Assessing mathematical processes in teaching practices

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Validating an instrument to evaluate the Teaching of Mathematics through Processes

Abstract

The aim of this study is to validate an instrument to evaluate the Teaching of Mathematics through Mathematical Processes using a Structural Equations Model. To that end, we have administered the instrument to 95 in-service Spanish teachers and we have also analysed the presence of mathematical processes (problem solving, reasoning and proof, communication, connections and representation) in teaching practice.

The descriptive statistics obtained through a quantitative study show that all the items perform similarly in each of the processes, obtaining medium to high scores. A change in this trend is only detected in some of the items of the mathematical process "connections", which measure if mathematical knowledge is related to other disciplines. The results obtained from the exploratory factor analysis show a high coefficient for all the processes (higher than 0.72), as well as a significant p-value; and the results obtained from the confirmatory factor analysis show an internal consistency of the items of each construct, with values greater than 0.8.

Keywords: Mathematics Teaching, Mathematical Processes, classroom observation instrument, validation analysis, Structural Equation Model, Early Years and Primary Education.

Validating an instrument to evaluate the Teaching of Mathematics through Processes

The analysis of teacher practice remains a hot topic of research in mathematics education. In recent years, several monographs in international journals have focused on issues related to mathematics teachers' practice, learning, and professional development. In the *Journal of Mathematics Teacher Education*, for example, Jones and Pepin (2016) provide a state-of-the-art review of relevant literature about mathematical tasks and tools, understood as key resources in mathematics teaching and mathematics teacher Education. In another example, Karsenty and Sherin (2017) highlight the use of video in professional development programmes as a catalyst for mathematics teachers' reflection on their teaching and on their students' learning. In *ZDM Mathematics Education*, Skott, Van Zoest and Gellert (2013) promote discussion about the relationships between the theoretical assumptions of research conducted on teachers' knowledge, beliefs, and identity; and in a later monograph, Chapman and An (2017) identify and describe different themes emerging from relevant contributions in university-based teacher education 'programmes' for in-service and pre-service teachers, to illustrate the nature and effectiveness of the programmes and implications for teacher Education. According to Llinares (2018), this series of monographs provides knowledge that allows us to advance in our understanding of the practice of the mathematics teacher as a field of scientific research, opening channels that enhance communication between theory and practice, and between research and teacher training.

This study aims to contribute to broadening this knowledge about mathematics teachers' practice in a specific mathematics education research agenda: the systematic incorporation of mathematical processes (MPs) in teaching practice, given the important role that such processes have in the development of students' mathematical competence.

Specifically, our focus is on developing an instrument to evaluate teaching with an emphasis on MPs (thinking and doing), rather than on math content (memorizing definitions and procedures).

We adopt the approach advanced by NCTM (2014), which indicates that MPs highlight the ways in which thinking and doing takes place. In other words, they are the tools that mathematics provides to engage students in meaningful learning through individual and collaborative experiences that promote their ability to make sense of mathematical ideas and to reason mathematically.

In order to carry out an objective analysis of the degree to which early years teachers include MPs in their teaching practice, [Authors' reference] designed and validated an assessment tool to analyse the presence of MPs in teaching practice. In this new study, our aim is to continue advancing in the development of an instrument for evaluating the teaching of mathematics through processes. According to this, the aim of this work is to validate an instrument through a Structural Equation Model (González-Montesinos & Backhoff, 2010) to evaluate the Teaching of Mathematics through Processes. The instrument has been called ETMAP, which is the acronym for "Evaluating the Teaching of Mathematics through Processes". To this end, we have administered the instrument to 95 Spanish teachers of Early Years and Primary Education. With this objective, we have structured this article into different sections: (1) literature review, (2) Instrument and data collection, (3) Results, organised into three stages: descriptive analysis of the data; Exploratory Factor Analysis and Confirmatory Factor Analysis; using a Structural Equation Model; and, finally, we have concluded with section (4), discussing the results along with the limitations of the study and future prospects.

Literature review

The training of maths teachers is a field of research that has seen considerable changes in recent years in relation to both the questions studied, as well as in relation to the search for research methods different from those employed traditionally. In this sense, different lines of research are actively aiming to analyse the knowledge underlying teachers' practices, the suitability of teachers' practices, discourse analysis, the responses given to students, and the management of problem solving, among other aspects. This growing interest, which seeks to improve our understanding of learning and teaching processes and professional development, has been clearly demonstrated through the different research groups working to share their studies, such as Working Group Research on the Psychology of Mathematics Teacher Development in PME (Psychology of Mathematics Education).

As we have noted, our purpose is to advance knowledge about the incorporation of MPs into the teaching practice of early years teachers and, more specifically, on the management they carry out in order to teach content through MPs. In this sense, one of the ten recommendations of the Joint Position Statement on Early Childhood Education (NAEYC & NCTM, 2002, p. 5) is precisely to "use curricula and teaching practices that strengthen childhood processes for solving problems and reasoning, as well as those for the representation, communication and connection of mathematical ideas".

Regarding problem solving, for example, several researchers have noted using problem solving as a process in order to promote higher level thinking and reasoning. In 1945, George Pólya published the book *How To Solve It* which identifies four basic principles of problem solving (Pólya, 1945): understand the problem; devise a plan (translate); carry out the plan (solve); look back (check and interpret). Fan and Zhu (2007) talk about a framework for problem solving modified from Pólya's four-step problem-solving strategy: the list they produce includes developing a plan, carrying out the plan and/or modifying the plan if necessary, and ending with seeking alternative solutions and checking for reasonableness. Students good at problem solving do all of these things.

According to Schoenfeld (1994), successful problem solvers are agile users of what he calls the tools and logic of mathematics. That ability is improved through the solving of “good problems.” Schoenfeld defines a good problem:

> Good problems can introduce students to fundamental ideas and to the importance of mathematical reasoning and proof. Good problems can serve as starting points for serious explorations and generalizations. Their solutions can motivate students to value the processes of mathematical modelling and abstraction and develop students’ competence with the tools and logic of mathematics. (p. 60).

So, to be good at problem solving a student must exhibit the following: 1) show confidence in solving problems; 2) demonstrate persistence when encountering a difficult problem and refuse to give up; 3) when given an unfamiliar problem, know what to do and be able to switch strategies if one is not working; and 4) have an unofficial list of problem solving strategies to call upon when solving problems.

Costa and Kallick (2000) say that as students increase in their problem solving ability, they become more flexible in their thinking. They consider, express or paraphrase other points of view, can state several ways of solving the same problem, and evaluate the merits of more than one course of action. Students who have this habit of mind in place become systems thinkers. They analyse and scrutinise parts, but also shift their perspective to the big picture.

From this point of view, problem solving and reasoning are considered to be the central axes of mathematics. For this reason, teaching practices that promote competence in these processes in childhood are consistent with international reports on mathematics education (NCTM, 2000; Kilpatrick, Swafford, & Findell, 2001; National Research Council, 2009). From this perspective, Cobb et al. (1991) and Trafton and Harman (1997) indicate that teachers must be committed to the idea that working on problem solving at early ages will help students develop skills and basic strategies, along with higher level thinking skills. This data was supported some years later by Clements and Sarama (2011, p. 968), who stated that "very young children have the potential to learn mathematics that is complex and sophisticated". In order to promote such learning, early years teachers must cultivate and develop a mathematical willingness to propose problems; i.e. to generate new questions in a variety of contexts, such as from daily routines or the mathematical situations that emerge from stories (NCTM, 2000).

Mathematical reasoning, as indicated above, is another central axis in mathematics teaching practices. In this regard, Russell (1999, p. 1) indicates that it "is essentially about the development, justification, and use of mathematical generalizations". Subsequently, Research Report 17 of the National Council for Curriculum and Assessment (NCCA, 2014a, p. 67) suggests that "helping children to understand their thinking and assisting them to express it to others is central to the learning of mathematics". For Brandsford, Brown, and Cocking (1999), young children can start to reason on the basis of their experiences, and this skill is developed when children are encouraged to formulate conjectures, when they are given time to find proof to confirm or refute these, and when they are given the opportunity to explain or justify their ideas. To do this, teachers should provide physical material so that students can manipulate objects, compare them and end up making generalizations about them (NCTM, 2000). In view of this, particular emphasis should be placed on the development of general statements that deepen students’ understanding by clarifying what is true or false (Lannin, Barker, & Townsend, 2007; Lannin, Ellis, Elliott, & Zbiek, 2011). In order to foster this process, Carpenter and Levi (1999) highlight the importance of positing questions in the context of debates that, in line with [Authors’ reference], enable young students to explain, argue or justify actions carried out and propositions made, as well as to check the results of such actions and propositions, rather than simply demonstrating or validating them. From this perspective, reasoning and proof involve justifying statements made (“why do you think that is true?”); discovering (“what do you think will happen now?”); justifying propositions (“why does this work?”); and carrying out inductive reasoning based on one’s own experience.

Within this framework, communication is another essential mathematical process, since it also plays a role in the development of mathematical thinking (Ellerton, Clements, & Clarkson, 2000; Whitin and Whitin, 2003; Klibanoff, Levine, Huttenlocher, Hedges, & Vasilyeva, 2006) and, for this reason, its importance is recognized in policy statements and curricular documents (NCCA, 2014b). For Ginsburg (2009), talking about mathematical thinking and engaging in reasoning, justification and argumentation are central to mathematics education for all children aged 3 to 8 years’ old. According to the NRC report:

> Children must learn to describe their thinking (reasoning) and the patterns they see, and they must learn to use the language of mathematical objects, situations and notation. Children’s informal mathematical experiences, problem solving,

explorations, and language provide bases for understanding and using this formal mathematical language and notation (2009, p. 43)

From this perspective, a “math-talk learning community”, in which all children have opportunities to describe their thinking through processes of interaction, dialogue and negotiation in the mathematics classroom, has the potential to improve children’s mathematical language. In these “math-talk communities”, questions emerge as one of the most suitable measurement instruments [Authors’ reference], precisely because they enable thinking to develop from initial levels of awareness, on the basis of what is already known or what can already be done, to higher levels of consciousness in which the child becomes more aware of the ways in which they can learn better (Mercer, 2001). The NRC (2009) also points to the importance of children using language to make connections across different domains of mathematics, and across mathematics, other learning areas, and everyday life.

For Clements, Sarama, and DiBase (2004), the process of establishing connections deserves particular attention, since it enables a coherent system of mathematical knowledge to be built, as well as helping to improve knowledge of the wide applicability of mathematics. The idea of connections within mathematics receives considerable treatment in the NRC (2009), where it is stated that ‘every mathematical idea is embedded in a long chain of related ideas’ (p. 48). [Authors’ reference] propose two main principles of mathematical connections in the early years: a) connections between different blocks of mathematics content and between mathematical content and processes (intradisciplinary connections); and b) connections between mathematics and other areas of knowledge and the environment (interdisciplinary connections). In a subsequent study, [Authors’ reference] propose three types of mathematical connections that Early Childhood Education teachers should consider to develop the connective intelligence of children between 3 and 6 years’ old: a) conceptual connections, which produce links between different mathematical content; b) teaching connections, which link different mathematical concepts through active methodologies and by relating mathematical experiences with other subjects; and c) practical connections, which relate mathematics with the environment.

Regarding the representation of mathematical ideas, which is the last mathematical process proposed by the NCTM (2000), it is worth noting that among the forms of representation that children use to organise and convey their thinking we find concrete manipulatives, mental models, symbolic notation, tables, graphs, number lines, stories, and drawings (Langrall, Mooney, Nisbet, & Jones, 2008).

Duval (1995) develops the idea of “semiotic representation”, where two main points must be taken in account: 1) semiotic representations are related with particular sign systems (e.g. language, algebra, graphs.); 2) a semiotic representation within a sign system can be converted into an “equivalent” representation within another semiotic system, but this “new equivalent” representation may have different meanings to the subject who uses them. Janvier (1987) shows that external symbolic representations influence as well as reflect the internal representations of the mathematical knowledge possessed by learners.

The gradual development of the external representation of mathematical ideas and procedures goes from the specific to the abstract (Freudenthal, 1973). In this sense, early years teachers should try to ensure that initial representations are specific, based on objects or drawings, and using natural language; followed by pictorial representations using tables or diagrams; and, finally, conventional representations using abstract symbols, which are constantly changing depending on the semiotic system where they are being used, according to Duval (1995, 1998a, 1998b).

Although the development of representation generally goes from the specific to the abstract, the teaching/learning process is two-way as opposed to one-way. In other words, it goes from the specific to the abstract and then back from the abstract to the specific again; even though the purpose is always the same: using symbols that represent an object or a real situation meaningfully. Through interactions with different types of representations with the teacher and with other students, children develop their own mental images of mathematical ideas (NCTM, 2000). These internal representations enable them to advance in the learning of mathematics. Furthermore, the gradual acquisition of the representation of mathematical ideas and procedures increases their capacity to form and interpret physical, social and mathematical phenomena. In other words, it enables them to create models and to interpret reality. From this point of view, Dubal (1995, p. 15) indicates that "there is no knowledge that someone can mobilize without a representation activity".

Few studies have been carried out in which an analysis has been carried out of the MPs present in teaching practices in the early years. From this perspective, [Authors' reference] carried out a mixed method, multiple case study analysing the teaching practice of 12 Chilean teachers in situations in which they were teaching numbers to children from 4 to 8 years' old. In this pilot study, the analysis was carried out using the data from 48 audio-visual recordings of semi-structured interviews lasting an average of 15 minutes. On the one hand, the quantitative data show that the percentages relating to the presence of the indicators observed in each process and in the different cases studied were very low, in contrast to the cut-off percentages of each mathematical process obtained during the validation process through the expert review using Angoff method (Angoff, 1971): problem solving (24% in contrast to 70%), reasoning and proof (26.9% in contrast to 60%), communication (20.2% in contrast to

80%); connections (10.6% in contrast to 80%); representation (18.3% in contrast to 70%). On the other hand, the main qualitative data obtained from the audio-visual recordings, processed through the Elan 4.6.1 annotation tool, indicates the following aspects in relation to the management of MPs in teaching practice:

- Problem solving: little contextualization in relation to the children's lives; problem solving with the possibility of different solutions is not promoted; insufficient use of specific material; this makes it difficult to gain the children's interest and curiosity.
- Reasoning and proof: weaknesses in the promotion of the testing out of conjectures related to daily life; the questions posited do not generate argumentation and testing out; absence of feedback in response to the different reasoning advanced by the children.
- Communication: a silent environment is encouraged, without the interaction that would reinforce learning; the exchange of mathematical ideas that encourage numerical notation is not reinforced; no distinction is made in the use of precise vocabulary; the giving of information is prioritised over and above communication that supports the discovery of numbers.
- Connections: teaching/learning practices are disconnected from other mathematical content; practices are decontextualized from other childhood contexts; there is a predominance of artificial mathematics activities within the classroom; informal practices carried out by the children are not given sufficient attention.
- Representation: scarce use of materials to represent mathematical ideas; total absence of reflections on the notion of number in relation to symbolic representations; the use of example patterns (models) to support the construction of the notion of number is not observed sufficiently.

On the basis of the literature review carried out, the objective of this study is to analyse the level to which MPs are implemented in the teaching practice of 95 in-service Early Childhood and Primary Education teachers and, in addition, to validate the instrument through a Structural Equation Model (González-Montesinos & Backhoff, 2010).

Method

Instrument and data collection

The questionnaire applied consists of 35 items grouped into sevens, resulting in five groups which we call processes: problem solving, reasoning and proof, communication, connections and representation. All items were measured with a 5-point Likert scale, where level 3 was set to neutral, the scores above 3 were taken as positive, and those below 3 were considered negative scores (see Appendix A).

The data were collected by 20 evaluators who assessed the occurrence or implementation in the classroom of the *teaching and learning processes in mathematics* developed in the context of the subject "Mathematical Innovation and Research in Early Childhood Education" of the Master's Degree in Innovation and Research in Early Childhood and Primary Education of the University of Murcia (Spain). The data collection was carried out through interviews and direct observation of the teaching practice. In more detail, of a total of 10,072 early childhood and primary teachers in the Region of Murcia who work in public schools, a total of 1,500 collaborate each year with the Faculty of Education of the University of Murcia (Spain), receiving students for internships. The 21 students taking the subject "teaching and learning processes in mathematics" were instructed to identify the presence or absence of the five MPs and it was considered that they would be able to evaluate the presence or absence of these processes in the teaching practice of five teachers who were randomly assigned to them, from the teachers who had collaborated on the Master's programme at some point during the last two years. Thus, 105 teachers were randomly selected from those who collaborate with the faculty, and, after being properly instructed, 21 students were assigned the role of evaluators: from now on we will refer to them as "evaluators". Finally, the number of evaluators was reduced to 20, as one of them left the course before its conclusion. Each evaluator was assigned five teachers and instructed to attend three of their classes. Each evaluator attended classes of 5 teachers, except for two who attended classes of 6. After the three classes they discussed the presence or absence of the 5 MPs with the teachers, and the teacher completed the questionnaire in the presence of the student.

Of the 102 questionnaires conducted, seven were incomplete and were therefore eliminated, thus resulting in a total sample of 95 surveys, 47% (45) of which were conducted in Early Childhood Education centres and 53% (50) in Primary Education centres.

Of the total sample, only 13% (12) were men. This percentage is significantly reduced if we distinguish by educational level, with male teachers in Early Childhood Education representing a mere 4.4% (2) of the total compared to 95.6% (43) of female teachers. It is also observed that there are no differences by gender in terms of degree of experience ($Z = 1.42, p > 0.05$) or level of education ($Z = 0.746, p > 0.05$).

Results

A statistical analysis was carried out in three phases: descriptive analysis, Exploratory Factor Analysis and Confirmatory Factor Analysis, using a Structural Equation Model.

The open source statistical package (GPL) R, version 3.3.4 (R Core Team, 2017), has been used for all analyses. In recent years, the international scientific community has chosen R as the *lingua franca* of data analysis. It is well established in universities all over the world and is gradually being introduced in companies [Authors’ reference]. We have used the R 'lavaan' package (Rosseel, 2012) for the construction of the Structural Equation Model, the 'psych' package for Exploratory Factorial Analysis, and the 'likert' library for descriptive analysis.

Structural Equation Analysis is a technique used to evaluate models with complex relationships between variables, as well as models that include latent variables, measured from two or more observable variables (Sallán, Fernande, Simo, Lordan, & Gonzalez-Prieto, 2012), and to verify whether a model expressing a certain relationship between observable variables fits the empirical data (Schumacker & Lomax, 2004). On the other hand, Confirmatory Factorial Analysis is a special case within SEM techniques that enables the evaluation of the validity of a construction of measurement scales, which is defined as the metric property that ensures that the data or values resulting from the application of an instrument are effectively interpretable as a manifestation of the latent features measured, according to substantive theory and in the design context of the corresponding measurement model (González-Montesinos & Backhoff, 2010).

Descriptive analysis of the data

Regarding the analysis of the educational processes, we observe that, in general, for each of the blocks, all the items have a similar behaviour, accumulating high-medium scores, as can be seen in Tables 1 to 6. We only observe a change of tendency in some items of the “connections” process. These particular items assess if the mathematical knowledge worked on is related to or linked with other disciplines, such as artistic expression, musical expression or psychomotor skills.

We have thus found that items v3.3 and v3.4 accumulate proportions of 50% and 40% in the low levels, and that for items v3.5 and v3.6, the highest proportion of responses are found at lower and more medium levels than the rest of the items, although the high response ratios remain high (50%).

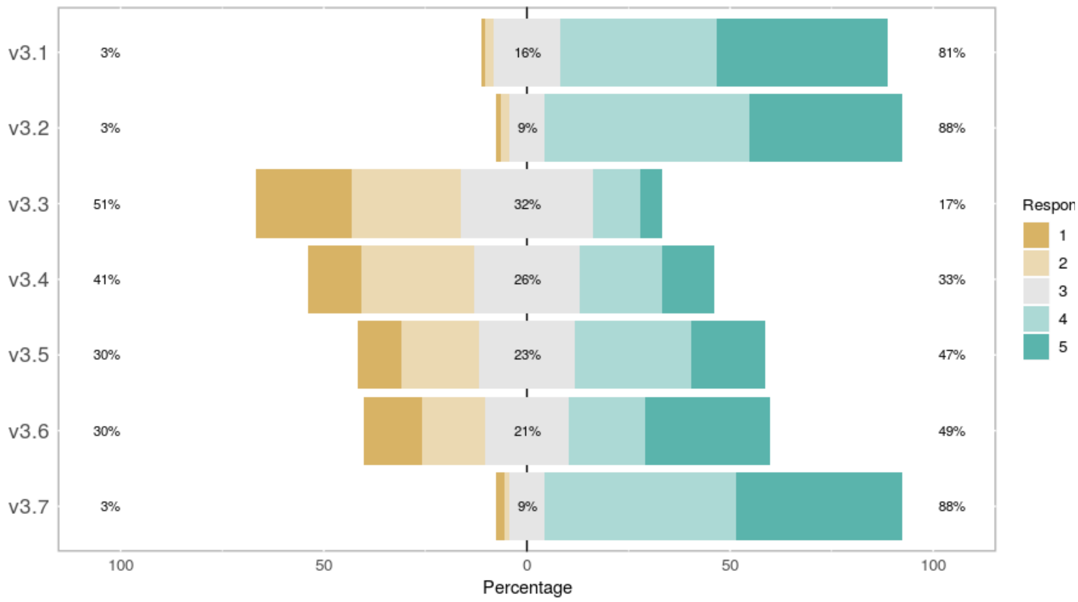

Figure 1. Frequencies of the answers in block 3: connections process. Low scores indicate low presence of the indicator in teaching practice. The percentages on the right, centre, and left of each bar indicate the percentage of responses considered low, medium, and high item presence.

The cause of these differences is found when disaggregating the items by educational level, since it is observed that in children's education, when working with mathematics content, these items are related to other more artistic or manipulative disciplines that serve to bring these contents closer to a younger and more informal student body.

Table 1

Descriptive statistics block 1 (Problem Solving)

Item	n	low	neutral	high	mean	sd	r_{i-t}
v1.1	89	1.05	9.47	89.47	4.32	0.70	0.70
v1.2	89	3.16	11.58	85.26	4.30	0.80	0.778
v1.3	89	2.15	8.60	89.25	4.20	0.69	0.76
v1.4	89	6.38	18.09	75.53	4.03	0.89	0.79
v1.5	89	1.06	18.09	80.85	4.18	0.787	0.68
v1.6	89	1.08	8.60	90.32	4.26	0.67	0.70
v1.7	89	3.16	22.11	74.74	4.12	0.84	0.81

Table 2

Descriptive statistics block 2 (Reasoning and Proof)

Item	n	low	neutral	high	mean	sd	r_{i-t}
v2.1	91	4.30	19.35	76.34	4.06	0.85	0.66
v2.2	91	5.26	23.16	71.58	4.00	0.97	0.84
v2.3	91	7.45	15.96	76.60	4.09	0.94	0.81
v2.4	91	2.13	9.57	88.30	4.307	0.74	0.75
v2.5	91	7.37	24.21	68.42	3.88	0.98	0.76
v2.6	91	4.21	14.74	81.05	4.21	0.88	0.78
v2.7	91	11.70	37.23	51.06	3.50	0.96	0.73

Table 3

Descriptive statistics block 3 (Connections)

Item	n	low	neutral	high	mean	sd	r_{i-t}
v3.1	89	3.23	16.13	80.65	4.179	0.87	0.54
v3.2	89	3.23	8.60	88.17	4.20	0.79	0.43
v3.3	89	50.54	32.26	17.20	2.46	1.15	0.80
v3.4	89	40.86	25.81	33.33	2.90	1.23	0.78
v3.5	89	29.79	23.40	46.81	3.20	1.27	0.80
v3.6	89	29.67	20.88	49.45	3.35	1.43	0.78
v3.7	89	3.23	8.60	88.17	4.25	0.84	0.63

Table 4

Descriptive statistics block 4 (Communication)

Item	n	low	neutral	high	mean	sd	r_{i-t}
v4.1	91	4.30	11.83	83.87	4.15	0.80	0.58

v4.2	91	4.30	13.98	81.72	4.07	0.83	0.71
v4.3	91	7.37	10.53	82.11	4.07	0.98	0.73
v4.4	91	9.47	21.05	69.47	3.80	1.07	0.71
v4.5	91	6.38	13.83	79.79	4.098	0.95	0.82
v4.6	91	1.05	6.32	92.63	4.57	0.72	0.65
v4.7	91	3.16	21.05	75.79	3.96	0.82	0.69

Table 5

Descriptive statistics block 5 (Representations)

Item	n	low	neutral	high	mean	sd	r_{i-t}
v5.1	88	5.32	15.96	78.72	3.98	0.92	0.63
v5.2	88	4.26	17.02	78.72	4.09	0.88	0.60
v5.3	88	9.47	20.00	70.53	3.86	1.01	0.60
v5.4	88	1.08	12.90	86.02	4.34	0.74	0.76
v5.5	88	2.11	13.68	84.21	4.17	0.78	0.74
v5.6	88	5.43	10.87	83.70	4.13	0.95	0.73
v5.7	88	8.51	27.66	63.83	3.80	0.96	0.73

In addition to the above, the item-total correlation indices in each block (or construct) have been positive for all questions, with values between 0.433 and 0.841, all of which are above the minimum required (0.3). This means that all items, in greater or lesser quantities, contribute to the measurement model and do so in the same direction.

Exploratory factorial analysis

In order to ensure the internal consistency and unidimensionality of the scales, an Exploratory Factorial Analysis was carried out for each construct using Kaser-Mayer-Olkin's criterion and Bartlett's test of sphericity. A high KMO coefficient was obtained for all the blocks (in the worst case higher than 0.72) as well as a significant p-value, which confirms that the items of each block make up a single factor; that is, the items are measuring the same theoretical concept.

Confirmatory factor analysis

A Confirmatory Factor Analysis was then carried out according to the structural equations approach to assess the construct validity of the measurement scales. Construct

Validity refers to the ability of a measurement tool to actually measure the theoretical concept being studied; the extent to which the conceptual definitions match the operational definitions (Wolf, Joye, Smith, & Fu, 2016). It is necessary to quantify the degree to which inferences can legitimately be made from the operationalizations in our study to the theoretical constructs on which those operationalizations were based. We start from a conceptual model composed of five constructs measured through 7 scales, which represents the relationships between the set of latent or theoretical factors and the variables observed under the following conditions (González-Montesinos & Backhoff, 2010):

(1) Conditional independence: Each latent factor influences a set of observed variables, which are independent from each other, but conditioned by the latent variable that determines them.

(2) The latent factor can be quantified by means of a conceptual structure based on the existence of theoretical constructs that generate a causal influence on the valuations of the participants.

For the estimation of the parameters of the Structural Equations Model, the Diagonal Weighted Least Squares (DWLS) has been used, which produces robust standard errors and adjustment to the existing lack of normality in the variables that comprise the model, since it is a questionnaire with Likert (7) scales (Beaujean, 2014).

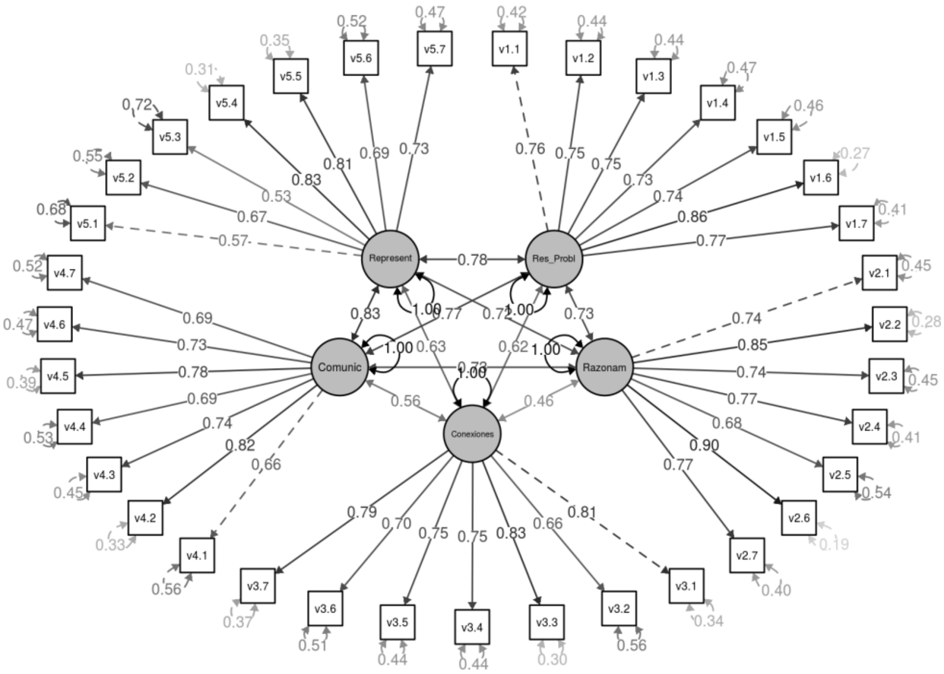

Figure 2. Model of Structural Equations with estimated coefficients and errors in each item. The circles represent the latent variable (construct) and the squares represent the observed variables that constitute the dimensionality of the instrument.

Before explaining in depth the model of structural equations that we present in this work, we provide a detailed account of the study process: in the first instance, we defined a "raw" model in which we introduced all the items of the questionnaire without separating by groups, with the objective of finding out if the items of any of the 5 processes had a notable influence on a theoretical construct that was called "Teaching-learning". None of the groups stood out from the rest and we also obtained reliability indices (nfi=0.892, cfi=0.935, tli=0.931) which, although sufficient, could be improved. We also obtained an RMSEA value of 0.119 (p=0), which would indicate that the proposed model does not fit well with the data. In a second approach, we decided to define the joint model composed of the 5 processes, establishing for each process a latent variable composed of its seven teaching practices. In this way we formed five independent but related groups. We consider that this scheme makes much more sense, remaining as follows:

```
modEAs <-
' Problem solving =~ v1.1 + v1.2 + v1.3 + v1.4 + v1.5 + v1.6 + v1.7
Reasoning and proof =~ v2.1 + v2.2 + v2.3 + v2.4 + v2.5 + v2.6 + v2.7
Connections =~ v3.1 + v3.2 + v3.3 + v3.4 + v3.5 + v3.6 + v3.7
Communication =~ v4.1 + v4.2 + v4.3 + v4.4 + v4.5 + v4.6 + v4.7
Representations =~ v5.1 + v5.2 + v5.3 + v5.4 + v5.5 + v5.6 + v5.7'
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With this model we obtained higher factorial loads and better adjustment indices, which are detailed below. This is the final model with which we will work, since we tried with a last model, equal to the previous one, but in which we eliminated the variable v5.3, because it has a much lower factorial load than the rest. However, the model barely improved so we decided to keep it in the model because of its relevance to the study.

The relationships between latent and observed variables can be interpreted as coefficients of a multiple regression, which show the influence of each construct on its items, so that if the latent factor increases one unit, the items increase according to the weight of their coefficients.

The goodness of fit of the CFA is given by the hypothesis that the difference between the matrix derived from the data and the matrix reproduced by the model is not statistically significant (González-Montesinos & Backhoff, 2010). To measure the fit, the Root Mean Square Error of Approximation (*RMSEA*) has been used, resulting in a value of 0.075, indicating that the questionnaire measurement model and teachers' responses have a reasonable fit.

In addition, the following incremental fit indices (Hooper, Coughlan, & Mullen, 2008) were used: the Normed-fit Index (NFI), Comparative Fit Index (CFI) and the

Tucker-Lewis Index (TLI), all of them resulting in values very close to 1, as we can see in the table.

Table 6

Model goodness adjustment indices

Indicator	value	ideal value	good fit
nfi	0.93	0.9	yes
cfi	0.97	0.9	yes
tli	0.97	0.9	yes

Based on these indices, we can accept the validity of the construct and affirm that the model is relevant to understand the relations between the MPs referred to by the 5 constructs.

Complementarily, and as a measure of internal consistency of the items of each construct, the composite reliability index was calculated for each of the constructs that make up the SEM, obtaining values greater than 0.8 (Varela Mallou & Lévy Mangin, 2006).

Discussion and conclusions

This study has validated the ETMAP classroom observation instrument for evaluating the teaching of mathematics through processes and, in addition, has analysed the level of implementation of MPs in the teaching practice of 95 Early Years and Primary Education teachers, in line with the recommendations made by different organisations (NCTM, 2000, 2014; NAEYC & NCTM, 2002; National Research Council, 2009) and authors (Brandsford et al., 1999; Clements et al., 2004; Ginsburg, 2009; Clements and Sarama, 2011, among others), who highlight the importance of developing mathematics curricula at early ages using MPs, with an emphasis on thinking and doing rather than memorizing definitions and procedures.

An exploratory factorial analysis has been carried out to ensure the internal consistency and unidimensionality of the scales. The obtaining of a high KMO coefficient has confirmed that the items of each block make up one single factor. Furthermore, the CFA based on a structural equation model (Wolf et al., 2016) has enabled us to verify the good fit of the model used to measure the questionnaire as well as the teachers' responses.

Regarding the presence of MPs, our study has been carried out with 95 Spanish teachers of Early Years (3-6 year olds) and Primary Education (6-12 year olds) and the data obtained show medium-high scores in all MPs, except in the process of "connections".

In general, these data indicate that the participants in our study tend to consider MPs in their practice in a fairly systematic way, both in Early Years and Primary Education. As such, these results are in line with the recommendations of Cobb et al. (1991), Trafton and Hartman (1997) and Clements and Sarama (2011), regarding the inclusion of problem solving in mathematics education at early ages. Specifically, these authors suggest that working on problem solving helps students to develop basic skills and strategies as well as higher-level thinking abilities.

Our results in relation to the presence of reasoning and proof in teaching practice show that teachers carry out practices which tend to encourage students to formulate conjectures, giving them time to confirm or refute them, as well as providing them with opportunities to explain and justify their mathematical ideas, in the way proposed by Brandsford et al. (1999). Although most results on reasoning and proof items accumulate high-medium scores, the low presence of item 2.7 (gives feedback with concrete material allowing divergent thinking) is worrying. Based on these data, it could be interpreted that teachers are not interested in promoting critical mathematical thinking or do not know how to do it, which can have negative consequences for achieving mathematically literate citizens, that is, citizens that in addition to using comprehensive and effective mathematical knowledge in all situations of daily life, critically analyse the information (Aizikovitsh & Cheng, 2015). According to these authors, this is very important in teaching practices, because when students have the opportunity to think critically in mathematics in a context of interaction, negotiation and dialogue, they make reasoned decisions or judgments about what to do and think. In other words, students consider the criteria or grounds for a thoughtful decision and do not simply guess or apply a rule without assessing its relevance.

In this way, communication in the mathematics classroom acquires vital importance, as pointed out by Ginsburg (2009), with his allusion to the "math-talk learning community". For the NRC (2009), the use of language is important in order to make connections between the different domains of mathematics. Nevertheless, although the results obtained in our study show that teachers tend to foster communication in the mathematics classroom, low scores have been obtained in relation to the presence of connections in teaching practice. This contradiction is mainly due to the effect of teaching connections [Authors' reference], since in some items of the "connections" process, the items which assess if the mathematics knowledge worked on is related to and linked with other disciplines, such as artistic expression, musical expression or psychomotor skills, obtain low percentages of 50% and 40%. The reasons for these differences could be because: a) teachers find excuses not to implement them (Nolan, 2012); b) it is hard to connect maths to these areas (because of the nature of music/literature/arts); c) teachers don't have enough knowledge to empower connections through teaching practices; d) Spanish teachers in Early years and Primary Education are generalists, and their subject background may influence their decisions on items relating to “connections”; e) or maybe because of the different planning and management of mathematical practices in the two educational stages analysed. While in Early Years practice we observe a tendency to work on mathematics connections with other disciplines, in order to promote the teaching of mathematics based on situations exploring the environment or involving manipulation and experimentation with material and games, it seems that this tendency is not continued in the same way in Primary Education.

Additionally, other aspects that must be kept in mind to interpret the low results are the connections items in the ETMAP classroom observation instrument. Considering that mathematics is everywhere and can be connected to almost anything, items 3.1 and 3.2 refer to inner-mathematical connections; items 3.3, 3.4 and 3.5 cover the "cultural area"; item 3.6 covers psychomotor skills (e.g. gesturing how movements look like in dance or sports); and item 3.7 covers everyday life. According to this, the three items in the cultural area make up for almost half (3/7) of the connections dimension, which makes this dimension heavily biased towards the "cultural area". This fails to consider other possible connections with other fundamental areas such as nature, biology, equity, health, technology or sustainability, which are also squeezed into the “everyday life” in the same way as music, literature or arts. Consequently, the results are probably biased towards the “cultural area” (regardless of all cultural areas) and this aspect should therefore be kept in mind to improve the instrument.

Finally, the data of our study seem to confirm that teachers consider representation as a necessary process in mathematics practice, since this item also obtains medium-high scores. Authors such as Langrall et al. (2008) have already highlighted the importance of working on the representation of mathematical ideas from early ages, since this allows for the creation of models and the interpretation of reality.

An exploratory factorial analysis has been carried out to ensure the internal consistency and unidimensionality of the scales. The obtaining of a high KMO coefficient has confirmed that the items of each block make up one single factor. Furthermore, the CFA based on a structural equation model (Wolf et al., 2016) has enabled us to verify the good fit of the model used to measure the questionnaire, as well as the teachers’ responses.

Limitations and future prospects

One of the main limitations of this study is that data were collected by twenty different evaluators. While they all received the same training to be able to assess the appearance or presentation of mathematics teaching and learning processes, we cannot be entirely sure that the data collection carried out through the interviews and the direct observation of the teaching practice was identical. Future studies could limit the number of evaluators and include other kinds of recording techniques, such as video recording class sessions in order to be able to triangulate the data.

The data obtained, taken alongside the data obtained in future studies with wider sample sizes, should provide the basis for designing intervention programmes that help to improve mathematics teaching practice, by including MPs in a systematic way, in order to foster the professional development of early years mathematics teachers in line with 21st century demands.

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Appendix A

Questionnaire items grouped by blocks

Block 1	PROBLEM SOLVING Indicators
1.1	Raises problematic situations using different types of support (oral, concrete, pictorial).
1.2	Contextualizes the problematic situations within the daily life of the students.
1.3	Proposes various types of problem situations.
1.4	Asks questions that generate research and exploration to solve the problem.
1.5	Allows children to use concrete and/or pictorial material with oral support for problem solving.
1.6	Keeps children engaged in the problem solving process.
1.7	Promotes discussion around problem solving strategies and outcomes.
Blok 2	REASONING AND PROOF indicators
2.1	Invites students to make conjectures.
2.2	Allows the students themselves to discover, analyse and propose different ways of resolution.
2.3	Asks students to explain, justify or argue the strategies or techniques they used during the resolution.
2.4	Asks students questions to develop their answers.
2.5	Encourages students to check out conjectures from everyday life.
2.6	Promotes support for mathematical reasoning.
2.7	Gives feedback with concrete material allowing divergent thinking.
Blok 3	CONNECTIONS Indicators
3.1	Considers students' everyday mathematical experiences to move towards more formal mathematics.
3.2	Makes connections between different mathematical knowledge.
3.3	Develops mathematical activities linked to musical contexts.
3.4	Works on mathematics by linking it to children's literature.
3.5	Relates mathematics to artistic expression.
3.6	Generates mathematical knowledge through contexts linked to psychomotricity.
3.7	Encourages students to apply mathematical knowledge to everyday situations.
Blok 4	COMMUNICATION Indicators
4.1	Places more emphasis on communication in the classroom than the delivery of unidirectional information.
4.2	Encourages interaction with others to learn and understand mathematical ideas.
4.3	Encourages the exchange of mathematical ideas through oral, gestural, graphic, concrete and/or symbolic language.
4.4	Asks the child to explain their strategies and responses with appropriate mathematical language
4.5	Encourages students to respect the way they think and express their points of view on mathematical knowledge.
4.6	Encourages attentive listening to the views of others.
4.7	Intervenes mostly through questions rather than explanations.
Blok 5	REPRESENTATION indicators

5.1	Asks children to talk, listen and reflect on mathematics to move towards symbolic representation.
5.2	Uses specific materials as resources to represent mathematical ideas.
5.3	Uses exemplary models (schemes, among others) to show ways of solving problem situations.
5.4	Works with children on specific representations (drawings, etc.).
5.5	Works with children on pictorial representations (signs, etc.).
5.6	Works with children on symbolic representations (conventional notation).
5.7	Shows two-way work (from specific to abstract and from abstract to concrete.