{n-1}} \cdot \chi_{[1,\infty)},$$

which satisfies the linear combination

$$f_{n+2}(x) = \frac{1}{2} f_{n+1}(x) + \frac{1}{2} f_n(x),$$

for all $x \in [0, \infty)$ and $n \geq 1$. Notice that $f_1(x)$ and $f_2(x)$ are not in $L^1(\mu)$. Additionally, $\lim_{n\to\infty} f_n(x) = 0 =: f(x)$ for all $x \in [0, \infty)$; although

$$\int_{\mathbb{X}} f_n \, d\mu = \frac{1}{n} \int_{\mathbb{X}} \chi_{[1,\infty)} \, d\mu = +\infty \neq 0 = \int_{\mathbb{X}} f \, d\mu.$$

2.3 Example

In this section, we see the utility of Theorem A by considering a sequence of non-negative measurable functions that are not monotonic, so we cannot apply the Monotone Convergence Theorem, but that verifies an inequality of Copson's type and therefore, Theorem A can be applied.

To do so, we need to consider the ternary Cantor set which is denoted by $\mathcal{C}$. Recall that such set is built by deleting the open middle third from a set of line segments in an iterative way. For instance, we start removing the subinterval $\left(\frac{1}{3}, \frac{2}{3}\right)$ from the unit interval $I = [0, 1]$. Then, from the remaining intervals $C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$, we remove again the corresponding middle thirds obtaining the four subintervals $C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$. If we continue with that process, the n-th set will be given by

$$C_n = \frac{C_{n-1}}{3} \cup \left(\frac{2}{3} + \frac{C_{n-1}}{3}\right),$$

for $n \geq 1$. From here, the ternary Cantor set is defined as the set of points of the unit interval that are not removed at any step of the previous process. In this sense, we can define it as

$$\mathcal{C} = \bigcap_{n=1}^{\infty} C_n.$$

Now, we can build a sequence of non-negative measurable functions (f_n), $f_n : I \to I$ for every $n \geq 1$. The sequence is given by

$$f_n(x) = \begin{cases} 0 & \text{if } x \in C_n, \\ \frac{1}{2}\left(f_{n-1}\left(\frac{1}{2}\right) + f_{n-2}\left(\frac{1}{2}\right)\right) & \text{if } x \in I \setminus C_n. \end{cases}$$

for every $n \geq 3$ where the initial terms, f_1 and f_2, are

$$f_1(x) = \begin{cases} 0 & \text{if } x \in C_1 \\ 1 & \text{if } x \in I \setminus C_1 \end{cases} \quad \text{and} \quad f_2(x) = \begin{cases} 0 & \text{if } x \in C_2 \\ \frac{1}{2} & \text{if } x \in I \setminus C_2 \end{cases}.$$

The reader can find represented functions f_1, f_2 and f_3 in Figure 2.1.

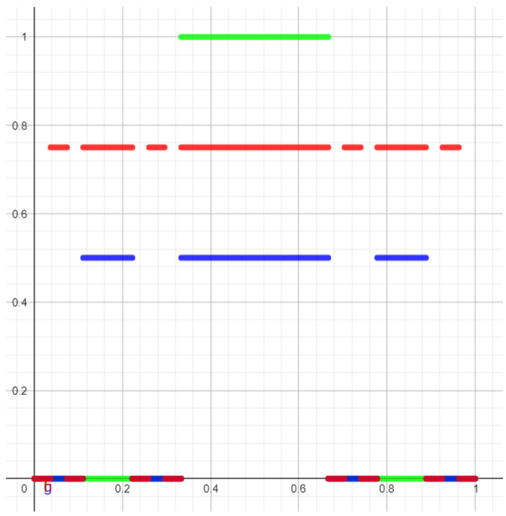


Figure 2.1: Sequence of functions $f_n(x)$, $n = 1, 2, 3$. f_1 in green, f_2 in blue and f_3 in red.

Observe that in $I \setminus C_n$ we have defined the sequence as a linear difference equation

$$x_{n+2} = \frac{1}{2}x_{n+1} + \frac{1}{2}x_n,$$

where $x_1 = f_1\left(\frac{1}{2}\right) = 1$, $x_2 = f_2\left(\frac{1}{2}\right) = \frac{1}{2}$ and $x_n = f_n\left(\frac{1}{2}\right)$, $n \geq 3$.

Consequently, we can compute the general term of the sequence, that is,

$$x_n = \frac{2}{3} - \frac{2}{3}\left(-\frac{1}{2}\right)^n, \; n \geq 1.$$

Hence, the sequence of functions are defined as:

$$f_n(x) = \begin{cases} 0 & \text{if } x \in C_n, \\ \frac{2}{3} - \frac{2}{3}\left(-\frac{1}{2}\right)^n & \text{if } x \in I \setminus C_n. \end{cases}.$$

It is easy to see that such sequence (f_n) is not monotonic. However, it verifies the Copson's inequality

$$f_{n+2}(x) \geq \frac{1}{2} f_{n+1}(x) + \frac{1}{2} f_n(x), \tag{2.6}$$

for all $x \in [0,1]$ and every $n \geq 1$. Indeed, we consider three different cases to show it.

- If $x \in C_{n+2}$ or $x \in C_{n+1} \setminus C_{n+2}$, it follows that $f_n(x) = f_{n+1}(x) = 0$ and $f_{n+2}(x) \geq 0$, so (2.6) holds trivially.

- If $x \in C_n \setminus C_{n+1}$, then $f_n(x) = 0$, $f_{n+1}(x) = \frac{2}{3} - \frac{2}{3} \cdot \left(-\frac{1}{2}\right)^{n+1}$ and $f_{n+2}(x) = \frac{2}{3} - \frac{2}{3} \cdot \left(-\frac{1}{2}\right)^{n+2}$. In this case, (2.6) reads as $\frac{1}{3} + \frac{2}{3} \cdot \left(-\frac{1}{2}\right)^{n+1} \geq 0$, which is true for every $n \geq 1$.

- If $x \in I \setminus C_n$, we have $f_i(x) = \frac{2}{3} - \frac{2}{3} \cdot \left(-\frac{1}{2}\right)^i$ for $i = n, n+1, n+2$. For such functions, (2.6) is equivalent to $(-1)^{n+1} + (-1)^n \geq 0$ for all $n \geq 1$ which is obviously true.

Therefore, the conditions of Theorem A are satisfied and we can assure the existence of a measurable function f which will be the limit of the sequence (f_n). Finally, we will compute such function f.

If $x \in \mathcal{C}$, the convergence to $f(x) = 0$ is direct. On the other hand, if $x \notin \mathcal{C}$, there exists n_0, which depends on x, such that $x \notin C_n$ for all $n \geq n_0$. Thus, $f_n(x) = x_n$, where x_n is defined as $x_n = \frac{2}{3} - \frac{2}{3} \cdot \left(-\frac{1}{2}\right)^n$. Notice that $\lim_{n\to\infty} x_n = \frac{2}{3}$, so

$$\lim_{n\to\infty} f_n(x) = f(x) := \begin{cases} 0 & \text{if } x \in \mathcal{C}, \\ \frac{2}{3} & \text{if } x \notin \mathcal{C}. \end{cases}.$$

Then, by considering the Lebesgue measure in the unit interval I, we can apply Theorem A to conclude that

$$\lim_{n\to\infty} \int_I f_n \, d\mu = \int_I f \, d\mu = \int_0^1 \frac{2}{3} \, d\mu = \frac{2}{3}.$$

2.4 Conclusions

In this chapter, we have focused on the dynamical property of convergence by giving a generalization of the Monotone Convergence Theorem following the ideas presented by Copson in his seminal work [26]. In concrete, we have exchanged the monotonic condition over the sequence of non-negative measurable functions (f_n) by a suitable linear combination of Copson type, see Theorem A. It is interesting to highlight how the nature of its proof is essentially arithmetic, as Lemmas (1)-(2) show us, and it is strongly based on the Monotone and Dominated Convergence Theorems.

Also, we have extended the generalization presented in Theorem A to the scenario of general measurable functions $f_n : \mathbb{X} \to [-\infty, \infty]$, not necessarily non-negative. To achieve this, we needed to add the extra hypothesis of L^1 integrability of the initial functions $f_1, \ldots, f_k$.

On the other hand, in [63] we presented a survey on different generalizations and extensions of Copson's Theorem, where some proposals of future lines of research on the topic can be found. For instance, to obtain new results concerning sufficient conditions stated in the form of inequalities involving two or more sequences which ensure the convergence of a sequence, in the line of the following theorem from [100].

Theorem 12. *[100] Let $f(x_1, x_2, \ldots, x_k)$ be a continuous real function on $\mathbb{R}^k$ which satisfies the following conditions:*

1. *f is non-decreasing in each variable and increasing in the first one,*
2. *$f(x, x, \ldots, x) \leq x$ for every $x \in \mathbb{R}$.*

If (a_n) is a sequence bounded from below and satisfies the inequality

$$a_{n+k} \leq f(a_{n+k-1}, a_{n+k-2}, \ldots, a_n) + b_n,$$

where (b_n) is a sequence of real numbers such that $\sum_{n=0}^{\infty} |b_n| < \infty$, then it converges.

Next, in the following chapters, we leave the general scenario of the sequences of functions, to focus on the frame of autonomous difference equations. In this direction, we continue studying diverse dynamical properties of their solutions in order to understand their behaviour in the long-term. Specifically, in the next chapter, we deal with the dynamical property of global periodicity.

Chapter 3

Global periodicity

In the former chapter, we have studied the dynamical property of convergence for a more general scenario since the analysis was focused on the frame of sequences of functions. Now, we concentrate on autonomous difference equations and we tackle with their dynamics. In the last decades, autonomous difference equations have appeared to hold significant relevance in the modelling of realistic phenomena and have been useful to describe a huge variety of applications in diverse fields such as biology, physics, economics or engineering. Therefore, the study of different dynamical properties of autonomous difference equations is crucial for the understanding of the behaviour of the models described by them. For instance, see [98, 34].

In this direction, in order to achieve the main goal of this dissertation, the analysis of diverse relevant dynamical properties for autonomous difference equations, the objective of this chapter is to tackle the problem of global periodicity for a particular family of difference equations of third order. Recall that an autonomous difference equation of k-order is globally periodic if every solution generated by it is periodic. In this case, it is known (see [84]) that the set of periods is bounded and, therefore, we can guarantee the existence of a positive integer p that is the least common multiple of the periods. In this line, we will say that the difference equation is globally periodic of period p, or, that it is a p-cycle.

Furthermore, every autonomous difference equation can be related to a discrete dynamical system by the associated map $F : \mathbb{R}^k \to \mathbb{R}^k$ given by

$$F(x_1, \dots, x_k) = \big(x_2, \dots, x_k, f(x_k, \dots, x_2, x_1)\big).$$

In terms of global periodicity, if the equation is a p-cycle, its associated dynamical system F satisfies $F^p \equiv \mathrm{Id}|_{\mathbb{R}^k}$, where $\mathrm{Id}|_{\mathbb{R}^k}$ denotes the identity map over $\mathbb{R}^k$. The reader interested in different tools that can be applied in order to deal with this problem is referred to [21].

As examples of p-cycles, the main families that can be found in the literature are rational cycles, [22]; and potential cycles, [18]. Probably, the most popular p-cycle is the 5-cycle $x_{n+1} = \frac{1+x_n}{x_{n-1}}$, known as Lyness' cycle, that receives its name after R.C. Lyness, who reported a collection of papers concerning the existence of cycles, [74, 75, 76]. It should be mentioned that, in an implicit way, this cycle was already known by Gauss, who dealt with it while working in the spherical geometry of the *pentagramma mirificum*, a spherical pentagram formed by five successively orthogonal great-circle arcs. The reader interested in the construction of the pentagram and its relation with the cycle can consult [27]. Other examples of well-known cycles are Todd's Equation $x_{n+3} = \frac{x_{n+2}+x_{n+1}+1}{x_n}$, an 8-cycle; or the 9-cycle $x_{n+2} = |x_{n+1}| - x_n$, see [28].

In the case of rational cycles, in [22] the authors considered the general difference equation of order k

$$x_{n+k} = \frac{A_1 x_n + A_2 x_{n+1} + \ldots + A_k x_{n+k-1} + A_0}{B_1 x_n + B_2 x_{n+1} + \ldots + B_k x_{n+k-1} + B_0}, \tag{3.1}$$

with initial conditions $(x_1, x_2, \ldots, x_k) \in (0, \infty)^k$, and $\sum_{i=0}^{k} A_i > 0$, $\sum_{i=0}^{k} B_i > 0$, $A_i \geq 0$, $B_i \geq 0$, and $A_1^2 + B_1^2 \neq 0$. They were able to prove that for $k \in \{1, 2, 3, 4, 5, 7, 9, 11\}$ any globally periodic recurrence of the family (3.1) is equivalent to one of the following difference equations

$$x_{n+1} = x_n, \quad x_{n+1} = \frac{1}{x_n}, \quad x_{n+2} = \frac{x_{n+1}}{x_n}, \quad x_{n+2} = \frac{1 + x_{n+1}}{x_n}, \quad x_{n+3} = \frac{1 + x_{n+2} + x_{n+1}}{x_n}.$$

It is still an open problem to know if the above result holds for any k.

Another scenario in which the problem of global periodicity has been studied is the second order difference equation

$$x_{n+2} = \frac{f(x_{n+1})}{x_n}, \tag{3.2}$$

with $f \in C^1\big([0, \infty), [0, \infty)\big)$. Mestel, [81], obtained all the p-cycles for that equation for $p \leq 5$. In his study, which was based on the resolution of functional equations, he proved that Equation (3.2) never exhibits 1-cycles or 2-cycles; the unique 3-cycle is given by $f(x) = \frac{C}{x}$ for some constant $C \in (0, \infty)$; the unique 4-cycle is given by $f(x) = \tilde{C}$ for some constant $\tilde{C} \in (0, \infty)$; and the 5-cycles are given by $f(x) = A \cdot (A^\alpha + x^\alpha)^{1/\alpha}$, where $\alpha \in (0, \infty)$ and $A \in (0, \infty)$.

Additionally, for more examples of p-cycles, the reader is referred to the monograph [40]; for a brief historical digression of the problem of global periodicity and open problems on the topic see [61]; and for different approaches to the problem see [21].

Concerning the techniques that can be employed to attack the problem of global periodicity, there exist diverse approaches. Mainly, the resolution of functional equations (for instance, see [7] and [81]); the application of techniques of discrete dynamical systems (for example, consult [17] and [95]); or the use of direct arguments of real analysis (for instance, see [2] and [104]).

One of the main problems related to global periodicity is the quest of families of p-cycles displaying a certain typology. In this sense, we dedicate this chapter to the analysis of a concrete family of difference equations in order to find new p-cycles. The work we develop here is mainly based on [68] and [70].

This chapter deeps in the advance of the global periodicity problem for the autonomous difference equation

$$x_{n+3} = x_i f(x_j, x_k), \tag{3.3}$$

where $i, j, k \in \{n, n+1, n+2\}$ are pairwise distinct, $f : (0, \infty)^2 \to (0, \infty)$ is continuous, and the initial conditions are positive real numbers, when $p = 6$ (it is worth-mentioning the appropriateness of working in the positive real line $(0, \infty)$, since while thinking in modelling and real applications, a lot of variables do not take negative values; in this line, for instance, consult the monograph on positive dynamical systems [56]). In concrete, firstly, in Section 3.1 we illustrate some real models displaying such form and we establish some general considerations concerning the equation. Then, Section 3.2 deals with the proof of the non-existence of 6-cycles when we assume the extra condition of f being symmetric, that is, $f(x, y) = f(y, x)$ for all $x, y > 0$.

Theorem B. *There are no* 6*-cycles of third order having the form* $x_{n+3} = x_i f(x_j, x_k)$*, whenever* $f : (0, \infty)^2 \to (0, \infty)$ *is a symmetric continuous map, and* $i, j, k \in \{n, n+1, n+2\}$ *are pairwise distinct.*

As a next step, in Section 3.3, we leave the symmetric condition and focus on the particular case where f separates variables, which means that $f(x, y) = g(x)h(y)$, for every $x, y > 0$, where $g, h : (0, \infty) \to (0, \infty)$ are continuous maps. In this case, we see that the unique 6-cycle displaying such form is given by the potential one, $x_{n+3} = x_n \left(\frac{x_{n+2}}{x_{n+1}}\right)^2$.

Theorem C. *Let us consider Equation* (3.17)*. The unique* 6*-cycle displaying such form is given by*

$$x_{n+3} = x_n \left(\frac{x_{n+2}}{x_{n+1}}\right)^2.$$

In fact, Subsection 3.3.1 shows that the unique potential 6-cycle exhibiting the form of Equation (3.3) is the one found in the case of separation of variables. Finally, Section 3.4 establishes some open problems and further lines of research related to the topic.

3.1 The family $x_{n+3} = x_i f(x_j, x_k)$

In this chapter we focus on the third order family of difference equations (3.3)

$$x_{n+3} = x_i f(x_j, x_k),$$

where $i, j, k \in \{n, n+1, n+2\}$ are pairwise distinct, $f : (0, \infty)^2 \to (0, \infty)$ is continuous, and the initial conditions are positive real numbers. Our main objective is to study the dynamical property of global periodicity.

As a first stage, relative to the dynamics of Equation (3.3), we can guarantee the existence of equilibrium points whenever it is globally periodic of period p. Indeed, in [46], the authors prove that any periodic homeomorphism defined on a space homeomorphic to $\mathbb{R}^n$ has fixed points if $n \leq 4$. In this sense, observe that being an equilibrium point of (3.3) is equivalent to be a fixed point of the associated discrete dynamical system $F : (0, \infty)^3 \to (0, \infty)^3$ given by

$$F(x_1, x_2, x_3) = \big(x_2, x_3, x_i f(x_j, x_k)\big).$$

This equivalence is convenient, since in the field of discrete dynamical systems, as it was pointed out before, it is well-known that if the corresponding equation is a p-cycle, $F^p \equiv \mathrm{Id}|_{(0,\infty)}$ holds. Thus, F is a periodic homeomorphism and, in our case, F is defined in a homeomorphic space to $\mathbb{R}^3$, so we get that F possesses a fixed point or, equivalently, the difference equation (3.3) has equilibrium points.

Lemma 3. *The set of equilibrium points of the difference equation* (3.3) *is nonempty.*

It is relevant to mention that there exists in the literature several applied models displaying a similar form of (3.3). Among others, we can highlight the Baumol-Wolf productivity model, [98], or the Pielou's discrete model in Ecology, [89].

On the one hand, the Baumol-Wolf equation is an economical model that takes information about the research and development (R&D) of diverse sectors and establishes different levels of production in the time period according to the corresponding information. In concrete, if x_n denotes the output level of the R&D sector in each period of time, the model is given by

$$x_{n+k} = x_{n+k-1}\left(1 + g\left(h\left[\sum_{i=1}^{k} f_i(x_{n+k-i})\right]\right)\right),$$

where $g, h, f_i, i \in \{1, \ldots, j\}$, are appropriate continuous real functions (consult [98] and [12] for the original model). On the other hand, Pielou's difference equation in Ecology is

$$N_{t+k} = \frac{aN_{t+k-1}}{1 + bN_{t-1}},$$

with k a non-negative integer, a and b positive real parameters and N_t representing the population density at the n-th time step (see [89]). Moreover, in [16], the reader can find a huge variety of rational difference equations exhibiting the form of (3.3).

Even more, we can find economical models described by a system of nonlinear difference equations where each one of the equations is of the form (3.3). For instance, in [98] the author deals with an economical model in which the consumer demand is determined by May's exponential variations of the Lotka-Volterra polynomial equations. Concretely, the model is given by

$$x_{n+1}^i = x_n^i \cdot \exp\left(\alpha_i - \sum_{j=1}^{m} c_{i,j} x_n^j\right), \quad i = 1, \ldots, m,$$

where we have m goods, products or commodities; x^i denotes the consumer demand for the i-th good, $i = 1, \ldots, m$; and $\alpha_i, c_{i,j}$ are constants depending on a total budget allocation and the unit prices of the goods.

In [7], the authors studied Equation (3.3) and determined the existence of the unique p-cycles for $p \leq 5$. In concrete, they proved the following:

- The unique 3-cycle of the form (3.3) is given by

$$x_{n+3} = x_n.$$

- The unique 4-cycle exhibiting the form (3.3) is

$$x_{n+3} = x_n \cdot \frac{x_{n+2}}{x_{n+1}}.$$

- The unique 5-cycles displaying the form (3.3) are given by

$$x_{n+3} = x_n \cdot \left(\frac{x_{n+2}}{x_{n+1}}\right)^{\phi} \quad \text{and} \quad x_{n+3} = x_n \cdot \left(\frac{x_{n+2}}{x_{n+1}}\right)^{\varphi},$$

where ϕ and φ are the roots of $\lambda^2 - \lambda - 1 = 0$.

The techniques they employed are basically the direct computation of the orbits generated by (3.3) for $p \leq 4$; and the resolution of functional equations for $p = 5$.

As a next step, we establish some general considerations concerning Equation (3.3) that will be useful in the sequel.

3.1.1 General considerations

Firstly, recall that an equilibrium point of a difference equation is a constant solution $\bar{x}$. In this sense, for Equation (3.3), this yields to the relation $\bar{x} = \bar{x} f(\bar{x}, \bar{x})$, that is, $f(\bar{x}, \bar{x}) = 1$. Hence, we are able to describe the set of equilibrium points of (3.3) by the closed set

$$\mathcal{F}_1 := \{x > 0 : f(x, x) = 1\}.$$

Observe that belonging to $\mathcal{F}_1$ is equivalent to be a fixed point of the associated discrete dynamical system F given by $F : (0, \infty)^3 \to (0, \infty)^3$,

$$F(x_1, x_2, x_3) = \big(x_2, x_3, x_i f(x_j, x_k)\big),$$

where $i, j, k \in \{n, n+1, n+2\}$ are pairwise distinct. Therefore, since we are assuming that (3.3) is a 6-cycle, by Lemma 3, we get the non-emptiness of the closed set $\mathcal{F}_1$.

Lemma 4. $\mathcal{F}_1 \neq \emptyset$.

Furthermore, we can assume without loss of generality that $1 \in \mathcal{F}_1$ and $f(1,1) = 1$. Indeed, take $\bar{x} \in \mathcal{F}_1$ and make the change of variables $w_l = \frac{x_l}{\bar{x}}$. Then, Equation (3.3), can be rewritten as

$$\bar{x}w_{n+3} = \bar{x}w_i f(\bar{x}w_j, \bar{x}w_k),$$

and we get

$$w_{n+3} = w_i f(\bar{x}w_j, \bar{x}w_k) = w_i \tilde{f}(w_j, w_k).$$

Now, observe that for $\bar{w} = 1$, we obtain $1 = \tilde{f}(1,1) = f(\bar{x}, \bar{x})$, so $\bar{w} = 1$ is an equilibrium point. To sum up, from now on we assume without loss of generality that

$$f(1,1) = 1 \quad \text{with} \quad 1 \in \mathcal{F}_1. \tag{$\blacklozenge$}$$

On the other hand, taken a point $x > 0$, we understand by fiber map $f_x : (0,\infty) \to (0,\infty)$ the continuous map defined as $f_x(y) = f(x,y)$, for all $y > 0$. Moreover, we will use the notation f_x^n, $n \geq 1$, to denote the iteration $(f_x \circ f_x \circ \underbrace{\ldots}_{n \text{ times}} \circ f_x)(\cdot)$. Now, we will see that f_x is bijective for some cases of Equation (3.3).

Lemma 5. *Consider $x_{n+3} = x_{n+2}f(x_{n+1}, x_n)$ or $x_{n+3} = x_{n+1}f(x_{n+2}, x_n)$ and assume that they are 6-cycles. Then, in both cases, the fiber map f_x is bijective for all $x > 0$.*

Proof. As a first step, take $x_{n+3} = x_{n+2}f(x_{n+1}, x_n)$ and assume that it is a 6-cycle. Fix $x > 0$ and set the initial conditions $x_1 = x, x_2 = 1, x_3 = y$. Now, since the value of y is arbitrarily taken, we can show the surjectivity of f_x by applying the fact that the difference equation is a 6-cycle:

$$y = x_3 = x_9 = x_8 f(x_7, x_6) = x_2 f(x_1, x_6) = f(x, x_6).$$

Next, suppose that $f(x,y) = f(x,z)$ for some $y, z > 0$. Observe that the initial conditions $y_1 = y, y_2 = x, y_3 = 1$, and $z_1 = z, z_2 = x, z_3 = 1$ generate the same solution, namely, $y_4 = f(x,y) = f(x,z) = z_4$, $y_5 = y_4 f(y_3, y_2) = z_4 f(z_3, z_2) = z_5$, ... and so on. Then, $z = z_7 = y_7 = y$ and we obtain the injectivity of f_x.

On the other hand, to prove that the fiber map f_x is bijective for the difference equation $x_{n+3} = x_{n+1}f(x_{n+2}, x_n)$ is equivalent and is omitted. Here, the surjectivity follows by fixing a point $x > 0$ and considering the initial conditions $x_1 = 1, x_2 = x, x_3 = y$; and the injectivity by assuming $f(x,u) = f(x,v)$ for some $u, v > 0$ and seeing that the initial conditions $u_1 = u, u_2 = 1, u_3 = x$ and $v_1 = v, v_2 = 1, v_3 = x$ generate the same solution under the equation. □

Remark 1. *Notice that Lemma 5 can be generalized to any p-cycle, $p \geq 4$. Moreover, the bijective condition for the fiber map f_x follows without the additional condition on the symmetry of f.*

The following result concerning monotonicity is immediate.

Lemma 6. *Let $\varphi : (0,\infty) \to (0,\infty)$ be an increasing (decreasing) homeomorphism. Then:*

(a) φ^3 is an increasing (decreasing) homeomorphism.

(b) The map $g(x) := \frac{1}{\varphi(x)}$ is a decreasing (increasing) homeomorphism.

In the particular case where the continuous map f separates variables, that is, Equation (3.3) is of the form

$$x_{n+3} = x_i g(x_j) h(x_k), \tag{$\bullet$}$$

with $i, j, k \in \{n, n+1, n+2\}$ are pairwise distinct, $g, h : (0,\infty)^2 \to (0,\infty)$ are continuous, and the initial conditions are positive real numbers, we can be more precise about the equilibrium points of the equation. Concretely, notice that $\bar{x}$ is an equilibrium point of $x_{n+3} = x_i g(x_j) h(x_k)$ if and only if $\bar{x} = \bar{x} g(\bar{x}) h(\bar{x})$, that is, $g(\bar{x})h(\bar{x}) = 1$. So, we can describe the set of equilibrium points by the closed set

$$\mathcal{F}_2 = \{x > 0 : g(x)h(x) = 1\}.$$

Moreover, taking $f(x_j, x_k) = g(x_j)h(x_k)$, Lemma 4 implies the non-emptiness character of $\mathcal{F}_2$.

Lemma 7. *$\mathcal{F}_2 \neq \emptyset$.*

On the other hand, similarly to the general case, by suitable changes of variables, we can assume without loss of generality that $1 \in \mathcal{F}_2$ and $g(1) = h(1) = 1$. Indeed, by Lemma 7 we can take a point $\bar{x} \in \mathcal{F}_2$. As a first step, rewrite Equation ($\bullet$) as

$$x_{n+3} = x_i \left(\frac{g(x_j)}{g(\bar{x})} \right) \big(g(\bar{x}) h(x_k) \big) =: x_i \widetilde{g}(x_j) \widetilde{h}(x_k).$$

Here, we have $\widetilde{g}(\bar{x}) = \widetilde{h}(\bar{x}) = 1$. Therefore, unless otherwise stated, in the sequel we assume

$$g(1) = h(1) = 1 \text{ with } 1 \in \mathcal{F}_2. \tag{$\clubsuit$}$$

Finally, as a direct consequence of Lemma 5, we can derive that the map h is an homeomorphism for two cases of the difference equation with separation of variables.

Lemma 8. *Consider $x_{n+3} = x_{n+2} g(x_{n+1}) h(x_n)$ or $x_{n+3} = x_{n+1} g(x_{n+2}) h(x_n)$ and assume that they are 6-cycles. Then, in both cases, h is a homeomorphism.*

3.2 Non-existence of 6-cycles for symmetric maps

The target of this section is to prove the non-existence of 6-cycles exhibiting the form of Equation (3.3), $x_{n+3} = x_i f(x_j, x_k)$, with $i, j, k \in \{n, n+1, n+2\}$ pairwise distinct and $f : (0,\infty)^2 \to (0,\infty)$ being a continuous map whenever we assume the extra condition of f being symmetric, which means that $f(x, y) = f(y, x)$, for every $x, y > 0$. This section is based on [68].

In the sequel, unless otherwise stated, we assume that Equation (3.3) is a 6-cycle. Notice that if we denote the initial conditions that generate a solution under (3.3) by x_1, x_2 and x_3, being a 6-cycles means, in particular, that $x_1 = x_7$, $x_2 = x_8$ and $x_3 = x_9$. Therefore, whenever we employ an argument *by global periodicity*, we mean that such relations hold.

An argument **by global periodicity** purports that $x_1 = x_7, x_2 = x_8$ and $x_3 = x_9$.

The structure of the section is the following: we analyze the three possible configurations of the difference equation in Subsections 3.2.1-3.2.3. In each case, we see that there are no 6-cycles displaying the form of (3.3) with f being a continuous symmetric map and we gather the whole study in the main result of the section, namely, Theorem B.

3.2.1 The case $x_{n+3} = x_{n+2} f(x_{n+1}, x_n)$

In the present subsection, we focus on the difference equation of third order

$$x_{n+3} = x_{n+2} f(x_{n+1}, x_n), \tag{3.4}$$

where $f : (0,\infty)^2 \to (0,\infty)$ is a continuous map. Recall that, unless otherwise stated, we assume that (3.4) is a 6-cycle.

Firstly, we show some general results concerning Equation (3.4) without assuming the symmetry of f.

Lemma 9. *Consider Equation (3.4). Then, the following holds:*

(a) The inverse map of $f_x(\cdot)$ is given by

$$\varphi_x(z) = z f(1, x) \cdot f(z, 1) \cdot f\big(z f(1, x), z\big).$$

(b) If $f(\alpha, \beta) = f(\beta, \alpha) = 1$ for some $\alpha, \beta \in (0,\infty)$, then $\alpha = \beta$.

Proof. We start proving (a). To do so, set the initial conditions $x_1 = x$, $x_2 = 1$ and $x_3 = z$, which generate under (3.4) the terms $x_4 = z f(1, x)$, $x_5 = z f(1, x) f(z, 1)$ and $x_6 = z f(1, x) \cdot f(z, 1) \cdot f\big(z f(1, x), z\big)$. On the other hand, since (3.4) is a 6-cycle, by global periodicity,

$$x_3 = z = x_9 = x_8 f(x_7, x_6) = x_2 f(x_1, x_6) = f(x, x_6),$$

which implies

$$z = f\big(x,\, zf(1,x)\cdot f(z,1)\cdot f(zf(1,x),z)\big) = f_x\big(zf(1,x)\cdot f(z,1)\cdot f(zf(1,x),z)\big).$$

Next, for Part (b), take $x_1 = \alpha, x_2 = \beta, x_3 = \alpha$. These initial conditions generate under Equation (3.4) the terms $x_4 = \alpha, x_5 = \alpha, x_6 = \alpha f(\alpha,\alpha)$. Again, since (3.4) is a 6-cycle, by global periodicity, we obtain

$$\alpha = x_1 = x_7 = x_6 f(x_5, x_4) = \alpha f(\alpha,\alpha) f(\alpha,\alpha) = \alpha\big(f(\alpha,\alpha)\big)^2.$$

Thus, $f(\alpha,\alpha) = 1$. Similarly, by considering the initial conditions $y_1 = \beta, y_2 = \alpha, y_3 = \beta$, we achieve that $f(\beta,\beta) = 1$. Lastly, take the initial conditions $z_1 = \alpha, z_2 = \alpha, z_3 = \beta$. By (3.4), we get $z_4 = \beta, z_5 = \beta, z_6 = \beta, \ldots, z_j = \beta$ for all $j \geq 3$. Therefore, since, in particular, being a 6-cycle implies $z_1 = z_7$, we conclude that $\alpha = \beta$. □

Lemma 10. *Consider Equation (3.4). The following relations hold:*

(a) $xf(x,1)\cdot f(x,x)\cdot f\big(xf(x,1),x\big) = 1$, *for all* $x > 0$.

(b) $y = f\big(x,\, y\cdot f(1,x)\cdot f(y,1)\cdot f(yf(1,x),y)\big)$ *for all* $x, y > 0$.

Proof. As a first step, since by hypothesis the recursion (3.4) is a 6-cycle, by global periodicity, it holds:

$$x_3 = x_9 = x_8 f(x_7, x_6) = x_2 f(x_1, x_6). \tag{3.5}$$

Now, for Part (a) consider the initial conditions $y_1 = 1, y_2 = y_3 = x$. By (3.4), we get $y_4 = xf(x,1), y_5 = xf(x,1)\cdot f(x,x)$, $y_6 = xf(x,1)\cdot f(x,x)\cdot f\big(xf(x,1),x\big)$. Hence, by (3.5)

$$x = y_3 = y_2 f(y_1, y_6) = xf\big(1, xf(x,1)\cdot f(x,x)\cdot f(xf(x,1),x)\big),$$

and, therefore,

$$1 = f\big(1, xf(x,1)\cdot f(x,x)\cdot f\left(xf(x,1),x\right)\big). \tag{3.6}$$

Now, bearing in mind, by Lemma 9, that the fiber map $f_1(\cdot)$ is bijective, from (♦) and (3.6) we deduce

$$xf(x,1)\cdot f(x,x)\cdot f\left(xf(x,1),x\right) = 1.$$

Finally, for Part (b), we iterate the initial conditions $x_1 = x, x_2 = 1, x_3 = y$ under (3.4) to obtain $x_4 = y\cdot f(1,x)$, $x_5 = y\cdot f(1,x)\cdot f(y,1)$ and $x_6 = y\cdot f(1,x)\cdot f(y,1)\cdot f\big(yf(1,x),y\big)$. Then, from (3.5), it follows

$$y = x_3 = x_2 f(x_1, x_6) = f(x, x_6) = f\big(x,\, y\cdot f(1,x)\cdot f(y,1)\cdot f(yf(1,x),y)\big).$$

□

Now, recall that the nonempty set $\mathcal{F}_1 = \{x > 0 : f(x,x) = 1\}$ represents the set of equilibrium points of the studied difference equation. The following result gives a relation for the equilibrium points of (3.4).

Lemma 11. *Let $x \in \mathcal{F}_1$, then $\frac{1}{x} = f\big(x, f(1,x)\big)$.*

Proof. Take $x_1 = x_2 = x$, $x_3 = 1$, and generate the remaining terms of the 6-cycle, that are, $x_4 = f(x,x) = 1, x_5 = f(1,x), x_6 = f(1,x) \cdot f(1,1)$. Therefore, by global periodicity,

$$1 = x_3 = x_9 = x_8 f(x_7, x_6) = x_2 f(x_1, x_6) = x f\big(x, f(1,x)\big),$$

and the result follows. □

Now, after establishing some general properties for the 6-cycles displaying the form of (3.4), we assume the extra condition of f being symmetric. In this particular case, we have the following properties:

Lemma 12. *Consider (3.4) and assume that f is a continuous symmetric map. Then,*

(a) If $f(x,y) = 1$, it follows that $x = y$.

(b) $f(x,x) = 1$ for all $x > 0$, which means, $\mathcal{F}_1 = (0, \infty)$.

(c) $\frac{1}{x} = f\big(x, f(1,x)\big)$ for all $x > 0$.

Proof. For Part (a), assume $f(x,y) = 1$. Since f is a symmetric continuous map, $f(x,y) = f(y,x) = 1$ and by Lemma 9-(b), we get $x = y$.

Next, for Part (c), set the initial conditions $x_1 = x_2 = 1$, $x_3 = x$ and apply (♦) in order to get the following terms: $x_4 = x$, $x_5 = xf(x,1)$, $x_6 = xf(x,1)f(x,x)$. Next, since (3.4) is a 6-cycle,

$$1 = x_2 = x_8 = x_7 f(x_6, x_5) = f\big(xf(x,1) \cdot f(x,x), xf(x,1)\big).$$

Therefore, we can apply Part (a) to obtain $xf(x,1) \cdot f(x,x) = xf(x,1)$, that yields to $f(x,x) = 1$. Observe that this allows us to deduce $\mathcal{F}_1 = (0, \infty)$.

Finally, Part (d) follows directly by Part (c) and Lemma 11. □

Once we have seen some properties related to Equation (3.4), we can prove the main result of this subsection.

Proposition 2. *There are no 6-cycles of third order having the form $x_{n+3} = x_{n+2} f(x_{n+1}, x_n)$, whenever $f : (0,\infty)^2 \to (0,\infty)$ is a symmetric continuous map.*

Proof. Take an arbitrary positive real number x. On the one hand, since f is a symmetric map, by Lemma 12-(c), it follows that $f(z,z) = 1$ for all $z > 0$. Thus, Lemma 10-(a) implies

$$xf(1,x) \cdot f\big(xf(1,x), x\big) = 1. \tag{3.7}$$

On the other hand, applying Lemma 10-(b) with $x = y$, we get

$$x = f\big(x,\, xf(1,x) \cdot f(1,x) \cdot f(xf(1,x), x)\big). \tag{3.8}$$

Next, if we replace (3.7) into (3.8), we obtain $x = f\big(x, f(1,x)\big)$. However, from Lemma 12-(d), we deduce that $x = \frac{1}{x}$, so $x = 1$, which contradicts that x was arbitrarily chosen. □

Remark 2. *It should be highlighted that the non-existence of* 6*-cycles displaying the form of* (3.4) *has been proved under the extra assumption of the symmetry of* f*. In this sense, it is an open problem to determine if the result remains true or not for the general case of any continuous map* f*.*

3.2.2 The case $x_{n+3} = x_{n+1}f(x_{n+2}, x_n)$

As a next step, we study in detail the third-order difference equation

$$x_{n+3} = x_{n+1}f(x_{n+2}, x_n), \tag{3.9}$$

with $f : (0,\infty)^2 \to (0,\infty)$ being a continuous symmetric map.

Recall that the set of equilibrium points, $\mathcal{F}_1$, is nonempty and the assumption of Equation (3.9) being globally periodic of period 6.

Lemma 13. *Suppose that* $f : (0,\infty)^2 \to (0,\infty)$ *is continuous and symmetric. Then,*

(a) $f(x,x) = 1$ *for all* $x > 0$*, or equivalently* $\mathcal{F}_1 = (0,\infty)$*.*

(b) If $f(x,y) = 1$*, then* $x = y$*.*

Proof. For Part (a), take an arbitrary point $x > 0$ and consider the initial conditions $x_1 = 1$, $x_2 = x, x_3 = 1$. By (3.9), $x_4 = xf(1,1) = x$, $x_5 = f(x,x)$ and $x_6 = xf\big(f(x,x),1\big)$. Thus, by global periodicity,

$$x = x_2 = x_8 = x_6 f(x_7, x_5) = x_6 f(x_1, x_5) = xf\big(f(x,x),1\big) \cdot f\big(1, f(x,x)\big),$$

which implies $1 = f\big(f(x,x),1\big) \cdot f\big(1, f(x,x)\big)$, or, by the symmetry of f, $1 = \big[f_1\big(f(x,x)\big)\big]^2$, that means, $1 = f_1\big(f(x,x)\big)$. Finally, by Lemma 5, we apply that $f_1(\cdot)$ is a bijective map and that $1 = f(1,1)$, to achieve $f(x,x) = 1$.

For Part (*b*), we assume $f(x,y) = 1$. Nevertheless, by Part (a), $f(x,x) = 1$ and due to the bijectivity of the fiber map f_x obtained in Lemma 5, it follows that $x = y$. □

Finally, we prove the result concerning the non-existence of 6-cycles displaying the form (3.9).

Proposition 3. *There are no* 6*-cycles of third order having the form* $x_{n+3} = x_{n+1}f(x_{n+2}, x_n)$*, whenever* $f : (0,\infty)^2 \to (0,\infty)$ *is a continuous symmetric map.*

Proof. We proceed by contradiction. Assume that the difference equation (3.9) is a 6-cycle and set the initial conditions $x_1 = x, x_2 = x_3 = 1$. Then, the rest of the cycle is given by

$$x_4 = f(1,x) = f_1(x), \quad x_5 = f\big(1, f_1(x)\big) = f_1^2(x), \quad x_6 = f_1(x) f\left(1, f_1^2(x)\right) = f_1(x) \cdot f_1^3(x).$$

Now, the global periodicity guarantees that

$$1 = x_3 = x_9 = x_7 f(x_8, x_6) = x_1 f(x_2, x_6) = x f_1\left(f_1(x) \cdot f_1^3(x)\right).$$

Thus, $\frac{1}{x} = f_1\big(f_1(x) \cdot f_1^3(x)\big)$ and, since $g(x) = \frac{1}{x}$ is a decreasing function, we deduce that $f_1\big(f_1(x) \cdot f_1^3(x)\big)$ is decreasing too. Next, we distinguish two cases depending on the monotonicity of $f_1(x)$:

- If $f_1(x)$ is increasing, by Lemma 6, f_1^3 is also increasing. Hence, $f_1(x) \cdot f_1^3(x)$ and $f_1\big(f_1(x) \cdot f_1^3(x)\big)$ are increasing too, a contradiction.

- If $f_1(x)$ is decreasing, by Lemma 6, f_1^3 is decreasing too. Thus, $f_1(x)\cdot f_1^3(x)$ is decreasing and $f_1\big(f_1(x) \cdot f_1^3(x)\big)$ is an increasing map, which yields to another contradiction.

In conclusion, we derive the non-existence of 6-cycles of the form $x_{n+3} = x_{n+1} f(x_{n+2}, x_n)$ whenever f is a continuous symmetric map. □

Remark 3. *As it happened in the previous case, the possible existence of* 6*-cycles of the form* $x_{n+3} = x_{n+1} f(x_{n+2}, x_n)$ *when* f *is not a symmetric map is still an open problem.*

3.2.3 The case $x_{n+3} = x_n f(x_{n+2}, x_{n+1})$

Lastly, we analyze the remaining case, that is,

$$x_{n+3} = x_n f(x_{n+2}, x_{n+1}), \tag{3.10}$$

with $f : (0,\infty)^2 \to (0,\infty)$ a continuous map. Recall that $f(1,1) = 1$ by (♦). As usual, we assume that Equation (3.10) is a 6-cycle. Firstly, we establish some general properties for any continuous map f. Again, the set of equilibrium points of (3.10) is given by $\mathcal{F}_1 \neq \emptyset$. Moreover, in the sequel we employ $\operatorname{Im}(\cdot)$ to denote the image of a map and $\langle\ ,\ \rangle$ to represent either an open interval $(\ ,\ )$, or a compact one $[\ ,\ ]$.

Lemma 14. *Consider Equation* (3.10)*. Then,* $1 \in \operatorname{Im}(f)$ *and* $\operatorname{Im}(f)$ *adopts one of the following forms*

$$\left[m, \frac{1}{m}\right], \quad \left(m, \frac{1}{m}\right), \quad or \quad (0,\infty),$$

where $0 < m < 1$.

Proof. As a first step, notice that we can assume that f is non-constant. Otherwise, since $f(1,1) = 1$, consequently, $f \equiv 1$. This would yield to the 3-cycle $x_{n+3} = x_n$, which contradicts the fact that (3.10) is globally periodic of period 6. Furthermore, observe that the non-emptiness of $\mathcal{F}_1$ implies that $1 \in \mathrm{Im}(f)$.

As a second step, consider arbitrary initial conditions $x_1 = x, x_2 = y, x_3 = z$, that generate under (3.10) the terms $x_4 = xf(z,y)$, $x_5 = yf(x_4,z)$ and $x_6 = zf(x_5,x_4)$. Then, by global periodicity,

$$z = x_3 = x_9 = x_6 f(x_8,x_7) = x_6 f(x_2,x_1) = zf(x_5,x_4) \cdot f(y,x),$$

and consequently, $1 = f(x_5,x_4) \cdot f(y,x)$.

Next, by using a direct argument of continuity and connectivity, if $f(y,x) \leq 1$, it is immediate to deduce that the closed interval $\left[f(y,x), \frac{1}{f(y,x)}\right]$ is included in $\mathrm{Im}(f)$ (the case where $f(y,x) > 1$ is analogous and it is left in charge of the reader). Finally, let

$$\mu := \sup\{f(x,y) : x,y > 0\}.$$

Obviously, $\mu > 1$ and, with a similar reasoning as in the above paragraph, it is direct to see that $\mathrm{Im}(f) = \left\langle \frac{1}{\mu}, \mu \right\rangle$, where μ can be finite or infinite. □

In the following result, we use $f_1(\cdot)$ and $f^1(\cdot)$ to denote the fiber maps $f(1,\cdot)$ and $f(\cdot,1)$, respectively.

Lemma 15. *Consider Equation* (3.10). *It holds* $1 \in \mathrm{Im}(f_1) \cap \mathrm{Im}(f^1)$.

Proof. It is a direct consequence of the fact $f(1,1) = 1$ given by (♦). □

Lemma 16. *Let* $(\alpha,\beta) \in (0,\infty)^2$ *satisfy* $f(\alpha,\beta) = f(\beta,\alpha) = 1$. *Then* $\alpha, \beta \in \mathcal{F}_1$.

Proof. Take the initial conditions $x_1 = x_2 = \alpha$, $x_3 = \beta$. By (3.10), we have $x_4 = x_5 = \alpha$, and $x_6 = \beta f(\alpha,\alpha)$. Now, by global periodicity,

$$\beta = x_3 = x_9 = x_6 f(x_8,x_7) = x_6 f(x_2,x_1) = \beta\big[f(\alpha,\alpha)\big]^2,$$

and we deduce that $1 = \big[f(\alpha,\alpha)\big]^2$, which implies $f(\alpha,\alpha) = 1$ and $\alpha \in \mathcal{F}_1$. On the other hand, by taking $y_1 = y_2 = \beta, y_3 = \alpha$, we can proceed analogously to obtain $\beta \in \mathcal{F}_1$. □

Now, we leave the general case and focus on Equation (3.10) when we assume the extra condition of f being symmetric.

Lemma 17. *Let* $f : (0,\infty)^2 \to (0,\infty)$ *be a symmetric continuous map. If* $f(1,z) = 1$ *for all* $z > 0$, *then* $f(x,y) = 1$ *for all* $x, y > 0$.

Proof. Set $x_1 = y, x_2 = x, x_3 = 1$ and iterate them under (3.10) to achieve $x_4 = yf(1,x) = y$, $x_5 = x_2 f(x_4, x_3) = xf(y, 1) = x$ and $x_6 = x_3 f(x_5, x_4) = f(x, y)$. Hence, by global periodicity,

$$1 = x_3 = x_9 = x_6 f(x_8, x_7) = f(x,y)f(x_2,x_1) = f(x,y)f(x,y) = \big[f(x,y)\big]^2$$

and $f(x, y) = 1$. □

Observe that Lemma 17 implies that if Equation (3.10) is a 6-cycle, then the fiber map $f_1(\cdot)$ cannot be constantly 1, because, otherwise, we would have the 3-cycle $x_{n+3} = x_n$.

Lemma 18. *Let $f : (0,\infty)^2 \to (0,\infty)$ be a symmetric continuous map. Then, for each $x > 0$ there exists a $z = z(x) > 0$ such that $f\big(x, z(x)\big) = 1$.*

Proof. Let us consider $x_1 = x, x_2 = x_3 = 1$ and obtain $x_4 = xf(1, 1) = x$, $x_5 = f(x, 1)$, and $x_6 = f\big(f(x, 1), x\big)$ under Equation (3.10), where we have used that $f(1, 1) = 1$. Now, by global periodicity,

$$1 = x_3 = x_9 = x_6 f(x_8, x_7) = x_6 f(x_2, x_1) = f\big(f(x,1), x\big) \cdot f(1, x).$$

Therefore, $1 = f_x\big(f(1, x)\big) \cdot f_x(1)$ and the continuity of f_x allows us to induce the existence of a point $z = z(x)$ such that $1 = f_x(z) = f\big(x, z(x)\big)$. □

Lemma 19. *Let $f : (0,\infty)^2 \to (0,\infty)$ be a symmetric continuous map. Then, $\mathcal{F}_1 = (0,\infty)$.*

Proof. Given an arbitrary $x > 0$, by Lemma 18, there exists a $z = z(x) > 0$ such that $f\big(x, z(x)\big) = 1$. Now, applying the symmetry of f and Lemma 16, we deduce that $f\big(z(x), z(x)\big) = f(x, x) = 1$. □

Lemma 20. *Let $f : (0,\infty)^2 \to (0,\infty)$ be a symmetric continuous map. Then,*

$$f(x,y) \cdot f\big(x, yf(x,y)\big) = 1 \ \textit{for all} \ x, y > 0.$$

In particular, $f_1(y) \cdot f_1\big(yf_1(y)\big) = 1$, for all $y > 0$.

Proof. Take $x_1 = x, x_2 = x_3 = y$. Bearing in mind Lemma 19, we get that the following terms of the solution under Equation (3.10) are $x_4 = x$, $x_5 = yf(x, y)$ and $x_6 = yf\big(yf(x, y), x\big)$. So, by global periodicity,

$$y = x_3 = x_9 = x_6 f(x_8, x_7) = yf\big(yf(x,y), x\big) \cdot f(y, x),$$

and the result follows. □

Lemma 21. *Assume that $f : (0,\infty)^2 \to (0,\infty)$ is a symmetric continuous map. Then, for all $y > 0$ it holds*

$$\frac{1}{f(1,y)} = f\left(y, \frac{1}{f(1,y)}\right).$$

Proof. Let us denote $z := f(1, y)$. As a first step, take the initial conditions $x_1 = x_2 = 1$, $x_3 = y$ and iterate them under (3.10) to obtain $x_4 = z$, $x_5 = f(z, y)$, $x_6 = yf\big(z, f(z, y)\big)$. Hence, applying that (3.10) is a 6-cycle, we get the following equalities:

$$
\begin{aligned}
1 &= x_1 = x_7 = x_4 f(x_6, x_5) = zf\big(yf\big(z, f(z, y)\big), f(z, y)\big),\\
1 &= x_2 = x_8 = x_5 f(x_7, x_6) = x_5 f(x_1, x_6) = f(z, y) \cdot f\big(1, yf\big(z, f(z, y)\big)\big),\\
y &= x_3 = x_9 = x_6 f(x_8, x_7) = x_6 f(x_2, x_1) = yf\big(z, f(z, y)\big) \cdot f(1, 1) = yf\big(z, f(z, y)\big).
\end{aligned}
$$

Observe that, bearing in mind the last equation, $1 = f\left(z, f(z, y)\right)$, we can rewrite the other two as

$$
1 = zf\big(y, f(z, y)\big), \tag{3.11}
$$
$$
1 = f(z, y) \cdot f(1, y), \tag{3.12}
$$

respectively.

From (3.12), the symmetry of f, and the fact that $z = f(1, y)$,

$$
\frac{1}{f(1, y)} = f(z, y) = f(y, z) = f\big(y, f(1, y)\big).
$$

Therefore, if we substitute $f(y, z) = f\big(y, f(1, y)\big)$ by $\frac{1}{f(1,y)}$ in (3.11), we obtain $\frac{1}{z} = f\big(y, \frac{1}{f(1,y)}\big)$, that is, $\frac{1}{f(1,y)} = f\big(y, \frac{1}{f(1,y)}\big)$. □

Lemma 22. *Assume that* $f : (0, \infty)^2 \to (0, \infty)$ *is a symmetric continuous map. Given* $\alpha > 0$, *let* β *be a fixed point of the fiber map* $f_\alpha(\cdot)$. *Then:*

(a) $f\big(\alpha, \beta^2\big) = \frac{1}{\beta}$.

(b) $f\big(1, \beta^2\big) = 1$.

Proof. Set the initial conditions $x_1 = \alpha, x_2 = x_3 = \beta$. Now, we compute the following terms of the sequence generated by such initial conditions under (3.10). To do so, we take into account that $f(\beta, \beta) = \beta$ by Lemma 19 and that β is a fixed point for the fiber map $f_\alpha(\cdot)$. Then, $x_4 = \alpha$, $x_5 = \beta f(\alpha, \beta) = \beta^2$, and $x_6 = \beta f(\beta^2, \alpha)$. As a next step, by global periodicity,

$$
\begin{aligned}
\alpha &= x_1 = x_7 = x_4 f(x_5, x_6) = \alpha f\big(\beta^2, \beta f(\beta^2, \alpha)\big),\\
\beta &= x_3 = x_9 = x_6 f(x_7, x_8) = x_6 f(x_1, x_2) = \beta f(\beta^2, \alpha) f(\alpha, \beta) = \beta^2 f(\beta^2, \alpha).
\end{aligned}
$$

Notice that the second equation yields to $\frac{1}{\beta} = f\left(\alpha, \beta^2\right)$ and Part (a) is proved. Moreover, the first relation implies $1 = f\big(\beta^2, \beta f(\beta^2, \alpha)\big)$, or equivalently, $1 = f(\beta^2, \beta f(\alpha, \beta^2))$ from the symmetry of f. So, if we apply Part (a), we get $1 = f\left(\beta^2, \beta \cdot \frac{1}{\beta}\right) = f(\beta^2, 1)$ and finally, again from the symmetry of f, $f(1, \beta^2) = 1$. □

Now, let us define the non-empty set

$$F := \left\{ \frac{1}{f(1,x)} : x > 0 \right\}. \tag{3.13}$$

Also, we say that $z \in F^2$ if and only if $z = u^2$ for some $u \in F$. The next result provides some properties concerning F.

Lemma 23. *F is connected, with non-empty interior, and $f_1|_{F^2} \equiv 1$.*

Proof. On the one hand, the continuity of f allows us to induce the connectivity of F. On the other hand, notice that F cannot be the singleton $\{1\}$. Indeed, $f_1 \equiv 1$ and Lemma 17 would imply that $f(u,v) = 1$ for all u, v. This fact reduces (3.10) into the 3-cycle $x_{n+3} = x_n$.

Finally, we show that $f(1,z) = 1$ for all $z \in F^2$. Let $z \in F^2$. This means that

$$z = \frac{1}{\big(f(1,x)\big)^2}, \quad \text{for some } x > 0.$$

By Lemma 21, we get $f\left(x, \frac{1}{f(1,x)}\right) = \frac{1}{f(1,x)}$, and if we apply Lemma 22 with $\alpha = x$ and $\beta = \frac{1}{f(1,x)}$, it yields that

$$f\left(1, \frac{1}{\big(f(1,x)\big)^2}\right) = 1,$$

which means, $f(1,z) = 1$. □

Lemma 24. *Assume that $f : (0,\infty)^2 \to (0,\infty)$ is a symmetric continuous map. Let $u, v \in (0,\infty)$, $u < v$, satisfy $f(1,u) = f(1,v) = 1$. Then, there exists $w \in (u,v)$ such that $f(1,w) = 1$.*

Proof. We proceed by *reductio ad absurdum.* To do so, define the map $g(x) := xf(1,x), x > 0$ and observe that $g(u) = u, g(v) = v$. Now, we distinguish two cases:

- Assume that $f(1,x) > 1$ for all $x \in (u,v)$. Here, we have $g(x) > u$ in (u,v). Then, since $g(x)$ is continuous, we can assure the existence of a point z sufficiently close to u such that $u < g(z) < v$, that is, $u < zf_1(z) < v$ (see Figure 3.1).

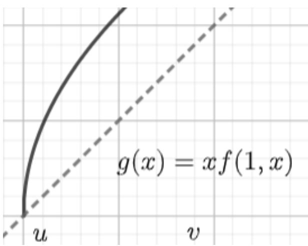


Figure 3.1: u is a repeller from the right for the map $g(x) = xf_1(x)$.

  In conclusion, $f\big(1, zf(1,z)\big) > 1$, and $f(1,z) \cdot f\big(1, zf_1(z)\big) > 1$, and we derive a contradiction due to Lemma 20.

- Assume that $f(1,x) < 1$ for all $x \in (u,v)$. Now, we take a point y in (u,v) sufficiently close to v so that $g(y) > u$. Thus, $yf(1,y) > u$ and $yf(1,y) < v \cdot 1 = v$, which imply $u < yf_1(y) < v$ (see Figure 3.2).

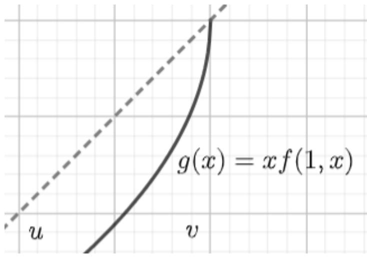


Figure 3.2: v is an attractor from the left for the map $g(x) = xf_1(x)$.

Again, we achieve a contradiction with Lemma 20, since $f(1,y) \cdot f\big(1, yf_1(y)\big) < 1$.

□

Now, after establishing some properties for Equation (3.10) in order to be globally periodic of period 6 whenever f is a symmetric continuous map, we show the non-existence of 6-cycles displaying the form of (3.10).

Proposition 4. *There are no 6-cycles of third order having the form $x_{n+3} = x_n f(x_{n+2}, x_{n+1})$, whenever $f : (0,\infty)^2 \to (0,\infty)$ is a symmetric continuous map.*

Proof. Consider the sets F and F^2 given by (3.13) and recall that f_1 denotes the fiber map $f(1,\cdot)$.

We begin by claiming that $F = \mathrm{Im}(f_1)$. Indeed, if $w \in \mathrm{Im}(f_1)$, by Lemma 14, $\frac{1}{w} \in \mathrm{Im}(f_1)$. Moreover, Lemma 20 implies $w \in \mathrm{Im}(f_1)$ if and only if $w \in F$. As a consequence, $w \in F$ if and only if $\frac{1}{w} \in F$ and, since $1 \in \mathrm{Int}(F) \neq \emptyset$ and F is connected by Lemma 23, we achieve

$$F = \langle \alpha, \omega \rangle = \mathrm{Im}(f_1), \tag{3.14}$$

where $0 \leq \alpha < 1$, $\omega = \frac{1}{\alpha}$ (we understand that $\omega = \infty$ if $\alpha = 0$). Also, $F^2 = \big\langle \alpha^2, \frac{1}{\alpha^2} \big\rangle$, and, necessarily $0 < \alpha < 1$; otherwise, if $\alpha = 0$, we will have $F = F^2 = (0,\infty)$ and, by Lemma 23, $f_1 \equiv 1$. Then, Lemma 17 implies $f(x,y) = 1$ for all $x, y > 0$ and Equation (3.10) reduces to the 3-cycle $x_{n+3} = x_n$, deriving a contradiction.

Next, we focus on the main step of the proof, that is, to show that $f_1(x) = 1$ for all $x > 0$. This will derive again a contradiction and the result will follow.

First, notice by Lemma 23 that $f_1(z) = 1$ for all $z \in F^2 = \big\langle \alpha^2, \frac{1}{\alpha^2} \big\rangle$. So, since $0 < \alpha < 1$, the continuity of f_1 gives us that at least $f_1(x) = 1$ for all $x \in \big[\alpha^2, \frac{1}{\alpha^2}\big]$. Now, we demonstrate the existence of a point $z_1 \in (0, \alpha^2)$ such that $f_1(z_1) = 1$. To do so, we assume the contrary and distinguish two cases:

- If $f_1(x) \in (1, \omega\rangle$ for all $x \in (0, \alpha^2)$ (we use (3.14)), we can choose a point y such that $0 < yf(1, y) \leq y \cdot \omega < \alpha^2$ (it suffices to take $y < \alpha^3$). On the other hand, Lemma 20 implies $1 = f(1, y) \cdot f\big(1, yf(1, y)\big)$, and therefore,

$$f\big(1, yf(1, y)\big) = \frac{1}{f(1, y)} < 1, \quad \text{with } 0 < yf(1, y) < \alpha^2,$$

  which contradicts our hypothesis on $f_1(x) > 1$ in $(0, \alpha^2)$.

- If $f_1(x) \in [\alpha, 1)$ for all $x \in (0, \alpha^2)$, we get $xf(1, x) < x < \alpha^2$. Now, we can proceed analogously to the previous case and apply Lemma 20 to achieve

$$f\big(1, yf(1, y)\big) = \frac{1}{f(1, y)} > 1, \quad \text{with } 0 < yf(1, y) < \alpha^2,$$

  which is a contradiction.

In conclusion, there exists at least a point $z_1 \in (0, \alpha^2)$ such that $f(1, z_1) = 1$.

As a next step, we show that $f_1(x) = 1$ for all $x \in [z_1, \alpha^2]$. Firstly, Lemma 24 ensures the existence of a point z_2 in (z_1, α^2) such that $f_1(z_2) = 1$. Moreover, observe that if we apply repeatedly Lemma 24, we find two points, $z_3 \in (z_1, z_2)$ and $z_4 \in (z_2, \alpha^2)$ such that $f_1(z_j) = 1, j = 3, 4$. Now, an argument of density allows us to conclude that $f_1(x) = 1$ for all $x \in [z_1, \alpha^2]$. In this sense, notice that if a subsequence (z_n) accumulates in an interior point, we can apply the continuity of f_1 and Lemma 24 in order to extend the result to new points x for which $f_1(x) = 1$.

Let us denote $m := \inf\{x > 0 : f(1, x) = 1\}$. We claim that $m = 0$. Indeed, on the one hand, $m < z_1$ from the above discussion. On the other hand, if on the contrary $m > 0$, which would imply $f_1(m) = 1$ by the continuity of the fiber map, we can reason analogously to the existence of a point $z_1 \in (0, \alpha^2)$ such that $f_1(z_1) = 1$, in order to prove the existence of a point $q < m$ satisfying $f_1(q) = 1$, which contradicts the fact that m is the infimum. Then, we can conclude that $m = 0$ and apply the argument of density explained previously to prove that $f_1(\alpha) = 1$ for all $x \in (0, \alpha^2)$. In fact, we already know that $f_1(x) = 1$ for all $x \in (0, \omega^2)$ (see Figure 3.3).

Now, we focus on the interval (ω^2, ∞). In fact, we see that there exists a point $w_1 > \omega^2$ such that $f_1(w_1) = 1$. Indeed, we already know that $\mathrm{Im}(f_1) = \langle \alpha, \omega \rangle$ and that $f_1 \equiv 1$ in the interval $(0, \omega^2]$, with $\alpha < 1 < \omega$. Then, there exist values $u, v \in (\omega^2, \infty)$ such that $f(1, u) < 1$ and $f(1, v) > 1$. If we apply the continuity of f_1, we achieve the existence of a point $w_1 > \omega^2$ satisfying $f_1(w_1) = 1$. Moreover, we can apply an argument of density based on Lemma 24 to obtain $f_1(x) = 1$ for all (ω^2, w_1).

Denote $M := \sup\{x > 0 : f(1, x) = 1\}$. We can proceed in a similar way as in the infimum case m to conclude that $M = \infty$. Otherwise, if $M < \infty$, Lemma 24 and an argument of density imply that $f_1 \equiv 1$ in $(0, M]$. However, since $\mathrm{Im}(f_1) = \langle \alpha, \omega \rangle$, we would have a point $x > M$ with $f_1(x) = 1$, which would contradict the definition of M.

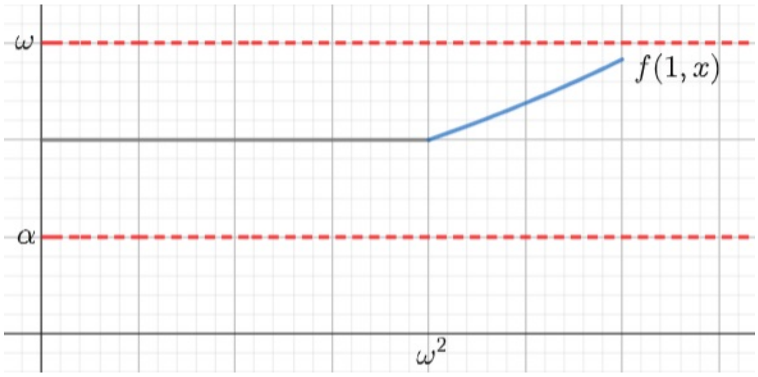


Figure 3.3: At least, $f(1,x)=1$ in $(0,\omega^2]$.

Finally, given an arbitrary point $z>0$, from our study we can find two values x, y such that $x<z<y$, $f_1(x)=f_1(y)=1$. Also, we can apply repeatedly Lemma 24 to obtain a sequence (q_n) of positive values tending to z with $f_1(q_n)=1$. Then, since f_1 is a continuous map, $f(1,z)=1$. In conclusion, $f_1\equiv 1$ on $(0,\infty)$. Notice that this means that Equation (3.10) reduces to $x_{n+3}=x_n$, a 3-cycle. This contradiction ends the proof. $\square$

For Equation (3.10), if we do not assume the symmetry of f, we are able to find a 6-cycle of potential form as we have seen in Section 3.1, namely,

$$x_{n+3}=x_n\left(\frac{x_{n+2}}{x_{n+1}}\right)^2. \tag{3.15}$$

In this sense, it will be interesting to study the existence of more 6-cycles displaying the form $x_{n+3}=x_nf(x_{n+2},x_{n+1})$ whenever f is not a symmetric map. Also, in the case that there exist more 6-cycles of such form, will they be topological conjugate to (3.15)?

As a finishing touch, we state the main result of this section by gathering together Propositions 2-3-4.

Theorem B. *There are no 6-cycles of third order having the form $x_{n+3}=x_if(x_j,x_k)$, whenever $f:(0,\infty)^2\to(0,\infty)$ is a symmetric continuous map, and $i,j,k\in\{n,n+1,n+2\}$ are pairwise distinct.*

3.3 Existence of 6-cycles with separation of variables

In the previous section, we have seen that there are no 6-cycles displaying the form of the family of third order $x_{n+3}=x_if(x_j,x_k)$, with $i,j,k\in\{n,n+1,n+2\}$ pairwise distinct if we assume the extra condition of f being a symmetric map. Nevertheless, if we suppress such assumption, we find the potential cycle

$$x_{n+3}=x_n\left(\frac{x_{n+2}}{x_{n+1}}\right)^2, \tag{3.16}$$

which in fact, it is the unique potential 6-cycle of the form (3.3). In this sense, it arises naturally the question of the existence of more 6-cycles or, if on the contrary, the potential cycle is the only one that exists.

Here, we advance on the problem by considering the particular case of f separating variables. This means that we study the existence of 6-cycles of third order exhibiting the form

$$x_{n+3} = x_i g(x_j) h(x_k), \tag{3.17}$$

with $i, j, k \in \{n, n+1, n+2\}$ pairwise distinct, and $g, h : (0, \infty) \to (0, \infty)$ continuous maps. In concrete, we prove that the unique 6-cycle of the form (3.17) is, indeed, the potential one given by (3.16). Therefore, in the sequel, unless otherwise stated, we assume that Equation (3.17) is a 6-cycle.

In the literature we can find some results concerning the problem of global periodicity in autonomous difference equations that separates variables. For instance, in [8], the authors show that for the second order difference equation $x_{n+2} = f(x_{n+1}, x_n) = \sigma(x_n)\rho(x_{n+1})$, with σ and ρ being continuous maps, there exists a unique 3-cycle given by $f(x, y) = \frac{C}{xy}$ for some real constant $C > 0$, whose proof is based on studying the functional equation $x = f\big(f(y, x), y\big)$ and analyzing the fiber maps $f(\cdot, z)$ and $f(z, \cdot)$. Moreover, in [9] such result is generalized by obtaining every $(k+1)$-cycle for the difference equation of order k, $x_{n+k} = f_1(x_n) f_2(x_{n+1}) \cdot \ldots \cdot f_k(x_{n+k-1})$, where $f_j : (0, \infty) \to (0, \infty)$ are continuous maps. Concretely, it was shown that the equation is a $(k+1)$-cycle if and only if

- k is even and
$$x_{n+k} = \frac{C}{x_n x_{n+1} \ldots x_{n+k-1}}, \quad \text{for some } C > 0;$$
- k is odd and either
$$x_{n+k} = \frac{\tilde{C}}{x_n x_{n+1} \ldots x_{n+k-1}}, \quad \text{for some } \tilde{C} > 0;$$
  or
$$x_{n+k} = \frac{\prod_{j=1}^{(k+1)/2} x_{n+2(j-1)}}{\prod_{j=1}^{(k-1)/2} x_{n+2j-1}}.$$

The present section, which is based on [70], is organized as follows: as a first step, in Subsection 3.3.1, we prove that (3.16) is the unique potential 6-cycle displaying the form of (3.3). Then, we deeply study the different configurations of Equation (3.17) in Subsections 3.3.2-3.3.4. In concrete, we see that in two of the possible configurations there are no 6-cycles displaying the form of the studied difference equation, while in the other one, namely, $x_{n+3} = x_n g(x_{n+2}) h(x_{n+1})$, the unique 6-cycle that exists is the potential one given by (3.16). The analysis developed in those subsections will be gather in Theorem C.

3.3.1 Potential 6-cycles

In the general case, where no extra assumptions are made over f, it is unknown how many 6-cycles exist exhibiting the form $x_{n+3} = x_i f(x_j, x_k)$. However, following the ideas of [19], in [68], the authors are able to see the unique potential 6-cycle of such form. To show it, take the third order difference equation of potential form $x_{n+3} = x_{n+2}^{\alpha_2} x_{n+1}^{\alpha_1} x_n^{\alpha_0}$, where $\alpha_i, i = 1, 2, 3$, are real numbers, and linearize it by the change of variables $y_j = \log(x_j)$. After that, we achieve the difference equation $y_{n+3} = \alpha_2 y_{n+2} + \alpha_1 y_{n+1} + \alpha_0 y_n$, whose associated characteristic equation is given by $\lambda^3 - \alpha_2\lambda^2 - \alpha_1\lambda - \alpha_0 = 0$.

Since we are looking for 6-cycles, we need that every root, $\lambda_1, \lambda_2, \lambda_3$, of the associated characteristic polynomial, $p(\lambda) = \lambda^3 - \alpha_2\lambda^2 - \alpha_1\lambda - \alpha_0$, lies on the boundary of the unit disk, $|\lambda_j| = 1$ for $j = 1, 2, 3$, and they have to be simple. Furthermore, since $p(\lambda)$ is a polynomial of third degree, one of its roots must be real and the other two must be conjugated complex 6-roots of the unity. Bearing in mind that the 6-roots of the unity are

$$\begin{array}{lll} \lambda_1 = 1, & \lambda_2 = \frac{1}{2} + i\frac{\sqrt{3}}{2}, & \lambda_3 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \\ \lambda_4 = -1, & \lambda_5 = \bar{\lambda}_3 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}, & \lambda_6 = \bar{\lambda}_2 = \frac{1}{2} - i\frac{\sqrt{3}}{2}, \end{array}$$

in order to obtain 6-cycles, we have to analyze the possible combinations of such roots. In this sense, we have four possible cases:

(1) $\{\lambda_1, \lambda_2, \lambda_6\}$: In this case, the characteristic polynomial is given by

$$p(\lambda) = (\lambda - 1) \cdot \left(\lambda - \frac{1}{2} - i\frac{\sqrt{3}}{2}\right) \cdot \left(\lambda - \frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = \lambda^3 - 2\lambda^2 + 2\lambda - 1,$$

thus, $\alpha_0 = 1$, $\alpha_1 = -2$ and $\alpha_2 = 2$. In conclusion, the corresponding difference equation is

$$x_{n+3} = x_n \left(\frac{x_{n+2}}{x_{n+1}}\right)^2.$$

(2) $\{\lambda_1, \lambda_3, \lambda_5\}$: Here, the characteristic polynomial is

$$p(\lambda) = (\lambda - 1) \cdot \left(\lambda + \frac{1}{2} - i\frac{\sqrt{3}}{2}\right) \cdot \left(\lambda + \frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = \lambda^3 - 1,$$

so we obtain a 3-cycle instead of a 6-cycle, that is, $x_{n+3} = x_n$. Observe that this happens because, indeed, λ_1, λ_3 and λ_5 are 3-roots of the unity too.

(3) $\{\lambda_4, \lambda_2, \lambda_6\}$: These roots yield to the characteristic polynomial

$$p(\lambda) = (\lambda + 1) \cdot \left(\lambda - \frac{1}{2} - i\frac{\sqrt{3}}{2}\right) \cdot \left(\lambda - \frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = \lambda^3 + 1,$$

which implies the 6-cycle

$$x_{n+3} = \frac{1}{x_n}.$$

(4) $\{\lambda_4, \lambda_3, \lambda_5\}$: Now, $\alpha_0 = -1$, $\alpha_1 = \alpha_2 = -2$, since

$$p(\lambda) = (\lambda+1)\cdot\left(\lambda+\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)\cdot\left(\lambda+\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) = \lambda^3+2\lambda^2+2\lambda+1,$$

and we obtain the 6-cycle:

$$x_{n+3} = \frac{1}{x_n(x_{n+1}x_{n+2})^2}.$$

We gather the previous reasoning in the following result.

Proposition 5. *The unique 6-cycles of third order and potential form are given by*

$$x_{n+3} = x_n\left(\frac{x_{n+2}}{x_{n+1}}\right)^2, \quad x_{n+3} = \frac{1}{x_n(x_{n+1}x_{n+2})^2}, \quad x_{n+3} = \frac{1}{x_n}.$$

Therefore, we derive the following:

Corollary 2. *The unique potential 6-cycle of the form $x_{n+3} = x_i f(x_j, x_k)$, with $i, j, k \in \{n, n+1, n+2\}$ pairwise distinct and $f : (0,\infty)^2 \to (0,\infty)$ continuous, is given by*

$$x_{n+3} = x_n\left(\frac{x_{n+2}}{x_{n+1}}\right)^2.$$

Once that we have established the unique potential 6-cycle displaying the form of (3.3), we center on the particular case of separation of variables by studying its different configurations in the following subsections.

3.3.2 The case $x_{n+3} = x_{n+2}g(x_{n+1})h(x_n)$

Firstly, we deal with the third order difference equation

$$x_{n+3} = x_{n+2}\, g(x_{n+1})h(x_n), \tag{3.18}$$

and we see that there are no 6-cycles displaying such form. To do so, we assume that Equation (3.18) is globally periodic of period 6 and we will arrive to a contradiction. As a first step, we establish some properties that must be verified by the set of equilibrium points $\mathcal{F}_2$. Recall by (♣) that $g(1) = h(1) = 1$ with $1 \in \mathcal{F}_2$. After that, we distinguish two cases depending on the cardinality of $\mathcal{F}_2$ and, finally, we gather the whole study in Theorem 13.

Recall that h is a homeomorphism by Lemma 8. This implies that the map is strictly monotonic, a fact that will be useful later on. Next, we set some properties for $\mathcal{F}_2$.

Lemma 25. *Assume that Equation (3.18) is a 6-cycle. Let $u, x \in \mathcal{F}_2$, with $u \neq x, u < x$. Then:*

(a) $x = ug(u)h\Big(xg(x)h(u)\Big)$.

(b) $ug(u)h(x) \in \mathcal{F}_2$. *In fact, if we put* $x_0 = x$, *then the sequence* $(x_k)_k$ *defined recursively as* $x_{k+1} = ug(u)h(x_k)$, $k \geq 0$, *is contained in* $\mathcal{F}_2$. *Moreover,* $x_1 \notin \{x, u\}$.

(c) $xg(x)h(u) \in \mathcal{F}_2$. *In fact, if we put* $u_0 = u$, *then the sequence* $(u_k)_k$ *defined recursively as* $u_{k+1} = xg(x)h(u_k)$, $k \geq 0$, *is contained in* $\mathcal{F}_2$. *Moreover,* $u_1 \notin \{x, u\}$.

(d) It holds $\frac{u}{x_n} = \frac{g(x_n)}{g(x_{n+1})} = \frac{h(x_{n+1})}{h(x_n)}$ *for all* $n \geq 0$.

(e) If h *is increasing,* $x_1, u_1 \in (u, x)$.

(f) If h *is decreasing, then* $x_1 < u < x_2 < x$ *and* $u < u_1 < x < u_2$, *where* x_1, x_2, u_1, u_2 *are defined in Part (b) and Part (c).*

(g) If h *is decreasing, then* $\inf \mathcal{F}_2 = 0$ *and* $\sup \mathcal{F}_2 = \infty$. *In fact,* $\mathcal{F}_2 = (0, \infty)$.

Proof. Before proving the statements of the lemma, recall that $g(z)h(z) = 1$ for every $z \in \mathcal{F}_2$. In particular, $g(z)h(z) = 1$ if $z \in \{x, u\}$.

To see Part (a), take the initial conditions $z_1 = z_2 = u$, $z_3 = x$, which generate under Equation (3.18) the terms $z_4 = x$, $z_5 = z_6 = xg(x)h(u)$. Now, since the equation is a 6-cycle,

$$x = z_3 = z_9 = z_8\, g(z_7)\, h(z_6) = z_2\, g(z_1)\, h(z_6) = u\, g(u)\, h\Big(xg(x)h(u)\Big).$$

Now, for Part (b), set $z_1 = z_2 = x$, $z_3 = u$. Thus, $z_4 = u$, $z_5 = z_6 = ug(u)h(x)$. Again, by global periodicity,

$$x = z_2 = z_8 = z_7 g(z_6)h(z_5) = z_1 g(z_6)h(z_5) = x \cdot g\big(ug(u)h(x)\big) \cdot h\big(ug(u)h(x)\big).$$

So $1 = g\big(ug(u)h(x)\big) \cdot h\big(ug(u)h(x)\big)$ and we obtain that $ug(u)h(x) \in \mathcal{F}_2$. We can repeat the reasoning with the points $x_1 := ug(u)h(x)$ and u (set $z_1 = z_2 = x_1$ and $z_3 = u$) and see that $x_2 := ug(u)h(x_1)$ belongs to $\mathcal{F}_2$ too. If we continue the process, one can see by induction that x_{k+1} defined recursively as $x_{k+1} = ug(u)h(x_k), k \geq 0$ is a point in $\mathcal{F}_2$.

On the other hand, we show that $x_1 \notin \{x, u\}$. To do so, take $x = x_1 = ug(u)h(x)$ and generate a sequence of initial conditions $z_1 = z_2 = x$, $z_3 = u$ in order to obtain, by global periodicity,

$$u = z_3 = z_9 = z_8 g(z_7)h(z_6) = x \cdot g(x) \cdot h\big(ug(u)h(x)\big) = xg(x)h(x) = x,$$

which is a contradiction.

Furthermore, suppose $ug(u)h(x) = x_1 = u$ and set $z_1 = z_2 = x$, $z_3 = u$. Now, the sequence generated under Equation (3.18) is given by $(z_j) = (x, x, u, u, u, u, \ldots)$, which yields to the same contradiction as before, $x = u$, due to global periodicity.

Part (c) is analogous to Part (b) and is omitted. The main difference resides in considering the initial conditions (u, u, x).

Next, to show (d), set $x_0 = x$ and $x_{n+1} = ug(u)h(x_n)$ for all $n \geq 0$. From (b), we know that initial conditions (x, x, u), yield to the relation $u = xg(x)h(x_1)$. Therefore,

$$x_1 = ug(u)h(x) = \big(xg(x)h(x_1)\big) \cdot g(u)h(x) = xh(x_1)g(u) \cdot \big(g(x)h(x)\big) = xh(x_1)g(u),$$

and $uh(x) = xh(x_1)$. Hence, $\frac{u}{x} = \frac{h(x_1)}{h(x)}$. Now, since $x_j \in \mathcal{F}_2, j = 0, 1$, then $h(x_j) = \frac{1}{g(x_j)}, j = 0, 1$, and we get

$$\frac{u}{x_0} = \frac{h(x_1)}{h(x_0)} = \frac{g(x_0)}{g(x_1)}.$$

Finally, we can repeat the reasoning with the initial conditions (x_n, x_n, u), for all $n \geq 1$ and obtain

$$\frac{u}{x_n} = \frac{h(x_{n+1})}{h(x_n)} = \frac{g(x_n)}{g(x_{n+1})}.$$

For Part (e), assume that h is an increasing map. Then, $u < x$ implies

$$x_1 = ug(u)h(x) \geq ug(u)h(u) = u,$$

but $x_1 \neq u$, so $x_1 > u$. Moreover, $x_1 < x$. Indeed, assume the contrary, that is $u < x < x_1 = ug(u)h(x)$. By Part (a), we can exchange the roles of u and x and apply that h is increasing with $u < x < x_1$, in order to obtain

$$u = xg(x) \cdot h\big(ug(u)h(x)\big) = xg(x)h(x_1) \geq xg(x)h(x) = x,$$

so $u > x$, a contradiction. To sum up, $u < x_1 < x$. The proof of the case $u < u_1 < x$ is similar and we omit it.

Next, to see Part (f), assume h is decreasing. As a first step, we claim that $x_2 \neq u$. Notice that, otherwise, the equality $u = x_2 = ug(u)h(x_1)$ would yield to $g(u)h(x_1) = 1$, and Part (d) would imply

$$\frac{u}{x} = \frac{h(u)}{h(x)} = \frac{\frac{1}{g(u)}}{h(x)},$$

that is, $ug(u)h(x) = x$. Setting $x_1 = ug(u)h(x)$, we would get $x_1 = u = x$, a contradiction.

Next, since $u < x$, we have that x_1 verifies the inequality

$$x_1 = ug(u)h(x) \leq ug(u)h(u) = u,$$

and we derive $x_1 < u < x$. Now, bearing in mind that $x_2 = ug(u)h(x_1)$, we deduce that $x_2 \geq ug(u)h(u) = u$ and $x_2 > u$. Furthermore, $x_2 < x$, since on the contrary, by Part (d) and the monotonic character of h, we would get

$$1 < \frac{u}{x_1} = \frac{h(x_2)}{h(x_1)} \leq \frac{h(x_1)}{h(x_1)} = 1,$$

a contradiction.

Finally, to see Part (g), we start proving that $\inf \mathcal{F}_2 = 0$ and $\sup \mathcal{F}_2 = \infty$. Let us proceed by contradiction. Firstly, suppose that $\inf \mathcal{F}_2 =: a > 0$. By the continuity of the maps g and h, $a = \inf \mathcal{F}_2 \in \mathcal{F}_2$. Thus, we could apply the same reasoning as in Part (f) with $u = a < x$ and we would obtain values smaller than a that belong to $\mathcal{F}_2$, which is not possible. Analogously, we can prove $\sup \mathcal{F}_2 = \infty$. In conclusion, for any two values of $\mathcal{F}_2$, by Part (f), we obtain another value of $\mathcal{F}_2$ between them. This density in $\mathcal{F}_2$ implies that $\mathcal{F}_2 = (0, \infty)$.

□

Once that we have settled some properties of the set of equilibrium points $\mathcal{F}_2$, we distinguish several cases depending on the cardinality of $\mathcal{F}_2$ and the monotonic character of h (recall that h is monotonic since it is a homeomorphism).

The case Card $(\mathcal{F}_2) \geq 2$

To demonstrate the non-existence of 6-cycles displaying the form of Equation (3.18) whenever Card $(\mathcal{F}_2) \geq 2$, we analyze two cases depending on the monotonicity of h.

- h **is increasing:** Here, we prove the non-existence of 6-cycles by applying several properties of Lemma 25.

Proposition 6. *Let h be an increasing homeomorphism and* Card$(\mathcal{F}_2) \geq 2$*. Then, the Equation* (3.18) *cannot be a* 6*-cycle.*

Proof. Consider two different points in $\mathcal{F}_2$, x, y, with $x < y$. We apply Lemma 25-(b)-(c)-(e) in order to obtain that, $xg(x)h(y)$ and $yg(y)h(x)$ belong to $\mathcal{F}_2$; they are different from x, y; and they are in the open interval (x, y), respectively.

Furthermore, $y = x\,g(x)\,h\big(yg(y)h(x)\big)$ due to the increasing character of h, the inequalities $x < yg(y)h(x) < y$ and Lemma 25-(a). From here, we have

$$y = x\,g(x) \cdot h\big(yg(y)h(x)\big) < x\,g(x)\,h(y),$$

so $y < xg(x)h(y)$, which contradicts our previous observation on the location of the value $xg(x)h(y)$ in the open interval (x, y). □

- h **is decreasing:** Firstly, observe that by Lemma 25-(g), $\mathcal{F}_2 = (0, \infty)$; let us rewrite Equation (3.18) as

$$x_{n+3} = x_{n+2}\frac{g(x_{n+1})}{g(x_n)} \quad \text{or} \quad x_{n+3} = x_{n+2}\frac{h(x_n)}{h(x_{n+1})}.$$

Bearing this in mind, we see the non-existence of 6-cycles in the case Card $(\mathcal{F}_2) \geq 2$ and h decreasing proceeding by contradiction and by analyzing a certain functional equation.

Lemma 26. *The functional equation*

$$x = \varphi\left(\frac{x}{\varphi(x)}\right), \quad x > 0, \tag{3.19}$$

has no solutions in the family of continuous maps defined from $(0, \infty)$ *into itself.*

Proof. Let $\varphi : (0, \infty) \to (0, \infty)$ be a solution of (3.19). Firstly, observe that, due to (3.19), it is direct that $\text{Im}(\varphi) = (0, \infty)$. In particular, $1 \in \text{Im}(\varphi)$. Moreover, we claim that $\varphi(1) = 1$, and $\varphi(x) \neq x$ if $x \neq 1$. Indeed, being $\varphi^{-1}(\{1\}) \neq \emptyset$, suppose that $\varphi(z) = 1$ for some $z > 0$. Thus, $z = \varphi\left(\frac{z}{\varphi(z)}\right) = \varphi(z) = 1$, which proves the first claim. Also, if $\varphi(x) = x$ for some $x \neq 1$, we would have $x = \varphi\left(\frac{x}{\varphi(x)}\right) = \varphi(1) = 1$, which is a contradiction that proves the second claim.

According to the above properties, we analyze three different possibilities for the solutions of (3.19) (see Figure 3.4):

(i) Assume $x < \varphi(x) < 1$. Here, if $x < 1$, then $\frac{x}{\varphi(x)} < 1$. Thus,

$$x = \varphi\left(\frac{x}{\varphi(x)}\right) > \frac{x}{\varphi(x)}.$$

Therefore, $\varphi(x) > 1$ for all $x < 1$, a contradiction.

(ii) Consider $\varphi(x) < x < 1$. Now, if we take a point $z > 1$, we have $\varphi(z) > z$, since otherwise, $\varphi(w) < w$ for all $w \neq 1$ and then

$$w = \varphi\left(\frac{w}{\varphi(w)}\right) < \frac{w}{\varphi(w)}.$$

This would imply $\varphi(w) < 1$ for all $w \neq 1$, which contradicts the fact that $\text{Im}(\varphi) = (0, \infty)$. In this sense, if $z > 1$, we get $\frac{z}{\varphi(z)} < 1$ and, by hypothesis, $z = \varphi\left(\frac{z}{\varphi(z)}\right) < 1$, which is, again, a contradiction

(iii) Suppose $\varphi(x) > 1$ for all $x < 1$. In this case, $\frac{x}{\varphi(x)} < 1$ for all $x < 1$. Therefore, $x = \varphi\left(\frac{x}{\varphi(x)}\right) > 1$, and we have another contradiction.

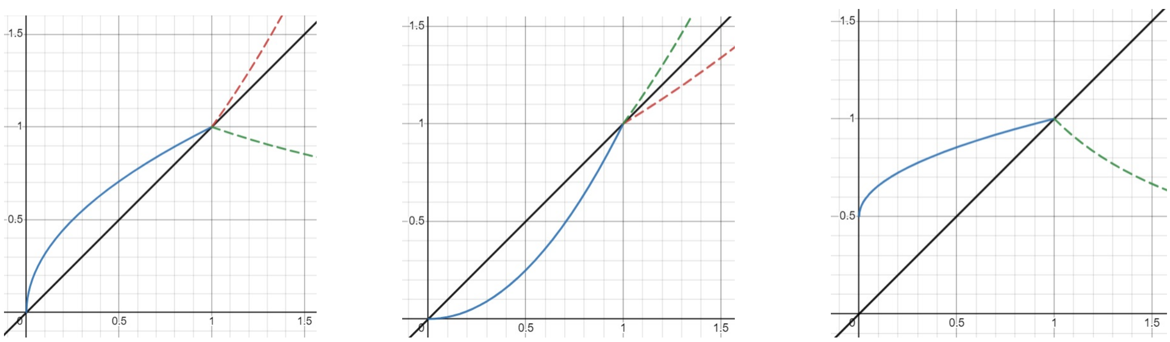


Figure 3.4: Examples of the solution φ for the Cases (i)-(iii).

□

Corollary 3. *The functional equation* (3.19) *has no solutions in the family* $\mathcal{H}$ *of homeomorphisms* φ *from* $(0, \infty)$ *into itself.*

Proposition 7. *Assume that* Card $(\mathcal{F}_2) \geq 2$ *and* h *is a decreasing homeomorphism. Then there are no* 6*-cycles displaying the form of Equation (3.18).*

Proof. Firstly, recall that $\mathcal{F}_2 = (0, \infty)$ and $g(x) = \frac{1}{h(x)}$ for all $x > 0$ by Lemma 25.

Take the initial conditions $x_1 = x_2 = 1, x_3 = x$, with $x > 0$ arbitrarily taken. If we iterate such points under Equation (3.18), we obtain $x_4 = x$, $x_5 = xg(x) = \frac{x}{h(x)}$ and $x_6 = \frac{x}{h(x)} g(x)h(x) = \frac{x}{h(x)}$. Therefore, applying that the recurrence is a 6-cycle and the fact that $\mathcal{F}_2 = (0, \infty)$, we achieve

$$x = x_3 = x_9 = x_8 g(x_7)h(x_6) = x_2 g(x_1)h(x_6) = h\left(\frac{x}{h(x)}\right),$$

that is,

$$x = h\left(\frac{x}{h(x)}\right) \text{ for all } x > 0. \tag{3.20}$$

In conclusion, from Corollary 3 we derive the non-existence of 6-cycles exhibiting the form of Equation (3.18). □

The case Card $(\mathcal{F}_2) = 1$

In the sequel, we analyze the case where $\mathcal{F}_2$ is a singleton. Recall by (♣) that we can assume without loss of generality that $1 \in \mathcal{F}_2$, so we study the case $\mathcal{F}_2 = \{1\}$.

Now, we begin by establishing some relations that will be useful later.

Lemma 27. *Assume Equation (3.18) is a* 6*-cycle, then for all* $x > 0$ *it holds:*

(a) $h^{-1}(x) = x\big(g(x)\big)^2 h(x)$.

(b) $h^{-1}(x) = \frac{1}{g\big(xg(x)\big)h(x)}$.

(c) $1 = g\left(\frac{1}{g\big(xg(x)\big)h(x)}\right) h\big(xg(x)\big)$.

Proof. Consider an arbitrary point $x > 0$. By Lemma 8, we know that h is a homeomorphism, so we can take the initial terms $x_1 = h^{-1}(x), x_2 = x_3 = 1$, in order to obtain $x_4 = x_5 = x$ and $x_6 = xg(x)$. Now, from the fact that Equation (3.18) is a 6-cycle, we get the three statements:

$$\begin{aligned} h^{-1}(x) &= x_1 = x_7 = x_6 g(x_5)h(x_4) = x\big(g(x)\big)^2 h(x), \\ 1 &= x_2 = x_8 = x_7 g(x_6)h(x_5) = x_1 g(x_6)h(x_5) = h^{-1}(x) g\Big(xg(x)\Big)h(x), \\ 1 &= x_3 = x_9 = x_8 g(x_7)h(x_6) = x_2 g(x_1)h(x_6) = g\left(h^{-1}(x)\right) h\big(xg(x)\big). \end{aligned}$$

□

In the following result, recall that Fix(h) denotes the set of fixed points of h, that is, the set of points $x > 0$ that verifies $h(x) = x$.

Lemma 28. *Assume that Equation (3.18) is a 6-cycle. Then:*

(a) $x \in \mathrm{Fix}(h)$ *if and only if* $\big(g(x)\big)^2 = \frac{1}{x}$.

(b) If $x \in \mathrm{Fix}(h)$, *then* $\sqrt{x} \in \mathrm{Fix}(h)$. *Furthermore,* $x = 1$ *is the unique fixed point of* h.

Proof. Part (a) follows directly from Lemma 27-(a), so we limit to prove Part (b). To do so, let $x \in \mathrm{Fix}(h)$. Then, $h(x) = x = h^{-1}(x)$ and we can apply Lemma 27-(b) in order to obtain $x = \frac{1}{g\big(xg(x)\big)x}$, or

$$x^2 g\big(xg(x)\big) = 1. \tag{3.21}$$

Now, Part (a) gives $g(x) = \sqrt{\frac{1}{x}}$, or equivalently, $xg(x) = \sqrt{x}$. Also, if we replace it into (3.21), we achieve $x^2 g\left(x\sqrt{\frac{1}{x}}\right) = 1$, that implies, $g\big(\sqrt{x}\big) = \frac{1}{x^2}$. Next, by Lemma 27-(c), and the previous considerations,

$$\begin{aligned} 1 &= g\left(\frac{1}{g\big(xg(x)\big)h(x)}\right) h\big(xg(x)\big) = g\left(\frac{1}{g\big(\sqrt{x}\big)h(x)}\right) h\big(\sqrt{x}\big) \\ &= g\left(\frac{1}{\frac{1}{x^2}x}\right) h\big(\sqrt{x}\big) = g\,(x)\, h\big(\sqrt{x}\big) = \sqrt{\frac{1}{x}} h\big(\sqrt{x}\big). \end{aligned}$$

As a consequence, $h\big(\sqrt{x}\big) = \sqrt{x}$ and $\sqrt{x} \in \mathrm{Fix}(h)$.

As a final step, from Part (a), $g(x) = \frac{1}{\sqrt{x}}$, allows us to infer $g\big(\sqrt{x}\big) = \frac{1}{\sqrt{\sqrt{x}}}$. Then, bearing in mind that $g\big(\sqrt{x}\big) = \frac{1}{x^2}$ too, we can equal both expressions, $\frac{1}{\sqrt{\sqrt{x}}} = \frac{1}{x^2}$, and obtain $x = x^8$, whose unique positive real solution is $x = 1$. □

Lemma 29. *Let Equation (3.18) be a 6-cycle. Then, for all* $x > 0$,

$$h(x) = \frac{1}{g(x)} \sqrt{\frac{1}{xg\big(xg(x)\big)}}, \tag{3.22}$$

and

$$xg(x) = h^{-1}\left(\frac{1}{g\big(h^{-1}(x)\big)}\right). \tag{3.23}$$

Proof. Take an arbitrary point $x > 0$ and set $x_1 = x_2 = 1$, and $x_3 = x$ to obtain, under the iteration of Equation (3.18), the terms $x_4 = x,\ \ x_5 = xg(x),\ \ x_6 = x\big(g(x)\big)^2 h(x)$. Then, by global periodicity,

$$\begin{aligned} &1 = x_1 = x_7 = x_6 g(x_5) h(x_4) = x\big(g(x)\big)^2 h(x) \cdot g\big(xg(x)\big) h(x), \\ &1 = x_2 = x_8 = x_7 g(x_6) h(x_5) = x_1 g(x_6) h(x_5) = g\Big(x(g(x))^2 h(x)\Big) \cdot h\big(xg(x)\big). \end{aligned}$$

From the first equality, we deduce

$$h(x) = \frac{1}{g(x)} \sqrt{\frac{1}{xg\,(xg(x))}};$$

while from the second one and the fact that h is a homeomorphism,

$$xg(x) = h^{-1}\left(\frac{1}{g\big(x(g(x))^2h(x)\big)}\right).$$

Finally, by Lemma 27-(a), we get $xg(x) = h^{-1}\left(\frac{1}{g(h^{-1}(x))}\right)$. $\square$

Once we have established some auxiliary results concerning Equation (3.18) whenever we assume that it is a 6-cycle with the set of equilibrium points, $\mathcal{F}_2$, being a singleton, we distinguish several cases depending on the monotonicity of the map h, namely,

(a) h is increasing, with $h(x) > x$ if $0 < x < 1$.

(b) h is increasing, with $h(x) < x$ if $0 < x < 1$.

(c) h is decreasing.

Observe that, since $\mathcal{F}_2 = \{1\}$, for the rest of values $x \neq 1$ it must be either $g(x)h(x) < 1$ for all $x \in (0,1)$ or the reverse inequality $g(x)h(x) > 1$ for all $x \in (0,1)$. Moreover, recall that we are assuming that Equation (3.18) is a 6-cycle.

Case (a): h is increasing, with $h(x) > x$ if $0 < x < 1$ (see Figure 3.5).

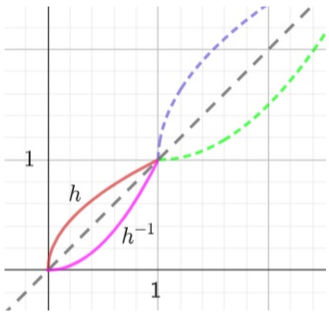


Figure 3.5: Case h increasing, $h(x) > x$ if $0 < x < 1$.

We differentiate two subcases depending on the value of $g(x)h(x)$ for all $x \in (0,1)$.

(a.1) Let $g(x)h(x) > 1$ for all $x \in (0,1)$. By Lemma 27-(a), we have

$$h^{-1}(x) = x\big(g(x)\big)^2h(x),$$

and we can apply the symmetry of h and h^{-1} with respect to the diagonal $y = x$ to get

$$x > h^{-1}(x) = x\left(g(x)\right)^2 h(x) = xg(x)\cdot\big(g(x)h(x)\big) > xg(x),$$

which means, $g(x) < 1$ for all $(0,1)$. Therefore, $h(x) > 1$ for every $x \in (0,1)$ and we derive a contradiction.

(a.2) Assume $g(x)h(x) < 1$, for all $x \in (0,1)$. Analogously to the previous case, we derive $h^{-1}(x) = x\big(g(x)\big)^2h(x) < xg(x)$. Moreover, since $h(x) > x$ in $(0,1)$, we get

$$h^{-1}(x) = x\big(g(x)\big)^2h(x) > x^2\big(g(x)\big)^2.$$

This two inequalities imply that $xg(x) < 1$ for every $x \in (0,1)$.

On the other hand, Equation (3.23) yields to

$$h^{-1}\left(\frac{1}{g(h^{-1}(x))}\right) < 1 \text{ for every } x \in (0,1).$$

Therefore, $\frac{1}{g(h^{-1}(x))} < 1$ or $g\big(h^{-1}(x)\big) > 1$ in $(0,1)$. Now, since $h^{-1}|_{(0,1)}$ is a homeomorphism from the open unit interval into itself, we get $g(w) > 1$ for all $w \in (0,1)$, and, hence, $x < xg(x) < 1$ in $(0,1)$.

Next, by the hypothesis of this case and Lemma 27-(b), we have $1 < g\big(xg(x)\big)h(x)$. As a final step, the monotonic character of h applied to $x < xg(x) < 1$ implies

$$1 < g\big(xg(x)\big) \cdot h(x) < g\big(xg(x)\big) \cdot h\big(xg(x)\big) < 1$$

due to our assumption $h(z)g(z) < 1$ for every $z \in (0,1)$. We obtain a contradiction.

In summary, we gather the previous analysis in the following result.

Proposition 8. *Consider Equation (3.18) with $\mathcal{F}_2 = \{1\}$, and suppose that h is increasing, with $h(x) > x$ for every $x \in (0,1)$. Then, the equation cannot be a 6-cycle.*

Case (b): h is increasing, with $h(x) < x$ if $0 < x < 1$ (interchange the roles of h and h^{-1} in Figure 3.5). Again, since $\mathcal{F}_2 \cap (0,1) = \emptyset$, we distinguish two subcases.

(b.1) Assume $g(x)h(x) > 1$ for all $x \in (0,1)$. Here, in $(0,1)$, $h(x) < 1$, so $g(x) > 1$. Moreover, the increasing character of h and Lemma 27-(a) imply

$$x > h(x) = \frac{h^{-1}(x)}{x\big(g(x)\big)^2} > \frac{x}{x\big(g(x)\big)^2} = \frac{1}{\big(g(x)\big)^2}$$

from where we deduce $x\big(g(x)\big)^2 > 1$. On the other hand, $xg(x) > h(x)g(x) > 1$, and Equation (3.23) yields to

$$xg(x) = h^{-1}\left(\frac{1}{g\big(h^{-1}(x)\big)}\right) > 1,$$

which implies $g\big(h^{-1}(x)\big) < 1$ for all $0 < x < 1$. Now, setting $w = h^{-1}(x)$ and bearing in mind that h^{-1} is an increasing homeomorphism with $h^{-1}(1) = 1$, we deduce that $g(w) < 1$ for all $w \in (0,1)$, contrary to our initial hypothesis.

(b.2) Let $g(x)h(x) < 1$ for all $x \in (0,1)$. We apply Lemma 27-(a) to get

$$x < h^{-1}(x) = x\big(g(x)\big)^2 h(x) = xg(x) \cdot \big(g(x)h(x)\big) < xg(x),$$

and, therefore, $g(x) > 1$ in $(0,1)$. Now,

- If $xg(x) > 1$ for some $x < 1$, then Equation (3.23) yields to $h^{-1}\left(\frac{1}{g(h^{-1}(x))}\right) > 1$. Thus, $g(h^{-1}(x)) < 1$. Take $w = h^{-1}(x)$, with $w < 1$ to deduce that $g(w) < 1$, a contradiction.

- If $xg(x) < 1$ for some $x < 1$, then Lemma 27-(b) implies $g\big(xg(x)\big)h(x) > 1$. Then, since h is an increasing map, $1 < g\big(xg(x)\big)h(x) < g\big(xg(x)\big)h\big(xg(x)\big) < 1$, a contradiction.

Therefore, we can derive that $xg(x) = 1$ for all $x \in [0,1]$. Then, by Lemma 27-(b) we get $h^{-1}(x) = \frac{1}{h(x)}$, and therefore, $h(x)h^{-1}(x) = 1$ for all $x \in [0,1]$, which is impossible since $h(x) < 1$ and $h^{-1}(x) < 1$ for all $x \in [0,1]$.

In conclusion, we can summon the previous study into the following result.

Proposition 9. *Consider Equation* (3.18) *with* $\mathcal{F}_2 = \{1\}$ *and suppose that* h *is increasing, with* $h(x) < x$ *for every* $x \in (0,1)$*. Then, the equation cannot be a* 6*-cycle.*

Case (c): Assume that h is a decreasing map. Observe that in this case, $h(x) > 1$ for every $0 < x < 1$ and $h(x) < 1$ for every $x > 1$ (see Figure 3.6).

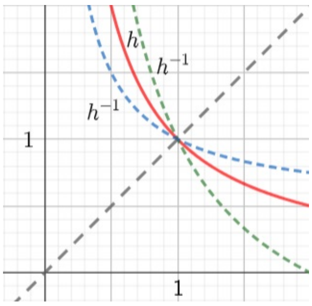


Figure 3.6: Case h decreasing.

As usual, we differentiate two subcases, namely, depending on the value of $g(x)h(x)$ in the open interval $(0,1)$. However, before proceeding with the analysis of each case, it should be highlighted that the symmetry of the inverse map with respect to the diagonal and the fact that h is decreasing imply that h^{-1} is decreasing too, with $h^{-1}(x) > 1$ for every $x < 1$.

(c.1) Let us assume that $g(x)h(x) < 1$ for all $x \in (0,1)$. Since $h(x) > 1$ in such interval, necessarily $g(x) < 1$ in it. Thus, $xg(x) < x < 1$ in $(0,1)$.

As a first step, we claim that, indeed, $g(z) < 1$ for all $z \neq 1$. To prove it, observe that Equation (3.23) and the inequality $xg(x) < 1$ in $(0,1)$ imply that

$$1 > xg(x) = h^{-1}\left(\frac{1}{g\big(h^{-1}(x)\big)}\right) \quad \text{for all } 0 < x < 1.$$

Hence, since h^{-1} is decreasing, $\frac{1}{g\left(h^{-1}(x)\right)} > 1$ for every $x \in (0,1)$. Moreover, $h^{-1}|_{(0,1)}$ is a homeomorphism from $(0,1)$ to $(1,\infty)$, therefore, setting $w = h^{-1}(x)$, we have $g(w) < 1$ for all $w > 1$, which ends the claim.

Next, we distinguish two subcases depending on the value of $g(x)h(x)$ in $(1,\infty)$:

(c.1.1) If $g(z)h(z) > 1$ in $(1,\infty)$, we deduce that $h(z) > 1$ for all $z > 1$ due to the previous claim and we achieve a contradiction.

(c.1.2) If $g(z)h(z) < 1$ in $(1, \infty)$, applying Lemma 27-(b) to an arbitrary point $z > 1$, we deduce

$$1 > h^{-1}(z) = \frac{1}{g\big(zg(z)\big)h(z)},$$

that is equivalent to $g\big(zg(z)\big)h(z) > 1$. Now, being $h(z) < 1$ implies $g\big(zg(z)\big) > 1$ for each $z > 1$, which contradicts the fact that $g(u) \leq 1$ for all $u \in (0, \infty)$.

In conclusion, Case (c.1) derives in a contradiction and we reject this case.

(c.2) Assume $g(x)h(x) > 1$ for every $x \in (0, 1)$. Firstly, observe that if $ug(u) = 1$ for some $u < 1$, then Lemma 27-(b) implies that

$$h^{-1}(u) = \frac{1}{g\big(ug(u)\big) \cdot h(u)} = \frac{1}{h(u)}$$

and $h(u) \cdot h^{-1}(u) = 1$, which is not possible. Consequently, either $xg(x) > 1$ or $xg(x) < 1$ for every $x \in (0, 1)$. Let us analyze both scenarios:

(c.2.1) Let us assume that $xg(x) > 1$ for every $x \in (0, 1)$. Necessarily, $g(x) > 1$ in such interval. Furthermore, Equation (3.23) allows us to deduce that

$$1 < xg(x) = h^{-1}\left(\frac{1}{g\big(h^{-1}(x)\big)}\right),$$

and, consequently, $\frac{1}{g\left(h^{-1}(x)\right)} < 1$ and $g\big(h^{-1}(x)\big) > 1$ for every $x < 1$. In this situation, due to the fact that $h^{-1}|_{(0,1)}$ is a homeomorphism from $(0, 1)$ onto $(1, \infty)$, we have that $g(w) > 1$ for all $w > 1$ and, therefore, $g(u) \geq 1$ for every $u > 0$.

As a final step, we apply Lemma 27-(b) to an arbitrary point $x < 1$ in order to obtain the contradiction $1 = h^{-1}(x) \cdot h(x) \cdot g\big(xg(x)\big) > 1$.

(c.2.2) Let us assume that $xg(x) < 1$ for every $x \in (0, 1)$. As a first step, we see that $g(w) < 1$ for every $w > 1$. To do so, apply Equation (3.23) to an arbitrary point $x < 1$ and the fact that h^{-1} is decreasing to deduce that $\frac{1}{g\left(h^{-1}(x)\right)} > 1$ or $g\big(h^{-1}(x)\big) < 1$. Now, since the point was arbitrarily taken and h^{-1} is a homeomorphism from $(0, 1)$ onto $(1, \infty)$, we derive that $g(w) < 1$ for all $w > 1$ as we wanted to see.

Secondly, observe that $g(x) \neq 1$ for every $x < 1$, since otherwise, if for some $\widetilde{x} < 1$, $g\big(\widetilde{x}\big) = 1$, by Lemma 27-(b) we would have

$$h^{-1}\big(\widetilde{x}\big) = \frac{1}{g\big(\widetilde{x}g\big(\widetilde{x}\big)\big) \cdot h\big(\widetilde{x}\big)} = \frac{1}{g\big(\widetilde{x}\big) \cdot h\big(\widetilde{x}\big)} = \frac{1}{h\big(\widetilde{x}\big)},$$

which would imply $h\big(\widetilde{x}\big)h^{-1}\big(\widetilde{x}\big) = 1$, that is a contradiction since $h(x), h^{-1}(x) > 1$ for every $x \in (0, 1)$.

Bearing this in mind, we distinguish two scenarios depending on the value of $g(x)$ in $(0, 1)$.

(∗) Let us assume $g(x) < 1$ for every $x \in (0,1)$. Observe that since we have seen that $g(w) < 1$ for every $w > 1$, we have that $g(u) \leq 1$ for every $u > 0$. Thus, since h, h^{-1} are decreasing maps, if we apply Lemma 27-(b) we obtain the contradiction

$$1 = h^{-1}(w) \cdot g\big(wg(w)\big) \cdot h(w) < 1.$$

(∗∗) Let us assume that $g(x) > 1$ for every $x \in (0,1)$. Since $g(w) < 1$ for all $w > 1$, we deduce that $g(w)h(w) < 1$ if $w > 1$. Then, we have two possibilities:

- If $wg(w) > 1$ for some $w > 1$, then $g\big(wg(w)\big) < 1$ and Lemma 27-(b) implies

$$h^{-1}(w) = \frac{1}{g\big(wg(w)\big) \cdot h(w)} > \frac{1}{h(w)} > 1,$$

which is impossible for values $w > 1$.

- If $wg(w) \leq 1$ for some $w > 1$, by Equation (3.23), we have

$$1 \geq wg(w) = h^{-1}\left(\frac{1}{g\big(h^{-1}(w)\big)}\right).$$

Thus $\frac{1}{g\left(h^{-1}(w)\right)} \geq 1$ and consequently $g\big(h^{-1}(w)\big) \leq 1$, which is impossible because $h^{-1}(w) < 1$ and we had supposed that $g(x) > 1$ in $(0,1)$.

Therefore, we have also obtained that this case cannot provide 6-cycles exhibiting the form of (3.18). We gather the conclusions obtained from the analysis of cases (c.1) and (c.2) in the following result:

Proposition 10. *Consider Equation (3.18) with $\mathcal{F}_2 = \{1\}$. Assume that h is decreasing. Then, the equation cannot be a 6-cycle.*

As a result of putting together Propositions 6 and 7, for the case in which Card $(\mathcal{F}_2) \geq 2$, and Propositions 8, 9 and 10, for the case Card $(\mathcal{F}_2) = 1$, we obtain the main result of the present subsection.

Theorem 13. *There are no 6-cycles displaying the form*

$$x_{n+3} = x_{n+2}\, g(x_{n+1})\, h(x_n),$$

with $g, h : (0,\infty) \longrightarrow (0,\infty)$ continuous.

3.3.3 The case $x_{n+3} = x_{n+1} g(x_{n+2}) h(x_n)$

Now, the research delves into the existence of 6-cycles of the form

$$x_{n+3} = x_{n+1}\, g(x_{n+2}) h(x_n), \tag{3.24}$$

with $g, h : (0, \infty) \to (0, \infty)$ continuous maps. Recall that by (♣) we can assume without loss of generality that $1 \in \mathcal{F}_2$, with $g(1) = h(1) = 1$.

Firstly, we establish some preliminaries results that will be useful to prove the main theorem of this subsection, namely, Theorem 14.

Lemma 30. *If Equation (3.24) is a 6-cycle, then* $h^{-1}(x) = \frac{1}{h\left(xg^2(x)\right)}$ *for all* $x > 0$.

Proof. Set an arbitrary $x > 0$ and consider the initial conditions $x_1 = h^{-1}(x)$, $x_2 = x_3 = 1$. If we apply Equation (3.24), we achieve $x_4 = x$, $x_5 = g(x)$ and $x_6 = xg^2(x)$. Thus, since the recurrence is a 6-cycle, $x_{n+6} = x_n$ for all $n \geq 1$, we get

$$1 = x_3 = x_9 = x_7 g(x_8) h(x_6) = x_1 g(x_2) h(x_6) = h^{-1}(x) \cdot h\big(xg^2(x)\big),$$

from where the result follows. □

Let us define $\psi(x) := x\, g\big(g(x)\big)$, for all $x > 0$. From the previous lemma and the fact that h is a homeomorphism, we have

$$h^{-1}(x) = \frac{1}{h\big(\psi(x)\big)}, \quad \text{for all } x > 0, \tag{3.25}$$

which can be rewritten as

$$\psi(x) = h^{-1}\left(\frac{1}{h^{-1}(x)}\right), \quad \text{for all } x > 0, \tag{3.26}$$

due to the fact that h, h^{-1} are homeomorphisms. This expression allows us to establish the monotonic character of ψ.

Lemma 31. *If Equation (3.24) is a 6-cycle, then* $\psi : (0, \infty) \to (0, \infty)$ *is a decreasing homeomorphism.*

Proof. Firstly, ψ is a homeomorphism from $(0, \infty)$ into itself since h^{-1} is one. Secondly, if h^{-1} is increasing (respectively, decreasing), then $\frac{1}{h^{-1}}$ is decreasing (increasing), and therefore, $h^{-1}\left(\frac{1}{h^{-1}}\right)$ is decreasing in both cases. □

We are now ready to prove the main result of this section and demonstrate the non-existence of 6-cycles displaying the form of (3.24).

Theorem 14. *There are no 6-cycles exhibiting the form*

$$x_{n+3} = x_{n+1}\, g(x_{n+2})\, h(x_n),$$

with $g, h : (0, \infty) \longrightarrow (0, \infty)$ *continuous maps.*

Proof. Assume Equation (3.24) is a 6-cycle. Bearing in mind that $g^2(x) = \frac{\psi(x)}{x}$, we can apply Lemma 31 to deduce that $g^2(x)$ is a decreasing homeomorphism from $(0, \infty)$ into itself. Nevertheless, it is easy to see that $g : (0, \infty) \to (0, \infty)$ is a homeomorphism too, either increasing or decreasing, but, in both cases, we obtain that g^2 is an increasing map, a contradiction. □

3.3.4 The case $x_{n+3} = x_n g(x_{n+2}) h(x_{n+1})$

Finally, we devote our study to the case

$$x_{n+3} = x_n\, g(x_{n+2})\, h(x_{n+1}), \tag{3.27}$$

with $g, h : (0, \infty) \to (0, \infty)$ being continuous maps. Here, in contrast with the other cases analyzed previously, we prove the existence of a 6-cycle displaying the form (3.27). In concrete, we arrive to the potential cycle

$$x_{n+3} = x_n \left(\frac{x_{n+2}}{x_{n+1}}\right)^2.$$

Furthermore, we are able to show that such cycle is, indeed, the unique 6-cycle of the form (3.27).

As in the other cases, recall that the non-emptiness character of $\mathcal{F}_2$ let us assume that $1 \in \mathcal{F}_2$ with $g(1) = h(1) = 1$ (see (♣)). Moreover, g^n denotes the composition $g \circ \underbrace{\ldots}_{n \text{ times}} \circ g$.

Firstly, we give a sufficient condition on the maps g, h in order that Equation (3.27) is a 6-cycle. To do so, we will apply the following result from [18].

Lemma 32. *Consider the system of functional equations*

$$f\left(\frac{x}{f(x)}\right) = \frac{1}{f(x)} \quad \textit{and} \quad f^2(x) = \big(f(x)\big)^2,$$

where $f : (0, \infty) \to (0, \infty)$ is a continuous map. Then, their solutions are given by $f(x) = 1$ or $f(x) = x^2$ for all $x > 0$.

Proposition 11. *Assume Equation (3.27) is a 6-cycle. If $g(x)h(x) = 1$ for all $x > 0$, then the 6-cycle must be*

$$x_{n+3} = x_n \left(\frac{x_{n+2}}{x_{n+1}}\right)^2.$$

Proof. As a first step, it should be highlighted that the condition $g(x)h(x) = 1$ for all $x > 0$ is equivalent to the fact $\mathcal{F}_2 = (0, \infty)$.

Now, we take an arbitrary point $x > 0$ and set the initial conditions $x_1 = 1$, $x_2 = x_3 = x$. If we apply the recurrence, we get $x_4 = 1$, $x_5 = xh(x)$ and $x_6 = xg\big(xh(x)\big)$. Then, by global periodicity,

$$x = x_3 = x_9 = x_6 g(x_8) h(x_7) = x_6 g(x_2) h(x_1) = xg\big(xh(x)\big)g(x).$$

Thus, applying $h(x) = \frac{1}{g(x)}$, we obtain

$$g\left(\frac{x}{g(x)}\right) = \frac{1}{g(x)}, \quad \text{for all } x > 0. \tag{3.28}$$

On the other hand, we iterate the initial conditions $x_1 = x$, $x_2 = x_3 = 1$ under Equation (3.27) in order to obtain $x_4 = x$, $x_5 = g(x)$ and $x_6 = h(x) \cdot g\big(g(x)\big)$. Again, by global periodicity,

$$1 = x_3 = x_9 = x_6 g(x_8) h(x_7) = x_6 g(x_2) h(x_1) = h(x) \cdot g\big(g(x)\big) \cdot h(x).$$

So, $g^2(x) = \frac{1}{h(x)h(x)}$, but bearing in mind that $h(x) = \frac{1}{g(x)}$, we deduce

$$g^2(x) = \big[g(x)\big]^2, \quad \text{for all } x > 0. \tag{3.29}$$

Therefore, by Lemma 32, the unique solutions of the system of functional equations (3.28)-(3.29) are $g(x) = 1$ and $g(x) = x^2$ for all $x > 0$. The first one gives rise to the 3-cycle $x_{n+3} = x_n$; while the second one yields to the potential 6-cycle $x_{n+3} = x_n \left(\frac{x_{n+2}}{x_{n+1}}\right)^2$. □

Once we have established such sufficient condition, we return to the general case, that is, without assuming $g(x)h(x) = 1$ for all $x > 0$.

Lemma 33. *Assume Equation* (3.27) *is a* 6*-cycle. Then, for all* $x > 0$,

$$h(x) = \sqrt{\frac{1}{g^2(x)}}. \tag{3.30}$$

Proof. From the proof of Proposition 11, we know that the sequence generated from the initial conditions $x_1 = x$, $x_2 = x_3 = 1$ and the property of global periodicity give rise to $\big(h(x)\big)^2 = \frac{1}{g^2(x)}$, which implies (3.30). □

The above result allows us to characterize the set of fixed points of the continuous map g.

Lemma 34. *Assume that Equation* (3.27) *is a* 6*-cycle. Then, the set of fixed points of* g *is the singleton* $\{1\}$, $\mathrm{Fix}(g) = \{1\}$.

Proof. Since $g(1) = 1$, we know that $\mathrm{Fix}(g) \neq \emptyset$. Suppose that $z \in \mathrm{Fix}(g)$ for some $z > 0$. Then, $g^2(z) = z$ and applying (3.30), we get $h(z) = \frac{1}{\sqrt{z}}$.

Now, take the initial conditions $x_1 = \sqrt{z}$, $x_2 = z, x_3 = 1$ to obtain, under Equation (3.27), $x_4 = \sqrt{z} \cdot h(z) = 1$, $x_5 = z$, and $x_6 = g(z) = z$. Then, by global periodicity,

$$z = x_2 = x_8 = x_5 g(x_7) h(x_6) = x_5 g(x_1) h(x_6) = z g\big(\sqrt{z}\big) h(z) = z g\big(\sqrt{z}\big) \frac{1}{\sqrt{z}},$$
$$1 = x_3 = x_9 = x_6 g(x_8) h(x_7) = x_6 g(x_2) h(x_1) = z g(z) h\big(\sqrt{z}\big) = z^2 h\big(\sqrt{z}\big).$$

From the first relation we get $g\big(\sqrt{z}\big) = \sqrt{z}$, while from the second one we deduce $h\big(\sqrt{z}\big) = \frac{1}{z^2}$.

Moreover, $g\big(\sqrt{z}\big) = \sqrt{z}$ and (3.30) yield to

$$h(\sqrt{z}) = \sqrt{\frac{1}{g\big(g(\sqrt{z})\big)}} = \sqrt{\frac{1}{\sqrt{z}}}.$$

Therefore, $h(\sqrt{z}) = \frac{1}{z^2}$ and $h(\sqrt{z}) = \sqrt{\frac{1}{\sqrt{z}}}$, which allow us to conclude that $\frac{1}{z^8} = \frac{1}{z}$ and $z = 1$. □

Lemma 35. *Assume Equation (3.27) is a 6-cycle. Then, for all $x > 0$,*

$$1 = \sqrt{\frac{1}{g^3(x)}} \cdot g\left(\sqrt{g^2(x)}\right), \tag{3.31}$$

and

$$\left[g(x)\right]^2 = \sqrt{g^2\left(\sqrt{g^2(x)}\right)}. \tag{3.32}$$

Proof. Take the initial conditions $x_1 = x$, $x_2 = x_3 = 1$. Their iteration under Equation (3.27) gives us $x_4 = x$, $x_5 = g(x)$ and $x_6 = h(x) \cdot g^2(x)$. Then, by an argument of global periodicity, we get

$$\begin{aligned} x = x_1 = x_7 = x_4 g(x_6) h(x_5) &= x\, g\big(h(x)g^2(x)\big) \cdot h\big(g(x)\big), \\ 1 = x_2 = x_8 = x_5 g(x_7) h(x_6) = x_5 g(x_1) h(x_6) &= g(x)\, g(x) \cdot h\Big(h(x)g^2(x)\Big). \end{aligned}$$

From the first equality, $g\big(h(x)g^2(x)\big) \cdot h\big(g(x)\big) = 1$, and applying (3.30) twice, the first one with $g(x)$ instead of x, we deduce

$$1 = \sqrt{\frac{1}{g^3(x)}} \cdot g\left(\sqrt{g^2(x)}\right),$$

and we obtain (3.31). On the other hand, from the second equality we conclude that $1 = \left[g(x)\right]^2 h\Big(h(x)g(g(x))\Big)$. Now, applying (3.30) twice, we obtain (3.32):

$$\begin{aligned} 1 &= \left[g(x)\right]^2 h\left(\sqrt{\frac{1}{g\big(g(x)\big)}} \cdot g\big(g(x)\big)\right) = \left[g(x)\right]^2 h\left(\sqrt{g^2(x)}\right) \\ &= \left[g(x)\right]^2 \sqrt{\frac{1}{g^2\left(\sqrt{g^2(x)}\right)}}\,. \end{aligned}$$

□

Lemma 36. *Assume Equation (3.27) is a 6-cycle. Then, the following holds:*

(a) If $g^2(x) = 1$ for some $x > 0$, then $g(x) = 1$.

(b) If $g(x) = 1$ for some $x > 0$, then $x \in \mathcal{F}_2$.

(c) If $h(x) = 1$ for some $x > 0$, then $x \in \mathcal{F}_2$.

(d) If $x \in \mathcal{F}_2$, then $g\left(\frac{x}{g(x)}\right) = \frac{1}{g(x)}$.

Proof. Suppose $g^2(x) = 1$ for some $x > 0$. Then (3.32) implies $[g(x)]^2 = 1$ and Part (a) follows. On the other hand, if $g(x) = 1$, (3.30) and $g(1) = 1$ imply $h(x) = 1$. Hence, $h(x)g(x) = 1$ and $x \in \mathcal{F}_2$, obtaining Part (b). Similarly, if $h(x) = 1$, (3.30) yields to $g^2(x) = 1$ and we can apply Part (a) to get $x \in \mathcal{F}_2$, which proves Part (c). Finally, for Part (d), consider the initial conditions $x_1 = 1$, $x_2 = x_3 = x$ where $x \in \mathcal{F}_2$. In the proof of Proposition 11 we have seen that we arrive to (3.28). □

Lemma 37. *Assume Equation (3.27) is a 6-cycle. Then, for all $x > 0$,*

$$g^3(x) = \sqrt{g^2\left(\sqrt{g^2\left(\sqrt{g^2(x)}\right)}\right)}, \tag{3.33}$$

and

$$g^3(x) = \sqrt{g^2\left([g(x)]^2\right)}. \tag{3.34}$$

Proof. The result follows by a direct application of Lemma 35. Firstly, by (3.31), we get $g^3(x) = \left[g\left(\sqrt{g^2(x)}\right)\right]^2$, and using (3.32), with $\sqrt{g^2(x)}$ instead of x, we achieve (3.33):

$$g^3(x) = \sqrt{g^2\left(\sqrt{g^2\left(\sqrt{g^2(x)}\right)}\right)}.$$

Secondly, to obtain (3.34), simply combine Equations (3.33) and (3.32). □

In order to simplify the notation, we denote

$$\varphi(x) := \sqrt{x}, \ \ x > 0.$$

Notice, that this allows us to rewrite (3.33) as

$$g^3(x) = \varphi\left(g^2\left(\varphi(g^2(\varphi(g^2(x))))\right)\right) \Longleftrightarrow g^3 = \left[\varphi \circ g^2\right]^3. \tag{3.35}$$

Now, we reduce the problem to an appropriate functional equation whose solutions are the candidates to 6-cycles displaying the form of Equation (3.27). To do so, recall that if (3.27) is a 6-cycle, then according to (3.31) and (3.35) we have

$$\varphi \circ g^3 = g \circ \varphi \circ g^2, \tag{3.36}$$
$$g^3 = \varphi \circ g^2 \circ \varphi \circ g^2 \circ \varphi \circ g^2, \tag{3.37}$$

where $\varphi(x) = \sqrt{x}$, for all $x > 0$. On the one hand, from (3.36) and (3.37) we have

$$\begin{aligned} g^3 &= \varphi \circ g \circ \left(g \circ \varphi \circ g^2\right) \circ \varphi \circ g^2 = \varphi \circ g \circ \left(\varphi \circ g^3\right) \circ \varphi \circ g^2 \\ &= (\varphi \circ g)^2 \circ g \circ \left(g \circ \varphi \circ g^2\right) = (\varphi \circ g)^2 \circ g \circ \left(\varphi \circ g^3\right) \\ &= (\varphi \circ g)^2 \circ \left(g \circ \varphi \circ g^2\right) \circ g = (\varphi \circ g)^2 \circ \left(\varphi \circ g^3\right) \circ g = (\varphi \circ g)^3 \circ g^3. \end{aligned}$$

So, we can conclude that

$$g^3 = (\varphi \circ g)^3 \circ g^3. \tag{3.38}$$

On the other hand, we can also gather the compositions into the form

$$\begin{aligned} g \circ g \circ g &= \varphi \circ g \circ \left(g \circ \varphi \circ g^2\right) \circ \varphi \circ g^2 = \varphi \circ g \circ \left(\varphi \circ g^3\right) \circ \varphi \circ g^2 \\ &= \varphi \circ \left(g \circ \varphi \circ g^2\right) \circ g \circ \varphi \circ g^2 = \varphi \circ \left(\varphi \circ g^3\right) \circ g \circ \varphi \circ g^2 \\ &= \varphi \circ \left(\varphi \circ g^3\right) \circ \left(g \circ \varphi \circ g^2\right) = \varphi \circ \varphi \circ g^3 \circ \left(\varphi \circ g^3\right) \\ &= \varphi^2 \circ g^2 \circ \left(g \circ \varphi \circ g^2\right) \circ g = \varphi^2 \circ g^2 \circ \left(\varphi \circ g^3\right) \circ g \\ &= \varphi^2 \circ g \circ \left(g \circ \varphi \circ g^2\right) \circ g^2 = \varphi^2 \circ \left(g \circ \varphi \circ g^2\right) \circ g^3 \\ &= \varphi^2 \circ \left(\varphi \circ g^3\right) \circ g^3 = \varphi^3 \circ g^3 \circ g^3. \end{aligned}$$

Therefore,

$$g^3 = \varphi^3 \circ g^3 \circ g^3. \tag{3.39}$$

Taking into account that $\varphi^3(x) = \sqrt[8]{x}$ and that its inverse is $\phi(x) = x^8$, Equation (3.39) can be written as

$$\phi \circ g^3 = g^3 \circ g^3. \tag{3.40}$$

If we put $g^3 = f$, we can think about g^3 as a solution of the functional equation

$$\phi \circ f = f \circ f. \tag{3.41}$$

In our case, $\phi(x) = x^8$ is given and the unknown is the map f.

In the sequel, we consider the set

$$\mathcal{S} := \operatorname{Im}\left(g^3\right) = \{g^3(x) : x > 0\}.$$

Observe that $\mathcal{S} \neq \emptyset$ since $1 \in \mathcal{S}$ as long as $g(1) = 1$.

Lemma 38. *Assume that g^3 satisfies the functional equation (3.41). If $y \in \mathcal{S} = \operatorname{Im}(g^3)$, then $g^3(y) = y^8$ and $y^8 \in \mathcal{S}$.*

Proof. Suppose that $y \in \mathcal{S}$. Therefore, there exists a point $z > 0$ such that $g^3(z) = y$. Thus, by (3.40),

$$\phi\big(g^3(z)\big) = g^3\big(g^3(z)\big),$$

so $\phi(y) = g^3(y)$ and $y^8 = g^3(y)$. □

We claim that the set $\mathcal{S}$ is either the singleton $\{1\}$, or the interval $(0, \infty)$. Indeed, from Lemma 38 and the intermediate value property applied to the continuous map g^3, we deduce that $\mathcal{S}$ must be one of the following sets: $\mathcal{S} = \{1\}$, $\mathcal{S} = (0, \infty)$, $\mathcal{S} = (0, 1]$ or $\mathcal{S} = [1, \infty)$. However, it is easy to see that $(0, 1] \subseteq \mathcal{S}$ if and only if $[1, \infty) \subseteq \mathcal{S}$, discarding the two last cases.

Lemma 39. *Assume that g^3 satisfies the functional equation (3.41). Then, $(0,1] \subseteq \mathcal{S}$ if and only if $[1,\infty) \subseteq \mathcal{S}$.*

Proof. Firstly, let us assume that $(0,1] \subseteq \mathcal{S}$. We apply Lemma 38 to obtain $g^3(y) = y^8$ for every $y \in (0,1]$. Now, from (3.38), we deduce $y^8 = (\varphi \circ g)^3(y^8)$, or, by the change of variables $x = y^8$, $x = (\varphi \circ g)^3(x)$ for all $x \in (0,1]$. Also, from the definition of $\varphi(x)$, it is direct to conclude that $\varphi \circ g$ is an increasing homeomorphism with $(\varphi \circ g)^3|_{(0,1]} = \mathrm{Id}|_{(0,1]}$. This yields to $(\varphi \circ g)|_{(0,1]} = \mathrm{Id}|_{(0,1]}$ and, therefore, $g(x) = x^2$ for all $x \in (0,1]$. Then, applying (3.30), we derive that $h(y) = \sqrt{\frac{1}{g^2(y)}} = \sqrt{\frac{1}{y^4}} = \frac{1}{y^2}$. Consequently, $h(y)g(y) = 1$ for all $(0,1]$, and $(0,1] \subseteq \mathcal{F}_2$.

On the other hand, from Lemma 36-(d), we have $g\left(\frac{y}{g(y)}\right) = \frac{1}{g(y)}$ for all $y \in (0,1]$, that is, $g\left(\frac{1}{y}\right) = \frac{1}{y^2}$ and we conclude $g(u) = u^2$ for all $u = \frac{1}{y} > 1$. This implies that $g(v) = v^2$ for all $v > 0$, and it holds $\mathcal{S} = (0,\infty)$.

Next, to see the other implication, the reasoning is analogous. Assume that $[1,\infty) \subseteq \mathcal{S}$. Here, we can repeat every step to lead to $g(y) = y^2$ if $y \geq 1$. Then, as in such case, we find that $y \in \mathcal{F}_2$ if $y \geq 1$, and $g\left(\frac{1}{y}\right) = \frac{1}{y^2}$. Hence, $g(v) = v^2$ for all $v > 0$, and again, we arrive to $\mathcal{S} = (0,\infty)$. ∎

In conclusion, we only have to analyze two different scenarios:

Case (i): $\mathcal{S} = \{1\}$. Suppose that $g^3(x) = 1$ for all $x > 0$. Then, by Lemmas 37 and 36(a), we have that $g(x) = 1$ for all $x > 0$. Moreover, (3.30) implies that $h(x) = 1$ for all $x > 0$. In conclusion, we obtain the 3-cycle $x_{n+3} = x_n$.

Case (ii): $\mathcal{S} = (0,\infty)$. Take an arbitrary $w \in (0,\infty)$. Then, there exists a point $z = z(w)$ such that $g^3(z) = w$ and, by (3.40), $w^8 = g^3(w)$. From here, it can be easily seen that g is an increasing homeomorphism. Indeed, g is surjective since $\mathcal{S} = (0,\infty)$; it is injective, since, by Lemma 38, g^3 it is; and it is increasing since, otherwise, g^3 would contradict Lemma 38.

We can proceed as in Lemma 39, in order to obtain $g(w) = w^2$ for all $w > 0$. Finally, (3.30) provides the value of $h(x)$, namely,

$$h(x) = \sqrt{\frac{1}{g^2(x)}} = \sqrt{\frac{1}{g(x^2)}} = \sqrt{\frac{1}{x^4}} = \frac{1}{x^2},$$

$x > 0$. Then, we conclude that the difference equation (3.27) is the potential 6-cycle

$$x_{n+3} = x_n \left(\frac{x_{n+2}}{x_{n+1}}\right)^2.$$

We can gather the previous analysis in the following result:

Proposition 12. *Consider Equation (3.27), with $g, h : (0,\infty) \to (0,\infty)$ continuous. Then, the unique 6-cycle is given by*

$$x_{n+3} = x_n \left(\frac{x_{n+2}}{x_{n+1}}\right)^2.$$

Finally, we use the change of variables presented at the end of Subsection 3.1.1 to get rid of the condition $g(1) = h(1) = 1$. In this sense, take $\gamma(y) = y^2$ and $\eta(y) = \frac{1}{y^2}$ and consider the 6-cycle

$$y_{n+3} = y_n \left(\frac{y_{n+2}}{y_{n+1}}\right)^2 = y_n \gamma(y_{n+2})\eta(y_{n+1}),$$

and set $y_l = \frac{x_l}{\bar{x}}$, where $\bar{x} \in \mathcal{F}_2$. Hence, the above equation transforms into

$$x_{n+3} = x_n \gamma\left(\frac{x_{n+2}}{\bar{x}}\right)\eta\left(\frac{x_{n+1}}{\bar{x}}\right) = x_n \left(\frac{x_{n+2}}{\bar{x}}\right)^2 \cdot \frac{1}{\left(\frac{x_{n+1}}{\bar{x}}\right)^2} = x_n \left(\frac{x_{n+2}}{x_{n+1}}\right)^2.$$

In conclusion, we can state the main result of this section:

Theorem 15. *The unique* 6*-cycle displaying the form of Equation* (3.27)*, where the maps* $g, h : (0, \infty) \to (0, \infty)$ *are continuous, is given by*

$$x_{n+3} = x_n \left(\frac{x_{n+2}}{x_{n+1}}\right)^2.$$

Now, once that we have studied each possible configuration of Equation (3.17), we can gather the obtained results, namely Theorem 13, 14 and 15, in the following Theorem, which is the main result of the Section.

Theorem C. *Let us consider Equation* (3.17)*. The unique* 6*-cycle displaying such form is given by*

$$x_{n+3} = x_n \left(\frac{x_{n+2}}{x_{n+1}}\right)^2.$$

3.4 Conclusions and open problems

This chapter has dealt with the analysis of global periodicity for a concrete family of difference equations of third order, namely,

$$x_{n+3} = x_i f(x_j, x_k),$$

where $i, j, k \in \{n, n+1, n+2\}$ are pairwise distinct, $f : (0, \infty)^2 \to (0, \infty)$ is a continuous map and the initial conditions are positive real numbers. The existence of p-cycles for $p \leq 5$ was already known, see [7], so we have advanced in this problem by focusing on the existence of 6-cycles displaying such form. In Section 3.2, we have proved the non-existence of 6-cycles in the case where the map f is symmetric, that is, $f(x, y) = f(y, x)$, for every $x, y > 0$; and in Section 3.3, we have shown that the unique 6-cycle that exists in the particular case where the map f separates variables, $f(x, y) = g(x)h(y)$, with $g, h : (0, \infty) \to (0, \infty)$ continuous, is the potential cycle

$$x_{n+3} = x_n \left(\frac{x_{n+2}}{x_{n+1}}\right)^2.$$

The techniques employed have been the direct computation of the orbits generated by (3.3), the analysis of the fiber maps $f(\cdot, z)$ and $f(z, \cdot)$, the resolution of functional equations and the study of the monotonic character of f or of the maps g, h in the case of separation of variables.

A natural problem that arises is the determination of the existence of 6-cycles exhibiting the form $x_{n+3} = x_i f(x_j, x_k)$ whenever the map f does not separates variables. Furthermore, it will also be of interest to attack the problem of the existence of p-cycles for $p > 6$.

On the other hand, we can consider a generalization of the family of third order to an order $k \geq 4$:

$$x_{n+k} = x_{j_1} f(x_{j_2}, \ldots, x_{j_k}),$$

where $j_1, \ldots, j_k \in \{n, \ldots, n+k-1\}$ are pairwise distinct and $f : (0, \infty)^{k-1} \to (0, \infty)$ is a continuous map. If we consider the extra assumption of $f(x_1, \ldots, x_{k-1}) = f\big(\sigma(x_1, \ldots, x_{k-1})\big)$ for all $x_1, \ldots, x_{k-1} \in (0, \infty)$, where σ is any non-trivial permutation of the variables, are there p-cycles exhibiting such form? As a first step, the problem can be addressed in the particular case $k = 4$ and $p \leq 6$.

Moreover, the same questions related to the existence of p-cycles can be proposed for similar families of difference equations of third order such as

$$x_{n+3} = \frac{1}{x_i} f(x_j, x_k), \quad \text{or} \quad x_{n+3} = x_i^{\alpha} f(x_j, x_k), \quad \text{where } \alpha \in \mathbb{R}.$$

As far as we are concerned, the only advances made in this direction are gather in [18] where the authors studied the particular case

$$x_{n+3} = \frac{1}{x_n} f(x_{n+2}, x_{n+1}),$$

being $f : (0, \infty) \times (0, \infty)$ a continuous map. Concretely, they were able to prove that there are not 3-cycles and 5-cycles exhibiting such form and that the unique 4-cycle is given by $f(x, y) = \frac{C}{xy}$ for some constant $C > 0$.

The dynamical property of global periodicity is a rigid one and discovering tangible models demonstrating this behaviour is a challenging endeavor. In the next chapter, we will continue with the evolution of the study of the dynamics of autonomous difference equations. Concretely, we will delve with the property of periodicity, the study of the accumulation points for the non-periodic solutions of the models and the invariance of the equations.

Chapter 4

Periodicity, accumulation points and invariance

With the aim of continuing with the analysis of dynamical properties of autonomous difference equations, this chapter is devoted to study periodicity, the accumulation points of non-periodic solutions and the invariance of the difference equations. Previously, we have already studied the dynamical property of global periodicity, but such dynamical property is a strong one and it is challenging to find real models exhibiting this behaviour. In this sense, it is natural to carry on our study with the search of periodic solutions and their associated periods.

The study of the class of max-type difference equations is suitable for transitioning from the analysis of global periodicity to periodicity due to the dynamical properties that manifest in this type of equations. For instance, the family of generalized max-type Lyness' difference equations

$$x_{n+1} = \frac{\max\{x_n^k, A\}}{x_n^l x_{n-1}},$$

where A is a positive real number, the exponents k and l are integer numbers and the initial conditions are positive, provides a variety of equations with periodic solutions. Also, the reciprocal difference equation

$$x_{n+1} = \max\left\{\frac{A_0}{x_n}, \dots, \frac{A_k}{x_{n-k}}\right\},$$

where the parameters $A_0, \dots, A_k$, are real numbers and the initial conditions are nonzero real numbers, is characterized by having solutions that are eventually periodic.

Furthermore, the class of max-type difference equations has a lot of applications in diverse fields. For example, in automatic control theory, [6, 83, 90]; biology, [20, 42, 54, 97]; or economics, see Hicks Equation in [98]. For a survey on the topic, consult [69].

Here, we focus on the k-order autonomous difference equation

$$x_{n+k} = \max\{x_{n+k-1}, x_{n+k-2}, \dots, x_{n+1}, 0\} - x_n, \tag{4.1}$$

which appeared for the first time in [39], where the author proposed the problem of demonstrating that every sequence (x_n) generated by (4.1) from monotonic initial conditions was periodic of period $3k-1$. The solution of such problem was provided by David Callan among others.

Later on, in [29], the authors tackle with some properties of (4.1). In concrete, they show the boundedness character of the solutions and that the equation is not globally periodic for $k \geq 4$. However, they prove that the set of periods is unbounded.

In the particular cases $k = 2$ and $k = 3$, Equation (4.1) is globally periodic of periods 5 and 8, respectively. However, for $k = 4$ such property does not hold and it arises naturally the problem of determining the dynamics of Equation (4.1) in such particular case. In this direction, in this chapter we focus on the complete description of the dynamics of the fourth order max-type difference equation

$$x_{n+4} = \max\{x_{n+3}, x_{n+2}, x_{n+1}, 0\} - x_n, \tag{4.2}$$

with arbitrary real initial conditions. Therefore, we deep into the analysis of (4.2) and fully established the behaviour of their solutions. Specifically, as a first step, we study the existence of periodic solutions and we provide a complete description of the set of periods of (4.2), $\mathrm{Per}(F_4)$.

Theorem D. *Consider Equation (4.2) and let* $\mathrm{Per}(F_4)$ *be its set of periods. Then*

$$\mathrm{Per}(F_4) = \{1, 8, 11\} \bigcup \big\{10 \cdot a + 11 \cdot b \mid \gcd(a, b) = 1, a \geq 1, b \geq 2a + 1\big\}.$$

This is developed in Section 4.1. Furthermore, apart from the fact that the set of periods is unbounded, which was proved in [29], we show that 1674 is the biggest natural number not included in it. Then, Section 4.3 deals with the behaviour of the non-periodic solutions. In this line, we fully determine their accumulation point sets that, in fact, are configured as proper compact intervals of the real line.

Theorem E. *Let* (x_1, x_2, x_3, x_4) *be arbitrary real initial conditions that generate a non-periodic orbit* (x_n) *under Equation (4.2). Then, the set of accumulation points of* (x_n) *is a compact interval. Even more, the tuple* (x_1, x_2, x_3, x_4) *is equivalent to some tuple of initial conditions* (x, y, z, w)*, with* $x = \max\{x_n : n \geq 1\}$*,* $x \geq w \geq y \geq z \geq 0$*, and* $\frac{w-z}{x} \in \mathbb{R} \setminus \mathbb{Q}$*, and the orbit accumulates in the compact interval* $\big[\min\{w - x, -z\}, x\big]$*.*

Next, Section 4.4 focuses on the invariants of Equation (4.2); concretely, we give a new first integral of the discrete dynamical system associated to Equation (4.2) and comment on the possible existence of another first integral for the system based on numerical simulations. Finally, Section 4.5 is devoted to gather some open problems and forthcoming lines of research related to the topic. This chapter is based on [65], [66] and [62].

4.1 Periodic solutions

This section, that is mainly based on [65], is focused on the periodic solutions of Equation (4.2). Its structure is as follows: firstly, we give some properties concerning the general Equation (4.1) and the particular Equation (4.2). Straightaway, we present possible ways to describe the periodic orbits and determine the set of periods of (4.2), as well as possible realizations of such periods.

4.1.1 Properties for the general case

We begin by establishing some properties for the general case, Equation (4.1),

$$x_{n+k} = \max\{x_{n+k-1}, x_{n+k-2}, \ldots, x_{n+1}, 0\} - x_n,$$

with arbitrary real initial conditions.

Observe that we can consider its associate dynamical system

$$F_k(x_1, x_2, \ldots, x_k) = \big(x_2, \ldots, x_k, \max\{x_2, \ldots, x_k, 0\} - x_1\big).$$

The above map, $F_k : \mathbb{R}^k \to \mathbb{R}^k$, is a bijection whose inverse is given by

$$F_k^{-1}(y_1, y_2, \ldots, y_k) = \big(\max\{y_1, y_2, \ldots, y_{k-1}, 0\} - y_k, y_1, \ldots, y_{k-1}\big).$$

From this fact, it is direct to infer that every eventually periodic sequence of (4.1) is, indeed, periodic.

Proposition 13. *Every eventually periodic sequence of Equation (4.1) is periodic.*

Now, concerning the periods of Equation (4.1), we can easily see the following result.

Proposition 14. *Equation (4.1) has a unique equilibrium point, namely, $x^* = 0$. In particular, $1 \in \mathrm{Per}(F_k)$ for all $k \geq 2$.*

Furthermore, we can deep into the existence of other periods for Equation (4.1).

Proposition 15. *Consider Equation (4.1). Then, the following statements hold:*

(a) $2k \in \mathrm{Per}(F_k)$ *for all* $k \geq 4$.

(b) For $k \geq 3$, $2 \in \mathrm{Per}(F_k)$ if and only if k is odd. Moreover, a sequence (x_n) has period 2 if and only if for some $a > 0$, either $x_{2j-1} = a$ and $x_{2j} = 0$ for all $j \geq 1$, or $x_{2j-1} = 0$ and $x_{2j} = a$ for all $j \geq 1$.

Proof. To see Part (a), take initial conditions $(x_1, \ldots, x_k) = (0, x, 0, x, x, \ldots, x)$, with $x > 0$. Next, by Equation (4.1), we obtain the following $2k$ terms of the sequence

$$\underbrace{0, x, 0, x, x, \ldots, x,}_{k \text{ terms}} \ \underbrace{x, 0, x, 0, 0, \ldots, 0,}_{k \text{ terms}} \ \underbrace{0, x, 0, x, x, \ldots, x,}_{k \text{ terms}} \ldots$$

Let (x_n) be a periodic sequence of period q being such period a divisor of $2k$. Obviously, $q \notin \{1, 2, k\}$ as $x > 0$. Even, $q \neq k - 1$ since $2k$ is not divisible by $(k - 1)$ for $k \geq 4$. Now, if $q = 2k$, the proof follows. On the other hand, $q \leq k - 2$ and we have $(x_4, x_5, \ldots, x_k, x_{k+1}) = (x, x, \ldots, x)$, which is a string of a periodic sequence with length $k - 2$. This allows us to deduce that $(x_n) = (x)_n$, which would imply $x = 0$, a contradiction. Hence, $q = 2k$.

Finally, for Part (b), let us distinguish two cases:

- If k is odd with $k \geq 3$, set the initial conditions $(x, 0, x, 0, \ldots, x, 0, x, 0, x)$, with $x > 0$. Then, it is direct to see that they generate a 2-periodic sequence.

- If $k = 2m$ is even, then $2 \notin \text{Per}(F_{2m})$. Indeed, take $(x_1, x_2, \ldots, x_{2m-1}, x_{2m}) = (a, b, a, b, \ldots, a, b)$ and suppose that they provide a periodic sequence of period 2, with $a \geq b$. Thus, $a = x_1 = x_{2m+1} = \max\{a, b, 0\} - a = \max\{a, 0\} - a$, so $2a = \max\{a, 0\}$ and we deduce $a = 0$. In an analogous way, $b = x_2 = x_{2m+2} = \max\{a, b, 0\} - b = -b$, so $b = 0$. In conclusion, $a = b = 0$ and we would obtain the equilibrium point $x^* = 0$, which contradicts our hypothesis.

The proof of the characterization of 2-periodic sequences is immediate and we omit it. □

Next, we see that the greatest term of a non-trivial periodic sequence must be positive.

Proposition 16. *Let (x_n) be a periodic solution of Equation (4.1) of period p. Then, $m = \max\{x_j : j = 1, \ldots, p\} \geq 0$. Moreover, if $m = 0$, then the solution is the equilibrium point $x^* = 0$.*

Proof. Notice that, since (x_n) is a periodic sequence, we can assume, without loss of generality, that $x_1 = \max\{x_j : j = 1, \ldots, p\}$. Let us suppose that $x_1 < 0$ and derive a contradiction. We consider the inverse difference equation and we get

$$x_1 = \max\{x_p, x_{p-1}, \ldots, x_{p-(k-2)}, 0\} - x_{p-(k-1)},$$

where the indexes are taken mod(p) if $p < k$. Since x_1 is the greatest term of the sequence and $x_1 < 0$, it follows that $x_i < 0$ for every $i = 1, \ldots, p$, so

$$\max\{x_p, x_{p-1}, \ldots, x_{p-(k-2)}, 0\} = 0,$$

and, therefore, $x_1 + x_{p-(k-1)} = 0$. However, this is impossible due to the fact that every term of the sequence is negative. Moreover, it is direct to check that assuming $m = 0$ implies $(x_n) = (0, 0, 0, \ldots)$. □

Considering the above result, in the sequel we will always assume that for every periodic sequence (x_n), $x_1 = \max\{x_j : j = 1, \ldots, p\} \geq 0$.

Proposition 17. *Let (x_n) be a p-periodic solution of Equation (4.1) with $p \geq 2$. Then*

$$x_j \geq 0, \text{ for } j = 1, \ldots, k, \text{ and } x_{p-j} \geq 0, \text{ for } j = 0, \ldots, k-2.$$

Furthermore, $x_{k+1} \leq 0$ and $x_{p-k+1} \leq 0$. (Every index is taken $\mathrm{mod}(p)$*).*

Proof. Applying the periodic character of the sequence, for $j = 2, \ldots, k$, we obtain

$$\begin{aligned} x_j &= \max\{x_{j-1}, \ldots, x_1, x_p, x_{p-1}, \ldots, x_{p-(k-j-1)}, 0\} - x_{p-(k-j)} \\ &= x_1 - x_{p-(k-j)} \geq 0. \end{aligned}$$

Now, using the inverse of Equation (4.1), we achieve the second set of inequalities.

$$\begin{aligned} x_p &= \max\{x_1, x_2, \ldots, x_{k-1}, 0\} - x_k = x_1 - x_k \geq 0, \\ x_{p-1} &= \max\{x_p, x_1, \ldots, x_{k-2}, 0\} - x_{k-1} = x_1 - x_{k-1} \geq 0, \\ &\ldots \\ x_{p-k+2} &= \max\{x_{p-k+3}, x_{p-k+4}, \ldots, x_p, x_1, 0\} - x_2 = x_1 - x_2 \geq 0. \end{aligned}$$

Finally,

$$\begin{aligned} x_{k+1} &= \max\{x_k, x_{k-1}, \ldots, x_2, 0\} - x_1 \leq 0, \\ x_{p-k+1} &= \max\{x_{p-k+2}, x_{p-k+3}, \ldots, x_p, 0\} - x_1 \leq 0. \end{aligned}$$

□

Now, concerning the boundedness of the solutions of Equation (4.1), we find the following result in [29, Theorem 12]:

Proposition 18. *Every solution (x_n) of Equation (4.1) is bounded.*

Finally, observe that multiplying by a positive constant α the set of initial conditions that generate a periodic sequence does not interfere in the periodic character of the sequence. Indeed, the proof of this fact resides on taking out the factor α in the equation.

Proposition 19. *Let (x_n) be a periodic solution of Equation (4.1). Then, the sequence $(\alpha \cdot x_n)$ with $\alpha > 0$ is periodic with the same period.*

4.1.2 Properties for the fourth order equation

Now, we focus on the particular case of order $k = 4$ and we establish some properties concerning the max-type difference equation (4.2):

$$x_{n+4} = \max\{x_{n+3}, x_{n+2}, x_{n+1}, 0\} - x_n.$$

As a first step, we can derive the following result from Proposition 17.

Corollary 4. *Let (x_n) be a p-periodic sequence of (4.2). If there exists a $j \in \{1, \ldots, p\}$ such that $x_j = x_1$, then x_{j+1}, x_{j+2} and x_{j+3} are non-negative and x_{j+4} is non-positive.*

In addition, in [39] it was proved that monotonic initial conditions yield to periodic solutions. Indeed, for the general case, Equation (4.1), the author shows that the solutions generated from monotonic initial terms are periodic of period $3k - 1$. So, for $k = 4$, this implies the existence of 11-cycles. Moreover, it is direct to see that four consecutive monotonic terms in the orbit of a solution of Equation (4.2) imply that it is an 11-cycle too, since if they are not the initial values of the orbit, the sequence would be eventually periodic and we would get an 11-cycle from Proposition 13.

Proposition 20. *Let (x_n) be a solution of Equation (4.2). If there exist four consecutive monotonic terms, then the solution is an 11-cycle.*

As a next step, from Proposition 15, we know that $8 \in \text{Per}(F_4)$. In this sense, we present a characterization of 8-cycles.

Proposition 21. *Consider Equation (4.2). A solution (x_n) is periodic of period 8 if and only if*

$$(x_n) = (\ldots, x, 0, x, \alpha, 0, x, 0, x - \alpha, x, 0, x, \alpha, 0, x, 0, x - \alpha, \ldots), \tag{4.3}$$

with $x > 0$ and $\alpha \in [0, x]$.

Proof. We begin by proving the sufficiency. Set $x > 0$, $\alpha \in [0, x]$ and take initial conditions $(x_1, x_2, x_3, x_4) = (x, 0, x, \alpha)$. If we compute the subsequent terms of the sequence x_j, $j = 5, \ldots, 12$, by Equation (4.2), we obtain that (x_n) is periodic, being the period a divisor of 8. However, since $x \neq 0$, we cannot have period 1. Also, Proposition 15 excludes period 2; and if the period were 4, we would achieve a contradiction because we would have $x = x_1 = x_5 = 0$. In conclusion, the period of the sequence is 8.

Thereafter, we prove the necessity. Suppose that (x_n) is an 8-cycle generated by Equation (4.2). From Proposition 16, we know that $x_1 = \max\{x_n : n \geq 1\} > 0$. Moreover, from Proposition 17, we have $x_5 \leq 0$, while the remaining terms are non-negative. Furthermore,

$$x_8 = x_1 - x_4; \quad x_7 = x_1 - x_3; \quad x_6 = x_1 - x_2. \tag{4.4}$$

Since $x_8 = \max\{x_7, x_6, x_5, 0\} - x_4 = x_1 - x_4$, we deduce that $x_1 = \max\{x_7, x_6, x_5, 0\}$. Nevertheless, $x_5 \leq 0$ and $x_1 > 0$, so we only have two possibilities:

(i) Suppose $x_1 = x_6$. From (4.4) we get $x_2 = 0$. Moreover, $x_5 = x_1 - x_1 = 0$ and $0 = x_5 = \max\{x_4, x_3, x_2, 0\} - x_1 = \max\{x_4, x_3\} - x_1$, so $x_1 = \max\{x_4, x_3\}$.

• If $x_1 = x_3$, then $x_7 = 0$ due to (4.4). Also, we can choose x_4 arbitrarily, with $0 \leq x_4 \leq x_1$, and $x_8 = x_1 - x_4$. Setting $x_4 = \alpha \in [0, x_1]$, we achieve the sequence

$$(x_1, 0, x_1, \alpha, 0, x_1, 0, x_1 - \alpha, \ldots),$$

which is an 8-cycle.

• If $x_1 = x_4$, we get $x_8 = 0$ and taking an arbitrary $\beta \in [0, x_1]$, with $x_3 = \beta$, we have the 8-cycle

$$(x_1, 0, \beta, x_1, 0, x_1, x_1 - \beta, 0, \ldots).$$

(ii) Suppose $x_1 = x_7$. From (4.4) we have $x_3 = 0$. Analogously to the previous case, $x_5 = 0$ and $x_1 = \max\{x_2, x_4\}$.

• If $x_1 = x_2$, from (4.4) we get $x_6 = \max\{x_5, x_4, x_3, 0\} - x_2 = x_4 - x_1 = 0$, and we obtain $x_4 = x_1$, $x_8 = x_1 - x_4 = 0$, so we achieve the 8-cycle

$$(x_1, x_1, 0, x_1, 0, 0, x_1, 0, \ldots).$$

• If $x_1 = x_4$, from (4.4), $x_8 = 0$ and $x_5 = 0$. We set $x_2 = \gamma \in [0, x_1]$. Observe that $x_6 = x_1 - \gamma$ and we have the 8-cycle

$$(x_1, \gamma, 0, x_1, 0, x_1 - \gamma, x_1, 0, \ldots).$$

□

It merits the attention to highlight that the above proposition gives us the unique periodic solutions whose terms are all non-negative.

Proposition 22. *Let (x_n) be a periodic sequence of Equation (4.2), with $x_n \geq 0$ for all $n \geq 0$. Then, either the sequence is the equilibrium point or it is an 8-cycle given by (4.3).*

Proof. Firstly, notice that Proposition 17 implies that $x_j \geq 0$ for $j = 1, 2, 3, 4$, and $j = p-2, p-1, p$. Also, $x_5 \leq 0$, but since we are assuming that every term of the sequence is non-negative, we deduce that $x_5 = 0$. Now, since $x_5 = \max\{x_4, x_3, x_2, 0\} - x_1$, we have $x_1 = \max\{x_2, x_3, x_4\}$. In this sense, we analyze the different possibilities:

(i) Suppose that $x_1 = x_2$. Here, $x_6 = \max\{x_5, x_4, x_3, 0\} - x_2 = \max\{x_4, x_3\} - x_1$. However, $x_1 = \max\{x_j : j = 1, \ldots, p\} \geq 0$, so $x_6 \leq 0$ and again, $x_6 = 0$ from our hypothesis, thus we get $x_1 = x_3$ or $x_1 = x_4$.

• If $x_1 = x_3$, then $x_4 = x_1$, otherwise, $x_7 = \max\{x_6, x_5, x_4, 0\} - x_3 = x_4 - x_1 < 0$. Hence, setting $x_1 = x$, the initial conditions are (x, x, x, x), with $x \geq 0$ and they generate the sequence $(x, x, x, x, 0, 0, 0, -x, 0)$. This yields to $x = 0$ and we have the equilibrium point.

- If $x_1 = x_4$, take $x_1 = x = x_2 = x_4$ and $x_3 = y$. From here, we get the sequence $(x, x, y, x, 0, 0, x, -y, -y, \ldots)$ and we derive $y = 0$, obtaining an 8-cycle.

(ii) Suppose that $x_1 = x_3$. Hence, $x_6 = \max\{x_5, x_4, x_3, 0\} - x_2 = x_3 - x_2 = x_1 - x_2$, and $x_7 = \max\{x_6, x_5, x_4, 0\} - x_3 = \max\{x_1 - x_2, x_4\} - x_1 \leq 0$, thus $x_7 = 0$ and $\max\{x_1 - x_2, x_4\} = x_1$.

- If $x_1 - x_2 = x_1$, then $x_2 = 0$ and we achieve the 8-cycle generated by $(x, 0, x, y)$, with $0 \leq y \leq x$.

- If $x_1 = x_4$, set the initial conditions $x_1 = x_3 = x_4 = x$ and $x_2 = y$, with $y \leq x$. From here we generate the sequence $(x, y, x, x, 0, x - y, 0, -y, \ldots)$, so $y = 0$ and we get an 8-cycle.

(iii) Suppose that $x_1 = x_4$. Now, $x_6 = x_1 - x_2$ and $x_7 = x_1 - x_3$. This implies that $x_8 = \max\{x_1 - x_3, x_1 - x_2\} - x_1 \leq 0$, so $x_8 = 0$ and $x_1 = \max\{x_1 - x_3, x_1 - x_2\}$.

- If $x_1 = x_1 - x_2$, then $x_2 = 0$ and the initial conditions $(x_1, 0, x_3, x_1)$ generate an 8-cycle.

- If $x_1 = x_1 - x_3$, then $x_3 = 0$ and $(x_1, x_2, 0, x_1)$ provide another 8-cycle.

□

The following result gathers the previous study by establishing the first periods in $\text{Per}(F_4)$.

Proposition 23. *It holds* $\text{Per}(F_4) \cap [1, 11] = \{1, 8, 11\}$.

Proof. We already know that $x^* = 0$ is an equilibrium point of Equation (4.2); the initial conditions $(x, 0, x, \alpha)$, with $x > 0$ and $\alpha \in [0, x]$, generate an 8-cycle; and monotonic initial conditions, with $x_1 \neq 0$, provide an 11-cycle. Therefore, we have $\{1, 8, 11\} \in \text{Per}(F_4)$.

We see that, indeed, they are the only periods in $[1, 11]$. To do so, suppose that (x_n) is a periodic sequence with period $p \leq 10$. Recall that $x_1 = \max\{x_j : j \geq 1\} > 0$. We distinguish several cases:

- Suppose that $p \leq 7$. Proposition 17 implies that every term of the sequence is non-negative, so, by Proposition 22, either $p = 1$ or $p = 8$. Thus, $p = 1$ and we get the equilibrium.

- Suppose that $p = 9$. From Proposition 17, the fifth and sixth term of the sequence are non-positive, while the remaining terms are non-negative. Furthermore, applying

Equation (4.2),

$$\begin{aligned} x_8 &= \max\{x_7, x_6, x_5, 0\} - x_4 = x_7 - x_4, \\ x_7 &= \max\{x_6, x_5, x_4, 0\} - x_3 = x_4 - x_3. \end{aligned}$$

From here, we have $x_8 = x_7 - x_4 = (x_4 - x_3) - x_4 = -x_3$, so $x_3 + x_8 = 0$, with $x_3, x_8 \geq 0$, and, therefore, $x_3 = x_8 = 0$. On the other hand, if we consider the inverse of the difference equation, $x_8 = \max\{x_9, x_1, x_2, 0\} - x_3 = x_1 - x_3$. However, this would yield to $x_1 = 0$. Then, we obtain the equilibrium, $p = 1$, and we achieve a contradiction.

- Suppose that $p = 10$. From Proposition 17, x_5 and x_7 are non-positive, and the other terms of the sequence are non-negative. Now, $x_7 = \max\{x_6, x_5, x_4, 0\} - x_3 = \max\{x_6, x_4\} - x_3 \leq 0$ and we have $\max\{x_6, x_4\} \leq x_3$, in particular, $x_4 \leq x_3$. By periodicity acting backwards, $x_4 = \max\{x_3, x_2, x_1, 0\} - x_{10} = x_1 - x_{10}$, thus, $x_{10} = x_1 - x_4$. Analogously, $x_9 = x_1 - x_3$ and $x_8 = x_1 - x_2$. Since $x_5 \leq 0$ and $x_5 = \max\{x_6, x_7, x_8, 0\} - x_9 = \max\{x_6, x_8\} - x_9$, we have $x_8 \leq x_9$, or equivalently, $x_3 \leq x_2$. In conclusion, $x_4 \leq x_3 \leq x_2 \leq x_1$ and we get monotonic initial conditions that will provide an 11-cycle by Proposition 20, a contradiction.

□

As a final step, we define an equivalence relation in $\mathbb{R}^4$, which will be very useful in the sequel. Observe that, since the associate dynamical system F_4 of (4.2) is a bijection, for given initial conditions, we can build a unique sequence $(x_n)_{n\in\mathbb{Z}}$.

Definition 4. *Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^4$. We will say that $\boldsymbol{x} = (x_1, x_2, x_3, x_4) \sim \boldsymbol{y} = (y_1, y_2, y_3, y_4)$ if and only if $\boldsymbol{x}$ and $\boldsymbol{y}$ generate under Equation (4.2) and its inverse the same sequences $(x_n)_{n\in\mathbb{Z}}$ and $(y_n)_{n\in\mathbb{Z}}$ up to a shift. In particular, $\boldsymbol{x} \sim \boldsymbol{y}$ if $\boldsymbol{x}$ and $\boldsymbol{y}$ generate the same periodic sequence under Equation (4.2).*

It must be highlighted that $\sim$ is an equivalence relation. For instance, $(x, y, z, y) \sim (x, z, z, y)$ with $x > y > z > 0$, since under Equation (4.2) the tuple evolves as follows:

$$\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{y}, y - x, 0, y - z, -z, x - z, x - z, x - y, \boldsymbol{x}, \boldsymbol{z}, \boldsymbol{z}, \boldsymbol{y}.$$

4.1.3 Characterization of the set of periods. Proof of Theorem D

This section is devoted to characterize the set of periods of Equation (4.2), $\mathrm{Per}(F_4)$. To achieve it, we determine the possible configurations of the initial conditions that generate a periodic sequence under the equation and, furthermore, we establish its associate periods. We develop the study by parts:

Part I: We analyze the movement of a set of initial conditions that generate a periodic sequence. Concretely, we show that there exists five Cases C_i, $i = 1, \dots, 5$, and that the orbit of a solution visits these cases in a concrete way (see Figure 4.1). In addition, the orbit's movement through the different cases allows us to determine the period of the solution (by blocks of ten or eleven elements) in most situations.

Part II: We study the scenario where periodicity is achieved in the middle of the process described in Figure 4.1 and not in a Case C_i. Here, where periodicity holds between cases, we precise the tuples of initial conditions that yield to this situation. These tuples are called *controversial cases* and the equivalence relation given by Definition 4 enables us to reduce this casuistic into two classes of equivalence, namely, $(x, y, 0, z)$ and $(x, z, y, 0)$, where $x \geq y \geq z \geq 0$ and $x > z$.

Part III: This part delves into the analysis of the two classes of equivalence mentioned above. Specifically, we give the period associated to their periodic orbits.

Part IV: We study the set of initial conditions that verify the restrictions of more than once Case C_i. For instance, (x, z, z, y) with $x > y > z \geq 0$ satisfies Cases C_4 and C_5. This scenario, which we refer as *intersection between cases*, is reduced to trivial cases (periods 1, 8 or 11) or to the controversial cases studied in the preceding parts.

Part V: Once that the problematic possibilities have been studied in the previous steps, we focus on those initial conditions whose orbits are described unambiguously by Figure 4.1. This property is called *Condition U*.

Part VI: We collect the diverse possibilities analyzed in the other parts in the form of a Main Theorem concerning the set of periods of Equation (4.2), namely, Theorem D.

Part I - The routes of periodic solutions and their periods:

In this first part, we assume that the solution (x_n) is periodic of period p, with $p \geq 12$. Our target is to describe the solution of a tuple of initial conditions (x_1, x_2, x_3, x_4) under Equation (4.2). Recall that, since the solution is periodic, we can assume without loss of generality, that $x_1 = \max\{x_n : n \geq 1\}$. Under this assumption, we begin showing that, after ten or eleven iterations, we obtain indefinitely the term x_1.

Proposition 24. *Let (x_n) be a periodic sequence of period p, with $x_1 = \max\{x_j : 1 \leq j \leq p\}$. Then either $x_{11} = x_1$ or $x_{12} = x_1$. In both cases, x_j, x_{j+1}, x_{j+2}, x_{j+3} are non-negative for $j = 11$ in the first case and for $j = 12$ in the second case.*

Proof. As a first step, recall that x_2, x_3, x_4 are non-negative by Proposition 17. Furthermore, if we have monotonic initial conditions, the solution (x_n) will be the equilibrium point if $x_1 = 0$; or an 11-cycle (recall Proposition 20). Due to this, in the sequel we assume that $x_1 > 0$ and distinguish several cases:

(i) Suppose $x_1 \geq x_2 \geq x_4 \geq x_3 \geq 0$. Then, by the iteration of Equation (4.2), we get

$$\begin{aligned}
x_5 &= x_2 - x_1 \leq 0, \\
x_6 &= \max\{x_5, x_4, x_3, 0\} - x_2 = x_4 - x_2 \leq 0, \\
x_7 &= \max\{x_6, x_5, x_4, 0\} - x_3 = x_4 - x_3, \\
x_8 &= \max\{x_7, x_6, x_5, 0\} - x_4 = x_7 - x_4 = -x_3 \leq 0, \\
x_9 &= \max\{x_8, x_7, x_6, 0\} - x_5 = x_7 - x_5 = x_4 - x_3 - x_2 + x_1, \\
x_{10} &= \max\{x_9, x_8, x_7, 0\} - x_6 = x_9 - x_6 = x_1 - x_3, \\
x_{11} &= \max\{x_{10}, x_9, x_8, 0\} - x_7 = x_{10} - x_7 = x_1 - x_4, \\
x_{12} &= \max\{x_{11}, x_{10}, x_9, 0\} - x_8 = x_1 - x_3 + x_3 = x_1.
\end{aligned}$$

Moreover, using $x_1 \geq x_2 + x_3 - x_4 \geq 0$, we also find that the following three terms are non-negative:

$$x_{13} = x_2 + x_3 - x_4; \quad x_{14} = x_3; \quad x_{15} = x_4.$$

(ii) Assume that $x_1 \geq x_3 \geq x_4 \geq x_2 \geq 0$. Here, the subsequent terms are $x_5 = x_3 - x_1 \leq 0$, $x_6 = \max\{x_5, x_4, x_3, 0\} - x_2 = x_3 - x_2$, and

$$x_7 = \max\{x_6, x_5, x_4, 0\} - x_3 = \max\{x_3 - x_2, x_4, 0\} - x_3.$$

– If additionally $x_3 \geq x_2 + x_4$, then we continue

$$\begin{aligned}
x_7 &= \max\{x_6, x_5, x_4, 0\} - x_3 = x_3 - x_2 - x_3 = -x_2 \leq 0, \\
x_8 &= \max\{x_7, x_6, x_5, 0\} - x_4 = x_6 - x_4 = x_3 - x_2 - x_4 \geq 0, \\
x_9 &= \max\{x_8, x_7, x_6, 0\} - x_5 = x_3 - x_2 - x_3 + x_1 = x_1 - x_2, \\
x_{10} &= \max\{x_9, x_8, x_7, 0\} - x_6 = x_1 - x_2 - x_3 + x_2 = x_1 - x_3, \\
x_{11} &= \max\{x_{10}, x_9, x_8, 0\} - x_7 = x_9 - x_7 = x_1 - x_2 + x_2 = x_1;
\end{aligned}$$

even more, the next three terms are also non-negative (recall that $x_3 \geq x_2 + x_4$):

$$\begin{aligned}
x_{12} &= \max\{x_{11}, x_{10}, x_9, 0\} - x_8 = x_1 - x_8 = (x_1 - x_3) + (x_2 + x_4) \geq 0, \\
x_{13} &= \max\{x_{12}, x_{11}, x_{10}, 0\} - x_9 = x_1 - x_1 + x_2 = x_2, \\
x_{14} &= \max\{x_{13}, x_{12}, x_{11}, 0\} - x_{10} = x_1 - x_1 + x_3 = x_3.
\end{aligned}$$

– Otherwise, if $x_3 \leq x_2 + x_4$, we get

$$\begin{aligned}
x_7 &= \max\{x_6, x_5, x_4, 0\} - x_3 = x_4 - x_3 \leq 0, \\
x_8 &= \max\{x_7, x_6, x_5, 0\} - x_4 = x_6 - x_4 = x_3 - x_2 - x_4 \leq 0, \\
x_9 &= \max\{x_8, x_7, x_6, 0\} - x_5 = x_6 - x_5 = x_3 - x_2 - x_3 + x_1 = x_1 - x_2, \\
x_{10} &= \max\{x_9, x_8, x_7, 0\} - x_6 = x_9 - x_6 = x_1 - x_2 - x_3 + x_2 = x_1 - x_3, \\
x_{11} &= \max\{x_{10}, x_9, x_8, 0\} - x_7 = (x_1 - x_4) + (x_3 - x_2) \geq 0, \\
x_{12} &= \max\{x_{11}, x_{10}, x_9, 0\} - x_8 = x_{11} - x_8 = x_1 \geq 0, \\
x_{13} &= \max\{x_{12}, x_{11}, x_{10}, 0\} - x_9 = x_1 - x_9 = x_1 - (x_1 - x_2) = x_2, \\
x_{14} &= \max\{x_{13}, x_{12}, x_{11}, 0\} - x_{10} = x_1 - x_1 + x_3 = x_3, \\
x_{15} &= \max\{x_{14}, x_{13}, x_{12}, 0\} - x_{11} = x_1 - x_{11} = x_4 + x_2 - x_3 \geq 0;
\end{aligned}$$

and x_{13}, x_{14}, x_{15} are non-negative.

(iii) Let $x_1 \geq x_3 \geq x_2 \geq x_4 \geq 0$. By the iteration of (4.2),

$$x_5 = x_3 - x_1 \leq 0, \qquad x_6 = x_3 - x_2, \qquad x_7 = \max\{x_3 - x_2, x_4\} - x_3.$$

Again, in this situation we must distinguish two scenarios, the procedure is similar, so we limit to indicate the results.

– If $x_3 \geq x_2 + x_4$, then we get $x_7 = -x_2 \leq 0$; $x_8 = x_3 - x_2 - x_4$; $x_9 = x_1 - x_2$; $x_{10} = x_1 - x_3$; $x_{11} = x_1$, and the subsequent three terms, x_{12}, x_{13}, x_{14}, are non-negative: $x_{12} = (x_1 - x_3) + (x_2 + x_4)$; $x_{13} = x_2$; $x_{14} = x_3$.

– If $x_3 \leq x_2 + x_4$, and hence $0 \leq x_1 - x_2 - x_4 + x_3 \leq x_1$, we have $x_7 = x_4 - x_3 \leq 0$; $x_8 = x_3 - x_2 - x_4 \leq 0$; $x_9 = x_1 - x_2$; $x_{10} = x_1 - x_3$; $x_{11} = x_1 - x_2 - x_4 + x_3$; $x_{12} = x_1$; and $x_{13} = x_2$; $x_{14} = x_3$; $x_{15} = x_2 + x_4 - x_3$.

(iv) Let $x_1 \geq x_4 \geq x_2 \geq x_3 \geq 0$. For the computation of the following terms, we apply that $x_1 - x_3 \geq x_1 + x_2 - x_3 - x_4 \geq 0$, $x_2 - x_3 - x_4 \leq 0$ and $x_1 + x_2 \geq x_3 + x_4$. Then, it is direct to check that $x_5 = x_4 - x_1 \leq 0$; $x_6 = x_4 - x_2$; $x_7 = x_4 - x_3$; $x_8 = -x_3 \leq 0$; $x_9 = x_1 - x_3$; $x_{10} = x_1 + x_2 - x_3 - x_4$; $x_{11} = x_1 - x_4$; $x_{12} = x_1$ and $x_{13} = x_3$; $x_{14} = x_4 + x_3 - x_2$; $x_{15} = x_4$.

(v) Finally, suppose that $x_1 \geq x_4 \geq x_3 \geq x_2 \geq 0$. It is immediate to see that $x_5 = x_4 - x_1 \leq 0$; $x_6 = x_4 - x_2$; $x_7 = x_4 - x_3$; $x_8 = -x_2 \leq 0$; $x_9 = x_1 - x_2$; $x_{10} = x_1 - x_4$; $x_{11} = x_1 - x_2 - x_4 + x_3$; $x_{12} = x_1$ and $x_{13} = x_2$; $x_{14} = x_4$; $x_{15} = x_2 + x_4 - x_3$.

□

It is worth mentioning that if the orbit (x_n) were an 11-cycle, then we would have four consecutive monotonic terms as can be easily deduced from the inspection of the proof of the above result. For instance, in case (v), if the initial conditions generate an 11-cycle, then $x_{14} = x_4$ and $x_{15} = x_2 + x_4 - x_3$ imply $x_2 = x_3 = x_4$, so (x_1, x_2, x_2, x_2) are four consecutive monotonic terms. Therefore, we can deduce the following result.

Corollary 5. *Let (x_n) be a solution of Equation (4.2). Then the solution is an 11-cycle if and only if there exist four consecutive monotonic terms.*

Also, observe that in cases (ii) and (iii) of the proof of Proposition 24, the inequality $x_2 + x_4 \leq x_3$, or the reverse one, is the condition that establishes if $x_{11} = x_1$ or $x_{12} = x_1$, respectively. Bearing this in mind, we can sum up the analyzed cases for initial conditions in the following five:

Case 1 (C_1): $x_1 \geq x_2 \geq x_4 \geq x_3$.

Case 2 (C_2): $x_1 \geq x_3 \geq \max\{x_2, x_4\}$ with $x_3 \geq x_2 + x_4$.

Case 3 (C_3): $x_1 \geq x_3 \geq \max\{x_2, x_4\}$ with $x_3 \leq x_2 + x_4$.

Case 4 (C_4): $x_1 \geq x_4 \geq x_2 \geq x_3$.

Case 5 (C_5): $x_1 \geq x_4 \geq x_3 \geq x_2$.

In addition, from the proof of Proposition 24, we can elaborate a diagram, Figure 4.1, that shows the diverse connections that exist between the Cases C_i, $i = 1, \ldots, 5$. Specifically, an arrow from Case C_i to Case C_k expresses that if we begin with initial conditions (x_1, x_2, x_3, x_4) verifying the inequalities of Case C_i, then after $j = 10$ or $j = 11$ iterations, the new tuple of non-negative terms $(x_{j+1}, x_{j+2}, x_{j+3}, x_{j+4})$ satisfies the conditions of Case C_k. Furthermore, over each arrow we indicate the conditions that force the movement of the orbit to one case or another.

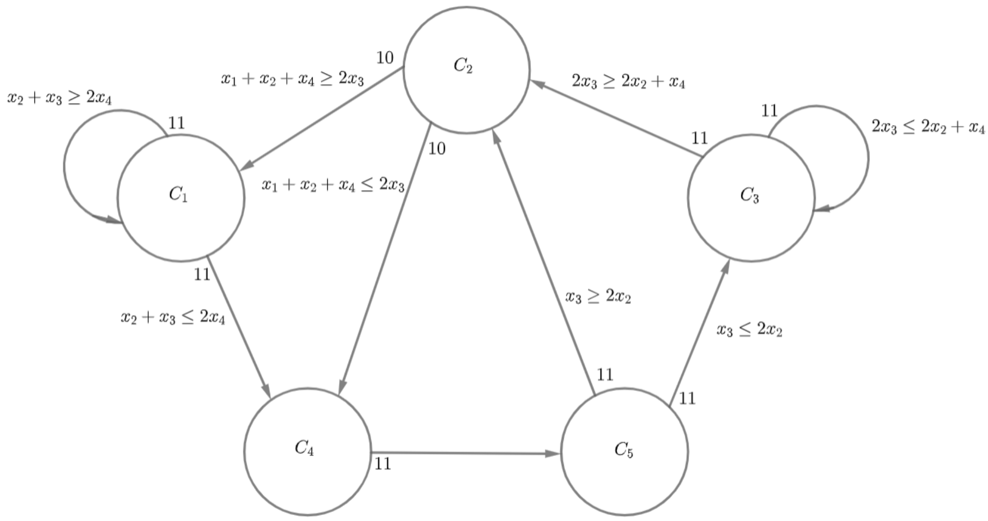


Figure 4.1: The movement of the Cases C_i while iterating Equation (4.2).

Moreover, apart from the diagram developed in Figure 4.1, we gather the information related to the evolution of a tuple (x_1, x_2, x_3, x_4) under Equation (4.2) for each Case C_i in Table 4.1. Again, such information comes from the inspection of the proof of Proposition 24.

$C_1:$	$x_1 \geq x_2 \geq x_4 \geq x_3 \geq 0$	$\stackrel{11}{\longrightarrow}$	$(x_{12} = x_1, x_2 + x_3 - x_4, x_3, x_4)$
$C_2:$	$x_1 \geq x_3 \geq \max\{x_2, x_4\} \geq 0$ $x_3 \geq x_2 + x_4$	$\stackrel{10}{\longrightarrow}$	$(x_{11} = x_1, x_1 - x_3 + x_2 + x_4, x_2, x_3)$
$C_3:$	$x_1 \geq x_3 \geq \max\{x_2, x_4\} \geq 0$ $x_3 \leq x_2 + x_4$	$\stackrel{11}{\longrightarrow}$	$(x_{12} = x_1, x_2, x_3, x_2 + x_4 - x_3)$
$C_4:$	$x_1 \geq x_4 \geq x_2 \geq x_3 \geq 0$	$\stackrel{11}{\longrightarrow}$	$(x_{12} = x_1, x_3, x_4 + x_3 - x_2, x_4)$
$C_5:$	$x_1 \geq x_4 \geq x_3 \geq x_2 \geq 0$	$\stackrel{11}{\longrightarrow}$	$(x_{12} = x_1, x_2, x_4, x_2 + x_4 - x_3)$

Table 4.1: Evolution of a tuple (x_1, x_2, x_3, x_4) in the different cases.

After presenting Table 4.1, we are going to justify the diagram of Figure 4.1. Firstly, assume that we are in C_1. Observe that $x_{12} \geq x_{13}$, since $x_1 + x_4 \geq x_2 + x_3$. Thus,

- If $x_2 + x_3 \geq 2x_4$, then $x_{12} \geq x_{13} \geq x_{15} \geq x_{14}$ and we are in C_1.
- If $x_2 + x_3 \leq 2x_4$, then $x_{12} \geq x_{15} \geq x_{13} \geq x_{14}$, since $x_3 - x_4 \geq 0$. Here, we are in C_4.

Secondly, let the tuple verify the conditions of C_4. Now, $x_4 - x_2 \geq 0$, so $x_{14} \geq x_{13}$. In addition, $x_3 - x_2 \leq 0$, so $x_{15} \geq x_{14}$. In conclusion, $x_{12} \geq x_{15} \geq x_{14} \geq x_{13}$ and we are in C_5.

Now, suppose that we are in C_5. Hence, $x_{12} \geq x_{14} \geq x_{13}$, and since $x_2 - x_3 \leq 0$, we get $x_{14} \geq x_{15}$. Therefore,

- If $x_3 \geq 2x_2$, then $x_4 \geq 2x_2 + x_4 - x_3$, which yields to $x_{14} \geq x_{13} + x_{15}$ and we are in C_2.
- If $x_3 \leq 2x_2$, then $x_4 \leq 2x_2 + x_4 - x_3$. Hence, $x_{14} \leq x_{13} + x_{15}$ and we are in C_3.

As a next step, assume that the tuple is in C_3. In this case, $x_{12} \geq x_{14} \geq \max\{x_{13}, x_{15}\}$. Therefore,

- If $2x_3 \geq 2x_2 + x_4$, then $x_3 \geq 2x_2 + x_4 - x_3$. So $x_{14} \geq x_{13} + x_{15}$ and we are in C_2.
- If $2x_3 \leq 2x_2 + x_4$, then $x_3 \leq 2x_2 + x_4 - x_3$. Therefore $x_{14} \leq x_{13} + x_{15}$ and we are in C_3 again.

Finally, let the tuple be in C_2. After the corresponding iterations, we have $x_{11} \geq x_{14} \geq x_{13}$. Therefore,

- If $x_1 + x_2 + x_4 \geq 2x_3$, then $x_1 - x_3 + x_2 + x_4 \geq x_3$, which implies $x_{12} \geq x_{14}$. So, $x_{11} \geq x_{12} \geq x_{14} \geq x_{13}$ and we are in C_1.

- If $x_1 + x_2 + x_4 \leq 2x_3$, then $x_1 - x_3 + x_2 + x_4 \leq x_3$, thus $x_{12} \leq x_{14}$. So, $x_{11} \geq x_{14} \geq x_{12} \geq x_{13}$ and we are in C_4.

After providing a justification for the diagram, it becomes crucial to ensure that any set of initial conditions (x_1, x_2, x_3, x_4) belongs only to one of the five classes C_i. Furthermore, there exist two options for the way in which periodicity can be achieved: either doing a cycle along the diagram starting and ending in the same case, or getting again the initial conditions while passing from one case to another one in Diagram 4.1. This last scenario is called *controversial cases* and it is studied in the next part.

Part II - Controversial cases:

In this part, we analyze what we call *controversial cases*. In general, a tuple of initial conditions that generate a cycle will start in some Case C_i and, after travelling through Diagram 4.1, we will achieve periodicity in the same case. However, it is possible that under certain conditions over the initial terms, periodicity will hold in the middle of the process defined by the diagram. In this sense, we employ the expression *controversial case* to represent those tuples of initial conditions that will give rise to a cycle in the middle of the process. Bearing this in mind, in the sequel, to analyze these contentious cases, we assume that the initial conditions of the periodic sequence (x_n) are (x_1, x_2, x_3, x_4), with $x_1 = \max\{x_n : n \geq 1\}$. Travelling by Diagram 4.1, after ten or eleven iterations, these initial conditions become into (z_1, z_2, z_3, z_4) belonging to some Case C_i. We will suppose that, at some point of the movement in Diagram 4.1, the new conditions (z_1, z_2, z_3, z_4) in Case C_i go to the tuple (w_1, w_2, w_3, w_4) living in Case C_k, and in the transition from C_i to C_k, we get the initial conditions (x_1, x_2, x_3, x_4). Observe that, by Proposition 24, $z_1 = x_1$.

In this sense, we write $C_{i,j}$ to denote that a tuple (z_1, z_2, z_3, z_4) satisfying the Case C_i achieves periodicity in the term z_j, $j = 2, \ldots, 11$ (if $i = 1, 3, 4, 5$), or $j = 2, \ldots, 10$ (if $i = 2$), that is, $z_j = x_1$, $z_{j+1} = x_2$, $z_{j+2} = x_3$ and $z_{j+3} = x_4$.

We proceed to analyze each case. Since the procedure is the same for every $C_{i,j}$, we only develop the reasoning for C_1. For the remaining cases we limit to specify the results obtained. In the sequel, x, y, z are real numbers such that $x \geq y \geq z \geq 0$.

- Case C_1: If the tuple (z_1, z_2, z_3, z_4) verifies the restrictions of C_1, by Proposition 24, we know how it evolves under Equation (4.2):

$$\begin{aligned}
z_5 &= z_2 - x_1 \leq 0, \quad z_6 = z_4 - z_2 \leq 0, \quad z_7 = z_4 - z_3, \\
z_8 &= -z_3 \leq 0, \quad z_9 = z_4 - z_3 - z_2 + x_1, \quad z_{10} = x_1 - z_3, \quad z_{11} = x_1 - z_4, \\
z_{12} &= x_1, \quad z_{13} = z_2 + z_3 - z_4, \quad z_{14} = z_3, \quad z_{15} = z_4.
\end{aligned}$$

Taking into account that the first term, $z_1 = x_1$, is positive, we could reach periodicity in $z_2, z_3, z_4, z_7, z_9, z_{10}$ or z_{11}. Let analyze the diverse scenarios.

$C_{1,2}$: $z_2 = x_1$; $z_3 = x_2$; $z_4 = x_3$; $z_2 - x_1 = x_4$, which yield to $(z_1, z_2, z_3, z_4) = (x_1, x_1, x_2, x_3)$. This is a controversial case, (x, x, z, y).

$C_{1,3}$: $z_3 = x_1$; $z_4 = x_2$; $z_5 = z_2 - x_1 = x_3 \leq 0$; $z_6 = z_4 - z_2 = x_4 \leq 0$. Observe that the two last inequalities imply $x_3 = 0$ and $x_4 = 0$. Then the initial conditions of the sequence are $(x_1, x_2, 0, 0)$, which generate an 11-cycle due to the monotonicity of the initial terms.

$C_{1,4}$: $z_4 = x_1$; $z_2 - x_1 = x_2 \leq 0$; $z_4 - z_2 = x_3 \leq 0$; $z_4 - z_3 = x_4$. By the second and third expressions, we have $x_2 = x_3 = 0$, so the initial conditions reduce to $(x_1, 0, 0, x_4)$ and we obtain a controversial case, $(x, 0, 0, y)$.

$C_{1,7}$: $z_7 = z_4 - z_3 = x_1$; $-z_3 = x_2 \leq 0$; $z_4 - z_3 - z_2 + x_1 = x_3$; $x_1 - z_3 = x_4$. It is easy to solve the system in order to obtain the initial conditions $(x_1, 0, x_3, x_1)$, which is an 8-cycle by Proposition 21.

$C_{1,9}$: $z_9 = z_4 - z_3 - z_2 + x_1 = x_1$; $x_1 - z_3 = x_2$; $x_1 - z_4 = x_3$; $x_1 = x_4$. From here, we get $(z_1, z_2, z_3, z_4) = (x_1, x_2 - x_3, x_1 - x_2, x_1 - x_3)$. Furthermore, since we are in C_1, the inequalities $z_1 \geq z_2 \geq z_4 \geq z_3$ hold and we get $x_2 = x_1$. This allows us to reduce the expression of the tuple into $(x_1, x_1 - x_3, 0, x_1 - x_3)$. In this way, we have achieved another controversial case, namely, $(x, y, 0, y)$.

$C_{1,10}$: $z_{10} = x_1 - z_3 = x_1$; $x_1 - z_4 = x_2$; $x_1 = x_3$; $z_2 + z_3 - z_4 = x_4$. Those equations imply that $(z_1, z_2, z_3, z_4) = (x_1, x_1 + x_4 - x_2, 0, x_1 - x_2)$, which is a controversial case of the type $(x, y, 0, z)$.

$C_{1,11}$: $z_{11} = x_1 - z_4 = x_1$; $x_1 = x_2$; $z_2 + z_3 - z_4 = x_3$; $z_3 = x_4$. By these equations, we deduce $z_4 = 0$ and $z_2 = x_3 - x_4$. Hence, the tuple is of the form $(x_1, x_3 - x_4, x_4, 0)$, but as they verify the Case C_1 we have $x_4 = 0$. So, the terms of the given tuple are monotonic and they generate an 11-cycle by Proposition 20.

• Case C_2 : We apply Proposition 24 in order to compute the subsequent terms of the tuple (x_1, z_2, z_3, z_4):

$$\begin{aligned} z_5 &= z_3 - x_1 \leq 0, \quad z_6 = z_3 - z_2, \quad z_7 = -z_2 \leq 0, \\ z_8 &= z_3 - z_2 - z_4, \quad z_9 = x_1 - z_2, \quad z_{10} = x_1 - z_3, \\ z_{11} &= x_1, \quad z_{12} = x_1 - z_3 + z_2 + z_4, \quad z_{13} = z_2, \quad z_{14} = z_3. \end{aligned}$$

In this case, the positive terms are z_2, z_3, z_4, z_6, z_8, z_9, z_{10}, so we could have the cycle in any of them. In the sequel, we indicate what happens in each scenario.

$C_{2,2}$: We obtain that the initial conditions are monotonic and, therefore, we have an 11-cycle.

$C_{2,3}$: We achieve a controversial case, that is, $(x, 0, z, y)$.

$C_{2,4}$: The tuple reduces to $(x_1, 0, x_1, 0)$, which is an 8-cycle.

$C_{2,6}$: We arrive to an 8-cycle generated from $(x_1, 0, x_3, x_1)$.

$C_{2,8}$: The initial conditions are $(x_1, x_1, 0, x_1)$ and we get an 8-cycle.

$C_{2,9}$: We obtain the controversial case $(x, 0, y, z)$.

$C_{2,10}$: We have an 11-cycle due to the monotonicity of the initial conditions.

• Case C_3: Let us suppose that the tuple (x_1, z_2, z_3, z_4) is in C_3. By Proposition 24, the sequence evolves as:

$$
\begin{aligned}
z_5 &= z_3 - x_1 \leq 0, \quad z_6 = z_3 - z_2, \quad z_7 = z_4 - z_3 \leq 0, \\
z_8 &= z_3 - z_2 - z_4 \leq 0, \quad z_9 = x_1 - z_2, \quad z_{10} = x_1 - z_3, \\
z_{11} &= x_1 - z_4 + z_3 - z_2, \\
z_{12} &= x_1, \quad z_{13} = z_2, \quad z_{14} = z_3, \quad z_{15} = z_4 + z_2 - z_3.
\end{aligned}
$$

As usual, we only focus on the positive terms, since they are the only ones where the cycle can occur, that is, $z_2, z_3, z_4, z_6, z_9, z_{10}, z_{11}$. We gather what happens for each positive term.

$C_{3,2}$: The initial conditions are monotonic, so we have an 11-cycle.

$C_{3,3}$: The initial conditions are $(x, y, 0, z)$, a controversial case .

$C_{3,4}$: We arise to the controversial case $(x, 0, y, 0)$.

$C_{3,6}$: The tuple reduces to $(x_1, 0, 0, x_1)$, which is an 8-cycle.

$C_{3,9}$: We obtain the controversial case $(x, 0, z, y)$.

$C_{3,10}$: The terms of (x_1, z_2, z_3, z_4) are monotonic, so we get an 11-cycle.

$C_{3,11}$: The controversial case (x, x, z, y) takes place.

• Case C_4: If the tuple (x_1, z_2, z_3, z_4) satisfies the conditions from C_4, then, by Proposition 24, the following terms of the sequence are:

$$
\begin{aligned}
z_5 &= z_4 - x_1 \leq 0, \quad z_6 = z_4 - z_2, \quad z_7 = z_4 - z_3, \quad z_8 = -z_3 \leq 0, \\
z_9 &= x_1 - z_3, \quad z_{10} = x_1 + z_2 - z_3 - z_4, \quad z_{11} = x_1 - z_4, \\
z_{12} &= x_1, \quad z_{13} = z_3, \quad z_{14} = z_4 + z_3 - z_2, \quad z_{15} = z_4.
\end{aligned}
$$

As a next step, we take the positive terms, z_2, z_3, z_4, z_6, z_7, z_9, z_{10}, z_{11}, and analyze when they are equal to x_1 generating a periodic orbit.

$C_{4,2}$: The tuple reduces to (x, x, y, x), which is a controversial case.

$C_{4,3}$: We obtain the equality $(z_1, z_2, z_3, z_4) = (x_1, x_1 - x_4, 0, x_1)$, that is an 8-cycle.

$C_{4,4}$: We obtain the controversial case $(x, 0, z, y)$.

$C_{4,6}$: The tuple $(z_1, z_2, z_3, z_4) = (x_1, 0, 0, x_1)$, which is an 8-cycle.

$C_{4,7}$: We obtain an 8-cycle because $(z_1, z_2, z_3, z_4) = (x_1, x_4, 0, x_1)$.

$C_{4,9}$: We get a controversial case of the type $(x, z, 0, y)$.

$C_{4,10}$: The corresponding equalities give rise to a controversial case, $(x, y, 0, y)$.

$C_{4,11}$: The terms z_1, z_2, z_3 and z_4 are monotonic, so we get an 11-cycle.

- Case C_5: By Proposition 24, we know that the given tuple evolves as follows:

$$\begin{aligned} z_5 &= z_4 - x_1 \leq 0, \quad z_6 = z_4 - z_2, \quad z_7 = z_4 - z_3, \quad z_8 = -z_2 \leq 0, \\ z_9 &= x_1 - z_2, \quad z_{10} = x_1 - z_4, \quad z_{11} = x_1 - z_2 - z_4 + z_3, \\ z_{12} &= x_1, \quad z_{13} = z_2, \quad z_{14} = z_4, \quad z_{15} = z_2 + z_4 - z_3. \end{aligned}$$

We proceed similarly to the previous cases and we center on the positive terms z_2, z_3, z_4, z_6, z_7, z_9, z_{10} and z_{11}. For each situation we have the following:

$C_{5,2}$: We achieve the tuple $(x, y, x, 0)$, a controversial case.

$C_{5,3}$: This case implies the controversial case $(x, x, 0, y)$.

$C_{5,4}$: The initial conditions are of the form $(x, 0, y, z)$, a controversial case.

$C_{5,6}$: We obtain an 8-cycle, since $(z_1, z_2, z_3, z_4) = (x_1, 0, x_1 - x_2, x_1)$.

$C_{5,7}$: We get another 8-cycle due to $(z_1, z_2, z_3, z_4) = (x_1, 0, 0, x_1)$.

$C_{5,9}$: We obtain $(x, 0, y, z)$, which is a controversial case.

$C_{5,10}$: The terms of the given tuple are monotonic and we get an 11-cycle.

$C_{5,11}$: It yields to the controversial case $(x, x, 0, y)$.

Once we have analyzed all the possibilities concerning the controversial cases, we gather the obtained results (recall that $x \geq y \geq z \geq 0$):

$$\begin{array}{ccccc} (x,x,z,y) & (x,0,0,y) & (x,y,0,y) & (x,y,0,z) & (x,x,0,y) \\ (x,0,y,0) & (x,x,y,x) & (x,0,z,y) & (x,z,0,y) & (x,0,y,z) \end{array}$$

However, bearing in mind the equivalence relation established in Definition 4, we can reduce the number of controversial cases.

Proposition 25. *Let x, y, z be real numbers such that $x \geq y \geq z \geq 0$. Then we have the following relations:*

$$(x,x,z,y) \sim (x,z,y,0); \quad (x,x,y,x) \sim (x,y,x,0); \quad (x,x,0,y) \sim (x,0,y,0).$$

Proof. The result follows by the computation of each tuple under Equation (4.2). As an example, if $x_1 = x$, $x_2 = x$, $x_3 = z$ and $x_4 = y$, then $x_5 = 0$ and $(x_1, x_2, x_3, x_4) \sim (x_2, x_3, x_4, x_5)$, or equivalently, $(x, x, z, y) \sim (x, z, y, 0)$. The remaining cases can be sum up as follows: $(x, x, y, x, 0); (x, x, 0, y, 0)$. □

This result enables us to reduce the list of controversial cases to nine:

$$(x,0,0,y); \quad (x,y,0,y); \quad (x,0,y,0); \quad (x,y,x,0);$$

$$(x,z,y,0); \quad (x,y,0,z); \quad (x,z,0,y); \quad (x,0,y,z); \quad (x,0,z,y).$$

Now, it must be highlighted that the tuples formed by two positive variables (the cases in the first line of the previous list) are included in the other cases with three non-negative variables. In concrete, $(x,0,0,y)$ is of the form $(x,z,0,y)$ with $z=0$; $(x,y,0,y)$ is of the form $(x,z,0,y)$ with $z=y$; $(x,0,y,0)$ is like the initial conditions $(x,0,y,z)$ with $z=0$; and $(x,y,x,0)$ behaves as $(x,z,y,0)$ with $x=y$. This allows us to reduce the controversial cases to:

$$(x,y,0,z);\quad (x,z,0,y);\quad (x,0,y,z);\quad (x,0,z,y);\quad (x,z,y,0). \tag{4.5}$$

Furthermore, thanks to Definition 4, we can relate some of the previous tuples. Firstly, notice that $(x,z,0,y)$ and $(x,0,y,z)$ generate the same sequence

$$\boldsymbol{x, z, 0, y,}\ y-x, y-z, y, 0, x, x-y+z, x-y, x, 0,$$

$$y-z, y, y-x, y, z, 0, x, x-y, x-z,\ \boldsymbol{x, 0, y, z}.$$

Thus, we get $(x,z,0,y)\sim(x,0,y,z)$.

Secondly, consider the initial conditions $(x,0,z,y)$ and iterate them under Equation (4.2) in order to obtain

$$\boldsymbol{x, 0, z, y,} y-x, y, y-z, 0, x, x-y, x-y+z, \boldsymbol{x, 0, y, y-z}.$$

Then, $(x,0,z,y)\sim(x,0,y,y-z)$, and as $y\geq y-z$, the new tuple $(x,0,y,y-z)$ has the form $(x,0,y,z')$, with $x\geq y\geq z'\geq 0$. So, if we know the behaviour of the tuple $(x,0,y,z')$ under Equation (4.2), we will know the evolution of $(x,0,z,y)$ too. In addition, we can proceed similarly with the tuple $(x,y,0,z)$, since after 11 iterations, it becomes into $(x,y-z,0,z)$; if $y-z\leq z$, then $(x,y,0,z)\sim(x,z',0,y')$; otherwise, repeating the process, we arrive to $(x,y-2z,0,z)$ after 11 iterations. It can be easily seen by induction that the tuple $(x,y-mz,0,z)$ with $m\geq 0$ and $y-mz>z$, becomes $\big(x,y-(m+1)z,0,z\big)$ after 11 iterations. Hence, at some point, the second term will be less than the fourth and we will get the previous equivalence. In conclusion, we have the equivalence $(x,y,0,z)\sim(x,z',0,y')$ for some $x\geq y'\geq z'$ and if we know the behaviour of $(x,z',0,y')$ under Equation (4.2), we will know the evolution of $(x,y,0,z)$.

Definitely, in view of (4.5) and the previous considerations, we only need to study in detail the controversial cases $(x,y,0,z)$ and $(x,z,y,0)$, since from their behaviour we can deduce the evolution of the remaining cases.

Part III - Analysis of the controversial cases $(x,y,0,z)$ and $(x,z,y,0)$:

In the previous part we studied in detail the so-called controversial cases. There, we were able to see that the complete casuistic could be reduced to two particular tuples of non-negative terms, namely, $(x,y,0,z)$ and $(x,z,y,0)$, where x,y,z are non-negative real numbers verifying $x\geq y\geq z\geq 0$. In the sequel, we study the evolution of such tuples under Equation (4.2).

III.a - Case $(\boldsymbol{x,y,0,z})$: Firstly, in order to avoid an 11-cycle, we must impose $z > 0$; otherwise, the initial conditions will be $(x, y, 0, 0)$ and they will be monotonic (see Proposition 20). Furthermore, we assume $x > z$, since, on the contrary, the tuple reduces to $(x, x, 0, x)$ giving rise to an 8-cycle (see Proposition 21). To sum up, to avoid the trivial cases (periods 1, 8 or 11), we will assume $x \geq y \geq z > 0$ with $x > z$.

Lemma 40. *Consider the tuple $(x, tx + y - sz, 0, z)$, with $x \geq tx + y - sz \geq 0$, $t \geq 0$ and $s \geq 0$.*

(a) If $tx + y - sz > z$, then after 11 iterations we will have $\big(x, tx + y - (s + 1)z, 0, z\big)$.

(b) If $tx + y - sz \leq z$, then after 32 iterations the tuple will evolve to the following terms $\big(x, (t + 1)x + y - (s + 1)z, 0, z\big)$.

Proof. We start with the case $tx + y - sz > z$. If we iterate $(x, tx + y - sz, 0, z)$ under Equation (4.2), the sequence continues as

$$\boldsymbol{x, tx + y - sz, 0, z}, (t-1)x + y - sz, (s+1)z - tx - y, z, 0,$$

$$(s+1)z - (t-1)x - y, x, x - z, \boldsymbol{x, tx + y - (s+1)z, 0, z}.$$

On the other hand, for the case $tx + y - sz \leq z$, we obtain

$$\boldsymbol{x, tx + y - sz, 0, z}, z - x, -tx - y + (s+1)z, z, 0, x,$$

$$(1+t)x + y - (s+1)z, x - z, x, 0, -tx - y + (s+1)z, z, z - x, z,$$

$$tx + y - sz, 0, x, x - z, (1-t)x - y + sz, x, 0, z, tx + y - sz, z - x,$$

$$z, 0, -tx - y + (s+1)z, x, x - z, \boldsymbol{x, (1+t)x + y - (s+1)z, 0, z}.$$

□

Observe that the controversial case $(x, y, 0, z)$ is a particular case of the more general tuple $(x, tx + y - sz, 0, z)$ with $s = t = 0$, so its evolution under Equation (4.2) is precisely given by the previous result.

Remark 4. *We inspect the proof of Lemma 40 to check the possibility of achieving periodicity in the middle of the process, following the same idea as the one developed while studying the existence of controversial cases in Part II.*

On the one hand, it is direct to check that periodicity cannot hold in the middle of the process for (a), that is, to recover the tuple $(x, y, 0, z)$. Nevertheless, in (b), we could achieve periodicity in the twenty second iteration. From here we get

$$(1-t)x - y + sz = x; \quad x = y; \quad 0 = 0; \quad z = z.$$

Then, the initial conditions are $(x, x, 0, z)$, with $(1-t)x - x + sz = x$, which implies $sz = (t+1)x$. This case will require a special analysis as we will see in Proposition 26. □

After highlighting the special case $(x, x, 0, z)$, we recover the study of the evolution of the tuple $(x, y, 0, z)$. The following result establishes a necessary condition in order to achieve periodicity.

Lemma 41. *Assume that the initial conditions $(x, y, 0, z)$ generate a periodic sequence under Equation (4.2), with $x \geq y \geq z > 0$, $x > z$. Then $\frac{z}{x} \in \mathbb{Q}$.*

Proof. Take the initial conditions $(x, y, 0, z)$ and apply Lemma 40. From here, in order to have periodicity, we get that there exists a pair of natural numbers $t, s \geq 1$ for which $tx + y - sz = y$ holds (in fact, there exist an infinity of values t, s holding the equality). As a direct consequence, we get $\frac{z}{x} = \frac{t}{s} \in \mathbb{Q}$. □

Now, we simplify the tuple $(x, y, 0, z)$ dividing each component by x, that is, $\left(1, \frac{y}{x}, 0, \frac{z}{x}\right)$. According to Lemma 41 and Proposition 19, we can go further by considering $\frac{z}{x} = \frac{q}{p}$ with $\gcd(p, q) = 1$. Finally, multiplying each term of the tuple $\left(1, \frac{y}{x}, 0, \frac{q}{p}\right)$ by p, we get $(p, \bar{y}, 0, q)$, where $\bar{y} = p \cdot \frac{y}{x}$ (observe that $\bar{y}$ is not necessarily a rational number). After developing this simplification, we show that the condition $\frac{z}{x} \in \mathbb{Q}$ is not only necessary for periodicity, but sufficient too. In the sequel, we will use the notation $\{\cdot\}$ and $\lfloor \cdot \rfloor$ to denote the fractional part and the integer part of a number, respectively.

Proposition 26. *Let $p, q \in \mathbb{Z}$, $\bar{y} \in \mathbb{R}$. Assume $p \geq \bar{y} \geq q > 0$ and $\gcd(p, q) = 1$. The initial conditions $(p, \bar{y}, 0, q)$ generate under Equation (4.2) a periodic sequence. Also, the period is given by $N = (p + q) \cdot 11 + q \cdot 10$.*

Proof. Observe that a direct consequence of Lemma 40 (with $x = p$, $y = \bar{y}$, $z = q$ and $t = s = 0$) is that after the corresponding block of 11 or 32 iterations, the second term of the tuple is the only one that varies. Let us denote by (z_1, z_2, z_3, z_4) the tuple after each block of 11 or 32 iterations, then we claim that $z_2 = \{\bar{y}\} + j$ for some $j \in \{0, 1, \ldots, p\}$. Certainly, in the first time, $z_2 = \bar{y} = \{\bar{y}\} + \lfloor y \rfloor$ and the property holds for $j = \lfloor \bar{y} \rfloor \in \{0, 1, \ldots, p\}$ (observe that $j = p$ is possible if $\{\bar{y}\} = 0$ and $p = \bar{y}$). As a next step, since $\bar{y} \geq q$, we have two possibilities:

- If $\bar{y} > q$, from Lemma 40, $\bar{y}$ goes to $z_2 = \bar{y} - q < p$.
- If $\bar{y} = q$, then, again by Lemma 40, $z_2 = p + \bar{y} - q = p$.

In both cases, $z_2 = \{\bar{y}\} + j$ for an appropriate $0 \leq j \leq p$. By induction, one can easily see that after the corresponding block of 11 or 32 iterations established in Lemma 40, we get that the new element z_2 has the wanted form, which ends the claim.

Consequently, z_2 can only take a finite number of values. From this, jointly with the fact that every eventually periodic sequence is, in fact, periodic, see Proposition 13, we can guarantee that the value $z_2 = \bar{y}$ will be repeated and we will obtain a periodic sequence.

Now, we compute its period. To do so, we distinguish two cases depending on the values of p and $\bar{y}$ as was emphasized in Remark 4.

- Assume $p > \bar{y}$. Here, periodicity is achieved after a finite concatenation of complete blocks of 11 and 32 iterations. Suppose that we apply α times a block of 11 iterations and β times a block of 32 in order to obtain periodicity. According to Lemma 40, the second term is of the form $\alpha \cdot p + \bar{y} - (\alpha+\beta) \cdot q$ and the equality $\alpha \cdot p + \bar{y} - (\alpha+\beta) \cdot q = \bar{y}$ will hold. From here, we get $\alpha \cdot p = (\alpha + \beta) \cdot q$, so α is a multiple of q and $(\alpha + \beta)$ is a multiple of p from the fact that $\gcd(p, q) = 1$. However, since the integer part of the second term, $\lfloor z_2 \rfloor$, varies in the set $\{0, 1, \ldots, p\}$, then $\alpha + \beta \leq p + 1$. Hence, $\alpha + \beta = p$ (notice that $p \geq 2$), so $\alpha \cdot p = p \cdot q$ and we deduce that $\alpha = q$, because p and q are coprime numbers.

  In conclusion, the periodic sequence is determined by the concatenation of $(p - q)$ blocks of 11 iterations and q blocks of 32, being its period $N = (p - q) \cdot 11 + q \cdot 32$, or, equivalently,

  $$N = (p + q) \cdot 11 + q \cdot 10.$$

- Assume $p = \bar{y}$. Recall that this leads us to the special case $(p, p, 0, q)$ highlighted in Remark 4. Therefore, the tuple will evolve following a concatenation of $r = t$ blocks of 32 iterations and $m = s - t$ blocks of 11 iterations; and then, in the middle of the following block of 32 iterations, periodicity is achieved (notice that we are assuming that we have not recovered the initial conditions before the s blocks); additionally, $s \cdot q = (t + 1) \cdot p$, which yields to $\frac{q}{p} = \frac{t+1}{s}$. Then, since $\gcd(p, q) = 1$, we deduce that $t + 1 \geq q$, $s \geq p$.

  $\star$ If $s > p$, then $t + 1 > q$. This means that there are s tuples, with $s = s' \cdot p$, $s' > 1$, $s \geq 2p$, that display the form $\left(p, \widetilde{t}p + \bar{y} - \widetilde{s}q, 0, q\right)$, where each term $\widetilde{t}p + \bar{y} - \widetilde{s}q$ is of the type $\bar{y} + j$, with $j \in \{0, 1, \ldots, p\}$. Since $s \geq 2p > p + 1$ if $p \geq 2$ (notice that $p = 1$ gives the 8-cycle generated from the initial conditions $(1, 1, 0, 1)$), there exist $\widetilde{s}, \widetilde{t}$, with $\widetilde{s} \in \{1, \ldots, p\}$, $\widetilde{t} \leq \widetilde{s}$, such that $\widetilde{y} + \widetilde{t}p - \widetilde{s}q = \widetilde{y}$. Nevertheless, this is a contradiction because $\widetilde{s} < s$ and we have assumed that periodicity was not achieved before the application of the s blocks of iterations.

  In conclusion, $s = p$, $t + 1 = q$, and the period is

  $$\begin{aligned} N &= 32 \cdot (q-1) + 11 \cdot (p - q + 1) + 21 = (2 \cdot 11 + 10) \cdot q + 11 \cdot (p - q) \\ &= (p + q) \cdot 11 + q \cdot 10. \end{aligned}$$

This concludes the proof of the result. □

Remark 5. *If we focus on the coefficients of the decomposition of the period N, that is, $a = q$ and $b = p + q$ with $\gcd(p, q) = 1$ and $p > q$, one can check that the conditions $b \geq 2a+1$*

and $\gcd(a,b) = 1$ *hold. Furthermore, every number of the form* $\widetilde{N} = 10 \cdot a + 11 \cdot b$*, with* $b \geq 2a+1$ *and* $\gcd(a,b) = 1$*, can be written as* $\widetilde{N} = 10 \cdot q + 11 \cdot (p+q)$ *by setting* $q = a$ *and* $p = b - a$*, where* $\gcd(p, p+q) = 1$ *and* $p + q \geq 2q + 1$*.*

Corollary 6. *Given the initial conditions* $(x, y, 0, z)$*, with* $x \geq y \geq z > 0$ *and* $x > z$*, generate a periodic sequence under Equation (4.2) if and only if* $\frac{z}{x} \in \mathbb{Q}$*. In that case, the period is* $N = 10 \cdot a + 11 \cdot b$ *for some natural numbers* a, b *where* $\gcd(a,b) = 1$ *and* $b \geq 2a+1$*.*

III.b - Case $(x, z, y, 0)$: Now, we study the evolution of the other controversial case that must be analyzed, namely, the set of initial conditions $(x, z, y, 0)$, with x, y, z non-negative real numbers satisfying the relations $x \geq y \geq z \geq 0$. Firstly, in order to avoid the trivial periods, that is, 1, 8 or 11, we assume $y > z > 0$. Indeed, if $y = z$, we would have monotonic initial conditions that would yield to an 11-cycle (see Proposition 20). On the other hand, if $z = 0$, the controversial case reduces to $(x, 0, y, 0)$. If we iterate such tuple under Equation (4.2), if $x - y \geq y$, after ten iterations, we obtain $(x, x-y, 0, y)$ (notice that by Definition 4, the tuples $(x, 0, y, 0)$ and $(x, x-y, 0, y)$ are equivalent); otherwise, if $x - y < y$, after twenty one iterations we get $(x, 0, -x+2y, y)$ (again $(x, 0, y, 0)$ and $(x, 0, -x+2y, y)$ are equivalent). Then, in the first case, the obtained tuple exhibits the form $(x, y', 0, z')$; while in the second case we have a tuple of the form $(x, 0, z', y')$, with $x \geq y' \geq z' \geq 0$. Notice that both cases have been analyzed in Part II and we can exclude them in the following study. To sum up, we are considering the set of initial conditions $(x, z, y, 0)$ with $x \geq y > z > 0$.

As a next step, we iterate those initial conditions ten times under Equation (4.2):

$$\boldsymbol{x, z, y, 0}, y-x, y-z, -z, y-z, x-z, x-y, \boldsymbol{x, x-(y-z), z, y},$$

and we obtain the equivalent tuple $\big(x, x-(y-z), z, y\big)$. Let us see how these terms evolve.

Lemma 42. *Consider the tuple* $\big(x, tx - s(y-z), z, y\big)$*, verifying* $x \geq y > z > 0$*,* $x \geq tx - s(y-z) \geq 0$ *and* t, s *being non-negative integers.*

(a) If $tx - s(y-z) \geq y$*, then after a block of 11 iterations, the considered tuple becomes into* $\big(x, tx - (s+1)(y-z), z, y\big)$*.*

(b) If $tx - s(y-z) < y$ *and* $tx - (s-1)(y-z) \geq y$*, then after 22 iterations the tuple becomes into* $\big(x, z, y, tx - s(y-z)\big)$*.*

Proof. We begin with the case $tx - s(y-z) \geq y$. Take the tuple $\big(x, tx - s(y-z), z, y\big)$ and iterate it under Equation (4.2):

$$\boldsymbol{x, tx - s(y-z), z, y}, (t-1)x - s(y-z), (s+1)y - tx - sz, y-z, -z,$$

$$(1-t)x + (s+1)(y-z), x-z, x-y, \boldsymbol{x, tx - (s+1)(y-z), z, y}.$$

Now, let us suppose that $tx - s(y-z) < y$ and $tx - (s-1)(y-z) \geq y$, then, the following terms of the considered tuple are:

$$\boldsymbol{x, tx - s(y-z), z, y}, y-x, (s+1)y - tx - sz, y-z, -z, x-z,$$

$$(t+1)x + (s-1)z - (s+1)y, x-y, x, z, -tx + (s+1)y - (s-1)z, y, y-x,$$

$$y-z, tx - sy + (s-1)z, -z, x-z, x-y, (1-t)x + sy - sz, \boldsymbol{x, z, y, tx - s(y-z)}.$$

□

Lemma 43. *Consider the tuple* $\big(x, z, y, tx - s(y-z)\big)$*, verifying* $x \geq y > z > 0$*,* $y > tx - s(y-z) \geq 0$ *and* t, s *being non-negative integers.*

(a) If $tx - (s+1)(y-z) \geq 0$*, then after a block of 11 iterations, the tuple becomes into* $\big(x, z, y, tx - (s+1)(y-z)\big)$*.*

(b) If $tx - (s+1)(y-z) < 0$*, then after a block of 10 iterations, the tuple becomes into* $\big(x, (t+1)x - (s+1)(y-z), z, y\big)$*.*

Proof. Firstly, assume $tx - (s+1)(y-z) \geq 0$ and iterate the terms $\big(x, z, y, tx - s(y-z)\big)$ under Equation (4.2):

$$\boldsymbol{x, z, y, tx - s(y-z)}, y-x, y-z, tx - (s+1)y + sz, -tx + (s+1)(y-z), x-z,$$

$$x-y, (1-t)x + (s+1)y - (s+1)z, \boldsymbol{x, z, y, tx - (s+1)(y-z)}.$$

On the other hand, if $tx - (s+1)(y-z) < 0$, then the tuple evolves as follows:

$$\boldsymbol{x, z, y, tx - s(y-z)}, y-x, y-z, -z, -tx + (s+1)(y-z), x-z, x-y,$$

$$\boldsymbol{x, (t+1)x - (s+1)(y-z), z, y}.$$

□

Remark 6. *It must be highlighted that the above results, Lemmas 42 and 43, jointly describe how the tuple* $\big(x, x-(y-z), z, y\big)$ *evolves under Equation (4.2). Indeed, notice that if we have four consecutive terms satisfying the conditions of Lemma 42(b), then, after a block of* 22 *iterations, we will obtain a tuple verifying the conditions of Lemma 43. In addition, if the conditions of Lemma 43(b) hold for a certain tuple, then, after a block of* 10 *iterations, the new terms will verify the conditions of Lemma 42.* □

Remark 7. *Analogously to Remark 4, we must check the possibility of achieving periodicity in the middle of the blocks of iterations described in the above Lemma, that is, if at some point of its evolution, it is possible to recover the terms* $\big(x, x-(y-z), z, y\big)$ *(notice that if* $\big(x, x-(y-z), z, y\big)$ *generate a periodic sequence, then the initial conditions* $(x, z, y, 0)$ *generate a periodic sequence with the same period due to Proposition 13).*

By the inspection of the proof of Lemmas 42 and 43, it is easy to see that periodicity cannot hold in the middle of the blocks of iterations given by Lemma 42(a), or Lemma 43. Therefore, we only have to focus on Lemma 42(b), where after the eleventh iteration, we could obtain periodicity if the following equalities hold

$$x = x; \qquad z = x - (y - z); \qquad -tx + (s+1)x - (s-1)z = z; \qquad y = y.$$

From the second equality, we deduce $x = y$, *so the initial conditions* $\big(x, x-(y-z), z, y\big)$ *reduce to* (x, z, z, x). *By iterating this terms three times under Equation (4.2), we get the equivalent tuple* $(x, 0, x-z, x-z)$, *which behaves as the set of initial conditions* $(x, 0, y', y')$ *with* $x > y' > 0$. *However, this tuple is a particular case of the controversial one* $(x, 0, \widetilde{y}, \widetilde{z})$, *with* $\widetilde{y} = y' = z'$ *and we saw in Part II that this case is covered by the analysis of the controversial case* $(x, \widetilde{y}, 0, \widetilde{z})$ *with* $x \geq \widetilde{y} \geq \widetilde{z} \geq 0$ *and* $x > \widetilde{z}$. *Therefore, we have already studied this possibility and we do not have to worry about achieving periodicity in the middle of the process.* □

Once that we have described precisely the evolution of the initial conditions $(x, z, y, 0)$ in Lemma 42, Lemma 43 and Remark 6, we deal with the period of the sequence. To do so, we differ two scenarios depending on the fact whether the initial conditions are rational numbers or not.

Case A: The initial conditions $(x, z, y, 0)$ are rational numbers. This allows us to multiply those terms by the least common multiple of their denominators in order to obtain integer numbers. Hence, by Proposition 19, we can assume without loss of generality that x, y, z are integers.

Put $d := \gcd(x, y-z)$. Recall that, after ten iterations, the initial conditions $(x, z, y, 0)$ evolve to $\big(x, x-(y-z), z, y\big)$. Then, dividing by d, we obtain $\left(\frac{x}{d}, \frac{x-(y-z)}{d}, \frac{z}{d}, \frac{y}{d}\right)$, or equivalently, (p, q, z', y'), where $y' = \frac{y}{d}$, $z' = \frac{z}{d}$ and $\gcd(p, q) = 1$. Observe that p and q are integer numbers with p being the greatest term of the tuple.

Proposition 27. *Under the previous considerations, the initial conditions* (p, q, z', y'), *with* $\gcd(p, q) = 1$ *and* $q = p - (y' - z')$, *generate under Equation (4.2) a periodic sequence. Furthermore, the period is given by* $N = (2p - q) \cdot 11 + (p - q) \cdot 10$.

Proof. As a first step, observe that the tuple (p, q, z', y') verifies the conditions of Lemma 42 with $x = p$, $y = y'$, $z = z'$ and $t = s = 1$, since $q = p - (y' - z')$. Moreover, we know how it will evolve under Equation (4.2) due to Lemmas 42 and 43.

Set $b_0^{(2)} := q$, where the superscript (2) indicates that such term is placed in the second position of the corresponding tuple. Now, depending on whether the inequality $p-(y'-z') \geq y'$ holds or not, we will apply Lemma 42(a) or (b). In any case, after the corresponding block of 11 or 22 iterations, we get a new tuple $\big(p, b_1^{(2)}, z', y'\big)$ or $\big(p, z', y', b_1^{(4)}\big)$, with $b_1^{(2)} =$

$p - 2(y' - z') = q - (y' - z')$ and $b_1^{(4)} = q$ (observe that we use (2) or (4) to indicate if the term occupies the second or fourth term of the tuple). If we continue this process, we can build a sequence $b_j^{(i_j)}$, with $j \geq 1$, $i_j \in \{2, 4\}$, where the associated tuple is $\left(p, b_j^{(2)}, z', y'\right)$ if $i_j = 2$; or $\left(p, z', y', b_j^{(4)}\right)$ if $i_j = 4$.

As a next step, from the conditions of Lemmas 42 and 43 and the fact $y' - z' = \frac{y-z}{d} \in \mathbb{N}$, we can deduce that the terms of the sequence $\left(b_j^{(i_j)}\right)$ belong to the set $\mathcal{V}$, where

$$\mathcal{V} := \left\{0^{(2)}, 0^{(4)}, 1^{(2)}, 1^{(4)}, \ldots, p^{(2)}, p^{(4)}\right\}.$$

Notice that $b_0^{(2)}$ satisfies the claim and, proceeding by induction, we can see that any $b_j^{(i_j)}$ satisfies it too. For example, if we are in Lemma 42(a), then $b_j^{(2)} = tp - (s+1)(y' - z')$ for some natural numbers $t, s \geq 1$ and $b_j^{(2)} > p$ would imply $tp - s(y' - z') > p + y' - z' > p$, a contradiction with the initial hypothesis of Lemma 42.

Now, we claim that $b_j^{(i_j)} \notin \left\{0^{(2)}, p^{(4)}\right\}$, which means that we cannot have the value 0 in the second position of the corresponding tuple; nor the value p in the fourth position. Once more, this follows directly from the hypothesis of Lemmas 42 and 43. For example, if we are in Lemma 42(b), after the corresponding block of 22 iterations, we will have the tuple $\left(p, z', y', tp - s(y' - z')\right)$. If $tp - s(y' - z') = p > y'$, it will contradict the hypothesis of Lemma 42(b). The other cases are analogous and we omit them.

In conclusion, $b_j^{(i_j)} \in \mathcal{V} \setminus \left\{0^{(2)}, p^{(4)}\right\}$ and there can only exist a finite number of tuples $\left(p, b_j^{(2)}, z', y'\right)$ or $\left(p, z', y', b_j^{(4)}\right)$. This implies that after a certain number of iterations we will repeat one of those tuples and we will have periodicity. Concretely, due to the fact that $\mathrm{Card}\left(\mathcal{V} \setminus \left\{0^{(2)}, p^{(4)}\right\}\right) = 2p$, where Card denotes the cardinality of a set, periodicity will be achieved after applying at most $2p$ times Lemmas 42 and 43.

Once that we have proved that the sequence generated from the initial conditions (p, q, z', y'), when they are rational numbers, under Equation (4.2) is periodic, we compute its period.

Let us denote by α the amount of blocks of 11 iterations described by Lemma 42(a); β the number of blocks of 22 iterations characterized by Lemma 42(b); γ the quantity of blocks of 11 iterations outlined by Lemma 43; and δ the totality of blocks of 10 iterations detailed in Lemma 43(b), needed to obtain periodicity.

We start claiming that $\beta = \delta \geq 1$. Indeed, observe that when Lemma 42(b) applies, the only change that is made is the translation of the term $b_j^{(2)}$ from the second position of the tuple to the fourth, but its value does not change. Similarly, when Lemma 43(b) applies, the fourth term $b_j^{(4)}$ changes its position to the second place of the tuple, but its value does not vary. Therefore, we have the same amount of blocks of each type, since every change of position must be balanced. Moreover, the initial tuple (p, q, z', y') verifies the hypothesis of Lemma 42, so at some point we will apply Part (b) having $\beta \geq 1$.

Now, bearing in mind that the orbit of the sequence is described by a concatenation of blocks, see Remark 6, we will obtain periodicity when

$$(\delta+1)p-(\alpha+\gamma+\delta+1)(y'-z')=q=p-(y'-z').$$

This equation implies that

$$\delta\cdot p=(\alpha+\gamma+\delta)\cdot(y'-z').$$

Now, since $\gcd(p,q)=1$ and $q=p-(y'-z')$, we have $\gcd(p,y'-z')=1$. Therefore, $(\alpha+\gamma+\delta)$ is a multiple of p, and δ is a multiple of $(y'-z')$. Indeed, since $q=p-(y'-z')$, δ is a multiple of $(p-q)$ too. However, as $b_j^{(i_j)}$ varies in the set $\mathcal{V}\setminus\left\{0^{(2)},p^{(4)}\right\}$ of cardinality $2p$, and $\alpha+\beta+\gamma+\delta$ represents the amount of times that $b_j^{(i_j)}$ changes its value or/and position, we get $\alpha+\beta+\gamma+\delta\leq 2p$. Also, $\beta\geq 1$ implies that $\alpha+\gamma+\delta<2p$, and jointly with the fact that $\alpha+\gamma+\delta$ is a multiple of p, we obtain $\alpha+\gamma+\delta=p$. Hence, $\delta\cdot p=p\cdot(p-q)$ and $\delta=p-q$.

In summary, the periodic sequence will be determined by α blocks of 11 iterations; $\beta=\delta$ blocks of 22 iterations; γ blocks of 11 iterations; and δ blocks of 10 iterations; which implies that the period is

$$\begin{aligned}N &= \alpha\cdot 11+\delta\cdot 22+\gamma\cdot 11+\delta\cdot 10=(\alpha+2\delta+\gamma)\cdot 11+\delta\cdot 10\\ &= (p+p-q)\cdot 11+(p-q)\cdot 10=(2p-q)\cdot 11+(p-q)\cdot 10.\end{aligned}$$

□

Case B: At least one of the initial conditions $(x,z,y,0)$ is not rational. Recall, by Lemmas 42 and 43, that periodicity will be achieved when the equality

$$x-(y-z)=tx-s(y-z)$$

holds for a certain $t,s\geq 0$. From here we deduce that $\frac{x}{y-z}=\frac{s-1}{t-1}\in\mathbb{Q}$, and, consequently, $\frac{x}{y-z}\in\mathbb{Q}$ is a necessary condition in order to achieve periodicity. Let us consider the irreducible fraction, $\frac{x}{y-z}=\frac{p}{m}$, with $\gcd(p,m)=1$. By Proposition 19, we can vary the initial conditions by multiplying them by a positive constant

$$\begin{aligned}\big(x,x-(y-z),z,y\big) &\longrightarrow \left(\frac{x}{y-z},\frac{x}{y-z}-1,\frac{z}{y-z},\frac{y}{y-z}\right)\\ &\longrightarrow \left(\frac{p}{m},\frac{p}{m}-1,\frac{z}{y-z},\frac{y}{y-z}\right)\longrightarrow(p,p-m,z',y'),\end{aligned}$$

with $z'=m\cdot\frac{z}{y-z}$, $y'=m\cdot\frac{y}{y-z}$ and $\gcd(p,m)=1$, which implies $\gcd(p,p-m)=1$. Furthermore, $y'-z'=m\cdot\frac{y}{y-z}-m\cdot\frac{z}{y-z}=m\in\mathbb{N}$. Hence, if we denote by $q=p-m$, the initial conditions are (p,q,z',y') with $\gcd(p,q)=1$ and $y'-z'\in\mathbb{N}$; and we can apply the same reasoning as in Proposition 27 to obtain that the period of the sequence is

$$N=(2p-q)\cdot 11+(p-q)\cdot 10.$$

Remark 8. *Firstly, observe that the coefficients of the decomposition of the period* N*,* $a = p - q$ *and* $b = 2p - q$ *satisfy the conditions* $b \geq 2a + 1$ *and* $\gcd(a, b) = 1$*. Moreover, for* a, b *natural numbers such that* $b \geq 2a + 1$ *and* $\gcd(a, b) = 1$ *every number of the form* $\widetilde{N} = 10 \cdot a + 11 \cdot b$ *can be written as* $\widetilde{N} = 10 \cdot (p - q) + 11 \cdot (2p - q)$ *by setting* $p = b - a$ *and* $q = b - 2a$*.*

In summary, the Case III.b reduces to the following result.

Corollary 7. *Given the initial conditions* $(x, z, y, 0)$*, with* $x \geq y > z > 0$*, generate a periodic sequence under Equation* (4.2) *if and only if* $\frac{x}{y-z} \in \mathbb{Q}$*. In this case, the period of the sequence is* $N = 10 \cdot a + 11 \cdot b$*, for some natural numbers* a, b*, with* $\gcd(a, b) = 1$ *and* $b \geq 2a + 1$*.*

Part IV - Intersection between cases:

Another problematic situation that must be analyzed in detail is the possibility of having initial conditions verifying more than one Case C_i at the same time. Diagram 4.1 gives us the movement of an orbit under Equation (4.2), but we need such movement to be unique in order to study the possible periodic character of the orbit and its associated period. In this sense, Part IV delves into the analysis of this situation that we have called *intersection between cases.*

As a first step, let us remember the five Cases C_i:

Case 1 (C_1): $x_1 \geq x_2 \geq x_4 \geq x_3$.

Case 2 (C_2): $x_1 \geq x_3 \geq \max\{x_2, x_4\}$ with $x_3 \geq x_2 + x_4$.

Case 3 (C_3): $x_1 \geq x_3 \geq \max\{x_2, x_4\}$ with $x_3 \leq x_2 + x_4$.

Case 4 (C_4): $x_1 \geq x_4 \geq x_2 \geq x_3$.

Case 5 (C_5): $x_1 \geq x_4 \geq x_3 \geq x_2$.

Observe that the conditions of two different Cases C_i could be verified by the same tuple (x_1, x_2, x_3, x_4) since the inequalities that characterized them are not strict. Specifically, this happens when at least one of the terms are repeated or when $x_3 = x_2 + x_4$ holds, which means that the tuple verifies the Cases C_2 and C_3 simultaneously. Nevertheless, in this last situation, the initial conditions will be of the form $(x, z, y, y - z)$. If we compute the following terms of the sequence generated by such initial conditions under Equation (4.2), we find

$$\boldsymbol{x, z, y, y - z}, y - x, y - z, -z, 0, x - z, x - y, \boldsymbol{x, x, z, y}.$$

Then, by Definition 4, we obtain that $(x, z, y, y-z) \sim (x, x, z, y)$ which, in turn, is equivalent to $(x, z, y, 0)$ by Proposition 25, but we have already studied this controversial case in Part III. So, we only focus on the tuples of initial conditions that have repeated terms. In the sequel, we assume $x > y > z \geq 0$.

- **All four terms are identical:**

  ∘ (x, x, x, x) : The initial conditions are monotonic and we get an 11-cycle by Proposition 20.

- **Three terms are equal:**

  ∘ (x, x, x, y) and (x, y, y, y): They generate 11-cycles due to the monotonicity of the terms by Proposition 20.

  ∘ (x, x, y, x): It is a controversial case. See Part II.

  ∘ (x, y, x, x): The sequence generated from this initial conditions follows as x, y, x, x, 0, $x - y$. Then $(x, y, x, x) \sim (x, x, 0, x - y)$, which behaves as the controversial case $(x, x, 0, y')$.

- **Two terms are equal:**

  ∘ (x, x, y, z), (x, y, y, z), (x, y, z, z) and (x, x, y, y): In the four cases we have monotonic initial conditions and, hence, we obtain 11-cycles.

  ∘ (x, y, x, z), (x, x, z, y) and (x, z, y, x): They are controversial cases, already analyzed in Part II.

  ∘ (x, y, z, x): By computing the following terms under Equation (4.2), we get, x, y, z, x, 0, $x - y$, $x - z$. So $(x, y, z, x) \sim (x, 0, x - y, x - z)$, which behaves as the controversial case $(x, 0, z', y')$.

  ∘ (x, z, x, y): The sequence evolves as x, z, x, y, 0, $x - z$. Thus $(x, z, x, y) \sim (x, y, 0, x - z)$. If $y \geq x - z$, then $(x, y, 0, x - z)$ behaves as the controversial case $(x, y', 0, z')$; otherwise, if $y < x - z$, then $(x, y, 0, x - z)$ behaves as the controversial case $(x, z', 0, y')$.

  ∘ (x, y, x, y): By iterating the initial conditions, we have x, y, x, y, $0, x - y$. Therefore, $(x, y, x, y) \sim (x, y, 0, x - y)$. If $y \geq x - y$, then $(x, y, 0, x - y)$ behaves as the controversial case $(x, y', 0, z')$; on the other hand, if $y < x - y$, then it behaves as $(x, z', 0, y')$, which is a controversial case too.

  ∘ (x, y, y, x): We iterate the tuple, x, y, y, x, $0, x - y, x - y$. Then $(x, y, y, x) \sim (x, 0, x - y, x - y)$, which is of the type of the controversial case $(x, 0, y', y')$.

  ∘ (x, y, z, y), (x, z, z, y), (x, z, y, y) and (x, z, y, z): We have not analyzed these cases yet, so we proceed to their study.

The remaining cases that must be studied are (x, y, z, y), (x, z, z, y), (x, z, y, y) and (x, z, y, z). However, we begin by showing that these tuples are equivalent.

Proposition 28. *Let x, y, z be real numbers such that $x > y > z \geq 0$. Then*

$$(x, y, z, y) \sim (x, z, z, y) \sim (x, z, y, y) \sim (x, z, y, z).$$

Proof. The proof follows by the iteration of Equation (4.2) over those terms.

$$\begin{array}{llllllll}
\boldsymbol{x,y,z,y,} & y-x, & 0, & y-z, & -z, & x-z, & x-z, & x-y, \\
\boldsymbol{x,z,z,y,} & y-x, & y-z, & y-z, & -z, & x-z, & x-y, & x-y, \\
\boldsymbol{x,z,y,y,} & y-x, & y-z, & 0, & -z, & x-z, & x-y, & x-z, \quad \boldsymbol{x,z,y,z}.
\end{array}$$

□

The above result enables us to reduce these tuples to only one, for instance, (x, y, z, y). However, it is easy to see that these initial conditions are, indeed, a particular case of the controversial one $(x, z, y, 0)$. As a matter of fact, by the simplifications developed in Part III, $(x, z, y, 0)$ and $\big(x, x-(y-z), z, y\big)$ are equivalent. Observe that, if $x+z = 2y$, we obtain the particular case (x, y, z, y) and we can apply Corollary 7.

In conclusion, if a set of initial conditions verifies the restrictions of more than one Case C_i, $i = 1, \ldots, 5$, that generate a periodic sequence, we can guarantee that its period, N, is either $1, 8, 11$, or follows the pattern $N = 10 \cdot a + 11 \cdot b$, with $\gcd(a, b) = 1$ and $b \geq 2a + 1$.

Part V - Condition U:

In this part, we deal with those periodic sequences whose orbits are unambiguously described by the Diagram in Figure 4.1. This means that if we consider a set of initial conditions (x_1, x_2, x_3, x_4) and we iterate them under Equation (4.2), then, after the corresponding blocks of 10 or 11 iterations, every obtained tuple will only verify the restrictions of one of the Cases C_i, $i = 1, 2, 3, 4, 5$. Observe that if this holds, it will exist a unique well-defined cycle. Bearing this in mind, in the sequel we assume the following condition:

> **Condition U:** A tuple of initial conditions (x_1, x_2, x_3, x_4) only verifies one Case C_i and have exactly a unique movement according to Diagram 4.1.

In conclusion, Condition U implies that there is no ambiguity while going from one Case C_i to another C_j. Nevertheless, if at some point of the orbit the corresponding tuple verifies the conditions of more than one case, then we would be dealing with one of the intersection between cases studied in the previous part.

To illustrate Condition U, let us take the initial conditions $x_1 = 15$, $x_2 = 6$, $x_3 = 2$ and $x_4 = 7$. Observe that $(15, 6, 2, 7)$ verifies the conditions of Case C_4. If we compute this tuple under Equation (4.2), we get a 54-cycle:

$$\begin{array}{l}
\mathbf{15}, \mathbf{6}, \mathbf{2}, \mathbf{7}, -8, 1, 5, -2, 13, 12, 8, \\
\mathbf{15}, \mathbf{2}, \mathbf{3}, \mathbf{7}, -8, 5, 4, -2, 13, 8, 9, \\
\mathbf{15}, \mathbf{2}, \mathbf{7}, \mathbf{6}, -8, 5, -1, -1, 13, 8, 14, \\
\mathbf{15}, \mathbf{2}, \mathbf{7}, \mathbf{1}, -8, 5, -2, 4, 13, 8, \\
\mathbf{15}, \mathbf{11}, \mathbf{2}, \mathbf{7}, -4, -4, 5, -2, 9, 13, 8, \mathbf{15}, \mathbf{6}, \mathbf{2}, \mathbf{7}.
\end{array}$$

As it can be easily seen, after the corresponding block of 10 or 11 iterations, the new tuple satisfies only one case. Indeed, $(15, 6, 2, 7)$ is in C_4; $(15, 2, 3, 7)$ satisfies the conditions of Case C_5; $(15, 2, 7, 6)$ verifies the restrictions of Case C_3; $(15, 2, 7, 1)$ is in C_2; and $(15, 11, 2, 7)$ belongs to C_1.

After establishing Condition U and inspecting the Diagram in Figure 4.1, one must analyze the possibility of having an orbit that always stays in the same case. Notice that this can only happen for the Cases C_1 and C_3. The following result answers the question negatively, except with the trivial case of 11-cycles.

Proposition 29. *Let (x_n) be a periodic sequence of period $p \geq 12$. Assume that the initial conditions satisfy Cases C_1 or C_3. Then, there exists an $m > 1$ such that after $11 \cdot m$ iterations, the new tuple $(x_{1+11m}, x_{2+11m}, x_{3+11m}, x_{4+11m})$ does not satisfy the same inequalities as the initial terms in C_1 or C_3.*

Proof. Due to the analogy between cases, we only develop the reasoning for the Case C_1. Let us proceed by contradiction. Assume that the tuples $(x_{1+11j}, x_{2+11j}, x_{3+11j}, x_{4+11j})$ verify the conditions of C_1 for every $j \in \mathbb{N} \cup \{0\}$, that is, $x_{1+11j} \geq x_{2+11j} \geq x_{4+11j} \geq x_{3+11j}$, for every $j \in \mathbb{N} \cup \{0\}$. Now, it is easy to see by induction that, after m blocks of 11 iterations, the tuple will be of the form $\big(x_1, x_2 - m(x_4 - x_3), x_3, x_4\big)$. In order to have periodicity, for some $m \geq 1$, the equality $x_2 - m(x_4 - x_3) = x_2$ must hold. However, this implies that $x_3 = x_4$, which give rise to monotonic initial conditions. Definitely, we would have an 11-cycle or the equilibrium point depending if $x_1 > 0$ or $x_1 = 0$, respectively. □

From now on, in order to study the movement of a periodic orbit under Equation (4.2), we consider Figure 4.1 as an oriented graph, $G = (V, U)$, where $V = \{C_1, C_2, C_3, C_4, C_5\}$ is a finite set and $U \subset V \times V$. The elements of V are the vertices of the oriented graph G and each element $(C_i, C_j) \in U$ is called an arrow from C_i to C_j. Thus, the elements of U are the arrows of G. A path that always visits the same vertex is called a loop (notice that our graph G only admits two loops, one in C_1, and another in C_3; also, the number of consecutive loops is finite by Proposition 29). A route is a circuit that visits each vertex once, except the possibility of having a loop. We denote them by R_i. Also, in view of Diagram 4.1, we assume, without loss of generality, that the initial conditions verify the restrictions of Case C_4, and then periodicity holds when the corresponding tuple verifies again the conditions of the same case. Our graph only admits the following routes:

$R_1 : C_4 \longrightarrow C_5 \longrightarrow C_2 \longrightarrow C_1 \to \ldots \to C_1 \longrightarrow C_4.$

$R_2 : C_4 \longrightarrow C_5 \longrightarrow C_2 \longrightarrow C_4.$

$R_3 : C_4 \longrightarrow C_5 \longrightarrow C_3 \to \ldots \to C_3 \longrightarrow C_2 \longrightarrow C_1 \to \ldots \to C_1 \longrightarrow C_4.$

$R_4 : C_4 \longrightarrow C_5 \longrightarrow C_3 \to \ldots \to C_3 \longrightarrow C_2 \longrightarrow C_4.$

Moreover, with the aim of determining the period of the sequence, we need to compute the length of the different routes, which are the number of iterations needed to go from the initial conditions until the final step. In our case,

$$|R_1| = 43 + 11 \cdot m, \quad |R_2| = 32, \quad |R_3| = 54 + 11 \cdot (m+n), \quad |R_4| = 43 + 11 \cdot n,$$

where $m \geq 0$ and $n \geq 0$ are the times that the vertices C_1 and C_3 are repeated in the possible loops.

In addition, it is easy to see how the initial conditions (x_1, x_2, x_3, x_4), with $x_1 \geq x_4 \geq x_2 \geq x_3$, will evolve under each route R_i:

$$\begin{aligned}
R_1 : (x_1, x_2, x_3, x_4) &\implies \big(x_1, x_1 + x_2 - (m+2) \cdot (x_4 - x_3), x_3, x_4\big);\\
R_2 : (x_1, x_2, x_3, x_4) &\implies \big(x_1, x_1 + x_2 - (x_4 - x_3), x_3, x_4\big);\\
R_3 : (x_1, x_2, x_3, x_4) &\implies \big(x_1, x_1 + x_2 - (m+n+3) \cdot (x_4 - x_3), x_3, x_4\big);\\
R_4 : (x_1, x_2, x_3, x_4) &\implies \big(x_1, x_1 + x_2 - (n+2) \cdot (x_4 - x_3), x_3, x_4\big).
\end{aligned}$$

In order to illustrate the reasoning that has to be made to conclude the above assertions, we are going to specify the process for the route R_1. In this sense, consider the initial conditions (x_1, x_2, x_3, x_4) verifying $x_1 \geq x_4 \geq x_2 \geq x_3 \geq 0$. We refer to Table 4.1 to elaborate the following:

$$\begin{aligned}
C_4 : (x_1, x_2, x_3, x_4) &\xrightarrow{11} (x_1, x_3, x_4 + x_3 - x_2, x_4) \in C_5\\
&\xrightarrow{11} (x_1, x_3, x_4, x_2) \in C_2\\
&\xrightarrow{10} (x_1, x_1 + x_2 + x_3 - x_4, x_3, x_4) \in C_1\\
&\xrightarrow{11} \big(x_1, x_1 + x_2 + 2(x_3 - x_4), x_3, x_4\big) \in C_1\\
&\xrightarrow{11} \dots\\
&\xrightarrow{11} \big(x_1, x_1 + x_2 + (m+1)(x_3 - x_4), x_3, x_4\big) \in C_1\\
&\xrightarrow{11} \big(x_1, x_1 + x_2 + (m+2)(x_3 - x_4), x_3, x_4\big) \in C_4,
\end{aligned}$$

where m denotes the amount of loops that occur in C_1. Notice that $x_1 + x_2 + jx_3 \geq (j+1)x_4$ for $j = 1, \dots, m$ and $x_1 + x_2 + (m+1)x_3 \leq (m+2)x_4$.

Now, we proceed in two steps. Firstly, we focus on the evolution of a tuple of initial conditions by concatenation of routes and we determine how the tuple will finish at the end of the circuit and the condition that must be fulfilled to have periodicity. In the second step, we deal with the periods of the periodic sequences satisfying Condition U.

Step 1 : Consider the set of initial conditions (x_1, x_2, x_3, x_4) satisfying $x_1 \geq x_4 \geq x_2 \geq x_3$, that is, Case C_4. Assume that they generate a periodic orbit under Equation (4.2) verifying Condition U. Observe that the cycle determined by the periodic sequence will be formed by a concatenation of routes R_i. In this sense, if we denote by A_i the number of times that each route R_i appears, $i = 1, 2, 3, 4$, we can specify the accurate evolution of the initial terms along the cycle. Beforehand, in the evolution of the tuple in each route R_i, observe that the second term is the only one that varies at the end of the routes. So, for the sake of brevity, in the sequel we only write the evolution of the second term. The study of this fact, which is based on induction, is mechanical and is omitted.

As a first step, we begin studying the evolution of the initial conditions after A_1 routes of R_1. In this scenario, x_2 becomes into

$$\tilde{x}_2 = A_1 \cdot x_1 + x_2 - (m_1 + \ldots + m_{A_1} + 2 \cdot A_1) \cdot (x_4 - x_3),$$

being m_i $(i = 1, \ldots, A_1)$ the number of loops in C_1 that occurs in each route.

Secondly, the evolution after A_2 routes of R_2 transforms x_2 into

$$\tilde{x}_2 = A_2 \cdot x_1 + x_2 - A_2 \cdot (x_4 - x_3).$$

Next, we focus on the case of A_3 routes of the type R_3. Let denote $\tilde{m}_i$ $(i = 1, \ldots, A_3)$ and $\tilde{n}_j$ $(j = 1, \ldots, A_3)$ as the numbers of loops in C_1 and C_3, respectively. Then, x_2 evolves to

$$\tilde{x}_2 = A_3 \cdot x_1 + x_2 - (\tilde{m}_1 + \ldots + \tilde{m}_{A_3} + \tilde{n}_1 + \ldots + \tilde{n}_{A_3} + 3 \cdot A_3) \cdot (x_4 - x_3).$$

Finally, we study the case of A_4 routes of the type R_4. In what follows, n_i $(i = 1, \ldots, A_4)$ denotes the number of loops in C_3 that occurs in each route. Then, x_2 becomes into

$$\tilde{x}_2 = A_4 \cdot x_1 + x_2 - (n_1 + \ldots + n_{A_4} + 2 \cdot A_4) \cdot (x_4 - x_3).$$

Once we have detailed each particular case, we gathered the above analysis in order to give the evolution of the tuple in the general case, which means, that the cycle is determined by a finite concatenation of routes R_i.

Corollary 8. *If the periodic sequence (x_n) follows A_1 routes R_1, A_2 routes R_2, A_3 routes R_3 and A_4 routes R_4, the second term will end as:*

$$\tilde{x}_2 = (A_1 + A_2 + A_3 + A_4) \cdot x_1 + x_2 - (H + 2A_1 + 2A_4 + A_2 + 3 \cdot A_3) \cdot (x_4 - x_3),$$

where $H = m_1 + \ldots + m_{A_1} + n_1 + \ldots + n_{A_4} + \tilde{m}_1 + \ldots + \tilde{m}_{A_3} + \tilde{n}_1 + \ldots + \tilde{n}_{A_3}$.

Notice that the order of the routes does not influence in the period of the sequence. For example, the final effect of R_1, R_3, R_3, R_4 is the same as R_4, R_3, R_1, R_3.

Next, periodicity will be achieved if the second term equals x_2, so by Corollary 8,

$$(A_1 + A_2 + A_3 + A_4) \cdot x_1 + x_2 - (H + 2A_1 + 2A_4 + A_2 + 3 \cdot A_3) \cdot (x_4 - x_3) = x_2,$$

or, equivalently,

$$x_1 = \frac{H + 2A_1 + 2A_4 + A_2 + 3A_3}{A_1 + A_2 + A_3 + A_4} \cdot (x_4 - x_3). \tag{4.6}$$

Then, taking into account the length of each route R_i, and that Condition U is verified, the period of the sequence, N, is determined by the minimal concatenation of routes needed to achieve periodicity (A_j routes R_j, $j = 1, 2, 3, 4$):

$$\begin{aligned} N &= A_1 \cdot |R_1| + A_2 \cdot |R_2| + A_3 \cdot |R_3| + A_4 \cdot |R_4| \\ &= (A_1 + A_2 + A_3 + A_4) \cdot 10 + (3A_1 + 2A_2 + 4A_3 + 3A_4 + H) \cdot 11. \end{aligned}$$

We call $A := A_1+A_2+A_3+A_4$ and $B := 3A_1+2A_2+4A_3+3A_4+H$, so $N = 10\cdot A+11\cdot B$ with $A, B \in \mathbb{N}$.

Step 2 : Finally, this step delves into the possible periods of Equation (4.2) and we provide a necessary condition for a number to belong to its set of periods, $\text{Per}(F_4)$. To accomplish that, we base our study in the evolution of an orbit through the different routes R_i developed previously. Nevertheless, before doing so, we homogenize the notation by denoting

$$\delta_1 = m + 2, \quad \delta_2 = 1, \quad \delta_3 = m + n + 3, \quad \delta_4 = n + 2,$$

where as usual m and n represent the number of loops that take place in C_1 and C_3, respectively. In this direction, after a route R_i, the second term of the initial conditions will evolve to

$$\tilde{x}_2 = x_1 + x_2 - \delta_i(x_4 - x_3). \tag{4.7}$$

Furthermore, we can give the length of each route R_i in terms of the above δ_i; concretely,

$$|R_i| = 32 + 11 \cdot (\delta_i - 1), \quad i = 1, 2, 3, 4.$$

Proposition 30. *Let p, q be natural numbers, with $q \geq 2p + 1$ and $\gcd(p, q) = 1$. Let (x_1, x_2, x_3, x_4) be a tuple of real numbers verifying Condition U such that $x_1 = \frac{q-p}{p}(x_4 - x_3)$ and $x_1 > x_4 > x_2 > x_3$ (Case C_4). Then, these initial conditions determine under Equation (4.2) a periodic sequence described by p routes R_i. Furthermore, its period is $N = 10\cdot p+11\cdot q$.*

Proof. Let us consider the set of initial conditions (x_1, x_2, x_3, x_4) verifying Condition U and let us focus on its iterations under Equation (4.2). After A routes R_i, recall that, from Step 1, the second term is the only one that varies with respect to the initial tuple; concretely, x_2 evolves to

$$Ax_1 + x_2 - \sum_{j=1}^{A} \delta^j (x_4 - x_3),$$

where δ^j denotes the δ_i associated to the j-th route of the cycle, with $j = 1, \ldots, A$. Bearing this in mind, the corresponding sequence (x_n) generated from those initial conditions will be periodic if the equality $Ax_1 + x_2 - \sum_{j=1}^{A} \delta^j (x_4 - x_3) = x_2$ holds, or, equivalently,

$$Ax_1 = \sum_{j=1}^{A} \delta^j (x_4 - x_3).$$

Now, applying the hypothesis $x_1 = \frac{q-p}{p}(x_4 - x_3)$, jointly with the inequality $x_4 > x_3$ (otherwise, $x_4 = x_2 = x_3$ and we would have monotonic initial conditions yielding to an 11-cycle), we get

$$A \cdot \frac{q-p}{p} = \sum_{j=1}^{A} \delta^j. \tag{4.8}$$

It is direct to see that A is a multiple of p. Indeed, if $p = 1$, then the claim is trivial. On the contrary, if $p \neq 1$, observe that in Equation (4.8), the right-hand term, $\sum_{j=1}^{A} \delta^j$ is a natural number, while in the left-hand term, the factor $\frac{q-p}{p}$ is not, since $\gcd(p, q) = 1$. Therefore, we deduce that A is a multiple of p. Furthermore, we show that $A = p$, but, previously, we ensure the periodic character of the sequence.

To see that the sequence is periodic, we focus again on how the initial conditions evolve under each route R_i. Observe that, independently of the route, we start and end in the Case C_4, so its conditions must be fulfilled. In concrete, the second term of the corresponding tuple must be less than or equal to the fourth, that is,

$$x_1 + x_2 - \delta_i(x_4 - x_3) \leq x_4.$$

In the same way as before, we apply that $x_1 = \frac{q-p}{p}(x_4 - x_3)$, add and subtract the third term x_3 to get

$$\frac{q-p}{p}(x_4 - x_3) + x_2 - (\delta_i + 1)(x_4 - x_3) - x_3 \leq 0.$$

If we divide by $x_4 - x_3$ and we denote $\mu := \frac{x_2 - x_3}{x_4 - x_3} \in (0, 1)$, then, for every route R_i, we obtain

$$\frac{q-p}{p} + \mu - 1 \leq \delta_i. \tag{4.9}$$

Next, we find an upper bound for the integer δ_i, $i = 1, \ldots, 4$. Again, to do so, we analyze the diverse routes R_i. Firstly, we deal with R_1 and R_3. In both cases, let us consider the tuple obtained at the end of the route, $\big(x_1, x_1 + x_2 - \delta_i(x_4 - x_3), x_3, x_4\big)$. If we do 11 iterations backwards, it is easy to see that such tuple comes from

$$\big(x_1, x_1 + x_2 - (\delta_i - 1)(x_4 - x_3), x_3, x_4\big),$$

which verifies the conditions from Case C_1. Hence, the restrictions from this case must hold and, in particular, the second term must be greater than or equal to the fourth, which implies

$$x_1 + x_2 - (\delta_i - 1)(x_4 - x_3) \geq x_4.$$

Now, we proceed similarly to inequality (4.9) and we get

$$\frac{q-p}{p}(x_4 - x_3) + x_2 - (\delta_i - 1)(x_4 - x_3) - x_4 \geq 0,$$

$$\frac{q-p}{p}(x_4 - x_3) - \delta_i(x_4 - x_3) + x_2 - x_3 \geq 0,$$

and dividing by $x_4 - x_3$,

$$\delta_i \leq \frac{q-p}{p} + \mu. \tag{4.10}$$

Moreover, for R_2, $\delta_i = \delta_2 = 1$ and $\frac{q-p}{p} \geq 1$ due to the condition $q \geq 2p+1$, so inequality (4.10) is valid for this route too.

Finally, we focus in the route R_4. Here, from the final tuple

$$\big(x_1, x_1 + x_2 - \delta_i(x_4 - x_3), x_3, x_4\big),$$

we iterate backwards 10 times to get $\big(x_1, x_3, x_4, x_2 - (\delta_i - 1)(x_4 - x_3)\big)$ in Case C_2. Again, we iterate backwards to obtain the tuple in Case C_3, that is,

$$\big(x_1, x_3, x_4, x_2 - (\delta_i - 2)(x_4 - x_3)\big).$$

Now, the restrictions of Case C_3 imply that the sum of the second and fourth term must be greater than or equal to the third, so

$$x_2 - (\delta_i - 2)(x_4 - x_3) + x_3 \geq x_4.$$

This implies that

$$x_2 + x_4 - x_3 \geq \delta_i(x_4 - x_3),$$

but, since $x_1 > x_2$, we get $x_1 + x_4 - x_3 \geq \delta_i(x_4 - x_3)$ and, proceeding similarly to the previous cases, we obtain

$$\frac{q-p}{p}(x_4 - x_3) + x_4 - x_3 \geq \delta_i(x_4 - x_3),$$

and

$$\frac{q-p}{p} + 1 \geq \delta_i. \tag{4.11}$$

In conclusion, for every route R_i, we can unify inequalities (4.9), (4.10) and (4.11) as

$$\frac{q-p}{p} + \mu - 1 \leq \delta_i \leq \frac{q-p}{p} + 1.$$

Observe that δ_i can only admit two possible values, since it is a natural number and $\mu \in (0,1)$. Concretely, $\delta_i = \lfloor \frac{q-p}{p} \rfloor$ or $\delta_i = \lfloor \frac{q-p}{p} \rfloor + 1$. Hence, after a route R_i, considering (4.7), and the hypothesis $x_1 = \frac{q-p}{p}(x_4 - x_3)$, the second term x_2 evolves either to

$$\tilde{x}_2 = x_2 + \left(\frac{q-p}{p} - \left\lfloor \frac{q-p}{p} \right\rfloor\right)(x_4 - x_3), \tag{4.12}$$

or either to

$$\tilde{x}_2 = x_2 + \left(\frac{q-p}{p} - \left\lfloor \frac{q-p}{p} \right\rfloor - 1\right)(x_4 - x_3). \tag{4.13}$$

Now, we apply the fact that $n - \lfloor n \rfloor = \{n\}$, for every $n \in \mathbb{R}$. Then,

$$\frac{q-p}{p} - \left\lfloor \frac{q-p}{p} \right\rfloor = \left\{\frac{q-p}{p}\right\} = \frac{r}{p},$$

where r is the rest of the division of $q-p$ by p. This guarantees the simplification of (4.12) and (4.13) as

$$\tilde{x}_2 = x_2 + \frac{r}{p}(x_4 - x_3) \quad \text{or} \quad \tilde{x}_2 = x_2 - \left(1 - \frac{r}{p}\right)(x_4 - x_3). \tag{4.14}$$

At this point, a natural question arises, how do we know if after a route the term $\tilde{x}_2$ is defined by one expression or the other? The answer resides in the fact that the new tuple $(x_1, \tilde{x}_2, x_3, x_4)$ is in Case C_4, so, in particular, $x_4 \geq \tilde{x}_2 \geq x_3$. However, these inequalities are only true for one and only one of the two expressions in (4.14). Indeed:

- If $x_4 \geq x_2 + \frac{r}{p}(x_4 - x_3)$, then $x_3 \geq x_2 - \left(1 - \frac{r}{p}\right)(x_4 - x_3)$. Thus, the conditions of Case C_4 are not verified by the second expression in (4.14). Observe that $x_2 + \frac{r}{p}(x_4 - x_3) \geq x_3$ holds trivially.

- If $x_2 + \frac{r}{p}(x_4 - x_3) > x_4$ (notice that in this case the first expression in (4.14) does not satisfy the conditions of Case C_4), then $x_2 + \frac{r}{p}(x_4 - x_3) > x_4 - x_3 + x_3$; and $x_2 - \left(1 - \frac{r}{p}\right)(x_4 - x_3) > x_3$. Consequently, the second option verifies the conditions of Case C_4, since $x_4 \geq x_2 - \left(1 - \frac{r}{p}\right)(x_4 - x_3)$ holds trivially.

In conclusion, after one route R_1, we have

$$\tilde{x}_2 = \begin{cases} x_2 + \frac{r}{p} \cdot d, & \text{if } x_2 + \frac{r}{p} \cdot d \leq x_4, \\ x_2 + \frac{r}{p} \cdot d - d, & \text{otherwise,} \end{cases} \tag{4.15}$$

where $d = x_4 - x_3$. Thus, after p routes, the second term $\tilde{x}_2$ evolves to $x_2 + p \cdot \frac{r}{p} \cdot d - \alpha \cdot d$, with $\alpha \in \mathbb{Z}$ and $0 \leq \alpha \leq p$, where α represents the number of times that $\tilde{x}_2$ takes the second value of (4.15) after one route. Moreover, since the final tuple is in Case C_4, we have

$$x_3 \leq x_2 + r \cdot d - \alpha \cdot d \leq x_4,$$

or, equivalently,

$$\alpha \cdot d \leq x_2 - x_3 + r \cdot d \leq (\alpha + 1) \cdot d.$$

Dividing by d,

$$\alpha \leq \frac{x_2 - x_3}{d} + r \leq \alpha + 1,$$

with $0 \leq \frac{x_2 - x_3}{d} \leq 1$. From here, since α and r are integers, we get $\alpha = r$, except when $x_2 = x_3$, where apart from $\alpha = r$, it can also be hold $\alpha = r - 1$. Nevertheless, in this last case, $x_2 = x_3$ would imply that the set of initial conditions would satisfy simultaneously Cases C_4 and C_5, which is a contradiction with the hypothesis of the ordering of the initial conditions. So $\alpha = r$.

In this scenario, after p routes, the second term will verify $x_2 + p \cdot \frac{r}{p} \cdot d - rd = x_2$, which means that the initial conditions generate a periodic sequence after p routes. Additionally, since we have seen that the amount of routes A needed to achieve periodicity must be a multiple of p, we have $A = p$.

As a final step, we determine the period of the sequence. To carry out this task, recall that the length of each route R_i is given by $32 + 11 \cdot (\delta_i - 1)$. Therefore, as we have obtained periodicity after p routes R_i, the period must be

$$N = \sum_{j=1}^{p} \big(32 + 11 \cdot (\delta^j - 1)\big) = 21p + 11 \sum_{j=1}^{p} \delta^j, \tag{4.16}$$

where δ^j denoted the δ_i associated to the j-th route of the cycle. This sum can be computed easily, because after $A = p$ routes, the second term is $p \cdot x_1 + x_2 - \sum_{j=1}^{p} \delta^j (x_4 - x_3)$ and if we equal such expression to x_2 in order to have periodicity, we get

$$p \cdot x_1 = \sum_{j=1} \delta^j (x_4 - x_3).$$

Then,

$$p \cdot \frac{q-p}{p}(x_4 - x_3) = \sum_{j=1}^{p} \delta^j (x_4 - x_3) \quad \text{and} \quad \sum_{j=1}^{p} \delta^j = q - p.$$

Finally, we substitute it in (4.16) and we achieve that the period of the sequence is

$$N = 21 \cdot p + 11 \sum_{j=1}^{p} \delta^j = 21 \cdot p + 11 \cdot (q - p) = 10 \cdot p + 11 \cdot q.$$

□

The description made in Proposition 30 provides a necessary condition for a certain number to be in the set of periods, $\mathrm{Per}(F_4)$, under the restriction that Condition U holds and the period is achieved through a concatenation of routes R_i.

Proposition 31. *Let N be a natural number. If N is the prime period of a periodic sequence described by routes R_i and whose initial conditions verify Condition U, then N admits a decomposition $N = 10 \cdot a + 11 \cdot b$ with $a, b \in \mathbb{N}$ such that $\gcd(a, b) = 1$ and $b \geq 2a + 1$.*

Proof. Let us consider a periodic sequence, (x_n), described by routes R_i and whose initial conditions verify Condition U. Let N be its prime period. By Corollary 8 and Equation (4.6), we can assume that N admits a decomposition of the form $N = 10 \cdot a + 11 \cdot b$, with $a, b \in \mathbb{N}$. Specifically, $a = A_1 + A_2 + A_3 + A_4$ and $b = 3A_1 + 2A_2 + 4A_3 + 3A_4 + H$, where $A_i, i = 1, \ldots, 4$, and H are the ones described in (4.6). So, our target is to prove that a and b are coprime numbers verifying $b \geq 2a + 1$.

Firstly, we can rewrite $b = 2a + A_1 + A_4 + 2A_3 + H$. We claim that $A_1 + A_4 + 2A_3 + H \geq 1$. Indeed, since $A_i \geq 0$ for $i = 1, \ldots, 4$, and $H \geq 0$, if $A_1 + A_4 + 2A_3 + H = 0$, it would mean that the cycle is only formed by A_2 routes R_2. However, in this scenario, the initial conditions would evolve to $\big(x_1, A_2 \cdot x_1 + x_2 + A_2 \cdot (x_3 - x_4), x_3, x_4\big)$, as can be easily seen by induction. Hence, periodicity would hold if $A_2 \cdot x_1 + x_2 + A_2 \cdot (x_3 - x_4) = x_2$, and we would deduce $x_1 + x_3 = x_4$. Bearing in mind that x_1 is the maximum element of the periodic sequence and that the initial conditions are non-negative, it follows $x_3 = 0$ and $x_1 = x_4$. In conclusion, the initial conditions would reduce to $(x_1, x_2, 0, x_1)$, which generate an 8-cycle by Proposition 21, or the equilibrium point if $x_1 = 0$. Therefore, $A_1 + A_4 + 2A_3 + H \geq 1$ and we get $b \geq 2a + 1$.

As a final step, we show that a and b are coprime numbers reasoning by contradiction. In this sense, suppose that $\gcd(a, b) = d > 1$. Then, there exist natural numbers p, q such that $a = d \cdot p$ and $b = d \cdot q$. Regarding that the orbit of (x_n) is a concatenation of routes R_i, Equation (4.6) implies that $x_1 = \frac{b-a}{a}(x_4 - x_3)$. Thus, the initial conditions are $\big(\frac{b-a}{a}(x_4 - x_3), x_2, x_3, x_4\big)$. Nevertheless, $\frac{b-a}{a} = \frac{d \cdot q - d \cdot p}{d \cdot p} = \frac{q-p}{p}$, and the initial terms read as $\Big(\frac{q-p}{p}(x_4 - x_3), x_2, x_3, x_4\Big)$, with $x_4 \geq x_2 \geq x_3 \geq 0$. Here, $\gcd(p, q) = 1$ and $q \geq 2p + 1$, since $b \geq 2a + 1$. Then, from Proposition 30, it follows that those initial conditions generate under Equation (4.2) a periodic sequence whose period is $N_1 = 10 \cdot p + 11 \cdot q$. Observe that $N_1 < N$, since $d \neq 1$ and $d \cdot N_1 = 10 \cdot p \cdot d + 11 \cdot q \cdot d = 10 \cdot a + 11 \cdot b = N$. So, (x_n) is periodic with period $N_1 < N$, but we had supposed that N was the prime period, a contradiction. Definitely, $\gcd(a, b) = 1$ is a necessary condition for $N \in \mathrm{Per}(F_4)$. $\square$

Part VI - Proof of Theorem D

We finish this section by proving its main result, namely, Theorem D. In it, we provide a precise description of the set of periods $\mathrm{Per}(F_4)$ for Equation (4.2). Observe that we can do that since in the previous parts we have described the diverse routes determined by Figure 4.1, the controversial cases, and the intersection between Cases C_i.

Theorem D. *Consider Equation* (4.2) *and let* $\mathrm{Per}(F_4)$ *be its set of periods. Then*

$$\mathrm{Per}(F_4) = \{1, 8, 11\} \bigcup \big\{10 \cdot a + 11 \cdot b \mid \gcd(a, b) = 1, a \geq 1, b \geq 2a + 1\big\}.$$

Proof. As a first step, from Proposition 23, we have $\mathrm{Per}(F_4) \cap [1, 11] = \{1, 8, 11\}$. Furthermore, in Part II, III, IV and V, we have developed the precise description of $\mathrm{Per}(F_4)$ for periods $N > 11$ by analyzing the controversial cases, the intersection between cases, and the initial conditions satisfying Condition U.

Specifically, Corollaries 6 and 7 establish the behaviour of the periodic sequences generated by the controversial cases $(x, y, 0, z)$ and $(x, z, y, 0)$, with $x \geq y \geq z > 0$, respectively. In addition, Part IV deals with the intersection between cases, where we were able to reduce that scenario to the casuistic of controversial cases. Moreover, Proposition 31 focuses on the periods of the periodic sequences generated by initial conditions holding Condition U.

Finally, by Proposition 30, Remarks 5 and 8, jointly with Proposition 23, we conclude that $\mathrm{Per}(F_4)$ is precisely

$$\{1, 8, 11\} \bigcup \big\{10 \cdot a + 11 \cdot b \mid \gcd(a, b) = 1, a \geq 1, b \geq 2a + 1\big\}.$$

This ends the proof of the theorem.

□

We must emphasize that we have not only made a precise description of $\mathrm{Per}(F_4)$, rather we have obtained a way to generate cycles of any period in $\mathrm{Per}(F_4)$. Indeed, given a period $N \in \mathrm{Per}(F_4)$, it is enough to consider its decomposition $N = 10 \cdot a + 11 \cdot b$, with $\gcd(a, b) = 1$ and $b \geq 2a + 1$, since Proposition 30 or Propositions 26 or 27 provide us the appropriate information to choose the correct initial conditions.

4.2 The maximum of the complementary of the set of periods

The characterization of the set of periods, $\mathrm{Per}(F_4)$, of Equation (4.2) has been completely done in the previous section. In this sense, we have been able to see that the periods are appropriate combinations of multiples of 10 and 11, see Theorem D.

Let us consider the set $\mathrm{Per}(F_4)$, which in this section is renamed as $\mathcal{A}$ for convenience

$$\mathcal{A} = \big\{10 \cdot a + 11 \cdot b \mid \gcd(a, b) = 1, a \geq 1, b \geq 2a + 1\big\}.$$

Here we ask for the existence of a number M such that every integer greater than M belongs to $\mathcal{A}$. In this direction, we see that such number exists and it is $M = 1674$. This means that every integer number greater than 1674 is a period of Equation (4.2). The work presented in this section is based on [66].

Before tackling the mentioned problem, observe that if we had not had the restriction $b \geq 2a + 1$, we would have been facing the Diophantic Frobenius Problem. Specifically, this problem, that is also known as Coin Problem, lies in determining the largest number that cannot be represented as a linear combination with positive coefficients from a given set $(a_1, \ldots, a_n)$ of natural numbers with $\gcd(a_1, \ldots, a_n) = 1$. Particularly, for two coprime numbers (a_1, a_2), the largest number is $(a_1 - 1)(a_2 - 1) - 1$ (consult [91]). In our case, since $a_1 = 10$ and $a_2 = 11$, such number would be 89. Nevertheless, the additional restriction $b \geq 2a + 1$ hinders the problem.

We begin by studying the prime numbers and the multiples of 11 that belong to $\mathcal{A}$. In this line, let us denote the set of prime numbers by $\mathcal{P}$. Now, we form a list gathering the elements of $\mathcal{P}$ until 401 and we encircle those admitting a decomposition $p = 10 \cdot a + 11 \cdot b$ with $\gcd(a, b) = 1$ and $b \geq 2a + 1$: $2, 3, 5, 7$, $\textcircled{11}$, $13, 17, 19, 23, 29, 31, 37, 41$, $\textcircled{43}$,

$47, 53, 59, 61, 67, 71, 73, 79, 83,\ 89,\ \textcircled{97},\ 101, 103,\ \textcircled{107},\ \textcircled{109},\ 113, 127,\ \textcircled{131},\ 137,\ \textcircled{139},\ 149,$
$\textcircled{151},\ 157,\ \textcircled{163},\ 167,\ \textcircled{173},\ 179, 181, 191,\ \textcircled{193},\ \textcircled{197},\ 199, 211, 223,\ \textcircled{227},\ \textcircled{229},\ 233,\ \textcircled{239},\ \textcircled{241},$
$\textcircled{251},\ \textcircled{257},\ \textcircled{263},\ \textcircled{269},\ \textcircled{271},\ 277,\ \textcircled{281},\ \textcircled{283},\ \textcircled{293},\ \textcircled{307},\ \textcircled{311},\ \textcircled{313},\ \textcircled{317},\ \textcircled{331},\ \textcircled{337},\ \textcircled{347},$
$\textcircled{349},\ \textcircled{353},\ \textcircled{359},\ \textcircled{367},\ \textcircled{373},\ \textcircled{379},\ \textcircled{383},\ \textcircled{389},\ \textcircled{397},\ \textcircled{401}.$

For example, $383 \in \mathcal{A}$ since $383 = 20 + 363 = 10 \cdot 2 + 11 \cdot 33$, whereas $277 \notin \mathcal{A}$ as the decompositions $277 = 90 + 187 = 10 \cdot 9 + 11 \cdot 17$ and $277 = 200 + 77 = 10 \cdot 20 + 11 \cdot 7$ are not allowed, since $b < 2a + 1$ in both cases. It must be underlined that we have highlighted every prime number greater than 277 as the following result confirms.

Proposition 32. *If $p \in \mathcal{P}$, with $p \geq 281$, then $p \in \mathcal{A}$.*

Proof. Notice that, following the previous list, we can assume that $p \geq 401$. Observe that p can be written in the form $p = (p - 10m) + 10m$, where m is the smallest positive integer (in fact, unique, with $m \in \{1, 2, \ldots, 11\}$) holding that $p - 10m$ is a multiple of 11. Then $p = 11 \cdot \left(\frac{p-10m}{11}\right) + 10 \cdot m$. Let $d = \gcd\left(\frac{p-10m}{11}, m\right)$. Hence, on the one hand, p is a multiple of d since d divides $\frac{p-10m}{11}$ and m; on the other hand, $d \leq m$, and p being prime with $p > 12$, imply that $d = 1$. Finally, we have to prove that $\frac{p-10m}{11} \geq 2m + 1$. This is equivalent to see that $p \geq 32m + 11$, which obviously occurs because $m \leq 11$ and $p \geq 363$. $\square$

As a next step, we analyze the multiples of prime numbers that might belong to $\mathcal{A}$.

Proposition 33. *Let $p \in \mathcal{P} \cap \mathcal{A}$, with $p \geq 43$. Assume that $p = 10a + 11b$, with $\gcd(a, b) = 1$, $b \geq 2a + 1$, $a \geq 1$. Then:*

(a) $pq \in \mathcal{A}$ for all $q \geq 1$ with $\gcd(p, q) = 1$ and $aq \geq 12, (b - 2a)q \geq 33$. In particular, $pq \in \mathcal{A}$ for all $q \geq 33$ with $\gcd(p, q) = 1$.

(b) $p^k q \in \mathcal{A}$ for all $k \geq 2$ and for all $q \geq 1$ with $\gcd(p, q) = 1$.

Proof. We start proving Part (a). Firstly, notice that for every integer r, with $aq \geq 1 + 11r$, we can write

$$pq = (10a + 11b) \cdot q = 10 \cdot (aq) + 11 \cdot (bq) = 10 \cdot (aq - 11r) + 11 \cdot (bq + 10r). \qquad (4.17)$$

Take $r = 1$, that is, $pq = 10 \cdot (aq - 11) + 11 \cdot (bq + 10)$. Obviously, $bq + 10 \geq 2\,(aq - 11) + 1$. Let $d_1 = \gcd\,(aq - 11, bq + 10)$. Observe that d_1 divides pq. If d_1 contains some divisor q' of q, then $aq - 11$ and $bq + 10$ will be multiples of q'; then, q' will be a common divisor of 10 and 11, so $q' = 1$. Consequently, d_1 only contains divisors of p, and being p prime, we deduce that $d_1 \in \{1, p\}$. Now, we analyze both cases:

- If $d_1 = 1$, it is enough to impose the additional property of being $aq \geq 12$.

- If $d_1 = p$, we can deduce that d divides aq and bq because $\gcd(p, q) = 1$ and d divides p. Therefore, according to the above comment on the divisors of q, d is a common divisor of a and b. As a final step, bearing in mind that $\gcd(a, b) = 1$, we conclude that $d = 1$, and $pq \in \mathcal{A}$ whenever $aq \geq 1 + 11p$.

Next, we improve our bound for the values q by using the following approach. We take $r = -1$ in (4.17) to set $pq = 10 \cdot (aq + 11) + 11 \cdot (bq - 10)$. Observe that we must have $bq - 10 \geq 2aq + 23$, that is, $bq \geq 2aq + 33$. Put $d_2 = \gcd(aq + 11, bq - 10)$. Analogously to the above study for d_1, it is easy to deduce that $d_2 \in \{1, p\}$.

- If $d_2 = 1$, the proof follows by requiring $aq \geq 12$ and $bq \geq 2aq + 33$ (in particular, if $q \geq 33$, we simultaneously get the two inequalities).

- If $d_2 = p$, we would obtain $aq - 11 = pu_1$ and $aq + 11 = pu_2$ for some positive integers u_1, u_2, and then $22 = p(u_2 - u_1)$, which would imply that p divides 2 or 11, contrary to the fact that $p \geq 43$.

Now, we proceed to prove Part (b). For $k \geq 2$, we write

$$p^k q = pp^{k-1} q = (10a + 11b)p^{k-1} q = 10 \cdot (ap^{k-1} q) + 11 \cdot (bp^{k-1} q),$$

or, equivalently,

$$p^k q = 10 \cdot (ap^{k-1} q - 11) + 11 \cdot (bp^{k-1} q + 10).$$

Denote $a' = ap^{k-1}q - 11, b' = bp^{k-1}q + 10$. It is direct to check that $b' \geq 2a' + 1$. Set $d = \gcd(a', b')$. We show that $d = 1$, which will end the proof.

If d contains some divisor q' of q in its factorization, then a' and b' are multiples of q', thus q' is a common divisor of 10 and 11. Therefore, $q' = 1$ and d divides p^k. If p divides d, then p would divide both a' and b', and consequently 10 and 11 are multiples of p (notice that $k - 1 \geq 1$), thus $p = 1$, a contradiction and we conclude that $d = 1$. □

Notice that, although $p = 11$ is a period of Equation (4.2), the previous result excludes it from the set $\mathcal{A}$. Now, we establish the numbers of the form $11^k q$, $k \geq 1$, and $\gcd(11, q) = 1$, that belong to $\mathcal{A}$.

Proposition 34. *The following statements hold true:*

(a) $11^k q \in \mathcal{A}$ *for all* $k \geq 3$ *and for all* $q \geq 1$ *with* $\gcd(11, q) = 1$.

(b) $11^2 q \in \mathcal{A}$ *for all* $q \geq 3$ *with* $\gcd(11, q) = 1$. *Furthermore,* 11^2 *and* $11^2 \cdot 2$ *do not belong to* $\mathcal{A}$.

(c) $11q \in \mathcal{A}$ *whenever* $\gcd(11, q) = 1$ *and*

$$q \in \{1\} \cup \big(\{q : q \geq 33\} \setminus \{43, 54, 76, 120\}\big).$$

Proof. We begin dealing with statements (a) and (b). In the cases $k \geq 3$ or $k = 2, q \geq 3$, the inequalities $11^{k-1}q - 10 > 0$ and $11^{k-1}q - 10 \geq 23$ are true, so we can decompose the numbers of the form $11^k q$ as follows:

$$11^k q = 11 \cdot \left(11^{k-1}q - 10\right) + 10 \cdot 11.$$

Therefore, it is easily seen that $\gcd\left(11, 11^{k-1}q - 10\right) = 1$ and $11^k q \in \mathcal{A}$. In addition, $11^2 = 110 + 11$ and $11^2 \cdot 2 = 110 + 132 = 220 + 20$, from where we deduce that those numbers do not belong to $\mathcal{A}$ and we have completely proved Part (a) and (b).

Now we proceed with Part (c). To do so, we decompose the numbers of the form $11q$ and study several cases.

- Assume $1 \leq q \leq 10$. Here, the only decomposition allowed is $11q = 11q + 10 \cdot 0$. Hence, these numbers are excluded from $\mathcal{A}$ with the exception of 11.

- Suppose $11 \leq q \leq 32$. Then, we can decompose the elements of the form $11q$ in three different ways: $0 + 11q$, $10 \cdot 11 + 11(q-10)$, and $10 \cdot 22 + 11(q-20)$; nevertheless, none of them are admissible as elements of $\mathcal{A}$, either because $\gcd(0, q) = q > 1$ or because the inequality $b \geq 2a + 1$ does not hold. In conclusion, $\mathcal{A}$ does not have any elements from $\{11q : 11 \leq q \leq 32\}$.

- Consider $q \geq 33$. We claim that every number of the form $11q$ belongs to $\mathcal{A}$, except for $q \in \{43, 54, 76, 120\}$. Indeed, we start showing that $11q \in \mathcal{A}$ whenever $q \geq 353$. In this sense, if q is a multiple of 11, it follows from Part (a) and (b); otherwise, we can write the decomposition $11q = 10 \cdot 121 + 11 \cdot (q - 110)$. Observe that $q - 110 \geq 2 \cdot 121 + 1$, since $q \geq 353$; and that $\gcd(121, q - 110) = 1$, due to the fact that q and 11 are coprime numbers. As a second step, we deal with the range $33 \leq q \leq 352$. Consider the decomposition $11q = 10 \cdot 11 + 11 \cdot (q - 10)$, which is allowed since $q - 10 \geq 23$. If $q - 10$ is not a multiple of 11, then $\gcd(11, q - 10) = 1$ and we finish. Otherwise, we have to study the case $q - 10 = 11m$, $33 \leq q \leq 352$. From here, we have the following list of candidates of the form $q = 10 + 11m$ to belong to $\mathcal{A}$:

$$\begin{aligned} C_q^0 : \;=\; & \big\{43, 54, 65, 76, 87, 98, 109, 120, 131, 142, 153, 164, 175, 186, 197, \\ & 208, 219, 230, 241, 252, 263, 274, 285, 296, 307, 318, 329, 340, 351\big\}. \end{aligned}$$

  For these elements $q \in C_q^0$, except 43, 54 (we need $q \geq 65$ in order to obtain elements of $\mathcal{A}$), we consider the decomposition $11q = 10 \cdot 22 + 11 \cdot (q - 20)$ and set $d_0 = \gcd(22, q - 20)$. Observe that if q is an odd number, then $q - 20$ so is, and we deduce that $d_0 \in \{1, 11\}$. In this direction, if d_0 is a multiple of 11, then 11 divides $q - 20$ or d_0 divides $11m - 10$; so 10 is a multiple of d_0, which is impossible. To sum up, $11q$ is an admissible number if q is odd, $65 \leq q \leq 353$. Therefore, we discard odd numbers q in C_q^0, except 43, and the set is reduced to

$$C_q^1 := \{43, 54, 76, 98, 120, 142, 164, 186, 208, 230, 252, 274, 296, 318, 340\}.$$

Now, we look into the even numbers of C_q^1, of type $54 + 22j, 0 \leq j \leq 13$, and consider the decomposition

$$11q = 10 \cdot 33 + 11 \cdot (q - 30) = 10 \cdot 33 + 11 \cdot (24 + 22j),\ j = 0, 1, \ldots, 13.$$

It must be highlighted that, since $q - 30 \geq 2 \cdot 33 + 1$ must be satisfied, we have to consider $q \geq 97$. If $d_1 = \gcd(33, 24 + 22j)$, since $24 + 22j = 11 \cdot (2 + 2j) + 2$ and $33 = 3 \cdot 11$, we infer that $d_1 \in \{1, 3\}$. Thus, if $24 + 22j$ is not a multiple of 3, we obtain that $d_1 = 1$ and the corresponding value $11q$ belongs to $\mathcal{A}$. Notice that this occurs when $j \notin \{0, 3, 6, 9, 12\}$. To sum up the previous reasoning, $11q$ is an element of $\mathcal{A}$, except, maybe, if q belongs to

$$C_q^2 := \{43, 54, 76, 120, 186, 252, 318\}.$$

As a next step, we consider the decomposition $11q = 10 \cdot 55 + 11 \cdot (q - 50)$. Here we need to assume that $q \geq 161$, so it only applies for the numbers $186, 252, 318$, where $q - 50$ is $136, 202, 268$, respectively. In the three cases we have $\gcd(55, q - 50) = 1$, so they belong to $\mathcal{A}$. This reduces the set C_q^2 into

$$C_q^3 := \{43, 54, 76, 120\}.$$

Finally, we check directly that neither of the possible decompositions $10a + 11b$ of the elements of C_q^3 satisfy simultaneously $\gcd(a, b) = 1$ and $b \geq 2a + 1$.

$$\begin{aligned}
11 \cdot 43 = 473 &= 10 \cdot 11 + 11 \cdot 33 = 10 \cdot 22 + 11 \cdot 23 \\
&= 10 \cdot 33 + 11 \cdot 13 = 10 \cdot 44 + 11 \cdot 3, \\
11 \cdot 54 = 594 &= 10 \cdot 11 + 11 \cdot 44 = 10 \cdot 22 + 11 \cdot 34 \\
&= 10 \cdot 33 + 11 \cdot 24 = 10 \cdot 44 + 11 \cdot 14 = 10 \cdot 55 + 11 \cdot 4, \\
11 \cdot 76 = 836 &= 10 \cdot 11 + 11 \cdot 66 = 10 \cdot 22 + 11 \cdot 56 = 10 \cdot 33 + 11 \cdot 46 \\
&= 10 \cdot 44 + 11 \cdot 36 = 10 \cdot 55 + 11 \cdot 26 = 10 \cdot 66 + 11 \cdot 16 \\
&= 10 \cdot 77 + 11 \cdot 6, \\
11 \cdot 120 = 1320 &= 10 \cdot 11 + 11 \cdot 110 = 10 \cdot 22 + 11 \cdot 100 = 10 \cdot 33 + 11 \cdot 90 \\
&= 10 \cdot 44 + 11 \cdot 80 = 10 \cdot 55 + 11 \cdot 70 = 10 \cdot 66 + 11 \cdot 60 \\
&= 10 \cdot 77 + 11 \cdot 50 = 10 \cdot 88 + 11 \cdot 40 = 10 \cdot 99 + 11 \cdot 30 \\
&= 10 \cdot 110 + 11 \cdot 20 = 10 \cdot 121 + 11 \cdot 10.
\end{aligned}$$

□

Now, we can deduce the biggest multiple of 11 not belonging to $\mathcal{A}$.

Corollary 9. *It holds that* $1320 = \max\{11 \cdot n : n \geq 1, 11 \cdot n \notin \mathcal{A}\}$.

After outlining the key properties of prime numbers and conducting a thorough examination of the multiples of 11 belonging to $\mathcal{A}$, our focus shifts to demonstrating that the set

$$\mathcal{NA} := \mathbb{N} \setminus \mathcal{A}$$

is bounded. Our goal, in fact, is to determine the maximum value within $\mathcal{NA}$. To do so, we divide the set of natural numbers, not multiple of 11, in ten different classes,

$$\mathcal{C}_m := \{10m + 11k,\ k \geq 0\},$$

where we fix the value $m \in \{1, 2, \ldots, 10\}$. For each class $\mathcal{C}_m$, we prove the boundedness of $\mathcal{NA} \cap \mathcal{C}_m$, and from the inspection of each subset $\mathcal{NA} \cap \mathcal{C}_m$ we deduce the maximum of $\mathcal{NA}$. To develop the examination of those classes, we employ the following fundamental fact:

> Given a natural number N not being a multiple of 11, there exists a unique $m \in \{1, 2, \ldots, 10\}$ such that $N - 10m$ is a multiple of 11.

The class $\mathcal{C}_1$: Observe that every number n that belongs to the class $\mathcal{C}_1$ can be written as $n = 10 + 11b$. Thus, we have $a = 1$ and the condition of a and b being coprime always holds. In this direction, we only have to force that $b \geq 2a + 1 = 3$. This implies that $n \in \mathcal{A}$ for all $n \geq 43$, and, therefore, $n = 10, 21, 32$ belong to $\mathcal{NA}$, which allow us to deduce that

$$N_1 := \max\{\mathcal{NA} \cap \mathcal{C}_1\} = 32.$$

The class $\mathcal{C}_2$: The class $\mathcal{C}_2$ is formed by the numbers n that can be decompose as $n = 20 + 11b, b \geq 0$. So, $a = 2$ and we distinguish two cases depending on the parity of b.

- Assume that b is odd, with $b \geq 5$. In this case, $n \in \mathcal{A}$ because $\gcd(a, b) = 1$. Additionally, we get that 31 and 53 belong to $\mathcal{NA}$.

- Let b be an even number, $b = 2j$, with $j \geq 1$ (if $j = 0$ we have $n = 20 \in \mathcal{NA}$). Notice that here $\gcd(a, b) = 2$, so we need to handle another decomposition:

  $$n = 20 + 11b = 130 + 11b - 110 = 130 + 11 \cdot (b - 10) = 10 \cdot 13 + 11 \cdot (2j - 10).$$

  Taking into account the condition $b \geq 2a+1$, we have that the previous decomposition will be valid if $2j - 10 \geq 27$. Hence, for $j \leq 18$, every number of the form $n =$

$130 + 11 \cdot (2j - 10)$ belong to $\mathcal{NA}$. For instance, $416 \in \mathcal{NA}$ since it admits the following decompositions

$$416 = 10 \cdot 2 + 11 \cdot 36 = 10 \cdot 13 + 11 \cdot 26 = 10 \cdot 24 + 11 \cdot 16 = 10 \cdot 35 + 11 \cdot 6,$$

and neither verifies simultaneously the two conditions $\gcd(a', b') = 1, b' \geq 2a' + 1$ on the coefficients of $10a' + 11b'$.

Let us continue with the study of the decomposition $n = 10 \cdot 13 + 11 \cdot (2j - 10)$, $j \geq 19$. Firstly, if $j - 5$ is not a multiple of 13, we can ensure that n belongs to $\mathcal{A}$. Otherwise, $j = 5 + 13k$, with $k \geq 2$ and, consequently, we can write $n = 10 \cdot 13 + 11 \cdot (26k)$. Here, 26 divides n, so we must modify the decomposition as $n = 10 \cdot 35 + 11 \cdot (26k - 20)$.

⋆ If 5 nor 7 divide $26k - 20$, we get that the coefficients are coprime. Therefore, if $26k - 20 \geq 2 \cdot 35 + 1 = 71$, then $n \in \mathcal{A}$. It is direct that such inequality holds for $k \geq 4$, and the remaining cases, $k = 2, 3$, yield to the numbers $n = 702, 988$, which belong to $\mathcal{NA}$.

⋆ If 5 or 7 divide $26k - 20$, with $k \geq 5$, observe that we could divide n by $26 \cdot 5$ or $26 \cdot 7$. Moreover, realize that in the sequence

$$\left\{x_r^{(2)} : r \geq 0\right\} = \{13 + 11r : r \geq 0\}$$

we find $x_{36}^{(2)} = 409$, a prime number in $\mathcal{A}$. If in the decompositions

$$n = 10 \cdot (13 + 11r) + 11 \cdot (26k - 10r) \tag{4.18}$$

we are not able to achieve $\gcd(13 + 11r, 26k - 10r) = 1$ for $r = 2, \dots, 36$, at least we know that n can be divided by 26, 5 or 7, 79 and 409, so $n = 409 \cdot t$, with $t \geq 26 \cdot 79$. Since 409 is a prime number, by Proposition 33, we deduce that $n \in \mathcal{A}$. In order to carry out this reasoning, for $r = 36$ we need that in (4.18), either $26k - 360 \geq 2 \cdot (409) + 1$ if $\gcd(409, 26k - 360) = 1$ or $26k - 360 \geq 1$, otherwise; in both cases, it is enough to take $k \geq 46$.

For $5 \leq k \leq 45$, consider $r = 6$, $n = 10 \cdot 79 + 11 \cdot (26k - 60)$. Observe that $\gcd(79, 26k - 60) \neq 1$ if and only if $k = 57 + 79s$ with $s \in \mathbb{N}$. Furthermore, $26k - 60 \geq 2 \cdot 79 + 1$ is equivalent to $k \geq 9$. Hence, for $9 \leq k \leq 45$, we will have $n \in \mathcal{A}$.

As a final step, for $5 \leq k \leq 8$, as we assumed that we can divide $26k - 20$ by 5 or 7, the only possible value is $k = 5$. Therefore, $n = 1560$ and neither of its possible decompositions verify simultaneously the two required conditions $\gcd(a', b') = 1$ and $b' \geq 2a' + 1$.

This concludes the case b even in $\mathcal{C}_2$.

To sum up the above study, we have achieved that

$$N_2 := \max\{\mathcal{NA} \cap \mathcal{C}_2\} = 1560.$$

The class $\mathcal{C}_3$: Let $n \in \mathcal{C}_3$, which means that we can decompose it as $n = 30 + 11b$, $b \geq 0$. Since we need $b \geq 2a + 1 = 7$, we can discard from the set $\mathcal{A}$ the numbers $30, 41, 52, 63, 74, 85, 96$, that correspond to $b = 0, \ldots, 6$. So, we begin by assuming $a = 3$ and $b \geq 7$.

- If b is not a multiple of 3, we have that a and b are coprime, so every number of the form $n = 30 + 11b$ belongs to $\mathcal{A}$.

- If b is a multiple of 3, $b = 3j$, with $j \geq 3$, we can rewrite the decomposition of n as $n = 10 \cdot 14 + 11 \cdot (3j - 10)$. Then, imposing the condition $3j - 10 \geq 2 \cdot 14 + 1$, we get $j \geq 13$. It is direct to check that the numbers corresponding to $3 \leq j \leq 12$, $n \in \{129, 162, 195, 228, 261, 294, 327, 360, 393, 426\}$, are in $\mathcal{NA}$.

  Assume $\gcd(14, 3j - 10) = 1$, with $j \geq 13$. Then, $n \in \mathcal{A}$. Observe that such condition holds for $j = 13$ and the odd numbers $j \geq 15$ such that $j \neq 15 + 14u$, $u \geq 0$. Nevertheless, if $j \geq 14$ is even or $j = 15 + 14k$, with $k \geq 0$, we deduce that n is a multiple of 2 or 7, and we have to look for other decompositions of n.

  In this direction, we can vary the coefficients in order to obtain the successive decompositions

  $$n = 10 \cdot (14 + 11r) + 11 \cdot (3j - 10 - 10r),\ r \geq 0. \tag{4.19}$$

  ⋆ Take $r = 1$. Then, $n = 10 \cdot 25 + 11 \cdot (3j - 20)$. The numbers n admitting such decomposition will belong to $\mathcal{A}$ if $3j - 20 \geq 51$, which yields to $j \geq 24$. Now, since $j = 2k$ with $k \geq 7$; or $j = 15 + 14k$, with $k \geq 0$; we need to analyze the cases $j = 14, 15, 16, 18, 20, 22$. These values correspond to the numbers $n \in \{492, 525, 558, 624, 690, 756\}$, and it can be easily checked that they belong to $\mathcal{NA}$.

  As a next step, we focus on the case $j \geq 24$. Firstly, if $3j - 20$ cannot be divided by 5, we have that $n = 30 + 33j$ belongs to $\mathcal{A}$. In this sense, since $j = 2k$, with $k \geq 12$; or $j = 15 + 14k$, with $k \geq 1$; 5 divides $3j - 20$ if $k = 5w$, with $w \geq 3$; or $j = 15 + 70w$, with $w \geq 1$, where a new decomposition will be needed.

  ⋆ Now, we consider $r = 3$, so $n = 10 \cdot 47 + 11 \cdot (3j - 40)$. Here, we have the inequality $3j - 40 \geq 2 \cdot 47 + 1$ and we deduce that $j \geq 45$. Observe that this forces us to analyze apart the remaining values of j, namely, $j = 30, 40$, which correspond to the numbers $n = 1020, 1350$, respectively. It is easy to see that neither of them belong to $\mathcal{A}$.

In the sequel, we consider $j = 10w$, with $w \geq 5$; or $j = 15 + 70w$, with $w \geq 1$. Notice that $r = 3$ implies that $g := \gcd(47, 3j - 40) \in \{1, 47\}$. If $g = 1$, $n = 30 + 33j$ will be in $\mathcal{A}$. Otherwise, 47 divides $3j - 40$ and this happens where $30w - 40$, $w \geq 5$, is a multiple of 47; or when $3 \cdot (15 + 70w) - 40$, $w \geq 1$ so is; that is, whenever either $3w - 4$, $w \geq 5$, or $1 + 42w$, $w \geq 1$, are a multiple of 47. The first case implies $w = 17 + 47u$, $u \geq 0$, and the second case yields to $w = 19 + 47u$, $u \geq 0$. Therefore, either $j = 10 \cdot (17 + 47u) = 170 + 470u, u \geq 0$, or $j = 15 + 70 \cdot (19 + 47u) = 1345 + 3290u, u \geq 0$.

For such values of j, we use the new decomposition

$$n = 10 \cdot (14 + 11r) + 11 \cdot (3j - 10 - 10r). \tag{4.20}$$

Now, analogously to the previous case, by considering the sequence

$$\left\{x_r^{(3)} : r \geq 0\right\} = \{14 + 11r : r \geq 0\},$$

we find the prime number $x_{27}^{(3)} = 311$ in $\mathcal{A}$. From here, if in (4.20) we are not able to attain a value $r = 1, 2, \ldots, 27$, such that $\gcd(14 + 11r, 3j - 10 - 10r) = 1$, at least we know that n is a multiple of 3; 2 or 7; 47 and 311, so $n = 311 \cdot t$, with $t \geq 6 \cdot 47$. In conclusion, since 311 is a prime number, from Proposition 33, we get $n \in \mathcal{A}$.

On the other hand, if we want to apply the same reasoning for $r = 27$, we need that in (4.19) either $3j - 10 - 10 \cdot 27 \geq 2 \cdot 311 + 1$ if $\gcd(311, 3j - 10 - 10 \cdot 37) = 1$ or $3j - 10 - 10 \cdot 27 \geq 1$; otherwise, in both cases, it is enough to take $j \geq 301$. Regarding that $j = 170 + 470u$, $u \geq 0$, or $j = 1345 + 3290u$, $u \geq 0$, this choice is always appropriate, except for the value of $j = 170$, which give rise to $n = 30 + 33 \cdot 170 = 5640$. This number belongs to $\mathcal{A}$ too because $5640 = 10 \cdot 91 + 11 \cdot 430$, with $\gcd(91, 430) = 1$ and $430 \geq 2 \cdot 91 + 1 = 183$. This finishes the case $b = 3j$ in $\mathcal{C}_3$.

In conclusion, from the above study, we have

$$N_3 := \max\{\mathcal{NA} \cap \mathcal{C}_3\} = 1350.$$

The class $\mathcal{C}_4$: Our target is to deeply analyze the elements of $\mathcal{NA}$ which display the form of the class $\mathcal{C}_4 = \{40 + 11k, k \geq 0\}$, and to determine the maximum of $\mathcal{NA} \cap \mathcal{C}_4$. To do so, let us consider $n \in \mathcal{C}_4$, that is, $n = 40 + 11b$, $b \geq 0$.

It is straightforward to establish the first elements of $\mathcal{C}_4$, that are $n = 40$, 51, 62, 73, 84, 95, 106, 117, 128. Observe that such numbers correspond to $b = 0, \ldots, 8$, thus, the condition $b \geq 2a + 1$ is not satisfied and they belong to $\mathcal{NA}$. Therefore, from now on we consider $b \geq 9$.

- Assume that b is odd with $b \geq 9$. It is direct that $n = 40 + 11b \in \mathcal{A}$, since the conditions $\gcd(a, b) = 1$ and $b \geq 2a + 1$ are assured.

- Assume that b is even, that is, $b = 2j$, $j \geq 5$. Here, the coefficients a and b are not coprime, so we need to find another suitable decomposition for the elements of $\mathcal{C}_4$. As a first step, we consider $n = 10 \cdot 15 + 11 \cdot (2j - 10)$. Observe that the condition $2j - 10 \geq 31$ will hold for $j \geq 21$. Hence, if $5 \leq j \leq 20$, every element $40 + 22j$, i.e., $\{150, 172, 194, 216, 238, 260, 282, 304, 326, 348, 370, 392, 414, 436, 458, 480\}$ belong to $\mathcal{NA}$. As a next step, we assume that $j \geq 21$. In this case, if $2j - 10$ is neither a multiple of 3 nor a multiple of 5, the number n belongs to $\mathcal{A}$. Otherwise, either $j = 5k, k \geq 5$ if 5 divides $2j - 10$, or $j = 8 + 3k,\ k \geq 5$, if $2j - 10$ is a multiple of 3. In an analogous way to the reasoning developed for $\mathcal{C}_2$ and $\mathcal{C}_3$, we can write n as follows:

$$n = 10 \cdot (15 + 11r) + 11 \cdot (2j - 10 - 10r),\ r \geq 0. \tag{4.21}$$

Take $r = 2$, then $n = 10 \cdot 37 + 11 \cdot (2j - 30)$ will be in $\mathcal{A}$ if $2j - 30 \geq 75$, which means, $j \geq 53$. Recall that we are considering that $j = 5k, k \geq 5$ or $j = 8 + 3k, k \geq 5$, so the decomposition cannot be executed if $j = 25, 30, 35, 40, 45, 50$, or $j = 23, 26, 29, 32, 35, 38, 41, 44, 47, 50$. In those cases, their corresponding n are elements of $\mathcal{NA}$. In concrete, the sets $\{590, 700, 810, 920, 1030, 1140\}$ and $\{612, 678, 744, 810, 876, 942, 1008, 1074, 1140\}$ are contained in $\mathcal{NA}$.

Now, we focus on the case $j \geq 53$. If additionally, $2j - 30$ is coprime with 37, every number will be in $\mathcal{A}$. On the contrary, if 37 divides $2j - 30$, then either 37 divides $10k-30$, with $k \geq 11$, when $j = 5k$; or 37 divides $6k-14$, with $k \geq 15$, when $j = 8+3k$. In the first situation, we can deduce that 37 divides $k - 3$, so $k = 40 + 37u$, $u \geq 0$; while in the second scenario, 37 divides $3k - 7$, so $k = 27 + 37u$, $u \geq 0$. Therefore, we carry on our study by considering $j \geq 53$ and $j = 5k = 200 + 185u, u \geq 0$, or $j = 8 + 3k = 89 + 111u, u \geq 0$.

It is easy to see that there are prime numbers included in $\mathcal{A}$ that are terms of the sequence

$$\left\{x_r^{(4)} : r \geq 0\right\} = \{15 + 11r : r \geq 0\}.$$

For instance, $x_{32}^{(4)} = 367$. As a consequence, if in the decomposition (4.21) we are not able to find a value $r = 1, \ldots, 32$, such that $\gcd(15+11r, 2j-10-10r) = 1$, at least we can guarantee that every number n can be divided by 37, 59, and 367, so $n = 367 \cdot t$, with $t \geq 37 \cdot 59$. Thus, from Proposition 33 we can deduce that $n \in \mathcal{A}$. To properly develop this reasoning, for $r = 32$ we need that in (4.21) either $2j - 10 - 10 \cdot 32 \geq 2 \cdot 367 + 1$ if $\gcd(367, 2j - 10 - 10 \cdot 32) = 1$ or $2j - 10 - 10 \cdot 32 \geq 1$, otherwise; in both cases, it is enough to take $j \geq 533$. As $j = 200 + 185u, u \geq 0$, or $j = 89 + 111u, u \geq 0$, we can choose $j \geq 533$ except for the values $j = 89, 200, 311, 385, 422$. For them, the corresponding numbers $n = 40 + 22j$ are $1998, 4440, 6882, 8510, 9324$, belonging to $\mathcal{A}$. This finishes the case where b is even.

In conclusion,

$$N_4 := \max\{\mathcal{NA} \cap \mathcal{C}_4\} = 1140.$$

The class $\mathcal{C}_5$: The class that we are going to analyze is defined by $\mathcal{C}_5 = \{50 + 11k,\ k \geq 0\}$. Let us consider an element $n \in \mathcal{C}_5$, that exhibits the form $n = 50 + 11b$, $b \geq 0$.

As a first step, since the condition $b \geq 2a + 1 = 11$ must be satisfied, we can easily compute the first values of $\mathcal{C}_5$ that belong to $\mathcal{NA}$. Such numbers correspond to $b = 0, 1, \ldots, 10$, and they are $n = 50$, 61, 72, 83, 94, 105, 116, 127, 138, 149, 160, respectively. In the sequel, we assume $b \geq 11$ and distinguish two cases.

- Assume that b is not divisible by 5. In this case, every number $n = 50 + 11b$, with $b \geq 11$ and b not being a multiple of 5, belongs to $\mathcal{A}$.

- Assume that b is a multiple of 5, that means, $b = 5j$, with $j \geq 3$. In this situation, we need to find another decomposition that verifies the conditions needed. In this sense, we use $n = 10 \cdot 16 + 11 \cdot (5j - 10)$.

  $\star$ Assume that j is odd. Hence, $\gcd(16, 5j - 10) = 1$. On the other hand, we need $5j - 10 \geq 33$ or $j \geq 9$. This implies that $n = 50 + 55j \in \mathcal{A}$ for all $j \geq 9$. Additionally, for $j = 3, 5, 7$, the corresponding elements, $n = 215$, 325, 435, are in $\mathcal{NA}$.

  $\star$ Assume that j is even with $j \geq 4$. Thus, $j = 2k$ and $b = 10k$, $k \geq 2$. Bearing this in mind, we use the new decomposition $n = 10 \cdot 27 + 11 \cdot (5j - 20)$. On the one hand, if $5j - 20$ cannot be divided by 3, then n will be in $\mathcal{A}$ whenever $j \geq 15$. For the remaining values of j, $j = 6, 8, 12, 14$, with $j - 4$ not being a multiple of 3, we obtain $n = 50 + 55j \in \{380, 490, 710, 820\}$, all of them in $\mathcal{NA}$. On the other hand, if $j = 2k, k \geq 2$, $j - 4$ is a multiple of 3, hence $k = 2 + 3u$, and $j = 4 + 6u$, $u \geq 0$. Therefore, $n = 10 \cdot 27 + 11 \cdot (30u)$, with $u \geq 0$. As a next step, we consider the decomposition $n = 10 \cdot 49 + 11 \cdot (30u - 20)$. If $3u - 2$ is not divisible by 7, we can guarantee that $n \in \mathcal{A}$ if $30u - 20 \geq 99$, that is, if $u \geq 4$. In conclusion, if $30u - 20 = 5j - 40$ is not a multiple of 7, with $u \geq 4$, then n is in $\mathcal{A}$. Notice that we have not considered the values $u = 0, 1, 2$ (for which $3u - 2$ is not a multiple of 7), or the corresponding values $n = 50 + 55 \cdot (4 + 6u) \in \{270, 600, 930\}$, which are in $\mathcal{NA}$. Otherwise, if $3u - 2$ is a multiple of 7, with $u \geq 3$, we have $u = 3 + 7\ell$, with $\ell \geq 0$. This yields to $j = 4 + 6u = 22 + 42\ell$, $\ell \geq 0$. In particular, $j \geq 22$. Notice that we can write $n = 50 + 55j$ in the following forms:

$$n = 10 \cdot (27 + 11r) + 11 \cdot (30u - 10r),\ r \geq 0. \tag{4.22}$$

  The sequence

$$\left\{x_r^{(5)} : r \geq 0\right\} = \{27 + 11r : r \geq 0\}$$

contains at least a prime number included in $\mathcal{A}$, namely $x_{22}^{(5)} = 269$. Hence, if in the decompositions (4.22) we cannot obtain a value $r = 1, \ldots, 22$ such that $\gcd(27 + 11r, 30u - 10r) = 1$, at least we know that n can be divided by $3, 7$ and 71, as well as 269, so $n = 269 \cdot t$, with $t \geq 21 \cdot 71$. Then, applying Proposition 33, we get that $n \in \mathcal{A}$. To the end of developing a suitable reasoning, for $r = 22$ we need that in (4.22) either $30u - 10 \cdot 22 \geq 2 \cdot 269 + 1$ if $\gcd(269, 30u - 10 \cdot 22) = 1$ or $30u - 10 \cdot 22 \geq 1$, otherwise; in both cases, it is enough to take $u \geq 26$. As $u = 3 + 7\ell$, we achieve it by taking $\ell \geq 4$. For the remaining cases, $\ell = 0, 1, 2, 3$, we have $n = 1260, 3570, 5880, 8190$. Among them, $3570, 5880, 8190 \in \mathcal{A}$. On the contrary, $1260 \in \mathcal{NA}$, since $1260 = 10 \cdot 5 + 11 \cdot 110 = 10 \cdot 16 + 11 \cdot 100 = 10 \cdot 27 + 11 \cdot 90 = 10 \cdot 38 + 11 \cdot 80 = 10 \cdot 49 + 11 \cdot 70 = 10 \cdot 60 + 11 \cdot 60 = \cdots$; and we have finished the analysis of the class $\mathcal{C}_5$.

In conclusion, from the above study we conclude that

$$N_5 := \max\{\mathcal{NA} \cap \mathcal{C}_5\} = 1260.$$

The class $\mathcal{C}_6$: We proceed to analyze the class $\mathcal{C}_6 = \{60 + 11k,\ k \geq 0\}$ in order to determine the maximum value of $\mathcal{C}_6$ belonging to $\mathcal{NA}$. Firstly, as usual, for $b = 0, 1, \ldots, 12$, the condition $b \geq 2a + 1$ does not hold and the corresponding terms belong to $\mathcal{NA}$. In what follows we assume that $a = 6$ and $b \geq 13$.

- Assume that, simultaneously, b is not a multiple of 2 nor 3. Under those assumptions it is direct to see that $60 + 11b \in \mathcal{A}$.

- Assume that either $b = 2j$ for $j \geq 7$, or $b = 3j$ for $j \geq 5$. In this situation, we have to look for another decomposition, namely, $n = 10 \cdot 17 + 11 \cdot (b - 10)$.

  ⋆ If $b - 10$ is not a multiple of 17, the coefficients a, b in the decomposition are coprime. So, we only need to impose that the inequality $b - 10 \geq 2 \cdot 17 + 1$ holds, and this happens when $b \geq 45$.

  (i) If, besides, $b = 2j$ for $j \geq 7$, it yields that $n \in \mathcal{A}$ for $j \geq 23$; and, since $2j - 10$ is not a multiple of 17, we have to exclude the values corresponding to $7 \leq j \leq 21$, that are, 214, 236, 258, 280, 302, 324, 346, 368, 390, 412, 434, 456, 478, 500, 522. In conclusion, those numbers belong to $\mathcal{NA}$. Observe that, even, for $j = 22$ we get $n = 544 \in \mathcal{NA}$.

  (ii) If, in addition, $b = 3j$, with $j \geq 5$, then $n \in \mathcal{A}$ for $j \geq 15$ (notice $b \geq 45$). Moreover, if we impose that $3j - 10$ is not divisible by 17, the discarded values are $j = 5, 6, 7, 8, 10, 11, 12, 13, 14$, with associate numbers $n =$ 225, 258, 291, 324, 390, 423, 456, 489, 522 which are included in $\mathcal{NA}$.

  ⋆ If $b - 10$ is a multiple of 17, we need to analyze two subcases: either $b = 2j$, $j \geq 7$, or $b = 3j$ with $j \geq 5$. For the first case, notice that $2j - 10$ is divisible by 17 if and only if $j-5$ is not a multiple of 17. In conclusion, $j = 22 + 17u, u \geq 0$. In the second case, $3j - 10$ is a multiple of 17 if and only if $j = 9 + 17u, u \geq 0$.

Observe that we can write $n = 60 + 11b$ as follows:

$$n = 10 \cdot (6 + 11r) + 11 \cdot (b - 10r),\ r \geq 0. \tag{4.23}$$

It is easy to see that the sequence

$$\left\{x_r^{(6)} : r \geq 0\right\} = \{6 + 11r : r \geq 0\}$$

contains at least a prime number that belongs to $\mathcal{A}$, namely $x_{17}^{(6)} = 193$. Therefore, if we cannot find a value $r \in \{1, \dots, 17\}$, such that $\gcd(6 + 11r, b - 10r) = 1$ in (4.23), at least we know that n is a multiple of 2 or 3, 17 and 61, as well as 193, from where we derive that $n = 193 \cdot t$, with $t \geq 34 \cdot 61$. In this sense, by Proposition 33, we achieve that $n \in \mathcal{A}$. In the particular case $r = 17$, we need that in (4.23) either $b - 10 \cdot 17 \geq 2 \cdot 193 + 1$ if $\gcd(193, b - 10 \cdot 17) = 1$ or $b - 10 \cdot 17 \geq 1$, otherwise; in both cases, it is enough to consider $b \geq 557$.

Now, bearing in mind the subcases highlighted previously:

(a) If $j = 22 + 17u, u \geq 0$, then $b = 2j = 44 + 34u$, and $b \geq 557$ for $u \geq 16$; hence, $n = 544 + 374u \in \mathcal{A}$. For the values $0 \leq u \leq 15$, we get their corresponding n, namely, 544, 918, 1292, 1666, 2040, 2414, 2788, 3162, 3536, 3910, 4284, 4658, 5032, 5406, 5780, 6154. It is immediate to check that $544, 918 \in \mathcal{NA}$, and by using an algorithmic routine implemented in our personal computer, one can see that the remaining values can be decomposed in the form $10a' + 11b'$, where the pairs (a', b') are given respectively by

$$\begin{gathered}(39, 82), (39, 116), (61, 130), (39, 184), (39, 218), (61, 232), (61, 266),\\ (39, 320), (61, 334), (39, 388), (39, 422), (61, 436), (39, 490), (39, 524).\end{gathered}$$

(b) If $j = 9 + 17u, u \geq 0$, then $b = 3j = 27 + 51u$, and $b \geq 557$ for $u \geq 11$; in this situation, $n = 357 + 561u \in \mathcal{A}$. For $0 \leq u \leq 10$, we can compute the values of n, that are 357, 918, 1479, 2040, 2601, 3162, 3723, 4284, 4845, 5406, 5967. Here, $357, 918 \in \mathcal{NA}$, whereas the other numbers belong to $\mathcal{A}$ and its respective pairs (a', b') in the decomposition $10a' + 11b'$ are given by

$$(28, 109), (61, 130), (28, 211), (61, 232), (28, 313), (61, 334), (28, 415), (61, 436), (28, 517).$$

To sum up the previous study, we have achieved that

$$N_6 := \max\{\mathcal{NA} \cap \mathcal{C}_6\} = 918.$$

The class $\mathcal{C}_7$: Let us consider the class $\mathcal{C}_7 = \{70 + 11k,\ k \geq 0\}$ and take $n = 70 + 11b,\ b \geq 0$. Firstly, since $a = 7$, we need to force $b \geq 15$ in order to satisfy $b \geq 2a + 1$. From here, we obtain the first elements of $\mathcal{NA}\cap\mathcal{C}_7$, which correspond with $b\leq 14$, and they are $n=70,\ 81,\ 92,\ 103,\ 114,\ 125,\ 136,\ 147,\ 158,\ 169,\ 180,\ 191,\ 202,\ 213,\ 224$. In the sequel, $b\geq 15$.

- Assume that b is not a multiple of 7. Here, $\gcd(a,b)=1$ and $n=70+11b\in\mathcal{A}$.
- Assume that b is a multiple of 7, that is, $b=7j$, with $j\geq 3$. For this case, we consider the decomposition $n=10\cdot 18+11\cdot(7j-10)$, whenever $7j-10\geq 37$, or $j\geq 7$. For the remaining values $j=3,\dots,6$, we obtain $n=301,378,455,532$, that belong to $\mathcal{NA}$. Next, we assume that $j\geq 7$ and we analyze different scenarios:

  $\star$ If $7j-10$, $j\geq 7$, is not divisible by 2 nor 3, we get that $n=180+11\cdot(7j-18)\in\mathcal{A}$, since $\gcd(18,7j-10)=1$ and $7j-10\geq 37$.

  $\star$ If $7j-10$ is either divisible by 2 or 3, we need to deep on our study since the condition $\gcd(a,b)=1$ is not fulfilled. To this regard:

  (a) If $7j-10$ is even with $j\geq 7$, then $j=8+2u, u\geq 0$.

  (b) If $7j-10$ is multiple of 3 with $j\geq 7$, then $j=7+3u, u\geq 0$.

  In what follows, we consider $n=10\cdot 29+11\cdot(7j-20)$, which provides numbers in $\mathcal{A}$ if $7j-20\geq 59$, or $j\geq 12$. For $j=7,\dots,11$, we only need to check the values $j=7,8,10$, due to (a) and (b), for which the corresponding values are $609,686,840$, that belong to $\mathcal{NA}$.

  If, additionally, we assume that $7j-20$ is not divisible by 29, we achieve that $n=10\cdot 29+11\cdot(7j-20)\in\mathcal{A}$ for all $j\geq 12$. Otherwise, if we assume that $7j-20$ is a multiple of 29, the restrictions (a) and (b) yield either to: (i) $7\cdot(8+2u)-20$ is a multiple of 29, that is, 29 divides $18+7u$, so $u=14+29\ell$, $\ell\geq 0$; or (ii) $7\cdot(7+3u)-20$ is a multiple of 29, that is, $u=29\ell$, $\ell\geq 0$. As usual, we write n as follows:

$$n=10\cdot(18+11r)+11\cdot(7j-10-10r),\ r\geq 0. \tag{4.24}$$

  Now, the sequence

$$\left\{x_r^{(7)}:r\geq 0\right\}=\{18+11r:r\geq 0\}$$

  contains prime numbers that belong to $\mathcal{A}$, for example, $x_{11}^{(7)}=139$. So, if in (4.24) we are not able to find a value $r\in\{1,\dots,11\}$, such that $\gcd(18+11r,7j-10-10r)=1$, at least we know that n can be divided by 7, 2 or 3, by 29, 73 and by 139, thus $n=139\cdot t$, with $t\geq 14\cdot 29$. Therefore, from Proposition 33, we get that $n\in\mathcal{A}$. To justify our reasoning, for $r=11$ it is necessary that in (4.24) either $7j-10-10\cdot 11\geq 2\cdot 139+1$ if $\gcd(139,7j-10-10\cdot 11)=1$ or $7j-10-10\cdot 11\geq 1$, otherwise; in both cases, it suffices to take $j\geq 57$. As a final step,

(i) if $j = 8 + 2u = 36 + 58\ell$, $\ell \geq 0$, it is enough to set $\ell \geq 1$ to guarantee that n is in $\mathcal{A}$; the unique exception appears for $\ell = 0$ or $j = 36$, and for this value we obtain $n = 2842$, a new number in $\mathcal{A}$;

(ii) if $j = 7 + 3u = 7 + 87\ell$, $\ell \geq 0$, we set again $\ell \geq 1$ to ensure $j \geq 57$; here, the exception is given by $\ell = 0$, or $j = 7$, and for this value we obtain $n = 609$, which is not in $\mathcal{A}$.

This ends the inspection of the set $\mathcal{NA} \cap \mathcal{C}_7$. We have obtained

$$N_7 := \max \{\mathcal{NA} \cap \mathcal{C}_7\} = 840.$$

The class $\mathcal{C}_8$: Here, we focus on the class $\mathcal{C}_8 = \{80 + 11k,\ k \geq 0\}$. Our main goal is to analyze the existence of values of $\mathcal{NA}$ and to determine the maximum value in $\mathcal{NA} \cap \mathcal{C}_8$. To execute that, take $n \in \mathcal{C}_8$, $n = 80 + 11b$, $b \geq 0$. For this class, $a = 8$, so we have to impose the condition $b \geq 17$ in order to satisfy the inequality $b \geq 2a + 1$. This allows us to establish the first elements of $\mathcal{NA} \cap \mathcal{C}_8$. In concrete, they are $n = 80$, 91, 102, 113, 124, 135, 146, 157, 168, 179, 190, 201, 212, 223, 234, 245, 256. From now on, we consider $b \geq 17$ and distinguish two cases depending on the parity of b.

- Assume that b is odd. In this situation a and b are coprime, so directly $n = 80 + 11b \in \mathcal{A}$ for all $b \geq 17$.

- Assume that b is even, with $b = 2j$, $j \geq 9$. Now, we consider the new decomposition

$$n = 10 \cdot 19 + 11 \cdot (2j - 10). \tag{4.25}$$

As a first step, if we impose $b' \geq 2a' + 1$, with $a' = 19$ and $b' = 2j - 10$, we get $j \geq 25$. Hence, for $9 \leq j \leq 24$, we obtain the following elements of $\mathcal{NA}$: 278, 300, 322, 344, 366, 388, 410, 432, 454, 476, 498, 520, 542, 564, 586, 608. As a consequence, we assume that $b = 2j$ with $j \geq 25$ and we proceed to inspect (4.25). Notice that if $2j - 10$ is not divisible by 19, we can guarantee that $n \in \mathcal{A}$ for all $j \geq 25$. Nevertheless, if $2j - 10$ is a multiple of 19, that is, $j = 24 + 19u$, $u \geq 1$, we are forced to carry on with the search of appropriate decompositions. For instance,

$$n = 10 \cdot (19 + 11r) + 11 \cdot (2j - 10 - 10r),\ r \geq 0. \tag{4.26}$$

In an analogous way as in the preceding classes, we consider the following sequence

$$\left\{x_r^{(8)} : r \geq 0\right\} = \{19 + 11r : r \geq 0\}$$

which includes prime numbers that are in $\mathcal{A}$, for example, $x_8^{(8)} = 107$. This implies that, if in the decompositions (4.26) we cannot find a value $r = 1, \ldots, 8$, such that gcd$(19+11r, 2j-10-10r)=1$, at least we know that n can be divided by 2, 19 and 107, thus $n=107\cdot t$, with $t\geq 38$. Therefore, Proposition 33 implies that $n\in\mathcal{A}$. Indeed, for $r=8$, we need that in (4.26) either $2j-10-10\cdot 8\geq 2\cdot 107+1$ if $\gcd(107, 2j-10-10\cdot 8)=1$ or $2j-10-10\cdot 8\geq 1$, otherwise; in both cases, it enough to set $j\geq 153$. Bearing in mind that $j=24+19u$, $u\geq 1$, the inequality $j\geq 153$ holds when $u\geq 7$, and then we can set up that $n=10\cdot 19+11\cdot(2j-10)$ is in $\mathcal{A}$. In addition, for $1\leq u\leq 6$, we get the numbers $n=1026$, 1444, 1862, 2280, 2698, 3116, 3534. In this list, it can be easily verified that $1026\in\mathcal{NA}$, whereas the other numbers are elements of $\mathcal{A}$.

To sum up, from the previous study we can establish the maximum value in $\mathcal{NA}\cap\mathcal{C}_8$. In concrete,

$$N_8 := \max\{\mathcal{NA}\cap\mathcal{C}_8\} = 1026.$$

The class $\mathcal{C}_9$: Now, we analyze the set $\mathcal{C}_9=\{90+11k,\ k\geq 0\}$ in order to obtain the maximum value in $\mathcal{NA}\cap\mathcal{C}_9$. Let $n\in\mathcal{C}_9$, $n=90+11b$, $b\geq 0$. Here, $a=9$, so we need $b\geq 19$ to satisfy the condition $b\geq 2a+1$. This implies that, for $b=0,1,\ldots,18$, their corresponding values n belong to $\mathcal{NA}$. Thus, we start our analysis with $b\geq 19$.

- If b is not a multiple of 3, then $n=10\cdot 9+11\cdot b\in\mathcal{A}$ for all $b\geq 19$.
- If $b=3j$, with $j\geq 7$, consider the decomposition $n=10\cdot 31+11\cdot(3j-20)$. We distinguish two cases.

  $\star$ If, additionally, 31 does not divide $3j-20$, we get that n belongs to $\mathcal{A}$ if $3j-20\geq 63$ or $j\geq 28$. Thus, we must analyze the remaining values $7\leq j\leq 28$. In this line, we find that 321, 354, 387, 420, 453, 486, 519, 552, 585, 618, 684, 750, 816, 882, 915, 948 belong to $\mathcal{NA}$, and on the other hand $\{651, 717, 783, 849, 981, 1014\}\subset\mathcal{A}$, with associate pairs (a', b') given respectively by $(20,47)$, $(20,53)$, $(20,59)$, $(20,71)$, $(31,64)$.

  $\star$ If $3j-20$ is a multiple of 31, $j\geq 7$, it can be deduced that $j=17+31u$, $u\geq 0$. In this case, we rewrite the corresponding decomposition as

$$n=10\cdot(9+11r)+11\cdot(3j-10r),\ r\geq 0. \tag{4.27}$$

  Observe that the sequence

$$\left\{x_r^{(9)} : r\geq 0\right\}=\{9+11r : r\geq 0\}$$

  includes prime numbers which are in $\mathcal{A}$, for example $x_8^{(9)}=97$. In this direction, if in the decompositions (4.27) we are not able to find a value $r=1,\ldots,8$, such that $\gcd(9+11r, 3j-10r)=1$, at least we know that n can be divided by 3, 31, as well as 97, so $n=97\cdot t$, with $t\geq 93$. Then, Proposition 33 implies $n\in\mathcal{A}$.

For $r = 8$ we need that in (4.27) either $3j - 10 \cdot 8 \geq 2 \cdot 97 + 1$ if $\gcd(97, 3j - 10 \cdot 8) = 1$ or $3j - 10 \cdot 8 \geq 1$, otherwise; in both cases, it suffices to take $j \geq 92$. Bearing in mind that $j = 17 + 31u$, $u \geq 0$, the inequality $j \geq 92$ holds when $u \geq 3$. Here, the numbers n will be in $\mathcal{A}$. Moreover, for $u = 0, 1, 2$, we have $n = 651, 1674, 2697$, being 651 and 2697 in $\mathcal{A}$. As a final step, 1674 provides us the maximum of $\mathcal{NA} \cap \mathcal{C}_9$. Indeed, $1674 = 10 \cdot 9 + 11 \cdot 144 = 10 \cdot 20 + 11 \cdot 134 = 10 \cdot 31 + 11 \cdot 124 = 10 \cdot 42 + 11 \cdot 114 = 10 \cdot 53 + 11 \cdot 104 = \ldots$, and $104 < 2 \cdot 53 + 1$.

To sum up, we have proved that

$$N_9 := \max\{\mathcal{NA} \cap \mathcal{C}_9\} = 1674.$$

The class $\mathcal{C}_{10}$: Finally, we focus on the set $\mathcal{C}_{10} = \{100 + 11k,\ k \geq 0\}$. Let $n \in \mathcal{C}_{10}$, $n = 100 + 11b$, $b \geq 0$. Here, $a = 10$ and we need $b \geq 21$ in order to satisfy the condition $b \geq 2a + 1$. Hence, for $b = 0, \ldots, 20$, we obtain the following elements of $\mathcal{NA}$: $n = 100$, 111, 122, 133, 144, 155, 166, 177, 188, 199, 210, 221, 232, 243, 254, 265, 276, 287, 298, 309, 320.

Now, we consider the decomposition $n = 10 \cdot 32 + 11 \cdot (b - 20)$, with $b \geq 21$.

- Assume that b is odd. Notice that, in this case, $\gcd(32, b - 20) = 1$, which allows us to deduce that $n \in \mathcal{A}$ if the condition $b - 20 \geq 2 \cdot 52 + 1$ is satisfied. This implies to set $b \geq 85$. Using a computer, we determine whether the excluded values $b = 21, \ldots, 83$, generate elements belonging to $\mathcal{A}$ or the set $\mathcal{NA}$. For these values we obtain $\{375, 485, 595, 705\} \subset \mathcal{NA}$ and the rest of the elements are in $\mathcal{A}$, whose respective pairs (a', b') are given by

$$(10,21),(10,23),(10,27),(10,29),(10,31),(10,33),(10,37),(10,39),(10,41),(10,43),$$
$$(10,47),(10,49),(10,51),(10,53),(10,57),(10,59),(10,61),(10,63),(21,55),(10,67),$$
$$(10,69),(10,71),(10,73),(21,65),(10,77),(10,79),(10,81),(10,83).$$

- Assume that b is even, i.e., $b = 2j$ with $j \geq 11$. Analogously to the previous case, we assume that $b \geq 86$ and consider alone the exceptions $b = 22 + 2 \cdot s$, $0 \leq s \leq 31$. From the corresponding values of $n = 10 \cdot 32 + 11 \cdot (b - 20)$, only $\{694, 716, 760, 782, 848, 892, 914, 958, 1024\} \subset \mathcal{A}$, with respective pairs (a', b') given by

$$(21,44),(21,46),(21,50),(21,52),(21,58),(21,62),(21,64),(21,68),(21,74),$$

and with the rest of the elements included in $\mathcal{NA}$. As a next step, we restrict to the case $b = 2j \geq 86$, or $j \geq 43$. In this direction, we consider the combination $n = 10 \cdot 43 + 11 \cdot (b - 30)$, and distinguish two cases:

$\star$ If $b - 30 = 2j - 30$ is not a multiple of 43, then $\gcd(43, b - 30) = 1$ and we finish if we guarantee that $2j - 30 \geq 87$, or $j \geq 58$. Notice that this implies that we have to study separately the values $43 \leq j \leq 57$. For them, we find

$\{1068, 1134, 1200, 1266, 1288, 1332\} \subset \mathcal{NA}$ and the remaining elements belong to $\mathcal{A}$ with the following respective pairs (a', b'):

$$(21, 76), (21, 80), (21, 82), (21, 86), (21, 88), (21, 92), (21, 94), (21, 100), (21, 104).$$

$\star$ If $b - 30 = 2j - 30$ is a multiple of 43, then $j - 15$ so is, then $j = 58 + 43u$, $u \geq 0$. Now, we consider

$$n = 10 \cdot (10 + 11r) + 11 \cdot (b - 10r),\ r \geq 0. \tag{4.28}$$

Take the sequence

$$\left\{x_r^{(10)} : r \geq 0\right\} = \{10 + 11r : r \geq 0\}.$$

Observe that such sequence contains several prime numbers which are in $\mathcal{A}$. In concrete, we focus on $x_9^{(10)} = 109$. Bearing this in mind, if we cannot find a value $r = 1, \ldots, 9$, in the decompositions (4.28) such that $\gcd(10 + 11r, 2j - 10r) = 1$ for the number $n = 10 \cdot 10 + 11 \cdot (2j)$, with $j \geq 58$, $j = 58 + 43u, u \geq 0$, at least we know that n can be divided by 2 and 43, as well as 109, so $n = 109 \cdot t$, with $t \geq 86$. Therefore, by Proposition 33, we conclude that $n \in \mathcal{A}$. To effectively guide our reasoning for $r = 9$, we need that in (4.28) either $2j - 10 \cdot 9 \geq 2 \cdot 109 + 1$ if $\gcd(109, 2j - 10 \cdot 9) = 1$ or $2j - 10 \cdot 9 \geq 1$; otherwise, in both cases, it will be enough to consider $j \geq 155$. Taking into account that $j = 58 + 43u$, $u \geq 0$, the inequality $j \geq 155$ holds when $u \geq 3$. For this scenario, the numbers n will be in $\mathcal{A}$. For the remaining values of u, $u \in \{0, 1, 2\}$, we get $n \in \{1376, 2322, 3268\}$, which present the following respective decompositions $10 \cdot 21 + 11 \cdot 106$, $10 \cdot 65 + 11 \cdot 152$, and $10 \cdot 21 + 11 \cdot 278$. Thus they are in $\mathcal{A}$ and the discussion regarding the even numbers concludes.

In conclusion, we have proved that

$$N_{10} := \max\{\mathcal{NA} \cap \mathcal{C}_{10}\} = 1332.$$

4.2.1 The summary of the elements of the set of periods

Finally, in this subsection we establish the maximum value of $\mathcal{NA}$. In this direction, we denote by N_m the maximum value in $\mathcal{NA} \cap \mathcal{C}_m$, $1 \leq m \leq 10$, and by the previous study we obtain:

$$\begin{aligned} &N_1 = 32, \quad N_2 = 1560, \quad N_3 = 1350, \quad N_4 = 1140, \quad N_5 = 1260, \\ &N_6 = 918, \quad N_7 = 840, \quad N_8 = 1026, \quad N_9 = 1674, \quad N_{10} = 1332. \end{aligned}$$

Moreover, by Proposition 34 and Corollary 9, we know that the maximum value of $\mathcal{NA}$ being a multiple of 11 is $N_{11} = 1320$. In conclusion,

$$M = \max\{\mathcal{NA}\} = \max\{N_m : 1 \leq m \leq 11\} = \boxed{1674}.$$

In addition, apart from determining the maximum value of $\mathcal{NA}$, we obtain all the elements in $\mathcal{A}$ by using a mathematical software. We gather them in the following table:

Intervals	Numbers in $\mathcal{A}$
$n \in [1, 100]$	$1, 8, 11, 43, 54, 65, 75, 76, 87, 97, 98$
$n \in [101, 200]$	$107, 109, 118, 119, 120, 131, 139, 140, 141, 142, 151, 153,$ $161, 163, 164, 171, 173, 175, 182, 183, 184, 185, 186, 193, 197$
$n \in [201, 300]$	$203, 204, 205, 206, 207, 208, 217, 219, 226, 227, 229, 230, 235, 237, 239,$ $241, 246, 247, 248, 249, 250, 251, 252, 257, 259, 263, 267, 268, 269, 271,$ $272, 273, 274, 279, 281, 283, 285, 289, 290, 292, 293, 295, 296, 299$
$n \in [301, 400]$	$303, 305, 307, 311, 312, 313, 314, 315, 316, 317, 318, 323, 329, 331, 332,$ $333, 334, 335, 336, 337, 338, 339, 340, 343, 345, 347, 349, 351, 353, 355,$ $356, 358, 359, 361, 362, 363, 365, 367, 369, 371, 373, 374, 376, 377, 379,$ $381, 382, 383, 384, 385, 389, 391, 395, 396, 397, 398, 399, 400$
$n \in [401, 500]$	$[401, 500] \setminus \{408, 410, 412, 414, 416, 420, 423, 426, 430,$ $432, 434, 435, 436, 452, 453, 454, 455, 456, 458, 473,$ $474, 476, 478, 480, 485, 486, 490, 492, 496, 498, 500\}$
$n \in [501, 600]$	$[501, 600] \setminus \{518, 519, 520, 522, 525, 532, 540, 542, 544, 546,$ $552, 558, 562, 564, 584, 585, 586, 590, 594, 595, 600\}$
$n \in [601, 700]$	$[601, 700] \setminus \{606, 608, 609, 612, 618, 624,$ $628, 650, 672, 678, 684, 686, 690, 700\}$
$n \in [701, 800]$	$[701, 800] \setminus \{702, 705, 710, 738, 744, 750, 756\}$
$n \in [801, 900]$	$[801, 900] \setminus \{804, 810, 820, 826, 836, 840, 870, 876, 882\}$
$n \in [901, 1000]$	$[901, 1000] \setminus \{915, 918, 920, 930, 936, 942, 948, 980, 988\}$
$n \in [1001, 1100]$	$[1001, 1100] \setminus \{1002, 1008, 1020, 1026, 1030, 1068, 1074\}$
$n \in [1101, 1300]$	$[1101, 1300] \setminus \{1134, 1140, 1200, 1260, 1266, 1274, 1288\}$
$n \in [1301, 1700]$	$[1301, 1700] \setminus \{1320, 1332, 1350, 1560, 1674\}$
$n > 1674$	All the values

Table 4.2: Elements of the set $\mathcal{A}$.

Remark 9. *The elements of Table 4.2 can be computed using the algebraic platform of free access, GAP (see [38], a system for computational discrete algebra), and the corresponding package for calculating the numbers of $\mathcal{A}$ (see [30]); the reader with an interest in algorithmic procedures can calculate them by executing the provided instructions:*

```
s:=NumericalSemigroup(10,11);
belong:=x->ForAny(Factorizations(x,s),p->p[2]>=2*p[1]+1 and Gcd(p)=1);
Filtered(Intersection(s,[0..2000]),belong);
```

□

To sum up, in the present section we have proved that the set of periods of the fourth order max-type difference equation, Equation (4.2), is unbounded. Furthermore, we have been able to determine the greatest number that does not belong to the set of periods, that is, $M = 1674$.

4.3 Accumulation points of the non-periodic solutions

Once that the periodic character of Equation (4.2) has been analyzed, and the set of periods and its associate periodic orbits have been established, we center on the non-periodic solutions of the equation. In [29], the authors proved the existence of non-periodic solutions, as well as its boundedness character. In this section, we go further and study in detail the dynamics of the non-periodic orbits. In this direction, we analyze the set of accumulation points of a non-periodic solution (x_n) under the iteration of Equation (4.2), and we see that such solutions are dense in a compact interval of the real line. Concretely, we prove the following result, which is the main Theorem of this section.

Theorem E. *Let (x_1, x_2, x_3, x_4) be arbitrary real initial conditions that generate a non-periodic orbit (x_n) under Equation (4.2). Then, the set of accumulation points of (x_n) is a compact interval. Even more, the tuple (x_1, x_2, x_3, x_4) is equivalent to some tuple of initial conditions (x, y, z, w), with $x = \max\{x_n : n \geq 1\}$, $x \geq w \geq y \geq z \geq 0$, and $\frac{w-z}{x} \in \mathbb{R} \setminus \mathbb{Q}$, and the orbit accumulates in the compact interval $\big[\min\{w - x, -z\}, x\big]$.*

The organization of this section, which is based on [62], is as follows: firstly, we give some preliminaries results concerning the boundedness character of the solutions and the existence of periodic solutions of arbitrarily large periods surrounding every non-periodic orbit. As a next step, we describe the evolution of non-negative tuples in order to determine the accumulation points of a solution (x_n). Here, recall that we can assume without loss of generality that any tuple of initial conditions (x_1, x_2, x_3, x_4) is equivalent to some tuple (x, y, z, w) satisfying the restrictions of Case C_4. Finally, we describe the accumulation points of the non-periodic solutions of Equation (4.2) by distinguishing two scenarios: on the one hand, we show that the non-negative terms of the orbit accumulate in the interval $[0, x]$, while the non-positive elements are dense in the interval $\big[\min\{w - x, -z\}, 0\big]$. Such study allows us to prove Theorem E.

4.3.1 First results

We begin by focusing on the boundedness character of the solutions of Equation (4.2). As it has been already said, it was proved in [29] that every solution (x_n) of the equation is bounded. Moreover, the authors showed that

$$|x_n| \leq M := \max\{|x_j| : 1 \leq j \leq 12\}, \quad \text{for all } n \geq 1.$$

In fact, one can see that such bound M is related to the maximum of the positive terms in the corresponding solution, that is, $M = \max\{x_n\} \geq 0$. Indeed, assume that a solution of the equation is generated by the iteration of (4.2) over the initial conditions (x_1, x_2, x_3, x_4), where $x_i, i = 1, \ldots, 4$, are arbitrary real numbers. Now, take x_j such that $|x_j| = M$. Thus, if $x_j \geq 0$, the result follows; while if $x_j < 0$, we have

$$\begin{aligned} x_{j+4} &= \max\{x_{j+1}, x_{j+2}, x_{j+3}, 0\} - x_j \\ &= \max\{x_{j+1}, x_{j+2}, x_{j+3}, 0\} + |x_j| \geq M \geq 0, \end{aligned}$$

and we deduce $x_{j+4} = M$, reaching the maximum of the sequence with a positive term.

In this line, bearing in mind Definition 4, where an equivalence relation was established, we can assume, without loss of generality, that the first term of the sequence, x_1, is the maximum term of the solution. Notice that this holds, since if j_{max} is the index that satisfies

$$x_{j_{max}} = \max\{|x_j| : 1 \leq j \leq 12\} \geq |x_n| \text{ for all } n \geq 1,$$

we can consider the shifted sequence generated by $y_1 = x_{j_{max}}$; $y_2 = x_{j_{max}+1}$; $y_3 = x_{j_{max}+2}$; $y_4 = x_{j_{max}+3}$; and the equivalence relation guarantees that the tuple (y_1, y_2, y_3, y_4) generates under Equation (4.2) the same solution as (x_1, x_2, x_3, x_4). Then, we can state the following:

Claim 1. *We can assume that $x_1 = \max\{x_n : n \geq 1\}$ for every sequence (x_n) generated by Equation (4.2).*

Observe that the above claim implies that the initial conditions of the solution, x_1, x_2, x_3, x_4, are non-negative terms by Proposition 17.

Next, we determine a necessary and sufficient condition in order to have a non-periodic orbit, which follows from the argument that gave rise to Equation (4.6).

Proposition 35. *Let (x_1, x_2, x_3, x_4) be a tuple of initial conditions, with $x_1 = \max\{x_n\}$, and holding the restrictions of Case C_4. Let (x_n) be the corresponding solution generated by Equation (4.2). Then, the solution is non-periodic if and only if $\frac{x_4 - x_3}{x_1} \notin \mathbb{Q}$.*

Recall that the assumption of having initial conditions satisfying the restrictions of Case C_4 does not yield to a loss of generality.

The next result is based on Dirichlet's Theorem relative to Diophantine approximation which states as follows:

Theorem 16. *Given $\alpha \in \mathbb{R}$ and $N > 1$, there exist integers x, y with $1 \leq y \leq N$ and $|\alpha y - x| < \frac{1}{N}$. When α is irrational, there are infinitely many reduced fractions $\frac{x}{y}$ with $\left|\alpha - \frac{x}{y}\right| < \frac{1}{y^2}$.*

For a detailed proof of the theorem, consult [99]. Now, we show that each initial condition generating a non-periodic sequence has, arbitrarily close, initial conditions leading to periodic sequences whose set of periods is not bounded.

Proposition 36. *Let the tuple $\boldsymbol{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ generate a non-periodic orbit under Equation (4.2). Let $\mathcal{U} = \mathcal{U}(\boldsymbol{x})$ be an arbitrary neighbourhood of $\boldsymbol{x}$. Then, there are tuples in $\mathcal{U}$ that generate periodic sequences of arbitrarily large period.*

Proof. Let us rewrite, for the sake of commodity, the tuple $\boldsymbol{x}$ as $\boldsymbol{x} = (x, y, z, w)$, and assume, without loss of generality, that they verify the conditions of Case C_4. This implies $x \geq w \geq y \geq z \geq 0$, and $w - z > 0$, otherwise the tuple would be (x, w, w, w), which is an 11-cycle due to the monotonicity of the initial conditions by Proposition 20. Moreover, from Proposition 35, since this tuple generates a non-periodic solution, we have $\frac{x}{w-z} \in \mathbb{R} \setminus \mathbb{Q}$.

As a matter of fact, we can consider another tuple $\tilde{\boldsymbol{x}}$ in $\mathcal{U}$, arbitrarily close to $\boldsymbol{x}$, such that $\tilde{\boldsymbol{x}} = (\tilde{x}, \tilde{y}, \tilde{z}, \tilde{w})$ satisfies $\tilde{w} - \tilde{z} \neq 0$, $\tilde{x} > \tilde{w} > \tilde{w} - \tilde{z}$, and $\frac{\tilde{x}}{\tilde{w}-\tilde{z}} \in \mathbb{R} \setminus \mathbb{Q}$, so it is not restrictive to assume that the same tuple $\boldsymbol{x}$ verifies $x > w > w - z > 0$ too.

By Dirichlet's Theorem, Theorem 16, for a sufficiently small $\varepsilon > 0$ and the irrational number $\frac{x}{w-z}$, we obtain that there exist infinitely many reduced fractions $\frac{m}{n}$ such that

$$\left| \frac{x}{w-z} - \frac{m}{n} \right| < \frac{1}{n^2} < \frac{\varepsilon}{w-z}.$$

Furthermore, since $\frac{x}{w-z} > 1$, we can set $\frac{m}{n} > 1$.

On the other hand, set $p := n$ and $q := m + n$, so $\frac{m}{n} = \frac{q-p}{p}$. Observe that $\gcd(p, q) = \gcd(m, n) = 1$, and $q > 2p$ since $\frac{m}{n} > 1$. Now, applying Proposition 30 to the tuple $\left(\frac{m}{n}(w - z), y, z, w\right)$, we obtain that it generates a periodic sequence of period $10p + 11q = 10n + 11(m + n)$. Additionally, the tuples $\left(\frac{m}{n}(w - z), y, z, w\right)$ are close to $\boldsymbol{x}$, so they belong to $\mathcal{U}$, and they present arbitrarily large periods. □

Finally, one of the main tools used in the proof of Theorem E is a consequence of Kronecker's Theorem, which is stated for the sake of completeness (its proof can be consulted in [86, pp. 34-40]). Recall that the fractional part of a number is denoted by $\{\cdot\}$ and that $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. Thus, $\{x\} = x - \lfloor x \rfloor$. In addition, for any $\alpha \in \mathbb{R}$ we have $\{t + \alpha\} = \{\alpha\}$ for every $t \in \mathbb{Z}$, since $\lfloor t + \alpha \rfloor = \lfloor t \rfloor + \lfloor \alpha \rfloor$.

Theorem 17. ***Kronecker's Theorem*** *Let δ be an irrational number. Then, for each non-empty open subinterval U of $[0, 1]$, there is an $m \in \mathbb{N}$ such that $\{m \cdot \delta\} \in U$.*

L. Kronecker published the original version of this theorem in [57]. Also, in [44, Chapter XXIII] several ways for proving Theorem 17, including its generalization to higher dimensions, are developed. As a consequence of Kronecker's Theorem, we get the following result which will play a key role in the proof of Theorem E.

Corollary 10. *Let δ_1 be an irrational number and let δ_2 be an arbitrary real number. The set $S_{\delta_1} = \big\{\{s\delta_1 + \delta_2\} : s \in \mathbb{N}\big\}$ is dense in $[0, 1]$.*

4.3.2 Evolution by the routes R_i

The target of this part is to describe the evolution of a tuple of non-negative initial conditions under the different routes R_i, $i = 1, 2, 3, 4$. To do so, we consider initial conditions (x, y, z, w) with $x, y, z, w \in \mathbb{R}$ satisfying the restrictions of the Case C_4, that is, $x \geq w \geq y \geq z \geq 0$. Recall that the description of the orbit of the general case was made by the diagram of Figure 4.1 (see Table 4.1 too). Also, it must be highlighted that, in the proof of Proposition 24, we have already computed the evolution of the tuple. Nevertheless, for the sake of clarification, we will write precisely the terms that appear in the orbit in each case, so we are able to emphasize the non-positive terms and the linear combinations that appear in it. In this sense, we begin by studying the evolution of the first terms of an orbit through the different routes; and after that, we will repeat the process with a general tuple of the form $\big(x, tx + y - s(w - z), z, w\big)$, since this kind of terms are essential in the proof of the denseness.

First terms of an orbit through the routes: As a first step, we analyze the evolution of the initial conditions (x, y, z, w) under each route R_i, with $i = 1, 2, 3, 4$. By considering the proof of Proposition 24, we write the terms that appear in the orbit and highlight in bold format the non-positive terms. In addition, our emphasis is on the linear combinations of the form $tx + y - s(w - z)$.

• Route $R_1 : C_4 \to C_5 \to C_2 \to C_1 \to \ldots \to C_1 \to C_4$. We begin with (x, y, z, w) in C_4. Then, the orbit continues as:

$$\boldsymbol{w - x}, w - y, w - z, \boldsymbol{-z}, x - z, x + y - w - z, x - w, (x, z, w + z - y, w) \text{ in } C_5,$$

$$\boldsymbol{w - x}, w - z, y - z, \boldsymbol{-z}, x - z, x - w, x - y, (x, z, w, y) \text{ in } C_2,$$

$$\boldsymbol{w - x}, w - z, \boldsymbol{-z}, w - z - y, x - z, x - w, \big(x, x + y - (w - z), z, w\big) \text{ in } C_1,$$

$$\boldsymbol{y - (w - z)}, \boldsymbol{w - x - y + (w - z)}, w - z, \boldsymbol{-z}, -y + 2(w - z), x - z, x - w,$$

$$\big(x, x + y - 2(w - z), z, w\big).$$

If this tuple satisfies the restrictions of C_4 we have finished the route. Otherwise, we will have a loop in C_1 and the orbit will follow as

$$\boldsymbol{y - 2(w - z)}, \boldsymbol{w - x - y + 2(w - z)}, w - z, \boldsymbol{-z}, -y + 3(w - z), x - z, x - w,$$

$$\big(x, x + y - 3(w - z), z, w\big).$$

Again, if the tuple is in C_4, we have ended the route, otherwise we will have another loop in C_1. Assume that we have $m_1 \geq 0$ loops in C_1 (notice that $m_1 < \infty$ by Proposition 29). Then, the route will finish with the terms

$$\big(x, x + y - (m_1 + 2)(w - z), z, w\big) \text{ in } C_4.$$

In the middle of the process we will have the tuples

$$\big(x, x+y-j(w-z), z, w\big) \text{ in } C_1, \text{ with } j=1,\ldots,m_1+1.$$

Moreover, every time that the orbit passes through C_1 the following non-positive terms will appear

$$\boldsymbol{y-j(w-z)} \text{ and } \boldsymbol{w-x-y+j(w-z)}, \text{ with } j=1,\ldots,m_1+1.$$

• Route $R_2 : C_4 \to C_5 \to C_2 \to C_4$. We start with (x,y,z,w) in C_4. Then, the orbit follows as:

$$\boldsymbol{w-x}, w-y, w-z, \boldsymbol{-z}, x-z, x+y-w-z, x-w, (x,z,w+z-y,w) \text{ in } C_5,$$

$$\boldsymbol{w-x}, w-z, y-z, \boldsymbol{-z}, x-z, x-w, x-y, (x,z,w,y) \text{ in } C_2,$$

$$\boldsymbol{w-x}, w-z, \boldsymbol{-z}, w-z-y, x-z, x-w, \big(x, x+y-(w-z), z, w\big) \text{ in } C_4.$$

Observe that the second term of the initial conditions, y, has evolved to $x+y-(w-z)$ under a route R_2. Moreover, the only non-positive terms that take place in R_2 are $w-x$ and $-z$.

• Route $R_4 : C_4 \to C_5 \to C_3 \to \ldots \to C_3 \to C_2 \to C_4$. We start with (x,y,z,w) in C_4. Then, the orbit continues as:

$$\boldsymbol{w-x}, w-y, w-z, \boldsymbol{-z}, x-z, x+y-w-z, x-w, (x,z,w+z-y,w) \text{ in } C_5,$$

$$\boldsymbol{w-x}, w-z, y-z, \boldsymbol{-z}, x-z, x-w, x-y, (x,z,w,y) \text{ in } C_3,$$

$$\boldsymbol{w-x}, w-z, \boldsymbol{y-w}, \boldsymbol{-y+(w-z)}, x-z, x-w, x-y+(w-z), \big(x,z,w,y-(w-z)\big).$$

This tuple can verify either the Case C_2 or C_3. If we have a loop in C_3, the orbit continues as

$$\boldsymbol{w-x}, w-z, \boldsymbol{y-(w-z)-w}, \boldsymbol{-y+2(w-z)}, x-z, x-w, x-y+2(w-z),$$

$$\big(x,z,w,y-2(w-z)\big).$$

Again, the new tuple satisfies either the conditions of Case C_2 or those of Case C_3. Assume that we have $m_3 \geq 0$ loops in C_3 (by Proposition 29, $m_3 < \infty$). Then, after that reiterative process, we will achieve the tuple

$$\big(x,z,w,y-(m_3+1)(w-z)\big) \text{ in } C_2.$$

Observe that in the process we have obtained the non-positive terms

$$\boldsymbol{-y+j(w-z)} \text{ and } \boldsymbol{y-(j-1)(w-z)-w}, \text{ with } j=1,\ldots,m_3+1.$$

Finally, if we continue computing the terms, we end going from C_2 to C_4 as follows:

$$\boldsymbol{w-x}, w-z, \boldsymbol{-z}, -y+(m_3+2)(w-z), x-z, x-w,$$

$$\big(x, x+y-(m_3+2)(w-z), z, w\big) \text{ in } C_4.$$

• Route $R_3 : C_4 \to C_5 \to C_3 \to \ldots \to C_3 \to C_2 \to C_1 \to \ldots \to C_1 \to C_4$. The terms appearing in the evolution of this route only contain a combination of elements of the routes R_1 and R_4, and the analysis is omitted.

Remark 10. *Notice that, independently of the route R_i, the initial conditions (x, y, z, w) have evolved to $\big(x, x+y-n(w-z), z, w\big)$, where $n \in \mathbb{N}$ and $x+y-n(w-z) \geq 0$. Moreover, in the middle of the process we have obtained the non-negative terms $y-j(w-z)$, with $j = 1, \ldots, n-1$.*

Evolution of a general tuple through the routes: From the above study, we know that when an orbit of Equation (4.2) evolves through the routes R_i there appear *general tuples* of the form $\big(x, tx+y-s(w-z), z, w\big)$ where $t, s \in \mathbb{N}$ and $s \geq t$. Next, in order to clarify which terms appear in the orbit of general tuples, we describe the routes R_i again, but now when we begin with such a tuple.

• Route $R_1 : C_4 \to C_5 \to C_2 \to C_1 \to \ldots \to C_1 \to C_4$. Let us consider the terms $\big(x, tx+y-s(w-z), z, w\big)$ in C_4. Then, the orbit evolves as:

$$\boldsymbol{w-x}, w-tx-y+s(w-z), w-z, \boldsymbol{-z}, x-z, (t+1)x+y-s(w-z)-z-w, x-w$$

$$\big(x, z, w+z-tx-y+s(w-z), w\big) \text{ in } C_5,$$

$$\boldsymbol{w-x}, w-z, -z+tx+y-s(w-z), \boldsymbol{-z}, x-z, x-w, (1-t)x-y+s(w-z),$$

$$\big(x, z, w, tx+y-s(w-z)\big) \text{ in } C_2,$$

$$\boldsymbol{w-x}, w-z, \boldsymbol{-z}, -tx-y+(s+1)(w-z), x-z, x-w,$$

$$\big(x, (t+1)x+y-(s+1)(w-z), z, w\big) \text{ in } C_1,$$

$$\boldsymbol{tx+y-(s+1)(w-z)}, \boldsymbol{w-(t+1)x-y+(s+1)(w-z)}, w-z, \boldsymbol{-z},$$

$$-tx-y+(s+2)(w-z), x-z, x-w,$$

$$\big(x, (t+1)x+y-(s+2)(w-z), z, w\big).$$

This last tuple can either verify the conditions of C_4, and we would have ended the route, or verify again C_1. Let us assume that we have $m_1 \geq 0$ loops in C_1 (recall that $m_1 < +\infty$), then we will have the tuples

$$\big(x, (t+1)x+y-(s+j)(w-z), z, w\big), \text{ with } j = 1, \ldots, m_1+1,$$

verifying the Case C_1 and we will end the route with

$$\big(x,(t+1)x+y-(s+m_1+2)(w-z),z,w\big) \text{ in } C_4.$$

Moreover, we emphasize that in that process the following non-positive terms appear

$$\boldsymbol{tx+y-(s+j)(w-z)} \text{ and } \boldsymbol{w-(t+1)x-y+(s+j)(w-z)},\ j=1,\dots,m_1+1.$$

• Route $R_2 : C_4 \to C_5 \to C_2 \to C_4$. Let us consider the tuple $\big(x,tx+y-s(w-z),z,w\big)$ in C_4. Then, the orbit evolves as follows:

$$\boldsymbol{w-x}, w-tx-y+s(w-z), w-z, \boldsymbol{-z}, x-z, (t+1)x+y-s(w-z)-z-w, x-w$$

$$\big(x,z,w+z-tx-y+s(w-z),w\big) \text{ in } C_5,$$

$$\boldsymbol{w-x}, w-z, -z+tx+y-s(w-z), \boldsymbol{-z}, x-z, x-w, (1-t)x-y+s(w-z),$$

$$\big(x,z,w,tx+y-s(w-z)\big) \text{ in } C_2,$$

$$\boldsymbol{w-x}, w-z, \boldsymbol{-z}, -tx-y+(s+1)(w-z), x-z, x-w,$$

$$\big(x,(t+1)x+y-(s+1)(w-z),z,w\big) \text{ in } C_4.$$

Now, the non-negative linear combination $tx+y-s(w-z)$ has evolved to $(t+1)x+y-(s+1)(w-x)$ under a route R_2. Furthermore, the only non-positive terms that take place in R_2 are $w-x$ and $-z$.

• Route $R_4 : C_4 \to C_5 \to C_3 \to \dots \to C_3 \to C_2 \to C_4$. Let us consider the tuple $(x,tx+y-s(w-z),z,w)$ in C_4. Then, the orbit evolves as follows:

$$\boldsymbol{w-x}, w-tx-y+s(w-z), w-z, \boldsymbol{-z}, x-z, (t+1)x+y-s(w-z)-z-w, x-w$$

$$\big(x,z,w+z-tx-y+s(w-z),w\big) \text{ in } C_5,$$

$$\boldsymbol{w-x}, w-z, -z+tx+y-s(w-z), \boldsymbol{-z}, x-z, x-w, (1-t)x-y+s(w-z),$$

$$\big(x,z,w,tx+y-s(w-z)\big) \text{ in } C_3, \tag{4.29}$$

$$\boldsymbol{w-x}, w-z, \boldsymbol{tx+y-s(w-z)-w}, \boldsymbol{-tx-y+(s+1)(w-z)}, \tag{4.30}$$

$$x-z, x-w, (1-t)x-y+(s+1)(w-z), \big(x,z,w,tx+y-(s+1)(w-z)\big).$$

Now, we can be either in C_2 or in C_3. Assume that we have $m_3 \geq 0$ loops in C_3 ($m_3 < +\infty$). Then, the reiterative process in C_3 will end with the tuple

$$\big(x,z,w,tx+y-(s+m_3+1)(w-z)\big) \text{ in } C_2.$$

Apart from this tuple, in the middle, after each loop, we have obtained

$$\big(x,z,w,tx+y-(s+j)(w-z)\big) \text{ in } C_3, \text{ with } j=1,\dots,m_3.$$

Moreover, it should be highlighted that we have achieved the non-positive terms

$\boldsymbol{tx + y - (s+j)(w-z) - w}$ and $\boldsymbol{-tx - y + (s+j+1)(w-z)}$, with $j = 0, \dots, m_3$.

Next, once we have $\big(x, z, w, tx + y - (s + m_3 + 1)(w - z)\big)$ in C_2, the orbit continues as

$$\boldsymbol{w - x}, w - z, \boldsymbol{-z}, -tx - y + (s + m_3 + 2)(w - z), x - z, x - w,$$

$$\big(x, (t + 1)x + y - (s + m_3 + 2)(w - z), z, w\big) \text{ in } C_4.$$

- Route $R_3 : C_4 \to C_5 \to C_3 \to \dots \to C_3 \to C_2 \to C_1 \to \dots \to C_1 \to C_4$. The terms appearing through the evolution of this route are a combination of the elements appearing in the routes R_1 and R_4, and we omit the analysis.

Remark 11. *Notice that, independently of the route R_i, the tuple $\big(x, tx+y-s(w-z), z, w\big)$ has evolved to $\big(x, (t+1)x + y - (s+n)(w-z), z, w\big)$, where $n \in \mathbb{N}$ and $(t+1)x + y - (s + n)(w - z) \geq 0$. Moreover, in the middle of the process we have obtained the non-negative terms $tx + y - (s + j)(w - z)$, with $j = 1, \dots, n - 1$.*

4.3.3 Proof of Theorem E

After showing in the previous part how the orbits of a solution (x_n) of Equation (4.2) evolve by writing the specific terms of the solution, we proceed to prove the main theorem of this section. In this way, we split the proof in two parts: the first one focuses on the accumulation points obtained by the non-negative terms of a non-periodic solution; while the second one is devoted to study the accumulation points of the non-positive terms.

A - Density of the non-negative terms: Following the exhaustive description of the routes in the preceding subsections, and bearing in mind Remarks 10 and 11, we obtain the following result:

Lemma 44. *Let (x, y, z, w) be initial conditions with $x, y, z, w \in \mathbb{R}$ verifying the relation $x \geq w \geq y \geq z \geq 0$. For every $s \in \mathbb{N}$, there exists at least a $t \in \mathbb{N}$ with $t \leq s$, such that the linear combination $tx + y - s(w - z) \geq 0$ belongs to the orbit generated by such initial conditions under Eq. (4.2).*

Our purpose is to show that those non-negative terms $tx + y - s(w - z)$ that appear in the orbit are dense in the interval $[0, x]$. Firstly, observe that these terms that we are considering belong to the interval $[0, x]$, since $x \geq tx + y - s(w - z) \geq 0$ for every linear combination of such form.

Secondly, we divide by $x > 0$ the terms $tx + y - s(w - z)$ to simplify the study. Hence, we analyze the accumulation of $t + \alpha - s\sigma$, where $\alpha = \frac{y}{x}$ and $\sigma = \frac{w-z}{x}$. Notice that, in order to not achieve periodicity, from Proposition 35, the following condition

$$\sigma = \frac{w - z}{x} \in \mathbb{R} \setminus \mathbb{Q}, \tag{4.31}$$

must hold. On the contrary, at some moment we would find values of s, s', t, t' for which $t - s\sigma = t' - s'\sigma$, and we will repeat the corresponding tuples, hence achieving a periodic orbit.

Since $t + \alpha - s\sigma \in [0,1]$, it yields that $t + \alpha - s\sigma = \{t + \alpha - s\sigma\} = \{\alpha - s\sigma\}$. By Lemma 44, s goes through the natural numbers, so by Corollary 10 we obtain the density of the set of elements $t + \alpha - s\sigma$ in $[0,1]$ and, therefore, the density of the non-negative terms of an orbit in $[0, x]$.

B - Density of the non-positive terms: Finally, we deal with the density of the non-positive terms in the interval $\big[\min\{-z, w - x\}, 0\big]$. Before diving into the problem, we collect in Table 4.3 the non-positive terms that appear in each route R_i, $i = 1, 2, 3, 4$. Recall that in the development of the subsection devoted to the evolution of the orbits, such terms have been highlighted in bold type.

R_1 :	$w - x$ and $-z$ $tx + y - (s + j)(w - z)$ $w - (t + 1)x - y + (s + j)(w - z)$	$j \in \{1, \dots, m_1 + 1\}$
R_2 :	$w - x$ and $-z$	
R_3 :	The non-positive terms that appear in R_1 and R_4	
R_4 :	$w - x$ and $-z$ $tx + y - (s + j - 1)(w - z) - w$ $-tx - y + (s + j)(w - z)$	$j \in \{1, \dots, m_3 + 1\}$

Table 4.3: Non-positive terms in the routes R_i.

Now, we analyze the bounds of the non-positive terms of a solution of Equation (4.2).

Lemma 45. *Given initial conditions (x, y, z, w) with $x, y, z, w \in \mathbb{R}$ verifying the relation $x \geq w \geq y \geq z \geq 0$, then every non-positive term appearing in the corresponding orbit belongs to the interval*

$$\big[\min\{-z, w - x\}, 0\big].$$

Proof. We focus on the non-positive terms given by the linear combinations

$$tx + y - (s + j)(w - z),\ j = 1, \dots, m_1 + 1; \text{ and } tx + y - (s + j - 1)(w - z) - w,\ j = 1, \dots, m_3 + 1.$$

Firstly, the combination $tx + y - (s + j)(w - z)$ appears while going from

$$\big(x, (t + 1)x + y - (s + j)(w - z), z, w\big) \text{ in } C_1$$

to

$$\big(x, (t + 1)x + y - (s + j + 1)(w - z), z, w\big).$$

Observe that the tuple $\big(x, (t+1)x + y - (s+j)(w-z), z, w\big)$ satisfies the restrictions of Case C_1. Thus, it can be deduced that (see Table 4.1)

$$x \geq (t+1)x + y - (s+j)(w-z) \geq w,$$

or, equivalently,

$$0 \geq tx + y - (s+j)(w-z) \geq w - x.$$

In conclusion, every time that the orbit passes through Case C_1, the non-positive terms that appear belong to $[w-x, 0]$. In addition, it is direct to see that $tx + y - (s+j)(w-z)$ and $w - (t+1)x - y + (s+j)(w-z)$ are symmetric in the interval $[w-x, 0]$ for every $j = 1, \dots, m_1 + 1$.

Secondly, the combinations $tx + y - (s+j-1)(w-z) - w$ appear while going from

$$\big(x, z, w, tx + y - (s+j-1)(w-z)\big) \text{ in } C_3$$

to

$$\big(x, z, w, tx + y - (s+j)(w-z)\big).$$

Notice that the tuple $\big(x, z, w, tx + y - (s+j-1)(w-z)\big)$ satisfies the conditions from Case C_3, which means that (see again Table 4.1)

$$w \leq z + tx + y - (s+j-1)(w-z),$$

or

$$-z \leq tx + y - (s+j-1)(w-z) - w \leq 0.$$

To sum up, the non-positive terms that appear when the orbit passes through Case C_3 belong to the interval $[-z, 0]$. Moreover, it can be easily seen that the linear combinations $tx + y - (s+j-1)(w-z) - w$ and $-tx - y + (s+j)(w-z)$ are symmetric in the interval $[-z, 0]$. □

Next, we prove the denseness of the non-positive terms of the orbit in the interval $\big[\min\{w-x, -z\}, 0\big]$. To do so, we develop the following steps:

Step 1: We prove that the orbit of a non-periodic solution of Equation (4.2) passes through the five Cases C_i an infinite number of times.

Step 2: We analyze the non-positive terms that appear when the orbit passes through Case C_1 (routes R_1 and R_3) in order to show that they are dense in $[w-x, 0]$.

Step 3: We deal with the non-positive terms that appear when the orbit passes through Case C_3 (routes R_4 and R_3) and we see that they are dense in $[-z, 0]$.

Therefore, we will be able to collect these outcomes in order to demonstrate that the non-positive terms are dense in the interval $\big[\min\{w-x,-z\},0\big]$.

Step 1: Evolution of a non-periodic orbit through the Cases C_i. The first step consists in showing that the orbit of a non-periodic solution of Equation (4.2) must pass through every Case C_i infinitely many times.

We start claiming that the orbit cannot be configured by an infinite concatenation of routes R_2. Indeed, assume that we have a tuple of initial conditions (x,y,z,w) satisfying the restrictions of Case C_4, namely, $x\geq w\geq y\geq z\geq 0$. If the tuple begins evolving under a route R_2, it will end with the terms $\big(x,x+y-(w-z),z,w\big)$ verifying again the conditions of Case C_4, which means, $x\geq w\geq x+y-(w-z)\geq z\geq 0$. If we continue with another route R_2, we will achieve $\big(x,2x+y-2(w-z),z,w\big)$ with $x\geq w\geq 2x+y-2(w-z)\geq z\geq 0$. Notice that the unique term that changes after each route R_2 is the second one, where we are adding the non-negative constant $x-(w-z)$. This implies the existence of a natural number N such that $Nx+y-N(w-z)>w$ and we have a contradiction with the conditions of Case C_4.

In conclusion, as the orbit cannot be formed only by routes R_2, apart from the Cases C_2, C_4 and C_5, the orbit has to pass through C_1 or C_3 too. In the following result we show that, in fact, the orbit travels through the Cases C_1 and C_3 infinitely many times for each one of them.

Proposition 37. *Assume that the set of initial conditions (x,y,z,w) verifies the conditions of Case C_4. Then, the orbit must verify the Cases C_1 and C_3 infinitely many times.*

Proof. We proceed by *reductio ad absurdum* and divide the proof in two parts. The first part is based on the study of an orbit that, after certain iteration, does not pass through Case C_1, the second part analyze a similar scenario, but with Case C_3. In both situations, we derive the corresponding contradiction.

Firstly, assume that, after a certain iteration, the orbit does not pass through Case C_1. Notice that this means that a route R_2 or R_4 will take place. In both cases, at the end of the route, the tuple verifying the Case C_4, $\big(x,tx+y-s(w-z),z,w\big)$, satisfies (recall the conditions in Table 4.1)

$$w\geq tx+y-s(w-z),$$

or, equivalently,

$$s\geq\frac{tx+y-w}{w-z}. \tag{4.32}$$

Moreover, if we go backwards in the orbit, bearing in mind that we do not pass through Case C_1 in the routes R_2 or R_4, the tuple $\big(x,tx+y-s(w-z),z,w\big)$ in Case C_4 derives from the evolution of a tuple $\big(x,z,w,(t-1)x+y-(s-1)(w-z)\big)$, which verifies the conditions in Case C_2. Hence, due to the fact that the terms of a tuple in a certain Case C_i are always non-negative, we have

$$(t-1)x+y-(s-1)(w-z)\geq 0,$$

so,

$$s\leq \frac{(t-1)x+y}{w-z}+1. \tag{4.33}$$

Therefore, from (4.32) and (4.33), we obtain

$$\frac{tx+y-w}{w-z}\leq s\leq \frac{(t-1)x+y}{w-z}+1. \tag{4.34}$$

This inequality allows us to derive that, for every natural number t, there exists an integer s in the interval

$$I_1:=\left[\frac{tx+y-w}{w-z},\frac{(t-1)x+y}{w-z}+1\right].$$

Notice that the length of I_1 is $|I_1|=1-\frac{x-w}{w-z}\leq 1$. Also, it should be mentioned that there exists at most a value $\tilde{t}$ for which

$$\alpha=\frac{(\tilde{t}-1)x+y}{w-z}$$

is an integer number. Indeed, if α is an integer number, then

$$\frac{(\tilde{t}+p-1)x+y}{w-z}=\alpha+p\frac{x}{w-z}$$

must be irrational for any $p\in\mathbb{Z}\setminus\{0\}$ according to Proposition 35, since $\frac{x}{w-z}\in\mathbb{R}\setminus\mathbb{Q}$. To sum up, for every $t=\tilde{t}+p$ with $p\in\mathbb{Z}\setminus\{0\}$, the number $\frac{(t-1)x+y}{w-z}$ is irrational.

Now, to ensure that an integer $s\in I_1$ exists, we must force the following inequality

$$\left\{\frac{(t-1)x+y}{w-z}\right\}\leq 1-\frac{x-w}{w-z},$$

where $\{\cdot\}$ denotes the fractional part of a number as usual.

Indeed, suppose that $A:=\frac{(t-1)x+y}{w-z}$ does not belong to the set of integer numbers. Thus, $\lfloor A+1\rfloor\in[A,A+1]$, where $\lfloor\cdot\rfloor$ denotes the integer part of a number. Then, $\lfloor A+1\rfloor\in I_1$ if and only if $\frac{tx+y-w}{w-z}\leq\lfloor A+1\rfloor=A+1-\{A\}$, so

$$\{A\}\leq A-\frac{tx+y-w}{w-z}+1=1-\frac{x-w}{w-z}.$$

Here, we can apply Corollary 10 because $\frac{x}{w-z}$ is irrational and conclude that the set

$$\left\{\left\{\frac{tx+y}{w-z}\right\}\right\}_{t\in\mathbb{N},t\geq\tilde{t}}$$

is dense in $[0,1]$. Observe that this means that it will exist a natural number t such that

$$\left\{\frac{(t-1)x+y}{w-z}\right\}>1-\frac{x-w}{w-z},$$

and there will not exist the corresponding natural number s.

To sum up, inequality (4.34) cannot hold for every $t \in \mathbb{N}$ and, as a result, the route has to visit the Case C_1. In addition, it cannot only pass a finite number of times, since in this situation, after the last time that it passes, we would be able to apply the same reasoning and achieve a contradiction.

As a next step, we proceed in an analogous way to prove that the orbit passes infinitely many times through Case C_3. Assume the opposite, that is, after a certain iteration, the orbit does not pass through Case C_3 and therefore, eventually will be a concatenation of routes R_1 and R_2. For both routes, at the end of them, the tuple satisfying the conditions of Case C_4 will be of the form $\big(x, tx+y-s(w-z), z, w\big)$ where $t, s \in \mathbb{N}$ and $s \geq t$. This tuple will go to Case C_5 and will display the form $\big(x, z, z+w-tx-y+s(w-z), w\big)$ in Case C_5 (see Table 4.1). Taking into account the inequalities of the different cases, for every $t \in \mathbb{N}$, the third term of that tuple must be greater than or equal to the double of the second one, which yields to the following inequality

$$w - z \geq tx + y - s(w-z),$$

so,

$$s \geq \frac{tx+y}{w-z} - 1. \tag{4.35}$$

Furthermore, since the tuple $\big(x, tx+y-s(w-z), z, w\big)$ belongs to Case C_4, it holds that $z \leq tx+y-s(w-z)$, or, equivalently,

$$s \leq \frac{tx+y-z}{w-z}. \tag{4.36}$$

From (4.35) and (4.36), we get

$$\frac{tx+y}{w-z} - 1 \leq s \leq \frac{tx+y-z}{w-z}. \tag{4.37}$$

Hence, for every natural number t, we can guarantee the existence of a non-negative integer s in the interval

$$I_2 := \left[\frac{tx+y}{w-z} - 1, \frac{tx+y-z}{w-z}\right].$$

Observe that the length of such interval is $|I_2| = 1 - \frac{z}{w-z} \leq 1$. Furthermore, arguing as before, we can conclude that $\frac{tx+y}{w-z}$ can be an integer, or even a rational number, at most for a single value of $\tilde{t}$.

Next, in order to see that there exists an integer s belonging to the interval I_2, bearing in mind that $\frac{tx+y}{w-z}$ is not an integer number for $t > \tilde{t}$, we need

$$\frac{z}{w-z} \leq \left\{\frac{tx+y}{w-z}\right\}.$$

Indeed, if $B := \frac{tx+y}{w-z}$ is not an integer number, then $\lfloor B \rfloor \in [B-1, B]$. By construction, $\lfloor B \rfloor \in \big[B-1, B-\frac{z}{w-z}\big]$ if and only if $\lfloor B \rfloor \leq B - \frac{z}{w-z}$ or, equivalently, $\frac{z}{w-z} \leq B - \lfloor B \rfloor = \{B\}$.

Finally, since $\frac{x}{w-z}$ is irrational, by Corollary 10, the set

$$\left\{\left\{\frac{tx+y}{w-z}\right\}\right\}_{t\in\mathbb{N},t>\tilde{t}}$$

is dense in $[0,1]$. Therefore, there exists a natural number t such that

$$\frac{z}{w-z} > \left\{\frac{tx+y}{w-z}\right\}$$

and, consequently, it will not exist the corresponding natural number s.

Definitely, inequality (4.37) cannot hold for every $t \in \mathbb{N}$, which implies that the orbit must pass through C_3. Also, it must visit such case an infinite number of times, since if it only passes a finite number, after the last time, we could apply the same reasoning in order to achieve a contradiction. □

In conclusion, in this first step, we have proved that the non-periodic orbit generated by the iteration of Equation (4.2) over the initial conditions (x, y, z, w) satisfying the conditions of Case C_4, has to pass an infinite number of times through every Case C_i. •

Step 2: Density of the non-positive terms in $[w-x, 0]$. Here, we focus on the accumulation points of the non-positive terms that appear when the orbit passes through Case C_1. Recall that, by Proposition 37, the orbit will go through that case an infinite number of times. We only pay attention to the linear combinations displaying the form $tx + y - s(w - z) \leq 0$, since the other non-positive combinations that appear while passing through Case C_1, $w - (t + 1)x - y + s(w - z)$, are symmetric in $[w - x, 0]$. Observe that once we have the density of the first combinations, it will be enough.

Consider the initial conditions (x, y, z, w) verifying the restrictions of Case C_4. Every time that the orbit passes through Case C_1, that is, when a route R_1 or R_3 takes place, we will have a non-positive term of the form $\tilde{t}x + y - \tilde{s}(w - z) \leq 0$, with $\tilde{t}, \tilde{s} \in \mathbb{N}$. Let us consider the sequence formed by the non-positive linear combinations

$$\big(t_n x + y - s_n(w-z)\big)_n, \tag{4.38}$$

where $t_n, s_n \in \mathbb{N}$, $s_n \geq t_n$ and $s_{n+1} > s_n$ for every $n \geq 1$. It should be highlighted that the sequence of natural numbers $(s_n)_n$ is increasing, but we cannot apply Corollary 10, since it does not necessarily increase one by one. Therefore, in order to prove the density of sequence (4.38) in $[w - x, 0]$, we proceed as follows:

(a) We construct a more general sequence, where (4.38) is embedded, in order to have the coefficients of $(w - z)$ increasing one by one. In this sense, we take a sequence

$$\big(\tilde{t}_n x + y - n(w-z)\big)_n, \tag{4.39}$$

where every term $\tilde{t}_n x + y - n(w - z)$ belongs to $[-x, 0]$.

(b) We prove the density of (4.39) in $[-x, 0]$.

(c) We see that the terms of (4.39) that appear in (4.38) are in the interval $[w - x, 0]$, while the other terms are in $[-x, w - x]$.

Let us start with (a). We build the sequence (4.39). Observe that for every $n \in \mathbb{N}$, we have two possible scenarios:

- If the linear combination $t_n x + y - n(w - z)$ appears in (4.38), i.e., such that $s_n = n$, we take $\tilde{t}_n = t_n$.

- If $t_n x + y - n(w - z)$ does not belong to (4.38), by Lemma 44, we can ensure that for every $n \in \mathbb{N}$, there exists a natural number t such that $x \geq tx + y - n(w - z) \geq 0$, so $(t - 1)x + y - n(w - z)$ will be in $[-x, 0]$. Here, we set $\tilde{t}_n = t - 1$.

(b) We divide by x the terms of (4.39) in order to obtain the associated $\tilde{t}_n + \alpha - n\sigma \in [-1, 0]$, where $\alpha = \frac{y}{x}$ and $\sigma = \frac{w-z}{x} \in \mathbb{R}\setminus\mathbb{Q}$. This fact, jointly with $\{\tilde{t}_n + \alpha - n\sigma\} = \{\alpha - n\sigma\}$, enables us to apply Corollary 10 to deduce that $\left(\tilde{t}_n + \alpha - n\sigma\right)_n$ is dense in $[-1, 0]$ and, consequently, the sequence $\left(\tilde{t}_n x + y - n(w - z)\right)_n$ is dense in $[-x, 0]$.

(c) We claim that the terms of the subsequence (4.38) belong to $[w - x, 0]$, while the remaining terms of (4.39) are in the interval $[-x, w - x]$. Recall that we have already seen the first part of the claim in Lemma 45, so we only focus on the terms that do not appear in the non-periodic orbit.

Let us take a non-positive term $\tilde{t}_n x + y - n(w - z)$ of (4.39) that does not appear in (4.38). Due to the analysis developed in Subsection 4.3.2, we know that the positive linear combination $(\tilde{t}_n + 1)x + y - n(w - z) = tx + y - n(w - z)$ has to appear in some of the positive tuples of the orbit generated from (x, y, z, w). In concrete, $\big(x, tx + y - n(w - z), z, w\big)$ in Case C_1 or C_4; or $\big(x, z, w, tx + y - n(w - z)\big)$ in the Cases C_2 or C_3 .

- If $\big(x, tx + y - n(w - z), w, z\big)$ satisfies the conditions of Case C_1, one can see that $(t - 1)x + y - n(w - z) = \tilde{t}_n + y - n(w - z)$ belongs to the orbit, which contradicts our assumptions on $\tilde{t}_n$.

- If $\big(x, tx + y - n(w - z), w, z\big)$ belongs to the Case C_4, the inequality $x_4 \geq x_2$ must hold, thus $w \geq tx + y - n(w - z)$ and we obtain $-x + w \geq (t - 1)x + y - n(w - z)$, with $\tilde{t}_n = t - 1$.

- If $\big(x, z, w, tx + y - n(w - z)\big)$ verifies the restrictions of Cases C_2 or C_3, in particular, the third term of the tuple must be greater than or equal to the fourth term. Consequently, $w \geq tx + y - n(w - z)$, or, equivalently, $-x + w \geq (t - 1)x + y - n(w - z)$, with $\tilde{t}_n = t - 1$, as desired.

To sum up, we can divide the sequence $\big(\tilde{t}_n x + y - n(w-z)\big)_n$, that is dense in $[-x, 0]$, in two subsequences: the one formed by the non-positive terms that appear in the orbit, subsequence (4.38), that are in $[w-x, 0]$; and the other one formed by the terms that do not appear, which belong to the interval $[-x, w-x]$. This guarantees that the sequence $\big(t_n x + y - s_n(w-z)\big)_n$ of the non-positive terms that appear in the orbit is dense in the interval $[w-x, 0]$. •

Step 3: Density of the non-positive terms in $[-z, 0]$. Now, we proceed similarly to the previous case, but considering the non-positive terms that belong to the orbit while passing through the Case C_3. Specifically, we only center on the linear combinations of the form $tx + y - (s-1)(w-z) - w$, since the other non-positive terms that appear, $-tx - y + s(w-z)$, are symmetric to the first combinations in $[-z, 0]$.

Consider the initial conditions (x, y, z, w). The orbit generated by this tuple under Equation (4.2) passes through Case C_3 infinitely many times; recall that this happens when a route R_3 or R_4 takes place. Here, we set the sequence formed by the non-positive terms of the form $\tilde{t}x + y - \tilde{s}(w-z) - w$, that is,

$$\big(t_n x + y - s_n(w-z) - w\big)_n, \tag{4.40}$$

where $t_n, s_n \in \mathbb{N}$, $s_n \geq t_n$ and $s_{n+1} > s_n$ for every $n \geq 1$. Again, $(s_n)_n$ is an increasing sequence of natural numbers, but it does not necessarily increase one by one, so we construct a more general sequence for which (4.40) is a subsequence,

$$\big(\hat{t}_n x + y - n(w-z) - w\big)_n, \tag{4.41}$$

where every term $\hat{t}_n x + y - n(w-z) - w$ is in the interval $[-x, 0]$. Next, we develop three different steps analogously to the previous case:

(a) Given a natural number n, if there exists a linear combination in (4.40) of the form $t_n x + y - n(w-z)$, i.e. $s_n = n$, we take $\hat{t}_n = t_n$. Notice that such term appears by iterating a tuple of the orbit exhibiting the form $\big(x, z, w, tx+y-n(w-z)\big)$, which verifies the conditions of Case C_3. On the contrary, if such term does not exist in (4.40), by Lemma 44, we can ensure that for all $n \in \mathbb{N}$, there exists a $t \in \mathbb{N}$, such that $x \geq tx + y - n(w-z) \geq 0$. If $tx + y - n(w-z) - w < 0$, we set $\hat{t}_n = t$; otherwise, if $tx + y - n(w-z) - w \geq 0$, then $(t-1)x + y - n(w-z) - w \leq 0$ and we take $\hat{t}_n = t-1$. Notice that, in both cases, $\hat{t}_n x + y - n(w-z) - w \in [-x, 0]$.

(b) Let us divide by x the terms $\hat{t}_n x + y - n(w-z) - w$ in order to obtain $\hat{t}_n + \alpha - n\sigma - \delta$, where $\alpha = \frac{y}{x}$, $\sigma = \frac{w-z}{x} \in \mathbb{R} \setminus \mathbb{Q}$ and $\delta = \frac{w}{x}$. Hence, $-1 \leq \hat{t}_n + \alpha - n\sigma - \delta \leq 0$, and we get $\hat{t}_n + \alpha - n\sigma - \delta = \{\hat{t}_n + \alpha - n\sigma - \delta\} - 1$. Consequently, we can apply Corollary 10 to deduce the density of $\big(\hat{t}_n + \alpha - n\sigma - \delta\big)_n$ in the interval $[-1, 0]$ and, therefore, the density of the sequence (4.41) in $[-x, 0]$.

(c) Finally, we show that the terms of (4.41) that do not belong to the orbit are in the interval $[-x, -z]$. Moreover, recall that from the proof of Lemma 45 we already know that the terms of (4.41) that appear in the orbit, namely, the terms of the subsequence (4.40), belong to $[-z, 0]$.

Take the non-positive term $\hat{t}_n x + y - n(w-z) - w$ in (4.41) that does not belong to the orbit and let us analyze its evolution. As a first step, from Subsection 4.3.2, we know that a non-negative term of the form $tx + y - n(w-z)$ exists (see Lemma 44) and appears in some of the following scenarios: the tuple $\big(x, tx+y-n(w-z), z, w\big)$ in Case C_1 or C_4; or in the tuple $\big(x, z, w, tx+y-n(w-z)\big)$ in Case C_2 or C_3. Let us study in detail the diverse possibilities:

- If $(x_1, x_2, x_3, x_4) = \big(x, tx+y-n(w-z), z, w\big)$ satisfies the conditions of Case C_1, we have $x_1 \geq x_2 \geq x_4 \geq x_3 \geq 0$. This implies, in particular, that $tx+y-n(w-z)-w \geq 0$ and, consequently, by the definition of (4.41), $\hat{t}_n = t-1$. Moreover, $(t-1)x+y-n(w-z) \leq 0$ and $w - z \geq 0$. From here, $(t-1)x+y-(n+1)(w-z) \leq 0$, which is equivalent to the inequality $(t-1)x+y-n(w-z)-w \leq -z$, that implies $\hat{t}_n x+y-n(w-z)-w \leq -z$, as desired.

- If $(x_1, x_2, x_3, x_4) = \big(x, tx+y-n(w-z), z, w\big)$ is in Case C_4, the condition $x_2 \leq x_4$ holds, thus $tx+y-n(w-z)-w \leq 0$ and $\hat{t}_n = t$. Additionally, such tuple evolves to $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4) = \big(x, z, w+z-tx-y+n(w-z), w\big)$ in Case C_5. Regarding that our target is to prove the inequality $\hat{t}_n x+y-n(w-z)-w = tx+y-n(w-z)-w \leq -z$; once we are in Case C_5, we have two possibilities:

  ⋆ To pass from Case C_5 to C_3. This situation gives rise to the tuple $\big(x, z, w, tx + y-n(w-z)\big)$ that belongs to C_3. However, from Subsection 4.3.2, we have that the iteration of this tuple yields to the element $tx+y-n(w-z)-w$, which is contrary to the definition of $\hat{t}_n$.

  ⋆ To pass from Case C_5 to C_2. Here we obtain the tuple $\big(x, z, w, tx+y-n(w-z)\big)$ in C_2, which comes from the tuple

  $$(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4) = \big(x, z, z+w-tx-y+n(w-z), w\big) \ \text{ in } \ C_5$$

  verifying $\bar{x}_3 \geq 2\bar{x}_2$, (see Figure 5.5 and Table 4.1), that implies

  $$w+z-tx-y+n(w-z) \geq 2z,$$

  or, equivalently, $-z \geq tx+y-n(w-z)-w$.

- If $(x_1, x_2, x_3, x_4) = \big(x, z, w, tx+y-n(w-z)\big)$ satisfies the restrictions of Cases C_2 or C_3, we have that $x_3 \geq x_4$, which means, $w \geq tx+y-n(w-z)$, from where we get $tx+y-n(w-z)-w \leq 0$, and again, $\hat{t}_n = t$.

  ⋆ If the tuple is in C_2, in particular, $x_2+x_4 \leq x_3$ holds, so $x+tx+y-n(w-z) \leq w$, and we obtain the desired inequality $tx+y-n(w-z)-w \leq -z$.

$\star$ If the tuple is in C_3, from Subsection 4.3.2, we would have the non-positive element $tx+y-n(w-z)-w$, in contradiction with the definition of $\hat{t}_n$, which requires the non-existence of such non-positive terms.

To sum up, we have proved that we can divide the sequence $\big(\hat{t}_n x + y - n(w-z) - w\big)_n$, which is dense in $[-x, 0]$, in two subsequences: the one formed by the non-positive terms appearing in the orbit, subsequence (4.40), that are in $[-z, 0]$; and another one formed by those terms that do not belong to the orbit and that are in the interval $[-x, -z]$. In conclusion, we can guarantee the density of the non-positive terms that belong to the orbit, $\big(t_n x + y - s_n(w-z)\big)_n$, in the interval $[-z, 0]$. $\bullet$

C - Proof of Theorem E: Finally, we can gather the study developed in the previous subsections in order to prove Theorem E.

Theorem E. *Let (x_1, x_2, x_3, x_4) be arbitrary real initial conditions that generate a non-periodic orbit (x_n) under Equation (4.2). The, the set of accumulation points of (x_n) is a compact interval. Even more, the tuple (x_1, x_2, x_3, x_4) is equivalent to some tuple of initial conditions (x, y, z, w), with $x = \max\{x_n : n \geq 1\}$, $x \geq w \geq y \geq z \geq 0$, and $\frac{w-z}{x} \in \mathbb{R} \setminus \mathbb{Q}$, and the orbit accumulates in the compact interval $\big[\min\{w-x, -z\}, x\big]$.*

Proof. Take the initial conditions (x_1, x_2, x_3, x_4), where $x_i, i = 1, 2, 3, 4$, are real numbers. By Definition 4, we know that those terms are equivalent to a tuple of non-negative terms (x, y, z, w), where $x = \max\{x_n : n \geq 1\}$, and they satisfy the conditions from the Case C_4, namely, $x \geq w \geq y \geq z$. Assume that they generate a non-periodic orbit under Equation (4.2). Therefore, by Proposition 35, $\frac{w-z}{x} \in \mathbb{R} \setminus \mathbb{Q}$, and $x > z$ in order to avoid monotonic initial conditions that would yield to an 11-cycle.

As a final step, from the analysis developed previously, we deduce that the solution (x_n) is dense in a compact interval, more precisely, it accumulates in $\big[\min\{w-x, -z\}, x\big]$. $\square$

4.4 Invariance and first integrals

In this section, where the results we present are based on [62], we focus on the relation between Equation (4.2) and its associate discrete dynamical system $X_{n+1} = F(X_n)$, where $F : \mathbb{R}^4 \to \mathbb{R}^4$ is defined by

$$F(x, y, z, w) = \big(y, z, w, \max(0, y, z, w) - x\big).$$

In Section 4.3, where we have studied the non-periodic solutions of Equation (4.2), we have shown that their limit sets are closed intervals. These intervals can be regarded as projections of the limit sets of the corresponding trajectories, denoted by γ_X, having initial conditions $X \in \mathbb{R}^4$ within the discrete dynamical system linked to F. In this sense, it would be stimulating to determine the topological characteristics of the limit sets of its orbits

$$\omega_F(\gamma_X) = \{Y \in \mathbb{R}^4 : \exists\, (n_j)_j \subset \mathbb{N}, n_1 < n_2 < \ldots < n_j < \ldots, \lim_{j\to\infty} F^{n_j}(X) = Y\}.$$

Our numerical simulations indicate that these trajectories could densely fill closed curves of $\mathbb{R}^4$. Additionally, these curves may be simple, which means, lacking self-intersections. Investigating the topology of these limit sets or establishing, at least, the connectivity of $\omega_F(\gamma_X)$ would be of considerable interest.

In this direction, we present two examples. The first one considers the initial conditions $\left(2\sqrt{2}, 2, 0, 1\right)$, which, by Definition 4, are equivalent to the tuple $\left(2\sqrt{2}, 1, 0, 1\right)$ in Case C_4. We have computed 10^4 iterates of such tuple under F and in Figure 4.2 we show a pair of views of a projection of them into $\mathbb{R}^3$. Notice that Theorem E establishes that the solution (x_n) of Eq. (4.2) accumulates in the interval $\left[1-2\sqrt{2}, 2\sqrt{2}\right]$. The inspection of the iterates of F with this initial conditions indicates that the iterates fill a closed curve. The self-intersections that appear could be due to an effect of the projection of the curve into $\mathbb{R}^3$.

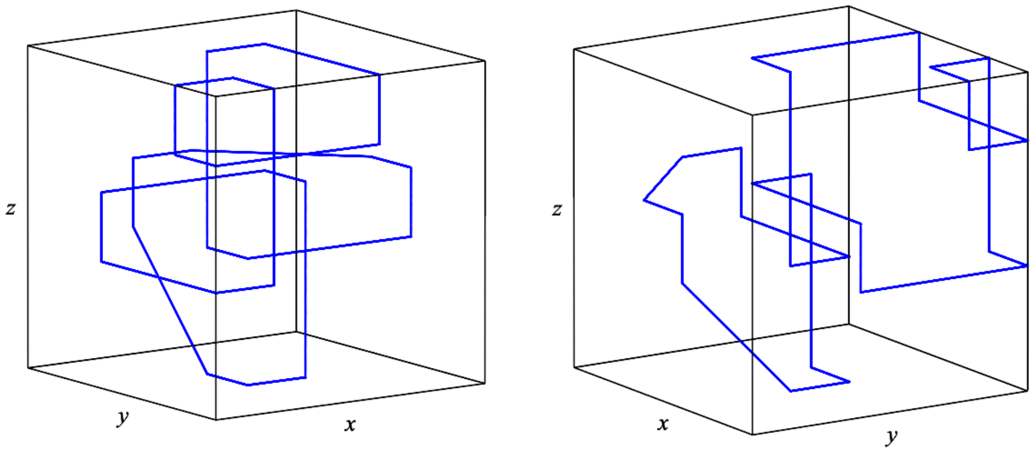


Figure 4.2: Two views of the orbit of the map F with initial conditions $(2\sqrt{2}, 2, 0, 1)$.

For another example, we consider the initial conditions $\left(\sqrt{2}+10\sqrt{3}, 1, 2, 0\right)$ and we iterate them 10^4 times under the map F. In Figure 4.3 we show some projections in $\mathbb{R}^3$ of those iterations. In this case, it is easy to check that the set of initial conditions is equivalent to $\left(\sqrt{2}+10\sqrt{3}, \sqrt{2}+10\sqrt{3}-17, 1, 2\right)$, which is in Case C_4. Therefore, from Theorem E, the sequence (x_n) generated by Equation (4.2) accumulates in the interval $\left[2-\sqrt{2}-10\sqrt{3}, \sqrt{2}+10\sqrt{3}\right]$. The inspection of this projection into $\mathbb{R}^3$ indicates that it is a simple closed curve.

Finally, we make a comment concerning the first integrals associated to the discrete dynamical system. In this line, recall that a *first integral* of the discrete dynamical system

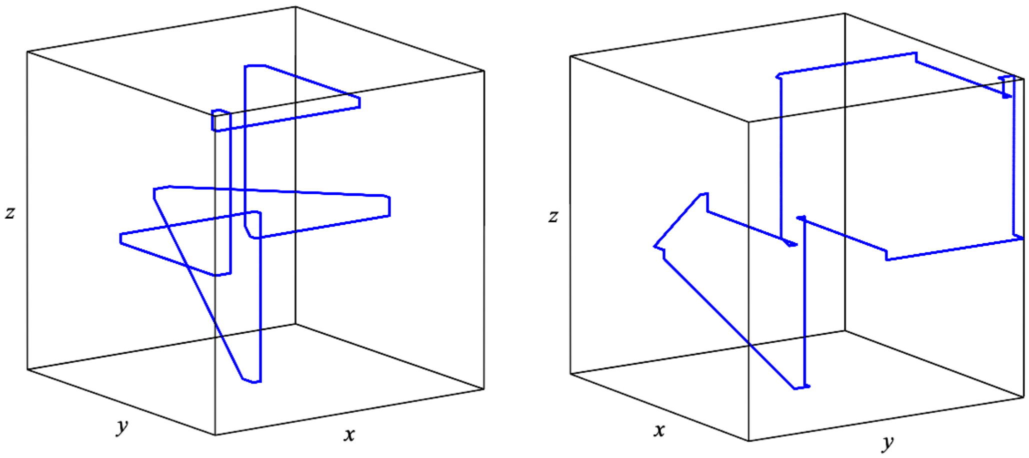


Figure 4.3: Two views of the orbit of the map F with initial conditions $\left(10\sqrt{3}+\sqrt{2},1,2,0\right)$.

in $\mathbb{R}^n$ generated by a map G is a non constant function in a nonempty open set $\mathcal{U} \subseteq \mathbb{R}^n$, $V : \mathcal{U} \to \mathbb{R}$, which is constant on the orbits, i.e.,

$$V(F(\boldsymbol{x})) = V(\boldsymbol{x}) \text{ for all } \boldsymbol{x} \in \mathcal{U}.$$

A set $V_1, \ldots, V_k$ of first integrals of G defined in an open set $\mathcal{U}$ are *functionally dependent* if there exists a real-valued function $R : \mathcal{U} \to \mathbb{R}$ not identically zero such that $R\big(V_1(\boldsymbol{x}), \ldots, V_k(\boldsymbol{x})\big) = 0$ for all $\boldsymbol{x} \in \mathcal{U}$. Otherwise, we say that they are *functionally independent*, [88, pages 84–85]. Also, we say that G is *completely integrable* if it has n functionally independent first integrals.

It is well-known, see [11], that Equation (4.2) has an invariant, which yields to the existence of a first integral of F,

$$\begin{aligned} V_1(x,y,z,w) = &\max(0,x,y,z,w) + \max(0,-x) + \max(0,-y) + \max(0,-z) \\ &+ \max(0,-w). \end{aligned}$$

From our simulations, it can be suggested that the map F could have exactly three functionally independent first integrals. Also, from [23, Theorem 1(b)], it is known that the map F cannot be completely integrable because it is not globally periodic. In light of our simulations, we found it intriguing to discover new first integrals of F (or invariants for (4.2)) functionally independent with V_1.

Proposition 38. *The function*

$$\begin{aligned} V_2(x,y,z,w) = &\max(0,x,y,z,w,x+w) + \max(0,x,y) + \max(0,y,z) + \max(0,z,w) \\ &- x - y - z - w \end{aligned}$$

is a first integral of the map F. In other words, it is an invariant function for the recurrence (4.2). Furthermore, there exist nonempty open sets in $\mathbb{R}^4$ where V_2 is functionally independent with V_1.

Proof. Our target is to prove that $\Delta V_2 = V_2\big(y,z,w,F(x,y,z,w)\big) - V_2(x,y,z,w) \equiv 0$. A straightforward computation shows that

$$\begin{aligned}\Delta V_2 =& \max\{0,y,z,w,\max\{y,z,w,0\}-x,\max\{y,z,w,0\}-x+y\}\\ &+\max\{0,w,\max\{y,z,w,0\}-x\}-\max\{0,x,y,z,w,x+w\}\\ &-\max\{0,y,z,w\}-\max\{0,x,y\}+2x.\end{aligned}$$

As a next step, we see that $\Delta V_2 \equiv 0$ when $x \geq 0$ and $x \geq \max\{0,y,z,w\}$. The remaining cases (namely, $0 \leq x \leq \max\{0,y,z,w\}$; $x < 0$ and $x \geq \max\{0,y,z,w\}$; and $x < 0$ and $x \leq \max\{0,y,z,w\}$) can be done analogously. Indeed, suppose that $x \geq 0$ and $x \geq \max\{0,y,z,w\}$, then

- $\max\big\{0,y,z,w,\max\{y,z,w,0\}-x,\max\{y,z,w,0\}-x+y\big\} = \max\{0,y,z,w\}$,
- $\max\big\{0,w,\max\{y,z,w,0\}-x\big\} = \max\{0,w\}$,
- $\max\{0,x,y,z,w,x+w\} = \max\{x,x+w\}$,
- $\max\{0,x,y\} = x$.

Thus

$$\Delta V_2 = \max\{0,w\} - \max\{x,x+w\} + x.$$

If $w \geq 0$, then $\Delta V_2 = w - x - w + x = 0$; if $w < 0$, then $\Delta V_2 = x - x = 0$. Therefore, $\Delta V_2 \equiv 0$.

Finally, in order to see that there exist open sets in which V_1 and V_2 are functionally independent, consider an initial condition in the open set $\mathcal{U} = \{(x,y,z,w) : x > w > y > z > 0\}$, that is, verifying the restrictions of Case C_4 with strict inequalities. A computation shows that, in this case,

$$V_1(x,y,z,w) = x \text{ and } V_2(x,y,z,w) = x + w - z,$$

which are obviously functionally independent. □

We think that it is worth mentioning how we have found the second invariant V_2. Observe that Equation (4.2) is the ultradiscretization, in the sense of [87], of the 4-th order Lyness' Equation

$$x_{n+4} = \frac{a + x_{n+1} + x_{n+2} + x_{n+3}}{x_n},$$

where a is a real number, or, equivalently, that F is the ultradiscretization of the 4-dimensional Lyness map

$$L_4(x,y,z,w) = \left(y,z,w,\frac{a+y+z+w}{x}\right).$$

It is known that L_4 has two functionally independent first integrals [37], see also [24, 101]:

$$H_1(x,y,z,w) = \frac{(a+x+y+z+w)(x+1)(y+1)(z+1)(w+1)}{xyzw},$$
$$H_2(x,y,z,w) = \frac{(a+x+y+z+w+xw)(1+x+y)(1+y+z)(1+z+w)}{xyzw}.$$

It can be easily seen that V_1 is the ultradiscrete version of H_1 and this fact encouraged us to obtained V_2 as the ultradiscretization of H_2. For the sake of completeness, the main formula for the ultradiscretization is given by

$$\lim_{\varepsilon \to 0^+} \varepsilon \log \left(e^{X/\varepsilon} + e^{Y/\varepsilon} \right) = \max\{X, Y\},$$

for arbitrary real numbers X and Y. The above formula allows us to transform the field of real numbers into the so-called "max-plus" algebra. In [87], the authors extended this concept by defining what they called "inversible max-plus algebra", which includes a correspondence for the subtraction.

Even though, as far as we know, it is not proved, the results we have presented here are in consonance with the conjecture proposed in [37], which states that the maximum number of functionally independent first integrals of L_k is $\lfloor (k+1)/2 \rfloor$. If this was correct, L_4 should not admit more than two functionally independent first integrals. However, our numerical simulations, as it was highlighted before, are compatible with the fact that the map F could have three functionally independent first integrals (compare the graphs of Figures 4.2 and 4.3 and the graph of the iterations of a map L_4 that is presented in Figure 1 of [24], where it is clearly intuited that the iterates of L_4 evolve over a 2-dimensional surface). The existence of a third first integral for the map F would show that the maps that come from the ultradiscretization of non–globally periodic rational maps can have more first integrals than the original rational maps (remember that in [87, Theorem 3.5] it is shown that globally periodic rational maps give rise to globally periodic ultradiscrete maps; in such a case, both maps will have as many first integrals as the phase space, according to the results in [23]).

4.5 Open questions and forthcoming lines of research

In this chapter, we have described the dynamics of Equation (4.2) by characterizing its periodic orbits and by determining the accumulation points of the non-periodic solutions. Therefore, a natural continuation of the problem is to study the dynamics of

$$x_{n+k} = \max\{x_{n+k-1}, \ldots, x_{n+1}, 0\} - x_n,$$

for $k \geq 5$, where we believe that the techniques developed here could be useful. As a first step, it would be interesting to focus on the particular case $k = 5$ and to try, at least, to prove if the set $\mathbb{N} \setminus \mathrm{Per}(F_k)$ is bounded.

On the other hand, we have given a precise description of the elements of $\mathrm{Per}(F_4)$ by determining the greatest number that does not belong to the set, namely, 1674, and by giving in detail all its elements, which are collected in Table 4.2. This study, developed in Section 4.2, was related to a coin problem with additional restrictions on the coefficients of the linear combinations, $\gcd(a, b) = 1$ and $b \geq 2a + 1$. As far as we know, this variation of the coin problem has not been studied in the literature before, so some interesting questions arise in this sense:

- To investigate the boundedness character of the complementary of semigroups generated by two coprime numbers, p and q, whose elements are formed through combinations of the type $a \cdot p + b \cdot q$, with a, b natural integers verifying $\gcd(a, b) = 1$. Additionally, in the bounded scenario, to derive a formula for the largest positive integer that cannot be represented under the assumption of being coprime a, b.

- To extend the prior problem in the case where p and q are not necessarily coprime numbers.

- To analyze the problem of adding extra conditions on the coefficients of the linear combinations, a and b, apart from the fact of being coprime, for instance, linear inequalities.

Moreover, in the sense of Section 4.4, we leave as an open problem the possibility of finding a new functionally independent first integral for the map F.

In conclusion the chapter has deepen in the study of diverse dynamical properties of autonomous difference equations. Specifically, we have dealt with the property of periodicity, the accumulation point sets for the solutions of autonomous difference equations and the property of invariance and its corresponding equivalence for dynamical systems, that is, the existence of first integrals. In the next chapter, we will continue with the analysis of other dynamical properties, namely, attraction and topological conjugacies.

Chapter 5

Topological conjugacies and attraction

The first time that the idea of a strange attractor appeared in the literature was in 1963 when the meteorologist E.N. Lorenz analyzed a system of three first-order differential equations, see [71]. In his study, he showed that the solutions of the system were unstable with respect to small variations, which means that minimal modifications of the initial conditions can produce significantly different dynamics.

This phenomenon encouraged the study of other systems exhibiting the same properties. In light of this, in 1976, M. Hénon discovered the first strange attractor for a discrete dynamical system. In concrete, he dealt with the two-dimensional mapping

$$\begin{cases} x_{n+1} &= y_n + 1 - a x_n^2 \\ y_{n+1} &= b x_n \end{cases}, \tag{5.1}$$

where a and b are positive real numbers. In [47], he presented simulations for the case $a = 1.4$ and $b = 0.3$ and showed that, depending on the initial conditions, every solution of Equation (5.1) either diverges to infinity or tends to a strange attractor which seems to be the product of a one-dimensional manifold by a Cantor set, see Figure 5.1.

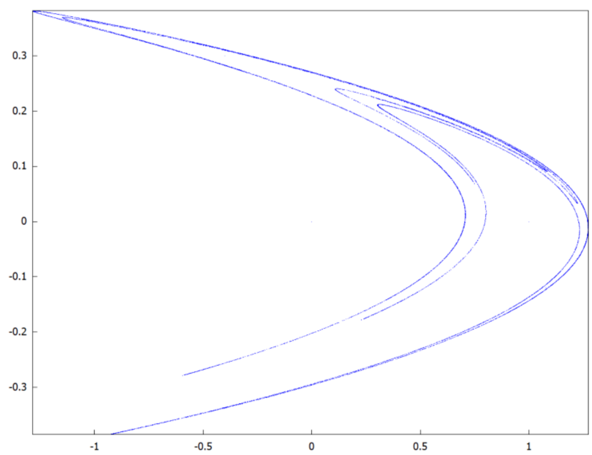


Figure 5.1: Hénon map for $a = 1.4$, $b = 0.3$. 20000 iterations.

Later on, in 1978, R. Lozi exchanged the quadratic term in Hénon's map by the absolute value function. Specifically, he studied the system of difference equations

$$\begin{cases} x_{n+1} &= 1 - a|x_n| + y_n \\ y_{n+1} &= bx_n \end{cases}, \tag{5.2}$$

where a and b are real numbers. In [72], he presented numerical simulations for the parametric values $a = 1.7$ and $b = 0.5$, where a strange attractor that seemed the product of parts of straight lines by a Cantor set appeared, see Figure 5.2.

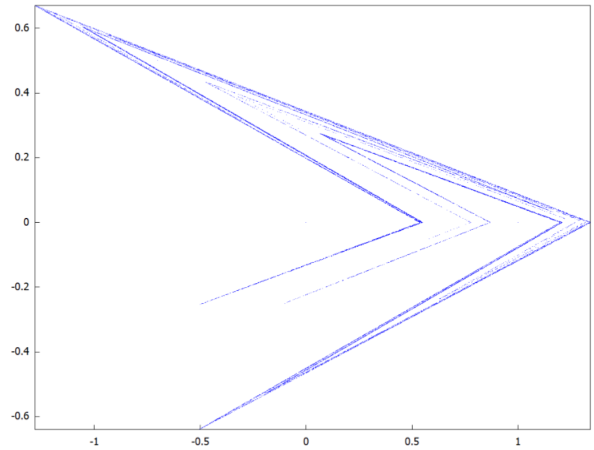


Figure 5.2: Lozi map for $a = 1.7$, $b = 0.5$. 20000 iterations.

It is worth mentioning that System (5.2) can be found in the literature in the following equivalent form:

$$\begin{cases} x_{n+1} &= 1 - a|x_n| + by_n \\ y_{n+1} &= x_n \end{cases}. \tag{5.3}$$

Also, notice that System (5.2) or (5.3) can be reduced into the second-order piecewise linear difference equation

$$x_{n+1} = 1 - a|x_n| + bx_{n-1}, \tag{5.4}$$

with a and b real numbers. In the sequel, we call Equation (5.4) as *Lozi map*.

It merits the attention to highlight that Lozi map gave rise to an abundant literature and that, indeed, the first analytical proof for the existence of a strange attractor was developed for this map. Concretely, in 1980, M. Misiurewicz, [82], established trapping regions for certain values of the parameters a and b, that is, nonempty sets that are mapped with its closure into its interior. This enabled the author to show the hyperbolic structure of the Lozi map and allowed him to deduce that the intersection of the images of the trapping regions were a strange attractor. In fact, the parametric region where such attractor appears is given by the following restrictions:

$$0 < b < 1; \quad a > b + 1; \quad 2a + b < 4; \quad b < \frac{a^2 - 1}{2a + 1}; \quad \sqrt{2}a > b + 2. \tag{5.5}$$

Moreover, in [10], the authors describe the basin of attraction of the strange attractor of Lozi map when the parameters a and b verify (5.5), that is, the set of points of the plane whose orbits tend to the strange attractor.

To illustrate some of the research developed after Lozi map, we can start by emphasizing the particular case of Equation (5.4) with $a = b = -1$, that is,

$$x_{n+1} = 1 + |x_n| - x_{n-1},$$

which is known as *Gingerbreadman equation*. In [31], R.L. Devaney considers the associated system

$$\begin{cases} x_{n+1} &= & 1 + |x_n| - y_n \\ y_{n+1} &= & x_n \end{cases}. \tag{5.6}$$

System (5.6) has a unique equilibrium point, $(1, 1)$, which is non-hyperbolic and his stability matrix has complex conjugated eigenvalues. Moreover, the existence of an hexagon where every solution, except the equilibrium, is periodic of period 6 was proved. Nevertheless, the author was able to show that the equilibrium is surrounded by infinitely many invariant polygons of arbitrarily large radius and that the regions between any of those consecutive polygons provide the equation with zones of instability. Therefore, System (5.6) is stable in some regions and chaotic in others.

Furthermore, M. Crampin studied the piecewise linear equation

$$x_{n+1} = |x_n| - x_{n-1}. \tag{5.7}$$

Such equation is globally periodic of period 9, see [28]. In addition, he noticed that the linear difference equations $x_{n+1} = x_n - x_{n-1}$ and $x_{n+1} = -x_n - x_{n-1}$, related to Equation (5.7) are also globally periodic of periods 6 and 3, respectively. In this direction, he proposed the study of the combination of two periodic linear difference equations through a piecewise linear equation.

In the literature, see for example [40], it can be found a connection between a generalization of Lozi map and a well-known class of difference equations named max-type equations. To have a general scope of the dynamics of diverse classes of difference equations with maximum, consult the survey [69].

In this sense, we propose the piecewise linear difference equation

$$y_{n+1} = \alpha|y_n| + \beta y_n + \gamma y_{n-1} + \delta, \tag{5.8}$$

where $\alpha, \beta, \gamma, \delta$ are real numbers with $\alpha \neq 0$. In the sequel, we refer to Equation (5.8) as *generalized Lozi map*. Observe that it is easy to recover the original equation, (5.4), by taking $\alpha = -a$, $\beta = 0$, $\gamma = b$ and $\delta = 1$.

The present chapter, which is mainly based on [67], delves into the relationship between Lozi maps and certain classes of max-type difference equations. In this sense, it is organized as follows: as a first step, we present in Section 5.1 a topological conjugation that allows us to link a generalized Lozi map with max-type difference equations.

Theorem F. *Consider the generalized Lozi map with $\delta = 0$,*

$$y_{n+1} = \alpha|y_n| + \beta y_n + \gamma y_{n-1}.$$

Then, it is topologically conjugate to the max-type equation

$$z_{n+1} = \frac{\max\{z_n^{2\alpha}, B\}}{z_n^{\alpha-\beta} \cdot z_{n-1}^{-\gamma}} \cdot B^{\frac{\alpha+\beta+\gamma-1}{-2\alpha}},$$

for all $B > 0$.

In particular, if $\alpha+\beta+\gamma-1 = 0$, then the generalized Lozi map is topologically conjugate to the max-type equation

$$z_{n+1} = \frac{\max\{z_n^{2\alpha}, B\}}{z_n^{\alpha-\beta} \cdot z_{n-1}^{-\gamma}},$$

for all $B > 0$.

Theorem G. *Consider the generalized Lozi map with $\delta \neq 0$,*

$$y_{n+1} = \alpha|y_n| + \beta y_n + \gamma y_{n-1} + \delta.$$

Then, either for $A > 1$ and $\frac{\alpha}{q} > 0$, or for $0 < A < 1$ and $\frac{\alpha}{q} < 0$, with $q \in \mathbb{R} \setminus \{0\}$, Equation (5.8) is topologically conjugate to

$$z_{n+1} = \frac{\max\{z_n^{2\alpha}, A^{\frac{-2\alpha p}{q}}\}}{z_n^{\alpha-\beta} \cdot z_{n-1}^{-\gamma}} \cdot A^{\frac{p(\alpha+\beta+\gamma-1)+\delta}{q}}, \quad \text{for all } \ p \in \mathbb{R}.$$

In particular, assuming that $\alpha+\beta+\gamma-1 \neq 0$, Equation (5.8) is topologically conjugate to:

- $z_{n+1} = \frac{\max\{z_n^{2\alpha}, B\}}{z_n^{\alpha-\beta} \cdot z_{n-1}^{-\gamma}}$, *for all $B > 1$, if $\frac{\delta}{\alpha+\beta+\gamma-1} > 0$.*
- $z_{n+1} = \frac{\max\{z_n^{2\alpha}, C\}}{z_n^{\alpha-\beta} \cdot z_{n-1}^{-\gamma}}$, *for all $0 < C < 1$, if $\frac{\delta}{\alpha+\beta+\gamma-1} < 0$.*

Moreover, if $\alpha + \beta + \gamma - 1 = 0$, additionally we get that Equation (5.8) is topologically conjugate to:

- $z_{n+1} = \frac{\max\{z_n^{2\alpha}, 1\}}{z_n^{\alpha-\beta} \cdot z_{n-1}^{-\gamma}} \cdot B$, *for all $B > 1$, if $\frac{\delta}{\alpha} > 0$.*
- $z_{n+1} = \frac{\max\{z_n^{2\alpha}, 1\}}{z_n^{\alpha-\beta} \cdot z_{n-1}^{-\gamma}} \cdot C$, *for all $0 < C < 1$, if $\frac{\delta}{\alpha} < 0$.*

The relevance of this relation resides in the connection of the dynamics of a concrete generalized Lozi map with a whole family of max-equations. In this line, it is worth mentioning that all the members of this family share the same dynamics as they are topologically conjugate to the same generalized Lozi map. To illustrate this, in Section 5.2, we determine the dynamics of a one-parametric family of max-type equations, namely,

$$x_{n+1} = \frac{\max\{x_n^3, A\}}{x_n x_{n-1}}, \quad A > 0,$$

by studying its corresponding topologically conjugate generalized Lozi map

$$y_{n+1} = \frac{3}{2}|y_n| + \frac{1}{2}y_n - y_{n-1}.$$

Later on, in Section 5.3, we focus on the particular case of the Lozi map for $a = b$. Recall that taking $\alpha = -a$, $\beta = 0$, $\gamma = b$ and $\delta = 1$ we can recover Lozi map from its generalization, so we can also apply the topological conjugation developed in Theorem G. Here, the dynamics of the equation is well-known for $|a| < \frac{1}{2}$, so we deal with the boundary cases and determine its dynamics for $a = \frac{1}{2}$ and $a = -\frac{1}{2}$.

Theorem H. *Given the difference equation*

$$x_{n+1} = 1 - \frac{1}{2}|x_n| + \frac{1}{2}x_{n-1},$$

its dynamics is given by:

- ***(a)*** *An equilibrium point,* $\overline{x} = 1$.
- ***(b)*** *A continuum of* 2*-periodic sequences* $(\ldots, x, y, x, y, \ldots)$ *with* $0 \leq x, y \leq 2$, $x + y = 2$.
- ***(c)*** *The rest of solutions converge to one of the* 2*-periodic solutions given in Part (b).*

Theorem I. *Given the difference equation*

$$x_{n+1} = 1 + \frac{1}{2}|x_n| - \frac{1}{2}x_{n-1},$$

its unique equilibrium point $\bar{x} = 1$ *is a global attractor.*

Then, Section 5.4 gathers some simulations and display some particular dynamics which encourage the consideration of some associated problems. Finally, we present some conclusions in Section 5.5.

5.1 The transformation

The main purpose of this section is to present a connection between the Lozi map and a family of max-type difference equations. Furthermore, we establish such connection for a generalization of the Lozi map. It should be mentioned that in [40], the authors introduce the following generalization:

$$y_{n+1} = \frac{k}{2}|y_n| + \left(\frac{k}{2} - l\right) y_n - m y_{n-1} + \delta, \tag{5.9}$$

where $k, l, m, \delta \in \mathbb{Z}$. Nevertheless, the restriction of the parameters to the integer set induce to propose a more extensive generalization. With this aim, we present the piecewise linear difference equation (5.8),

$$y_{n+1} = \alpha|y_n| + \beta y_n + \gamma y_{n-1} + \delta,$$

where $\alpha, \beta, \gamma, \delta$ are real numbers with $\alpha \neq 0$.

Now we transform the generalized Lozi map (5.8) into a family of max-type difference equations. To do so, we apply the change of variables

$$y_n = \log_A\left(z_n^q\right) + p,$$

where p and q are arbitrarily taken real numbers with $q \neq 0$, $A > 0$ arbitrary too, with $A \neq 1$, and $z_n > 0$ for all $n \geq -1$. Thus, we obtain

$$\log_A(z_{n+1}) = \frac{\alpha}{q}\big|\log_A(z_n^q \cdot P)\big| + \log_A(z_n^\beta \cdot z_{n-1}^\gamma) + \frac{\beta p + \gamma p + \delta - p}{q},$$

where $P = A^p$. Moreover, bearing in mind that $|z| = \max\{z, -z\}$, we can change the above expression into

$$\log_A(z_{n+1}) = \frac{\alpha}{q}\max\big\{\log_A(z_n^q \cdot P), -\log_A(z_n^q \cdot P)\big\} + \log_A(z_n^\beta \cdot z_{n-1}^\gamma) + \frac{\beta p + \gamma p + \delta - p}{q}. \quad (5.10)$$

As a next step, we exchange the maximum function and the logarithm. To do so, recall that the logarithm function $\log_A x$ is increasing (decreasing) when $A > 1$ ($0 < A < 1$); and we set an arbitrary q satisfying $\frac{\alpha}{q} > 0$, $\left(\frac{\alpha}{q} < 0\right)$. Then, for $A > 1$ and $\frac{\alpha}{q} > 0$,

$$\frac{\alpha}{q}\max\big\{\log_A(z_n^q \cdot P), -\log_A(z_n^q \cdot P)\big\} = \log_A\left(\max\left\{z_n^\alpha \cdot P^{\alpha/q}, \frac{1}{z_n^\alpha \cdot P^{\alpha/q}}\right\}\right);$$

and for $0 < A < 1$ and $\frac{\alpha}{q} < 0$, we obtain

$$\begin{aligned} \frac{\alpha}{q}\max\{\log_A(z_n^q \cdot P), -\log_A(z_n^q \cdot P)\} &= \min\left\{\log_A\left(z_n^\alpha \cdot P^{\alpha/q}\right), \log_A\left(\frac{1}{z_n^\alpha \cdot P^{\alpha/q}}\right)\right\} \\ &= \log_A\left(\max\left\{z_n^\alpha \cdot P^{\alpha/q}, \frac{1}{z_n^\alpha \cdot P^{\alpha/q}}\right\}\right). \end{aligned}$$

We collect the previous discussion in the following result.

Lemma 46. *Consider the generalized Lozi map*

$$y_{n+1} = \alpha|y_n| + \beta y_n + \gamma y_{n-1} + \delta,$$

with $\alpha, \beta, \gamma, \delta$ being real numbers and $\alpha \neq 0$. Let p and q be arbitrarily real numbers, and $A > 0$. Then, the equation is topologically conjugate to

$$\log_A(z_{n+1}) = \log_A\left(\frac{\max\left\{z_n^{2\alpha}, A^{-2\alpha p/q}\right\}}{z_n^{\alpha-\beta} \cdot z_{n-1}^{-\gamma}}\right) + \frac{p(\alpha + \beta + \gamma - 1) + \delta}{q}, \quad (5.11)$$

if $A > 1$ and $\frac{\alpha}{q} > 0$; or $0 < A < 1$ and $\frac{\alpha}{q} < 0$.

Observe that we have established a topological conjugation between the generalized Lozi map, (5.8), and Equation (5.11). Indeed, take the homeomorphism $\varphi : (0,\infty)^2 \to \mathbb{R}^2$ given by $\varphi(x,y) = \big(\phi(x), \phi(y)\big)$, where $\phi(z) = \log_A(z^q) + p$. Hence, $\phi^{-1}(w) = A^{\frac{w-p}{q}}$. As a consequence, if we consider the associated dynamical system $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined as $F(x,y) = \big(y, \alpha|y| + \beta y + \gamma x + \delta\big)$, and by $\tilde{F}$ the system $\tilde{F} : (0,\infty)^2 \to (0,\infty)^2$ given by

$$\tilde{F} = \left(y, \frac{\max\left\{y^{2\alpha}, A^{-\frac{2\alpha p}{q}}\right\}}{y^{\alpha-\beta} \cdot x^{-\gamma}} \cdot A^{\frac{p(\alpha+\beta+\gamma-1)+\delta}{q}} \right),$$

it is direct to check that $\tilde{F} = \varphi^{-1} \circ F \circ \varphi$ holds.

Now, we present some consequences of Lemma 46 depending on the value δ.

Theorem F. *Consider the generalized Lozi map with* $\delta = 0$*,*

$$y_{n+1} = \alpha|y_n| + \beta y_n + \gamma y_{n-1}.$$

Then, it is topologically conjugate to the max-type equation

$$z_{n+1} = \frac{\max\{z_n^{2\alpha}, B\}}{z_n^{\alpha-\beta} \cdot z_{n-1}^{-\gamma}} \cdot B^{\frac{\alpha+\beta+\gamma-1}{-2\alpha}}, \tag{5.12}$$

for all $B > 0$*.*

In particular, if $\alpha+\beta+\gamma-1 = 0$*, then the generalized Lozi map is topologically conjugate to the max-type equation*

$$z_{n+1} = \frac{\max\{z_n^{2\alpha}, B\}}{z_n^{\alpha-\beta} \cdot z_{n-1}^{-\gamma}}, \tag{5.13}$$

for all $B > 0$*.*

Proof. Firstly, assume that $\delta = 0$ and $\alpha+\beta+\gamma-1 \neq 0$. In this case, Equation (5.11) is equivalent to

$$\log_A(z_{n+1}) = \log_A\left(\frac{\max\left\{z_n^{2\alpha}, A^{-2\alpha p/q}\right\}}{z_n^{\alpha-\beta} \cdot z_{n-1}^{-\gamma}} \right) + \frac{p(\alpha+\beta+\gamma-1)}{q}.$$

If we set $p = 0$, we obtain

$$z_{n+1} = \frac{\max\{z_n^{2\alpha}, 1\}}{z_n^{\alpha-\beta} \cdot z_{n-1}^{-\gamma}}.$$

If, otherwise, $p \neq 0$, we can write the equation as follows

$$z_{n+1} = \frac{\max\left\{z_n^{2\alpha}, A^{\frac{-2\alpha p}{q}}\right\}}{z_n^{\alpha-\beta} \cdot z_{n-1}^{-\gamma}} \cdot A^{\frac{p(\alpha+\beta+\gamma-1)}{q}} = \frac{\max\left\{z_n^{2\alpha}, A^{\frac{-2\alpha p}{q}}\right\}}{z_n^{\alpha-\beta} \cdot z_{n-1}^{-\gamma}} \cdot A^{\left(\frac{-2\alpha p}{q}\right) \cdot \frac{\alpha+\beta+\gamma-1}{-2\alpha}}.$$

Notice that for $A > 1$ and $\frac{\alpha}{q} > 0$, since p is arbitrarily taken, once that q is fixed, having the same sign as α, we have that $A^{-\frac{2\alpha p}{q}} \in (0,\infty)$. Thus, if we put $B = A^{-\frac{2\alpha p}{q}}$, we achieve (5.12). For $0 < A < 1$ and $\frac{\alpha}{q} < 0$ we can proceed similarly.

On the other hand, if $\alpha+\beta+\gamma-1=0$, we can take an arbitrary p, and get

$$z_{n+1}=\max\left\{z_n^{2\alpha},A^{-2\alpha p/q}\right\}\cdot z_n^{\beta-\alpha}\cdot z_{n-1}^{\gamma}=\frac{\max\left\{z_n^{2\alpha},\left(\frac{1}{A}\right)^{\frac{2\alpha p}{q}}\right\}}{z_n^{\alpha-\beta}\cdot z_{n-1}^{-\gamma}}.$$

Again, due to the fact that p is arbitrarily chosen, if $A>1$ and $\frac{\alpha}{q}>0$, we know that $\left(\frac{1}{A}\right)^{\frac{2\alpha p}{q}}\in(0,\infty)$, and we obtain

$$z_{n+1}=\frac{\max\{z_n^{2\alpha},B\}}{z_n^{\alpha-\beta}\cdot z_{n-1}^{-\gamma}},$$

for all $B>0$. The same applies if $0<A<1$ and $\frac{\alpha}{q}<0$. □

Example 1. *Consider the piecewise linear difference equation*

$$y_{n+1}=\frac{5}{2}|y_n|-\frac{1}{2}y_n-2y_{n-1}.$$

By Theorem F, it is topologically conjugate to any max-type equation

$$z_{n+1}=\frac{\max\{z_n^5,B\}}{z_n^3\cdot z_{n-1}^2},\quad B>0.$$

□

Theorem G. *Consider the generalized Lozi map with $\delta\neq 0$,*

$$y_{n+1}=\alpha|y_n|+\beta y_n+\gamma y_{n-1}+\delta. \tag{5.14}$$

Then, either for $A>1$ and $\frac{\alpha}{q}>0$, or for $0<A<1$ and $\frac{\alpha}{q}<0$, with $q\in\mathbb{R}\setminus\{0\}$, Equation (5.14) is topologically conjugate to

$$z_{n+1}=\frac{\max\{z_n^{2\alpha},A^{\frac{-2\alpha p}{q}}\}}{z_n^{\alpha-\beta}\cdot z_{n-1}^{-\gamma}}\cdot A^{\frac{p(\alpha+\beta+\gamma-1)+\delta}{q}},\quad \textit{for all}\ \ p\in\mathbb{R}.$$

In particular, assuming that $\alpha+\beta+\gamma-1\neq 0$, Equation (5.14) is topologically conjugate to:

- $z_{n+1}=\frac{\max\{z_n^{2\alpha},B\}}{z_n^{\alpha-\beta}\cdot z_{n-1}^{-\gamma}}$*, for all $B>1$, if $\frac{\delta}{\alpha+\beta+\gamma-1}>0$.*
- $z_{n+1}=\frac{\max\{z_n^{2\alpha},C\}}{z_n^{\alpha-\beta}\cdot z_{n-1}^{-\gamma}}$*, for all $0<C<1$, if $\frac{\delta}{\alpha+\beta+\gamma-1}<0$.*

Moreover, if $\alpha+\beta+\gamma-1=0$, additionally we get that Equation (5.14) is topologically conjugate to:

- $z_{n+1}=\frac{\max\{z_n^{2\alpha},1\}}{z_n^{\alpha-\beta}\cdot z_{n-1}^{-\gamma}}\cdot B$*, for all $B>1$, if $\frac{\delta}{\alpha}>0$.*
- $z_{n+1}=\frac{\max\{z_n^{2\alpha},1\}}{z_n^{\alpha-\beta}\cdot z_{n-1}^{-\gamma}}\cdot C$*, for all $0<C<1$, if $\frac{\delta}{\alpha}<0$.*

Proof. Firstly, notice that from Lemma 46, it follows directly that Equation (5.14) is topologically conjugate to

$$z_{n+1} = \frac{\max\left\{z_n^{2\alpha}, A^{\frac{-2\alpha p}{q}}\right\}}{z_n^{\alpha-\beta} \cdot z_{n-1}^{-\gamma}} \cdot A^{\frac{p(\alpha+\beta+\gamma-1)+\delta}{q}}, \tag{5.15}$$

for $A > 1$ and $\frac{\alpha}{q} > 0$, or for $0 < A < 1$ and $\frac{\alpha}{q} < 0$, and for all $p \in \mathbb{R}$.

Furthermore, if $\alpha + \beta + \gamma - 1 \neq 0$, by setting $p = \frac{-\delta}{\alpha+\beta+\gamma-1}$, (5.15) reduces to

$$z_{n+1} = \frac{\max\left\{z_n^{2\alpha}, A^{\frac{2\alpha\delta}{q(\alpha+\beta+\gamma-1)}}\right\}}{z_n^{\alpha-\beta} \cdot z_{n-1}^{-\gamma}}.$$

Now, depending on the sign of $\frac{\delta}{\alpha+\beta+\gamma-1}$ and taking into account if $A > 1$ or $0 < A < 1$, we deduce that the corresponding generalized Lozi map is topologically conjugate to:

- The equations $z_{n+1} = \frac{\max\{z_n^{2\alpha}, B\}}{z_n^{\alpha-\beta} \cdot z_{n-1}^{-\gamma}}$, for all $B > 1$ when $\frac{\delta}{\alpha+\beta+\gamma-1} > 0$.
- The equations $z_{n+1} = \frac{\max\{z_n^{2\alpha}, C\}}{z_n^{\alpha-\beta} \cdot z_{n-1}^{-\gamma}}$, for all $0 < C < 1$ when $\frac{\delta}{\alpha+\beta+\gamma-1} < 0$.

Finally, if $\alpha + \beta + \gamma - 1 = 0$, then Equation (5.15) is converted into

$$z_{n+1} = \frac{\max\left\{z_n^{2\alpha}, A^{-\frac{2\alpha p}{q}}\right\}}{z_n^{\alpha-\beta} \cdot z_{n-1}^{-\gamma}} A^{\delta/q}.$$

So, if we set $p = 0$, we obtain the max-type difference equation

$$z_{n+1} = \frac{\max\{z_n^{2\alpha}, 1\}}{z_n^{\alpha-\beta} \cdot z_{n-1}^{-\gamma}} \cdot A^{\delta/q}.$$

Additionally, since A is arbitrary, depending on the interval where it belongs and the corresponding sign of q, we get:

- The equations $z_{n+1} = \frac{\max\{z_n^{2\alpha}, 1\}}{z_n^{\alpha-\beta} \cdot z_{n-1}^{-\gamma}} \cdot B$, for all $B > 1$ when $\frac{\delta}{\alpha} > 0$.
- The equations $z_{n+1} = \frac{\max\{z_n^{2\alpha}, 1\}}{z_n^{\alpha-\beta} \cdot z_{n-1}^{-\gamma}} \cdot C$, for all $0 < C < 1$ when $\frac{\delta}{\alpha} < 0$.

□

Example 2. *Take the difference equation*

$$y_{n+1} = \frac{3}{2}|y_n| + \frac{1}{2}y_n - y_{n-1} + \frac{1}{2}.$$

By Theorem G, it is topologically conjugate to the family of max-type difference equations

$$z_{n+1} = \frac{\max\{z_n^3, 1\}}{z_n \cdot z_{n-1}} \cdot C, \quad \textit{for all} \ \ 0 < C < 1.$$

□

Finally, once that we have established the connection that links the generalized Lozi map with certain families of difference equations with maximum, we focus on some conditions that allow us to connect in turn our proposal of generalized Lozi map with the one already presented in the literature, namely, Equation (5.9),

$$y_{n+1} = \frac{k}{2}|y_n| + \left(\frac{k}{2} - l\right) y_n - m y_{n-1} + \delta.$$

Firstly, observe that this equation is equivalent to Equation (5.8). Indeed, $\frac{k}{2} = \alpha$, $\frac{k}{2} - l = \beta$, $-m = \gamma$ and $\delta = \delta$ define a bijective relation between the coefficients of both equations. In particular, we focus on the conditions $\delta = k - 1 - l - m$, $\delta = -(k - 1 - l - m)$ and $\delta = 0$, since they appear in [40] when applying the change of variables to Equation (5.9) in order to obtain a generalization of the Lyness max-type difference equation, namely,

$$x_{n+1} = \frac{\max\{x_n^k, A\}}{x_n^l \cdot x_{n-1}^m}. \tag{5.16}$$

Observe that such relations yields to the following:

- $\delta = k - 1 - l - m \Leftrightarrow \delta = \alpha + \beta + \gamma - 1$;
- $\delta = -(k - 1 - l - m) \Leftrightarrow \delta = -(\alpha + \beta + \gamma) + 1$;
- $\delta = 0$.

This allows us to state the following:

Corollary 11. *[40] Let $k, l, m, \delta \in \mathbb{Z}$ and take the generalized Lozi map*

$$y_{n+1} = \frac{k}{2}|y_n| + \left(\frac{k}{2} - l\right) y_n - m y_{n-1} + \delta.$$

Then, it is topologically conjugate to:

- $x_{n+1} = \frac{\max\{x_n^k, A\}}{x_n^l \cdot x_{n-1}^m}$*, for all* $A > 0$*, if* $\delta = 0$*;*
- $x_{n+1} = \frac{\max\{x_n^k, A\}}{x_n^l \cdot x_{n-1}^m}$*, for all* $A > 1$*, if* $\delta = k - 1 - l - m$*;*
- $x_{n+1} = \frac{\max\{x_n^k, A\}}{x_n^l \cdot x_{n-1}^m}$*, for all* $0 < A < 1$*, if* $\delta = -(k - 1 - l - m)$*.*

The transformation that connects Equation (5.9) with Equation (5.16) can be found in [40]. Nevertheless, it must be highlighted that, on the one hand, here we have presented a more general change of variables and, on the other hand, the parameters can be arbitrary real numbers, and not be restricted to the integer set $\mathbb{Z}$. Furthermore, as far as we are concerned, it is the first time that the problem is treated from the point of view of topological conjugacies.

As a final comment, bearing in mind that the relations $\alpha = -a$, $\beta = 0$, $\gamma = b$ and $\delta = 1$ allow us to recover the original Lozi map from the generalized one, the above conditions, $\delta = \alpha + \beta + \gamma - 1$ and $\delta = -(\alpha + \beta + \gamma) + 1$, imply that $b - a = 2$ and $a = b$, respectively. Therefore, this motivates the analysis of Lozi map in the particular cases $a = b$ and $b - a = 2$. It should be noticed that, since those cases do not verify the conditions established in (5.5), it will not appear a strange attractor and, therefore, the dynamics are not expected to be complex. In the sequel, after illustrating the application of the transformation with a concrete max-type difference equation, we focus on the case $a = b$ and we present some advances on the problem in Subsection 5.3.

5.2 A family of difference equations with maximum

This section focuses on the dynamics of a concrete family of max-type difference equations. In concrete, we study the one-parametric family

$$x_{n+1} = \frac{\max\{x_n^3, A\}}{x_n x_{n-1}}, \quad A > 0, \tag{5.17}$$

which, by Theorem F, is topologically conjugate to the generalized Lozi map

$$y_{n+1} = \frac{3}{2}|y_n| + \frac{1}{2}y_n - y_{n-1}. \tag{5.18}$$

Beyond the description of the dynamics of the above equations, our target is to illustrate that, thanks to the transformations developed previously, the study of a concrete Lozi map is sufficient to determine the dynamics of a whole family of max-type difference equations. In this sense, we establish the dynamics of Equation (5.18) and derive the dynamics of the family (5.17).

The proof of the first result concerning the equilibrium points of (5.18) is straightforward and is omitted.

Proposition 39. *Equation* (5.18) *has infinitely many equilibrium points, namely,*

$$\mathcal{F} = \{\bar{x} : \bar{x} \geq 0\}.$$

Theorem 18. *Every non-trivial solution of Equation (5.18) diverges to ∞.*

Proof. Let us denote by (y_{-1}, y_0) the initial conditions of (5.18). We proceed by considering different cases depending on the values of (y_{-1}, y_0).

Case 1: If $0 \leq y_{-1} < y_0$, then, by induction, it is easy to see that

$$y_n = y_{-1} + (n+1)(y_0 - y_{-1}),$$

for all $n \geq 1$. Hence, $\lim_{n\to+\infty} y_n = +\infty$.

Case 2: If $y_{-1} \le 0 < y_0$ or $y_{-1} < 0 \le y_0$, then $y_1 = 2y_0 - y_{-1} > y_0 > 0$. Thus, we can apply Case 1 in order to deduce that the solution diverges to $+\infty$.

Case 3: If $y_{-1} \le y_0 \le 0$, $|y_{-1}| + |y_0| > 0$, then $y_1 = -\frac{3}{2}y_0 + \frac{1}{2}y_0 - y_{-1} = -y_0 - y_{-1} > 0$, and we can apply Case 2.

Case 4: Assume that $0 \le y_0 < y_{-1}$. Consider the value $\varepsilon = y_{-1} - y_0$ and fix the smallest non-negative integer N such that

$$y_0 - (N+1)\varepsilon < 0 \le y_0 - N\varepsilon.$$

Then, it is direct to see that

$$y_1 = 2y_0 - y_{-1} = y_0 - \varepsilon, \ldots, y_N = y_0 - N\varepsilon, y_{N+1} = y_0 + (N+1)\varepsilon,$$

and being $y_{N+1} < 0 \le y_N$, we obtain

$$y_{N+2} = \frac{3}{2}\big(-y_0 + (N+1)\varepsilon\big) + \frac{1}{2}\big(y_0 - (N+1)\varepsilon\big) - y_0 + N\varepsilon = -2y_0 + (2N+1)\varepsilon.$$

Now, if $y_{N+2} \ge 0$, we have the pair of new initial conditions (y_{N+1}, y_{N+2}), with $y_{N+1} < 0 \le y_{N+2}$ and we can apply Case 2. Otherwise, following the iteration we get

$$\begin{aligned} y_{N+3} &= \frac{3}{2}\big(2y_0 - (2N+1)\varepsilon\big) + \frac{1}{2}\big(-2y_0 + (2N+1)\varepsilon\big) - \big(y_0 - (N+1)\varepsilon\big) \\ &= y_0 - N\varepsilon \ge 0, \end{aligned}$$

and we can apply Case 2 to the new initial conditions $y_{N+2} < 0 \le y_{N+3}$.

Case 5: If $y_0 \le y_{-1} \le 0$, $|y_{-1}| + |y_0| > 0$, since $y_1 = -y_0 - y_{-1} > 0$, we can apply Case 2 to the initial conditions $y_0 \le 0 < y_1$.

Case 6: If $y_0 \le 0 < y_{-1}$ or $y_0 < 0 \le y_{-1}$, we have $y_1 = -y_0 - y_{-1}$. Now:

- If $y_{-1} + y_0 < 0$, then $y_0 \le 0 < y_1$ and we apply Case 2.
- If $y_{-1} + y_0 \ge 0$, then $y_0 < 0$, $y_1 \le 0$, and we can use Case 3 or Case 5 depending on whether $y_0 \le y_1 \le 0$ or $y_1 \le y_0 \le 0$, respectively.

□

Once that we have established the dynamics of the Lozi map (5.18), we can derive the dynamics of the corresponding max-type difference equation.

Corollary 12. *Given an arbitrary value $A \in (0, +\infty)$, consider the max-type difference equation*

$$x_{n+1} = \frac{\max\{x_n^3, A\}}{x_n x_{n-1}}.$$

For any arbitrary positive initial conditions (x_{-1}, x_0), either $x_{-1} = x_0 = \bar{x}$ is an equilibrium point, or the solution generated by them diverges to infinity. The stationary solution $(\bar{x}, \bar{x}, \bar{x}, \ldots)$ appears for all the values $\bar{x} \ge \sqrt[3]{A}$.

5.3 Lozi map for $a = b$

This section delves into the dynamics of Lozi map in the particular case $a = b$, that is,

$$x_{n+1} = 1 - a|x_n| + ax_{n-1}. \tag{5.19}$$

Recall that this case is of interest, since $a = b$ is one of the conditions that allow us to transform the generalized Lozi map into a family of max-type difference equations of the form (5.16), in concrete, a generalization of the Lyness difference equation with maximum.

As a first step, we determine the equilibrium points of (5.19). The proof is straightforward and is omitted.

Lemma 47. *The equilibrium points of Equation (5.19) are given by:*

(a) $\bar{x} = 1$ *if* $a \in \left(-\infty, \frac{1}{2}\right]$.

(b) $\bar{x} = 1$ *and* $\bar{x} = \frac{1}{1-2a}$ *if* $a \in \left(\frac{1}{2}, +\infty\right)$.

For the original Lozi map, (5.4), it is known that the condition $|a| + |b| < 1$ implies that the equation has a unique equilibrium point that is a global attractor, see [35, pages 181-182]. In the particular case $a = b$, this implies the following result.

Proposition 40. *Let* $a \in \left(-\frac{1}{2}, \frac{1}{2}\right)$. *Then, all the solutions of Equation (5.19) converge to the equilibrium point* $\bar{x} = 1$.

The above result establishes the parametric region where the equilibrium point, $\bar{x} = 1$, is an attractor for the map. In this sense, it is interesting to focus on the boundary values of the parameter a in such region, that is, the cases $a = \frac{1}{2}$ and $a = -\frac{1}{2}$. Therefore, in the sequel we deep in the analysis of the dynamics of the second-order piecewise linear difference equations

$$x_{n+1} = 1 - \frac{1}{2}|x_n| + \frac{1}{2}x_{n-1} \quad \text{and} \quad x_{n+1} = 1 + \frac{1}{2}|x_n| - \frac{1}{2}x_{n-1}.$$

Before studying those equations, we give some comments concerning the existence of periodic solutions of Lozi map in the case $a = b$. For $a \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, by Proposition 40, we know that $\bar{x} = 1$ is a global attractor; and for $|a| > \frac{1}{2}$, it is straightforward to compute the 2-periodic solutions of the equation.

Lemma 48. *Assume that* $|a| > \frac{1}{2}$. *Then, the only initial conditions* (x_{-1}, x_0) *which generate* 2*-periodic solutions are given by:*

$$(x_{-1}, x_0) = \left(\frac{1}{2a^2 - 2a + 1}, \frac{1 - 2a}{2a^2 - 2a + 1}\right), \quad or \quad (x_{-1}, x_0) = \left(\frac{1 - 2a}{2a^2 - 2a + 1}, \frac{1}{2a^2 - 2a + 1}\right),$$

with $a > \frac{1}{2}$.

In order to determine the stability of the 2-periodic points given in the above Lemma, we consider the associated map to Equation (5.19), $F(x, y) = (y, 1 - a|y| + ax)$ and compute its Jacobian matrix evaluated at the 2-periodic points:

$$DF\left(\frac{1}{2a^2 - 2a + 1}, \frac{1 - 2a}{2a^2 - 2a + 1}\right) = \begin{pmatrix} 0 & 1 \\ a & a \end{pmatrix},$$

and

$$DF\left(\frac{1 - 2a}{2a^2 - 2a + 1}, \frac{1}{2a^2 - 2a + 1}\right) = \begin{pmatrix} 0 & 1 \\ a & -a \end{pmatrix}.$$

Consequently,

$$DF^2\left(\frac{1}{2a^2 - 2a + 1}, \frac{1 - 2a}{2a^2 - 2a + 1}\right) = \begin{pmatrix} 0 & 1 \\ a & a \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ a & -a \end{pmatrix} = \begin{pmatrix} a & -a \\ a^2 & a - a^2 \end{pmatrix}.$$

This gives the characteristic equation $\lambda^2 + (a^2 - 2a)\lambda + a^2 = 0$, whose roots are

$$\lambda_1 = \frac{2a - a^2 + a\sqrt{a^2 - 4a}}{2} \quad \text{and} \quad \lambda_2 = \frac{2a - a^2 - a\sqrt{a^2 - 4a}}{2}.$$

Notice that the eigenvalues lie inside the unit circle if and only if the coefficients of the characteristic equation verify (consult [33, Th.2.37])

$$|-a^2 + 2a| < 1 + a^2 < 2.$$

Those inequalities hold for the range of values $a \in \left(\frac{1}{2}, 1\right)$, so we can establish the local asymptotical stability of the 2-periodic points whenever $\frac{1}{2} < a < 1$.

Now, we proceed to study the boundary cases $a = \frac{1}{2}$ and $a = -\frac{1}{2}$.

5.3.1 The upper boundary case

This subsection delves into the dynamics of the Lozi map in the particular case $a = \frac{1}{2}$,

$$x_{n+1} = 1 - \frac{1}{2}|x_n| + \frac{1}{2}x_{n-1}. \tag{5.20}$$

Beforehand, observe that, by applying Theorem G, Equation (5.20) is topologically conjugate to the max-type difference equation

$$x_{n+1} = x_n^{1/2} \cdot x_{n-1}^{1/2} \cdot \max\left\{\frac{1}{x_n}, A\right\}, \quad 0 < A < 1. \tag{5.21}$$

We start by establishing the equilibrium points and the 2-periodic points of Equation (5.20). The proof of the first result follows from Lemma 47.

Lemma 49. *The unique equilibrium point of (5.20) is* $\bar{x} = 1$.

Lemma 50. *Suppose that the initial condition $(x_{-1}, x_0) = (x, y)$ generates a 2-periodic solution of Equation (5.20). Then,*

$$0 \leq x, y \leq 2; \quad x + y = 2; \quad x \neq y.$$

Proof. Assume that the initial condition $(x_{-1}, x_0) = (x, y)$ generate a 2-periodic solution. By iterating the equation, we obtain the system

$$\begin{cases} x &= 1 - \frac{1}{2}|y| + \frac{1}{2}x \\ y &= 1 - \frac{1}{2}|x| + \frac{1}{2}y \end{cases}. \tag{5.22}$$

Notice that from this system we can derive the equations $\frac{1}{2}x = 1 - \frac{1}{2}|y|$ and $\frac{1}{2}y = 1 - \frac{1}{2}|x|$. If we subtract them, we get

$$x - y = |x| - |y|. \tag{5.23}$$

Bearing such equation in mind, we proceed by distinguishing the following cases:

- Assume $x, y \geq 0$ and take $x = \alpha \cdot y$, where $\alpha \geq 0$. Notice that $\alpha = 0$ yields to $x = 0$ and $y = 2$. In this case, it is direct to see that $(x_{-1}, x_0) = (0, 2)$ generates a 2-periodic solution. On the other hand, if $\alpha > 0$, we get $\frac{1}{2}x = 1 - \frac{1}{2} \cdot \frac{x}{\alpha}$; or equivalently, $x = \frac{2\alpha}{1+\alpha}$. Then, $y = \frac{x}{\alpha} = \frac{2}{1+\alpha}$. Here, $x + y = 2$ and the initial conditions $\left(\frac{2\alpha}{1+\alpha}, \frac{2}{1+\alpha}\right)$ generates a 2-periodic orbit too.

- Assume $x \geq 0, y \leq 0$. Equation (5.23) implies $y = 0$, so $x = 2$. Thus, the initial condition is $(2, 0)$, which is included in the above case.

- Assume $x \leq 0, y \geq 0$. The symmetry of Equation (5.23) allows us to proceed as in the above case.

- Assume $x \leq 0, y \leq 0$. Here, Equation (5.23) yields to $x = y$. Replacing this in (5.22), we obtain $\frac{1}{2}x = 1 + \frac{1}{2}x$, which is impossible.

□

If we apply the change of variables $y_n = \log_A\left(\frac{A}{x_n^k}\right)$, we can translate the above results to the corresponding max-type difference equation, Equation (5.21).

Corollary 13. *Suppose that the initial conditions $(x_{-1}, x_0) = (x, y)$ generate a 2-periodic solution of Equation (5.21). Then,*

$$x, y \in \left[A, \frac{1}{A}\right], \quad A \in (0,1) \quad \textit{with} \quad xy = 1, \; x \neq y.$$

As a next step, we see that any pair of initial conditions (x_{-1}, x_0) generates an asymptotically periodic orbit to an appropriate 2-periodic orbit $(x, y, x, y, \ldots)$ with $x + y = 2$.

A point worthy of mention is that the linearization technique does not work in this case, since the computation of the corresponding eigenvalues of the matrix DF^2, where $F(x,y) = (y, 1-\frac{1}{2}|y|+\frac{1}{2}x)$, gives us $\lambda_1 = 1$ and $\lambda_2 = \frac{1}{4}$. Therefore, we need to proceed by a different technique. In concrete, we distinguish diverse regions where the initial conditions can be located and we develop a case study.

Firstly, we show the existence of two invariant triangles in the plane.

Lemma 51. *Consider the triangles*

$$\Delta_\ell = \{(x,y): \ 0 \leq x, y \leq 2, \ x+y \leq 2\}$$

and

$$\Delta_u = \{(x,y): \ 0 \leq x, y \leq 2, \ x+y \geq 2\}.$$

Then, $F(\Delta_\ell) \subset \Delta_\ell$ *and* $F(\Delta_u) \subset \Delta_u$*, where* $F(x,y) = (y, 1-\frac{1}{2}|y|+\frac{1}{2}x)$*.*

Proof. Let $(x,y) \in \Delta_\ell$. Then, $F(x,y) = \left(y, 1-\frac{1}{2}y+\frac{1}{2}x\right)$, with $0 \leq y \leq 2$,

$$1-\frac{1}{2}y+\frac{1}{2}x \geq 1-\frac{1}{2}y \geq 1-1 = 0,$$

$$1-\frac{1}{2}y+\frac{1}{2}x \leq 1-\frac{1}{2}y+1 \leq 2-\frac{1}{2}y \leq 2,$$

and

$$y+\left(1-\frac{1}{2}y+\frac{1}{2}x\right) = 1+\frac{1}{2}x+\frac{1}{2}y \leq 1+\frac{x+y}{2} \leq 1+1 = 2.$$

In conclusion, $F(x,y) \in \Delta_\ell$.

The proof of the invariance of Δ_u by F is analogous and is omitted. □

Observe that the coordinates of the points in the invariant rectangle $\Delta_\ell \cup \Delta_u$ are always positive. This fact transforms Lozi map into a linear system and it is easy to see that every solution is asymptotically periodic to some 2-periodic point.

Proposition 41. *Let* $(x,y) \in \Delta_\ell \cup \Delta_u$*. Then,* $\left(F^n(x,y)\right)_{n\geq 0}$ *is asymptotically periodic to some 2-periodic point* (v,w)*.*

Proof. Take initial conditions $(x,y) \in \Delta_\ell \cup \Delta_u$. In this invariant regions, the coordinates of the iterates are always positive and Lozi map becomes $F(x,y) = (y, 1-\frac{1}{2}y+\frac{1}{2}x)$. Therefore, we have the linear difference equation $y_{n+1} = 1-\frac{1}{2}y_n+\frac{1}{2}y_{n-1}$, with $y_{-1} = y$, $y_0 = 1-\frac{1}{2}y+\frac{1}{2}x$. Here, we can obtain the general solution of the equation, which is given by $y_n = A(-1)^n + B\left(\frac{1}{2}\right)^n + 1$, for arbitrary constants $A, B \in \mathbb{R}$.

By imposing the initial conditions, we get $A = \frac{2y-x-1}{3}$ and $B = \frac{x+y-2}{3}$. Therefore, it is straightforward to see that

$$\lim_{n\to\infty} y_{2n} = \frac{2y-x-1}{3}+1 = \frac{2y-x+2}{3} \quad \text{and} \quad \lim_{n\to\infty} y_{2n+1} = -\frac{2y-x-1}{3}+1 = \frac{-2y+x+4}{3}.$$

This guarantees that the orbit converges to the 2-periodic sequence

$$\left(\ldots, \frac{2y-x+2}{3}, \frac{-2y+x+4}{3}, \frac{2y-x+2}{3}, \frac{-2y+x+4}{3}, \ldots\right).$$

□

As a consequence of this proposition, if some iterate of the orbit by F lies in Δ_ℓ or Δ_u, then the orbit will be asymptotically periodic to an appropriate 2-periodic point. In light of this, we focus on proving that for any initial condition (x, y) there exists a positive N such that $F^N(x, y) \in \Delta_\ell \cup \Delta_u$, since from here we are able to deduce that every orbit converges to a 2-periodic solution.

To do so, let us divide the plane into squares

$$C_{m,n} := [2m, 2m+2] \times [2n, 2n+2], \quad \text{for } m, n \in \mathbb{Z}.$$

We show that for any square, either there exists an $N \in \mathbb{Z}$ such that

$$F^N(C_{m,n}) \subset C_{0,0} \quad \text{or} \quad \cap_{p\geq 0} F^p(C_{m,n}) = \{(0,2),(2,0)\}.$$

In both cases, we derive that for any $(x, y) \in \mathbb{R}^2$, the orbit $\big(F^n(x, y)\big)_{n\geq 0}$ is asymptotically periodic to some 2-periodic point.

In the sequel, let us denote the upper half-plane and the lower half-plane by

$$H_u = \{(x, y) : y \geq 0\} \quad \text{and} \quad H_\ell = \{(x, y) : y \leq 0\}.$$

We begin by proving some facts concerning the application of F over the squares.

Lemma 52. *The map F transforms the square $C_{m,n}$ into a parallelogram and the vertices of $C_{m,n}$ go to the vertices of that parallelogram.*

Proof. It follows directly from the fact that the square $C_{m,n}$ is entirely located in H_u or in H_ℓ and the linearity of F. □

Lemma 53. *Let $A(C_{m,n})$ and $A\big(F(C_{m,n})\big)$ be the areas of the square $C_{m,n}$ and its image by F, respectively. Then $A\big(F(C_{m,n})\big) = \frac{1}{2}A(C_{m,n})$.*

Proof. Firstly, if the square is located in H_u, take $\big(x(u, v), y(u, v)\big) = \big(v, 1 - \frac{1}{2}v + \frac{1}{2}u\big)$; and if it is in H_ℓ, take $\big(x(u, v), y(u, v)\big) = \big(v, 1 + \frac{1}{2}v + \frac{1}{2}u\big)$.

The area of the image of $C_{m,n}$ under F is given by

$$A\big(F(C_{m,n})\big) = \int\int_{C_{m,n}} \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\ dv,$$

where $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$ denotes the absolute value of the determinant $\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$.

In both cases, $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ \frac{1}{2} & \pm\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$, and the result follows. □

In the same way, F transforms a triangle, T, entirely contained in H_u or H_l, in a new triangle whose vertices are the vertices of T by the action of F. Moreover, $A\big(F(T)\big) = \frac{1}{2}A(T)$.

From this point forward, our strategy consists in proving:

(a) Consider the region R_1 surrounding $C_{0,0}$, including $C_{0,0}$ itself,

$$R_1 := \bigcup_{i,j\in\{-1,0,1\}} C_{i,j} = [-2,4]\times[-2,4].$$

Firstly, we prove that R_1 is invariant. Secondly, we show that either the images $F^N(C_{i,j})$ are included in $C_{0,0}$ for some positive integer N or the intersection of the images not included in $C_{0,0}$ converges to the 2-periodic orbit $\{(0,2),(0,2)\}$.

(b) The image of any square $C_{i,j}$ for $i\geq 2$ or $j\geq 2$ is eventually contained in R_1.

For Part (a), we use the above lemmas. As a first step, we analyze the evolution of the segments

$$S_{\varepsilon,0} = \{(x,0) : 2\leq x\leq 2+\varepsilon\} \text{ and } S_{\varepsilon,2} = \{(x,2) : -\varepsilon\leq x\leq 0\},\ \varepsilon\in[0,1].$$

Lemma 54. *Given $\varepsilon\in[0,1]$, consider $F^n(S_{\varepsilon,0}), n\geq 0$. Then, there exists an N such that $F^N(S_\varepsilon)\subset C_{0,0}$.*

Proof. It suffices to study the evolution of the endpoints of $S_{\varepsilon,0}$. Realize that $F^n(S_{\varepsilon,0})$ are segments lying fully in either H_ℓ or H_u. Furthermore, it is direct that $F^{2n}(2,0)=(2,0)$ and $F^{2n+1}(2,0)=(0,2)$, being $(2,0)$ and $(0,2)$ in $C_{0,0}$. On the other hand,

$$\begin{aligned}(2+\varepsilon,0) &\xrightarrow{F} \left(0,2+\frac{\varepsilon}{2}\right) \xrightarrow{F} \left(2+\frac{\varepsilon}{2},-\frac{\varepsilon}{4}\right) \xrightarrow{F} \left(-\frac{\varepsilon}{4},2+\frac{\varepsilon}{8}\right) \xrightarrow{F} \left(2+\frac{\varepsilon}{8},-\frac{3}{16}\varepsilon\right)\\ &\xrightarrow{F} \left(-\frac{3}{16}\varepsilon,2-\frac{\varepsilon}{32}\right) \xrightarrow{F} \left(2-\frac{\varepsilon}{2},-\frac{5}{64}\varepsilon\right) \xrightarrow{F} \left(-\frac{5}{64}\varepsilon,2-\frac{7}{128}\varepsilon\right)\\ &\xrightarrow{F} \left(2-\frac{7}{128}\varepsilon,-\frac{3}{256}\varepsilon\right) \xrightarrow{F} \left(-\frac{3}{256}\varepsilon,2-\frac{17}{512}\varepsilon\right) \xrightarrow{F} \left(2-\frac{17}{512}\varepsilon,\frac{11}{1024}\varepsilon\right).\end{aligned}$$

At this point, observe that $F^{10}(2+\varepsilon,0) = \left(2-\frac{17}{512}\varepsilon,\frac{11}{1024}\varepsilon\right)\in C_{0,0}$, which ends the proof. □

Lemma 55. *Given $\varepsilon\in[0,1]$, consider $F^n(S_{\varepsilon,2}), n\geq 0$. Then, there exists an N such that $F^N(S_{\varepsilon,2})\subset C_{0,0}$.*

Proof. Firstly, $F^{2n}(0,2)=(0,2)$ and $F^{2n+1}(0,2)=(2,0)$, being those points in $C_{0,0}$. Next, the iterates of the endpoint $(-\varepsilon,2)$ are

$$(-\varepsilon,2) \xrightarrow{F} \left(2,-\frac{\varepsilon}{2}\right) \xrightarrow{F} \left(-\frac{\varepsilon}{2},2-\frac{\varepsilon}{4}\right) \xrightarrow{F} \left(2-\frac{\varepsilon}{4},-\frac{\varepsilon}{8}\right) \xrightarrow{F} \left(-\frac{\varepsilon}{8},2-\frac{3}{16}\varepsilon\right) \xrightarrow{F} \left(2-\frac{3}{16}\varepsilon,\frac{\varepsilon}{32}\right).$$

Now $\left(2-\frac{3}{16}\varepsilon,\frac{\varepsilon}{32}\right)\in C_{0,0}$ and the result follows. □

Secondly, we study the action of F under suitable neighbourhoods of the points $(0,2)$ and $(2,0)$. The proof is direct and we omit it. However, we foster the reader to follow the reasoning by focusing on Figure 5.3.

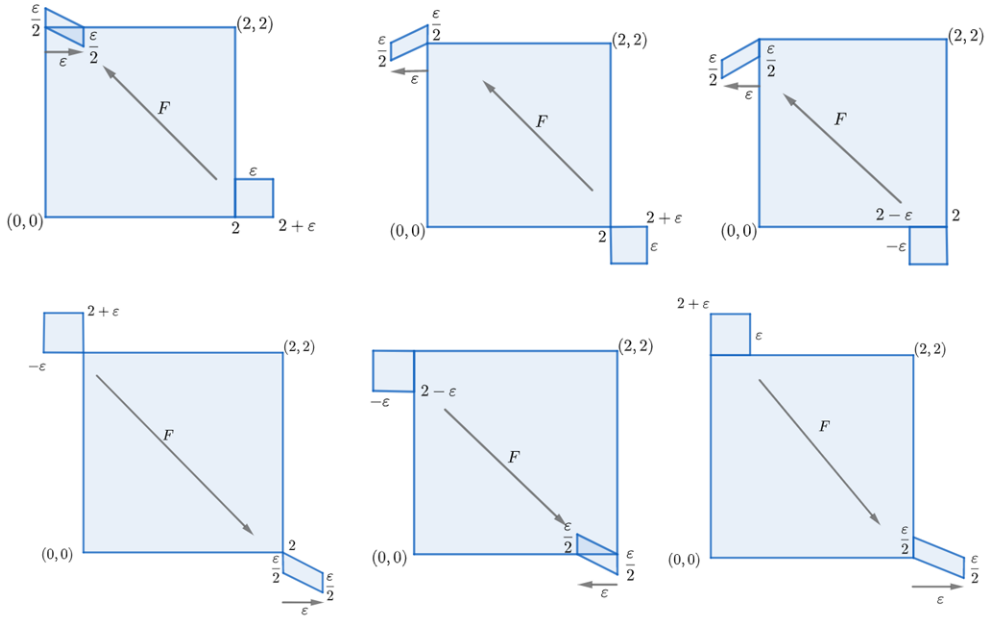


Figure 5.3: Evolution of neighbourhoods $Q_{j,\varepsilon}$ of $(2,0)$ (top) and neighbourhoods $W_{j,\varepsilon}(0,2)$ (bottom), $j \in \{1,2,3\}$, in Lemma 56.

Lemma 56. *Let $\varepsilon \in [0,1]$. Consider the squares $Q_{1,\varepsilon} = [2, 2+\varepsilon] \times [0,\varepsilon]$, $Q_{2,\varepsilon} = [2, 2+\varepsilon] \times [-\varepsilon, 0]$ and $Q_{3,\varepsilon} = [2-\varepsilon, 2] \times [-\varepsilon, 0]$. Then:*

(a.1) *$F(Q_{1,\varepsilon}) \subset C_{0,0} \cup T_{1,\varepsilon}$, where $T_{1,\varepsilon}$ is the triangle with vertices $(0,2)$, $\left(0, 2+\frac{\varepsilon}{2}\right)$, $(\varepsilon, 2)$.*

(a.2) *$F(Q_{2,\varepsilon})$ is the parallelogram with vertices $(0,2)$, $\left(0, 2+\frac{\varepsilon}{2}\right)$, $(-\varepsilon, 2)$, $\left(-\varepsilon, 2-\frac{\varepsilon}{2}\right)$.*

(a.3) *$F(Q_{3,\varepsilon})$ is the parallelogram with vertices $(0,2)$, $\left(0, 2-\frac{\varepsilon}{2}\right)$, $\left(-\varepsilon, 2-\frac{\varepsilon}{2}\right)$, $(-\varepsilon, 2-\varepsilon)$.*

Consider the squares $W_{1,\varepsilon} = [-\varepsilon, 0] \times [2, 2+\varepsilon]$, $W_{2,\varepsilon} = [-\varepsilon, 0] \times [2-\varepsilon, 2]$ and $W_{3,\varepsilon} = [0,\varepsilon] \times [2, 2+\varepsilon]$. Then:

(b.1) *$F(W_{1,\varepsilon})$ is the parallelogram with vertices $(2,0)$, $\left(2, -\frac{\varepsilon}{2}\right)$, $(2+\varepsilon, -\varepsilon)$, $\left(2+\varepsilon, -\frac{\varepsilon}{2}\right)$.*

(b.2) *$F(W_{2,\varepsilon}) \subset C_{0,0} \cup T_{2,\varepsilon}$, where $T_{2,\varepsilon}$ is the triangle with vertices $(2-\varepsilon, 0)$, $(2,0)$, $\left(2, -\frac{\varepsilon}{2}\right)$.*

(b.3) *$F(W_{3,\varepsilon})$ is the parallelogram with vertices $(2,0)$, $\left(2, \frac{\varepsilon}{2}\right)$, $(2+\varepsilon, 0)$, $\left(2+\varepsilon, -\frac{\varepsilon}{2}\right)$.*

We are currently able to demonstrate that every orbit within the region $R_1 = [-2,4] \times [-2,4]$ enters the square $C_{0,0}$. In the sequel we maintain the notation introduced in the above Lemmas.

Proposition 42. *For any point (x, y) in the region R_1, its orbit $\big(F^n(x, y)\big)_n$ eventually enters in the square $C_{0,0}$.*

Proof. The proof is developed by cases bearing in mind that $R_1 = \bigcup_{-1\leq i,j\leq 1} C_{i,j}$. Due to the analogies in the reasoning, we only give some of them.

- Assume that $(x, y) \in C_{1,1}$. Here, $F(C_{1,1}) \subset C_{1,0}$ is the parallelogram having vertices $(2, 1)$, $(2, 2)$, $(4, 1)$, $(4, 0)$, and the second iterate $F^2(C_{1,1})$ is the parallelogram with vertices $\big(1, \frac{5}{2}\big)$, $(2, 1)$, $\big(1, \frac{3}{2}\big)$, $(0, 3)$. Observe that $F^2(C_{1,1}) \subset C_{0,0} \cup W_{3,\varepsilon} \cup \tilde{T}$ with $\varepsilon = 1$ and $\tilde{T}$ being the triangle of vertices $\big(1, \frac{5}{2}\big)$, $\big(\frac{4}{3}, 2\big)$ and $(1, 2)$; additionally, $F^2(\tilde{T}) \subset C_{0,0} \cup W_{3,\varepsilon}$, since the vertices of $\tilde{T}$ by F^2 go to $\big(\frac{1}{4}, \frac{17}{8}\big)$, $\big(\frac{2}{3}, \frac{5}{3}\big)$ and $\big(\frac{1}{2}, \frac{7}{4}\big)$. Next, by Lemmas 54-56, we deduce that the images of $W_{3,\varepsilon}$ by F either enter to $C_{0,0}$, or either the intersection of such images approach to segments of type $S_{\varepsilon,0}$ or $S_{\varepsilon,2}$.

- Assume that $(x, y) \in C_{1,0}$. Now, $F(C_{1,0})$ is the parallelogram with vertices $(0, 2)$, $(0, 3)$, $(2, 2)$, $(2, 1)$ and, therefore, $F^2(C_{1,0})$ is a new parallelogram having vertices $(2, 0)$, $\big(3, -\frac{1}{2}\big)$, $(2, 1)$, $\big(1, \frac{3}{2}\big)$; since $F^2(C_{1,0}) \subset C_{0,0} \cup Q_{1,\varepsilon} \cup Q_{2,\varepsilon}$, with $\varepsilon = 1$, applying a similar reasoning to what was undertaken in the previous scenario leads us to the sought-after conclusion regarding the entry of the orbit into the square $C_{0,0}$.

- Assume that $(x, y) \in C_{-1,-1}$. In this case, $F(C_{-1,-1})$ is a parallelogram with vertices $(-2, -1)$, $(-2, 0)$, $(0, 1)$, $(0, 0)$; which can be decomposed into two triangles $T_1 \cup T_2$, with vertices $(-2, -1)$, $(-2, 0)$, $(0, 0)$ for T_1 and $(-2, 0)$, $(0, 0)$, $(0, 1)$ for T_2; then $F(T_2) \subset C_{0,0}$, while $F(T_1)$ is sent to the new triangle T_3 with vertices $\big(-1, -\frac{1}{2}\big)$, $(0, 0)$, $(0, 1)$. Due to the fact that T_3 intersects both half-planes H_ℓ and H_u, we need to decompose it again as $T_4 \cup T_5$, where T_4 has vertices $\big(-1, -\frac{1}{2}\big)$, $\big(-\frac{2}{3}, 0\big)$, $(0, 0)$, and T_5 has vertices $\big(-\frac{2}{3}, 0\big)$, $(0, 0)$, $(0, 1)$. Observe that $F(T_4)$ has vertices $\big(-\frac{1}{2}, \frac{1}{4}\big)$, $\big(0, \frac{2}{3}\big)$, $(0, 1)$ and it is direct to check that $F^2(T_4) \subset C_{0,0}$; while it is a simple matter to see that $F(T_5) \subset C_{0,0}$.

□

Now, we deal with Part (b). Here, we study the evolution of the images of the squares $C_{i,j}$. The idea is to generalize the process by induction in different levels. To do so, recall that the vertices of the squares are sent to the vertices of the new parallelograms.

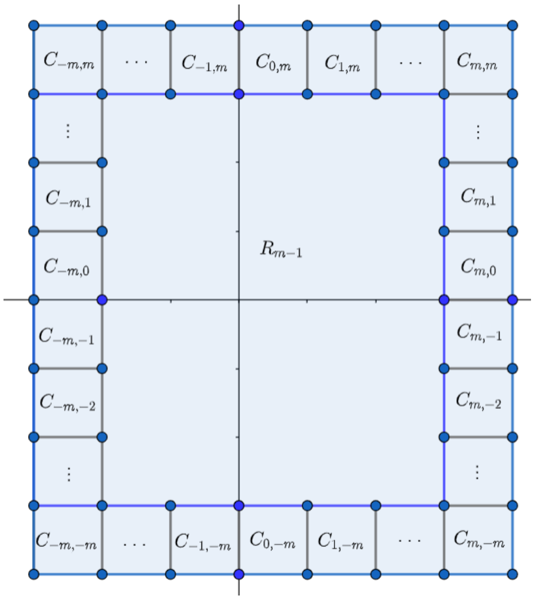


Figure 5.4: The squares $C_{m,n}$, $C_{n,m}$, $C_{-m,n}$ and $C_{n,-m}$ with $n \in [-m, m]$.

Beforehand, let us denote

$$R_t = \cup_{-t \le i,j \le t} C_{i,j}.$$

To proceed by induction, we assume that, eventually, the orbits generated by points from R_t will enter into the region R_{t-1}, with $1 \le t \le m-1$ and $m \ge 2$. Notice that this will yield to the convergence of the orbits to 2-periodic points in $C_{0,0}$.

In this direction, we begin analyzing the movements of the squares $C_{i,j}$. For each case, it is enough to study the images of the vertices of the squares. Due to this, we only prove the first case to illustrate the process.

Firstly, we focus on the squares $C_{m-j,m}$ and $C_{m,j}$ with $j = 0, 1, \ldots, m$.

Lemma 57. *Let $m \ge 2$ be a positive integer. For any value $j \in \{0, 1, \ldots, m\}$:*

(a) $F(C_{m-j,m}) \subset \begin{cases} C_{m,-\frac{j}{2}} & \text{if } j \text{ is even,} \\ C_{m,\frac{-j-1}{2}} \cup C_{m,\frac{-j+1}{2}} & \text{if } j \text{ is odd.} \end{cases}$

(b) $F(C_{m,j}) \subset \begin{cases} C_{j,\frac{m-j}{2}} & \text{if } m-j \text{ is even,} \\ C_{j,\frac{m-j-1}{2}} \cup C_{j,\frac{m-j+1}{2}} & \text{if } m-j \text{ is odd.} \end{cases}$

Proof. We start by evaluating the vertices of $C_{m-j,m}$ by F:

$$F(2m-2j, 2m) = (2m, -j+1),\ F(2m-2j, 2m+2) = (2m+2, -j),$$

$$F(2m-2j+2, 2m+2) = (2m+2, -j+1),\ F(2m-2j+2, 2m) = (2m, -j+2).$$

From here, it follows Part (a). We proceed analogously for Part (b):

$$F(2m, 2j) = (2j, 1+m-j),\ F(2m, 2j+2) = (2j+2, m-j),$$

$$F(2m+2,2j)=(2j,2+m-j),\ F(2m+2,2j+2)=(2j+2,1+m-j).$$

□

Next, the evolution of the squares $C_{-j,m}$ and $C_{m,-j}$, $j=0,1,\ldots,m$, is given in the following result.

Lemma 58. *Let $m\geq 2$ be a positive integer. For any value $j\in\{1,\ldots,m\}$:*

(a) $F(C_{-j,m})\subset\begin{cases} C_{m,\frac{-m-j}{2}} & \text{if } m+j \text{ is even,}\\ C_{m,\frac{-m-j-1}{2}}\cup C_{m,\frac{-m-j+1}{2}} & \text{if } m+j \text{ is odd.}\end{cases}$

(b) $F(C_{m,-j})\subset\begin{cases} C_{-j,\frac{m-j+1}{2}} & \text{if } m-j \text{ is odd,}\\ C_{-j,\frac{m-j}{2}}\cup C_{-j,\frac{m-j+2}{2}} & \text{if } m-j \text{ is even.}\end{cases}$

Finally, we deal with the squares $C_{-m,j}$ and $C_{j,-m}$ for $-m\leq j\leq m$.

Lemma 59. *Let $m\geq 2$ be a positive integer. For any value $j\in\{0,1,\ldots,m\}$:*

(a) $F(C_{-m,j})\subset\begin{cases} C_{j,\frac{-m-j}{2}} & \text{if } m+j \text{ is even,}\\ C_{j,\frac{-m-j-1}{2}}\cup C_{j,\frac{-m-j+1}{2}}, & \text{if } m+j \text{ is odd.}\end{cases}$

(b) $F(C_{-m,-j})\subset\begin{cases} C_{-j,\frac{-m-j+1}{2}} & \text{if } m+j \text{ is odd,}\\ C_{-j,\frac{-m-j}{2}}\cup C_{-j,\frac{-m-j+2}{2}}, & \text{if } m+j \text{ is even.}\end{cases}$

Lemma 60. *Let $m\geq 2$ be a positive integer. For any integer value $j\in[-m,m]$:*

$$F(C_{j,-m})\subset\begin{cases} C_{-m,\frac{-m+j+1}{2}} & \text{if } m-j \text{ is odd,}\\ C_{-m,\frac{-m+j}{2}}\cup C_{-m,\frac{-m+j+2}{2}}, & \text{if } m-j \text{ is even.}\end{cases}$$

It is relevant to highlight that for $j=-m$, we have $F(C_{-m,-m})\subset C_{-m,-m}\cup C_{-m,-m+1}$. This implies that a thorough discussion of this case is necessary to ensure that its iterates eventually enter R_{m-1}. This clarification is achieved through the examination of the images $F(C_{m,-j})$ and the tracking of parallelograms obtained in successive steps.

Lemma 61. *Let $m\geq 2$ be an arbitrary integer. It holds $F(C_{-m,-m})\subset C_{-m,-m}\cup C_{-m,-m+1}$. In particular:*

(a) $F(C_{-m,-m})$ is the parallelogram with vertices $(-2m+2,-2m+3)$, $(-2m+2,-2m+2)$, $(-2m,-2m+1)$, $(-2m,-2m+2)$.

(b) $F^2(C_{-m,-m})$ is a parallelogram with vertices $\left(-2m+1,-2m+\frac{3}{2}\right)$, $(-2m+2,-2m+2)$, $\left(-2m+3,-2m+\frac{7}{2}\right)$, $(-2m+2,-2m+3)$.

(c) $F^2(C_{-m,-m})\subset C_{-m,-m}\cup C_{-m,-m+1}\cup C_{-m+1,-m+1}$, and the part contained in $C_{-m,-m}\cup C_{-m,-m+1}$ is the triangle T_m with vertices $\left(-2m+1,-2m+\frac{3}{2}\right)$, $(-2m+2,-2m+2)$, $(-2m+2,-2m+3)$.

(d) The image of the triangle T_m, $F(T_m)$, is a new triangle with vertices $\left(-2m+\frac{3}{2},-2m+\frac{9}{4}\right)$, $(-2m+2,-2m+3)$, $\left(-2m+3,-2m+\frac{7}{2}\right)$. Therefore, $F(T_m)\subset C_{-m,-m+1}\cup R_{m-1}$.

We gather the above lemmas concerning the movements of the squares in the following result.

Proposition 43. *Let*

$$(x,y)\in\bigcup_{-m\le j\le m}\left(C_{j,m}\cup C_{m,j}\cup C_{j,-m}\cup C_{-m,j}\right),$$

with $m\geq 1$. Then, there exists a positive integer N such that $F^N(x,y)\in R_{m-1}$.

Proof. The proof follows directly from Proposition 42 and Lemmas 57-61. □

Consequently, we are able to establish the dynamics of Equation (5.20).

Theorem H. *Given the difference equation*

$$x_{n+1}=1-\frac{1}{2}|x_n|+\frac{1}{2}x_{n-1},$$

its dynamics is given by:

(a) *An equilibrium point, $\overline{x}=1$.*

(b) *A continuum of 2-periodic sequences $(\ldots,x,y,x,y,\ldots)$ with $0\leq x,y\leq 2$, $x+y=2$.*

(c) *The rest of solutions converge to one of the 2-periodic solutions given in Part (b).*

Taking into account the connection established between Lozi maps and difference equations with maximum, Theorem G, we can deduce the following:

Corollary 14. *Given the family of difference equations*

$$x_{n+1}=(x_n\cdot x_{n-1})^{\frac{1}{2}}\cdot\max\left\{\frac{1}{x_n},A\right\},$$

defined for any arbitrary positive real initial conditions, with $0<A<1$, its dynamics is given by:

(a) *An equilibrium point, $\overline{x}=1$.*

(b) *A continuum of 2-periodic sequences $\left(\ldots,x,\frac{1}{x},x,\frac{1}{x},\ldots\right)$, with $x\in\left[A,\frac{1}{A}\right]$.*

(c) *The rest of solutions converge to one of the 2-periodic solutions given in Part (b).*

Remark 12. *It is interesting to highlight that in [15], a continuum of 2-periodic sequences for arbitrary a and b verifying the constraints $a>0$ and either $a+b=1$ or $b-a=1$ was identified. In fact, they plot, fixing $b=0.5$, a superposition of twenty attractors for the values $a=0.1\cdot n$, with $n=1,\ldots,20$. In the particular case $a=0.5$, they show a segment line representing the attractor. Here, we have proved analytically this property of global attraction. It may be of some interest to study such problem for other values of a and b.*

5.3.2 The lower boundary case

Now, we take $a = -\frac{1}{2}$ and study the Lozi map

$$x_{n+1} = 1 + \frac{1}{2}|x_n| - \frac{1}{2}x_{n-1}. \tag{5.24}$$

Observe that, by Theorem G, the dynamics of Equation (5.24) can be translated to the max-type difference equation

$$x_{n+1} = \frac{\max\{x_n, C\}}{x_n^{1/2} \cdot x_{n-1}^{1/2}}, \text{ for all } 0 < C < 1.$$

Specifically, we see that the unique equilibrium point of the equation, $\bar{x} = 1$, is a global attractor. The strategy closely resembles the one developed in the case $a = \frac{1}{2}$, therefore we simply outline the proof.

As a first step, consider the map associated to Equation (5.24), namely,

$$F_{-\frac{1}{2}}(x, y) = \left(y, 1 + \frac{1}{2}|y| - \frac{1}{2}x\right).$$

If we compute the Jacobian of $F_{-\frac{1}{2}}$ and we evaluate it at the equilibrium, we get

$$DF_{-\frac{1}{2}}(1, 1) = \begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

In this way, the corresponding eigenvalues are a pair of complex conjugate numbers with modulo less than 1, $\lambda_{1,2} = \frac{1}{4} \pm \frac{\sqrt{7}}{4}i$. Thus, we can conclude that, at least, the equilibrium is locally asymptotically stable.

Next, we show that the equilibrium is an attractor too. To do so, we keep the division of the plane into squares, $C_{m,n} = [2m, 2m+2] \times [2n, 2n+2]$, made in the previous case. In a similar way, one can reason that it is always possible to descend from a region $R_m = \bigcup_{-m \leq i,j \leq m} C_{i,j}$ into $R_{m-1} = \bigcup_{-m+1 \leq i,j \leq m-1} C_{i,j}$, and check, by induction, that the orbit generated by any pair of points will eventually enter in $C_{0,0}$. The details are omitted since they are analogous to the case $a = \frac{1}{2}$.

Bearing this in mind, it is sufficient to study the behaviour of the solutions of Equation (5.24) in the square $C_{0,0}$. Firstly, it must be emphasized that $C_{0,0}$ is invariant under $F_{-\frac{1}{2}}$ (see Figure 5.5).

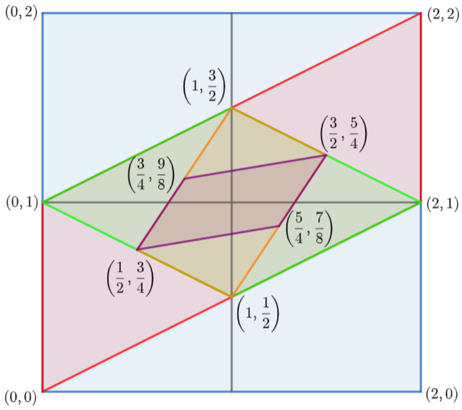


Figure 5.5: Dynamics of Equation (5.24) in the square $C_{0,0}$.

Thus, we move in the upper half-plane H_u, and consequently, the dynamics of $F_{-\frac{1}{2}}$ is governed by the linear difference equation $x_{n+1} = 1 + \frac{1}{2}x_n - \frac{1}{2}x_{n-1}$, whose solutions can be obtained explicitly. Indeed, given a pair of initial conditions $(x_{-1}, x_0) = (x, y) \in C_{0,0}$, the general solution is given by

$$x_n = \left(\frac{1}{2}\right)^{\frac{n+1}{2}} \cdot \left((x-1)\cos\big((n+1)\theta\big) + \left[\frac{(y-1)\sqrt{2}}{\sin(\theta)} + (1-x)\cot(\theta)\right] \sin\big((n+1)\theta\big) \right) + 1,$$

for $n \geq -1$, as can be easily checked (here, $\theta = \arctan(\sqrt{7})$). Then,

$$\lim_{n \to +\infty} x_n = 1 = \bar{x}.$$

To sum up, we have seen that every solution of Equation (5.24) will eventually enter the square $C_{0,0}$ and that the equilibrium attracts every solution in such region. Therefore, jointly with the fact that the equilibrium is locally asymptotically stable, we can conclude that, in fact, it is a global attractor.

Theorem I. *Given the difference equation*

$$x_{n+1} = 1 + \frac{1}{2}|x_n| - \frac{1}{2}x_{n-1},$$

its unique equilibrium point $\bar{x} = 1$ *is a global attractor.*

Corollary 15. *Given the family of difference equations*

$$x_{n+1} = \frac{\max\{x_n, C\}}{x_n^{1/2} \cdot x_{n-1}^{1/2}} \quad \textit{for all} \quad 0 < C < 1,$$

its unique equilibrium point $\bar{x} = 1$ *is a global attractor.*

5.4 Numerical simulations

The main interest of the transformation developed in Section 5.1 resides in the fact that the dynamics of a particular and single equation of Lozi type can be translated into a whole family of max-type difference equations. Consequently, the existence of a concrete dynamical property of the equation

$$x_{n+1} = \frac{\max\{x_n^k, A\}}{x_n^l, x_{n-1}^m},$$

for a specific value $0 < A < 1$ or $A > 1$, will be shared by the complete family. Due to this, we begin by illustrating this fact through numerical simulations. In concrete, we focus on an equation studied by Abu-Saris and Allan in [1]. In their paper, they proved the existence of a strange attractor for the equation

$$x_{n+1} = \frac{\max\{x_n^2, A\}}{x_n x_{n-1}},$$

in the particular case $A = 2.3$, and they also comment that *the solution in this case is also chaotic for certain values of A*. In fact, since a topological conjugation is a transitive relation, due to our study developed in Section 5.1, we can guarantee that the whole family $x_{n+1} = \frac{\max\{x_n^2, B\}}{x_n x_{n-1}}$, with $B > 1$, presents the *same* strange attractor, which is a homeomorphic copy of the attractor detected for $A = 2.3$, because such family of difference equations with maximum is topologically conjugate to

$$y_{n+1} = |y_n| - y_{n-1} - 1.$$

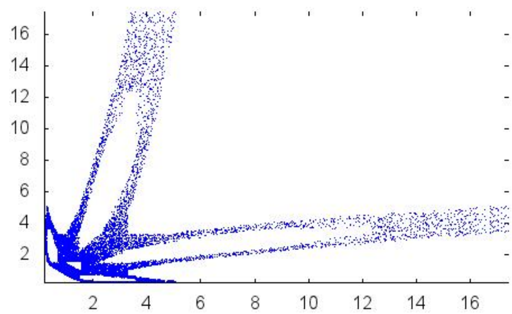


Figure 5.6: Reproduction of Abu-Saris Allan's figure 4 in [1].

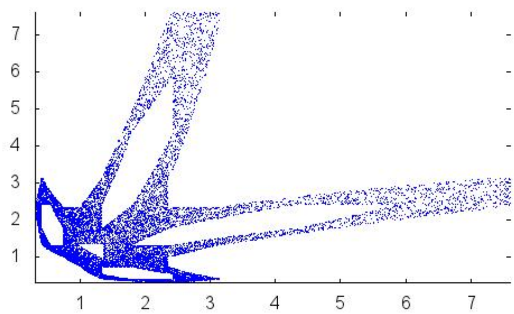


Figure 5.7: Simulation with $A = 1.8$ and initial conditions $x_{-1} = x_0 = 1.8^{0.8}$.

For $0 < A < 1$, the max-equation $x_{n+1} = \frac{\max\{x_n^2, A\}}{x_n x_{n-1}}$ is topologically conjugate to $y_{n+1} = |y_n| - y_{n-1} + 1$. Numerical simulations suggest that the same behaviour that the one exhibited for $A > 1$ takes place in this case (see Figure 5.8).

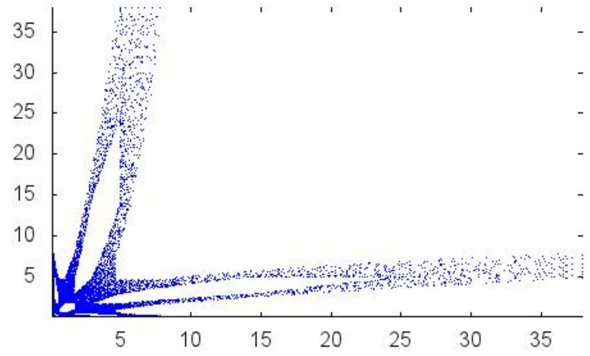


Figure 5.8: Simulation with $A = 0.35$ and two random initial conditions $x_{-1}, x_0 \in (0, 1)$.

In this sense, a question that arises naturally is to determine if the systems for $A > 1$ and $0 < A < 1$ are topologically conjugate.

Next, once that we have illustrated the power of the transformation that links max-type difference equations with generalized Lozi maps, we present some numerical simulations for the Lozi map in the particular case $a = b$ with $a \notin \left[-\frac{1}{2}, \frac{1}{2}\right]$. For every simulation we have taken the origin, $(x_{-1}, x_0) = (0, 0)$ as initial conditions.

Firstly, it is direct to see that the case $a = b = -1$ yields to the *Gingerbreadman equation* studied in [31]. In particular, taking $(0, 0)$ as initial conditions, we obtain a periodic solution of period 6. On the other hand, when $a = b < -1$, it seems that the orbit of $(0, 0)$ tends to infinity in a spiral movement as Figure 5.9 shows. It would be interesting to prove if any initial conditions verify this class of dynamics.

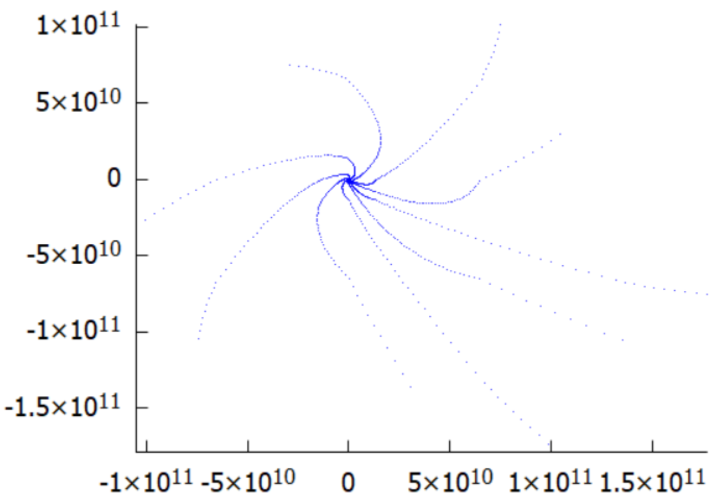


Figure 5.9: Case $a = b = -1.01$. A repulsive orbit.

For the parametric region $\left(-1, -\frac{1}{2}\right)$, it looks like the orbit generated by the origin is always trapped by an equilibrium point (see Figure 5.10). In this context, it would be interesting to analytically establish whether the equilibrium behaves as a global attractor. Employing a technique akin to the one developed in Section 5.3 might prove to be effective.

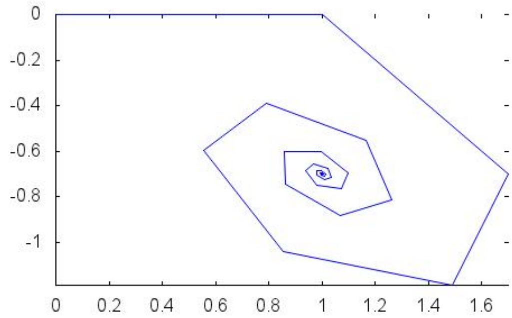


Figure 5.10: Case $a = b = -0.7$. The orbit of $(0, 0)$ goes to the equilibrium point.

Nevertheless, when the parameter takes a value $a > \frac{1}{2}$, it seems that the behaviour changes radically. In this case, two unstable equilibrium points appear, namely, $\bar{x} = 1$ and $\bar{x} = \frac{1}{1-2a}$ (see Lemma 47). Moreover, there exists a 2-periodic orbit (see Lemma 48) which seems to be a local attractor that attracts the origin. We illustrate in Figures 5.11 and 5.12 the cases $a = b = 0.99$ and $a = b = 0.9998$.

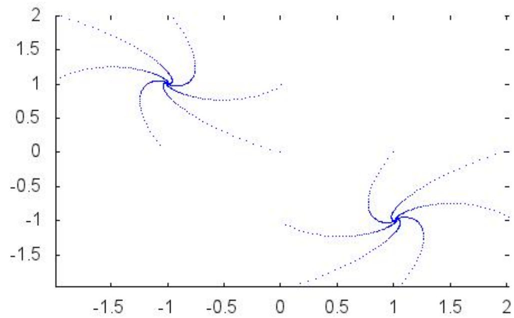


Figure 5.11: Case $a = b = 0.99$. The orbit of $(0, 0)$ goes to a 2-periodic orbit.

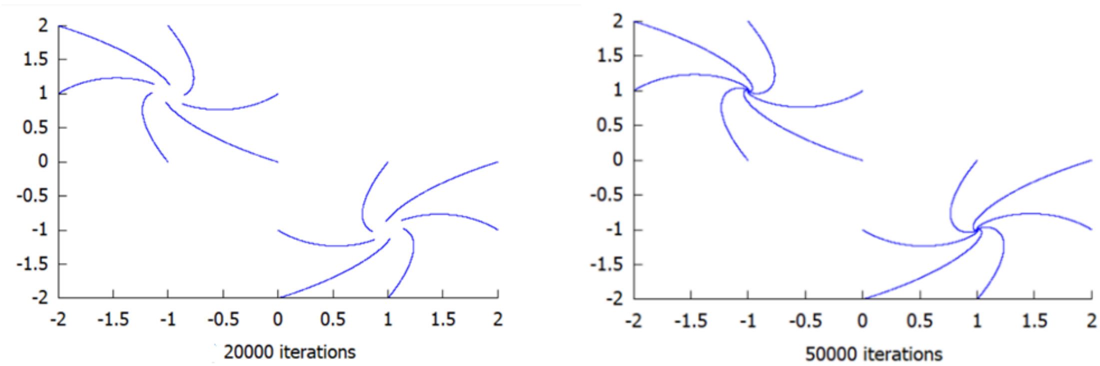
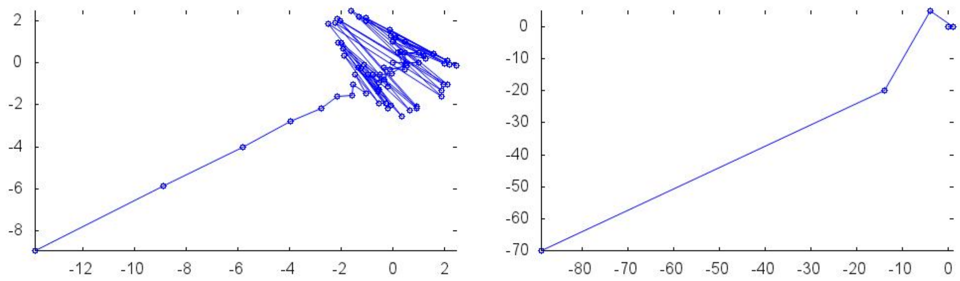


Figure 5.12: Case $a = b = 0.9998$.

Finally, for $a = b = 1$, the orbit generated by the origin is 12-periodic. Furthermore, when $a = b > 1$, the orbit remains for some time near to the origin and finally goes to infinity by the third quadrant. Also, when we increase the value of a, the exit from the neighbourhood of $(0, 0)$ is faster, see Figures 5.13a and 5.13b.



(a) Case $a = b = 1.01$.

(b) Case $a = b = 5$.

5.5 Conclusions and open problems

The connection established between the Lozi map and the difference equations with maximum enables us to extend the dynamics of a particular equation into an entire one-dimensional family through the variable transformations outlined in Section 5.1. This broadens the treatment of dynamical aspects for max-type equations, typically confined to properties related to periodicity and boundedness. In this context, max-type equations can be viewed as generators of complex dynamics, including attractors and omega-limit sets, making their exploration an intriguing avenue for future research. It is noteworthy that the Lozi map finds applications in various fields such as control theory, game theory, and synchronization theory, among others (see [35]), and these applications can be automatically translated to max equations.

Furthermore, we would like to emphasize that this connection allows us to profit from some techniques of differential equations or discrete dynamics, apart from the usual ones used in the literature that, in most cases, are strongly related to arguments of real analysis. For additional information on max-type equations, see [69], wherein the authors compile extensive information on max-type difference equations, their known dynamics as the techniques employed in related research.

Also, we would like to gather in this section some open problems related to the topic treated in this chapter. On the one hand, to determine analytically the dynamics of the Lozi map for $a = b$ when $|a| > \frac{1}{2}$, to verify if the behaviours suggested in the numerical simulations presented in Section 5.4 are true. On the other hand, to deeply study the other scenario deduced from the relation between the generalized Lozi map and max-type equations, namely, the case $b - a = 2$. Even, we could propose delving deeper into understanding the dynamics of generalized Lozi maps given by Equation (5.8) when $\delta = 0$ and $\alpha + \beta + \gamma = 1$.

Finally, it is noteworthy that Lozi map continues to serve as a source of inspiration for research across various fields, particularly in the realm of difference equations. As an example of this, we can find, recently published in the literature, a Special Issue concerning the Lozi map where 32 articles were proposed by 78 authors from 20 different countries. The reader interested might consult [73].

Chapter 6

Applications to population dynamics

In the field of Biology, and more specifically in population dynamics, difference equations are used to model population systems with non-overlapping generations. In this regard, in the last decades diverse discrete models have been analyzed deepening in the understanding of the long-term behaviour of the populations involved. For instance, see [5, 78, 92].

After studying different dynamical properties in the preceding chapters, we devote this final one to the application of autonomous difference equations for modelling host-parasitoid systems. In this sense, we pursue two goals. On the one hand, to illustrate the significance of difference equations as powerful tools to model real phenomena (apart from biological applications, there exist models based on autonomous difference equations in a huge variety of fields: physics, engineering, economics, control theory...; for applied models, consult, for example, [98]). On the other hand, to study other dynamical properties different from the ones analyzed in the previous chapters, for instance, stability, bifurcations or permanence. Also, it must me emphasized that in our study we provide a rigorous and analytical proof for the diverse dynamical aspects analyzed, not limiting to develop numerical simulations. In this direction, we prove accurately the occurrence of various bifurcation scenarios for the equilibrium points in a general case, since we consider an arbitrary probability function to define our model, which covers numerous examples already existing in the literature as we will see at the end of the chapter.

One of the first models that appeared in the literature is the well-known Nicholson-Bailey model, [85],

$$\begin{cases} H_{n+1} = rH_n e^{-bP_n}, \\ P_{n+1} = eH_n \left[1 - e^{-bP_n}\right], \end{cases} \tag{6.1}$$

where r and e are parameters representing a survival rate after the different life stages (larvae, pupae, and adult) for the host and parasitoid, respectively; and the parameter b represents the parasitoid's search efficiency or the probability of encountering a host during its search lifetime. In the sequel, (H_n) and (P_n) represent the host and parasitoid population's sequences, respectively.

Nevertheless, this first model appeared to have various inconsistencies. First, in the absence of parasitoids, the model is unrealistic because System (6.1) reduces to the linear difference equation $H_{n+1} = rH_n$. Moreover, if the host and parasitoids coexist, System (6.1) is unstable for the whole parametric range, which means that either the parasitoid or both populations will die out due to unstable oscillations, a phenomenon that occurs when the parasitoid population grows excessively in each generation, overshadowing the host population, see [32]. In order to solve such deficiencies, in [13] the authors modified System (6.1) by including density-dependent self-regulation by the prey. These variations gave rise to the well-known Beddington model

$$\begin{cases} H_{n+1} = H_n e^{r\left(1-\frac{H_n}{K}\right)-bP_n}, \\ P_{n+1} = eH_n\left[1-e^{-bP_n}\right]. \end{cases} \tag{6.2}$$

Here, the parameters r, e and b represent the same as in System (6.1) and K denotes the carrying capacity of the host. It is worth mentioning that, in the absence of the parasitoids, System (6.2) reduces to the Ricker law, [92], which its main properties have been gathered in Chapter 1,

$$H_{n+1} = H_n e^{r\left(1-\frac{H_n}{K}\right)}.$$

In [78], May shows that models including density dependence as System (6.2), display three clearly defined regimes of dynamic behaviour within the parametric space: stable equilibrium, bifurcation cycles and chaotic behaviour characterized by cycles of any period or aperiodic behaviour depending on the initial conditions of the system.

Observe that the previous systems consider the Poisson distribution, $f(x) = e^{-bx}$, to describe the fraction of hosts surviving parasitism and the same parameter b is used in both equations of the systems. In [50], the authors modified such models by taking different probability functions in the first and second equations. Concretely, the probability functions differ by a constant that multiplies the parasitoid population. In this direction, they proposed a more realistic model given by

$$\begin{cases} H_{n+1} = H_n e^{r\left(1-\frac{H_n}{K}\right)-bP_n} \\ P_{n+1} = eH_n\left[1-e^{-dP_n}\right]. \end{cases} \tag{6.3}$$

Notice that System (6.3) can be considered a generalization of the Beddington Model since for the particular case $b = d$ we recover System (6.2). This generalized model was analytically studied in [53], where the authors analyzed the local stability of the equilibrium points, determined the invariant manifolds for the extinction and exclusion equilibrium and analyzed numerically the possible occurrence of period-doubling, saddle-node and Neimark-Sacker bifurcations at the coexistence equilibrium.

In the present chapter, we study a more generic system by considering a general probability function instead of the Poisson distribution (this suggestion was made in [52]). The proposed system is the following

$$\begin{cases} H_{n+1} &= aH_n e^{r(1-H_n)} f(bP_n) \\ P_{n+1} &= cH_n\big(1-f(P_n)\big) \end{cases}, \tag{6.4}$$

where the parameters a, b, c, r are positive real numbers; the initial sizes of both populations are $H_0, P_0 \geq 0$. However, the probability function f must fulfilled some conditions that arise naturally from the relationship between hosts and parasitoids, in concrete, it must satisfy

$$\begin{aligned} &f \in C[0,\infty) \cap C^4(0,\infty),\ f(y) > 0,\ f'(y) < 0,\ f''(y) \geq 0 \text{ for } y > 0, \\ &\lim_{y\to 0^+} yf'(y) = 0, f(0) = 1, \text{and } f(\infty) = 0. \end{aligned} \tag{$\star$}$$

Observe that in the particular case $a = 1$ and $f(P) = e^{-P}$, we recover the system studied in [53], while for $b = 1$ and $f(P) = e^{-P}$, we achieve the model from [13].

Notice that System (6.4) can be reduced to the following difference equation of second order

$$P_{n+1} = \frac{aP_n\left(1 - f\left(P_n\right)\right) f\left(bP_{n-1}\right) e^{r\left(1-\frac{P_n}{c(1-f(P_{n-1}))}\right)}}{1 - f\left(P_{n-1}\right)},$$

if $P_0, P_1 \neq 0$. Instead of analyzing the dynamics of a two-dimensional system, one can explore the dynamics of the corresponding second-order difference equation

Our main objective in this chapter is to tackle with the rich dynamics of System (6.4) and outline the methods that can be employed to obtain results that can be applicable beyond this system type. In this direction, the structure is the following: firstly, we show the uniform boundedness of both populations. Then, Section 6.2 delves into the existence of the equilibrium points, where we see that System (6.4) has at most three equilibrium points of different type: extinction, exclusion and coexistence. As a next step, we analyze the local linearized stability of all the equilibrium points in Section 6.3. Moreover, in the case of the extinction and exclusion equilibrium we are able to give global stability results. For the coexistence equilibrium point, we only provide a local result due to the impossibility of determining a concrete expression for such point. The result establishing the casuistic about the local stability of the interior equilibrium point is the following:

Theorem J. *Assume that the coexistence equilibrium $E^* = (H^*, P^*)$ exists. Then,*

(a) E^* is locally asymptotically stable if and only if

$$rH^* + cH^*f'(P^*) - 2 < -cH^*f'(P^*)(1 - rH^*) - \frac{c\big(1 - f(P^*)\big)bH^*f'(bP^*)}{f(bP^*)} < 1.$$

(b) E^* is a repeller if and only if

$$rH^* + cH^*f'(P^*) - 2 < -cH^*f'(P^*)(1 - rH^*) - \frac{c\big(1 - f(P^*)\big)bH^*f'(bP^*)}{f(bP^*)}$$

and

$$1 < -cH^*f'(P^*)(1 - rH^*) - \frac{c\big(1 - f(P^*)\big)bH^*f'(bP^*)}{f(bP^*)}.$$

(c) E^* is a saddle if and only if

$$-cH^*f'(P^*)(1-rH^*) - \frac{c\big(1-f(P^*)\big)bH^*f'(bP^*)}{f(bP^*)} < rH^* + cH^*f'(P^*) - 2.$$

(d) E^* is non-hyperbolic if and only if

$$2 - cH^*f'(P^*)(2-rH^*) - rH^* - \frac{c\big(1-f(P^*)\big)bH^*f'(bP^*)}{f(bP^*)} = 0,$$

or

$$rH^* + cH^*f'(P^*) \leq 3 \quad \text{and} \quad -cH^*f'(P^*)(1-rH^*) - \frac{c\big(1-f(P^*)\big)bH^*f'(bP^*)}{f(bP^*)} = 1.$$

Sections 6.4 and 6.5 are devoted to the study of bifurcations. It is relevant to highlight that we do not only numerically verify the existence of bifurcations, but also analytically prove their occurrence, thus advancing the existing results in the literature. As a first step, we analyze the possible occurrence of period-doubling and transcritical bifurcations for the exclusion equilibrium, and after that, we determine the occurrence of period-doubling and Neimark-Sacker bifurcation for the coexistence equilibrium. This last bifurcation implies a change of stability of the interior equilibrium that creates a closed invariant curve that can be either stable or unstable.

Theorem K. *Assume that every assumption of Lemma 69 holds. Let $E^* = (H^*, P^*)$, $d(r_0)$ and $\alpha(r_0)$ be given by (6.6), (6.40) and (6.44), respectively. Then, the interior or coexistence equilibrium point $E^* = (H^*, P^*)$ undergoes a Neimark–Sacker bifurcation at $r = r_0$.*

- *Assume that $d(r_0) > 0$. If $\alpha(r_0) > 0$ ($\alpha(r_0) < 0$) then System (6.4) has an attracting (unstable) closed invariant curve when $r > r_0$ ($r < r_0$) and $r \approx r_0$.*
- *Assume that $d(r_0) < 0$. If $\alpha(r_0) > 0$ ($\alpha(r_0) < 0$) then System (6.4) has an attracting (unstable) closed invariant curve when $r < r_0$ ($r > r_0$) and $r \approx r_0$.*

Next, Section 6.6 gathers the study of the permanence of the system, where we see that host and parasitoid will coexist in the long-term in a specific parameter region.

Theorem L. *If $1 < ae^r < e^2$ and $-c\left(1 + \frac{\ln(a)}{r}\right) f'_+(0) > 1$, then System (6.4) is permanent.*

Theorem M. *Assume that $ae^r \geq e^2$ and $-cf'_+(0) > e^{-A\left(\frac{\ln(a)}{r}+1\right)-B}$, where A and B are given by (6.46) with $\rho_0 = h\left(\frac{ae^{r-1}}{r}\right)$, h being the Ricker map, $h(x) = axe^{r(1-x)}$, $x \geq 0$. Then, System (6.4) is permanent.*

In Section 6.7, we develop numerical simulations for particular probability functions, namely, $f(y) = \left(1 + \frac{y}{m}\right)^{-m}$, with $m > 0$; and $f(y) = e^{-y^m}$, with $0 < m \leq 1$. In both examples, every simulation is coherent with the analytical study developed in the preceding sections, thus illustrating how our model, which is more general by considering an arbitrary probability function, covers numerous existing examples in the literature. Finally, we conclude the chapter by outlining further lines of research related to the topic in Section 6.8.

Recall that all the basic notions and results related to stability and bifurcations used along the following sections can be consulted in Chapter 1.

6.1 Host and parasitoids populations' boundedness

In this section, we delve with the boundedness character of both populations (H_n, P_n) involved in the model. Concretely, we see that, in fact, they are uniformly bounded. To achieve that, we begin by considering the associated map to System (6.4), namely,

$$T\begin{pmatrix} H \\ P \end{pmatrix} = \begin{pmatrix} F(H,P) \\ G(H,P) \end{pmatrix} = \begin{pmatrix} aHe^{r(1-H)}f(bP) \\ cH\big(1-f(P)\big) \end{pmatrix}, \quad H, P \geq 0. \tag{6.5}$$

Proposition 44. *For any pair of initial conditions $(H_0, P_0) \in [0, +\infty)^2$, the sequences (H_n) and (P_n) are bounded.*

Proof. Beforehand, it is immediate that $H_n, P_n \geq 0$ for every $n \geq 0$, due to System (6.4) and the conditions ($\star$). So, we only have to see that both populations are bounded from above.

Consider the map $F(H,P) = aHe^{r(1-H)}f(bP)$. Observe that F is the product of the well-known Ricker map, $h(x) = axe^{r(1-x)}$, $x \geq 0$, by $f(bP)$. It is easy to see that $h(x)$ is bounded from above having its maximum at $\frac{1}{r}$, with $h\left(\frac{1}{r}\right) = \frac{ae^{r-1}}{r}$. Then, since $f(bP) \in [0,1]$ for every $P \geq 0$, by ($\star$) we can deduce that $F(H,P) = h(H)f(bP) \leq \frac{ae^{r-1}}{r}$. Therefore, the second equation of the model yields to

$$P_{n+1} = cH_n\big(1 - f(P_n)\big) \leq \frac{ace^{r-1}}{r},$$

for every $n > 0$ and we conclude $0 \leq P_n \leq \frac{ace^{r-1}}{r}$ for every $n > 0$.

Definitely, for any pair of initial conditions $(H_0, P_0) \in [0,+\infty) \times [0,+\infty)$, we have

$$(H_n, P_n) \in \left[0, \frac{ae^{r-1}}{r}\right] \times \left[0, \frac{ace^{r-1}}{r}\right],$$

for all $n > 0$. Furthermore, notice that T^2 maps $[0,+\infty) \times [0,+\infty)$ into $\left[0, \frac{ae^{r-1}}{r}\right] \times \left[0, \frac{ace^{r-1}}{r}\right]$, and $T\left(\left[0, \frac{ae^{r-1}}{r}\right] \times \left[0, \frac{ace^{r-1}}{r}\right]\right) \subseteq \left[0, \frac{ae^{r-1}}{r}\right] \times \left[0, \frac{ace^{r-1}}{r}\right]$. $\square$

6.2 Existence of equilibrium points

A point $\left(\bar{H}, \bar{P}\right)$ is an equilibrium point if it verifies the equation $T\begin{pmatrix}\bar{H}\\ \bar{P}\end{pmatrix} = \begin{pmatrix}\bar{H}\\ \bar{P}\end{pmatrix}$, that is, the system

$$\begin{cases}\bar{H} = a\bar{H}e^{r(1-\bar{H})}f(b\bar{P})\\ \bar{P} = c\bar{H}\left(1 - f(\bar{P})\right)\end{cases}. \tag{6.6}$$

Firstly, it is direct to see that $E_0 = (0, 0)$ always exists, so we will always have at least one equilibrium point that will be named as *extinction equilibrium*. In addition, any point of the form $(0, P)$ is eventually the extinction equilibrium since $T(0, P) = (0, 0)$. This means that the parasitoid cannot survive without the host.

As a next step, we focus on the possible existence of equilibrium points exhibiting the form $E_0^* = \left(H_0^*, 0\right)$ with $H_0^* > 0$, that is, an *exclusion equilibrium point*. In this case, system (6.6) reduces to the single equation $H_0^* = aH_0^*e^{r(1-H_0^*)}$, or equivalently,

$$1 = ae^{r(1-H_0^*)}.$$

From here, $H_0^* = 1 + \dfrac{\ln(a)}{r}$ and due to the fact that $H_0^* > 0$, we get that $ae^r > 1$ is a necessary and sufficient condition in order to have a parasitoid-free equilibrium, $E_0^* = \left(H_0^*, 0\right)$.

Lemma 62. *System (6.4) has a unique exclusion equilibrium point* $(H_0^*, 0) = \left(1 + \dfrac{\ln(a)}{r}, 0\right)$ *if and only if* $ae^r > 1$.

Finally, we deal with the existence of an interior equilibrium point, that we call *coexistence equilibrium*, $E^* = \left(H^*, P^*\right)$ with $H^*, P^* > 0$. Beforehand, we claim that $H^* < H_0^*$.

Lemma 63. *Assume that the coexistence and exclusion equilibrium points of System (6.4) exist. Then,* $H^* < H_0^*$.

Proof. Consider the map $Q(H) = \frac{1}{ae^{r(1-H)}}$. It is direct that $Q(0) > 0$ and $Q'(H) > 0$, so it is a positive increasing map on $(0, \infty)$.

On the other hand, from the first equation of (6.6), $H^* = aH^*e^{r(1-H^*)}f(bP^*)$, which implies $1 = ae^{r(1-H^*)}f(bP^*)$, or equivalently,

$$\frac{1}{Q(H^*)} = f(bP^*).$$

Notice that the same equality holds for H_0^* with $P^* = 0$. Next, applying that f is decreasing by ($\star$), we obtain

$$Q(H^*) = \frac{1}{ae^{r(1-H^*)}} = f(bP^*) < f(0) = 1 = \frac{1}{ae^{r(1-H_0^*)}} = Q(H_0^*).$$

In conclusion, $H^* < H_0^*$. □

For the next result we assume that $f'_+(0)$ exists, where f'_+ denotes the right-hand derivative of f.

Lemma 64. *Under the above assumption, it holds*

$$1 - f(P) \leq -f'_+(0)P, \quad \textit{for every } P > 0. \tag{6.7}$$

Proof. The result follows from the second order Maclaurin polynomial of $f(P)$ and the fact that $f''(x) \geq 0$ for all $x > 0$ by ($\star$). Indeed,

$$f(P) = 1 + f'_+(0)P + \frac{f''(\xi)}{2}P^2 \ \text{ for } P > 0, \ \xi \in (0, P).$$

Thus,

$$1 - f(P) = -f'_+(0)P - \frac{f''(\xi)}{2}P^2 \leq -f'_+(0)P, \ \text{ for } P > 0.$$

□

It must be emphasized that Lemmas 63 and 64 imply

$$P^* = cH^*\big(1 - f(P^*)\big) < -cH_0^* f'_+(0)P^*, \ \text{ hence } \ 1 < -cH_0^* f'_+(0).$$

Consequently, the coexistence equilibrium E^* will not exist if $1 + cH_0^* f'_+(0) \geq 0$. This allows us to derive a necessary condition for the existence of E^*, namely,

$$1 + cH_0^* f'_+(0) < 0 \text{ which, by Lemma 62, is equivalent to } -cf'_+(0)\left(1 + \frac{\ln(a)}{r}\right) > 1.$$

In fact, we see that such condition is sufficient too.

Theorem 19. *The following statements hold:*

(i) If $f'_+(0)$ exists, then a unique coexistence equilibrium point exists if and only if

$$-cf'_+(0)\left(1 + \frac{\ln(a)}{r}\right) > 1.$$

(ii) If $f'_+(0) = -\infty$, then a unique coexistence equilibrium point exists if and only if

$$1 + \frac{\ln(a)}{r} > 0.$$

Proof. Firstly, we focus on the first statement. Here, $H^*, P^* \neq 0$, so we can reduce (6.6) into

$$\begin{cases} 1 & = & ae^{r(1-H^*)}f(bP^*) \\ P^* & = & cH^*\big(1 - f(P^*)\big) \end{cases}. \tag{6.8}$$

Now, take $G_1(P) = \frac{P}{c\left(1-f(P)\right)}$, for $0 < P < \infty$. It is easy to see that

$$\lim_{P\to 0^+} G_1(P) = -\frac{1}{cf'_+(0)} \geq 0, \quad \lim_{P\to +\infty} G_1(P) = +\infty, \quad G'_1(P) = \frac{1 - f(P) + Pf'(P)}{c\big(1 - f(P)\big)^2}.$$

Let $K(P) = 1 - f(P) + Pf'(P)$ for all $P > 0$. Hence, $K'(P) = Pf''(P) > 0$, and by applying ($\star$) we obtain $\lim_{P\to 0^+} K(P) = 0$. Therefore, $K(P) > 0$ on $(0,\infty)$ and $G_1'(P) > 0$ for all $P > 0$, which allows us to conclude that $G_1(P)$ is a positive increasing function on $(0,\infty)$.

On the other hand, consider the first equation of (6.8) and take

$$G_2(P) = \frac{\ln(a) + \ln\big(f(bP)\big)}{r} + 1.$$

Observe that conditions ($\star$) imply

$$\lim_{P\to 0^+} G_2(P) = 1 + \frac{\ln(a)}{r}, \quad \lim_{P\to +\infty} G_2(P) = -\infty, \quad G_2'(P) = \frac{bf'(bP)}{rf(bP)} < 0.$$

Therefore, $G_2(P)$ is a decreasing function on $(0,\infty)$ and $G_1(P) = G_2(P)$ has a positive solution if and only if $\lim\limits_{P\to 0^+} G_1(P) < \lim\limits_{P\to 0^+} G_2(P)$, that is,

$$-\frac{1}{cf_+'(0)} < 1 + \frac{\ln(a)}{r},$$

from which the proof follows. The second statement follows from $\lim\limits_{P\to 0^+} G_1(P) = 0$. □

As a direct consequence of Lemma 62 and Theorem 19, we have the following result.

Corollary 16. *Assume that $ae^r \leq 1$. Then, System (6.4) has a unique equilibrium point: the extinction equilibrium E_0.*

6.3 Linearized stability

This section is devoted to analyze the linearized stability of the three equilibrium points of the system. Concerning local stability, basically we apply [58, Th.2.12]. For the sake of completeness, consider the system of first order difference equations given by

$$\begin{cases} x_{n+1} &= f(x_n, y_n) \\ y_{n+1} &= g(x_n, y_n) \end{cases}, \tag{6.9}$$

where f and g are given functions. We denote by $F : \mathbb{R}^2 \to \mathbb{R}^2$ the vector map

$$F(x,y) = \big(f(x,y), g(x,y)\big).$$

Also, let (x^*, y^*) represent a fixed point of F, where f and g are continuously differentiable functions at (x^*, y^*). Then, $J_F(x^*, y^*)$ will be the Jacobian matrix of F at (x^*, y^*), that is,

$$J_F(x^*, y^*) = \begin{pmatrix} \frac{\partial f}{\partial x}(x^*, y^*) & \frac{\partial f}{\partial y}(x^*, y^*) \\ \frac{\partial g}{\partial x}(x^*, y^*) & \frac{\partial g}{\partial y}(x^*, y^*) \end{pmatrix}.$$

Theorem 20. *[58, **Theorem 2.12**] Given System (6.9), the following holds:*

- *An equilibrium point (x^*, y^*) is locally asymptotically stable if and only if every solution of the characteristic equation*

$$\lambda^2 - \mathrm{Tr}\big(J_F(x^*, y^*)\big)\lambda + \mathrm{Det}\big(J_F(x^*, y^*)\big) = 0, \tag{6.10}$$

  lies inside the unit circle, that is, if and only if

$$\big|\mathrm{Tr}\big(J_F(x^*, y^*)\big)\big| < 1 + \mathrm{Det}\big(J_F(x^*, y^*)\big) < 2.$$

- *An equilibrium point (x^*, y^*) is locally a repeller if and only if every solution of the characteristic equation (6.10) lies outside the unite circle, that is, if and only if,*

$$\big|\mathrm{Tr}\big(J_F(x^*, y^*)\big)\big| < \big|1 + \mathrm{Det}\big(J_F(x^*, y^*)\big)\big| \quad \textit{and} \quad \big|\mathrm{Det}\big(J_F(x^*, y^*)\big)\big| > 1.$$

- *An equilibrium point (x^*, y^*) is locally a saddle point if and only if the characteristic equation (6.10) has one root that lies inside the unit circle and one root that lies outside the unit circle, that is, if and only if,*

$$\big|\mathrm{Tr}\big(J_F(x^*, y^*)\big)\big| > \big|1+\mathrm{Det}\big(J_F(x^*, y^*)\big)\big| \quad \textit{and} \quad \big(\mathrm{Tr}\big(J_F(x^*, y^*)\big)\big)^2 - 4\mathrm{Det}\big(J_F(x^*, y^*)\big) > 0.$$

- *An equilibrium point (x^*, y^*) is non-hyperbolic if and only if the characteristic equation (6.10) has at least one root that lies on the unit circle, that is, if and only if,*

$$\big|\mathrm{Tr}\big(J_F(x^*, y^*)\big)\big| = \big|1 + \mathrm{Det}\big(J_F(x^*, y^*)\big)\big|$$

  or

$$\mathrm{Det}\big(J_F(x^*, y^*)\big) = 1 \quad \textit{and} \quad \big|\mathrm{Tr}\big(J_F(x^*, y^*)\big)\big| \leq 2.$$

In this direction, we begin by computing the Jacobian matrix of the two-dimensional map T given by (6.5),

$$J_T(H, P) = \begin{pmatrix} ae^{r(1-H)} f(bP)(1 - rH) & abHe^{r(1-H)} f'(bP) \\ c\big(1 - f(P)\big) & -cHf'(P) \end{pmatrix}. \tag{6.11}$$

It is worth mentioning that in the case of the extinction and exclusion equilibrium we will not only establish the local stability, but also present global results under additional conditions. In the sequel, we assume that $f'_+(0)$ and $f''_+(0)$ exist.

6.3.1 Local and global stability of the extinction equilibrium

We focus on the extinction equilibrium point, namely, $E_0 = (0,0)$. As a first step, we evaluate the Jacobian (6.11) at such point,

$$J_T(E_0) = \begin{pmatrix} ae^r & 0 \\ 0 & 0 \end{pmatrix}.$$

It is immediate that the corresponding eigenvalues are $\lambda_1 = ae^r > 0$ and $\lambda_2 = 0$. Firstly, we study the case $\lambda_1 = ae^r = 1$ and establish that, in this situation, the equilibrium point is stable.

Lemma 65. *Let $E_0 = (0,0)$ be the extinction equilibrium for System (6.4). Assume that $ae^r = 1$. Then, E_0 is locally stable.*

Proof. Let us consider an $\varepsilon > 0$ and assume without loss of generality that $\varepsilon < 1$. Set $\delta < \varepsilon$ such that $f(P_0) > 1 - \frac{\varepsilon}{c}$ whenever $P_0 \in [0, \delta)$.

Now, take a pair of initial conditions, $(H_0, P_0) \in [0, \delta)^2$. From here, we have

$$\begin{aligned} H_1 &= H_0 e^{-rH_0} f(bP_0) \le H_0 < \delta, \\ P_1 &= cH_0\big(1 - f(P_0)\big) \le c \cdot \delta \cdot \frac{\varepsilon}{c} < \delta. \end{aligned}$$

Similarly, proceeding by induction, it can be easily seen that $H_n < \delta$ and $P_n < \delta$ for every $n \ge 0$. Consequently, we have shown that if the initial conditions are in the neighbourhood $[0, \delta)^2$, the whole orbit generated from them, $\big((H_n, P_n)\big)_n$ stays in the same region, so we can deduce, by definition, that the equilibrium is locally stable. □

Next, we analyze the remaining cases, namely, $\lambda_1 = ae^r \neq 1$.

Theorem 21. *For the extinction equilibrium point $E_0 = (0,0)$, the following holds:*

(a) If $ae^r < 1$, then it is locally asymptotically stable.

(b) If $ae^r > 1$, then it is a saddle point. Moreover, the local unstable manifold is located at the x-axis, while the local stable manifold is located at the y-axis. The global stable manifold is given by: $W^s = \{(H,P) : H = 0, 0 < P < \infty\}$, and the global unstable manifold is given by:

- $W^u = \big\{(H,P) : 0 \le H < 1 + \frac{\ln a}{r}, P = 0\big\}$, *if* $1 < ae^r \le e$.
- $W^u = \big\{(H,P) : 0 \le H \le \frac{ae^{r-1}}{r}, P = 0\big\}$, *if* $e < ae^r$.

Proof. The statements concerning the stability are straightforward. We restrict to the determination of the stable and unstable manifolds of E_0 whenever it is a saddle point. According to the Local Stable Manifold Theorem, Theorem 5, smooth invariant curves that are tangent in E_0 to the eigenvectors associated with the corresponding eigenvalues of the linearized problem exist. Bearing in mind that those eigenvectors are $v_1 = (1,0)$ for the unstable manifold and $v_2 = (0,1)$ for the stable manifold, by direct inspection, for a neighbourhood of the equilibrium point, it is immediate to see that the local unstable manifold is located at the x-axis, while the local stable manifold is located at the y-axis.

Finally, concerning the global manifold, $W^s = \{(H,P) : H = 0, 0 \leq P < \infty\}$ since $T(0,P) = (0,0)$ for every $P \geq 0$. On the other hand, for the unstable manifold, we need to distinguish two cases depending on the value of ae^r according to the properties of the Ricker map (see Chapter 1):

- If $1 < ae^r \leq e$, $W^u = \left\{(H,P) : 0 \leq H < 1 + \frac{\ln a}{r}, P = 0\right\}$.
- If $e < ae^r$, $W^u = \left\{(H,P) : 0 \leq H \leq \frac{ae^{r-1}}{r}, P = 0\right\}$.

□

Recall that, by Corollary 16, when $ae^r \leq 1$, the extinction equilibrium is the unique equilibrium point of System (6.4). Therefore, it is natural to study the possibility of having global stability. In this direction, we start proving that in this scenario E_0 is a global attractor.

Lemma 66. *Let (H_n, P_n) denote a solution of System (6.4). If $ae^r \leq 1$, then $\lim_{n\to\infty} (H_n, P_n) = E_0$.*

Proof. The first equation of System (6.4) guarantees that $H_{n+1} \leq aH_n e^{r(1-H_n)}$. Moreover, it is possible to majorize the sequence (H_n) with a sequence (z_n), which is the solution of the Ricker equation $z_{n+1} = az_n e^{r(1-z_n)}$. Indeed, let $h(z) = aze^{r(1-z)}$, and $z_{n+1} = h(z_n)$. It can be easily seen that the Ricker map h is an increasing function on $\left[0, \frac{1}{r}\right]$ and $h\left(\left[0, \frac{1}{r}\right]\right) \subseteq \left[0, \frac{1}{r}\right]$. Thus, if $z_0 = H_0 \in \left[0, \frac{1}{r}\right]$, then $H_n \leq z_n \leq 1/r$ for all $n > 0$.

One can see that $\lim_{n\to\infty} z_n = 0$ and this implies $\lim_{n\to\infty} H_n = 0$. Now, if $H_0 > 1/r$, then

$$H_1 = F(H_0, P_0) \leq aH_0 e^{r(1-H_0)} \leq \max_{z\in[0,\infty)} h(z) = ae^{r-1}/r \leq 1/r,$$

since $ae^r \leq 1$. Thus, we get that in this case $\lim_{n\to\infty} H_n = 0$. Next, we can choose $\varepsilon > 0$ sufficiently small, such that $-cf'_+(0)\varepsilon < 1$. For that ε there exists n_0 such that $H_n < \varepsilon$ for all $n > n_0$.

From the second equation of System (6.4) and applying Lemma 64, we have

$$P_{n+1} = cH_n(1 - f(P_n)) < -cf'_+(0)\varepsilon P_n.$$

Since $-cf'_+(0)\varepsilon < 1$, we obtain $P_n \to 0$ as $n \to \infty$. □

Finally, we see that in the case $ae^r \leq 1$ we can assure the global asymptotical stability of E_0.

Theorem 22. *Assume that $ae^r \leq 1$. Then, the extinction equilibrium E_0 is globally asymptotically stable.*

Proof. On the one hand, if $ae^r = 1$, Lemma 65 ensures that E_0 is locally stable. Moreover, by Lemma 66, we have that it is a global attractor. So, in this case the extinction equilibrium is globally asymptotically stable.

On the other hand, if $ae^r < 1$, Theorem 21 (a) gives us that E_0 is locally asymptotically stable and again, Lemma 66 guarantees that E_0 is a global attractor. Hence, it is globally asymptotically stable too. □

6.3.2 Local and global stability of the exclusion equilibrium

Now, we study the local stability of the exclusion equilibrium $E_0^* = (H_0^*, 0)$, where $H_0^* = 1 + \frac{\ln(a)}{r}$. Again, the main results concerning stability are a direct application of Theorem 20. Furthermore, we give a global result under appropriate assumptions.

As a first step, we evaluate the Jacobian matrix (6.11) in E_0^*,

$$J_T(E_0^*) = \begin{pmatrix} 1 - r - \ln(a) & \left(b + \frac{b\ln(a)}{r}\right) f'_+(0) \\ 0 & -c\left(1 + \frac{\ln(a)}{r}\right) f'_+(0) \end{pmatrix}. \tag{6.12}$$

By a direct inspection of $J_T(E_0^*)$, we get the corresponding eigenvalues $\lambda_1 = 1 - r - \ln(a)$ and $\lambda_2 = -c\left(1 + \frac{\ln(a)}{r}\right) f'_+(0)$. Observe that $\lambda_1 < 1$ and $\lambda_2 > 0$. Indeed, by Lemma 62, E_0^* exists if and only if $ae^r > 1$, which is equivalent to $1 + \frac{\ln(a)}{r} > 0$, implying $\lambda_1 < 1$.

Theorem 23. *Assume that the exclusion equilibrium point E_0^* exists. Then, for E_0^* the following holds:*

(a) If $r + \ln(a) - 2 < 0$ and $-c\left(1 + \frac{\ln(a)}{r}\right) f'_+(0) < 1$, then it is locally asymptotically stable.

(b) If $r + \ln(a) - 2 > 0$ and $-c\left(1 + \frac{\ln(a)}{r}\right) f'_+(0) > 1$, then it is a repeller.

(c) If

$$r + \ln(a) - 2 < 0 \text{ and } -c\left(1 + \frac{\ln(a)}{r}\right) f'_+(0) > 1$$

or

$$r + \ln(a) - 2 > 0 \text{ and } -c\left(1 + \frac{\ln(a)}{r}\right) f'_+(0) < 1,$$

then it is a saddle point. In the first case, the local stable and unstable manifolds are given by

$$W_1^s = \{(H, P) : 0 < H < \infty, P = 0\},$$

$$W_1^u = \{(H, P) : H = H_0^* + a_1 P + a_2 P^2 + O\big(P^3\big), H > 0, P > 0\},$$

where

$$a_1 = \frac{-bH_0^* f_+'(0)}{1 - rH_0^* + cH_0^* f_+'(0)},$$

and

$$\begin{aligned} a_2 &= \frac{b^2 H_0^* f_+'(0) \cdot \big[-2 + H_0^* r\big(2 - H_0^* r + 2cH_0^* f_+'(0)\big)\big]}{2\big(1 - H_0^* r + cH_0^* f_+'(0)\big)^2 \cdot \big[-1 + H_0^*\big(r + c^2 H_0^* (f_+'(0))^2\big)\big]} \\ &+ \frac{bH_0^* f_+''(0) \cdot \big[b(-1 + H_0^* r) - (-1+b)cH_0^* f_+'(0)\big] \cdot \big[-1 + H_0^*\big(r - cf_+'(0)\big)\big]}{2\big(1 - H_0^* r + cH_0^* f_+'(0)\big)^2 \cdot \big[-1 + H_0^*\big(r + c^2 H_0^* (f_+'(0))^2\big)\big]}, \end{aligned}$$

while in the second case, $W_2^s = W_1^u$ *and* $W_2^u = W_1^s$.

(d) *If* $r + \ln(a) - 2 = 0$ *or* $-c\left(1 + \frac{\ln(a)}{r}\right) f_+'(0) = 1$, *then it is a non-hyperbolic equilibrium.*

Proof. The results concerning stability follow by Theorem 20, thus we focus on determining the local manifolds in the case where the exclusion equilibrium is a saddle. To do so, we begin by shifting E_0^* to the origin by the change of variables $x_n = H_n - H_0^*$ and $y_n = P_n$, which allows us to transform System (6.4) into

$$\begin{cases} x_{n+1} = a(x_n + H_0^*)e^{r(-H_0^* - x_n + 1)} f(by_n) - H_0^* \\ y_{n+1} = c(x_n + H_0^*)\big(1 - f(y_n)\big) \end{cases}.$$

We consider

$$\begin{aligned} f_1(x,y) &= a(x + H_0^*)e^{r(-H_0^* - x + 1)} f(by) - H_0^*, \\ f_2(x,y) &= c(x + H_0^*)\big(1 - f(y)\big), \end{aligned}$$

and compute their Taylor series expansion about $(0,0)$ bearing in mind that $ae^{r(1-H_0^*)} = 1$,

$$\begin{aligned} f_1(x,y) &= (1 - rH_0^*)x + bH_0^* f_+'(0)y + (-2r + r^2 H_0^*)\frac{x^2}{2} + bf_+'(0)(1 - rH_0^*)xy \\ &\quad + b^2 H_0^* f_+''(0)\frac{y^2}{2} + O\big((|x| + |y|)^3\big), \\ f_2(x,y) &= -cH_0^* f_+'(0)y - cf_+'(0)xy - cH_0^* f_+''(0)\frac{y^2}{2} + O\big((|x| + |y|)^3\big). \end{aligned}$$

Next, we only study the case $r + \ln(a) - 2 < 0$ and $-c\left(1 + \frac{\ln(a)}{r}\right) f_+'(0) > 1$, since in the other scenario, namely, $r + \ln(a) - 2 > 0$ and $-c\left(1 + \frac{\ln(a)}{r}\right) f_+'(0) < 1$, the stable and unstable manifold interchange being $W_2^s = W_1^u$ and $W_2^u = W_1^s$.

Firstly, assume that the unstable manifold is given by $x = h(y) = a_1 y + a_2 y^2 + O(|y|^3)$. Due to the fact that the manifold is invariant, the equation

$$f_1\big(h(y), y\big) - h\big(f_2(h(y), y)\big) = 0$$

must be satisfied. In this sense, we get

$$\left[a_2\big(-1 + H_0^* r + (cH_0^* f_+'(0))^2\big) + a_1\left(-bf_+'(0) + bf_+'(0)H_0^* r + \frac{cH_0^* f_+''(0)}{2}\right) - \frac{b^2 H_0^* f_+''(0)}{2}\right] y^2,$$

$$= \big[-a_1(-1+rH_0^*) + bH_0^* f'_+(0) + a_1 c H_0^* f'_+(0)\big] y \ + a_1^2 \left(\frac{r(-2+H_0^* r)}{2} + c f'_+(0) \right) y^2 + O\big(|y|^3\big),$$

from where we obtain the values

$$\begin{aligned}
a_1 &= \frac{-bH_0^* f'_+(0)}{1 - rH_0^* + cH_0^* f'_+(0)}, \\
a_2 &= \frac{b^2 H_0^* f'_+(0)\big(-2 + H_0^* r(2 - H_0^* r + 2cH_0^* f'_+(0))\big)}{2\big(1 - H_0^* r + cH_0^* f'_+(0)\big)^2 \big(-1 + H_0^*(r + c^2 H_0^* (f'_+(0))^2)\big)} \\
&+ \frac{bH_0^* f''_+(0)\big(b(-1+H_0^* r) - (-1+b)cH_0^* f'_+(0)\big)\big(-1 + H_0^*(r - cf'_+(0))\big)}{2\big(1 - H_0^* r + cH_0^* f'_+(0)\big)^2 \big(-1 + H_0^*(r + c^2 H_0^* (f'_+(0))^2)\big)}.
\end{aligned}$$

It is relevant to highlight that the above coefficients a_1 and a_2 are well defined since the corresponding denominators cannot be zero. Indeed, since $\lambda_1 < -1$ and $\lambda_2 > 0$, we have $1 - rH_0^* + cH_0^* f'_+(0) = \lambda_1 - \lambda_2 \neq 0$; in addition, $-1 + H_0^*(r + c^2 H_0^*(f'_+(0))^2) = -\lambda_1 + \lambda_2^2 \neq 0$ for similar reasons.

In conclusion, the local unstable manifold is given by

$$W_1^u = \{(H,P) : H = H_0^* + a_1 P + a_2 P^2 + O(P^3), H > 0, P > 0\},$$

and the restriction of the mapping to the unstable invariant manifold is given locally by $y_{n+1} = f_2\big(h(y_n), y_n\big)$, that is,

$$y_{n+1} = -cH_0^* f'_+(0) y_n - \left(\frac{2ca_1 f'_+(0) + cH_0^* f''_+(0)}{2} \right) y_n^2 + O\left(|y_n|^3\right).$$

For the local stable manifold, according to the Local Stable Manifold Theorem, Theorem 5, there exists a smooth invariant curve tangent in E_0^* to the eigenvector associated with the corresponding eigenvalue of the linearized problem. Bearing in mind that for the stable manifold, an eigenvector is $v = (1,0)$, by direct inspection, for a neighbourhood of the equilibrium point, it is immediate to see that the local stable manifold, W_1^s, is contained in the x-axis. In addition, the dynamics restricted to the invariant manifold is given locally by

$$x_{n+1} = a(x_n + H_0^*) e^{r(-H_0^* - x_n + 1)} - H_0^*,$$

so undoing the change of variables, we return to the Ricker equation, $H_{n+1} = ae^r H_n e^{-rH_n}$. In Chapter 1, where we established some basic properties of Ricker map, we saw that $H_0^* = 1 + \frac{\ln(a)}{r}$ is a global attractor relative to the interval $(0,\infty)$ whenever $1 < ae^r < e^2$. Therefore, $W_1^s = \{(H,P) : 0 < H < \infty, P = 0\}$.

In the second case, $r + \ln(a) - 2 > 0$ and $-c\left(1 + \frac{\ln(a)}{r}\right) f'_+(0) < 1$, the unstable manifold becomes the stable one and opposite, $W_2^s = W_1^u$ and $W_2^u = W_1^s$. □

In the sequel, we deepen in the possibility of having global stability. In this direction, observe that $ae^r > 1$ must hold in order to ensure the existence of E_0^*. On the other hand, one of the conditions needed to have local asymptotical stability is $r + \ln(a) - 2 < 0$, which is equivalent to $ae^r < e^2$. Thus, we show that for $1 < ae^r < e^2$, under extra assumptions, the exclusion equilibrium is globally asymptotically stable relative to $(0,\infty) \times [0,\infty)$ (recall that the extinction equilibrium E_0 always exists). We see it by distinguishing two cases, namely, the case $1 < ae^r \le e$ and $e < ae^r < e^2$, corresponding to Theorem 24 and Theorem 26, respectively.

Lemma 67. *Let (H_n, P_n) denote the solution of System (6.4) with the initial condition $(H_0, P_0) \in \mathbb{R}^2_+$. If $1 < ae^r \le e$, then*

$$\limsup_{n\to\infty} H_n \le 1 + \frac{\ln(a)}{r}.$$

Moreover, if $-c\left(1 + \frac{\ln(a)}{r}\right) f'_+(0) < 1$, then $\lim_{n\to\infty}(H_n, P_n) = E_0^*$.

Proof. Let us consider the pair of initial conditions $(H_0, P_0) \in \mathbb{R}^2_+$. The first equation of System (6.4) yields to the inequality $H_{n+1} \le aH_n e^{r(1-H_n)}$. In this direction, we consider the Ricker map $h(z) = aze^{r(1-z)}$, and the difference equation $z_{n+1} = h(z_n)$. It is well-known that h is an increasing map on $\left(0, \frac{1}{r}\right]$ with $h\left(\left(0, \frac{1}{r}\right]\right) \subseteq \left(0, \frac{1}{r}\right]$. So,

- If $z_0 = H_0 \in \left(0, \frac{1}{r}\right]$ then $H_{n+1} \le z_{n+1} \le 1/r$, for $n \ge 0$. Furthermore, $1 + \frac{\ln(a)}{r}$ is a global attractor for the Ricker map when $1 < ae^r \le e$ (see [78]), which implies $\lim_{n\to\infty} z_n = 1 + \frac{\ln(a)}{r}$. Consequently, $\limsup_{n\to\infty} H_n \le 1 + \frac{\ln(a)}{r}$.
- If $H_0 > 1/r$, then $H_1 = F(H_0, P_0) \le h\left(\frac{1}{r}\right) = ae^{r-1}/r \le 1/r$, implying $\limsup_{n\to\infty} H_n \le 1 + \frac{\ln(a)}{r}$.

Next, assume that $-c\left(1 + \frac{\ln(a)}{r}\right) f'_+(0) < 1$. Thus, for every sufficiently small $\varepsilon > 0$, we have

$$-c\left(1 + \frac{\ln(a)}{r} + \varepsilon\right) f'_+(0) < 1. \tag{6.13}$$

Observe that for such ε there exists n_0, such that $H_n < 1 + \dfrac{\ln(a)}{r} + \varepsilon$ for all $n > n_0$. Then, by the second equation of System (6.4) and Lemma 64, we obtain

$$P_{n+1} \le -c\left(1 + \frac{\ln(a)}{r} + \varepsilon\right) f'_+(0) P_n.$$

Now, (6.13) implies that $P_n \to 0$ as $n \to \infty$, an we can deduce that for every $\varepsilon > 0$ there exists n_1, such that for all $n > n_1$, we have $P_n < \varepsilon$. Let $n_2 = \max\{n_0, n_1\}$. We can take $\varepsilon > 0$ sufficiently small, so that $1 < af(b\varepsilon)e^r \le e$. Therefore, the first equation of System (6.4) and the monotonicity of f yields to

$$H_{n+1} \ge af(b\varepsilon)H_n e^{r(1-H_n)} \text{ for all } n > n_2.$$

Let us consider the map $h_1(z) = af(b\varepsilon)ze^{r(1-z)}$, and $z_{n+1} = h_1(z_n)$. It is direct to check that h_1 is increasing on $(0, 1/r]$ and $h_1\big((0, 1/r]\big) \subseteq (0, 1/r]$. So, if $z_0 = H_0 \in (0, 1/r]$ then

$1/r \geq H_{n+1} \geq z_{n+1}$, for $n \geq 0$ and $\lim_{n\to\infty} z_n = 1 + \frac{\ln(af(b\varepsilon))}{r}$ (notice that $1 + \frac{\ln(af(b\varepsilon))}{r}$ is a fixed point for h_1 globally attractor relative to $(0, \infty)$, since $1 < af(b\varepsilon)e^r \leq e$). This implies $\liminf_{n\to\infty} H_n \geq 1 + \frac{\ln(af(b\varepsilon))}{r}$. If $H_0 > 1/r$, then $H_1 = F(H_0, P_0) \leq ae^{r-1}/r \leq 1/r$, implying $\liminf_{n\to\infty} H_n \geq 1 + \frac{\ln(af(b\varepsilon))}{r} > 0$. Letting $\varepsilon \to 0$, we obtain $\liminf_{n\to\infty} H_n \geq 1 + \dfrac{\ln(a)}{r}$. Therefore, $\lim_{n\to\infty} H_n = 1 + \dfrac{\ln(a)}{r}$. □

Now, we are able to establish a global stability result for the exclusion equilibrium whenever $1 < ae^r \leq e$ hold.

Theorem 24. *Assume that* $1 < ae^r \leq e$ *and* $-c\left(1 + \frac{\ln(a)}{r}\right) f'_+(0) < 1$. *Then,* E_0^* *is globally asymptotically stable relative to* $(0, \infty) \times [0, \infty)$.

Proof. On the one hand, the local asymptotical stability of E_0^* is ensured by Theorem 23. On the other hand, Lemma 67 gives us the global attraction of the equilibrium and the Ricker map governs the dynamics on the x-axis. Consequently, E_0^* is globally asymptotically stable. □

Once that the global asymptotical stability of the exclusion equilibrium has been established under certain assumptions in the previous theorem in the case $1 < ae^r \leq e$, we deal with the case $e < ae^r < e^2$. The reasoning is analogous to the preceding case: firstly, Theorem 25 states that the equilibrium is an attractor and, finally, Theorem 26 implies the global asymptotical stability of E_0^*.

Theorem 25. *Let* (H_n, P_n) *denote the solution of the System* (6.4) *with the initial condition* $(H_0, P_0) \in \mathbb{R}_+^2$. *If* $e < ae^r < e^2$, *and if* $-c\dfrac{ae^{r-1}}{r} f'_+(0) < 1$, *then* $\lim_{n\to\infty} (H_n, P_n) = E_0^*$.

Proof. As a first stage, we show that the parasitoid's population converges to zero as n tends to infinity. To achieve that, observe that the condition $e < ae^r < e^2$ is equivalent to $1 - \ln(a) < r < 2 - \ln(a)$ and, additionally, it implies $a < e^2$. Furthermore, Proposition 44 guarantees the boundedness character of the populations, so in particular $H_n \leq \frac{ae^{r-1}}{r}$ for $n > 0$. The mentioned facts, jointly with Lemma 64, allow us to deduce from the second equation of System (6.4) that

$$P_{n+1} \leq -c\frac{ae^{r-1}}{r} f'_+(0) P_n \text{ for } n \geq 1. \tag{6.14}$$

Consequently, since we are assuming that $-c\dfrac{ae^{r-1}}{r} f'_+(0) < 1$, from (6.14) we get that $P_n \to 0$ as $n \to \infty$.

Now, we prove that the host population converges to H_0^* as n tends to infinity. In this direction, notice that the convergence of the parasitoid population to zero implies that for every $\varepsilon > 0$ there exists n_0, such that for every $n > n_0$, we have $P_n < \varepsilon$. From here and the decreasing character of f, we obtain

$$aH_n e^{r(1-H_n)} f(b\varepsilon) \leq H_{n+1} \leq aH_n e^{r(1-H_n)} \text{ for } n > n_0. \tag{6.15}$$

We apply the inequalities from (6.15) twice in order to obtain

$$H_{n+2} \leq a^2 H_n e^{r\left(1-af(b\varepsilon)H_n e^{r(1-H_n)}\right)+r(1-H_n)} \text{ for } n > n_0 \tag{6.16}$$

and

$$H_{n+2} \geq a^2 (f(b\varepsilon))^2 H_n e^{r\left(1-aH_n e^{r(1-H_n)}\right)+r(1-H_n)} \text{ for } n > n_0. \tag{6.17}$$

Taking the above inequalities into account, we set the maps

$$h_1(z) = a^2(f(b\varepsilon))^2 z e^{r\left(1-aze^{r(1-z)}\right)+r(1-z)} \text{ and } h_2(z) = a^2 z e^{r\left(1-af(b\varepsilon)ze^{r(1-z)}\right)+r(1-z)},$$

which are compositions of Ricker maps, $h_{\alpha,\rho}(x) = \alpha x e^{\rho(1-x)}$, for suitable values of the parameters. Precisely,

$$h_1(z) = h_{ae^r(f(b\varepsilon))^2,r} \circ h_{ae^r,r} \text{ and } h_2(z) = h_{\frac{ae^r}{f(b\varepsilon)},r} \circ h_{ae^r f(b\varepsilon),r}. \tag{6.18}$$

The rest of the proof is based on analyzing the existence of fixed points for the maps $h_1(z)$ and $h_2(z)$ and on showing the convergence of the host population. In this sense, we develop the sequel in three different parts.

Part (a) - Fixed points of $h_2(z)$: We compute the derivative of $h_2(z)$,

$$h_2'(z) = a^2(rz-1)e^{-r\left(aze^{r-rz}f(b\varepsilon)+z-2\right)}\left(arze^{r-rz}f(b\varepsilon)-1\right).$$

One can easily check that $h_2'(z) = 0$ whenever $z = \frac{1}{r}$ or $arze^{r(1-z)}f(b\varepsilon) - 1 = 0$. Considering this, we set $g_2(z) = arze^{r(1-z)}f(b\varepsilon)$ and the following holds:

$$g_2(0) = 0, \quad \lim_{z\to+\infty} g_2(z) = 0, \quad g_2'(z) = -are^{r(1-z)}(rz-1)f(b\varepsilon),$$

$$g_2'(1/r) = 0, \quad \text{and} \quad g_2''(1/r) = -ae^{r-1}r^2 f(b\varepsilon) < 0.$$

Such analysis guarantees that $g_2(z) = 1$ has two positive solutions if and only if $g_2\left(\frac{1}{r}\right) = ae^{r-1}f(b\varepsilon) > 1$. Observe that since $e < ae^r < e^2$ holds, we can choose sufficiently small $\varepsilon > 0$ such that $e < af(b\varepsilon)e^r < e^2$, which guarantees that $g_2\left(\frac{1}{r}\right) > 1$. In conclusion, we have achieved the existence of two positive solutions of the equation $g_2(z) = 1$, which we denote by $z_1^{(2)}(\varepsilon)$ and $z_2^{(2)}(\varepsilon)$, verifying $z_1^{(2)}(\varepsilon) < 1/r < z_2^{(2)}(\varepsilon)$. As a consequence, we have three candidates to extreme points for $h_2(z)$, namely, $z = \frac{1}{r}$, $z_1^{(2)}(\varepsilon)$ and $z_2^{(2)}(\varepsilon)$. See Figure 6.1.

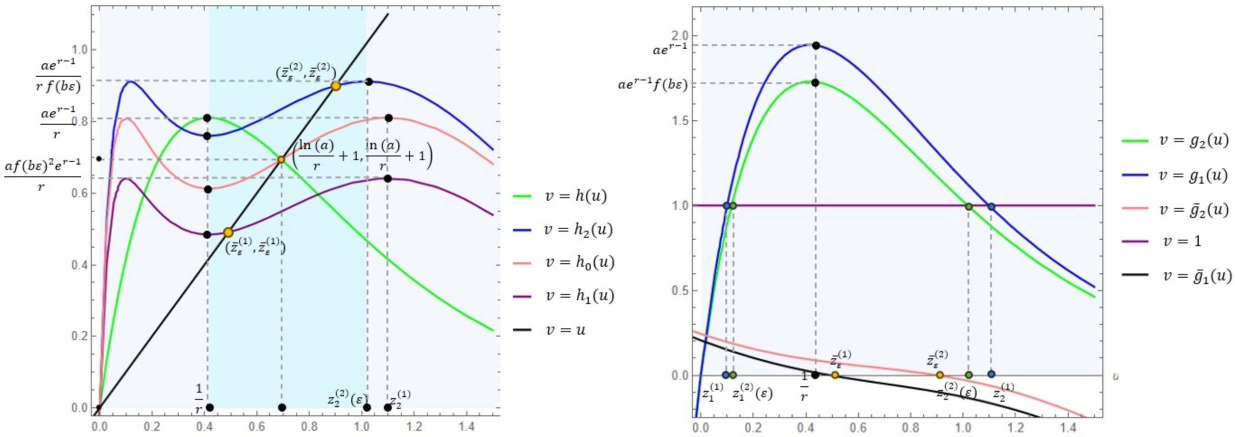


Figure 6.1: Graphs of the functions: $v = h(u) = aue^{r(1-u)}$, $v = h_1(u)$, $v = h_2(u)$, $v = h_0(u)$ (left); $g_1(u) = arue^{r-ru}$, $g_2(u) = arue^{r-ru}f(b\varepsilon)$, $\bar{g}_2(u) = \frac{2(\ln(a)+r)}{are^{r-ru}f(b\varepsilon)+r} - u$, $\bar{g}_1(u) = \frac{2(\ln(af(b\varepsilon))+r)}{are^{r-ru}+r} - u$ with $e < ae^r < e^2$ (right).

As a next step, we study if the candidates to extreme points really are and, in such case, if they are maxima or minima. Notice that $h_2(0) = 0$ and $\lim_{n\to+\infty} h_2(z) = 0$. Also, for $i = 1, 2$, from $g_2\big(z_i^{(2)}(\varepsilon)\big) = 1$ we have

$$e^{-rz_i^{(2)}(\varepsilon)} = \frac{e^{-r}}{arz_i^{(2)}(\varepsilon)f(b\varepsilon)}.$$

This yields to

$$h_2''\big(z_i^{(2)}(\varepsilon)\big) = -a^4r^2z_i^{(2)}(\varepsilon)e^{-3rz_i^{(2)}+4r-1}\big(rz_i^{(2)} - 1\big)^2 f(b\varepsilon)^2 < 0,$$

and $h_2\big(z_i^{(2)}(\varepsilon)\big) = \dfrac{ae^{r-1}}{rf(b\varepsilon)}$ for $i = 1, 2$. Moreover,

$$h_2''(1/r) = a^2re^{-ae^{r-1}f(b\varepsilon)+2r-2}\,(ae^rf(b\varepsilon) - e) > 0,$$

and

$$h_2(1/r) = \frac{a^2e^{-af(b\varepsilon)e^{r-1}+2r-1}}{r}.$$

From the above considerations, we deduce that $z = \frac{1}{r}$ is a minimum, while $z_1^{(2)}(\varepsilon)$ and $z_2^{(2)}(\varepsilon)$ are maxima for $h_2(z)$.

Now, observe that, for $z \neq 0$, being a fixed points of the map $h_2(z)$ is equivalent to

$$\bar{g}_2(z) := \frac{2(\ln(a) + r)}{are^{r-rz}f(b\varepsilon) + r} - z = 0. \tag{6.19}$$

Take the map $\bar{g}_2(z)$ and compute

$$\bar{g}_2'(z) = -\frac{a^2e^{2r}f(b\varepsilon)^2 - 2ae^{rz+r}(\ln(a) + r - 1)f(b\varepsilon) + e^{2rz}}{(ae^rf(b\varepsilon) + e^{rz})^2}.$$

From $\bar{g}_2'(z) = 0$, one obtains

$$(e^{rz})^2 - 2af(b\varepsilon)e^r e^{rz}(\ln(a) + r - 1) + a^2 f(b\varepsilon)^2 e^{2r} = 0.$$

Set $p_2(t) = t^2 - 2af(b\varepsilon)e^r t(\ln(a) + r - 1) + a^2 f(b\varepsilon)^2 e^{2r}$. Then the discriminant of the polynomial is

$$\text{Dis}(p_2(t)) = 4a^2 f(b\varepsilon)^2 e^{2r}(\ln(a) + r - 2)(\ln(a) + r) < 0,$$

and we have $p_2(e^{rz}) > 0$, which implies $\bar{g}_2'(z) < 0$ for all $z > 0$. From $\bar{g}_2(0) = \frac{2(\ln(a)+r)}{ae^r rf(b\varepsilon)+r} > 0$ and $\bar{g}_2(+\infty) = -\infty$, by the continuity of the map we get that there is exactly one positive solution $\bar{z}_\varepsilon^{(2)} > 0$ of the equation $\bar{g}_2(z) = 0$ which is a positive fixed point of the map h_2, i.e. $h_2\big(\bar{z}_\varepsilon^{(2)}\big) = \bar{z}_\varepsilon^{(2)}$. Furthermore, $h_2(z) > z$ for $0 < z < \bar{z}_\varepsilon^{(2)}$ and $h_2(z) < z$ for $z > \bar{z}_\varepsilon^{(2)}$.

Part (b) - Fixed points of $h_1(z)$: Here, we determine the existence of fixed points for the map $h_1(z)$ proceeding in an analogous way to Part(a). Firstly, we obtain the extreme points of the map. To do so, let us compute

$$h_1'(z) = a^2 f(b\varepsilon)^2(rz - 1)e^{-r\left(aze^{r-rz}+z-2\right)}\left(arze^{r-rz} - 1\right).$$

By equating $h_1'(z) = 0$, we get $z = \frac{1}{r}$ or $arze^{r(1-z)} = 1$. Take $g_1(z) = arze^{r-rz}$, which verifies

$$g_1(0) = 0, \quad \lim_{z\to+\infty} g_1(z) = 0, \quad g_1'(z) = -are^{r-rz}(rz - 1),$$
$$g_1'(1/r) = 0, \quad \text{and} \quad g_1''(1/r) = -ae^{r-1}r^2 < 0.$$

As a consequence, $z = \frac{1}{r}$ is a maximum for $g_1(z)$ and $g_1(z) = 1$ will have two solutions if and only if $g_1\left(\frac{1}{r}\right) = ae^{r-1} > 1$. However, recall that we are analyzing the case $e < ae^r < e^2$, so the condition $ae^{r-1} > 1$ always holds and we can guarantee the existence of two positive solutions $z_1^{(1)}$ and $z_2^{(1)}$ $\left(z_1^{(1)} < 1/r < z_2^{(1)}\right)$ of the equation $g_1(z) = 1$. This fact ensures that $h_1(z)$ has three candidates to extreme points, namely, $z = \frac{1}{r}$, $z_1^{(1)}$ and $z_2^{(1)}$. See Figure 6.1.

In the sequel, we determine if the candidates to extreme points are maxima or minima. In this direction, from $h_1'\big(z_i^{(1)}\big) = 0$ for $i = 1, 2$, we get

$$e^{-rz_i^{(1)}} = \frac{e^{-r}}{arz_i^{(1)}}.$$

Such equality implies $h_1''\big(z_i^{(1)}\big) = h_2''\big(z_i^{(2)}(\varepsilon)\big) < 0$, and $h_1\big(z_i^{(1)}\big) = \dfrac{af(b\varepsilon)^2 e^{r-1}}{r}$ for $i = 1, 2$. Moreover,

$$h_1''(1/r) = a^2 rf(b\varepsilon)^2 e^{-ae^{r-1}+2r-2}\left(ae^r - e\right) > 0,$$

and

$$h_1(1/r) = \frac{a^2 f(b\varepsilon)^2 e^{-ae^{r-1}+2r-1}}{r}.$$

In conclusion, $z_1^{(1)}$ and $z_2^{(1)}$ are maxima of $h_1(z)$ and $z = \frac{1}{r}$ is a minimum. To this regard, for $z \neq 0$, being a fixed point of the map $h_1(z)$ is equivalent to

$$\bar{g}_1(z) := \frac{2(\ln(af(b\varepsilon)) + r)}{are^{r-rz} + r} - z = 0. \tag{6.20}$$

Take $\bar{g}_1(z)$ and compute its derivative,

$$\bar{g}_1'(z) = -\frac{a^2e^{2r} - 2ae^{ry+r}(\ln(af(b\varepsilon)) + r - 1) + e^{2ry}}{(ae^r + e^{ry})^2}.$$

Since the discriminant of the polynomial

$$p_1(t) = a^2e^{2r} - 2ae^r t(\ln(af(b\varepsilon)) + r - 1) + t^2$$

is

$$\mathrm{Dis}(p_1(t)) = 4a^2e^{2r}(\ln(af(b\varepsilon)) + r - 2)(\ln(af(b\varepsilon)) + r) < 0,$$

we obtain $p_1(e^{rz}) > 0$, which yields to $\bar{g}_1'(z) < 0$, for all $z > 0$.

Also,

$$\bar{g}_1(0) = \frac{2(\ln(af(b\varepsilon)) + r)}{ae^r r + r} > 0 \quad \text{and} \quad \lim_{z\to+\infty} \bar{g}_1(z) = -\infty,$$

and applying the continuity of $\bar{g}_1(z)$, we obtain that there is exactly one positive solution $\bar{z}_\varepsilon^{(1)} > 0$ of the equation $\bar{g}_1(z) = 0$. This means that $\bar{z}_\varepsilon^{(1)}$ is a fixed point of $h_1(z)$. In addition, $h_1(z) > z$ for $0 < z < \bar{z}_\varepsilon^{(1)}$ and $h_1(z) < z$ for $z > \bar{z}_\varepsilon^{(1)}$, which ends Part (b).

Part (c) - Convergence of the host population: This final part delves into the convergence of the host population to $\frac{\ln(a)}{r} + 1$. Firstly, it is easy to see that

$$\lim_{\varepsilon\to 0} h_1(z) = \lim_{\varepsilon\to 0} h_2(z) = h_0(z) = a^2 z e^{r\left(1 - aze^{r(1-z)}\right) + r(1-z)} = \left(h_{ae^r,r} \circ h_{ae^r,r}\right)(z). \tag{6.21}$$

Additionally,

$$\lim_{\varepsilon\to 0} \bar{z}_\varepsilon^{(1)} = \lim_{\varepsilon\to 0} \bar{z}_\varepsilon^{(2)} = \bar{z} = \frac{\ln(a)}{r} + 1,$$

so $h(\bar{z}) = \bar{z}$, since the periodic points of $h_{ae^r,r}$ are reduced to the fixed points $z = 0$ and $z = \frac{\ln(a)}{r} + 1$ due to $e < ae^r < e^2$, and the case $z = 0$ is discarded from Equations (6.20) and (6.19).

As a next step, we show that $\bar{z}_\varepsilon^{(1)} = \bar{z}_\varepsilon^{(2)}$. To achieve this, observe that the assumption $e < ae^r < e^2$ implies that $\dfrac{1}{r} < \dfrac{\ln(a)}{r} + 1 < \dfrac{2}{r}$. Moreover,

$$z_1^{(1)} < z_1^{(2)}(\varepsilon) < 1/r < z_2^{(2)}(\varepsilon) < z_2^{(1)},$$

because $g_1(z) < g_2(z)$ for all $z > 0$.

From (6.21) we get $\lim_{\varepsilon\to 0} z_2^{(2)}(\varepsilon) = z_2^{(1)}$, and $\lim_{\varepsilon\to 0} h_2\big(z_2^{(2)}(\varepsilon)\big) = \lim_{\varepsilon\to 0} h_1\big(z_2^{(1)}\big) = h(z_2^{(1)}) = ae^{r-1}/r$. Furthermore, $h'\big(\ln(a)/r + 1\big) > 0$ yields to $1/r < \ln(a)/r + 1 < z_2^{(1)}$.

Now, it is straightforward to see that h_2 is non-decreasing in $I_2 = \left[1/r, z_2^{(2)}(\varepsilon)\right]$ and

$$h_2\left(I_2\right) = \left[h_2(1/r), h_2\big(z_2^{(2)}(\varepsilon)\big)\right].$$

Since $1/r < \lim_{\varepsilon\to 0} \bar{z}_\varepsilon^{(2)} = \ln(a)/r + 1 < \lim_{\varepsilon\to 0} z_2^{(2)}(\varepsilon) = z_2^{(1)}$, for all sufficiently small $\varepsilon > 0$ we have $1/r < \bar{z}_\varepsilon^{(2)} < z_2^{(2)}(\varepsilon)$.

Also, from $1/r < h_2(1/r) < h_2(\bar{z}_\varepsilon^{(2)}) = \bar{z}_\varepsilon^{(2)} < h_2(z_2^{(2)}(\varepsilon))$, we achieve $h_2(I_2) \subset I_2$. Take $l_0 = l_0^{(1)} = 1/r$ and $l_2 = l_0^{(2)} = z_2^{(2)}(\varepsilon)$, $l_{n+1}^{(1)} = h_2\big(l_n^{(1)}\big)$ and $l_{n+1}^{(2)} = h_2\big(l_n^{(2)}\big)$. Moreover, set $z_0 \in I_2$ and $z_{n+1} = h_2(z_n)$. From the monotonicity of h_2, it can be deduced that $l_1 \leq l_i^{(1)} \leq l_{i+1}^{(1)} \leq z_{i+1} \leq l_{i+1}^{(2)} \leq l_i^{(2)} < l_2$. If we let n tend to infinity, we have $l_1 \leq \lim_{n\to\infty} l_n^{(1)} = \lim_{n\to\infty} l_n^{(2)} = \bar{z}_2(\varepsilon) \leq l_2$, which implies

$$\lim_{n\to\infty} z_n = \bar{z}_\varepsilon^{(2)}.$$

Analogously, we can see that, for sufficiently small $\varepsilon > 0$, we have that h_1 is a non-decreasing function in $I_1 = \left[1/r, z_2^{(1)}\right]$ and $h_2\left(I_1\right) \subset I_1$, with $I_2 \subseteq I_1$. In addition, $\lim_{n\to\infty} z_n = \bar{z}_\varepsilon^{(1)}$ for all $z_0 \in I_1$ and $z_{n+1} = h_1(z_n)$. Therefore,

$$\bar{z}_\varepsilon^{(1)} = \lim_{n\to\infty} z_n = \bar{z}_\varepsilon^{(2)}.$$

Next, assume $0 < H_0 < \dfrac{1}{r}$. Set $h_3(z) = aze^{r(1-z)}f(b\varepsilon)$, $z_0 = H_0$, and $z_{n+1} = h_3(z_n)$. The map $h_3(z)$ is non-decreasing in $\left[0, \frac{1}{r}\right]$, which allows us to deduce that $z_{n+1} < H_{n+1}$ if $z_n < H_n < \frac{1}{r}$. Assume that $0 < H_n < \frac{1}{r}$ for all $n > 0$. We can choose $\varepsilon > 0$ such that $e < af(b\varepsilon)e^r < e^2$. It is well known that $\lim_{n\to\infty} z_n = \frac{\ln\left(af(b\varepsilon)\right)}{r} + 1$, which implies $\frac{1}{r} < \frac{\ln\left(af(b\varepsilon)\right)}{r} + 1 = \limsup_{n\to\infty} z_n \leq \limsup_{n\to\infty} H_n \leq \dfrac{1}{r}$, a contradiction. Thus, there exists $n_1 > n_0$ such that $\frac{1}{r} < H_{n_1} < \max_{z\in[0,1]} aze^{r(1-z)} = \dfrac{ae^{r-1}}{r} < \dfrac{ae^{r-1}}{rf(b\varepsilon)} = h_2(z_2^{(2)}(\varepsilon)) < z_2^{(2)}(\varepsilon)$, for sufficiently small $\varepsilon > 0$. If $H_0 > z_2^{(2)}(\varepsilon)$, then $H_1 < \max_{z\in[0,1]} aze^{r(1-z)} = \frac{ae^{r-1}}{r} < z_2^{(2)}(\varepsilon)$ for sufficiently small $\varepsilon > 0$. From the previous discussion, there exists $n_1 > n_0$ such that $H_{n_1} \in I_2$.

Finally, from (6.16) and (6.17) we obtain $h_1(H_n) \leq H_{n+2} < h_2(H_n)$ for all $n > n_0$. Let $z_0^{(1)} = z_0^{(2)} = H_{n_1}$. From the monotonicity of the map h_1 in I_1 and h_2 in I_2 we get

$$\frac{1}{r} < z_k^{(1)} \leq H_{n_1+2k} \leq z_k^{(2)} < z_2^{(1)}(\varepsilon),$$

for all $k > 0$. This implies

$$\bar{z}_1(\varepsilon) = \lim_{k\to\infty} z_k^{(1)} \leq \liminf_{k\to\infty} H_{n_1+2k} \leq \limsup_{k\to\infty} H_{n_1+2k} \leq \lim_{k\to\infty} z_k^{(2)} = \bar{z}_2(\varepsilon).$$

Letting $\varepsilon \to 0$ we obtain

$$\lim_{k\to\infty} H_{n_1+2k} = \frac{\ln(a)}{r} + 1.$$

Since $H_{n_1+2k+1} = F(H_{n_1+2k}, P_{n_1+2k})$, we achieve $\lim_{k\to\infty} H_{n_1+2k+1} = F\left(\frac{\ln(a)}{r}+1, 0\right) = \frac{\ln(a)}{r}+1$, which implies $\lim\limits_{k\to\infty} H_k = \dfrac{\ln(a)}{r}+1$. □

Theorem 26. *Assume that* $e < ae^r < e^2$, *and* $-c\dfrac{ae^{r-1}}{r}f'_+(0) < 1$. *Then,* E_0^* *is globally asymptotically stable relative to* $(0,\infty)\times[0,\infty)$.

Proof. The assumption $e < ae^r < e^2$ is equivalent to the inequality $1 < \ln a + r < 2$. Moreover, if we consider the Ricker map $h(x) = axe^{r(1-x)}$, we have

$$H_0^* = h(H_0^*) < h\left(\frac{1}{r}\right) = \frac{ae^{r-1}}{r}.$$

So the assumption $-c\dfrac{ae^{r-1}}{r}f'_+(0) < 1$ implies $-c\left(1+\frac{\ln a}{r}\right)f'_+(0) < 1$. Hence, by Theorem 23 we deduce that E_0^* is locally asymptotically stable. On the other hand, we have that the exclusion equilibrium is a global attractor relative to $(0,\infty)\times[0,\infty)$ by Theorem 25. In conclusion, E_0^* is globally asymptotically stable in such region. □

6.3.3 Local stability of the coexistence equilibrium

In this subsection, we consider the equilibrium point $E^* = (H^*, P^*) \in \mathbb{R}_+^2$ and we study its local stability. We achieve this by a direct application of Theorem 20. This strategy requires the computation of the trace and determinant of the Jacobian matrix of System (6.4) evaluated at E^*,

$$J := J_T(E^*) = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} = \begin{pmatrix} 1 - rH^* & \frac{bH^*f'(bP^*)}{f(bP^*)} \\ c\big(1 - f(P^*)\big) & -cH^*f'(P^*) \end{pmatrix}. \tag{6.22}$$

It should be mentioned that we have employed the equations satisfied by the interior equilibrium, namely, Equation (6.6) in order to simplify the expressions in (6.22). Moreover, it is direct to determine

$$\begin{aligned} \mathrm{Tr} &:= \mathrm{Tr}(J) = 1 - rH^* - cH^*f'(P^*), \\ \mathrm{Det} &:= \mathrm{Det}(J) = -cH^*f'(P^*)(1 - rH^*) - \frac{c\big(1 - f(P^*)\big)bH^*f'(bP^*)}{f(bP^*)}. \end{aligned}$$

We begin by establishing some preliminaries results concerning the expressions involved in the Jacobian (6.22), its trace and determinant.

Lemma 68. *Consider the matrix* (6.22). *Then, the following holds:*

(a) $A_1 < 1$, $B_1 < 0$, $C_1 > 0$ *and* $0 < D_1 < 1$.

(b) $\mathrm{Det} + 1 - \mathrm{Tr} > 0$.

(c) *If* $\mathrm{Det} < -1$, *then* $|1 + \mathrm{Det}| < |\mathrm{Tr}|$.

(d) *If* $|\mathrm{Tr}| > |1+\mathrm{Det}|$, *then* $(\mathrm{Tr})^2 - 4\mathrm{Det} > 0$. *In particular, this happens when* $\mathrm{Det} < -1$.

(e) $\mathrm{Tr} < 2$.

Proof. A direct inspection of (6.22) gives us $B_1 < 0$, $C_1 > 0$ and $D_1 > 0$. In addition, $A_1 = 1 - rH^* < 1$, since $r, H^* > 0$. Also, take

$$D_1 - 1 = -cH^* f'(P^*) - 1 = -\frac{P^*}{1 - f(P^*)} \cdot f'(P^*) - 1 = \frac{P^* f'(P^*) - f(P^*) + 1}{f(P^*) - 1},$$

where we have used that $P^* = cH^*\big(1 - f(P^*)\big)$. Consider the map $G(P) = P f'(P) - f(P) + 1$, that verifies $G'(P) = P f''(P) > 0$ for $P > 0$, and $\lim_{P\to 0^+} G(P) = 0$ as a consequence of Condition ($\star$). So $G(P) > 0$ for every $P > 0$ and we conclude $D_1 < 1$, which ends the proof of Part (a).

From here, we are able to deduce Part (b):

$$\mathrm{Det} + 1 - \mathrm{Tr} = A_1 D_1 - B_1 C_1 + 1 - A_1 - D_1 = (1 - A_1)(1 - D_1) - B_1 C_1 > 0.$$

Now, assume that $|1 + \mathrm{Det}| \geq |\mathrm{Tr}|$. If $\mathrm{Det} < -1$, then $1 + \mathrm{Det} < 0$ and we can reduce the previous inequality to $|\mathrm{Tr}| \leq -1 - \mathrm{Det}$, which yields to $1 + \mathrm{Det} \leq \mathrm{Tr} \leq -1 - \mathrm{Det}$. Nevertheless, this contradicts Part (b) and the statement from Part (c) follows.

As a next step, observe that if $|\mathrm{Tr}| > |1+\mathrm{Det}|$, then $\mathrm{Tr}^2 > (1+\mathrm{Det})^2$. So, $\mathrm{Tr}^2 - 4\mathrm{Det} > (1+\mathrm{Det})^2 - 4\mathrm{Det} = (1-\mathrm{Det})^2 \geq 0$ and we obtain Part (d). Finally, Part (e) follows directly from the fact that $\mathrm{Tr} = A_1 + D_1 < 1 + 1 = 2$. □

Now, we use the above relations to establish the main results concerning the local stability of the coexistence equilibrium.

Theorem J. *Assume that the coexistence equilibrium* $E^* = (H^*, P^*)$ *exists. Then,*

(a) E^* *is locally asymptotically stable if and only if*

$$rH^* + cH^* f'(P^*) - 2 < -cH^* f'(P^*)(1 - rH^*) - \frac{c\big(1 - f(P^*)\big) bH^* f'(bP^*)}{f(bP^*)} < 1.$$

(b) E^* *is a repeller if and only if*

$$rH^* + cH^* f'(P^*) - 2 < -cH^* f'(P^*)(1 - rH^*) - \frac{c\big(1 - f(P^*)\big) bH^* f'(bP^*)}{f(bP^*)}$$

and

$$1 < -cH^* f'(P^*)(1 - rH^*) - \frac{c\big(1 - f(P^*)\big) bH^* f'(bP^*)}{f(bP^*)}.$$

(c) E^* *is a saddle if and only if*

$$-cH^* f'(P^*)(1 - rH^*) - \frac{c\big(1 - f(P^*)\big) bH^* f'(bP^*)}{f(bP^*)} < rH^* + cH^* f'(P^*) - 2.$$

(d) E^* *is non-hyperbolic if and only if*

$$2 - cH^* f'(P^*)(2 - rH^*) - rH^* - \frac{c\big(1 - f(P^*)\big)bH^* f'(bP^*)}{f(bP^*)} = 0,$$

or

$$rH^* + cH^* f'(P^*) \leq 3 \quad \textit{and} \quad - cH^* f'(P^*)(1 - rH^*) - \frac{c\big(1 - f(P^*)\big)bH^* f'(bP^*)}{f(bP^*)} = 1.$$

Proof. **Part (a):** By Theorem 20, E^* will be locally asymptotically stable if and only if $|\text{Tr}| < 1 + \text{Det} < 2$. Firstly, the second inequality, $1 + \text{Det} < 2$, reads as

$$-cH^* f'(P^*)(1 - rH^*) - \frac{c\big(1 - f(P^*)\big)bH^* f'(bP^*)}{f(bP^*)} < 1.$$

Additionally, the remaining inequality $|\text{Tr}| < 1 + \text{Det}$ implies

$$-1 - \text{Det} < \text{Tr} < 1 + \text{Det}.$$

Notice that $\text{Tr} < 1 + \text{Det}$ follows by Lemma 68-(b) and $-1 - \text{Tr} < \text{Det}$ gives us

$$rH^* + cH^* f'(P^*) - 2 < -cH^* f'(P^*)(1 - rH^*) - \frac{c\big(1 - f(P^*)\big)bH^* f'(bP^*)}{f(bP^*)}.$$

Part (b): Theorem 20 establishes that E^* will be a repeller if and only if $|\text{Tr}| < |1+\text{Det}|$ and $|\text{Det}| > 1$. The second condition always holds by Lemma 68-(c). Moreover, such result allows us to reduce the other condition to $1 + \text{Det} > |\text{Tr}|$, which is equivalent to $-1 - \text{Det} < \text{Tr} < 1 + \text{Det}$, where the second inequality is ensured by 68-(b). In conclusion, E^* will be a repeller if and only if $\text{Det} > 1$ and $0 < \text{Tr} + 1 + \text{Det}$, from where we get

$$rH^* + cH^* f'(P^*) - 2 < -cH^* f'(P^*)(1 - rH^*) - \frac{c\big(1 - f(P^*)\big)bH^* f'(bP^*)}{f(bP^*)},$$

and

$$1 < -cH^* f'(P^*)(1 - rH^*) - \frac{c\big(1 - f(P^*)\big)bH^* f'(bP^*)}{f(bP^*)}.$$

Part (c): Again, by Theorem 20, E^* will be a saddle if and only if $|\text{Tr}| > |1 + \text{Det}|$ and $(\text{Tr})^2 - 4\text{Det} > 0$. The second inequality holds from Lemma 68-(d), so we only restrict to study the remaining inequality by distinguishing two different scenarios depending on the trace of the Jacobian.

(i) If $\text{Tr} \geq 0$, Lemma 68-(b) implies $\text{Det} + 1 > \text{Tr} \geq 0$ and the inequality $|\text{Tr}| > |1 + \text{Det}|$ will not hold.

(ii) If $\text{Tr} < 0$, we need to differ two cases:

- If $\text{Det} + 1 \geq 0 > \text{Tr}$, the inequality $|\text{Tr}| > |1 + \text{Det}|$ will read as $\text{Det} + 1 + \text{Tr} < 0$.
- If $0 \geq \text{Det} + 1 > \text{Tr}$, $|\text{Tr}| > |1 + \text{Det}|$ will read as $\text{Det} + 1 - \text{Tr} > 0$, which is true by Lemma 68-(a).

Definitely, E^* will be a saddle if and only if Det + 1 + Tr < 0, and this is equivalent to

$$-cH^*f'(P^*)(1-rH^*) - \frac{c\big(1-f(P^*)\big)bH^*f'(bP^*)}{f(bP^*)} < rH^* + cH^*f'(P^*) - 2.$$

Part (d): E^* will be non-hyperbolic if and only if $|\text{Tr}| = |1+\text{Det}|$, or Det = 1 and $|\text{Tr}| \leq 2$, see Theorem 20. Let us examine both cases:

(i) The equality $|\text{Tr}| = |1+\text{Det}|$ is equivalent to $\text{Tr} = 1+\text{Det}$ or $-\text{Tr} = 1+\text{Det}$. Notice that, due to Lemma 68-(b), the first scenario cannot hold, so we will have the non-hyperbolic character of the equilibrium if and only if $-\text{Tr} = 1+\text{Det}$, which means

$$2 - cH^*f'(P^*)(2-rH^*) - rH^* - \frac{c\big(1-f(P^*)\big)bH^*f'(bP^*)}{f(bP^*)} = 0.$$

(ii) Assume Det = 1 and $|\text{Tr}| \leq 2$. From Lemma 68-(e), Tr < 2, so the conditions that must be fulfill are given by Det = 1 and $-2 \leq \text{Tr}$, which are equivalent to

$$-cH^*f'(P^*)(1-rH^*) - \frac{c\big(1-f(P^*)\big)bH^*f'(bP^*)}{f(bP^*)} = 1,$$

and

$$rH^* + cH^*f'(P^*) \leq 3.$$

□

6.4 Bifurcations for the exclusion equilibrium

This section is devoted to the occurrence of bifurcations in the exclusion equilibrium point $E_0^* = \left(1 + \frac{\ln(a)}{r}, 0\right)$. Concretely, we show the appearance of a period-doubling bifurcation, see Subsection 6.4.1, and a transcritical bifurcation, see Subsection 6.4.2. To accomplish that, in the sequel we assume that $f'_+(0)$, $f''_+(0)$ and $f^{(3)}_+(0)$ exist.

Recall that the Jacobian matrix evaluated in E_0^* is given by (6.12). Moreover, the corresponding eigenvalues of $J_T(E_0^*)$ are $\lambda_1 = 1 - r - \ln(a) < 1$ and $\lambda_2 = -c\left(1 + \frac{\ln(a)}{r}\right) f'_+(0) = -cH_0^*f'_+(0) > 0$.

As a first step, we make the change of variables $x_n = H_n - H_0^*$, $y_n = P_n$ in order to shift the equilibrium to the origin. This transformation changes System (6.4) into

$$\begin{cases} x_{n+1} &= a\big(H_0^* + x_n\big)e^{r(1-H_0^*-x_n)}f(by_n) - H_0^*, \\ y_{n+1} &= c\big(H_0^* + x_n\big)\big(1 - f(y_n)\big). \end{cases} \tag{6.23}$$

Now, we proceed to study in detail two different scenarios concerning bifurcations in the exclusion equilibrium.

6.4.1 Period-doubling bifurcation

A period-doubling bifurcation corresponds to the creation of a period two cycle near the origin. This can occur when one of the eigenvalues of the Jacobian matrix is equal to -1. Bearing this in mind, we set the parameter r, the maximum growth rate of the host population, as the bifurcation parameter and we take $r = r_0$ such that $r_0 = \frac{2}{H_0^*(r_0)} = 2 - \ln(a) > 0$. Notice that this implies that $\lambda_1(r_0) = -1$. Also, in the sequel we assume that the second eigenvalue $\lambda_2(r_0)$ is different from one.

To prove the occurrence of this bifurcation, we apply center manifold theory in order to verify the conditions from [103, Section 20.1E]. Firstly, we consider the associated map, T_r, with (6.23),

$$T_r \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a\big(H_0^*(r)+x\big)e^{r(1-H_0^*(r)-x)}f(by) - H_0^*(r) \\ c\big(H_0^*(r)+x\big)\big(1-f(y)\big) \end{pmatrix} =: \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix}. \tag{6.24}$$

In order to apply the appropriate criteria from the literature, we compute the Taylor series expansions of $f_1(x,y)$ and $f_2(x,y)$ about the origin until order 4:

$$\begin{aligned} f_1(x,y) &= \big(1-rH_0^*(r)\big)x + bH_0^*(r)f_+'(0)y + \big(-2r + r^2H_0^*(r)\big)\frac{x^2}{2} + bf_+'(0)\big(1-rH_0^*(r)\big)xy \\ &\quad + b^2H_0^*(r)f_+''(0)\frac{y^2}{2} + \frac{1}{2}br\big(-2+rH_0^*(r)\big)f_+'(0)x^2y - \frac{1}{2}b^2\big(-1+rH_0^*(r)\big)f_+''(0)xy^2 \\ &\quad + \frac{1}{6}b^3H_0^*(r)f_+^{(3)}(0)y^3 + \frac{1}{6}\left(-r^2\left(-3+rH_0^*(r)\right)\right)x^3 + O\left((|x|+|y|)^4\right), \\ f_2(x,y) &= -cH_0^*(r)f_+'(0)y - cf_+'(0)xy - cH_0^*(r)f_+''(0)\frac{y^2}{2} - \frac{1}{2}cf_+''(0)xy^2 \\ &\quad - \frac{1}{6}cH_0^*(r)f_+^{(3)}(0)y^3 + O\left((|x|+|y|)^4\right). \end{aligned}$$

As a next step, we compute the Jacobian matrix of T_r at $(0,0)$,

$$J_{T_r}(0,0) = \begin{pmatrix} 1-rH_0^*(r) & bH_0^*(r)f_+'(0) \\ 0 & -cH_0^*(r)f_+'(0) \end{pmatrix},$$

and for the particular case $r = r_0$,

$$J_{T_{r_0}}(0,0) = \begin{pmatrix} -1 & bH_0^*(r_0)f_+'(0) \\ 0 & -cH_0^*(r_0)f_+'(0) \end{pmatrix}.$$

Observe that the eigenvalues of $J_{T_{r_0}}(0,0)$ are $\lambda_1 = -1$ and $\lambda_2 = -cH_0^*(r_0)f_+'(0)$. So, we can obtain their corresponding eigenvectors, namely,

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} \frac{bH_0^*(r_0)f_+'(0)}{1-cH_0^*(r_0)f_+'(0)} \\ 1 \end{pmatrix}.$$

It should be highlighted that due to the assumption $\lambda_2(r_0) \neq 1$, we can ensure that the denominator of v_2, $1 - cH_0^*(r_0)f_+'(0)$, is different from zero.

Let $r = r_0 + \mu$. Then, we have

$$J_{T_{r_0+\mu}}(0,0) = J_{T_{r_0}}(0,0) + \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \mu + \begin{pmatrix} J_{13} & J_{14} \\ J_{23} & J_{24} \end{pmatrix} \mu^2 + O\big(|\mu|^3\big),$$

where $\begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} = \left.\frac{\partial \left(J_{T_{r_0+\mu}}(0,0)\right)}{\partial \mu}\right|_{\mu=0}$ and $\begin{pmatrix} J_{13} & J_{14} \\ J_{23} & J_{24} \end{pmatrix} = \left.\frac{1}{2}\frac{\partial^2 \left(J_{T_{r_0+\mu}}(0,0)\right)}{\partial \mu^2}\right|_{\mu=0}$.
Therefore,

$$J_{11} = -1, \quad J_{12} = -\frac{b\ln(a)f'_+(0)}{(r_0)^2}, \quad J_{21} = 0, \quad J_{22} = \frac{c\ln(a)f'_+(0)}{(r_0)^2},$$
$$J_{13} = 0, \quad J_{14} = \frac{b\ln(a)f'_+(0)}{(r_0)^3}, \quad J_{23} = 0, \quad J_{24} = -\frac{c\ln(a)f'_+(0)}{(r_0)^3}.$$

This ensures the transformation of System (6.23) into

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = T_{r_0+\mu}\begin{pmatrix} x_n \\ y_n \end{pmatrix} = J_{T_{r_0}}(0,0)\begin{pmatrix} x_n \\ y_n \end{pmatrix} + \begin{pmatrix} g_1(x_n,y_n,\mu) \\ g_2(x_n,y_n,\mu) \end{pmatrix},$$

where

$$\begin{aligned}
g_1(x,y,\mu) &= \big(-2r_0 + r_0^2 H_0^*(r_0)\big)\frac{x^2}{2} + bf'_+(0)\big(1 - r_0H_0^*(r_0)\big)xy + b^2H_0^*(r_0)f''_+(0)\frac{y^2}{2} \\
&\quad -\frac{1}{6}r_0^2\big(-3 + r_0H_0^*(r_0)\big)x^3 + \frac{1}{2}br_0\big(-2 + r_0H_0^*(r_0)\big)f'_+(0)x^2y \\
&\quad +\left(-1 + r_0H_0^*(r_0) + \frac{1}{2}r_0^2\left(H_0^*\right)'(r_0)\right)x^2\mu \\
&\quad -bf'_+(0)\big(r_0(H_0^*)'(r_0) + H_0^*(r_0)\big)xy\mu - \frac{1}{2}b^2\big(-1 + r_0H_0^*(r_0)\big)f''_+(0)xy^2 \\
&\quad +\frac{1}{2}b^2\left(H_0^*\right)'(r_0)f''_+(0)y^2\mu + J_{11}x\mu + J_{12}y\mu + J_{13}x\mu^2 + J_{14}y\mu^2 \\
&\quad +\frac{1}{6}b^3H_0^*(r_0)f_+^{(3)}(0)y^3 + O\big((|x|+|y|+|\mu|)^4\big), \\
&= -bf'_+(0)xy + b^2H_0^*(r_0)f''_+(0)\frac{y^2}{2} + \frac{1}{6}r_0^2x^3 + \left(1 - \frac{1}{2}\ln(a)\right)x^2\mu - bf'_+(0)xy\mu \\
&\quad -\frac{1}{2}b^2f''_+(0)xy^2 + \frac{1}{2}b^2\left(H_0^*\right)'(r_0)f''_+(0)y^2\mu + J_{11}x\mu + J_{12}y\mu + J_{14}y\mu^2 \\
&\quad +\frac{1}{6}b^3H_0^*(r_0)f_+^{(3)}(0)y^3 + O\big((|x|+|y|+|\mu|)^4\big), \\
g_2(x,y,\mu) &= -cf'_+(0)xy - cH_0^*(r_0)f''_+(0)\frac{y^2}{2} + J_{22}y\mu - \frac{1}{2}cf''_+(0)xy^2 - \frac{1}{2}c(H_0^*)'(r_0)f''_+(0)y^2\mu \\
&\quad +J_{24}y\mu^2 - \frac{1}{6}cH_0^*(r_0)f_+^{(3)}(0)y^3 + O\big((|x|+|y|+|\mu|)^4\big),
\end{aligned}$$

where we have applied the condition $1 - r_0H_0^*(r_0) = -1$ to simplify the expressions of g_1 and g_2. In addition, for the sake of brevity, we write g_1 and g_2 as follows

$$\begin{aligned}
g_1(x,y,\mu) &= A_1x^2 + A_2xy + A_3y^2 + J_{11}x\mu + J_{12}y\mu + A_4x^3 + A_5x^2y + A_6x^2\mu \\
&\quad +A_7xy\mu + A_8xy^2 + A_9y^2\mu + A_{10}y^3 + J_{14}y\mu^2 + O\big((|x|+|y|+|\mu|)^4\big), \\
g_2(x,y,\mu) &= B_1xy + B_2y^2 + J_{22}y\mu + B_3xy^2 + B_4y^2\mu + B_5y^3 \\
&\quad +J_{24}y\mu^2 + O\big((|x|+|y|+|\mu|)^4\big).
\end{aligned}$$

As a next step, we consider the matrix formed by the eigenvectors v_1, v_2, that is,

$$A=\begin{pmatrix}1 & \frac{bH_0^*(r_0)f_+'(0)}{1-cH_0^*(r_0)f_+'(0)}\\ 0 & 1\end{pmatrix}=:\begin{pmatrix}1 & a_1\\ 0 & 1\end{pmatrix}.$$

Let $\begin{pmatrix}x_n\\ y_n\end{pmatrix}=A\cdot\begin{pmatrix}\tilde{x}_n\\ \tilde{y}_n\end{pmatrix}$, that is, $x_n=\tilde{x}_n+a_1\tilde{y}_n$ and $y_n=\tilde{y}_n$. Hence, we obtain

$$\begin{aligned}\begin{pmatrix}\tilde{x}_{n+1}\\ \tilde{y}_{n+1}\end{pmatrix} &= A^{-1}\begin{pmatrix}x_{n+1}\\ y_{n+1}\end{pmatrix}=A^{-1}J_{T_{r_0}}(0,0)\begin{pmatrix}x_n\\ y_n\end{pmatrix}+A^{-1}\begin{pmatrix}g_1(x_n,y_n,\mu)\\ g_2(x_n,y_n,\mu)\end{pmatrix}\\ &= A^{-1}J_{T_{r_0}}(0,0)A\begin{pmatrix}\tilde{x}_n\\ \tilde{y}_n\end{pmatrix}+A^{-1}\begin{pmatrix}g_1(\tilde{x}_n+a_1\tilde{y}_n,\tilde{y}_n,\mu)\\ g_2(\tilde{x}_n+a_1\tilde{y}_n,\tilde{y}_n,\mu)\end{pmatrix},\end{aligned}$$

or, equivalently,

$$\begin{pmatrix}\tilde{x}_{n+1}\\ \tilde{y}_{n+1}\end{pmatrix}=\begin{pmatrix}-1 & 0\\ 0 & -cH_0^*f_+'(0)\end{pmatrix}\begin{pmatrix}\tilde{x}_n\\ \tilde{y}_n\end{pmatrix}+\begin{pmatrix}\tilde{g}_1(\tilde{x}_n,\tilde{y}_n,\mu)\\ \tilde{g}_2(\tilde{x}_n,\tilde{y}_n,\mu)\end{pmatrix}, \tag{6.25}$$

where

$$\begin{aligned}\tilde{g}_1(x,y,\mu) &= A_1x^2+A_4x^3+(3a_1A_4+A_5)\,x^2y+A_6x^2\mu\\ &\quad+\big(A_3+a_1\left(A_2+a_1\left(A_1-B_1\right)-B_2\right)\big)y^2+\big(A_9+a_1\left(a_1A_6+A_7-B_4\right)\big)y^2\mu\\ &\quad+\big(A_{10}+a_1\left(A_8+a_1\left(a_1A_4+A_5-B_3\right)-B_5\right)\big)y^3\\ &\quad+\big(A_2+\mu(2a_1A_6+A_7)+a_1\left(2A_1-B_1\right)\big)xy\\ &\quad+\big(A_8+a_1\left(3a_1A_4+2A_5-B_3\right)\big)xy^2+J_{11}x\mu\\ &\quad+\big(J_{12}+a_1\left(J_{11}-J_{22}\right)\big)y\mu+(J_{14}-a_1J_{24})y\mu^2+O\big((|x|+|y|+|\mu|)^4\big),\\ \tilde{g}_2(x,y,\mu) &= B_1xy+B_3xy^2+(a_1B_1+B_2)\,y^2+B_4y^2\mu+(a_1B_3+B_5)y^3+J_{22}y\mu+J_{24}y\mu^2\\ &\quad+O\big((|x|+|y|+|\mu|)^4\big).\end{aligned}$$

Finally, we assume that the local center manifold in given by

$$\tilde{y}=h(\tilde{x},\mu)=h_1\tilde{x}^2+h_2\tilde{x}\mu+h_3\mu^2+O\big((|\tilde{x}|+|\mu|)^3\big),$$

and we apply $\tilde{y}_{n+1}=h(\tilde{x}_{n+1},\mu)$ to get

$$h\big(-\tilde{x}+\tilde{g}_1(\tilde{x},h(\tilde{x},\mu),\mu)\big)=-cH_0^*(r_0)f_+'(0)h(\tilde{x},\mu)+\tilde{g}_2\big(\tilde{x},h(\tilde{x},\mu),\mu\big). \tag{6.26}$$

By equating similar terms, we achieve $h_1=h_2=h_3=0$. Thus, $h(\tilde{x},\mu)=O\big((|\tilde{x}|+|\mu|)^3\big)$ and the dynamics of (6.25) on the center manifold is given by $x_{n+1}=G(x_n,\mu)$, where

$$\begin{aligned}G(x,\mu) &= -x+\tilde{g}_1\big(x,h(x,\mu),\mu\big)+O\big((|x|+|\mu|)^4\big)\\ &= -x+J_{11}x\mu+A_1x^2+A_6x^2\mu+A_4x^3+O\big((|x|+|\mu|)^4\big).\end{aligned}$$

Now, we are able to check the conditions from [103, Section 20.1E].

$$(i)\ \frac{\partial G}{\partial x}(0,0) = -1, \quad (ii)\ \frac{\partial G^2}{\partial \mu}(0,0) = 0, \quad (iii)\ \frac{\partial^2 G^2}{\partial x^2}(0,0) = 0,$$

$$(iv)\ \frac{\partial^2 G^2}{\partial x \partial \mu}(0,0) \neq 0, \quad (v)\ \frac{\partial^3 G^2}{\partial x^3}(0,0) \neq 0,$$

where $G^2(x,\mu) = G\big(G(x,\mu),\mu\big)$. Indeed:

(i) $\displaystyle \frac{\partial G}{\partial x}(0,0) = \frac{\partial G}{\partial x}(x,\mu)\Big|_{(0,0)} = -1.$

(ii) $\displaystyle \frac{\partial G^2}{\partial \mu}(0,0) = \frac{\partial}{\partial \mu} G\big(G(x,\mu),\mu\big)\Big|_{(0,0)} = \frac{\partial G}{\partial x}(0,0)\frac{\partial G}{\partial \mu}(0,0) + \frac{\partial G}{\partial \mu}(0,0) = 0.$

(iii) $\displaystyle \frac{\partial^2 G^2}{\partial x^2}(0,0) = \frac{\partial^2 G}{\partial x^2}(0,0)\left[\frac{\partial G}{\partial x}(0,0)\right]^2 + \frac{\partial G}{\partial x}(0,0)\frac{\partial^2 G}{\partial x^2}(0,0) = 0.$

(iv) $\displaystyle \frac{\partial^2 G^2}{\partial x \partial \mu}(0,0) = \frac{\partial G}{\partial x}(0,0)\left(2\frac{\partial^2 G}{\partial x \partial \mu}(0,0) + \frac{\partial G}{\partial \mu}(0,0)\frac{\partial^2 G}{\partial x^2}(0,0)\right) = -2J_{11} = 2 \neq 0.$ that $J_{11} = -1$. Therefore, $\displaystyle \frac{\partial^2 G^2}{\partial x \partial \mu}(0,0) = 2 \neq 0.$

(v) $\displaystyle \frac{\partial^3 G^2}{\partial x^3}(0,0) = \frac{\partial^3 G}{\partial x^3}(0,0)\left[\frac{\partial G}{\partial x}(0,0)\right]^3 + 3\frac{\partial G}{\partial x}(0,0)\left[\frac{\partial^2 G}{\partial x^2}(0,0)\right]^2 + \frac{\partial^3 G}{\partial x^3}(0,0)\frac{\partial G}{\partial x}(0,0) =$ $-12(A_1^2 + A_4)$.

In order to verify the fifth condition, $\displaystyle \frac{\partial^3 G^2}{\partial x^3}(0,0) \neq 0$, observe that $A_1 = 0$ implies $-12(A_1^2 + A_4) = -12A_4 = -2(\ln(a) - 2)^2$. So, if $a \neq e^2$, then $-12A_4 \neq 0$ and $(0,0)$ has period-doubling at $\mu = 0$, that is, $(H_0^*, 0)$ undergoes period-doubling bifurcation at $r = r_0$. Then, if $\mu = 0$, $G(x,0) = -x + A_4x^3 + O\big(|x|^4\big)$ and the Schwarzian derivative of $G_1(x) = -x + A_4x^3 + O\big(|x|^4\big)$ at $x = 0$ is given by $-6A_4 = -(\ln(a) - 2)^2 < 0$. Since $SG_1(0) < 0$, by Theorem 3, the origin is asymptotically stable and, consequently, the equilibrium $(H_0^*, 0)$ is asymptotically stable.

Now, we establish the stability of the period-two cycle that appears near the equilibrium point when the period-doubling bifurcation takes place. To do so, we apply Theorem 6, where we need to verify

$$\frac{\partial G}{\partial \mu}\frac{\partial^2 G}{\partial x^2} + 2\frac{\partial^2 G}{\partial x \partial \mu}\Big|_{(0,0)} \neq 0 \quad \text{and} \quad \frac{1}{2}\left(\frac{\partial^2 G}{\partial x^2}\right)^2 + \frac{1}{3}\frac{\partial^3 G}{\partial x^3}\Big|_{(0,0)} \neq 0.$$

Indeed, these conditions hold since

$$\frac{\partial G}{\partial \mu}\frac{\partial^2 G}{\partial x^2} + 2\frac{\partial^2 G}{\partial x \partial \mu}\Big|_{(0,0)} = 2J_{11} = -2 < 0$$

and

$$\frac{1}{2}\left(\frac{\partial^2 G}{\partial x^2}\right)^2 + \frac{1}{3}\frac{\partial^3 G}{\partial x^3}\Big|_{(0,0)} = 2A_4 = \frac{1}{3}\big(\ln(a) - 2\big)^2 > 0.$$

Therefore, we are able to conclude that the period two cycle near $(0,0)$ is stable for small $\mu > 0$. Furthermore, by [103, page 373], the appearance of the two-cycle takes place on the right side of $\mu = 0$ because

$$-\frac{\partial^3 G^2}{\partial x^3}(0,0)/\frac{\partial^2 G^2}{\partial x \partial \mu}(0,0) = -\frac{6A_4}{J_{11}} = (\ln(a) - 2)^2 > 0.$$

As an example, consider the function $f(y) = \left(1 + \frac{y}{m}\right)^{-m}$ with $m > 0$. In Figure 6.2 (i), it can be seen the birth of the period-doubling bifurcation at $r_0 = 3.108662624521611$.

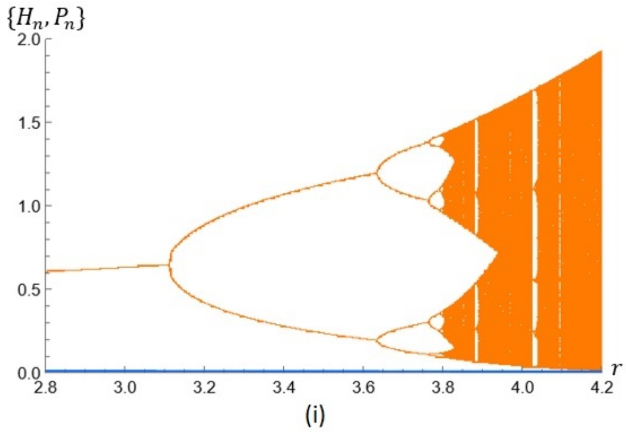


Figure 6.2: Period-doubling bifurcation for System (6.4) with $f(y) = \left(1 + \frac{y}{m}\right)^{-m}$ at $r_0 = 3.108662624521611$, $r \in (2.8, 4.2)$, $a = 0.33$, $b = 1.5$, $c = 1.2$, $m = 1.5$ and initial conditions $H_0 = 0.643$, $P_0 = 0.001$.

6.4.2 Transcritical bifurcation

A transcritical bifurcation occurs when two equilibrium points exchange their stability as a parameter is modified. Concretely, before the bifurcation, one equilibrium point is stable, while the other one in unstable, then for a concrete value of the bifurcation parameter, they collide passing the first one to be unstable and the other one to be stable (for more information consult [41, 59]).

In this scenario, we take the parasitoid maximum growth rate c as a bifurcation parameter and we set $c = c_0$, where c_0 is given by

$$c_0 = -\frac{1}{H_0^* f'_+(0)}.$$

Observe that for c_0, the eigenvalue $\lambda_2 = -c_0 H_0^* f'_+(0) = 1$. Moreover, we assume $r \neq 2 - \ln(a)$, so $\lambda_1 = 1 - r - \ln(a) \neq -1$.

Analogously to the period-doubling bifurcation, we proceed by applying center manifold theory. In this sense, we consider system (6.23) and its associated map T_c given by

$$T_c \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a\big(H_0^* + x\big)e^{r(1-H_0^*-x)} f(by) - H_0^* \\ c\big(H_0^* + x\big)\big(1 - f(y)\big) \end{pmatrix} =: \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix},$$

where the Taylor series expansions about $(0,0)$ are

$$\begin{aligned} f_1(x,y) &= (1-rH_0^*)x + bH_0^* f_+'(0)y + (-2r + r^2 H_0^*)\frac{x^2}{2} \\ &\quad + bf_+'(0)(1-rH_0^*)xy + b^2 H_0^* f_+''(0)\frac{y^2}{2} + O\big((|x|+|y|)^3\big), \\ f_2(x,y) &= -cH_0^* f_+'(0)y - cf_+'(0)xy - cH_0^* f_+''(0)\frac{y^2}{2} + O\big((|x|+|y|)^3\big). \end{aligned}$$

Here, we only have considered until order three since it is the order needed to satisfy the conditions from [103, Section 20.1D].

Now, we consider the Jacobian matrix of T_c evaluated at the origin,

$$J_{T_c}(0,0) = \begin{pmatrix} 1-rH_0^* & bH_0^* f_+'(0) \\ 0 & 1 \end{pmatrix},$$

and we substitute c for $c_0+\mu$, so for sufficiently small parameter μ, we get

$$J_{T_{c_0+\mu}}(0,0) = J_{T_{c_0}}(0,0) + \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}\mu + O\big(|\mu|^2\big),$$

where $\begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} = \left.\frac{\partial\left(J_{T_{c_0+\mu}}(0,0)\right)}{\partial\mu}\right|_{\mu=0}$. It is direct to see that $J_{11} = J_{12} = J_{21} = 0$ and $J_{22} = -H_0^* f_+'(0)$. From here, we can transform System (6.23) into

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = T_{c_0+\mu} = J_{T_{c_0}} + \begin{pmatrix} g_1(x_n,y_n,\mu) \\ g_2(x_n,y_n,\mu) \end{pmatrix},$$

where

$$\begin{aligned} g_1(x,y,\mu) &= (-2r+r^2H_0^*)\frac{x^2}{2} + bf_+'(0)(1-rH_0^*)xy + b^2H_0^*f_+''(0)\frac{y^2}{2} + O\big((|x|+|y|+|\mu|)^3\big), \\ g_2(x,y,\mu) &= -c_0 f_+'(0)xy - c_0 H_0^* f_+''(0)\frac{y^2}{2} - H_0^* f_+'(0)y\mu + O\big((|x|+|y|+|\mu|)^3\big). \end{aligned}$$

For the sake of brevity, we write g_1 and g_2 as follows

$$\begin{aligned} g_1(x,y,\mu) &= A_1x^2 + A_2xy + A_3y^2 + O\big((|x|+|y|+|\mu|)^3\big), \\ g_2(x,y,\mu) &= B_1xy + B_2y^2 + J_{22}y\mu + O\big((|x|+|y|+|\mu|)^3\big). \end{aligned}$$

As a next step, we compute the corresponding eigenvectors of $J_{T_c}(0,0)$. For $\lambda_1 = 1-rH_0^*$, the eigenvector is $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and for $\lambda_2 = 1$, we have $v_2 = \begin{pmatrix} \frac{bf_+'(0)}{r} \\ 1 \end{pmatrix}$. Thus, we set the matrix formed by those eigenvectors

$$B = \begin{pmatrix} 1 & \frac{bf_+'(0)}{r} \\ 0 & 1 \end{pmatrix} =: \begin{pmatrix} 1 & b_1 \\ 0 & 1 \end{pmatrix}.$$

Let $\begin{pmatrix} x_n \\ y_n \end{pmatrix} = B \begin{pmatrix} \tilde{x}_n \\ \tilde{y}_n \end{pmatrix}$, that is, $x_n = \tilde{x}_n + b_1\tilde{y}_n$ and $y_n = \tilde{y}_n$. Then, we have

$$\begin{aligned}\begin{pmatrix} \tilde{x}_{n+1} \\ \tilde{y}_{n+1} \end{pmatrix} &= B^{-1}J_{T_{c_0}}B\begin{pmatrix} \tilde{x}_n \\ \tilde{y}_n \end{pmatrix} + B^{-1}\begin{pmatrix} g_1(\tilde{x}_n + b_1\tilde{y}_n, \tilde{y}_n, \mu) \\ g_2(\tilde{x}_n + b_1\tilde{y}_n, \tilde{y}_n, \mu) \end{pmatrix} \\ &= \begin{pmatrix} 1 - rH_0^* & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \tilde{x}_n \\ \tilde{y}_n \end{pmatrix} + \begin{pmatrix} \tilde{g}_1(\tilde{x}_n, \tilde{y}_n, \mu) \\ \tilde{g}_2(\tilde{x}_n, \tilde{y}_n, \mu) \end{pmatrix},\end{aligned}$$

where

$$\begin{aligned}\tilde{g}_1(x, y, \mu) &= A_1x^2 + (A_2 + 2A_1b_1 - b_1B_1)\,xy + \big(A_3 + A_2b_1 + A_1b_1^2 - b_1\,(b_1B_1 + B_2)\big)\,y^2 \\ &\quad -b_1J_{22}y\mu + O\big((|x| + |y| + |\mu|)^3\big), \\ \tilde{g}_2(x, y, \mu) &= B_1xy + (b_1B_1 + B_2)\,y^2 + J_{22}y\mu + O\big((|x| + |y| + |\mu|)^3\big).\end{aligned}$$

Bearing this in mind, we can reduce the system as

$$\begin{cases} \tilde{x}_{n+1} = (1 - rH_0^*)\tilde{x}_n + \tilde{g}_1(\tilde{x}_n, \tilde{y}_n, \mu) \\ \tilde{y}_{n+1} = \tilde{y}_n + \tilde{g}_2(\tilde{x}_n, \tilde{y}_n, \mu) \end{cases}. \tag{6.27}$$

Next, let us assume that the local center manifold is given by $\tilde{x} = h(\tilde{y}, \mu) = h_1\tilde{y}^2 + h_2\tilde{y}\mu + h_3\mu^2 + O\big((|\tilde{y}| + |\mu|)^3\big)$. If we apply the center manifold equation to System (6.27), we get

$$(1 - rH_0^*)h(\tilde{y}, \mu) + \tilde{g}_1(h(\tilde{y}, \mu), \tilde{y}, \mu) = h\big(\tilde{y} + \tilde{g}_2\big(h(\tilde{y}, \mu), \tilde{y}, \mu\big), \mu\big). \tag{6.28}$$

The left-hand of (6.28) is

$$(1 - rH_0^*)h_1\tilde{y}^2 + (1 - rH_0^*)h_2\tilde{y}\mu + (1 - rH_0^*)h_3\mu^2 + \big(A_3 + A_2b_1 + A_1b_1^2 - b_1\,(b_1B_1 + B_2)\big)\,\tilde{y}^2$$
$$-b_1J_{22}\tilde{y}\mu + O\big((|\tilde{y}| + |\mu|)^3\big),$$

while the right-hand of (6.28) is

$$h_1\tilde{y}^2 + h_2\tilde{y}\mu + h_3\mu^2 + O\big((|\tilde{y}| + |\mu|)^3\big).$$

We equate similar terms in order to achieve

$$h_1 = \frac{b_1\,(A_1b_1 + A_2 - b_1B_1 - B_2) + A_3}{rH_0^*}, \quad h_2 = -\frac{b_1J_{22}}{rH_0^*}, \quad h_3 = 0.$$

Consequently,

$$\tilde{x} = h(\tilde{y}, \mu) = \frac{b_1\,(A_1b_1 + A_2 - b_1B_1 - B_2) + A_3}{rH_0^*}\tilde{y}^2 - \frac{b_1J_{22}}{rH_0^*}\tilde{y}\mu + O\big((|\tilde{y}| + |\mu|)^3\big),$$

and the dynamics on the center manifold is given by $\tilde{y}_{n+1} = G(\tilde{y}_n, \mu)$, where

$$\begin{aligned} G(y, \mu) &= y + \tilde{g}_2\big(h(y, \mu), y, \mu\big) + O\big((|y| + |\mu|)^3\big) \\ &= y + (B_2 + B_1b_1)y^2 + J_{22}y\mu + O\big((|y| + |\mu|)^3\big).\end{aligned}$$

Finally, we check that the map G satisfies the conditions from [103, page 365]:

$$(i)\ \frac{\partial G}{\partial y}(0,0)=1,\quad (ii)\ \frac{\partial G}{\partial \mu}(0,0)=0,\quad (iii)\ \frac{\partial^2 G}{\partial y \partial \mu}(0,0)\neq 0,\quad (iv)\ \frac{\partial^2 G}{\partial y^2}(0,0)\neq 0.$$

The first two conditions are easily checked. Moreover, conditions (iii) and (iv) hold too:

$$(iii)\ \ \frac{\partial^2 G}{\partial y \partial \mu}(0,0)=J_{22}=-H_0^* f_+'(0)>0,$$

$$(iv)\ \ \frac{\partial^2 G}{\partial y^2}(0,0)=2(B_2+B_1 b_1)=\frac{2bf_+'(0)}{H_0^* r}+\frac{f_+''(0)}{f_+'(0)}<0.$$

This establishes that $(0,0)$ undergoes a transcritical bifurcation at $\mu=0$, that is, $(H_0^*,0)$ has transcritical bifurcation at $c=c_0$. Furthermore, for $\mu=0$, the dynamics on the center manifold is given by the map

$$\tilde{y}_{n+1}=G(\tilde{y}_n,0)=\tilde{y}_n+(B_2+B_1 b_1)\tilde{y}_n^2+O\big(|\tilde{y}_n|^3\big).$$

Set $G_y(y)=y+(B_2+B_1b_1)y^2$. Since $G'(0)=1$ and $G''(0)=2(B_2+B_1b_1)<0$, the equilibrium of $G(y,0)$, and consequently, $(H_0^*,0)$, is unstable by Theorem 2.

We illustrate the occurrence of a transcritical bifurcation. To do so, we consider System (6.4) with the map $f(y)=\left(1+\frac{y}{m}\right)^{-m}$ and we produce its bifurcation diagram in (c,x,y) for particular parameters values, where (c,x) is in yellow and (c,y) is in blue colour. In Figure 6.3, one can see that the equilibrium E_0^* is asymptotically stable on the center manifold for $0\leq c<1.1145086254001821$, but when $c>1.1145086254001821$, such equilibrium becomes unstable. Precisely, Figure 6.3(i) shows the appearance of the transcritical bifurcation at $c=c_0=1.1145086254001821$, i.e., two fixed points exchange their stability at this value of the parameter c.

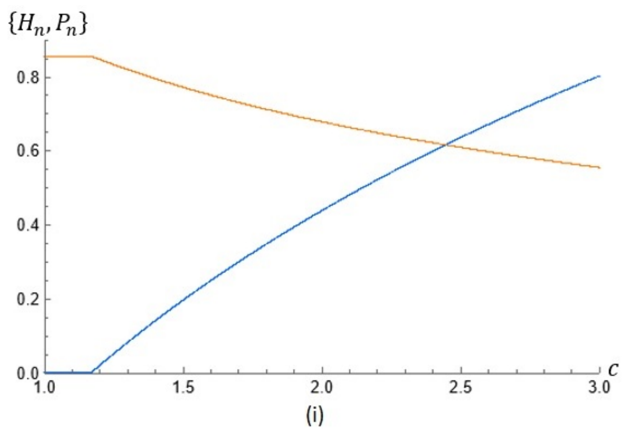


Figure 6.3: Transcritical bifurcation for $f(y)=\left(1+\frac{y}{m}\right)^{-m}$ at $c_0=1.1145086254001821$, $c\in(1.0,3.0)$, $a=0.75$, $b=0.9$, $r=2.0$, $m=1.7$ and $x_0=0.897$, $y_0=0.001$.

6.5 Bifurcations for the coexistence equilibrium

This section delves into the likely occurrence of bifurcations for the coexistence equilibrium point. Concretely, we show the manifestation of period-doubling and Neimark-Sacker bifurcations. To achieve this, we consider the host maximum growth rate r as a bifurcation parameter. Thus, by (6.6), the interior equilibrium $\big(H^*(r), P^*(r)\big) \in \mathbb{R}^2_+$ satisfies the equations

$$\begin{cases} H^*(r) &= aH^*(r)e^{r(1-H^*(r))}f\big(bP^*(r)\big) \\ P^*(r) &= cH^*(r)\big(1-f(P^*(r))\big) \end{cases}.$$

Here, the values $H^*(r)$ and $P^*(r)$ are nonzero, so from the above equations we can deduce

$$H^*(r) = 1 + \frac{\ln\big[af\big(bP^*(r)\big)\big]}{r} \quad \text{and} \quad P^*(r) = c\big(1-f(P^*(r)\big)\left(1+\frac{\ln\big[af\big(bP^*(r)\big)\big]}{r}\right), \tag{6.29}$$

where the logarithm is well-defined since $af\big(bP^*(r)\big) > 0$.

As in the preceding section, we make a change of variables in order to shift the equilibrium to the origin, $x_n = H_n - H^*(r)$ and $y_n = P_n - P^*(r)$. Thus, the corresponding system is given by

$$\begin{cases} x_{n+1} &= a\big(x_n + H^*(r)\big)e^{r(1-x_n-H^*(r))}f\big(b(y_n+P^*(r))\big) - H^*(r) \\ y_{n+1} &= c\big(x_n+H^*(r)\big)\cdot\big[1-f\big(y_n+P^*(r)\big)\big] - P^*(r) \end{cases}, \tag{6.30}$$

and its associated map $\mathbf{F}_r$ is

$$\mathbf{F}_r\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a\big(x+H^*(r)\big)e^{r(1-x-H^*(r))}f\big(b(y+P^*(r))\big) - H^*(r), \\ c\big(x+H^*(r)\big)\big(1-f(y+P^*(r))\big) - P^*(r). \end{pmatrix} =: \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}. \tag{6.31}$$

Now, in order the compute the Taylor series expansion of $\mathbf{F}_r$ about the origin, we obtain its Jacobian matrix evaluated at $(0,0)$,

$$\begin{aligned} J_{\mathbf{F}_r}(0,0) &= \begin{pmatrix} ae^{r(1-H^*(r))}\big(1-rH^*(r)\big)f\big(bP^*(r)\big) & abH^*(r)e^{r(1-H^*(r))}f'\big(bP^*(r)\big) \\ c - cf\big(P^*(r)\big) & -cH^*(r)f'\big(P^*(r)\big) \end{pmatrix} \\ &= \begin{pmatrix} 1-rH^*(r) & \frac{bH^*(r)}{f\left(bP^*(r)\right)}f'\big(bP^*(r)\big) \\ \frac{P^*(r)}{H^*(r)} & -cH^*(r)f'\big(P^*(r)\big) \end{pmatrix}. \end{aligned} \tag{6.32}$$

Thus, we have

$$\mathbf{F}_r\begin{pmatrix} x \\ y \end{pmatrix} = J_{\mathbf{F}_r}(0,0)\begin{pmatrix} x \\ vy \end{pmatrix} + \begin{pmatrix} \tilde{f}(x,y,r) \\ \tilde{g}(x,y,r) \end{pmatrix},$$

where

$$\begin{aligned}
\tilde{f}(x,y,r) &= x^2\left(\frac{1}{2}ae^{r-rH^*(r)}rf(bP^*(r))(-2+rH^*(r))\right)+y^2\left(\frac{1}{2}ab^2e^{r-rH^*(r)}H^*(r)f''(bP^*(r))\right)\\
&\quad -xy\big(abe^{r-rH^*(r)}(-1+rH^*(r))f'(bP^*(r))\big)\\
&\quad -x^3\left(\frac{1}{6}ae^{r-rH^*(r)}r^2f(bP^*(r))(-3+rH^*(r))\right)\\
&\quad +x^2y\left(\frac{1}{2}abe^{r-rH^*(r)}r(-2+rH^*(r))f'(bP^*(r))\right)\\
&\quad +y^3\left(\frac{1}{6}ab^3e^{r-rH^*(r)}H^*(r)f^{(3)}(bP^*(r))\right)\\
&\quad -xy^2\left(\frac{1}{2}ab^2e^{r-rH^*(r)}(-1+rH^*(r))f''(bP^*(r))\right)+O\big((|x|+|y|)^4\big),\\
\tilde{g}(x,y,r) &= -xy\left(cf'(P^*(r))\right)-y^2\left(\frac{1}{2}cH^*(r)f''(P^*(r))\right)-xy^2\left(\frac{1}{2}cf''(P^*(r))\right)\\
&\quad -y^3\left(\frac{1}{6}cH^*(r)f^{(3)}(P^*(r))\right)+O\big((|x|+|y|)^4\big).
\end{aligned}$$

6.5.1 Period-doubling bifurcation

The occurrence of a period-doubling bifurcation is related with the existence of an eigenvalue of the corresponding Jacobian matrix evaluated at the equilibrium equals to -1. From Theorem J, where we established the local stability of the interior equilibrium E^*, we know than this happens if and only if either $\mathrm{Tr}\big(J_{\mathbf{F}_{r_0}}\big)=-2$ and $\mathrm{Det}\big(J_{\mathbf{F}_{r_0}}\big)=1$, or

$$2-cH^*f'(P^*)(2-rH^*)-rH^*-\frac{c\big(1-f(P^*)\big)bH^*f'(bP^*)}{f(bP^*)}=0. \tag{6.33}$$

From the above equation, we set the bifurcation parameter $r=r_0$ such that

$$f'\big(bP^*(r_0)\big)=-\frac{f\big(bP^*(r_0)\big)\big(-2+H^*(r_0)\,r_0\big)\big(-1+cH^*(r_0)\,f'(P^*(r_0))\big)}{bc\big(-1+f(P^*(r_0))\big)H^*(r_0)}.$$

Now, we consider the Jacobian matrix (6.32) and we evaluate it at r_0,

$$J_{\mathbf{F}_{r_0}}=J_{\mathbf{F}_r}(0,0)|_{r=r_0}=\begin{pmatrix}1-r_0H^*(r_0) & \frac{H^*(r_0)bf'\left(bP^*(r_0)\right)}{f\left(bP^*(r_0)\right)}\\ c-cf\big(P^*(r_0)\big) & -cH^*(r_0)f'\big(P^*(r_0)\big)\end{pmatrix}.$$

Observe that this implies that the eigenvalues are $\lambda_1=-1$ and $\lambda_2=\mathrm{Tr}\big(J_{\mathbf{F}_{r_0}}\big)+1=2-r_0H^*(r_0)-cH^*(r_0)f'\big(P^*(r_0)\big)$. In the sequel, in order to avoid having the second eigenvalue λ_2 equals to ±1, we assume

$$\lambda_2=\mathrm{Tr}\big(J_{\mathbf{F}_{r_0}}\big)+1=-\mathrm{Det}\big(J_{\mathbf{F}_{r_0}}\big)\neq\pm1, \tag{6.34}$$

or, equivalently, $\mathrm{Tr}\big(J_{\mathbf{F}_{r_0}}\big)\notin\{-2,0\}$.

As a next step, take $r = r_0 + \mu$, so

$$J_{\mathbf{F}_{r_0+\mu}}(0,0) = J_{\mathbf{F}_{r_0}}(0,0) + \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \mu + \begin{pmatrix} J_{13} & J_{14} \\ J_{23} & J_{24} \end{pmatrix} \mu^2 + O\big(|\mu|^3\big),$$

where $\begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} = \left.\dfrac{\partial \left(J_{\mathbf{F}_{r_0+\mu}}(0,0)\right)}{\partial \mu}\right|_{\mu=0}$ and $\begin{pmatrix} J_{13} & J_{14} \\ J_{23} & J_{24} \end{pmatrix} = \left.\dfrac{1}{2}\dfrac{\partial^2 \left(J_{\mathbf{F}_{r_0+\mu}}(0,0)\right)}{\partial \mu^2}\right|_{\mu=0}$.
Therefore,

$$\begin{aligned}
J_{11} &= -H^*(r_0) - r_0\big(H^*\big)'(r_0), \\
J_{12} &= \frac{b\left(f'\left(bP^*\left(r_0\right)\right)\left(H^*\right)'\left(r_0\right) + bH^*\left(r_0\right)\left(P^*\right)'\left(r_0\right) f''\left(bP^*\left(r_0\right)\right)\right)}{f\left(bP^*\left(r_0\right)\right)} \\
&\quad -\frac{b^2 H^*\left(r_0\right) f'\left(bP^*\left(r_0\right)\right)^2 \left(P^*\right)'\left(r_0\right)}{f\left(bP^*\left(r_0\right)\right)^2}, \\
J_{21} &= -cf'\big(P^*(r_0)\big)\big(P^*\big)'(r_0), \\
J_{22} &= -c\big(H^*\big)'(r_0) f'\big(P^*(r_0)\big) - cH^*(r_0) f''\big(P^*(r_0)\big)\big(P^*\big)'(r_0),
\end{aligned}$$

and

$$\begin{aligned}
J_{13} &= -2(H^*)'\left(r_0\right) - r_0 (H^*)''\left(r_0\right), \\
J_{14} &= -\frac{b^2 f'\big(bP^*\left(r_0\right)\big)^2 \left[2(H^*)'\left(r_0\right)\left(P^*\right)'\left(r_0\right) + H^*\left(r_0\right)\left(P^*\right)''\left(r_0\right)\right]}{f\left(bP^*\left(r_0\right)\right)^2} \\
&\quad -\frac{3b^3 H^*\left(r_0\right) f'\left(bP^*\left(r_0\right)\right)\left(P^*\right)'\left(r_0\right)^2 f''\left(bP^*\left(r_0\right)\right)}{f\left(bP^*\left(r_0\right)\right)^2} \\
&\quad +\frac{b^2 \left[f''\left(bP^*\left(r_0\right)\right)\left(2(H^*)'\left(r_0\right)\left(P^*\right)'\left(r_0\right) + H^*\left(r_0\right)\left(P^*\right)''\left(r_0\right)\right)\right]}{f\left(bP^*\left(r_0\right)\right)} \\
&\quad +\frac{b^3 H^*\left(r_0\right)\left(P^*\right)'\left(r_0\right)^2 f^{(3)}\big(bP^*\left(r_0\right)\big)}{f\left(bP^*\left(r_0\right)\right)} + \frac{2b^3 H^*\left(r_0\right) f'\left(bP^*\left(r_0\right)\right)^3 \left(P^*\right)'\left(r_0\right)^2}{f\left(bP^*\left(r_0\right)\right)^3} \\
&\quad +\frac{bf'\left(bP^*\left(r_0\right)\right)\left(H^*\right)''\left(r_0\right)}{f\left(bP^*\left(r_0\right)\right)}, \\
J_{23} &= -c\left(\left(P^*\right)'\left(r_0\right)^2 f''\left(P^*\left(r_0\right)\right) + f'\left(P^*\left(r_0\right)\right)\left(P^*\right)''\left(r_0\right)\right), \\
J_{24} &= -c\left(f'\left(P^*\left(r_0\right)\right)\left(H^*\right)''\left(r_0\right) + H^*\left(r_0\right)\left(P^*\right)'\left(r_0\right)^2 f^{(3)}\left(P^*\left(r_0\right)\right)\right) \\
&\quad -cf''\left(P^*\left(r_0\right)\right)\left(2(H^*)'\left(r_0\right)\left(P^*\right)'\left(r_0\right) + H^*\left(r_0\right)\left(P^*\right)''\left(r_0\right)\right).
\end{aligned}$$

Bearing this in mind, we can transform System (6.30) into

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \mathbf{F}_{r_0+\mu}\begin{pmatrix} x_n \\ y_n \end{pmatrix} = J_{\mathbf{F}_{r_0}}\begin{pmatrix} x_n \\ y_n \end{pmatrix} + \begin{pmatrix} \tilde{f}_1(x_n, y_n, \mu) \\ \tilde{g}_1(x_n, y_n, \mu) \end{pmatrix},$$

where

$$
\begin{aligned}
\tilde{f}_1(x,y,\mu) &= x^2\left(\frac{1}{2}r_0\left(-2+H^*\left(r_0\right)r_0\right)\right)+y^2\left(\frac{b^2H^*\left(r_0\right)f''\left(bP^*\left(r_0\right)\right)}{2f\left(bP^*\left(r_0\right)\right)}\right)\\
&+xy\left(\frac{b\left(1-H^*\left(r_0\right)r_0\right)f'\left(bP^*\left(r_0\right)\right)}{f\left(bP^*\left(r_0\right)\right)}\right)+x\mu J_{11}+y\mu J_{12}\\
&+xy\mu\left(\frac{b^2\left(1-H^*\left(r_0\right)r_0\right)\left(P^*\right)'\left(r_0\right)f''\left(bP^*\left(r_0\right)\right)}{f\left(bP^*\left(r_0\right)\right)}\right)\\
&+xy\mu\left(-\frac{bf'\left(bP^*\left(r_0\right)\right)\left(H^*\left(r_0\right)+r_0(H^*)'\left(r_0\right)\right)}{f\left(bP^*\left(r_0\right)\right)}\right)\\
&+xy\mu\left(\frac{b^2\left(-1+H^*\left(r_0\right)r_0\right)f'\left(bP^*\left(r_0\right)\right)^2(P^*)'\left(r_0\right)}{f\left(bP^*\left(r_0\right)\right)^2}\right)\\
&-x^3\left(\frac{1}{6}r_0^2\left(-3+H^*\left(r_0\right)r_0\right)\right)\\
&+x^2y\left(\frac{br_0\left(-2+H^*\left(r_0\right)r_0\right)f'\left(bP^*\left(r_0\right)\right)}{2f\left(bP^*\left(r_0\right)\right)}\right)\\
&+x^2\mu\left(-1+H^*\left(r_0\right)r_0+\frac{1}{2}r_0^2(H^*)'\left(r_0\right)\right)\\
&-xy^2\left(\frac{b^2\left(-1+H^*\left(r_0\right)r_0\right)f''\left(bP^*\left(r_0\right)\right)}{2f\left(bP^*\left(r_0\right)\right)}\right)+x\mu^2J_{13}+y\mu^2J_{14}\\
&+y^3\left(\frac{b^3H^*\left(r_0\right)f^{(3)}\left(bP^*\left(r_0\right)\right)}{6f\left(bP^*\left(r_0\right)\right)}\right)+y^2\mu\left(\frac{b^3H^*\left(r_0\right)\left(P^*\right)'\left(r_0\right)f^{(3)}\left(bP^*\left(r_0\right)\right)}{2f\left(bP^*\left(r_0\right)\right)}\right)\\
&+y^2\mu\left(\frac{b^2\left(f\left(bP^*\left(r_0\right)\right)\left(H^*\right)'\left(r_0\right)-bH^*\left(r_0\right)f'\left(bP^*\left(r_0\right)\right)\left(P^*\right)'\left(r_0\right)\right)f''\left(bP^*\left(r_0\right)\right)}{2f\left(bP^*\left(r_0\right)\right)^2}\right)\\
&+O\big((|x|+|y|+|\mu|)^4\big),
\end{aligned}
$$

$$
\begin{aligned}
\tilde{g}_1(x,y,\mu) &= -y^2\left(\frac{1}{2}cH^*\left(r_0\right)f''\left(P^*\left(r_0\right)\right)\right)-xy\left(cf'\left(P^*\left(r_0\right)\right)\right)+x\mu J_{21}+y\mu J_{22}\\
&-xy^2\left(\frac{1}{2}cf''\left(P^*\left(r_0\right)\right)\right)-xy\mu\left(c(P^*)'\left(r_0\right)f''\left(P^*\left(r_0\right)\right)\right)+x\mu^2J_{23}+y\mu^2J_{24}\\
&-y^3\left(\frac{1}{6}cH^*\left(r_0\right)f^{(3)}\left(P^*\left(r_0\right)\right)\right)\\
&-y^2\mu\left(\frac{1}{2}c\left(\left(H^*\right)'\left(r_0\right)f''\left(P^*\left(r_0\right)\right)+H^*\left(r_0\right)\left(P^*\right)'\left(r_0\right)f^{(3)}\left(P^*\left(r_0\right)\right)\right)\right)\\
&+O\big((|x|+|y|+|\mu|)^4\big).
\end{aligned}
$$

For the sake of brevity, we rewrite those functions as

$$
\begin{aligned}
\tilde{f}_1(x,y,\mu) &= A_1x^2+A_2xy+A_3y^2+J_{11}x\mu+J_{12}y\mu+A_4x^3+A_5x^2y+A_6x^2\mu\\
&+A_7xy\mu+A_8xy^2+A_9y^2\mu+A_{10}y^3+J_{13}x\mu^2+J_{14}y\mu^2\\
&+O\big((|x|+|y|+|\mu|)^4\big),\\
\tilde{g}_1(x,y,\mu) &= B_1xy+B_2y^2+J_{21}x\mu+J_{22}y\mu+B_3xy^2+B_4y^2\mu+B_5y^3+B_6xy\mu\\
&+J_{23}y\mu^2+J_{24}y\mu^2\\
&+O\big((|x|+|y|+|\mu|)^4\big).
\end{aligned}
$$

We compute the eigenvector associated to the eigenvalues $\lambda_1 = -1$ and $\lambda_2 = \mathrm{Tr}(J_{\mathbf{F}_{r_0}})+1$, which are

$$v_1 = \begin{pmatrix} 1 \\ \frac{c(f(P^*(r_0))-1)}{1-cH^*(r_0)f'(P^*(r_0))} \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} \frac{2-rH^*(r_0)}{c-cf(P^*(r_0))} \\ 1 \end{pmatrix},$$

respectively.

Observe that, by (6.34), $\lambda_2 \neq -1$. Now, we take the matrix formed by such eigenvectors

$$A = \begin{pmatrix} 1 & \frac{2-rH^*(r_0)}{c-cf(P^*(r_0))} \\ \frac{c(f(P^*(r_0))-1)}{1-cH^*(r_0)f'(P^*(r_0))} & 1 \end{pmatrix} =: \begin{pmatrix} 1 & a_1 \\ a_2 & 1 \end{pmatrix}.$$

We claim that

$$\mathrm{Det}(A) = 1 - a_1 a_2 \neq 0.$$

Indeed, if $\mathrm{Det}(A) = 0$, then the eigenvectors v_1 and v_2 will be proportional, in contradiction with the fact that $\lambda_1 \neq \lambda_2$.

Now, set $\begin{pmatrix} x_n \\ y_n \end{pmatrix} = A \begin{pmatrix} \tilde{x}_n \\ \tilde{y}_n \end{pmatrix}$, that is, $x_n = \tilde{x}_n + a_1 \tilde{y}_n$ and $y_n = \tilde{y}_n + a_2 \tilde{x}_n$. We have

$$\begin{pmatrix} \tilde{x}_{n+1} \\ \tilde{y}_{n+1} \end{pmatrix} = A^{-1} J_{\mathbf{F}_{r_0}} A \begin{pmatrix} \tilde{x}_n \\ \tilde{y}_n \end{pmatrix} + A^{-1} \begin{pmatrix} \tilde{f}_1(\tilde{x}_n + a_1\tilde{y}_n, \tilde{y}_n + a_2\tilde{x}_n, \mu) \\ \tilde{g}_1(\tilde{x}_n + a_1\tilde{y}_n, \tilde{y}_n + a_2\tilde{x}_n, \mu) \end{pmatrix},$$

or, equivalently,

$$\begin{pmatrix} \tilde{x}_{n+1} \\ \tilde{y}_{n+1} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & \mathrm{Tr}(J_{\mathbf{F}_{r_0}}) + 1 \end{pmatrix} \begin{pmatrix} \tilde{x}_n \\ \tilde{y}_n \end{pmatrix} + \begin{pmatrix} \tilde{f}_2(\tilde{x}_n, \tilde{y}_n, \mu) \\ \tilde{g}_2(\tilde{x}_n, \tilde{y}_n, \mu) \end{pmatrix}, \tag{6.35}$$

where

$$\begin{aligned} \tilde{f}_2(x,y,\mu) \;=\;& y^2 \left(\frac{-A_3 + a_1\left(-A_2 + a_1\left(-A_1 + B_1\right) + B_2\right)}{-1 + a_1 a_2} \right) \\ &+x^2 \left(\frac{-A_1 + a_2\left(-A_2 + a_1 B_1 + a_2\left(-A_3 + a_1 B_2\right)\right)}{-1 + a_1 a_2} \right) \\ &-x^2 y \left(\frac{A_5 + a_2(2A_8 + 3a_2 A_{10} - a_1^2 a_2 B_3)}{-1 + a_1 a_2} \right) \\ &-x^2 y \left(\frac{a_1\big(3A_4 + a_2\big(2A_5 - 2B_3 + a_2\big(A_8 - 3B_5\big)\big)\big)}{-1 + a_1 a_2} \right) \\ &-xy^2 \left(\frac{A_8 + 3a_2 A_{10} + a_1^2\left(3A_4 + a_2\left(A_5 - 2B_3\right)\right) + a_1\left(2A_5 - B_3 + a_2\left(2A_8 - 3B_5\right)\right)}{-1 + a_1 a_2} \right) \\ &-y^3 \left(\frac{A_{10} + a_1\left(A_8 + a_1\left(a_1 A_4 + A_5 - B_3\right) - B_5\right)}{-1 + a_1 a_2} \right) \end{aligned}$$

$$
\begin{aligned}
& -x^3\left(\frac{A_4+a_2\left(A_5+a_2\left(A_8-a_1B_3+a_2\left(A_{10}-a_1B_5\right)\right)\right)}{-1+a_1a_2}\right)\\
& +x^2\mu\left(\frac{-A_6+a_2\left(-A_7+a_2\left(-A_9+a_1B_4\right)+a_1B_6\right)}{-1+a_1a_2}\right)\\
& +y^2\mu\left(\frac{-A_9+a_1\left(-A_7+B_4+a_1\left(-A_6+B_6\right)\right)}{-1+a_1a_2}\right)\\
& +xy\left(\frac{-A_2-2a_2A_3+a_1^2a_2B_1+a_1\left(-2A_1+B_1-a_2\left(A_2-2B_2\right)\right)}{-1+a_1a_2}\right)\\
& +xy\mu\left(\frac{-A_7-2a_2A_9+a_1^2a_2B_6+a_1\left(-2A_6-a_2\left(A_7-2B_4\right)+B_6\right)}{-1+a_1a_2}\right)\\
& +y\mu\left(\frac{-J_{12}+a_1\left(-J_{11}+a_1J_{21}+J_{22}\right)}{-1+a_1a_2}\right)\\
& +x\mu\left(\frac{-J_{11}+a_1J_{21}+a_2\left(-J_{12}+a_1J_{22}\right)}{-1+a_1a_2}\right)\\
& -x\mu^2\left(\frac{J_{13}+a_2J_{14}-a_1\left(1+a_2\right)J_{24}}{-1+a_1a_2}\right)\\
& +y\mu^2\left(\frac{-J_{14}+a_1\left(-J_{13}+\left(1+a_1\right)J_{24}\right)}{-1+a_1a_2}\right)\\
& +O\big((|x|+|y|+|\mu|)^4\big),
\end{aligned}
$$

$$
\begin{aligned}
\tilde{g}_2(x,y,\mu) \;=\; & x^2\left(\frac{a_2\left(A_1-B_1+a_2\left(A_2+a_2A_3-B_2\right)\right)}{-1+a_1a_2}\right)\\
& -y^2\left(\frac{-a_2\left(a_1\left(a_1A_1+A_2\right)+A_3\right)+a_1B_1+B_2}{-1+a_1a_2}\right)\\
& +x^3\left(\frac{a_2\left(A_4+a_2\left(A_5-B_3+a_2\left(A_8+a_2A_{10}-B_5\right)\right)\right)}{-1+a_1a_2}\right)\\
& +x^2y\left(\frac{a_2\left(A_5+a_1\left(3A_4+a_2\left(2A_5+a_2A_8-B_3\right)\right)-2B_3+a_2\left(2A_8+3a_2A_{10}-3B_5\right)\right)}{-1+a_1a_2}\right)\\
& +xy^2\left(\frac{-B_3+a_2\left(a_1^2\left(3A_4+a_2A_5\right)+A_8+3a_2A_{10}+2a_1\left(A_5+a_2A_8-B_3\right)-3B_5\right)}{-1+a_1a_2}\right)\\
& -y^3\left(\frac{-a_2\left(a_1\left(a_1\left(a_1A_4+A_5\right)+A_8\right)+A_{10}\right)+a_1B_3+B_5}{-1+a_1a_2}\right)\\
& +xy\left(\frac{-B_1+a_2\left(A_2+2a_2A_3+a_1\left(2A_1+a_2A_2-B_1\right)-2B_2\right)}{-1+a_1a_2}\right)\\
& +xy\mu\left(\frac{a_2\left(A_7+2a_2A_9-2B_4+a_1\left(2A_6+a_2A_7-B_6\right)\right)-B_6}{-1+a_1a_2}\right)\\
& +x^2\mu\left(\frac{a_2\left(A_6+a_2\left(A_7+a_2A_9-B_4\right)-B_6\right)}{-1+a_1a_2}\right)\\
& -y^2\mu\left(\frac{-a_2\left(a_1\left(a_1A_6+A_7\right)+A_9\right)+B_4+a_1B_6}{-1+a_1a_2}\right)\\
& +x\mu\left(\frac{-J_{21}+a_2\left(J_{11}+a_2J_{12}-J_{22}\right)}{-1+a_1a_2}\right)-y\mu\left(\frac{-a_2\left(a_1J_{11}+J_{12}\right)+a_1J_{21}+J_{22}}{-1+a_1a_2}\right)\\
& +y\mu^2\left(\frac{a_2\left(a_1J_{13}+J_{14}\right)-\left(1+a_1\right)J_{24}}{-1+a_1a_2}\right)+x\mu^2\left(\frac{a_2\left(J_{13}+a_2J_{14}\right)-\left(1+a_2\right)J_{24}}{-1+a_1a_2}\right)\\
& +O\big((|x|+|y|+|\mu|)^4\big).
\end{aligned}
$$

For convenience, we rewrite them as

$$\begin{aligned}\tilde{f}_2(x,y,\mu) &= \tilde{A}_1x^2+\tilde{A}_2xy+\tilde{A}_3y^2-\tilde{A}_4x^3-\tilde{A}_5x^2y+\tilde{A}_6x^2\mu+\tilde{A}_7xy\mu-\tilde{A}_8xy^2+\tilde{A}_9y^2\mu\\ &\quad -\tilde{A}_{10}y^3+\tilde{J}_{11}x\mu+\tilde{J}_{12}y\mu-\tilde{J}_{13}x\mu^2+\tilde{J}_{14}y\mu^2+O\big((|x|+|y|+|\mu|)^4\big).\\ \tilde{g}_2(x,y,\mu) &= \tilde{B}_1x^2+\tilde{B}_2xy-\tilde{B}_3y^2+\tilde{B}_4x^3+\tilde{B}_5x^2y+\tilde{B}_6x^2\mu+\tilde{B}_7xy\mu+\tilde{B}_8xy^2-\tilde{B}_9y^2\mu\\ &\quad -\tilde{B}_{10}y^3+\tilde{J}_{21}x\mu-\tilde{J}_{22}y\mu+\tilde{J}_{23}x\mu^2+\tilde{J}_{24}y\mu^2+O\big((|x|+|y|+|\mu|)^4\big).\end{aligned}$$

Applying center manifold theory, we assume that the center manifold is given by

$$\tilde{y}=h(\tilde{x},\mu)=h_1\tilde{x}^2+h_2\tilde{x}\mu+h_3\mu^2+O\big((|x|+|\mu|)^3\big).$$

In this sense, the center manifold equation holds

$$h\big(-\tilde{x}+\tilde{f}_2\big(\tilde{x},h(\tilde{x},\mu),\mu\big),\mu\big)=-\mathrm{Det}(J_{\mathbf{F}_{r_0}})\big(h_1\tilde{x}^2+h_2\tilde{x}\mu+h_3\mu^2\big)+\tilde{g}_2\big(\tilde{x},h(\tilde{x},\mu),\mu\big).$$

We equate similar terms in order to obtain the coefficients h_1, h_2, h_3,

$$h_1=\frac{\tilde{B}_1}{1+\mathrm{Det}(J_{\mathbf{F}_{r_0}})},\quad h_2=\frac{\tilde{J}_{21}}{\mathrm{Det}(J_{\mathbf{F}_{r_0}})-1}\quad\text{and}\quad h_3=0.$$

It is worth mentioning that h_1 and h_2 are well defined since $\mathrm{Det}(J_{\mathbf{F}_{r_0}})\neq\pm1$ by (6.34). Consequently, the center manifold is given by

$$\tilde{y}=\frac{\tilde{B}_1}{1+\mathrm{Det}(J_{\mathbf{F}_{r_0}})}\tilde{x}^2+\frac{\tilde{J}_{21}}{\mathrm{Det}(J_{\mathbf{F}_{r_0}})-1}\tilde{x}\mu+O\big((|\tilde{x}|+|\mu|)^3\big).$$

Hence, the dynamics of (6.35) on the center manifold is given by the map $\tilde{x}_{n+1}=G(\tilde{x}_n,\mu)$, where

$$\begin{aligned}G(x,\mu) &= -x+x\mu\tilde{J}_{11}+x\mu^2\left(-\tilde{J}_{13}+\frac{\tilde{J}_{12}\tilde{J}_{21}}{-1+\mathrm{Det}(J_{\mathbf{F}_{r_0}})}\right)+x^2\tilde{A}_1\\ &\quad +x^2\mu\left(\tilde{A}_6+\frac{\tilde{B}_1\tilde{J}_{12}}{1+\mathrm{Det}(J_{\mathbf{F}_{r_0}})}+\frac{\tilde{A}_2\tilde{J}_{21}}{-1+\mathrm{Det}(J_{\mathbf{F}_{r_0}})}\right)\\ &\quad +x^3\left(-\tilde{A}_4+\frac{\tilde{A}_2\tilde{B}_1}{1+\mathrm{Det}(J_{\mathbf{F}_{r_0}})}\right)+O\big((|x|+|\mu|)^4\big).\end{aligned}$$

Finally, we check that the conditions from [103, page 373] are verified:

$$(i)\ \frac{\partial G}{\partial x}(0,0)=-1,\quad (ii)\ \frac{\partial G^2}{\partial \mu}(0,0)=0,\quad (iii)\ \frac{\partial^2 G^2}{\partial x^2}(0,0)=0,$$

$$(iv)\ \frac{\partial^2 G^2}{\partial x\partial \mu}(0,0)\neq 0,\quad (v)\ \frac{\partial^3 G^2}{\partial x^3}(0,0)\neq 0,$$

where $G^2(x,\mu)=G\big(G(x,\mu),\mu\big)$. It is direct to see that conditions $(i)-(iii)$ hold. Moreover,

(iv) $\dfrac{\partial^2 G^2}{\partial x \partial \mu}(0,0) = -2\tilde{J}_{11} = \dfrac{2\left(J_{11} - a_1 J_{21} + a_2\left(J_{12} - a_1 J_{22}\right)\right)}{-1 + a_1 a_2}$. The condition $\tilde{J}_{11} \neq 0$ is equivalent to

$$J_{11} - a_1 J_{21} + a_2\left(J_{12} - a_1 J_{22}\right) \neq 0.$$

(v) $\dfrac{\partial^3 G^2}{\partial x^3}(0,0) = 12\left(-\tilde{A}_1^2 + \tilde{A}_4 - \dfrac{\tilde{A}_2 \tilde{B}_1}{1 + \operatorname{Det}(J_{\mathbf{F}_{r_0}})}\right) \neq 0.$

In conclusion, if $J_{11} - a_1 J_{21} + a_2\left(J_{12} - a_1 J_{22}\right) \neq 0$ and $12\left(-\tilde{A}_1^2 + \tilde{A}_4 - \frac{\tilde{A}_2 \tilde{B}_1}{1+\operatorname{Det}(J_{\mathbf{F}_{r_0}})}\right) \neq 0$, we will have the occurrence of a period-doubling bifurcation at $\mu = 0$, that is, $r = r_0$. If $\mu = 0$, then

$$G(x,0) = -x + \tilde{A}_1 x^2 + \left(-\tilde{A}_4 + \frac{\tilde{A}_2 \tilde{B}_1}{1 + \operatorname{Det}(J_{\mathbf{F}_{r_0}})}\right) x^3 + O\big((|x|)^4\big)$$

and $G'(0) = -1$, so the Schwarzian derivative of $G(x)$ at $x = 0$ is $6\left(-\tilde{A}_1^2 + \tilde{A}_4 - \frac{\tilde{A}_2 \tilde{B}_1}{1+\operatorname{Det}(J_{\mathbf{F}_{r_0}})}\right)$. By Theorem 2, the stability of the coexistence equilibrium is the following:

(i) If $\left(-\tilde{A}_1^2 + \tilde{A}_4 - \frac{\tilde{A}_2 \tilde{B}_1}{1+\operatorname{Det}(J_{\mathbf{F}_{r_0}})}\right) < 0$, then (H^*, P^*) is locally asymptotically stable.

(ii) If $\left(-\tilde{A}_1^2 + \tilde{A}_4 - \frac{\tilde{A}_2 \tilde{B}_1}{1+\operatorname{Det}(J_{\mathbf{F}_{r_0}})}\right) > 0$, then (H^*, P^*) is unstable.

As a final step, in order to determine the stability of the period-two solution that appears after the bifurcation, we apply Theorem 6. In concrete, if these conditions are verified:

$$\begin{aligned}
\left.\frac{\partial G}{\partial \mu}\frac{\partial^2 G}{\partial x^2} + 2\frac{\partial^2 G}{\partial x \partial \mu}\right|_{(0,0)} &= 2\tilde{J}_{11} \neq 0,\\
\tilde{d} := \left.\frac{1}{2}\left(\frac{\partial^2 G}{\partial x^2}\right)^2 + \frac{1}{3}\frac{\partial^3 G}{\partial x^3}\right|_{(0,0)} &= 2\tilde{A}_1^2 - 2\tilde{A}_4 + \frac{2\tilde{A}_2 \tilde{B}_1}{1 + \operatorname{Det}(J_{\mathbf{F}_{r_0}})} \neq 0,
\end{aligned}$$

then the sign of $\tilde{d}$ determines the stability and direction of the bifurcation of the orbits of period 2. The orbits are stable if $\tilde{d}$ is positive; if $\tilde{d}$ is negative, they are unstable.

6.5.2 Neimark-Sacker bifurcation

The Neimark-Sacker bifurcation (the discrete analogue to the Hopf bifurcation in continuous dynamical systems) occurs in discrete dynamical systems when an equilibrium point undergoes a change in its stability, giving rise to the emergence of a closed invariant curve. This phenomenon is characterized by a pair of complex eigenvalues with unit modulus. Depending on whether the bifurcation is supercritical or subcritical, the resulting closed invariant curve will be stable or unstable, respectively (consult [60] for a quick graphic understanding of this bifurcation).

In Part (d) of Theorem J, the second statement shows when the Jacobian matrix of System (6.4) evaluated at the interior equilibrium, $E^* = (H^*, P^*)$, will have a pair of complex eigenvalues with unit modulus. Concretely, this will happen when

$$rH^* + cH^* f'(P^*) \leq 3 \quad \text{and} \quad -cH^* f'(P^*)(1 - rH^*) - \frac{c\big(1 - f(P^*)\big) bH^* f'(bP^*)}{f(bP^*)} = 1,$$

hold. Bearing this in mind, we take the maximum growth rate of the host population r as a bifurcation parameter and set $r = r_0$ such that

$$f'\big(bP^*(r_0)\big) = \frac{e^{(-1+H^*(r_0))r_0}\left(-1 + cH^*(r_0)\left(-1 + H^*(r_0)\, r_0\right) f'\left(P^*(r_0)\right)\right)}{abP^*(r_0)}. \tag{6.36}$$

If we substitute such condition into the Jacobian (6.32), we obtain

$$J_{\mathbf{F}_{r_0}} = J_{\mathbf{F}_r}(0,0)\big|_{r=r_0} = \begin{pmatrix} 1 - r_0 H^*(r_0) & \frac{H^*(r_0)(cH^*(r_0)(r_0 H^*(r_0)-1) f'(P^*(r_0))-1)}{P^*(r_0)} \\ \frac{P^*(r_0)}{H^*(r_0)} & -cH^*(r_0) f'(P^*(r_0)) \end{pmatrix}. \tag{6.37}$$

In the sequel, we analyze the possible occurrence of a Neimark-Sacker bifurcation at the coexistence equilibrium of System (6.4). To achieve that, as a first step, we establish in Lemma 69 under which conditions there exist a pair of conjugate complex eigenvalues for the Jacobian $J_{\mathbf{F}_r(0,0)}$ for r close to r_0 with unit modulus at $r = r_0$. Then, Theorem K apply the mentioned Lemma and Theorem 7 in order to identify the occurrence of a Neimark-Sacker bifurcation at $r = r_0$. For the sake of brevity, in the following result, where we put H^* and P^* we mean $H^*(r_0)$ and $P^*(r_0)$, respectively.

Lemma 69. *Let $\boldsymbol{F}_r$ be defined by (6.31) and having a fixed point at $(0,0)$. Let $r_0 > 0$ satisfy the following:*

(a) $r_0 H^* + cH^* f'(P^*) = 1 - \mathrm{Tr}(J_{\boldsymbol{F}_{r_0}}) \leq 3$, *with* $\mathrm{Tr}(J_{\boldsymbol{F}_{r_0}}) \notin \{-1, 0\}$.

(b)

$$f'\big(bP^*\big) = \frac{e^{(-1+H^*)r_0}\big[-1 + cH^*\big(-1 + r_0 H^*\big) f'(P^*)\big]}{abP^*}. \tag{6.38}$$

(c)

$$f'\big(P^*\big) + \frac{r_0}{c} \cdot \frac{f\big(bP^*\big)\big[\ln\left(af\left(bP^*\right)\right) + r_0\big] - bP^* f'\big(bP^*\big)}{f\big(bP^*\big)\big[\ln\big(af(bP^*)\big) + r_0\big]^2} \neq 0. \tag{6.39}$$

Then, there are conjugate complex eigenvalues $\lambda(r)$, $\bar{\lambda}(r)$ of the Jacobian $J_{\boldsymbol{F}_r(0,0)}$ for r close to r_0, with modulus equal one at $r = r_0$, where

$$\lambda(r_0) = \frac{\mathrm{Tr}(J_{\boldsymbol{F}_{r_0}}) + i\sqrt{4 - \big(\mathrm{Tr}(J_{\boldsymbol{F}_{r_0}})\big)^2}}{2}, \; |\lambda(r_0)| = 1,$$

and $(\lambda(r_0))^k \neq 1$ for $k = 1, 2, 3, 4$. Moreover, $d(r_0) = \frac{d}{dr}|\lambda(r)|\big|_{r=r_0}$ is equal to

$$\frac{b^2 P^{*2}\cdot\ln\big(af(bP^*)\big)\cdot f''(bP^*)}{2f\big(bP^*(r_0)\big)\cdot\big[r_0\big(1+\ln\big(af\,(bP^*)\,\big)+r_0\big)+c\big(\ln\big(af\,(bP^*)\big)+r_0\big)\cdot f'\big(P^*\big)\big]}$$

$$-\frac{\ln\big(af(bP^*)\big)}{r_0\big[1+\ln\big(af(bP^*)\big)+r_0\big]+c\big[\ln\big(af(bP^*)\big)+r_0\big]\cdot f'\big(P^*\big)}$$

$$+\frac{c\big[\ln\big(af(bP^*)\big)\big)+r_0\big]\cdot\big[r_0(1+r_0)+\ln\big(af(bP^*)\big)\big(-1+2\ln\big(af(bP^*)\big)+3r_0\big)\big]\cdot f'\big(P^*\big)}{2r_0\cdot\big[r_0\big(1+\ln(af(bP^*))+r_0\big)+c\big(\ln(af(bP^*))+r_0\big)\cdot f'\big(P^*\big)\big]}$$

$$-\frac{c\ln\big(af(bP^*)\big)\cdot\big[-1+\ln\big(af(bP^*)\big)+r_0\big]\cdot\big[\ln\big(af(bP^*)\big)+r_0\big]\cdot P^* f''\big(P^*\big)}{2r_0\cdot\big[r_0\big(1+\ln(af(bP^*))+r_0\big)+c\big(\ln(af(bP^*))+r_0\big)\cdot f'\big(P^*\big)\big]}$$

$$-\frac{c^2\big[\ln\big(af(bP^*)\big)+r_0\big]^3\cdot\big[-1+\ln\big(af(bP^*)\cdot\big(\ln(af(bP^*))+r_0\big)\big]\cdot f'\big(P^*\big)^2}{2r_0^2\cdot\big[r_0\big(1+\ln(af(bP^*)+r_0\big)+c\big(\ln(af(bP^*)+r_0\big)\cdot f'\big(P^*\big)\big]}. \tag{6.40}$$

Proof. Let us assume that r_0 is a positive real number such that condition (6.38) holds. The characteristic equation of $J_{\mathbf{F}_{r_0}}(0,0)$ is given by $\lambda^2-\mathrm{Tr}\big(J_{\mathbf{F}_{r_0}}(0,0)\big)\lambda+\mathrm{Det}(J_{\mathbf{F}_{r_0}}(0,0))=0$, and we denote by $\lambda(r_0)$ and $\bar\lambda(r_0)$ its solutions. One can easily check that $\mathrm{Det}(J_{\mathbf{F}_{r_0}})=1$. In addition, the conditions from the second statement of Theorem J (d), jointly with Lemma 68 (e), imply $|\mathrm{Tr}(J_{\mathbf{F}_{r_0}})|<2$. Hence,

$$\lambda(r_0)=\frac{\mathrm{Tr}(J_{\mathbf{F}_{r_0}})+i\sqrt{4-\big(\mathrm{Tr}(J_{\mathbf{F}_{r_0}})\big)^2}}{2},\ |\lambda(r_0)|=1.$$

As r varies around r_0, $J_{\mathbf{F}_r}(0,0)$ varies continuously with respect to r. Thus, the eigenvalues of $J_{\mathbf{F}_r}(0,0)$ are conjugate complex for r close to r_0. Obviously, $\lambda(r_0)\neq\pm1$ (otherwise $|\mathrm{Tr}\big(J_{\mathbf{F}_{r_0}}\big)|=2$). Moreover, $\lambda(r_0)=\pm i$ if and only if $\mathrm{Tr}(J_{\mathbf{F}_{r_0}})=0$, and $(\lambda(r_0))^3=1$ if and only if $\mathrm{Tr}(J_{\mathbf{F}_{r_0}})=-1$. Now, we have $\big(\lambda(r_0)\big)^k\neq1$ for $k=1,2,3,4$ if and only if $\mathrm{Tr}(J_{\mathbf{F}_{r_0}})\neq0$ and $\mathrm{Tr}(J_{\mathbf{F}_{r_0}})\neq-1$.

Next, let $H^*=H^*(r)$ and $P^*=P^*(r)$ be functions of r. Due to the fact that $\lambda(r)\cdot\bar\lambda(r)=\mathrm{Det}(J_{\mathbf{F}_r})=1$, we have

$$\begin{aligned}
d(r_0) &= \frac{d}{dr}|\lambda(r)|\bigg|_{r=r_0}=\frac{d}{dr}\sqrt{\mathrm{Det}(J_{\mathbf{F}_r}(0,0))}\bigg|_{r=r_0}=\frac{1}{2}\frac{d\big(\mathrm{Det}(J_{\mathbf{F}_r}(0,0))\big)}{dr}\bigg|_{r=r_0}\\
&= \frac{1}{2}c\big[H^*\cdot\big(-1+r_0H^*\big)\cdot(P^*)'\,f''\big(P^*\big)\big] \qquad (6.41)\\
&\quad+\frac{1}{2}c\big[f'\big(P^*\big)\cdot\big(H^{*2}+(-1+2r_0H^*)\,(H^*)'\big)\big]\\
&\quad+\frac{1}{2}(-a)be^{-r_0(-1+H^*)}\cdot\big[bP^*\,(P^*)'\,f''\big(bP^*\big)\big]\\
&\quad+\frac{1}{2}(-a)be^{-r_0(-1+H^*)}\cdot\big[f'\big(bP^*\big)\cdot\big(-P^*\,(-1+H^*+r_0\,(H^*)')+(P^*)'\big)\big].
\end{aligned}$$

Since

$$G_1\big(P^*(r),r\big):=f\,(P^*(r))+\frac{rP^*(r)}{c\big[\ln\big(af\,(bP^*(r))\big)+r\big]}-1\equiv0,$$

and

$$H^*(r) = 1 + \frac{\ln\left[af\big(bP^*(r)\big)\right]}{r},$$

we have

$$(P^*)'(r) = -\frac{\dfrac{\partial G_1}{\partial r}\big(P^*(r), r\big)}{\dfrac{\partial G_1}{\partial P^*}\big(P^*(r), r\big)} = \frac{P^*(r)f\big(bP^*(r)\big)\ln\big(af\left(bP^*(r)\right)\big)}{D(r)}, \tag{6.42}$$

where

$$\begin{aligned} D(r) \;:=\; & brP^*(r)f'\big(bP^*(r)\big) - cf\big(bP^*(r)\big)f'\big(P^*(r)\big)\big[\ln\big(af\left(bP^*(r)\right)\big) + r\big]^2 \\ & -rf\big(bP^*(r)\big)\big[\ln\big(af\left(bP^*(r)\right)\big) + r\big]. \end{aligned}$$

and

$$(H^*)'(r) = \frac{br\left(P^*\right)'(r)\cdot f'\big(bP^*(r)\big) - f\big(bP^*(r)\big)\cdot\ln\big(af\left(bP^*(r)\right)\big)}{r^2 f\big(bP^*(r)\big)}. \tag{6.43}$$

Notice that by Hypothesis (6.39), $\frac{\partial G_1}{\partial P^*}\big(P^*(r), r\big) \neq 0$ for an r sufficiently close to r_0. By substituting (6.42) and (6.43) in (6.41), we obtain the expression $d(r_0)$. □

It must be highlighted that determining whether $d(r_0)$ is non-zero or not from the above expression is demanding. Nevertheless, in Section 6.7 we gather some numerical simulations for well-known parasitoid escape functions and every case shows that the expression differs from zero.

Finally, we determine the occurrence of a Neimark- Sacker bifurcation at $r = r_0$. As it was pointed out before, we apply the above Lemma and an analysis based on Theorem 7.

Theorem K. *Assume that every assumption of Lemma 69 holds. Let $E^* = (H^*, P^*)$, $d(r_0)$ and $\alpha(r_0)$ be given by (6.6), (6.40) and (6.44), respectively. Then, the interior or coexistence equilibrium point $E^* = (H^*, P^*)$ undergoes a Neimark–Sacker bifurcation at $r = r_0$.*

- *Assume that $d(r_0) > 0$. If $\alpha(r_0) > 0$ ($\alpha(r_0) < 0$) then System (6.4) has an attracting (unstable) closed invariant curve when $r > r_0$ ($r < r_0$) and $r \approx r_0$.*

- *Assume that $d(r_0) < 0$. If $\alpha(r_0) > 0$ ($\alpha(r_0) < 0$) then System (6.4) has an attracting (unstable) closed invariant curve when $r < r_0$ ($r > r_0$) and $r \approx r_0$.*

Proof. We begin rewriting System (6.30) by using Taylor series expansions

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 1 - r_0 H^*(r_0) & \frac{H^*(r_0)(cH^*(r_0)(r_0H^*(r_0)-1)f'(P^*(r_0))-1)}{P^*(r_0)} \\ \frac{P^*(r_0)}{H^*(r_0)} & -cH^*(r_0)\,f'(P^*(r_0)) \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} + \begin{pmatrix} f(x_n, y_n) \\ g(x_n, y_n) \end{pmatrix},$$

where

$$
\begin{aligned}
f(x,y) &= x^2\left(\frac{1}{2}r_0\left(-2+r_0H^*\left(r_0\right)\right)\right)+xy\left(\frac{-1+H^*\left(r_0\right)\left(r_0-c\left(-1+r_0H^*\left(r_0\right)\right)^2f'\left(P^*\left(r_0\right)\right)\right)}{P^*\left(r_0\right)}\right)\\
&\quad+y^2\left(\frac{1}{2}ab^2e^{r_0-r_0H^*(r_0)}H^*\left(r_0\right)f''\left(bP^*\left(r_0\right)\right)\right)-x^3\left(\frac{1}{6}r_0^2\left(-3+r_0H^*\left(r_0\right)\right)\right)\\
&\quad+x^2y\left(\frac{r_0\left(-2+r_0H^*\left(r_0\right)\right)\left(-1+cH^*\left(r_0\right)\left(-1+r_0H^*\left(r_0\right)\right)f'\left(P^*\left(r_0\right)\right)\right)}{2P^*\left(r_0\right)}\right)\\
&\quad-xy^2\left(\frac{1}{2}ab^2e^{r_0-r_0H^*(r_0)}\left(-1+r_0H^*\left(r_0\right)\right)f''\left(bP^*\left(r_0\right)\right)\right)\\
&\quad+y^3\left(\frac{1}{6}ab^3e^{r_0-r_0H^*(r_0)}H^*\left(r_0\right)f^{(3)}\left(bP^*\left(r_0\right)\right)\right)+O\big((|x|+|y|)^4\big),\\
g(x,y) &= -y^2\left(\frac{1}{2}cH^*\left(r_0\right)f''\left(P^*\left(r_0\right)\right)\right)-xy\left(cf'\left(P^*\left(r_0\right)\right)\right)-xy^2\left(\frac{1}{2}cf''\left(P^*\left(r_0\right)\right)\right)\\
&\quad-y^3\left(\frac{1}{6}cH^*\left(r_0\right)f^{(3)}\left(P^*\left(r_0\right)\right)\right)+O\big((|x|+|y|)^4\big).
\end{aligned}
$$

For the sake of simplicity, we write

$$
\begin{aligned}
f(x,y) &= A_{20}x^2+A_{11}xy+A_{02}y^2+A_{21}x^2y+A_{12}xy^2+A_{30}x^3+A_{03}y^3+O\big((|x|+|y|)^4\big),\\
g(x,y) &= B_{11}xy+B_{02}y^2+B_{12}xy^2+B_{03}y^3+O\big((|x|+|y|)^4\big).
\end{aligned}
$$

Let us denote

$$
J_{\mathbf{F}_{r_0}}(0,0) =: \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.
$$

Then, the eigenvalues of $J_{\mathbf{F}_{r_0}}$ are given by $\lambda=\mu+\omega i$ and $\bar{\lambda}=\mu-\omega i$, where $\omega>0$ and $\mu^2+\omega^2=1$. Let us consider

$$
P=\begin{pmatrix} a_{12} & 0 \\ \mu-a_{11} & -\omega \end{pmatrix},
$$

and compute its inverse

$$
P^{-1}=\frac{-1}{a_{12}\cdot\omega}\begin{pmatrix} -\omega & 0 \\ a_{11}-\mu & a_{12} \end{pmatrix}.
$$

By using the transformation

$$
\begin{pmatrix} x_n \\ y_n \end{pmatrix} = P\begin{pmatrix} \tilde{x}_n \\ \tilde{y}_n \end{pmatrix},
$$

we have

$$
\begin{pmatrix} \tilde{x}_{n+1} \\ \tilde{y}_{n+1} \end{pmatrix} = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix}\begin{pmatrix} \tilde{x}_n \\ \tilde{y}_n \end{pmatrix} + \begin{pmatrix} f_1(\tilde{x}_n,\tilde{y}_n) \\ g_1(\tilde{x}_n,\tilde{y}_n) \end{pmatrix},
$$

where $f_1(x,y)=f(x,y)/a_{12}$ and $g_1(x,y)=\left((\mu-a_{11})f(x,y)-a_{12}g(x,y)\right)/(a_{12}\omega)$. Substituting x by $a_{12}\tilde{x}$ and y by $(\mu-a_{11})\tilde{x}-\omega\tilde{y}$, we get

$$
\begin{aligned}
f_1(\tilde{x},\tilde{y}) =& \tilde{x}^2\left(\frac{(\mu-a_{11})^2A_{02}}{a_{12}}+(\mu-a_{11})A_{11}+a_{12}A_{20}\right)+\tilde{x}\tilde{y}\left(\frac{2\omega(-\mu+a_{11})A_{02}}{a_{12}}-\omega A_{11}\right)\\
&+\tilde{y}^2\left(\frac{\omega^2A_{02}}{a_{12}}\right)+\tilde{u}\tilde{y}^2\left(\omega^2\left(\frac{3(\mu-a_{11})A_{03}}{a_{12}}+A_{12}\right)\right)\\
&+\tilde{x}^2\tilde{y}\left(\omega\left(-\frac{3(\mu-a_{11})^2A_{03}}{a_{12}}+2(-\mu+a_{11})A_{12}-a_{12}A_{21}\right)\right)\\
&+\tilde{x}^3\left(\frac{(\mu-a_{11})^3A_{03}}{a_{12}}+(\mu-a_{11})^2A_{12}+(\mu-a_{11})a_{12}A_{21}+a_{12}^2A_{30}\right)\\
&-\tilde{y}^3\left(\frac{\omega^3A_{03}}{a_{12}}\right)+O\big((|\tilde{x}|+|\tilde{y}|)^4\big),
\end{aligned}
$$

$$
\begin{aligned}
g_1(\tilde{x},\tilde{y}) =& \tilde{x}^2\left(\frac{(\mu-a_{11})((\mu-a_{11})^2A_{02}+a_{12}((\mu-a_{11})(A_{11}-B_{02})+a_{12}(A_{20}-B_{11})))}{\omega a_{12}}\right)\\
&+\tilde{x}\tilde{y}\left(-\frac{2(\mu-a_{11})^2A_{02}}{a_{12}}-(\mu-a_{11})(A_{11}-2B_{02})+a_{12}B_{11}\right)+\tilde{y}^2\left(\frac{\omega(\mu-a_{11})A_{02}}{a_{12}}-\omega B_{02}\right)\\
&-\tilde{x}^2\tilde{y}\left(\frac{(\mu-a_{11})(3(\mu-a_{11})^2A_{03}+a_{12}((\mu-a_{11})(2A_{12}-3B_{03})+a_{12}(A_{21}-2B_{12})))}{a_{12}}\right)\\
&+\tilde{x}\tilde{y}^2\left(\omega\left(\frac{3(\mu-a_{11})^2A_{03}}{a_{12}}+(\mu-a_{11})(A_{12}-3B_{03})-a_{12}B_{12}\right)\right)+\tilde{x}^3\left[\frac{(\mu-a_{11})^4A_{03}}{\omega a_{12}}\right.\\
&\left.+\frac{(\mu-a_{11})((\mu-a_{11})^2A_{12}+a_{12}^2A_{30}-(\mu-a_{11})^2B_{03}+(\mu-a_{11})a_{12}(A_{21}-B_{12}))}{\omega}\right]\\
&+\tilde{y}^3\left(\omega^2\left(\frac{(-\mu+a_{11})A_{03}}{a_{12}}+B_{03}\right)\right)+O\big((|\tilde{x}|+|\tilde{y}|)^4\big).
\end{aligned}
$$

Finally, by Theorem 7, we are able to determine the direction of the Neimark-Sacker bifurcation by computing the following relation

$$
\alpha(r_0)=\mathrm{Re}\left[\frac{(1-2\lambda(r_0))(\bar{\lambda}(r_0))^2}{1-\lambda(r_0)}\xi_{11}\xi_{20}\right]+\frac{1}{2}|\xi_{11}|^2+|\xi_{02}|^2-\mathrm{Re}\big(\bar{\lambda}(r_0)\xi_{21}\big), \tag{6.44}
$$

where Re denotes the real part of a complex number and

$$
\begin{aligned}
\xi_{20} =& \frac{1}{8}\left\{(g_1)_{\tilde{x}\tilde{x}}-(g_1)_{\tilde{y}\tilde{y}}+2(g_2)_{\tilde{x}\tilde{y}}+i\left[(g_2)_{\tilde{x}\tilde{x}}-(g_2)_{\tilde{y}\tilde{y}}-2(g_1)_{\tilde{x}\tilde{y}}\right]\right\}\\
=&\frac{i\left(A_{11}\left(a_{11}\left(a_{11}-2\mu\right)+|\lambda|\right)+a_{12}\left(A_{20}\left(\bar{\lambda}-a_{11}\right)+B_{11}\left(a_{11}-\lambda\right)\right)-B_{02}(\lambda-a_{11})^2\right)}{4\omega}\\
&+\frac{iA_{02}\left(\lambda-a_{11}\right)^2\left(\bar{\lambda}-a_{11}\right)}{4a_{12}\omega},\\
\xi_{11} =& \frac{1}{4}\left\{(g_1)_{\tilde{x}\tilde{x}}+(g_1)_{\tilde{y}\tilde{y}}+i\left[(g_2)_{\tilde{x}\tilde{x}}+(g_2)_{\tilde{y}\tilde{y}}\right]\right\}\\
=&\frac{i\left((\bar{\lambda}-a_{11})\cdot\left(A_{11}\left(\mu-a_{11}\right)+a_{12}A_{20}+B_{02}(a_{11}-\lambda)\right)+a_{12}B_{11}(a_{11}-\mu)\right)}{4\omega}\\
&+\frac{iA_{02}\left(\lambda-a_{11}\right)\left(\bar{\lambda}-a_{11}\right)^2}{2a_{12}\omega},
\end{aligned}
$$

$$
\begin{aligned}
\xi_{02} &= \frac{1}{8}\left\{(g_1)_{\tilde{x}\tilde{x}} - (g_1)_{\tilde{y}\tilde{y}} - 2(g_2)_{\tilde{x}\tilde{y}} + i\left[(g_2)_{\tilde{x}\tilde{x}} - (g_2)_{\tilde{y}\tilde{y}} + 2(g_1)_{\tilde{x}\tilde{y}}\right]\right\} \\
&= \frac{i\left(\bar{\lambda} - a_{11}\right)\cdot\left(a_{12}\left(\left(\bar{\lambda} - a_{11}\right)\left(A_{11} - B_2\right) + a_{12}\left(A_{20} - B_{11}\right)\right) + A_{02}(\bar{\lambda} - a_{11})^2\right)}{4a_{12}\omega}, \\
\xi_{21} &= \frac{1}{16}\Big\{(g_1)_{\tilde{x}\tilde{x}\tilde{x}} + (g_1)_{\tilde{x}\tilde{y}\tilde{y}} + (g_2)_{\tilde{x}\tilde{x}\tilde{y}} + (g_2)_{\tilde{y}\tilde{y}\tilde{y}} \\
&\quad + i\Big[(g_2)_{\tilde{x}\tilde{x}\tilde{x}} + (g_2)_{\tilde{x}\tilde{y}\tilde{y}} - (g_1)_{\tilde{x}\tilde{x}\tilde{y}} - (g_1)_{\tilde{y}\tilde{y}\tilde{y}}\Big]\Big\} \\
&= \frac{i\left(\bar{\lambda} - a_{11}\right)\cdot\left(A_{12}\left(a_{11} - \lambda\right)\cdot\left(3a_{11} - 3\mu + i\omega\right) + a_{12}A_{21}(-3a_{11} + 3\mu + i\omega)\right)}{8\omega} \\
&\quad + \frac{3i\left(\bar{\lambda} - a_{11}\right)\cdot\left(a_{12}^2 A_{30} - B_{03}(\lambda - a_{11})^2\right)}{8\omega} + \frac{a_{12}B_{12}(a_{11} - \lambda)\cdot(-3ia_{11} + 3i\mu + \omega)}{8\omega} \\
&\quad + \frac{3iA_{03}\cdot\left(a_{11}(a_1 1 - 2\mu) + |\lambda|\right)^2}{8a_{12}\omega}.
\end{aligned}
$$

where $\lambda = \lambda(r_0)$ and $\bar{\lambda} = \mu - \omega i$.

□

As it happens with the computation of the expression $d(r_0)$, determining the sign of $\alpha(r_0)$ is challenging. However, the numerical simulations developed in Section 6.7 show that supercritical ($\alpha(r_0) > 0$), subcritical ($\alpha(r_0) < 0$), and Chenciner ($\alpha(r_0) = 0$) bifurcation appear for specific parameter values.

6.6 Permanence

From a biological point of view, one of the most relevant aspects resides in determining under what circumstances it can be assured the prolonged survival of all species. In this direction, this section focuses on analyzing the indefinite coexistence of the host and parasitoid populations without environmental changes, from where we will be able to deduce that they are ecologically permanent, see [3].

It should be highlighted that the previous study related to the stability of the equilibrium points is not enough to ensure the long term survival of the populations due to the fact that they are local properties. In this sense, if the interior equilibrium is unstable -see Theorem J (d)- it does not mean that any population will vanish; just like being locally asymptotical stable does not imply the permanence of the system in the long term. This study only allows us to conclude the behaviour of our populations in a neighbourhood of the equilibrium points and global results will be needed to describe the long term behaviour of the species.

Although we have presented global results for the extinction and exclusion equilibrium points in Section 6.3, giving a global result for the coexistence equilibrium is very demanding since we cannot establish explicitly the expression of the equilibrium. Therefore, to determine concrete conditions under which the interior equilibrium is globally asymptotical stable is a tough task. Here, we delve with the permanence of System (6.4) and we give under what circumstances both populations, host and parasitoid, will indefinitely coexist in the long term.

6.6.1 Notation and preliminary results

As a first step, we establish the mathematical terminology that will be used to analyze the permanence of the system. In this line, we use the notation from [55].

We say that an n species system is permanent if every orbit with nonzero initial conditions remain in a fixed, bounded region M of the phase space at a nonzero distance from the coordinate axes. Moreover, System (6.4) is considered permanent (for ∂X) if there is a compact set $M \subset X$ such that the minimum distance between M and ∂X is positive, where ∂X denotes the boundary of the set X. For every initial value in $intX$ (the interior of X) the orbits enter and remain in M. Also, a set M from a dynamical system (X, F) is called forward invariant if every orbit starting from a point of M remains in M, and by the omega limit set of M we understand the union $\Omega(M) = \cup_{x\in M}\Omega(x)$, where $\Omega(x)$ represents the accumulation points of the orbit of x under F.

Observe that the permanence of the system is compatible with the occurrence of any asymptotic behaviour, for instance, the appearance of strange attractors, chaotic orbits, the existence of a closed invariant curve (Neimark-Sacker bifurcation)... In addition, since the permanence covers every orbit with nonzero initial conditions, it is a global property, see [3].

Recall that, in Section 6.1, the uniform boundedness of System (6.4) was proved. Concretely, we showed that T^2 maps $[0,+\infty) \times [0,+\infty)$ into $[0, \frac{ae^{r-1}}{r}] \times [0, \frac{ace^{r-1}}{r}]$, and

$$T\left(\left[0, \frac{ae^{r-1}}{r}\right] \times \left[0, \frac{ace^{r-1}}{r}\right]\right) \subseteq \left[0, \frac{ae^{r-1}}{r}\right] \times \left[0, \frac{ace^{r-1}}{r}\right].$$

Hence, for System (6.4) the compact set $Y = [0, ae^{r-1}/r] \times [0, cae^{r-1}/r] \subset [0,+\infty) \times [0,+\infty)$ has the property that each orbit with nonzero initial values in $[0,+\infty) \times [0,+\infty)$ enters and remains in Y.

Now, we gather some results from [51], Theorem 27 and Corollary 17, that give sufficient conditions to ensure the permanence of a discrete dynamical system.

Theorem 27. *[51, Theorem 2.2] Consider the system*

$$Z_{n+1} = T(Z_n), Z_i \in \mathbb{R}^n_+, i \geq 0. \tag{6.45}$$

Assume that X is compact and that S is a compact subset of X with empty interior. Let S and $X \setminus S$ be forward invariant. Suppose that there is a continuous function $p : X \to \mathbb{R}_+$ which satisfies the following conditions

(a) $p(Z) = 0 \Leftrightarrow Z \in S$,

(b) $\sup_{n\geq 0} \liminf_{\substack{Z_0\to Z\\ Z_0\in X\setminus S}} \dfrac{p\big(T^n(Z_0)\big)}{p(Z_0)} > 1 \quad (Z \in S)$.

Then there is a compact forward invariant set M with distance $d(M,S) > 0$ which is such that every orbit in $X \setminus S$ enters and remains in M.

Corollary 17. *[51, Corollary 2.3] The conclusion of Theorem 27 remains true if, instead of (b), it is assumed that*

$$\sup_{n\geq 0} \liminf_{\substack{Z_0\to Z\\ Z_0\in X\setminus S}} \frac{p\big(T^n(Z_0)\big)}{p(Z_0)} > \begin{cases} 1, & Z \in \overline{\Omega(S)} \\ 0, & Z \in S \end{cases},$$

where $\Omega(S)$ is the omega limit set of S with respect to System (6.45).

Finally, we present a result concerning the averaging property for a discrete dynamical system, [49, Lemma 2.4]. This Lemma is inspired in the well-known conservation law noticed by Volterra in [102] that states that the averages of populations of prey and predator converge to the equilibrium point of the continuous system. Here, we define the concept of average population vector and give the result from [49] adapting them to System (6.4), where only two populations, (H_n) and (P_n), are involved.

Define the average population vector over n generations $\bar{X}_n = \big(\bar{H}_n, \bar{P}_n\big)$ by setting

$$\bar{H}_n = \frac{1}{n}\sum_{k=0}^{n-1} H_k \quad \text{and} \quad \bar{P}_n = \frac{1}{n}\sum_{k=0}^{n-1} P_k.$$

Lemma 70. *[49, Lemma 2.4] Assume that $H_n, P_n > 0$ for $n \geq 0$. Suppose that there are real numbers $b, b' > 0$, and a sequence $(k_j)_j$ diverging to infinity such that $b < H_{k_j} < b'$, $b < P_{k_j} < b'$ for every $j \geq 0$. Then, there are a subsequence, again denoted by $(k_j)_j$, and an equilibrium point X^* such that*

$$\lim_{j\to\infty} \bar{X}_{k_j} = X^*.$$

6.6.2 Permanence of the model

In the sequel, we assume that $f'_+(0)$ and $f''_+(0)$ exist and we will apply Theorem 27 and Corollary 17 in order to obtain sufficient conditions for the permanence of System (6.4).

For the compact set $Y = [0, ae^{r-1}/r] \times [0, cae^{r-1}/r]$, we take the following subsets:

$$S_1 = \{(H,P) \in Y \colon H = 0\} \quad \text{and} \quad S_2 = \{(H,P) \in Y \colon P = 0\}.$$

We define the following continuous functions $p_i : Y \to \mathbb{R}_+, i = 1, 2,$ by $p_1\big((H,P)\big) = H$ and $p_2\big((H,P)\big) = P$ (observe that p_i are continuous functions that satisfy condition (a) in Theorem 27). Set

$$\sigma_i(Z) = \sup_{n\geq 0} \liminf_{\substack{Z_0\to Z\\ Z_0\in X\setminus S}} \frac{p_i\big(T^n(Z_0)\big)}{p_i(Z_0)} \quad \text{for} \quad i = 1, 2,$$

where T is given by (6.5).

Lemma 71. *Under the above considerations, it holds*

$$\sigma_1(Z) = \sup_{n\geq 0} \prod_{i=0}^{n-1} \big(ae^r f(bv_i)\big) \quad \text{for every } Z \in S_1,$$

and

$$\sigma_2(Z) = \sup_{n\geq 0} \prod_{i=0}^{n-1} \big(-cu_i f'_+(0)\big) \quad \text{for every } Z \in S_2.$$

Proof. We can compute $\sigma_i(Z)$, $i = 1, 2$, by using the equations in System (6.4). Hence, if $Z \in S_1$,

$$\begin{aligned}
\sigma_1(Z) &= \sup_{n\geq 0} \liminf_{\substack{Z_0\to Z\\ Z_0\in X\setminus S}} \frac{p_1\big(T^n(Z_0)\big)}{p_1(Z_0)} = \sup_{n\geq 0} \liminf_{\substack{Z_0\to Z\\ Z_0\in X\setminus S}} \frac{p_1\big(T^n((H_0,P_0))\big)}{p_1\big((H_0,P_0)\big)} \\
&= \sup_{n\geq 0} \liminf_{\substack{Z_0\to Z\\ Z_0\in X\setminus S}} \frac{H_n}{H_0} = \sup_{n\geq 0} \liminf_{\substack{Z_0\to Z\\ Z_0\in X\setminus S}} \frac{H_n}{H_{n-1}} \cdots \frac{H_1}{H_0} \\
&= \sup_{n\geq 0} \liminf_{\substack{Z_0\to Z\\ Z_0\in X\setminus S}} \prod_{i=0}^{n-1} \frac{aH_i e^{r(1-H_i)} f(bP_i)}{H_i} = \sup_{n\geq 0} \prod_{i=0}^{n-1} \big(ae^r f(bv_i)\big),
\end{aligned}$$

and if $Z \in S_2$, considering that $\lim_{P_i\to 0} \frac{1-f(P_i)}{P_i} = -f'_+(0)$, we obtain

$$\begin{aligned}
\sigma_2(Z) &= \sup_{n\geq 0} \liminf_{\substack{Z_0\to Z\\ Z_0\in X\setminus S}} \frac{p_2\big(T^n(Z_0)\big)}{p_2(Z_0)} = \sup_{n\geq 0} \liminf_{\substack{Z_0\to Z\\ Z_0\in X\setminus S}} \frac{p_2\big(T^n((H_0,P_0))\big)}{p_2\big((H_0,P_0)\big)} \\
&= \sup_{n\geq 0} \liminf_{\substack{Z_0\to Z\\ Z_0\in X\setminus S}} \frac{P_n}{P_0} = \sup_{n\geq 0} \liminf_{\substack{Z_0\to Z\\ Z_0\in X\setminus S}} \frac{P_n}{P_{n-1}} \cdots \frac{P_1}{P_0} \\
&= \sup_{n\geq 0} \liminf_{\substack{Z_0\to Z\\ Z_0\in X\setminus S}} \prod_{i=0}^{n-1} \frac{cH_i(1-f(P_i))}{P_i} = \sup_{n\geq 0} \prod_{i=0}^{n-1} \big(-cu_i f'_+(0)\big),
\end{aligned}$$

where $Z_0 = (H_0, P_0)$, $Z = (u_0, v_0)$, $(H_i, P_i) = T^i\big((H_0, P_0)\big)$, and $(u_i, v_i) = T^i\big((u_0, v_0)\big)$. □

As a next step, after computing the functions σ_i, we obtain sufficient conditions for the permanence of the system. We do so by following the method used in [55].

Recall by Theorem 22 that the extinction equilibrium E_0 is globally asymptotically stable when $ae^r \leq 1$, therefore it makes no sense to study the permanence of the system in this scenario.

Theorem 28. *Assume that $ae^r > 1$. System (6.4) is permanent if the following condition holds:*

$$\sigma_2(Z) > 1 \quad \text{for any } Z \in \Omega\big(S_2 \setminus \{(0,0)\}\big).$$

Proof. Observe that the second equation from System (6.4) implies $\overline{\Omega(S_1)} = \Omega(S_1) = \{(0,0)\}$. As a consequence, for $\widetilde{Z} = (0,0)$ we have $\sigma_1\big(\widetilde{Z}\big) = \sup_{n\geq 0}(ae^r)^n > 1$. Furthermore, for any $Z \in S_1$, bearing in mind that $v_j = 0$ for $j \geq 1$, we have for $n \geq 1$ that

$$\sigma_1(Z) = \sup_{n\geq 0} \prod_{i=0}^{n-1} \big(ae^r f(bv_i)\big) = \sup_{n\geq 0} \, (ae^r)^n f(bv_0) \geq ae^r f(bv_0) > 0.$$

In conclusion, the assumptions of Corollary 17 are satisfied (notice also that $p_1(Z) = 0$ if and only if $Z \in S_1$), and consequently, there exists a compact $M_1 \subset Y$, such that the minimum distance between M_1 and S_1 is positive, and for every initial value in $Y \setminus S_1$ the orbits enter and remain in M_1.

Now, consider the behaviour of the orbits in M_1. From the hypothesis, $\sigma_2(Z) > 1$ for any

$$Z \in \overline{\Omega(S_2 \cap M_1)} = \Omega(S_2 \cap M_1) \subseteq \Omega\big(S_2 \setminus \{(0,0)\}\big).$$

Also, since $(0,0) \notin M_1$ and $d(M_1 \cap S_2, S_1) > 0$, it is direct to check that

$$\sigma_2(Z) = \sup_{n\geq 0} \big(-cf'_+(0)\big)^n \prod_{i=0}^{n-1} u_i > 0 \quad \text{for any } Z \in S_2 \cap M_1.$$

Now, take $X = M_1$, $S = S_2 \cap M_1$ and $p = p_2$, and apply Corollary 17 to ensure that there is a compact set $M_2 \subset M_1$ such that the minimum distance between M_2 and $S_2 \cap M_1$ is positive, and for every initial value in $M_1 \setminus S_2$ the orbits enter and remain in M_2. This completes the proof of the theorem. $\square$

Observe that the sufficient condition given in Theorem 28 is not easy to verify. Therefore, we provide alternative conditions that are easier to check. To achieve this, we distinguish two cases depending on the value of ae^r.

Theorem L. *If $1 < ae^r < e^2$ and $-c\left(1 + \frac{\ln(a)}{r}\right) f'_+(0) > 1$, then System (6.4) is permanent.*

Proof. It is well-known that $u^* = 1 + \frac{\ln(a)}{r}$ is a global attractor for the Ricker difference equation $u_{n+1} = au_n e^{r(1-u_n)}$, if $1 < ae^r < e^2$ (see Chapter 1). Therefore, on the set S_2, for $Z = \left(1 + \frac{\ln(a)}{r}, 0\right)$, we obtain

$$\sigma_2(Z) = \sup_{n\geq 0} \prod_{i=0}^{n-1} -cf'_+(0)u_i = \sup_{n\geq 0} \left(-cf'_+(0)\left(1 + \frac{\ln(a)}{r}\right)\right)^n.$$

Since $-c\left(1 + \frac{\ln(a)}{r}\right) f'_+(0) > 1$ and $1 < ae^r < e^2$, we have $\sigma_2(Z) > 1$. $\square$

Next, we focus on the case $ae^r \geq e^2$.

Lemma 72. *Let $\big((H_i,P_i)\big)_{i=0}^{\infty}$ be a solution of System (6.4) with $H_0>0$ and $P_0=0$. If $ae^r\geq e^2$, then*

$$0<h\left(ae^{r-1}/r\right)\leq \liminf_{i\to\infty} H_i\leq \limsup_{i\to\infty} H_i\leq ae^{r-1}/r,$$

where $h(x)=axe^{r(1-x)}$. Furthermore, $h(I)\subseteq I$ where $I=\big[h(ae^{r-1}/r),ae^{r-1}/r\big]$.

Proof. For $P_0=0$ and $H_0>0$, System (6.4) reduces to the Ricker map $H_{n+1}=h(H_n)=aH_ne^{r(1-H_n)}$. It is well-known that, if $1<ae^r<e^2$, then $\lim_{n\to\infty}H_n=H_0^*=1+\dfrac{\ln(a)}{r}$.

Assume that $ae^r\geq e^2$. Since $ae^r\geq e^2>e$, we get $1/r<H_0^*=h(H_0^*)\leq ae^{r-1}/r$. By the monotonicity of function h, we have $h(ae^{r-1}/r)<h(H_0^*)=H_0^*$. It is straightforward to check that $h(H)>H$ for $H\in(0,H_0^*)$, which implies $h^2(ae^{r-1}/r)>h(ae^{r-1}/r)$ and $h(I)\subset I$. If $h(ae^{r-1}/r)>1/r$, then for $J=[1/r,h(ae^{r-1}/r)]$ we have $h(J)=[h^2\left(ae^{r-1}/r\right),ae^{r-1}/r]\subset I$.

Now, assume that $H_0\in(0,1/r)$. Since $h(H)>H$ for $H\in(0,1/r)$, we achieve that $H_{n+1}>H_n$ if $H_n\in(0,1/r)$. If $H_n\in(0,1/r)$ for all $n>0$, then the sequence (H_n) is increasing and bounded from above by $1/r$, so it would converge to the equilibrium point $H_0^*>1/r$, which is a contradiction. Hence, there exists n_0 such that $H_{n_0}\geq 1/r$. Thus, for every $H_0\in(0,ae^{r-1}/r]$, there exists n_0 such that $H_n\in I$ for all $n>n_0$, i.e. $\Omega\big((H_n)_{n\geq 0}\big)\subset[h(ae^{r-1}/r),ae^{r-1}/r]$, from which the proof follows. □

As a next step, we employ the averaging property established in Lemma 70.

Lemma 73. *Let $\big((H_i,P_i)\big)_{i=0}^{\infty}$ be a solution of System (6.4) such that $H_0>0$. Suppose that there are real numbers $\rho_1,\rho_2>0$, and $n_0>0$ such that $\rho_1\leq H_i\leq\rho_2$, and $0\leq P_i\leq\rho_2$ for all $i>n_0$. Then, there exists a sequence (n_i) such that*

$$1+\frac{\ln(a)}{r}=\lim_{i\to\infty}\frac{\sum_{j=0}^{n_i-1}H_j}{n_i}-\frac{1}{r}\lim_{i\to\infty}\frac{\sum_{j=0}^{n_i-1}\ln\big(f(bP_j)\big)}{n_i}.$$

Proof. Let $\bar H_i=\dfrac{\sum_{j=0}^{i-1}H_j}{i}$ and $\bar P_i=\dfrac{\sum_{j=0}^{i-1}\ln\big(f(bP_j)\big)}{i}$. Due to the fact that the sequence $\big(\bar H_i\big)$ is bounded, by compactness there exists a convergent subsequence $\big(\bar H_{n_i}\big)$. Analogously, the boundedness of $\big(\bar P_{n_i}\big)$ implies the existence of a convergent subsequence, $\big(\bar P_{n_i}\big)$, again denoted by (n_i). Thus, we have

$$\begin{aligned}\frac{1}{n_i}\ln\frac{H_{n_i}}{H_0}&=\frac{1}{n_i}\sum_{j=1}^{n_i}\ln\frac{H_j}{H_{j-1}}=\frac{1}{n_i}\sum_{j=0}^{n_i-1}\ln\left(ae^{r(1-H_j)}f(bP_j)\right)\\&=\ln(a)+r-r\frac{\sum_{j=0}^{n_i-1}H_j}{n_i}+\frac{\sum_{j=0}^{n_i-1}\ln\big(f(bP_j)\big)}{n_i}=:\alpha(n_i).\end{aligned}$$

Observe that $\lim_{i\to\infty}\frac{1}{n_i}\ln\frac{H_{n_i}}{H_0}=0$, so the right-hand side of the above expression converges to zero too. This means that $\lim\limits_{i\to\infty}\alpha(n_i)=0$, which completes the proof of the lemma. □

Lemma 74. *Assume that* $0 < \rho_0 < ae^{r-1}/r$. *Let*

$$A = \frac{r\left(\ln\left(\frac{ae^{r-1}}{r}\right) - \ln(\rho_0)\right)}{ae^{r-1} - \rho_0 r},$$
$$B = \frac{ae^{r-1}\ln(\rho_0) - \rho_0 r \ln\left(\frac{ae^{r-1}}{r}\right)}{ae^{r-1} - \rho_0 r}$$

be the solutions of the system

$$\begin{cases} A\rho_0 + B = \ln(\rho_0) \\ Aae^{r-1}/r + B = \ln\left(ae^{r-1}/r\right). \end{cases} \tag{6.46}$$

Then, $\ln u \geq Au + B$ *for* $\rho_0 \leq u \leq ae^{r-1}/r$.

Proof. Let $f_1(u) = \ln u$, $f_2(u) = Au + B$ and $F(u) = f_1(u) - f_2(u)$ for $u > 0$. We have $F''(u) = f_1''(u) < 0$ for $u > 0$. Since system (6.46) holds, we find $F(\rho_0) = F(ae^{r-1}/r) = 0$; on the other hand, $F(0) = F(\infty) = -\infty$ and F is concave. So, it is easily seen that $F(u) \geq 0$, or $\ln u \geq Au + B$ for $\rho_0 \leq u \leq ae^{r-1}/r$. □

We finish this section by proving the permanence of System (6.4) when $ae^r \geq e^2$.

Theorem M. *Assume that* $ae^r \geq e^2$ *and* $-cf_+'(0) > e^{-A\left(\frac{\ln(a)}{r}+1\right)-B}$, *where* A *and* B *are given by* (6.46) *with* $\rho_0 = h\left(\frac{ae^{r-1}}{r}\right)$, h *being the Ricker map,* $h(x) = axe^{r(1-x)}$, $x \geq 0$. *Then, System* (6.4) *is permanent.*

Proof. Assume that $P_0 = 0$ and $H_0 > 0$. Lemma 72 ensures that

$$\Omega\big(S_2 \setminus \{(0,0)\}\big) \subset M_3 := \{(H,0) \in S_2 \colon H \in I\},$$

where $I = \left[h(ae^{r-1}/r), ae^{r-1}/r\right]$. Moreover, from Lemma 74 we get $\ln(H) \geq AH + B$ for $H \in I$, and taking into account Lemma 73, now with $P_j = 0$ for all $j \in \mathbb{N}$, we obtain for any $Z = (H_0, 0) \in M_3$, that there exists a subsequence (n_i) such that

$$1 + \frac{\ln(a)}{r} = \lim_{i\to\infty} \frac{\sum_{j=0}^{n_i-1} H_j}{n_i}.$$

Then, from here and the assumption of the theorem we have

$$-cf_+'(0) \lim_{i\to\infty} \exp\left[A\frac{\sum_{j=0}^{n_i-1} H_j}{n_i} + B\right] > 1.$$

Therefore, for any $Z \in M_3$, by using Lemma 74 for $\rho_0 = h(ae^{r-1}/r)$ and considering σ_2

from Lemma 71, we have

$$\begin{aligned}\sigma_2(Z) &= \sup_{n\geq 0}\prod_{i=0}^{n-1} -cf'_+(0)H_i \geq \sup_{j\geq 0}\left(-cf'_+(0)\left(\prod_{i=0}^{n_j-1} H_i\right)^{\frac{1}{n_j}}\right)^{n_j}\\ &= \sup_{j\geq 0}\left(-cf'_+(0)\exp\left[\frac{\sum_{i=0}^{n_j-1}\ln(H_i)}{n_j}\right]\right)^{n_j}\\ &\geq \sup_{j\geq 0}\left(-cf'_+(0)\exp\left[\frac{\sum_{i=0}^{n_j-1} AH_i+B}{n_j}\right]\right)^{n_j}\\ &= \sup_{j\geq 0}\left(-cf'_+(0)\exp\left[A\frac{\sum_{i=0}^{n_j-1} H_i}{n_j}+B\right]\right)^{n_j} > 1.\end{aligned}$$

Since $\Omega\big(S_2 \setminus \{(0,0)\}\big) \subset M_3$, the proof is a direct consequence of Theorem 28. □

6.7 Numerical simulations

Once that we have developed our theoretical study concerning the dynamics of System (6.4) for an arbitrary probability function f verifying the conditions ($\star$), we compute some numerical simulations for concrete functions, namely, $f(y) = \left(1+\frac{y}{m}\right)^{-m}$, with $m > 0$; and $f(y) = e^{-y^m}$, with $0 < m \leq 1$. Here, we see that both particular models exhibit rich and complex dynamics that can produce exciting structures such as rank-one and strange attractors [94]. Moreover, the numerical simulations that we are going to present are coherent with the theoretical results obtained in the preceding sections. We have arbitrarily selected the numerical values for simulations without relying on any specific research field. All the numerical simulations presented here have been developed using the mathematical software *Mathematica*.

6.7.1 Example 1

The first model that we are going to simulate is based on May's paper [80], where the probability function is $f(y) = \left(1+\frac{y}{m}\right)^{-m}$, with $m > 0$. Then, the corresponding generalized model is given by

$$\begin{cases} H_{n+1} = aH_n e^{r(1-H_n)}\left(1+\dfrac{bP_n}{m}\right)^{-m} \\ P_{n+1} = cH_n\left(1-\left(1+\dfrac{P_n}{m}\right)^{-m}\right)\end{cases}, \quad m>0, \tag{6.47}$$

where the parameters a, b and c are positive real numbers defined as in System (6.4) and the new parameter m represents the aggregation of parasitoid attacks. It is worth mentioning that if the value of m is close to zero, the parasitoid's attacks are concentrated in a concrete area, while for large values of m, the concentration of attacks is almost uniform [80]. Moreover, by straightforward calculations, it is easy to check that the probability function satisfies the conditions ($\star$):

$$f'(y) = -\left(\frac{y}{m}+1\right)^{-m-1} < 0 \text{ for } y > 0; \quad f(0) = 1;, \quad \lim_{y\to+\infty} f(y) = 0;$$

$$f'_+(0) = -1; \quad \lim_{y\to 0^+} yf'(y) = 0; \quad \text{and} \quad f''(y) = \frac{(m+1)\left(\frac{y}{m}+1\right)^{-m-2}}{m} > 0 \text{ for } y > 0.$$

Take the parameter values $a = 1.15$, $b = 1.1$ and $m = 1.2$. Then, by Theorem 19, if $0 < c < 1$ and $r \geq \frac{\ln(1.15)c}{1-c}$, System (6.47) has no interior equilibrium point. On the other hand, if $c \geq 1$, there exists a unique coexistence equilibrium point for each $r > 0$. Set $c = 1.62$, thus System (6.47) undergoes Neimark-Sacker bifurcation for $r_0 = 0.03695167557933803$. See Figure 6.4 (vii). In this scenario, the unique interior equilibrium is given by $(H^*, P^*) = (0.7002789797899277, 0.14611552210352347)$. The characteristic equation associated with the Jacobian matrix at (H^*, P^*) is $\lambda^2 - 1.85518\lambda + 1 = 0$. In addition, the roots of this equation are $\lambda, \bar{\lambda} = 0.92759 \pm 0.3736i$, with $|\lambda| = 1$, $d(r_0) = -0.288397 < 0$ and $\alpha(r_0) = 0.00323413 > 0$. By Theorem K, a unique stable invariant curve bifurcates from the coexistence equilibrium ($r < r_0$) when the parameter r crosses the bifurcation value $r_0 = 0.03695167557933803$ (supercritical Neimark–Sacker bifurcation). See Figure 6.4 (v)-(x). Next, this curve collapses. See Figure 6.4 (v)-(x) for the occurrence of Neimark-Sacker bifurcation and Figure 6.4 (xi)-(xii) for the collapse of the invariant curve.

On the other hand, for $r = r_1 = 2.50065315503643$ System (6.47) undergoes period-doubling bifurcation, see Figure 6.4 (iii). In this case, we can compute the precise coordinates of the unique interior equilibrium point of System (6.47) which is given by $(H^*, P^*) = (0.8847281447511166, 0.46754147816072716)$. The eigenvalues of the corresponding Jacobian matrix are $\lambda_1 = -1$, $\lambda_2 = -\text{Det}(J) = 0.482554$, which implies that the period-doubling bifurcation occurs at the bifurcation parameter $r_1 = 2.50065315503643$. Also, since $-\dfrac{\partial^3 G^2}{\partial u^3}(0,0)/\dfrac{\partial^2 G^2}{\partial u \partial \eta}(0,0) = 1.19645 > 0$, then the period two-points lie on the right side of $\eta = 0$. Additionally, we compute the stability type of this two-cycle:

$$\frac{1}{2}\left(\frac{\partial^2 G}{\partial u^2}\right)^2 + \frac{1}{3}\left(\frac{\partial^3 G}{\partial u^3}\right) = 0.42874 > 0.$$

Hence, condition $(F2)$ of Theorem 6 is satisfied and the production of a stable period two cycle near the origin for small $\eta > 0$ is guaranteed.

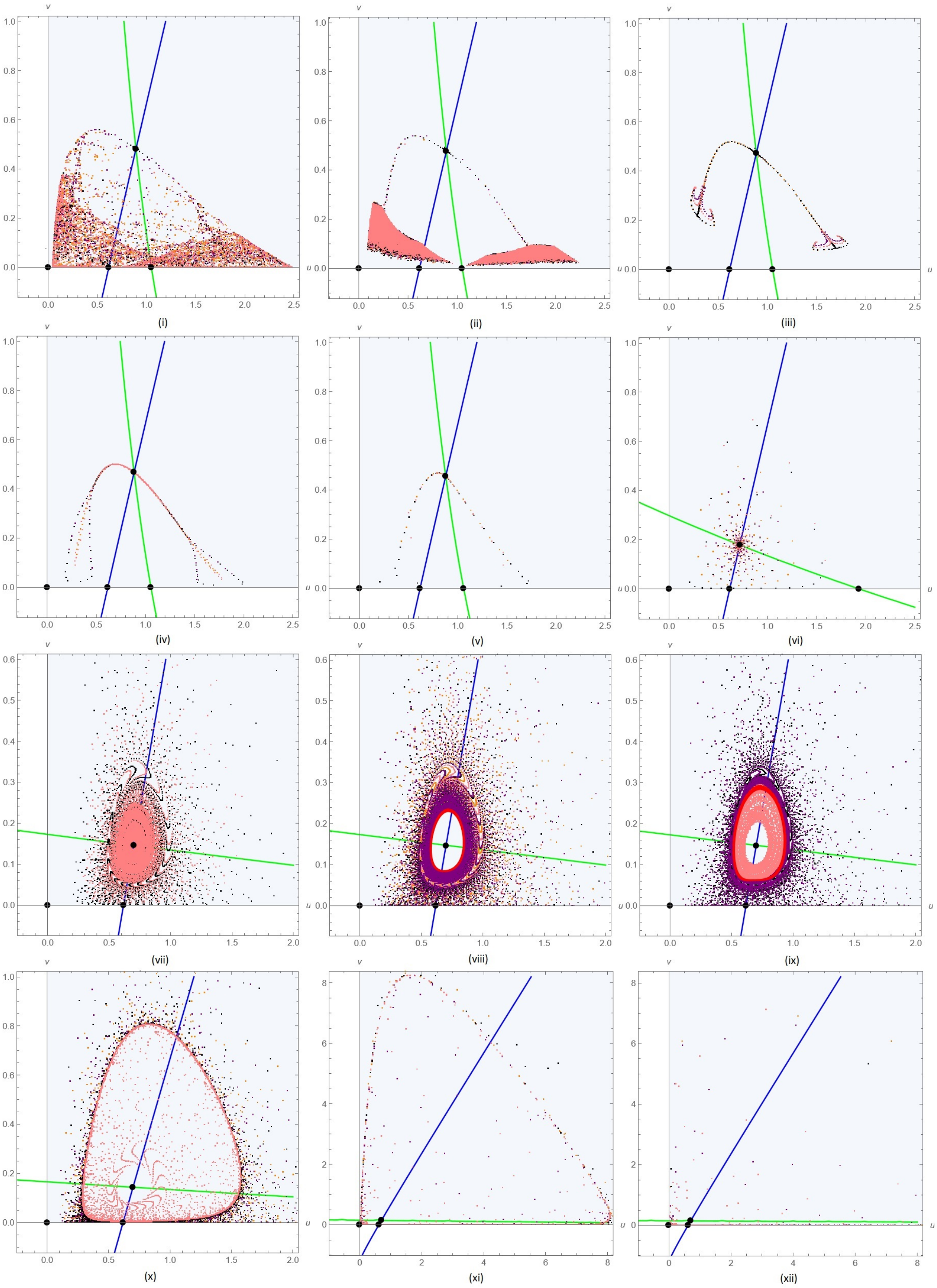


Figure 6.4: Trajectories (orange, pink, black and purple) and approximated invariant curve (red) for $a = 1.15$, $b = 1.1$, $c = 1.62$, $m = 1.2$ and (i) $r = 2.8$ (ii) $r = 2.7$ (iii) $r = 2.6$ (iv) $r = r_1 = 2.50065315503643$ (v) $r = 2.3$ (vi) $r = 0.15$ (vii) $r = r_0 = 0.03695167557933803$ (viii) $r = 0.0365$ (ix) $r = 0.036$ (x) $r = 0.03$ (xi) $r = 0.01$ and (xii) $r = 0.005$.

Now, in Figure 6.5 we gather the bifurcation diagram depending on the parameter r of System (6.47). Concretely, in Figure 6.5 (i), the occurrence of the supercritical Neimark-Sacker bifurcation is shown. In Figure 6.5 (ii), we can observe the occurrence of a period-doubling bifurcation. Initially, there is a stable interior equilibrium, followed by the emergence of a stable period two-cycle.

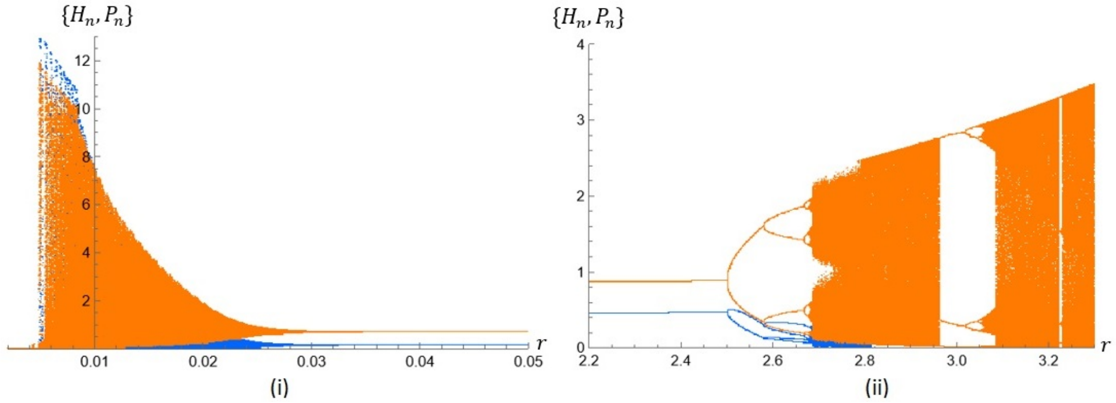


Figure 6.5: Bifurcation diagram for System (6.47) depending on r, the parameter values are $a = 1.15$, $b = 1.1$, $c = 1.62$, $m = 1.2$ and the initial conditions $(H_0, P_0) = (0.7, 0.15)$. In (i), $0 \leq r \leq 0.05$; and in (ii), $2.2 \leq r \leq 3.3$.

Furthermore, we have developed numerical simulations in $3D$ for different parametric spaces, Figure 6.6-6.7, where the emergence of the Neimark-Sacker and period-doubling bifurcations can be observed.

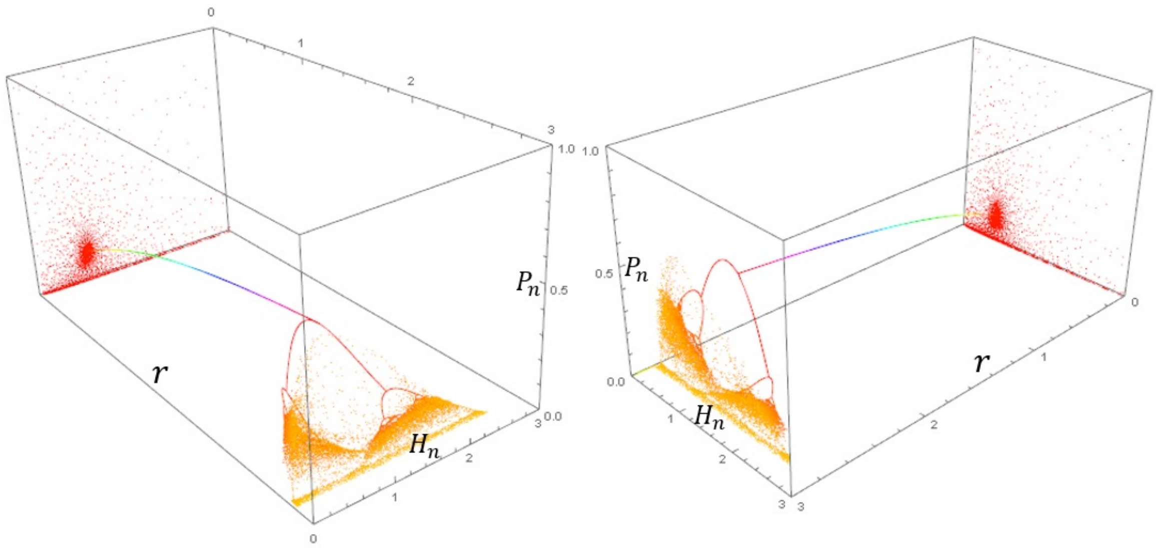


Figure 6.6: 3D bifurcation diagram in space $r - H_n - P_n$ for model (6.47) for $0 \leq r \leq 3.0$, $a = 1.15$, $b = 1.1$, $m = 1.2$, and $c = 1.62$ where initial conditions are $(H_0, P_0) = (1.8, 0.9)$.

Finally, we compute the approximate values of $r_0, d(r_0)$ and $\alpha(r_0)$ introduced in Subsection 6.5.2 and gather them in Table 6.1. It can be observed that there are values of other parameters $a = 1.52$, $b = 0.93$, $c = 0.6$, $m = 1.2$ such that $\alpha(r_0) \approx 0$. We obtain that $d(r_0) < 0$ for the chosen values. Then, since $\alpha(r_0)$ changes its sign, it implies the occurrence of the so-called Chenciner bifurcation (generalized Neimark-Sacker), [59]. This means that

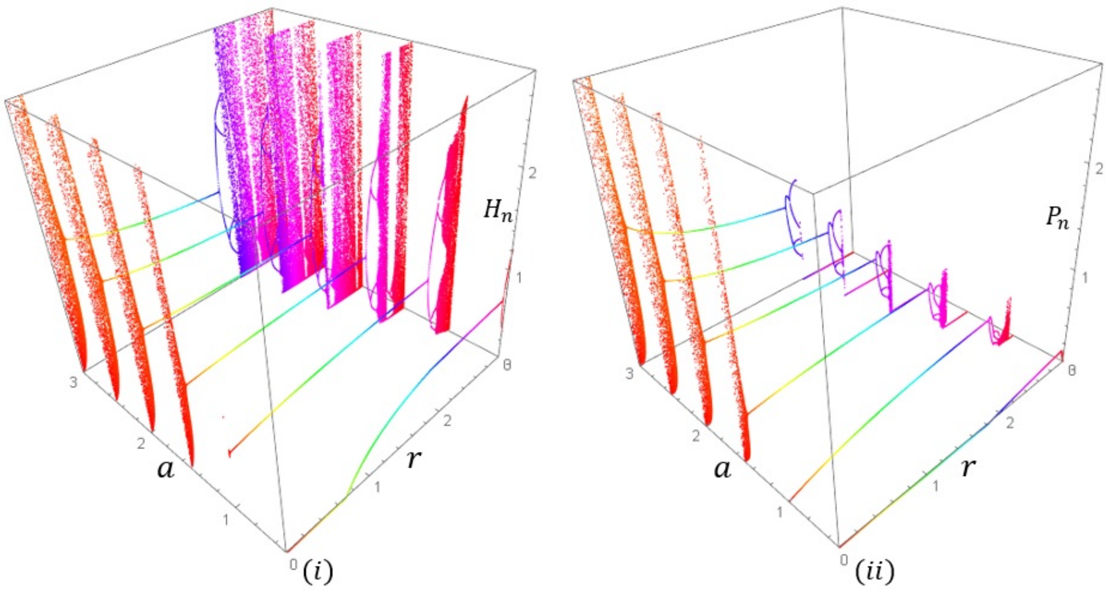


Figure 6.7: (i) 3D bifurcation diagram in the space $r - a - H_n$ for System (6.47) and $0 \leq r \leq 3.0$, $0.5 \leq a \leq 3.0$ with fixed $c = 1.62$ and $m = 1.2$, where the initial conditions are $(H_0, P_0) = (0.8, 0.3)$. (ii) 3D bifurcation diagram in the space $r - a - P_n$ for System (6.47) for the same parameter values.

we have supercritical and subcritical Neimark-Sacker bifurcation. In Figure 6.8 (iii), there are three significant features: a stable equilibrium and two invariant closed curves. The first curve encircling the equilibrium repels everything within it, while the second one attracts everything between and some orbits that begin outside. As the parameter a increases, the first closed curve decreases in size and eventually merges with the equilibrium, becoming a repeller. However, the second curve remains an attractor throughout this process. See Figure 6.8 (i)-(ii).

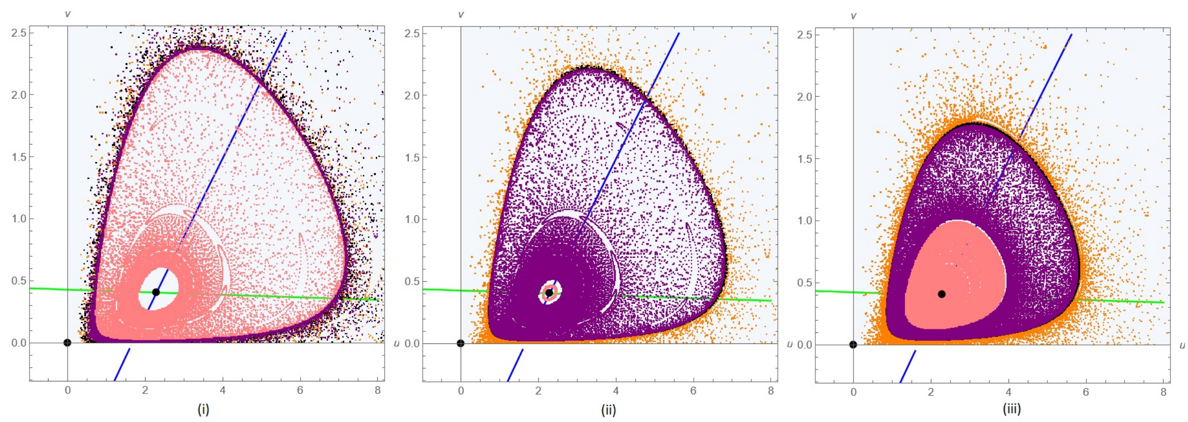


Figure 6.8: Trajectories (orange, pink, black and purple) for $b = 0.93$, $c = 0.6$, $m = 1.2$ and (i) $r_0 = 0.0092734$, $a = 1.502$ (ii) $r_0 = 0.0092734$, $a = 1.5$ and (iii) $r_0 = 0.0092734$, $a = 1.496$.

It should be highlighted that the values of the parameters a and r indicated in the caption of Figure 6.8 are slightly different from those in the table since the visualization of the Chenciner bifurcation requires the adjustment of such values in order to observe the two curves.

a	r_0	$H^*(r_0)$	$P^*(r_0)$	$d(r_0)$	$\alpha(r_0)$
1.47	0.00857975	2.39392	0.470847	-0.868	-0.0000887083
1.48	0.00879333	2.40849	0.480197	-0.868541	-0.0000714648
1.49	0.00900673	2.42303	0.48952	-0.869078	-0.0000543585
1.5	0.00921992	2.43754	0.498817	-0.86961	-0.000037401
1.51	0.00943284	2.452	0.508088	-0.870138	-0.0000206023
1.52	0.00964547	2.46643	0.517333	-0.870661	-0.000000397
1.53	0.00985777	2.48082	0.526551	-0.871179	0.0000124838
1.54	0.0100697	2.49518	0.535744	-0.871694	0.0000287562
1.55	0.0102813	2.5095	0.544911	-0.872204	0.0000448397
1.56	0.0104924	2.52378	0.554052	-0.872709	0.0000607289
1.57	0.0107031	2.53803	0.563168	-0.87321	0.0000764192
1.58	0.0109133	2.55224	0.572258	-0.873707	0.0000919068

Table 6.1: Approximate values of the coefficients r_0, $d(r_0)$, and $\alpha(r_0)$ for some values of a and $b = 0.93,\ c = 0.6,\ m = 1.2$.

6.7.2 Example 2

In this second example, we consider the probability function $f(y) = e^{-y^m}$, $0 < m \leq 1$, from [45], where the parameter m represents the mutual interference. For such map f, System (6.4) reads as

$$\begin{cases} H_{n+1} = aH_n e^{r(1-H_n)} e^{-(bP_n)^m} \\ P_{n+1} = cH_n\left(1 - e^{-P_n^m}\right) \end{cases}, \quad 0 < m \leq 1. \tag{6.48}$$

It should be emphasized that in their research, [45], Hassell and Varley highlighted the relevance of mutual interference in keeping the stability of host-parasitoid interaction and that its occurrence is common in host-parasitoid models. To show this, they reanalyzed data from various publications and proved that searching efficiency decreases exponentially as parasitoids' density increases. Incorporating this exponential law into the Nicholson-Bailey model, Hassell and Varley observed that interference can establish a stable equilibrium in an otherwise unstable model. This finding is discussed further in [36].

Now, it is easy to check that $f(y) = e^{-y^m}$ satisfies conditions ($\star$). Concretely, for

$0 < m < 1$, we obtain

$$f'(y) = -me^{-y^m}y^{m-1} < 0; \quad f(0) = 1; \quad \lim_{y\to+\infty} f(y) = 0; \quad f'_+(0) = -\infty;$$

$$\lim_{y\to 0^+} yf'(y) = 0; \quad \text{and} \quad f''(y) = m^2e^{-y^m}y^{2m-2} + (1-m)me^{-y^m}y^{m-2} > 0,$$

and for $m = 1$, we have

$$f'(y) = -e^{-y} < 0; \; f(0) = 1; \; f(\infty) = 0; \; f'_+(0) = -1; \; \lim_{y\to 0^+} yf'(y) = 0; \, f''(y) = e^{-y} > 0.$$

Take $a = 2.42$, $b = 1.7$, $c = 0.89$ and $m = 0.9$. For these values, the unique interior equilibrium is $(H^*, P^*) = (1.2953384748762256, 0.43264400517064194)$. The solutions of the characteristic equation $\lambda^2 - 1.15516\lambda + 1 = 0$ at (H^*, P^*) are $\lambda, \bar{\lambda} = 0.577579 \pm 0.816335i$ with $|\lambda| = 1$. Furthermore, $d(r_0) = -0.531228 < 0$ and $\alpha(r_0) = 0.192546 > 0$. By Theorem K, a unique and stable invariant curve bifurcates from the coexistence equilibrium ($r < r_0$) when the parameter r crosses the bifurcation value $r_0 = 0.4243497394800292$ (supercritical Neimark–Sacker bifurcation). See Figure 6.10 (vii)-(xii).

In Figure 6.9, we see the supercritical Neimark-Sacker bifurcation. Initially, in Figure 6.9 (i), there exists a stable closed invariant curve that encircles an unstable interior equilibrium. However, this curve gradually transforms and splits into multiple pieces, resulting in periodic solutions surrounded by the closed invariant curves in Figure 6.9 (iii). As time passes, these curves continuously expand and contract, leading to the emergence of rank-one attractors in Figure 6.9 (iv), which eventually culminate in forming a "large" attractor in Figure 6.9 (v). However, this attractor disappears in Figure 6.9 (vi).

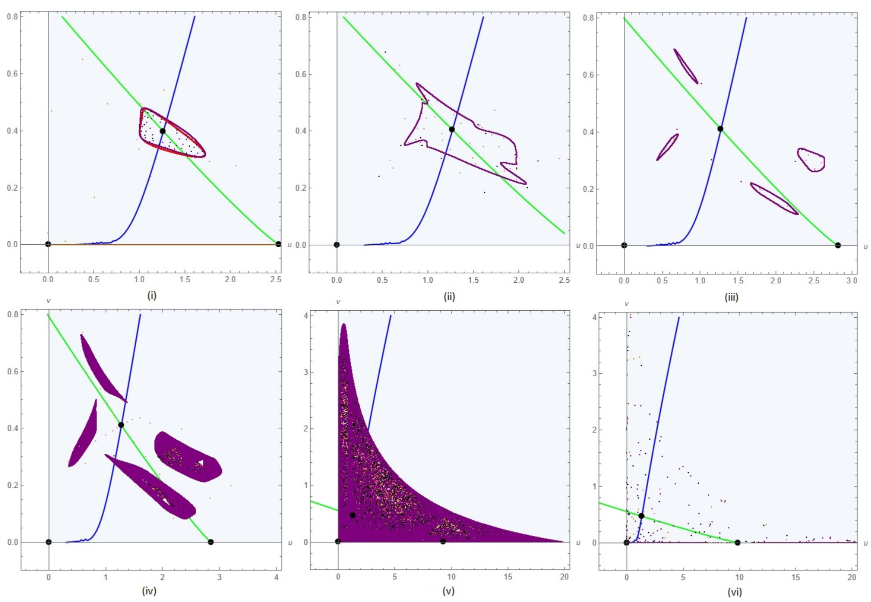


Figure 6.9: Trajectories (orange, pink, black and purple) and approximated invariant curve (red) for $a = 12$, $b = 5.6$, $c = 0.89$, $m = 0.9$ and (i) $r = 1.62$ (ii) $r = 1.48$ (iii) $r = 1.36$ (iv) $r = 1.34$ (v) $r = 0.3$ and (vi) $r = 0.28$.

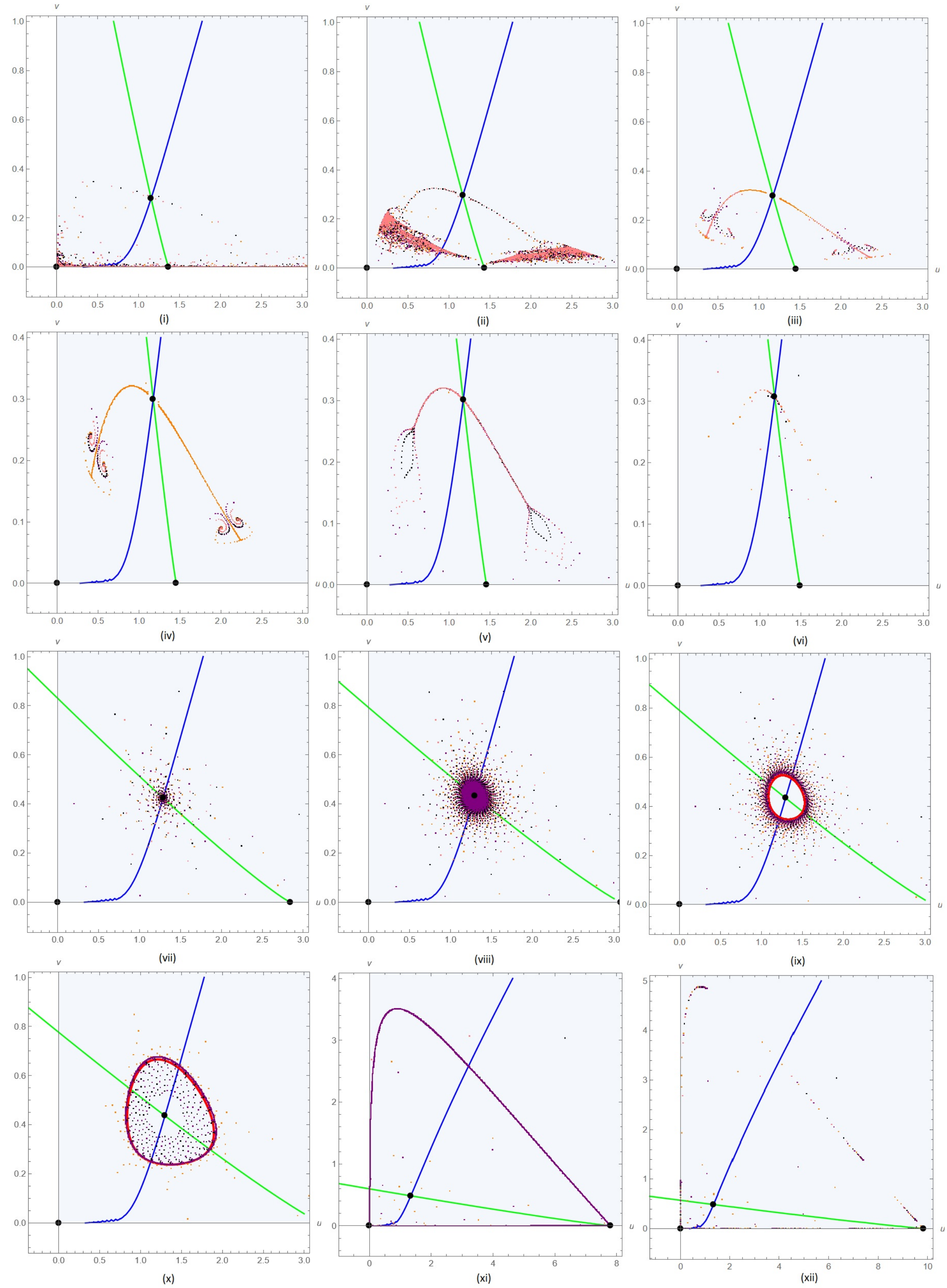


Figure 6.10: Trajectories (orange, pink, black and purple) and approximated invariant curve (red) for $a = 2.42$, $b = 1.7$, $c = 0.89$, $m = 0.9$ and (i) $r = 2.4$ (ii) $r = 2.03$ (iii) $r = 1.96$ (iv) $r = r_1 = 1.9413860286258822$ (v) $r = 1.92$ (vi) $r = 1.8$ (vii) $r = 0.5$ (viii) $r = r_0 = 0.4243497394800292$ (ix) $r = 0.42$ (x) $r = 0.4$ (xi) $r = 0.13$ and (xii) $r = 0.1$.

Next, for $r = r_1 = 1.9413860286258822$, System (6.48) undergoes period-doubling bifurcation. See Figure 6.10 (iv). For such bifurcation parameter, straightforward calculation shows that the unique interior equilibrium point of the system is given by $(H^*, P^*) = (1.1742011353973556, 0.3000322464476262)$. The eigenvalues of the corresponding Jacobian matrix are $\lambda_1 = -1,\ \lambda_2 = -\text{Det}(J) = 0.476708$. Moreover, it is easy to check that

$$-\frac{\partial^3 G^2}{\partial u^3}(0,0)/\frac{\partial^2 G^2}{\partial u \partial \eta}(0,0) = -0.271105 < 0,$$

so the period-two points lie on the left side of $\eta = 0$. In order to determine the stability type of this two-cycle we compute

$$\frac{1}{2}\left(\frac{\partial^2 G}{\partial u^2}\right)^2 + \frac{1}{3}\left(\frac{\partial^3 G}{\partial u^3}\right) = -0.121109 < 0.$$

Thus, condition $(F2)$ of Theorem 6 is fulfilled implying the production of an unstable period-two cycle near the origin for small $\eta < 0$.

In Figure 6.11, we provide the bifurcation diagrams for particular values of a, b, c and m. Figure 6.11(i) corresponds to the Neimark-Sacker bifurcation, while Figure 6.11(ii) illustrates the birth of the Neimark-Sacker and period-doubling bifurcations, with a stability regime in between. Figure 6.11(iii) shows a period-doubling bifurcation.

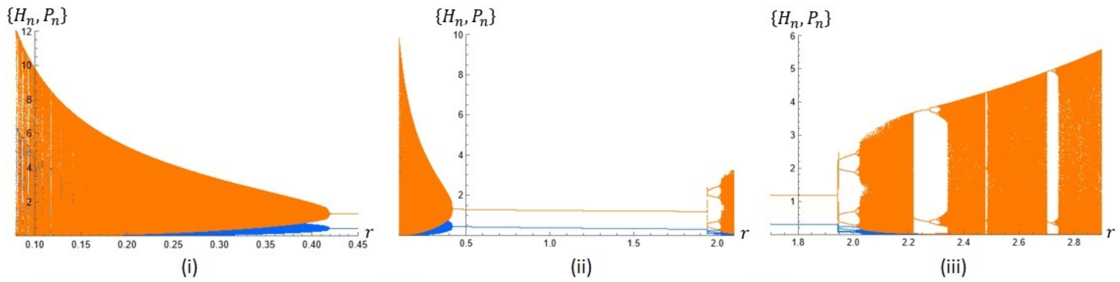


Figure 6.11: Bifurcation diagram depending on r with $a = 2.42$, $b = 1.7$, $c = 0.89$, $m = 0.9$ and initial conditions $(H_0, P_0) = (1.29, 0.43)$. In (i) and (iii), $0.08 \leq r \leq 0.45$; while in (ii), $1.7 \leq r \leq 2.9$.

Moreover, we provide a three-dimensional bifurcation diagram in Figure 6.13. In there, for fixed values of the parameters a, b, c and m, we can see the appearance of a Neimark-Sacker and period-doubling bifurcation as we vary the parameters r.

In the three-dimensional bifurcation diagram shown in Figure 6.13, we can see the Neimark-Sacker and period-doubling bifurcation as we vary the parameters r and a with the parameters b, c and m being constant. We plot diagrams in the $r - a - H_n$ space in Figure 6.13 (i), and the $r - a - P_n$ space in Figure 6.13 (ii).

Now, in Table 6.2, we have computed the values of $d(r_0) < 0$ and $\alpha(r_0)$ based on different values of the parameters a and r_0. Observe that the sign of the coefficient $\alpha(r_0)$ changes. For $a = 2.24$ and $r_0 = 0.0747272$, it is positive. Then, starting from $a = 2.25$ until

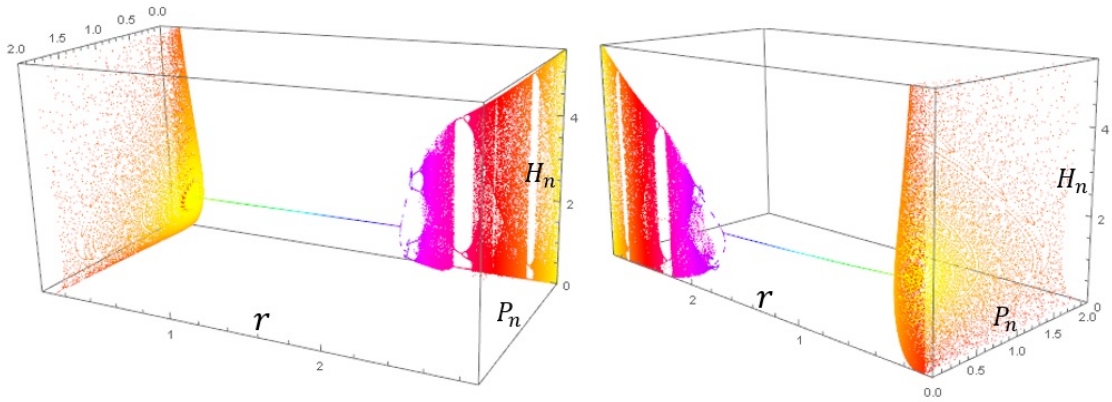
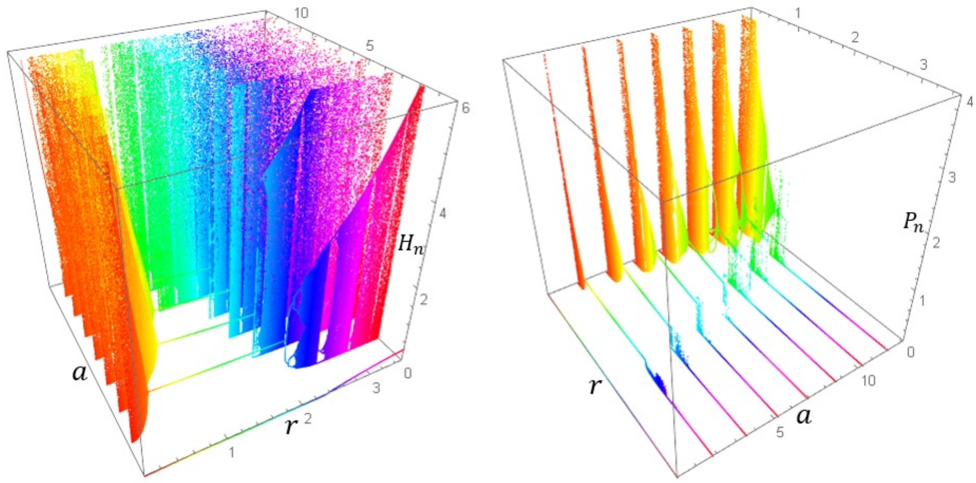


Figure 6.12: 3D bifurcation diagram in the space $r - H_n - P_n$ for System (6.48) for $0.1 \leq r \leq 2.9$, $a = 2.42$, $b = 1.7$, $m = 0.9$, $c = 0.89$ and initial conditions are $(H_0, P_0) = (1.5, 0.5)$.



(i) (ii)

Figure 6.13: 3D bifurcation diagrams for System (6.48) and $0.01 \leq r \leq 3.5$, $0.1 \leq a \leq 13.1$ with fixed $b = 1.7$, $c = 0.89$, $m = 0.9$ and initial conditions $(H_0, P_0) = (1.29, 0.43)$. In (i) the diagram is represented in the space $r - a - H_n$ and in (ii) in the space $r - a - P_n$.

$a = 2.39$ and the corresponding values of r_0, the sign of $\alpha(r_0)$ is negative. For $a = 2.24$, $b = 0.48$, $c = 0.89$, $m = 0.9$ at $r_0 \approx 0.0747272$, we have $\alpha(r_0) \approx 0$. Then, $\alpha(r_0) > 0$ and a Chenciner bifurcation occurs. See in Figure 6.14(iii)-(i). Concretely, there are two curves in the system - an unstable one and a stable one. The unstable curve surrounds the stable interior equilibrium (subcritical Neimark-Sacker bifurcation). On the other hand, the second curve attracts everything that starts between these two curves and even some orbits that begin outside. Since the solutions are uniformly bounded, they cannot go to infinity. When the first unstable curve collides with the interior equilibrium, it becomes a repeller (Figure 6.14(ii)). At the same time, the second curve becomes an attractor for all orbits that start inside (supercritical Neimark-Sacker bifurcation), Figure 6.14(ii).

a	r_0	$H^*(r_0)$	$P^*(r_0)$	$d(r_0)$	$\alpha(r_0)$
2.24	0.0747272	2.1446	1.42724	-0.663468	0.000200327
2.25	0.0759887	2.14966	1.43302	-0.664428	-0.000049704
2.26	0.0772448	2.15467	1.43875	-0.66538	-0.000237
2.27	0.0784954	2.15964	1.44443	-0.666321	-0.000372441
2.28	0.0797405	2.16458	1.45007	-0.667254	-0.000464894
2.29	0.0809803	2.16948	1.45565	-0.668177	-0.000521623
2.3	0.0822147	2.17434	1.4612	-0.669091	-0.000548601
2.31	0.0834438	2.17916	1.46669	-0.669997	-0.000550764
2.32	0.0846676	2.18394	1.47215	-0.670893	-0.000532201
2.33	0.0858862	2.18869	1.47755	-0.671781	-0.000496311
2.34	0.0870995	2.1934	1.48292	-0.67266	-0.000445931
2.35	0.0883076	2.19808	1.48824	-0.67353	-0.000383428
2.36	0.0895106	2.20272	1.49352	-0.674392	-0.000310792
2.37	0.0907085	2.20733	1.49876	-0.675246	-0.000229688
2.38	0.0919012	2.2119	1.50395	-0.676092	-0.000141522
2.39	0.0930889	2.21644	1.50911	-0.676929	-0.0000474772
2.4	0.0942716	2.22095	1.51422	-0.677759	0.0000514479
2.41	0.0954493	2.22542	1.5193	-0.678581	0.000154408

Table 6.2: Approximate values of the coefficients r_0, $d(r_0)$, and $\alpha(r_0)$ for some values of a and $b = 0.48,\ c = 0.89,\ m = 0.9$.

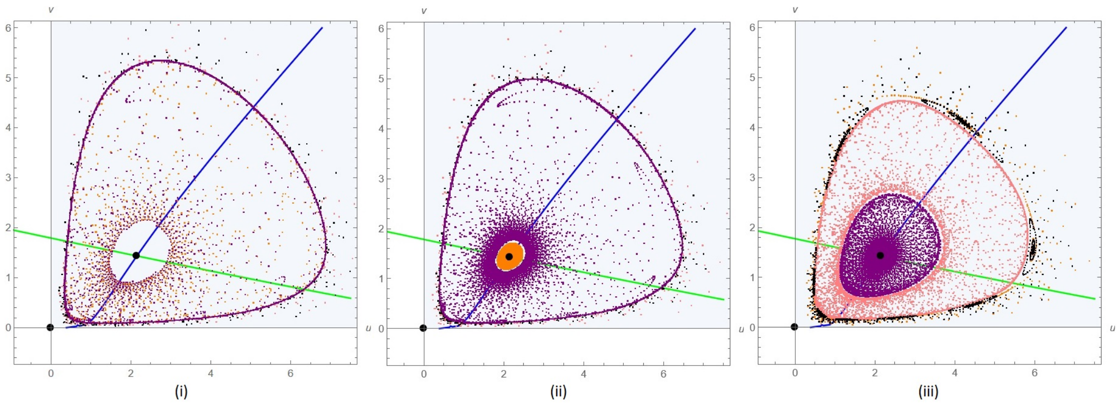


Figure 6.14: Trajectories (orange, pink, black and purple) for $b = 0.48$, $c = 0.89$, $m = 0.9$ and (i) $r = 0.0747272$, $a = 2.25$ (ii) $r = 0.0747272$, $a = 2.24$ and (iii) $r = 0.0747272$, $a = 2.23$.

It should be mentioned that the values for parameters a and r in the caption of Figure 6.14 are slightly different from those in the table. This is due to the fact that the visualization of the Chenciner bifurcation needs the adjustment of the values so both curves can be observed.

6.8 Conclusions

In this chapter, we have deeply analyzed the dynamics of a generalized Beddington host-parasitoid model with a general probability function satisfying some conditions that arise naturally from the relationship between hosts and parasitoids, namely, conditions ($\star$). Moreover, we have not consider exactly the same probability function in the host and parasitoid' equations from System (6.4), but a variation of it by introducing a multiplicative constant. The novelty of this work lies on the fact that our study for a more general system comprehends other models studied previously in the literature, as the one analyzed in [53].

Concretely, we have been able to prove the uniformly boundedness of both populations and to determine the existence of three different types of equilibria: extinction, exclusion and coexistence. Nevertheless, we could only give an explicit expression for the extinction and exclusion equilibrium points, for which we have established some global stability results and not only local ones. In the case of the interior equilibrium, it was impossible to give a concrete expression, which makes more challenging the study of the behaviour of the solutions near that point.

Concerning the occurrence of bifurcations, for the exclusion equilibrium point we have proved the occurrence of period-doubling and transcritical bifurcations. In the first case, for a particular value of the maximum growth rate of the host population r, the equilibrium bifurcates into a stable two-cycle. On the other hand, the transcritical bifurcation takes place when the parasitoids' maximum growth rate, c, reaches a specific boundary, which acts as an invasion boundary for the parasitoid populations. In addition, the study of the possible occurrence of bifurcations in the interior equilibrium shows the existence of the period-doubling and the Neimark-Sacker bifurcations. The period-doubling bifurcation indicates highly complex behaviour, while the Neimark-Sacker bifurcation shows that populations will oscillate around a mean value for the long term. It should be highlighted that, besides developing numerical simulations, we proof analytically the occurrence of the diverse bifurcations that appear in the model. This implies a significant advancement compared to specific models previously studied in the literature, where authors have limited themselves to conducting numerical simulations to visualize the occurrence of various types of bifurcations as it happens, for instance, in [53].

Moreover, although we could not establish a concrete result concerning the global stability of the interior equilibrium point due to the ignorance of a precise expression of such point, we have shown the permanence of the system. This fact implies that both population will coexist in the long term within some compact set located at a positive distance from the axes.

Finally, we have developed numerical simulations for concrete probability functions, namely, $f(y) = \left(1 + \frac{y}{m}\right)^{-m}$, with $m > 0$; and $f(y) = e^{-y^m}$, with $0 < m \leq 1$. In both examples we have illustrated for specific models the theoretical study developed in the preceding sections.

In the same direction as the one followed in this chapter, it will be of interest to study other population models with an arbitrary probability function, instead of concrete ones, in order to obtain more general results and to broaden the scope of the diverse results gathered in the literature.

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