UNIVERSIDAD DE MURCIA

ESCUELA INTERNACIONAL DE DOCTORADO

TESIS DOCTORAL

Zermelo's problem, Finsler spacetimes and applications

El problema de Zermelo, espacio-tiempos de Finsler y aplicaciones

D. **Enrique Pendás Recondo**

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El problema de Zermelo, espacio-tiempos de Finsler y aplicaciones

y dirigida por,

D./Dña. Miguel Ángel Javaloyes Victoria

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En Murcia, a 14 de diciembre de 2023

Fdo.: Enrique Pendás Recondo

Para mi familia, con cariño

Agradecimientos

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Mi agradecimiento final y más hondo está dirigido a mi familia. A vosotros dedico esta tesis, pues habéis sido siempre mi mayor apoyo. Quiero recordar especialmente a mis abuelos gallegos, que no han podido ver esta tesis completada, pero cuyo recuerdo perdurará siempre en mí. A mis abuelos asturianos, que con tanto cariño me cuidan y de los que tanto he aprendido. Soy mucho mejor persona, y más feliz, gracias a ellos. A Álvaro, que además de ser el mejor hermano del mundo, es también mucho mejor programador que yo, y me ha ayudado a perfeccionar los códigos informáticos relacionados con mi investigación. Esta tesis es mejor gracias a él. Y, por supuesto, a mis padres, a los que debo todo y de los que recibo un apoyo incondicional en cada aventura que comienzo. Gracias, porque me habéis acompañado siempre, todo el camino, y muchas veces cargándome sobre vuestros hombros. Si he conseguido llegar a la meta, es gracias a vosotros.

Enrique Pendás Recondo
Murcia, diciembre de 2023

Resumen

La presente tesis doctoral se compone de cuatro artículos de investigación, con un nexo común centrado en el *problema de Zermelo*, su resolución usando *espacio-tiempos de Finsler* y sus *aplicaciones* a la propagación de ondas y, en particular, de incendios forestales.

Antecedentes

El problema de navegación de Zermelo, punto de partida de esta tesis, consiste en determinar la trayectoria más rápida entre dos puntos fijos para un móvil que se mueve en presencia de una corriente, como un aeroplano en presencia de un viento. En general, el móvil viaja en una variedad diferenciable N—el *espacio*—y se asume que, en cada punto, los vectores que representan su velocidad máxima (teniendo en cuenta el viento) forman una esfera topológica Σ fuertemente convexa en el espacio tangente. En el caso en el que la velocidad es independiente del tiempo, Σ se puede interpretar como la *indicatriz* (vectores unitarios) de una *métrica de Finsler* F, que mide el *tiempo de viaje*: la F-longitud de una curva γ coincide con el tiempo que tarda el móvil en recorrer γ a velocidad máxima. Esto implica que la solución al problema de Zermelo (independiente del tiempo) es una pregeodésica de (N, F).

Si ahora consideramos el *espacio-tiempo* $M := \mathbb{R} \times N$, donde la proyección natural $t : M \to \mathbb{R}$ mide el tiempo absoluto (no relativista), y definimos la *métrica de Lorentz-Finsler* $G := dt^2 - F^2$, entonces todas las posibles trayectorias del móvil, recorridas a velocidad máxima, se corresponden con curvas luminosas en el *espacio-tiempo de Finsler* (M, G). Por tanto, fijados dos puntos $x_1, x_2 \in N$, el problema de Zermelo, consistente en hallar la trayectoria más rápida entre x_1 y x_2, es equivalente a encontrar la curva luminosa que, saliendo desde $(0, x_1) \in M$, llega primero a la línea vertical $t \mapsto (t, x_2)$. El *principio de Fermat* asegura que los puntos críticos del tiempo de viaje—y entre ellos, la solución al problema de Zermelo—son las pregeodésicas luminosas de (M, G).

Este marco espacio-temporal soluciona el caso dependiente del tiempo, en el que F se convierte en una métrica de Finsler en $\mathrm{Ker}(dt)$ pero G sigue siendo una métrica de Lorentz-Finsler en M. Así, mientras que las geodésicas de F no se pueden definir de la forma usual, las geodésicas de G satisfacen, formalmente, las mismas ecuaciones, haya dependencia temporal o no. Por otro lado, también se simplifica el caso de *viento fuerte*, donde la velocidad del viento supera la del móvil y, por consiguiente, Σ no contiene el vector cero. En esta situación, Σ es una *estructura finsleriana con viento* y define dos métricas en el espacio: una *métrica de Finsler cónica* y una *métrica de Finsler lorentziana*, ambas definidas en un dominio cónico—hay direcciones prohibidas, como moverse en contra del viento—. Esto supone una considerable complicación en el espacio, mientras que desde el punto de vista espacio-temporal, los puntos críticos del tiempo de viaje siguen siendo las pregeodésicas luminosas de (M, G).

Esta forma de resolver el problema de Zermelo mediante la *geometría de Finsler* ha generado, recientemente, aplicaciones a problemas físicos muy cotidianos. En esta tesis destacamos especialmente el trabajo pionero de Steen Markvorsen en la modelización de incendios forestales. Obviamente, existen varias diferencias notables entre el problema de Zermelo y la propagación de incendios. Por un lado, el fuego se comporta como una onda en términos de propagación—cumple el *principio de Huygens*—y se busca hallar el *frente del fuego* completo, no una única trayectoria entre dos puntos fijos. Por otro lado, la fuente inicial del incendio no suele ser un único punto—y aunque lo fuese, normalmente se detecta ya como un frente no puntual—. Sin embargo, existe también una similitud clave: cada punto del frente del fuego se puede ver como el punto más rápido de llegada de las trayectorias del fuego que salen de la fuente inicial. A esto hay que añadir, además, el hecho de que la propagación del fuego es altamente anisótropa debido, fundamentalmente, a los efectos del viento y la pendiente. De esta forma, la geometría de Finsler y, en particular, las técnicas usadas para resolver el problema de Zermelo, se vuelven una herramienta muy potente para modelizar este fenómeno.

De hecho, cuando la propagación infinitesimal del incendio—la indicatriz Σ dada por la velocidad del fuego—toma la forma de una elipse en cada punto (en general no centrada en el origen), Markvorsen demostró que las ecuaciones de las geodésicas de (N, F) son equivalentes a las conocidas como *ecuaciones de Richards*: un sistema de EDPs (ecuaciones en derivadas parciales) que utilizan varios simuladores de incendios actuales para calcular el frente del fuego.

Resultados

Partiendo de estos antecedentes con respecto al problema de Zermelo, y motivados por la aplicación a incendios forestales, nuestro interés en esta tesis se ha centrado en tres líneas de investigación principales, que describimos a continuación.

Propagación de ondas

La primera línea de investigación, desarrollada en los Artículos I y II, se centra en la siguiente variación del problema de Zermelo: en lugar de un móvil, supongamos que tenemos una onda que se propaga en todas las direcciones—o en general, cualquier fenómeno que cumpla el principio de Huygens—y queremos encontrar, dada la fuente inicial, la evolución del *frente de ondas* con el tiempo. En general, asumimos que la dimensión es arbitraria, la propagación es anisótropa (dependiente de la dirección) y reonómica (dependiente del tiempo), y el frente inicial S es una subvariedad compacta del espacio (de codimensión arbitraria).

La velocidad de la onda—variando en el espacio, tiempo y dirección—proporciona la indicatriz Σ de una métrica de Finsler F en $\mathrm{Ker}(dt)$ (esencialmente, una métrica dependiente del tiempo en N). Para tener en cuenta la dependencia temporal, consideramos el espacio-tiempo de Finsler (M, G) y asumimos que es *globalmente hiperbólico*. Entonces, aplicando el principio de Huygens, el frente de ondas en el instante $t_0 > 0$ viene dado por $\partial J^+(S) \cap \{t = t_0\}$, donde $J^+(S)$ es el *futuro causal* del frente inicial y $\{t = t_0\} := \{t_0\} \times N$ (véase la Figura 1.1).

Cualquier curva causal contenida en el *borde acronal* $\partial J^+(S)$ es una pregeodésica luminosa G-ortogonal a S y se interpreta como una trayectoria de la onda en el frente de ondas—de hecho, cualquier punto en $\partial J^+(S)$ es alcanzado por una de estas curvas—. Estas trayectorias minimizan el tiempo de propagación—son soluciones al problema de Zermelo desde S—mientras se mantengan en el frente, pero pueden salirse de él: si esto sucede, el último punto de la trayectoria en $\partial J^+(S)$ se denomina *punto de corte*. No obstante, los puntos de corte no empiezan a aparecer hasta después de un cierto tiempo global $\varepsilon > 0$. Es decir, $\partial J^+(S)$ está generado, antes del tiempo ε, por todas las pregeodésicas luminosas G-ortogonales a S, que además podemos calcular resolviendo un sistema de EDOs (ecuaciones diferenciales ordinarias): las ecuaciones de las geodésicas de (M, G). Esto nos proporciona un método eficiente de determinación del frente de ondas; esencialmente, cada pregeodésica calculada nos da un punto en el frente (en cada instante de tiempo) hasta que llega a su punto de corte, momento en el cual deberá descartarse.

En el caso de viento fuerte, $\partial J^+(S)$ sigue proporcionando el frente de ondas y podemos reducir su cálculo al caso anterior eligiendo, en lugar de observadores en reposo, observadores comóviles con el medio—esto es, esencialmente, un cambio de coordenadas, que genera una nueva descomposición de M como producto $\mathbb{R} \times N$—. La diferencia reside en que, ahora, las pregeodésicas luminosas que permanecen en $\partial J^+(S)$ minimizan el tiempo con respecto a los observadores comóviles, y solo aquellas trayectorias que van a favor del viento serán también minimizantes con respecto a los observadores en reposo.

Incendios forestales

La segunda línea de investigación, desarrollada en el Artículo III, se centra en aplicar el marco teórico general establecido en la primera línea para construir un modelo concreto que describa la evolución de un incendio forestal, teniendo en cuenta los efectos del viento y la pendiente—los principales responsables de la anisotropía—.

Para ello, en primer lugar hay que construir una métrica de Finsler cuya indicatriz Σ coincida con la propagación infinitesimal del fuego. En nuestro modelo, el efecto de la pendiente se modeliza por medio de una *métrica de Matsumoto* inversa, que favorece la dirección hacia arriba—el fuego se propaga más rápido cuesta arriba que cuesta abajo—. En presencia de viento, por otro lado, Σ toma la forma aproximada de una doble semi-elipse, que constituye un buen ajuste desde el punto de vista experimental.

La combinación de ambos efectos genera una métrica de Finsler F, con la que podemos construir la correspondiente métrica de Lorentz-Finsler $G = dt^2 - F^2$ y aplicar el sistema de EDOs obtenido en la primera línea de investigación. Esta forma de calcular la propagación del fuego presenta dos ventajas principales con respecto a simuladores de incendios actuales que utilizan las ecuaciones de Richards. La primera ventaja es la precisión y flexibilidad: las ecuaciones de Richards asumen que la propagación infinitesimal del incendio tiene una forma elíptica—la aproximación anisótropa más simple—, tanto con viento como con pendiente. En cambio, Σ adopta en nuestro modelo una forma más compleja, con una diferencia cualitativa entre los efectos generados por el viento y la pendiente. Además, incluso aunque Σ resultara ser empíricamente imprecisa, se podría adaptar a cualquier otra figura fuertemente convexa y el sistema de EDOs seguiría siendo el mismo, formalmente. La segunda ventaja es la eficiencia: desde un punto de vista computacional, resolver un sistema de EDOs (como las ecuaciones de las geodésicas) es más eficiente y sencillo que resolver un sistema de EDPs (como las ecuaciones de Richards).

En esta línea de investigación estudiamos también en detalle los puntos de corte, que pueden aparecer como puntos de intersección de dos trayectorias del fuego (en el espacio-tiempo), o como puntos focales del frente inicial. La detección de estos puntos tiene un papel fundamental en la simulación de incendios. Primero, porque necesitamos excluir las trayectorias que alcanzan sus puntos de corte para determinar de forma precisa el frente del fuego. Y segundo, porque desde un punto de vista práctico, estos puntos se corresponden con puntos de convergencia de las trayectorias del fuego, por lo que señalan las zonas más peligrosas del incendio. De hecho, una ventaja adicional de nuestro modelo es que simplifica enormemente el proceso de tratamiento y detección de los puntos de corte. Cuando calculamos el incendio resolviendo un sistema de EDPs, es necesario corregir el frente cada vez que hay un punto de corte, lo cual es complejo y costoso desde el punto de vista computacional. Cuando resolvemos un sistema de EDOs, por el contrario, cada trayectoria es independiente y podemos desecharla, una vez llega a su punto de corte, sin afectar a las demás.

Finalmente, este modelo se ha implementado computacionalmente en una serie de códigos informáticos, los cuales se pueden encontrar en

https://github.com/Physis10/wildfire_model.

Ley de Snell

En la tercera línea de investigación, desarrollada en el Artículo IV, buscamos extender aún más el problema de Zermelo y su aplicación a la propagación de ondas, asumiendo que la velocidad de la onda es discontinua en la interfaz de separación entre dos medios. Esto genera un efecto de refracción que, en el caso isótropo, viene descrito por la conocida *ley de Snell*.

En el caso anisótropo, el perfil de velocidades de la onda proporciona dos métricas de Finsler F_1, F_2 distintas, una en cada medio. Entonces, asumiendo independencia temporal, una trayectoria (parametrizada por el tiempo) que viaja de un medio a otro es un punto crítico del *funcional tiempo de viaje* si y solo si es una geodésica unitaria de F_1 en el primer medio, una geodésica unitaria de F_2 en el segundo medio, y satisface una determinada condición en el punto de contacto con la interfaz. Esta condición, que generaliza la ley de Snell clásica, establece un cambio de dirección en la curva, obteniendo así una *trayectoria refractada*.

De forma análoga, si consideramos una trayectoria (parametrizada por el tiempo) que vuelve al primer medio tras tocar la interfaz, entonces será un punto crítico—una *trayectoria reflejada*—si y solo si es una geodésica unitaria de F_1 que satisface, en el punto de contacto con la interfaz, una condición que generaliza la *ley de la reflexión* clásica.

En el caso en el que la onda viaja en $\mathbb{R}^2$, la interfaz es una línea recta y F_1, F_2 son normas euclidianas (esto es, la velocidad es isótropa), demostramos que estas leyes de Snell y de la reflexión generalizadas se reducen, en efecto, a las leyes clásicas. Cuando permitimos que la velocidad sea anisótropa pero homogénea—F_1, F_2 son métricas de Finsler constantes—, deducimos las condiciones de existencia de los *ángulos críticos* θ_c^+, θ_c^-, más allá de los cuales no existe refracción y la trayectoria incidente se refleja totalmente. Entre medias, para un ángulo de incidencia $\theta_1 \in (\theta_c^-, \theta_c^+)$, existe un único ángulo de refracción θ_2 que satisface la ley de Snell. Por otro lado, la existencia y unicidad de un ángulo de reflexión θ_3 que cumpla la ley de la reflexión está siempre garantizada.

Finalmente, a la hora de calcular el frente de ondas, este vendrá dado por las trayectorias que no solo son puntos críticos del tiempo de viaje, sino minimizadores globales—las soluciones del problema de Zermelo—. En el escenario anterior en $\mathbb{R}^2$, las trayectorias refractadas siempre son minimizadores globales, pero esto nunca se cumple para las trayectorias reflejadas usuales. En su lugar, entre dos puntos en el primer medio, la curva minimizante es o bien la línea recta que los une, o bien una trayectoria particular formada por tres segmentos, tal que satisface la ley de Snell en el primer punto de quiebre, la ley de la reflexión en el segundo, y el segmento intermedio está situado a lo largo de la interfaz. En conjunto, estos tres tipos de trayectorias minimizantes generan tres zonas distintas del frente de ondas total.

Conclusiones

Esta tesis presenta una serie de resultados, vinculados al problema de Zermelo, los espacio-tiempos de Finsler y sus aplicaciones a la propagación de ondas, que son relevantes tanto desde un punto de vista teórico como práctico.

Desde una perspectiva teórica, los Artículos I y II estudian las propiedades de minimización del tiempo de las geodésicas luminosas en espacio-tiempos de Finsler y presentan un marco geométrico donde se pueden interpretar como trayectorias de una onda en un contexto general: la onda puede ser anisótropa y reonómica en cualquier dimensión y con un frente inicial arbitrario—generalizando, por tanto, el problema clásico de Zermelo—. En este marco teórico, las velocidades de la onda proporcionan una métrica de Finsler en el espacio (y su correspondiente estructura de conos en el espacio-tiempo) y el cálculo del frente de ondas se reduce a resolver un sistema de EDOs—las ecuaciones de las geodésicas de una métrica de Lorentz-Finsler—. El Artículo IV completa esta descripción aportando las leyes de Snell y de la reflexión generalizadas, que proporcionan las trayectorias refractadas y reflejadas cuando la onda cruza la interfaz de separación entre dos medios.

Desde un punto de vista práctico, nos hemos centrado en la modelización de incendios forestales. En el Artículo II se demuestra que el citado sistema de EDOs, en el caso más simple en el que se asume una propagación del fuego elíptica, es equivalente al sistema de EDPs utilizado por simuladores de incendios actuales para calcular el frente del fuego. En el Artículo III desarrollamos un modelo más complejo, construyendo una métrica de Finsler concreta que tiene en cuenta la anisotropía generada por el viento y la pendiente. Este modelo todavía se encuentra en una fase inicial de desarrollo y necesita ser probado experimentalmente. No obstante, su verdadero valor reside, no tanto en la métrica específica—que puede ser fácilmente modificada—, sino en el novedoso uso de la geometría de Finsler, que simplifica la forma de calcular el frente y hace posible superar la restricción elíptica de manera eficiente.

Preface

This thesis is submitted in partial fulfillment of the requirements for the degree of *Philosophiae Doctor* in Mathematics at the University of Murcia. The research presented here was conducted mainly at the University of Murcia and, to a lesser extent, at the University of Granada and the Technical University of Denmark, under the supervision of professors Miguel Ángel Javaloyes Victoria and Miguel Sánchez Caja, along with the scientific collaboration with professor Steen Markvorsen.

The thesis is a collection of four research articles, presented in chronological order of writing (not publication). The articles are preceded by a prelude composed of two introductory chapters. The first one provides background information and motivation for the work, and includes a brief summary of the articles, relating them to each other. The second chapter describes the research goals and outlines the main results of the thesis, with some final conclusions. The first article is joint work with Miguel Ángel Javaloyes. The second and third articles are joint work with Miguel Ángel Javaloyes and Miguel Sánchez. The fourth article is joint work with Steen Markvorsen.

The thesis is written in English, except for the previous acknowledgments section "Agradecimientos" and summary chapter "Resumen", which are written in Spanish.

Enrique Pendás Recondo
Murcia, December 2023

List of Articles

Article I

M. Á. Javaloyes and E. Pendás-Recondo. Lightlike Hypersurfaces and Time-Minimizing Geodesics in Cone Structures. In: A. L. Albujer, M. Caballero, A. García-Parrado, J. Herrera and R. Rubio (eds.), *Developments in Lorentzian Geometry*, Springer Proceedings in Mathematics & Statistics, vol. 389, pp. 159–173, Springer Nature Switzerland AG, Cham, 2022.
DOI: 10.1007/978-3-031-05379-5_10.

Article II

M. Á. Javaloyes, E. Pendás-Recondo and M. Sánchez. Applications of cone structures to the anisotropic rheonomic Huygens' principle. *Nonlinear Analysis*, vol. 209 (2021), 112337 (29 pp.).
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Article III

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Article IV

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Prelude

Chapter 1

Introduction

This chapter describes the general topics to be covered in the thesis, providing background information, the starting motivation and a brief summary of each of the articles.

1.1 Background

1.1.1 Zermelo's problem

The starting point and connecting link of all the topics studied in this thesis is the so-called *Zermelo's navigation problem*, first proposed by the German mathematician Ernst Zermelo in 1931 (see [24]). In this problem, one seeks the fastest trajectory between two fixed points for a moving object that travels in the presence of a current.[1] This is the case, for example, of a travel by boat taking into account the water current, or a travel by airplane taking into account the wind. Zermelo himself solved the problem in dimension two using calculus of variations, obtaining the differential equation that provides the *time-minimizing* trajectory. In the past century, this problem became one of the classical problems of calculus of variations and, more recently, it has also been studied and solved using optimal control theory. Our interest, however, will focus on more geometric methods, mainly involving *Finsler geometry*. The use of Finsler geometry to solve Zermelo's problem was proposed in [20, 3] and since then, there has been an increasing interest to study this problem from a geometric point of view.[2]

In the simplest scenario, the moving object travels on some smooth manifold N—the *space*—and one assumes that its self-propelled maximum speed is smooth and isotropic (i.e., the same in all directions). This means that, at each point, the set of maximum velocity vectors forms a sphere in the tangent space, thus providing the *indicatrix* (unit vectors) Σ^0 of a Riemannian metric h. The current or wind, on the other hand, is modeled as a vector field W on N that displaces the self-propelled velocities. The indicatrix then becomes a non-centered sphere $\Sigma := \Sigma^0 + W$. A usual restriction at this point is to assume that the wind is *mild*, namely, its velocity is less than the one provided by the object's engine. Geometrically, this implies that the indicatrix Σ encloses the zero vector, giving rise to a *Randers metric*—a particular type of *Finsler metric*, which is, essentially, a smooth distribution of (non-necessarily symmetric) norms. This metric F is completely determined by Σ or, equivalently, by the so-called *Zermelo data* (h, W). The key point then is that F measures the *traveltime* in Zermelo's problem: the F-length of a curve γ coincides with the time spent by the object following γ at maximum speed. In conclusion, the solution to Zermelo's problem is a pregeodesic of (N, F).

[1] In general, we will refer to this current as *wind*.

[2] See [10, Section 2] for a complete review of the different variations of Zermelo's problem and how to solve them via Finsler geometry.

1.1.2 Fermat's principle

There is a connection between Zermelo's problem and *Fermat's principle*. Consider the *spacetime* $M := \mathbb{R} \times N$, where the natural projection $t : M \to \mathbb{R}$ measures the absolute (non-relativistic) time. It is possible to construct an auxiliary Lorentzian metric g on M such that all the possible trajectories of the object at maximum speed, when lifted to M, become lightlike curves of the Lorentzian spacetime (M, g)—resulting, specifically, in a *standard stationary spacetime*. So, fixing two points $x_1, x_2 \in N$, Zermelo's problem of finding the fastest travel between x_1 and x_2 is equivalent to finding the lightlike curve departing from $(0, x_1) \in M$ that arrives first at the vertical line $\ell_{x_2} : t \mapsto (t, x_2)$. Precisely, the relativistic Fermat's principle states that a lightlike curve γ is a critical point of the *traveltime functional*,[3] which measures the instant at which γ arrives at ℓ_{x_2}, if and only if γ is a pregeodesic of (M, g). It turns out, furthermore, that t-parametrized lightlike pregeodesics of (M, g) project onto unit-speed geodesics of (N, F). So we have the following conclusion, all in all: lightlike pregeodesics of (M, g) solve a more general Zermelo's problem, where we look not only for the fastest trajectory, but for any path that makes critical the traveltime. Obviously, the global minimum—the proper solution—will also be a lightlike pregeodesic. This correspondence between Zermelo's problem and Fermat's principle is fully developed in [5, Section 4] and [6, Section 7].

In addition, the spacetime viewpoint becomes crucial in the time-dependent Zermelo's problem, where we allow the wind and the self-propelled velocities of the object to depend on time. In this case, we have a time-dependent Randers metric on N and therefore, geodesics in the space cannot be defined in the usual way. In the spacetime M, on the contrary, everything remains the same because the Lorentzian metric g already incorporates the time coordinate, so lightlike pregeodesics satisfy formally the same ODE system,[4] whether or not g is time-dependent.

An additional constraint that can be removed in the basic setting is the assumption that the wind is mild. At the points where the wind is *strong*—namely, stronger than the object's engine—, the set of velocity vectors Σ becomes a non-centered sphere that does not enclose the zero vector. This is called a *wind Finslerian structure* and, instead of being the indicatrix of a single metric, now it defines a *conic Finsler metric* F (with positive definite fundamental tensor) and a *Lorentzian Finsler metric* F_l (with fundamental tensor of coindex 1), both defined in a conic set of directions—there are forbidden directions, e.g. going upwind, as the engine cannot overcome the wind. However, even in this case there is also an associated Lorentzian spacetime (M, g)—specifically, an *SSTK spacetime* (standard with a space-transverse Killing vector field)—such that its lightlike pregeodesics are again the critical points of the traveltime functional. This is studied in detail in [6] (in the time-independent case), where it also becomes clear, even without a time-dependence, that the spacetime viewpoint represents a great advantage over the classical description on the space. Indeed, in the spacetime we only have lightlike geodesics of (M, g) as critical points—although we must distinguish between those going downwind and those going upwind—, whereas in the space, these critical points project onto geodesics of (N, F) or (N, F_l) (up to some limit cases), which complicates the description and solution of the problem.

[3] Also called *arrival time functional* in the literature. When a curve is a minimum of this functional, we say indistinctly that it is *time-minimizing*, *first-arriving*, or that it *minimizes the propagation time* or *traveltime*.

[4] Ordinary differential equation system.

1.1.3 Finsler spacetimes

At the same time that Zermelo's problem was attracting the attention of Finslerian geometers, another line of research within Finsler geometry was developing. This focused on the generalization of Lorentzian geometry to include anisotropies and obtain, among other things, generalized versions of General Relativity (e.g., when we allow the speed of light to be anisotropic). This led to the definition of *Lorentz-Finsler metrics*, *Finsler spacetimes* and *cone structures*. There are several different definitions of these concepts in the literature; throughout this thesis we will follow [11] in this regard. Roughly speaking,[5] a Lorentz-Finsler metric is a (two-homogeneous) pseudo-Finsler one with fundamental tensor of coindex 1 (instead of being positive definite, as is the case with a proper Finsler metric) defined on a conic domain. The directions included in the closure of this domain are regarded as the future causal directions; the metric is positive in the interior—timelike directions—and can be smoothly extended as zero to the boundary—lightlike directions. A smooth manifold endowed with a Lorentz-Finsler metric is called a Finsler spacetime, and the future lightlike directions define what is called a cone structure (the generalization of lightcones in Lorentzian geometry). At each tangent space, the cone structure provides a cone that is non-necessarily isotropic—in the sense that a relativistic observer can measure different speeds of light in different directions.

Going back to Zermelo's problem, recall that we can identify the lightcones of a certain Lorentzian spacetime with the maximum velocities of a moving object, taking into account the wind. In this case, the lightcones do not model light propagation, but the traveling of a non-relativistic object. An observer at rest (an integral curve of $\frac{\partial}{\partial t}$) sees these lightcones as tilted in $\mathbb{R} \times N$, in order to account for the anisotropic travel, but notice that this is not a relativistic interpretation—N is not the relativistic restspace of this observer and t is an absolute time.

Now, recall also that one of the initial assumptions in Zermelo's problem was the isotropy of the self-propelled speed of the moving object (without wind). Finsler spacetimes allow us to remove this restriction. Indeed, assume that the set of self-propelled velocity vectors Σ^0 forms, instead of a sphere, any other (connected, compact) strongly convex hypersurface at each tangent space to N.[6] Then Σ^0 is directly the indicatrix of a Finsler metric F^0 and the inclusion of the wind—either displacing Σ^0 or generating a more sophisticated effect—defines a different Finsler metric F. So, in general (and particularly when the nature of the anisotropy is not known) one can start directly with an arbitrary Finsler metric F on N, which accounts for all the anisotropies present in the travel. In the same spirit as before, we want to construct a spacetime that associates the trajectories of the object on N with lightlike curves on M. However, the fact that F can be any Finsler metric makes it impossible to choose a Lorentzian metric in this case; instead, a Lorentz-Finsler metric is needed. The simplest Lorentz-Finsler metric that fulfills this role—and the one we will use throughout this thesis—is $G := dt^2 - F^2$. Since Fermat's principle is still valid in Finsler spacetimes (see [15]), the picture is, formally, the same as before: the critical points of the traveltime functional—and among them, the solution to Zermelo's problem—are

[5] Technical definitions are provided in the articles.

[6] This means that Σ^0 is, at each point, a topological sphere with positive definite second fundamental form or, equivalently, positive sectional curvature, with respect to any Euclidean scalar product on the tangent space to N.

exactly the lightlike pregeodesics of (M, G) or, equivalently, the *cone geodesics* of the corresponding cone structure. This even applies in the time-dependent case, when F is a time-dependent Finsler metric on N (or, equivalently, a Finsler metric on $\mathrm{Ker}(dt)$).

1.1.4 Applications

The recent increasing interest in the study of Finsler geometry has led to several applications to real-world physical problems, especially when there is an anisotropy involved (see, e.g., applications to sound waves [8, 9] or seismic waves [2, 23]). Among all the contributions, we highlight here the pioneering work by Steen Markvorsen in modeling the spread of wildfires (first proposed in [12] and then developed in [13]), which has become a central reference in this thesis. Observe that there are some significant differences between Zermelo's problem and the spread of wildfires. First, fire behaves like a wave in terms of propagation—technically, it satisfies *Huygens' principle*—and one looks for the entire *firefront*, not a single trajectory between two fixed points. And second, the initial source of the wildfire is usually not a single point, but a submanifold of the space—and even if it were a single point, after some time it would evolve into a non-point firefront, which can be regarded as the initial source. However, there is also a key similarity: each point of the firefront can be considered as the fastest arrival point of the fire trajectories that depart from the initial source. Added to this is also the fact that fire propagation is usually highly anisotropic, mainly due to the effects of the wind and the slope. Therefore, Finsler geometry and, in particular, the techniques used to solve Zermelo's problem, become a powerful tool to model this phenomenon.

In this regard, Markvorsen was able to identify the fire trajectories as Randers geodesics in a time-independent setting, assuming that, at each point, the infinitesimal wildfire spread—the indicatrix Σ given by the velocities of the fire—takes the form of a non-centered ellipse. This establishes an equivalence between the geodesic equations of the Randers metric defined by Σ and the so-called *Richards' equations*: a PDE system[7] developed for the first time in [16] (without using Finsler geometry) and famously known among the wildfire modeling community; in fact, current fire growth simulators such as Firesite [7] or Prometheus [22] still use these equations to compute the spread of wildfires. Subsequently, Markvorsen also studied the time-dependent case (again arriving at Richards' equations) using rheonomic Lagrangians (see [14]).

In the end, this application suggested a deeper connection between Zermelo's problem and the propagation, not just of wildfires, but of any anisotropic wave, and the use of Finsler spacetimes to model in a unified and simplified way the description of the time-dependent case. This provided the main starting motivation of the thesis.

1.2 Motivation

From this initial state of the art, and motivated by the wildfire application, our interest focused on the following variation to Zermelo's problem: instead of a moving object, suppose we have a wave that spreads in all directions. So, instead of fixing two points and searching for the time-minimizing trajectory, we fix only the starting point or region

[7] Partial differential equation system.

and the goal is to find the evolution of the *wavefront* over time. In fact, the previous work we have described suggests that we can consider the following generalizations:

- The dimension can be arbitrary.
- The propagation of the wave can be anisotropic (direction-dependent) and rheonomic (time-dependent).
- The wind can be strong (e.g., when the speed of the current with respect to the Earth is greater than the speed of the wave with respect to the current).
- The wave can be, in general, any physical phenomenon that satisfies Huygens' principle (such as wildfires, sound waves or seismic waves).
- The initial wavefront can be a submanifold of the space (as is usually the case in real-world applications).

Modeling the propagation of such a wave is, essentially, the main topic that connects all the articles presented in this thesis. As an introduction, we now describe briefly some of the initial ingredients and goals of this model (we will delve into the specific goals and results in the next chapter and, nevertheless, a more detailed review can be found in [10, Section 6]).

The velocity of the wave—varying at each point, time and direction—provides the indicatrix Σ of a time-dependent Finsler metric F (or a conic one, if the wind is strong) on the space N. Let $S \subset N$ be the initial wavefront, which is assumed to be a compact submanifold of N (usually a hypersurface). In order to account for the time-dependence, we also consider the Finsler spacetime (M, G) and assume it is *globally hyperbolic*: each $\{t = t_0\}$ is crossed exactly once by any inextendible causal curve.

Among all the possible trajectories of the wave—lightlike curves departing from S—, we are interested in those that generate the wavefront. If $S = \{p\}$, namely, the initial wavefront is a single point, then all the spacetime points that can be reached by the wave are given (by definition) by the *causal future* $J^+(p)$;[8] accordingly, all the spatial points the wave passes through are the projection of $J^+(p)$ on N. Following this reasoning, $\partial J^+(p)$ provides the outermost points reached by the wave, and therefore, $\partial J^+(p) \cap \{t = t_0\}$ represents the wavefront at each time $t_0 > 0$.[9] When S is a proper submanifold, we can apply Huygens' principle: each point of the initial wavefront can be regarded as an independent source of the wave, so the evolution of the wavefront is given now by the *achronal boundary* $\cup_{p \in S} \partial J^+(p) = \partial J^+(S)$. Note that a lightlike trajectory γ entirely contained in $\partial J^+(S)$ is time-minimizing: for any $p_0 = (t_0, x_0) \in \mathrm{Im}(\gamma)$, γ is the first causal curve from S to arrive at the vertical line $\ell_{x_0} : t \mapsto (t, x_0)$. In other words, the trajectories that generate the wavefront are the solutions to Zermelo's problem from S to each point of the wavefront (see Figure 1.1). The main goal then is to find a way to systematically compute these curves.

One final generalization, never considered before in this context, is to assume that the velocity of the wave is discontinuous at the separation interface between two different media. In this case, the time-minimizing curves that cross the interface—or,

[8] Note that all the causal futures are closed, due to the global hyperbolicity of (M, G).

[9] We will use the notation $\{t = t_0\} \coloneqq \{t_0\} \times N$ throughout the thesis.

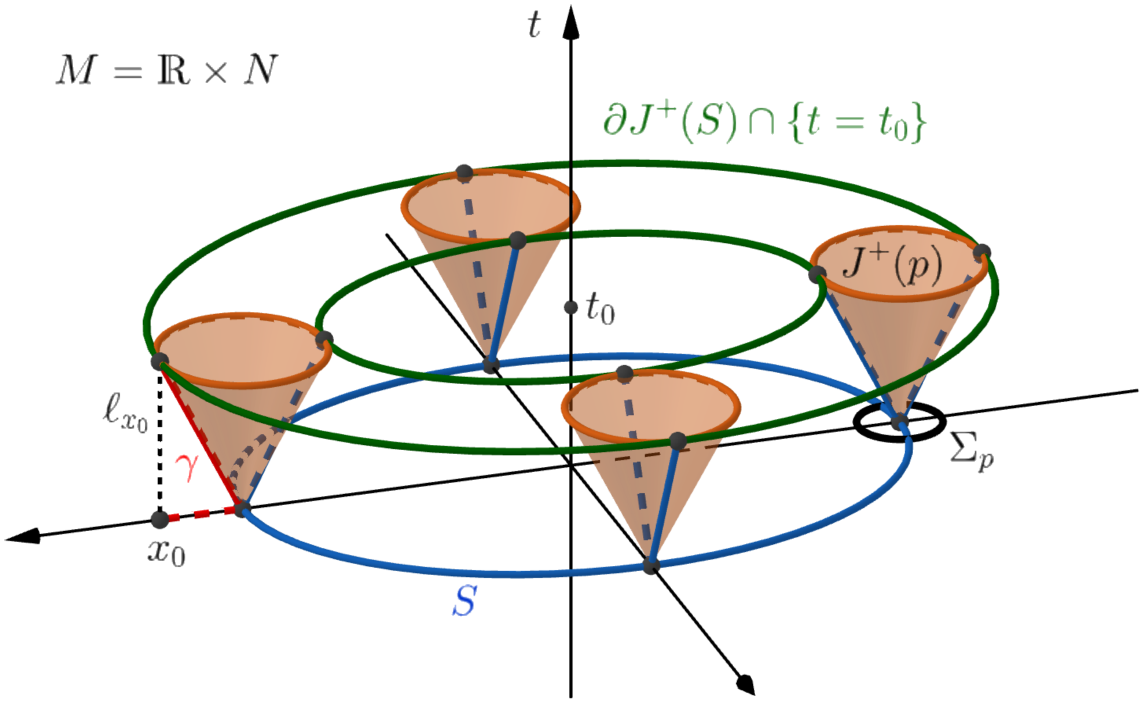

Figure 1.1: A simple representation of the wave evolution in dimension 3 (spatial dimension 2). The indicatrix Σ_p determines the velocity of the wave at p for every direction and the causal future $J^+(p)$ is the region in the spacetime that the wave can reach from p. The envelope $\partial J^+(S)$ of all the causal futures of points in the initial wavefront $S \subset \{t = 0\}$ is generated by time-minimizing curves from S, so $\partial J^+(S) \cap \{t = t_0\}$ represents the wavefront at any $t_0 > 0$. When S is a hypersurface of N, as in the picture, there are exactly two wavefronts; if it were an open domain of N, then $\partial J^+(S)$ would only provide the outward wavefront. A wave trajectory γ in $\partial J^+(S)$ is the solution to Zermelo's problem when traveling from S to some point $x_0 \in N$, i.e., γ is the first causal curve from S to reach $\ell_{x_0} : t \mapsto (t, x_0)$.

equivalently, the solution to Zermelo's problem between two points in different media—are refracted trajectories (broken at the interface) that should satisfy a generalized version of *Snell's law*—the classical law of refraction in physics, only valid for isotropic waves. Analogously, optimal trajectories bouncing off the interface should satisfy a generalized *law of reflection*. The goal then is to find these general laws so that, when considering a wave, we are able to compute both the refracted and reflected wavefronts.

1.3 Summary of Articles

Below is a brief overview of the articles contained in this thesis. The main results are discussed in more detail in the next chapter.

Article I introduces the first definitions regarding Lorentz-Finsler metrics, Finsler spacetimes and cone structures, and focuses on the study of lightlike hypersurfaces, achronal boundaries and time-minimizing cone geodesics within this setting. This provides some relevant properties in the framework of wave propagation, where the evolution of the wavefront is an achronal boundary and the individual wave trajectories are time-minimizing lightlike pregeodesics.

Article II establishes a general theory to model the propagation of any wave or physical phenomenon that satisfies Huygens' principle. The dimension can be arbitrary, the wave can be anisotropic and rheonomic, and the initial wavefront can be any compact submanifold of the space. The wave trajectories are characterized as lightlike pregeodesics—cone geodesics—of the Finsler spacetime departing orthogonally from the initial wavefront. This provides an ODE system (the geodesic equations) that enables us to compute the evolution of the wave over time. The case of wildfire propagation is used as a paradigmatic example, recovering Richards' equations when the indicatrices are ellipses. The case of strong wind is also studied at the end within both the general theory and the particular example of wildfires.[10]

Article III develops a specific model for wildfire propagation, taking into account the time-dependence and the anisotropies caused by the wind and the slope. The general theory established in Article II allows us to compute the wildfire spread once a Finsler metric is given. So, this article focuses on constructing a specific Finsler metric that matches the velocities of the fire. Unlike the examples in Article II, where the wind simply displaces the indicatrices, now the anisotropy is more sophisticated: the wind induces a double semi-elliptical growth, while the slope is modeled using a reverse Matsumoto metric. The advantages of this type of geometric models over current fire growth simulators are discussed at the end.

Article IV introduces another general version of Zermelo's problem, studying the case when the initial and final points belong to different anisotropic media. Critical points of the traveltime functional in this situation become refracted trajectories that break at the interface. This change in the direction of propagation is given by the generalized Snell's law, and the analogous generalized law of reflection is also obtained. The classical laws are recovered from these general versions when the wave is isotropic. We also study in detail the wave propagation and determination of the wavefront—given by the globally time-minimizing trajectories—in $\mathbb{R}^2$ with constant Finsler metrics.

[10] Article II references to this same thesis. All these references must be understood as references to Article I, published later, but whose results had already been obtained before and laid the foundations for the subsequent study carried out in Article II.

Chapter 2

Results and conclusions

This chapter describes the research goals of the thesis and outlines the main results obtained in the articles, along with some final conclusions.

2.1 Research goals

The thesis is divided into three main lines of research, each with its own general and specific goals, which we describe below.

2.1.1 Wave propagation

In the first line of research we aim to extend Zermelo's problem and the Finslerian techniques used to solve it, in order to establish a general theoretical framework to model the evolution of a wave (or any physical phenomenon that behaves as such). Specific goals:

1.1 Study the properties of achronal boundaries in Finsler spacetimes—the mathematical object that models the evolution of the wavefront—and the lightlike curves contained in them—the wave trajectories.

1.2 Allow the wave to propagate in a space of arbitrary dimension, its initial wavefront to be a submanifold of the space (of arbitrary codimension) and its velocity to be anisotropic and rheonomic.

1.3 Find the ODE system that governs the evolution of the wavefront over time, simplifying the PDE system provided by Huygens' principle.

1.4 Study the case of strong wind, when the wave is allowed to spread only in some specific directions.

2.1.2 Wildfires

The second line of research is dedicated to applying the theoretical framework developed in the first line to construct an explicit model that describes the evolution of a wildfire. Specific goals:

2.1 Find a Finsler metric whose indicatrix effectively models the infinitesimal growth of a wildfire, taking into account the effects of the wind and the slope—the main sources of anisotropy.

2.2 Apply the results of the first line of research to obtain an ODE system that provides the evolution of the firefront over time, improving some aspects of current wildfire simulators.

2.3 Study the properties of cut points—points of convergence of the fire trajectories—in Finsler spacetimes, which mark the most dangerous zones for firefighters.

2.4 Implement the model in a computer program, so that, once the parameters of the model have been provided, simulations of a specific wildfire can be carried out.

2.1.3 Snell's law

Finally, in the third line of research we seek to further extend Zermelo's problem and its application to wave propagation, assuming that the speed profile is discontinuous at the separation interface between two different media. Specific goals:

3.1 Find the critical points of the traveltime functional between two points that belong to different anisotropic media, obtaining the condition at the interface that generalizes the classical Snell's law (only valid in the isotropic case).

3.2 Analogously, generalize the classical law of reflection by finding the critical trajectories between two points in the same anisotropic media, with the restriction that the curves must touch the interface.

3.3 In particular, study the setting of $\mathbb{R}^2$ with constant (anisotropic) speed profiles, determining the conditions for the existence and uniqueness of refracted and reflected trajectories, and recovering the classical laws in the isotropic case.

3.4 Determine the refracted and reflected wavefronts for an anisotropic wave in $\mathbb{R}^2$, once it crosses the separation interface.

2.2 Main results

Below we present the main results of the thesis, divided into the three different lines of research and linking them to the goals described above. Articles I and II are dedicated to the first line of research, Article III to the second line, and Article IV to the third one. Bear in mind that the notation used in each article is slightly different, so we also highlight here the main differences that might lead to confusion.

2.2.1 Wave propagation

In Article I, $\mathcal{C}$ denotes a cone structure and (M, L) is an arbitrary Finsler spacetime compatible with it. When $\mathcal{C}$ is globally hyperbolic, the spacetime is topologically a product $M = \mathbb{R} \times D$ and the projection $t : M \to \mathbb{R}$ is a *temporal function*. In this setting, S denotes a compact hypersurface of M (with boundary) included in $\{t = 0\}$.[1] The main result of this article is Theorem 2, which states that the achronal boundary $\partial I^+(S) = \partial J^+(S)$ is a smooth *lightlike hypersurface* (defined and characterized in Section 3), at least for a small time $\varepsilon > 0$. Then, Proposition 4 shows that the only causal curves entirely contained in this hypersurface are all the cone geodesics—lightlike pregeodesics of (M, L)—departing L-orthogonally from ∂S (and pointing outwards).

[1] Note that in Article I, S is a hypersurface of M (not N), so $\partial J^+(S)$ only provides the wavefront pointing outwards in Figure 1.1.

In fact, Theorem 3 concludes that such a curve γ is time-minimizing, at least up to the time ε: for any $p_0 = (t_0, x_0) \in \mathrm{Im}(\gamma)$, with $t_0 < \varepsilon$, γ is the first causal curve from S to arrive at the vertical line $t \mapsto (t, x_0)$. This completes Goal 1.1.

In Article II, we first develop in Section 3 a general framework to model wave propagation. To this end, we use the globally hyperbolic Finsler spacetime $(M = \mathbb{R} \times N, G = dt^2 - F^2)$, where F is the Finsler metric on $\mathrm{Ker}(dt)$ whose indicatrix Σ represents the velocity vectors of the wave. The initial wavefront is $S_0 = \{0\} \times S$, where S is now assumed to be any compact embedded submanifold of codimension r in N. As explained in the previous chapter, the wavefront is given by $\partial J^+(S_0) \cap \{t = t_0\}$ at any time $t_0 > 0$. This complies with Goal 1.2.

Then, in Section 4 we show that the set $\nu(S_0)$ of unitary F-orthogonal directions to S_0 is a fiber bundle on S_0, with fiber diffeomorphic to the sphere $\mathbb{S}^{r-1}$, and define the *wavemap* $\hat{f} : [0, \infty) \times \nu(S_0) \to M$, which provides, for each point $s \in S$ and each F-orthogonal direction $u \in \mathbb{S}^{r-1}$, the unique t-parametrized cone geodesic $t \mapsto \hat{f}(t, s, u)$ with initial velocity $\hat{u} = (1, u)$ at $s_0 = (0, s)$. Namely, $\hat{f}$ provides all the cone geodesics G-orthogonal to S_0, and the *null cut function* $c : \nu(S_0) \to [0, \infty]$ is defined so that $c(s, u)$ is the time at which the corresponding cone geodesic $t \mapsto \hat{f}(t, s, u)$ leaves $\partial J^+(S_0)$—if $c(s, u) < \infty$, we say that $\hat{f}(c(s, u), s, u)$ is a *cut point*. Then, the wavefront can be characterized as

$$\partial J^+(S_0) = \{\hat{f}(t, s, u) : t \leq c(s, u), (s, u) \in \nu(S_0)\},$$

and Theorem 4.8 ensures that there exists a global time $\varepsilon > 0$ such that $c(s, u) > \varepsilon$ for all $(s, u) \in \nu(S_0)$,[2] which generalizes the results in Article I when S is a submanifold of arbitrary codimension. Indeed, up to the time ε, the image of the wavemap generates a smooth lightlike hypersurface that coincides with $\partial J^+(S_0)$, and every cone geodesic $t \mapsto \hat{f}(t, s, u)$ is time-minimizing—and entirely contained in $\partial J^+(S_0)$—until it reaches its cut point (necessarily beyond ε). This leads to the main result of the article, Theorem 4.11, which explicitly provides the ODE system that computes the wavemap at any time. In practice, the wavefront at $t = t_0$ is given by $(s, u) \mapsto \hat{f}(t_0, s, u)$, but one must discard the cone geodesics that have arrived at their cut points before t_0, since they no longer belong to the wavefront. The equivalent PDE system, written as orthogonality conditions, is provided in Theorem 4.14. This fulfills Goal 1.3. When $\dim(N) = 2$, $r = 1$ and Σ is a (smooth) field of ellipses, Theorem 5.5 shows that this PDE system reduces to Richards' equations—famously known for modeling wildfire propagation.

Finally, in Section 6 we tackle Goal 1.4 by considering the case of strong wind. Now the indicatrix Σ does not enclose the zero vector, but the wavefront is still given by $\partial J^+(S_0)$ and we can reduce its computation to the case of mild wind by choosing, instead of observers at rest $\frac{\partial}{\partial t}$, observers comoving with the medium $\frac{\partial}{\partial t} + W$ (essentially, a change of coordinates, which provides a new decomposition of M as a product $\mathbb{R} \times N$). This way, we obtain a proper Finsler metric F^0 (with indicatrix $\Sigma - W$) and the associated Lorentz-Finsler metric $G(\tau, v) = \tau^2 - F^0(v - \tau W)^2$, with which the ODE system in Theorem 4.11 can be computed to obtain the wavefront at any time. However, now the cone geodesics that remain in $\partial J^+(S_0)$ are time-minimizing with

[2] This is a key (and highly non-trivial) result, previously unknown even in Lorentzian geometry, to our knowledge.

respect to the comoving observers, and only the trajectories going downwind are still time-minimizing for the original observers at rest.

2.2.2 Wildfires

Based on the results obtained in Articles I and II, Article III develops a specific model for wildfire propagation. Here $\hat{N} \subset \mathbb{R}^3$ denotes the surface where the fire takes place, which can be seen as a graph $\hat{z}(x, y) = (x, y, z(x, y))$—using the natural coordinates (x, y, z) in $\mathbb{R}^3$—, and $N \subset \mathbb{R}^2$ is the projection of $\hat{N}$ on the xy-plane (i.e., the aerial view). Assuming that the velocities of the fire form a strongly convex oval at each point (enclosing the zero vector), this provides the indicatrix Σ of a Finsler metric F and we can consider the Finsler spacetime $(M = \mathbb{R} \times N, G = dt^2 - F^2)$, as above. Now the initial firefront S_0 is a simple closed curve in $\{t = 0\}$ enclosing a region B_0—the initial burned area—and we look for the outward firefront $\partial J^+(B_0)$.

First, Section 3 focuses on the construction of a suitable F, keeping in mind that Σ represents the infinitesimal wildfire spread. Without wind and slope, we assume that the propagation is isotropic, so Σ is a centered sphere whose radius $a + h$ may vary from one point to another (and over time), depending on the fuel and meteorological conditions. When there is a slope, we model Σ as the indicatrix of a reverse Matsumoto metric, modifying the sphere of radius h. In its usual version, this is a Finsler metric that effectively measures the walking time on a slope, favoring the travel downwards (see, e.g., [21]). Here we reverse this effect, since the fire moves faster upwards than downwards. In the presence of wind, on the other hand, the best experimental fitting for Σ is claimed to be a double semi-ellipse (see [1]). In our model, the sphere of radius a becomes a focus-centered ellipse with eccentricity ε—depending on the strength of the wind—oriented in the wind direction ϕ. The directional sum of this ellipse with the sphere of radius h generates the desired double semi-elliptical shape. Finally, the combined effect of the wind and the slope generates a Finsler metric (so long Σ remains strongly convex) whose explicit expression is given by Equation (3.12), thus completing Goal 2.1. This final metric depends only on the parameters a, h, ε, ϕ and the surface $z(x, y)$.

Then, in Section 4 we apply the results found in Article II to this particular case. The wavemap is now called *firemap* and Theorem 4.1 provides the ODE system that determines the fire trajectories. This way of computing the wildfire spread provides two main advantages over current fire growth simulators that use Richards' equations (such as Firesite [7] or Prometheus [22]), which are exemplified in Section 5. The first advantage is the accuracy and flexibility: Richards' equations assume that the infinitesimal growth of the wildfire is elliptical (the simplest anisotropic approximation), both with wind and slope, while in our model Σ adopts a more sophisticated shape, with a qualitative difference between the effects generated by the wind and the slope. Moreover, even if this Σ turned out to be empirically inaccurate, it could be easily modified to fit any other strongly convex pattern, and the ODE system would still be the same, formally. The second advantage is the efficiency: computationally speaking, solving an ODE system—the geodesic equations of a Lorentz-Finsler metric—is more efficient than solving a PDE system—Richards' equations. This fulfills Goal 2.2.

Cut points are studied in detail in Appendix A.1, following Goal 2.3. Specifically, Proposition A.1 states that cut points appear either as intersection points of two spacetime trajectories of the fire, or focal points of S_0.[3] The detection of these points plays a major role in wildfire simulation. First, because one needs to remove the trajectories that go beyond their cut points, in order to accurately determine the firefront. And second, because from a practical point of view, these points correspond to possible crossovers of the fire, so it is crucial to detect them beforehand. Section 5.2 shows examples of this identification process, which is greatly simplified in our model. Indeed, when solving a PDE system such as Richards' equations, the firefront must be corrected each time there is a cut point, and this process is both computationally complex and expensive. When solving the ODE system, on the other hand, each fire trajectory is computed independently and can be removed (or simply ignored) without affecting the others.

Lastly, the computational implementation of this model can be found in

https://github.com/Physis10/wildfire_model.

There are, on the one hand, codes for Mathematica that implement the model exactly as it is described in Article III, where the parameters—the inputs of the codes—need to be specified as smooth functions. These codes have been used to create the figures of the article. On the other hand, there are codes for Python where the model is discretized, which is a more realistic implementation. Namely, each input is now an array of data in a discrete domain, and some interpolations are subsequently carried out to compute the final trajectories. In `model.py`, the inputs are directly the parameters of the model—a, h, ε, ϕ and $z(x, y)$—, which have a physical meaning but they are not explicitly related to specific physical quantities. For instance, ε is associated with the strength of the wind, but the model does not specify the value of ε given the exact wind speed. This link between the mathematical model and a specific physical one is provided in `rothermel.py`, where the inputs are actual physical conditions and the parameters are calculated using Rothermel's model (see [17, 18]). In addition, `firesite.py` implements the model when the fire propagation is assumed to be elliptical ($h \equiv 0$) and the ellipse dimensions are calculated in the same way as in Farsite (which also uses Rothermel's model). These implementations complete Goal 2.4.

2.2.3 Snell's law

In Article IV, we have a manifold Q divided into two open subsets Q_1, Q_2 with the same boundary $\eta = \partial Q_1 = \partial Q_2$. Each subset is endowed with a different speed profile—anisotropic but time-independent—that do not match continuously at the interface η, giving rise to two different Finsler manifolds $(Q_1, F_1), (Q_2, F_2)$. Fixing two points $q_1 \in Q_1, q_2 \in Q_2$, $\mathcal{N}$ denotes the set of (regular) piecewise smooth curves $\gamma : [0, t_0] \to Q$ from q_1 to q_2 that cross η once at $\tau \in (0, t_0)$. Then, the traveltime one spends following a curve $\gamma \in \mathcal{N}$—with the prescribed speed profiles—is given by the Finsler length

$$T_{\mathcal{N}}[\gamma] = \int_0^\tau F_1(\gamma(t), \gamma'(t))dt + \int_\tau^{t_0} F_2(\gamma(t), \gamma'(t))dt,$$

and we look for the critical points of this traveltime functional.

[3] This generalizes the classical results in Riemannian geometry (see, e.g., [19, Proposition 4.1(1)]) and Lorentzian geometry (see, e.g., [4, Theorem 9.15]).

The main result of the article is Theorem 3.4, which states that a (time-parametrized) curve $\gamma \in \mathcal{N}$ is a critical point of $T_{\mathcal{N}}$ if and only if it is an F_1-unit speed geodesic from q_1 to $\gamma(\tau) \in \eta$, an F_2-unit speed geodesic from $\gamma(\tau)$ to q_2, and it satisfies the generalized Snell's law given by Equation (9)—which establishes a change of direction in the curve. These curves are *refracted trajectories* and generalize Zermelo's problem when the moving object (or wave) travels between two different anisotropic media, thus satisfying Goal 3.1.

In the same way, we can now fix two points $q_1, q_2 \in Q_1$ and consider the set $\mathcal{N}^*$ of curves $\gamma : [0, t_0] \to Q$ from q_1 to q_2 that remain entirely in $\overline{Q}_1$, touching η once at $\tau \in (0, t_0)$. The traveltime $T_{\mathcal{N}^*}$ is then the F_1-length and Theorem 4.1 characterizes its (time-parametrized) critical points—the *reflected trajectories*—as F_1-unit speed geodesics satisfying, at the break point $\gamma(\tau) \in \eta$, the generalized reflection law given by Equation (11). This fulfills Goal 3.2.

In the particular case of $Q = \mathbb{R}^2$ when η is a straight line and F_1, F_2 are Euclidean norms (i.e., the speed profiles are isotropic), we show in Examples 3.7 and 4.3, respectively, that the generalized Snell's and reflection laws reduce to the classical ones. When we allow the speed profiles to be anisotropic but homogeneous—F_1, F_2 are constant Finsler metrics (i.e., Minkowski norms)—, we derive in Section 5.1 the conditions for the existence and uniqueness of solutions. First, θ_c^+, θ_c^- denote the *critical angles*, beyond which there is no refraction and the incident trajectory is totally reflected. Lemma 5.1 determines the conditions for these critical angles to exist and Theorem 5.2 ensures that, for any angle of incidence $\theta_1 \in (\theta_c^-, \theta_c^+)$, there exists a unique angle of refraction θ_2 satisfying Snell's law. On the other hand, the existence of a unique angle of reflection θ_3—satisfying the reflection law—is always guaranteed by Theorem 5.3. This completes Goal 3.3.

Finally, given a wave that spreads in this setting—$\mathbb{R}^2$, a straight line as interface and constant Finsler metrics—, we seek to determine its wavefront, following Goal 3.4. However, this wavefront is given by the trajectories that are not only critical points of the traveltime functional, but in fact global minimizers—the proper solutions to Zermelo's problem. In Section 5.2 we deduce that the usual two-segment refracted trajectories are always global minimizers, but this is never the case for the usual reflected trajectories. Instead, given two points in Q_1, the time-minimizing trajectory is either the straight line, or a three-segment trajectory satisfying Snell's law at the first break point, the reflection law at the second one, and with the middle segment along η. In the end, each type of time-minimizing curves generates a different part of the whole wavefront. This way, we have the *standard wavefront* (straight lines in Q_1), the *refracted wavefront* (refracted trajectories in Q_2) and the *reflected wavefront* (three-segment trajectories in $\overline{Q}_1$), which are characterized in Theorem 5.8 and provide the entire evolution of the wave.[4] The intersection between the standard and reflected wavefronts generates a *cut locus*—the set of all cut points—, which is studied and characterized in Section 5.6.

[4]Even though Snell's and the reflection laws are derived in a time-independent setting, these laws only apply at a specific point and time, so they remain valid in the time-dependent case—although the overall trajectories change—and can be integrated into the general framework of wave propagation developed in Article II.

2.3 Conclusions

This thesis features a series of results, linked to Zermelo's problem, Finsler spacetimes and applications to wave propagation, that are relevant both from a theoretical and practical perspective.

From a theoretical point of view, Articles I and II study the time-minimizing properties of lightlike geodesics in Finsler spacetimes and present a geometric framework where they can be interpreted as wave trajectories in a general context: the wave can be anisotropic and rheonomic in any dimension and with an arbitrary initial wavefront—thus extending the classical Zermelo's problem. Within this setting, the wave velocities yield a Finsler metric on the space (and the corresponding cone structure in the spacetime) and the computation of the wavefront is reduced to solving an ODE system—the geodesic equations of a Lorentz-Finsler metric. In addition, this framework is completed in Article IV with the generalized Snell's and reflection laws, which provide the refracted and reflected trajectories when the wave crosses the interface between two different media.

From a practical point of view, our attention has been focused mainly on wildfires. Article II proves that the aforementioned ODE system, in the simplest case when the fire growth is assumed to be elliptical, is equivalent to the PDE system used by current fire growth simulators to compute the firefront. In Article III we develop a non-elliptical model, constructing a specific Finsler metric that accounts for the anisotropies generated by the wind and the slope. Even though this is a technical complication, it does not entail a computational difficulty, since the ODE system can be particularized to any Finsler metric.[5] This model is still in an early development stage and must be tested experimentally. Anyway, its true value lies not in the metric itself—which can be easily modified if it is not completely accurate—but in the novel use of Finsler geometry, which simplifies the computation of the firefront and allows us to effectively overcome the elliptical constraint.

This thesis also suggests some lines of research for future work:

- The study of cut points, which have proven to be one of the most interesting mathematical objects in this thesis, both from a theoretical and practical standpoint. Although we have derived some of their properties, they can still be further studied in the context of Finsler spacetimes and cone structures.

- The theoretical improvement of the general framework of wave propagation. For instance, we do not take into account possible wave obstacles (e.g., a lake in the case of wildfires), which would create a timelike boundary in the spacetime.

- Experimental testing and improvement of the wildfire model, which could be a first step towards the integration of Finslerian techniques into current fire growth simulators.

- More generalizations of Zermelo's problem. For instance, the case when the speed profile of the moving object is not strongly convex. This generates optimal zig-zag

[5]In fact, the computer programs in https://github.com/Physis10/wildfire_model solve this ODE system in a short lapse of time (irrelevant compared to the real-time spread of a wildfire).

trajectories, as seen in real-world examples such as sailboat navigation.[6]

- Lastly, the setting in Article IV can be generalized to the case where we have Finsler spacetimes instead of Finsler manifolds, thus extending Fermat's principle when there is a discontinuity in the spacetime.

[6]Some comments are made in this regard in Appendix A.2 of Article III, but within the context of wave propagation.

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Articles

Article I

Lightlike hypersurfaces and time-minimizing geodesics in cone structures

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Abstract

Some well-known Lorentzian concepts are transferred into the more general setting of cone structures, which provide both the causality of the spacetime and the notion of cone geodesics without making use of any metric. Lightlike hypersurfaces are defined within this framework, showing that they admit a unique foliation by cone geodesics. This property becomes crucial after proving that, in globally hyperbolic spacetimes, achronal boundaries are lightlike hypersurfaces under some restrictions, allowing one to easily obtain some time-minimization properties of cone geodesics among causal curves departing from a hypersurface of the spacetime.

Keywords: cone structures, cone geodesics, Lorentz-Finsler metrics, Finsler spacetimes, lightlike hypersurfaces, Fermat's principle, Zermelo's navigation problem.

Article II

Applications of cone structures to the anisotropic rheonomic Huygens' principle

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II

Abstract

A general framework for the description of classic wave propagation is introduced. This relies on a cone structure $\mathcal{C}$ determined by an intrinsic space Σ of velocities of propagation (point, direction and time-dependent) and an observers' vector field $\partial/\partial t$ whose integral curves provide both a Zermelo problem for the wave and an auxiliary Lorentz-Finsler metric G compatible with $\mathcal{C}$. The PDE for the wavefront is reduced to the ODE for the t-parametrized cone geodesics of $\mathcal{C}$. Particular cases include time-independence ($\partial/\partial t$ is Killing for G), infinitesimally ellipsoidal propagation (G can be replaced by a Lorentz metric) or the case of a medium which moves with respect to $\partial/\partial t$ faster than the wave (the "strong wind" case of a sound wave), where a conic time-dependent Finsler metric emerges. The specific case of wildfire propagation is revisited.

Keywords: Huygens' principle, Zermelo's navigation problem, wavefront, anisotropic medium, rheonomic Lagrangian, Lorentz-Finsler metrics and spacetimes, wildfire propagation, Analogue Gravity.

Article III

A general model for wildfire propagation with wind and slope

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Abstract

A geometric model for the computation of the firefront of a forest wildfire which takes into account several effects (possibly time-dependent wind, anisotropies and slope of the ground) is introduced. It relies on a general theoretical framework, which reduces the hyperbolic PDE system of any wave to an ODE in a Lorentz-Finsler framework. The wind induces a sort of double semielliptical fire growth, while the influence of the slope is modeled by means of a term which comes from the Matsumoto metric (i.e., the standard nonreversible Finsler metric that measures the time when going up and down a hill). These contributions make a significant difference from previous models because, now, the infinitesimal wavefronts are not restricted to be elliptical. Even though this is a technical complication, the wavefronts remain computable in real time. Some simulations of evolution are shown, paying special attention to possible crossovers of the fire.

Keywords: Finsler metrics and spacetimes, wildfire propagation, Matsumoto metric, nonelliptical fire growth.

III

Article IV

Snell's law revisited and generalized via Finsler geometry

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Abstract

We study the variational problem of finding the fastest path between two points that belong to different anisotropic media, each with a prescribed speed profile and a common interface. The optimal curves are Finsler geodesics that are refracted—broken—as they pass through the interface, due to the discontinuity of their velocities. This "breaking" must satisfy a specific condition in terms of the Finsler metrics defined by the speed profiles, thus establishing the generalized Snell's law. In the same way, optimal paths bouncing off the interface—without crossing into the second domain—provide the generalized law of reflection. The classical Snell's and reflection laws are recovered in this setting when the velocities are isotropic. If one considers a wave that propagates in all directions from a given ignition point, the trajectories that globally minimize the traveltime generate the wavefront at each instant of time. We study in detail the global properties of such wavefronts in the Euclidean plane with anisotropic speed profiles. Like the individual rays, they break when they encounter the discontinuity interface. But they are also broken due to the formation of cut loci—stemming from the self-intersection of the wavefronts—which typically appear when they approach a high-speed profile domain from a low-speed profile.

Keywords: Snell's law, Fermat's principle, Huygens' principle, Zermelo navigation, Finsler metrics, anisotropic discontinuous media, rays with least traveltime, wave propagation.

IV