Computation of multi-facility location Nash equilibria on a network under quantity competition

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Abstract

We deal with the location-quantity problem for competing firms when they locate multiple facilities and offer the same type of product. Competition is performed under delivered quantities that are sent from the facilities to the customers. This problem is reduced to a location game when the competing firms deliver the Cournot equilibrium quantities. While existence conditions for a Nash equilibrium of the location game have been discussed in many contributions in the literature, computing an equilibrium on a network when multiple facilities are to be located by each firm is a problem not previously addressed. We propose an integer linear programming formulation to fill this gap. The formulation solves the profit maximization problem for a firm, assuming that the other firms have fixed their facility locations. This allows us to compute location Nash equilibria by the best response procedure. A study with data of Spanish municipalities under different scenarios is presented and conclusions are drawn from a sensitivity analysis.

Keywords: Multi-facility location, Nash equilibria, Network optimization, Spatial Cournot competition.

1 Introduction

Location choice in spatial competition often deals with models based on a two-stage game. In the first stage, the competing firms select their facility locations, in the second stage, they compete on either price (Bertrand competition) or quantity (Cournot competition). There are hundreds of papers which have studied the Bertrand two-stage game for a variety of alternative pricing policies, aiming at finding whether there is a location equilibrium under different settings, and analyzing the resulting location patterns. A relatively smaller amount of research has dealt with the Cournot two-stage game. Although price competition is more common than quantity competition, firms compete on quantities in many markets in which finding a location Nash equilibrium could be of interest, as it happens in supply chain management (see [17, 20]). For a variety of models, the existence of a unique Nash equilibrium

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in the second stage is well known . Such equilibrium usually depends on where the facilities of the competing firms are located in the first stage. Then, the two-stage game reduces to a single stage location game, for which the existence and determination of location Nash equilibria have been investigated.

Most of the papers consider the customers distributed on a linear segment, a circumference, or a circle, where firms will locate their facilities (see for instance [2, 12, 15, 16, 27]). The existence of a Nash equilibrium for the location game has been proved in diverse models. Under Bertrand competition, two conclusions generally unanimous at equilibrium are: first, firms never agglomerate, second, each customer is served by a single firm. The reason for the former is that coincident locations of firms offering homogeneous product intensify price competition and drive profits to zero (see [19, 26]). The reason for the latter is that the customer is served by the firm offering the lowest price, which is the one with the minimum delivered cost (see [12]). Under Cournot competition, agglomeration is often found in linear markets, where firms usually agglomerate at the center of the market (see [14, 15]). However, agglomeration may not be found in some circular markets, where there may exist equilibria with firms locating equidistant from one another (see [2, 27]). For instance, see [6] for a recent review on agglomeration. On the other hand, under Cournot competition, contrary to Bertrand competition, at equilibrium each customer is served by all competing firms (see [1, 21]).

The location game has also been studied for spatially separated markets, mainly when the location space is a network. Under mill pricing, location Nash equilibria rarely exist. On a tree network, some conditions for existence are shown in [5, 30]. Under delivered pricing, a location Nash equilibrium exists if demand is fixed in each market (see [28]). In such a case, Nash equilibria can be found by minimizing the social cost. This problem has been solved on a planar space with single facilities (see [8]), and on a network when firms compete with multiple facilities (see [28]). If demand is price sensitive, the existence of a Nash equilibrium has not been proved for delivered pricing. However, under Cournot competition, the existence of Nash equilibria has been proved for price sensitive demand in both single and multi-facility location (see for instance [11, 18, 22, 31, 33]). Although it is known that Nash equilibria exist under quantity competition, few papers have investigated how to find such equilibria on a nonlinear location space. To our knowledge, Nash equilibria have been found for illustrative examples and some methods have been proposed to find such equilibria (see for instance [22, 32]). In [22] it is shown that the Nash equilibrium can be obtained by solving a variational inequality when firms locate their facilities on a discrete network. This method is also used in [23] to determine the reaction function for a leader when the followers play the Cournot game. In the previous two papers the authors consider a model where it is allowed the possibility that each firm can locate at every node, which means that the number of facilities to be located by each firm is not fixed. In [32] each firm locates one facility on a discrete network and full enumeration is used to determine the Nash equilibrium. This method cannot be extended to the multi-facility case due to the complexity of determining profit maximization locations by enumeration. We study the best response procedure to deal with multi-facility location when the number of facilities of each competing firm is fixed, and facility locations are points in a network, vertices or nodes. This requires to solve a network nonlinear optimization problem. Our main contribution is an integer linear programming formulation to solve the nonlinear multi-facility follower problem, that is to say, to find profit maximization locations for a firm, assuming that the other firms have fixed the location of their facilities. Previously, we show that optimal locations can be found at the nodes. The given formulation is used within the best response procedure, which requires to solve a sequence of follower problems in each iteration, to compute multi-facility location Nash equilibria. The proposed formulation allows to tackle large size problems, as it is shown by solving an illustrative example with real data from Spanish municipalities for different scenarios.

The paper is organized as follows. In Section 2, the location-quantity problem for multi-facility location is described. Basic hypotheses and notation are given, the equilibrium quantities are determined, and the problem is reduced to a location game. In Section 3, it is proved that optimal facility locations can be found at the nodes of the network. Then an integer linear programming formulation to determine the optimal locations is presented together with the best response procedure to find a Nash equilibrium. In Section 4, the illustrative example is solved for different values of the parameters. Finally, some conclusions are given in Section 5.

2 The location-quantity problem

Let $N = (V, E, l)$ be a network, with node set $V = \{v_k : k = 1, ..., n\}$, edge set $E = \{e : e = [v_k, v_j];\ v_k, v_j \in V\}$, and $l(e)$ being the length of edge e. Distance between two points a and b in the network is measured as the length of the shortest path linking the two points and it will be denoted by $d(a, b)$ (see [29]). We consider there is a set of consumption market $M = \{1, ..., m\}$ which are aggregated in the nodes $v_k,\ k = 1, ..., m$ (see [10] for demand point aggregation). Note that the network may contain some nodes on which no market is grouped, which occurs if there are some linking nodes with no customers around (the nodes $v_k,\ k = m + 1, ..., n$). There is a fixed number of firms which compete for demand of an homogeneous product with the aim of profit maximization. First, firms select the location of their facilities, then firms compete on the quantities delivered to each market. It is assumed that the profit of any firm in any market is independent of the profit obtained in any other market, and the unit delivered cost is independent of the quantity delivered.

The following notation will be used:

Indices

i, h	indeces of firms; $i, h = 1, \ldots, r$.
j	index of location candidates (in discrete location space) ; $j = 1, \ldots, n$.
k	index of demand nodes; $k = 1, \ldots, m$.

Data

f_i	number of facilities of firm i.
$L = V \bigcup E$	set of location candidates.
$M = \{1, 2, ..., m\}$	set of markets.
$d(x, k)$	distance between location x and demand node v_k; $x \in L$, $k \in M$.
$p_k(q) = \alpha_k - \beta_k q$	inverse demand function in market k; $k \in M$.
$pc_i(x)$	unit production cost of firm i at location x; $x \in L$.
$tc_i(x, k) = T(d(x, k))$	unit transportation cost of firm i from location x to market k; $x \in L$, $k \in M$.
$dc_i(x, k) = pc_i(x) + tc_i(x, k)$	unit delivered cost of firm i from location x to market k; $x \in L$, $k \in M$.

Decision variables

X_i	set of facility locations of firm i.
q_{ik}	quantity offered by firm i to market k.

Miscellaneous

$X = (X_1, X_2, ..., X_r)$	vector of locations for the facilities of the competing firms.
X_{-i}	vector of locations for the facilities of the firms other than i; $X = (X_i, X_{-i})$.
$Q = (q_{ik})$	matrix of all quantities delivered by the firms to the markets.
$Q_k = q_{1k} + q_{2k} + ... + q_{rk}$	total quantity delivered to market k.
$C_k(X_i) = \min\{dc_i(x, k) : x \in X_i\}$	minimum delivered cost from X_i to market k.

Once the locations X and the quantities Q are fixed, the profit made by firm i at market k is:

$$\Pi_{ik}(X, Q) = q_{ik}(p_k(Q_k) - c_{ik})$$

where c_{ik} is the unit delivered cost of firm i at market k. Note that the smaller c_{ik}, the greater profit will be obtained by firm i at market k. Then for profit maximization firm i will deliver the quantity q_{ik} from the facility with the minimum unit delivered cost, which means that it can be taken $c_{ik} = C_k(X_i)$. Therefore, the profit made by firm i at market k will be given by:

$$\Pi_{ik}(X, Q) = q_{ik}(p_k(Q_k) - C_k(X_i))$$

In the following two subsections, we briefly present some known results which will be used to formulate the multi-facility location problem for a firm assuming that the facilities of other firms have already been established.

2.1 Quantity competition

Once the vector of locations X is fixed, the firms will compete on quantities in each market. Each firm i maximizes its profit Π_{ik} by offering the quantity q_{ik} for which $\partial \Pi_{ik} / \partial q_{ik} = 0$. Hence, the following system of linear equations is obtained:

$$\alpha_k - \beta_k(q_{1k} + q_{2k} + ... + q_{rk}) - \beta_k q_{ik} = C_k(X_i), \qquad i = 1, 2, ...r.$$

The solution to the system of equations are the equilibrium quantities or Cournot quantities at market k, which are given by:

$$q_{ik}^* = \frac{1}{(r+1)\beta_k}(\alpha_k + \Sigma_{h\neq i} C_k(X_h) - rC_k(X_i)), \qquad i = 1, 2, ...r.$$

Notice that the equilibrium quantities depend on the location of the facilities of the competing firms.

The system of linear equations has a unique solution with positive quantities if $p_k(Q_k) > C_k(X_i)$ for all i. The condition $p_k(Q_k) > C_k(X_i)$ for all i is equivalent to,

$\alpha_k > (r+1)[\max\{C_k(X_i) : i = 1, 2, ..., r\}] - \sum_{i=1}^{r} C_k(X_i)$ (1).

We will assume that (1) holds. This is a reasonable assumption since the unit delivered cost to market k is usually small compared with the maximum price that customers in market k are willing to pay for the product.

2.2 Location competition

If the firms offer the equilibrium quantities q_{ik}^*, $i = 1, 2, ..., r$, at each market k, the location-quantity game reduces to a location game, where the profit of any firm i is given by:

$$\Pi_i(X) = \sum_{k=1}^{m} \Pi_{ik}(X, Q^*) = \sum_{k=1}^{m} \beta_k (q_{ik}^*)^2 = \frac{1}{(r+1)^2} \sum_{k=1}^{m} \frac{1}{\beta_k}(\alpha_k + \Sigma_{h\neq i} C_k(X_h) - rC_k(X_i))^2.$$

In the location game the firms will compete on location for profit maximization. A question of interest is whether there exists a Nash Equilibrium for this location game. We will use the abbreviation NE to refer indistinctly the singular or plural form, Nash Equilibrium or Equilibria, respectively. A NE is a set of locations for the firms such that no firm will increase its profit by changing its facilities to another locations if the locations of the other firms remain unchanged. In other words, a vector X^* is a NE if for any firm i it is verified that,

$$\Pi_i(X_i, X_{-i}^*) \leq \Pi_i(X_i^*, X_{-i}^*), \quad \forall X_i \in L.$$

A well known method to prove the existence of a NE in non-cooperative games is based on the best response function (see [4]). Given a vector X of facility locations for the firms, the best response of any firm i to the locations of its competitors X_{-i} is defined as,

$$R_i(X_{-i}) = \{\hat{X}_i : \Pi_i(\hat{X}_i, X_{-i}) \geq \Pi_i(X_i, X_{-i}), \quad \forall X_i \in L\}$$

The best response to the vector X is then defined as the following multi-function:

$$R(X) = (R(X_{-1}), R(X_{-2}), ..., R(X_{-r})).$$

It is verified that, X^* is a NE if and only if $X^* \in R(X^*)$.

The existence of NE is usually proved by generating a sequence of vectors $\{X^\nu : \nu = 1, 2, ...\}$, where $X_i^{\nu+1} \in R_i(X_1^{\nu+1}, ..., X_{i-1}^{\nu+1}, X_{i+1}^{\nu}, ..., X_r^{\nu})$, $i = 1, ..., r$, and showing that in some iteration ν_0 it will be verified that $X^{\nu_0+1} = X^{\nu_0}$. Then $X^{\nu_0} \in R(X^{\nu_0})$, and therefore X^{ν_0} is a NE. This method was used in [33] to prove the existence of a NE for the above mentioned location game, but no procedure has been given to find the best response $R_i(X_{-i})$ for any firm i, and therefore the problem of determining a NE has not been solved for multi-facility location.

In the following, we will show how to find optimal multi-facility locations for a firm when facility locations of its competitors have been fixed. This problem is known as *the follower problem* in the location literature. Then the best response method will be used to find a NE for the location game with multi-facilities.

3 Multi-facility location and NE

3.1 The follower problem

The best response of firm i to the locations of its competitors X_{-i}, $i = 1, ..., r$, is an optimal solution to the following optimization problem:

$$P_i(X_{-i}): \quad Maximize \quad \Pi_i(X_i, X_{-i}) = \frac{1}{(r+1)^2}\sum_{k=1}^{m}\frac{1}{\beta_k}(\alpha_k + \Sigma_{h\neq i}C_k(X_h) - rC_k(X_i))^2$$

$$s.t. \qquad \mid X_i \mid = n_i, X_i \subset L$$

Proposition 1 *If $pc_i(x)$ is a concave function when x varies along any edge in the network, and $T(d(x,k))$ is an increasing and concave function of distance, then there exists an optimal solution $\hat{X}_i$ to $P_i(X_{-i})$ so that $\hat{X}_i \subset V$, $i = 1, ..., r$.*

Proof: Let $X = (X_1, X_2, ..., X_r)$ be any set of $f_1 + f_2 + ... + f_r$ points on the network. If we consider that the points in X_{-i} are fixed, then we will prove that there exists a set V_i with f_i nodes such that $\Pi_i(X_i, X_{-i}) \leq \Pi_i(V_i, X_{-i})$. In fact, suppose that there is a point $x \in X_i$ which is not a node, x is on some edge $e = (v, v')$. $\Pi_i(X_i, X_{-i})$ can be seen as a function of x, assuming that the other points in X_i have also been fixed. Since for any k, the function $d(x, k)$ is concave when x varies along the edge (v, v') (see [29]), then $pc_i(x) + T(d(x, k))$ is also a concave function when x varies along the edge (v, v'). Due to the fact that a function given as the minimum of concave functions is also concave, it follows that $C_k(X_i)$ is a concave function, and $(\alpha_k + \Sigma_{h\neq i}C_k(X_h) - rC_k(X_i))^2$ is a convex function, when x varies along the edge (v, v'). Therefore, $\Pi_i(X_i, X_{-i})$ is a sum of convex functions which reaches its maximum value on (v, v') at any of the two nodes v or v'. As a consequence, by replacing each edge point $x \in X_i$ by $v(x) = v$ or $v(x) = v'$ we will obtain a set of nodes V_i such that $\Pi_i(X_i, X_{-i}) \leq \Pi_i(V_i, X_{-i})$. Therefore, there exists an optimal solution $\hat{X}_i$ to $P_i(X_{-i})$ so that $\hat{X}_i \subset V$, $i = 1, ..., r$. □

From the previous proposition, it can be guaranteed that there is a NE with facilities placed at different nodes of the network. Taking this assertion into account, a NE can be found by solving an iterative sequence of discrete location problems $P_i(X_{-i})$, where feasible solutions are reduced to sets of points $X_i \subset V$, $i = 1, ..., r$.

3.2 Integer linear programming formulation of problem $P_i(X_{-i})$

Each problem $P_i(X_{-i})$ can be formulated as a Binary Integer Linear Programming problem as follows:

Let $S_k(X_{-i}) = \alpha_k + \Sigma_{h\neq i}C_k(X_h)$, then the objective function of problem $P_i(X_{-i})$ can be expressed as,

$$\Pi_i(X_i, X_{-i}) = \frac{1}{(r+1)^2} \sum_{k=1}^{m} \frac{1}{\beta_k} (S_k(X_{-i})^2 - 2rS_k(X_{-i})C_k(X_i) + r^2 C_k(X_i)^2)$$

Since X_{-i} is fix, the function $\Pi_i(X_i, X_{-i})$ is a nonlinear decreasing function in $C_k(X_i)$, $k = 1, ..., m$. Both $C_k(X_i)$ and $C_k(X_i)^2$ can be expressed by a sum of linear functions by defining the following variables:

$$x_{ij} = \begin{cases} 1 & \text{if a facility of firm } i \text{ is located at node } v_j \\ 0 & \text{otherwise} \end{cases}$$

$$y_{ijk} = \begin{cases} 1 & \text{if market } k \text{ is served by firm } i \text{ from } v_j \\ 0 & \text{otherwise} \end{cases}$$

For simplicity, let s_{ijk} denote the unit delivered cost of firm i from node v_j to market k, $s_{ijk} = dc_i(v_j\ , k)$. A feasible solution is defined by $X_i = \{v_j : x_{ij} = 1\}$, where $\sum^n{}_{j=1} x_{ij} = f_i$. Then $C_k(X_i)$ and $C_k(X_i)^2$ are given as,

$$C_k(X_i) = \min\{s_{ijk} : v_j \in X_i\} = \min\{\sum_{j=1}^{n} y_{ijk}s_{ijk} : \sum_{j=1}^{n} y_{ijk} = 1, 0 \le y_{ijk} \le x_{ij}\}$$

$$C_k(X_i)^2 = \min\{s_{ijk}^2 : v_j \in X_i\} = \min\{\sum_{j=1}^{n} y_{ijk}s_{ijk}^2 : \sum_{j=1}^{n} y_{ijk} = 1, 0 \le y_{ijk} \le x_{ij}\}$$

For each k, if $C_k(X_i) = s_{ijk}$ ($C_k(X_i)^2 = s_{ijk}^2$) for some j, then $y_{ijk} = 1$ and $y_{ihk} = 0$ for $h \neq j$ is an optimal solution of the two previous minimization problems. In such a case it must be verified that, $x_{ih} = 0$ for any h such that $s_{ihk} < s_{ijk}$. Therefore, problem $P_i(X_{-i})$ is equivalent to the following Binary Integer Linear Programing ($BILP$) problem:

$$Maximize \quad \tfrac{1}{(r+1)^2} \textstyle\sum_{k=1}^{m} \tfrac{1}{\beta_k} (S_k(X_{-i})^2 - 2rS_k(X_{-i}) \sum_{j=1}^{n} y_{ijk}s_{ijk}$$

$$+r^2 \textstyle\sum_{j=1}^{n} y_{ijk}s_{ijk}^2)$$

s.a.

$$\textstyle\sum_{j=1}^{n} x_{ij} = f_i \quad ; \tag{1}$$

$$y_{ijk} \le x_{ij} \quad ; \quad j = 1, ..., n \quad k = 1, ..., m \tag{2}$$

$$\textstyle\sum_{j=1}^{n} y_{ijk} = 1 \quad ; \quad k = 1, ..., m \tag{3}$$

$$\textstyle\sum_{s_{ihk} < s_{ijk}} x_{ih} \le n_i(1 - y_{ijk}) \quad ; \quad j = 1, ..., n \quad k = 1, ..., m \tag{4}$$

$$x_{ij}, y_{ijk} \in \{0, 1\}; \quad j = 1, ..., n, \quad k = 1, ..., m \tag{5}$$

The objective function gives the profit of firm i. Constraint (1) indicates the number of facilities to be located by firm i. Constraints (2) guarantee that firm i can only deliver the product from nodes v_j where the firm opens a facility. Constraints (3) means that each market k will be served by firm i. Constraints (4) guarantee that firm i will deliver the product to each market k from the facility with the minimum delivered cost, $C_k(X_i) = \min\{s_{ijk} : x_{ij} = 1\}$, $k = 1, ..., m$. Constraints (5) require that variables x_{ij} and y_{ijk} are binary.

If constraints $y_{ijk} \in \{0, 1\}$ in (5) are replaced by constraints $y_{ijk} \geq 0$, the previous problem can be written as a Mixed Integer Linear Programming (*MILP*) problem.

Proposition 2 *There is an optimal solution with binary variables to the MILP problem.*

Proof: Let ($\bar{x}_{ij}$,$\bar{y}_{ijk}$) be an optimal solution to the *MILP* problem. Once the facility locations are fixed, the maximum profit of firm i from any market k is obtained by serving that market from the facility with the minimum delivered cost. Note that the equilibrium quantity of firm i at any market k is greater as long as the delivered cost is smaller. Consequently, at optimality it must be verified that:

$$\sum_{j=1}^{n} \bar{y}_{ijk} s_{ijk} = \min\{s_{ijk} : \bar{x}_{ij} = 1\}.$$

Then, for each market k, only variables y_{ilk} with $s_{ilk} = \min\{s_{ijk} : \bar{x}_{ij} = 1\}$ can be greater than 0 in the optimal solution. As constraints (3) have to be verified, if the minimum s_{ilk} is unique for all k, then ($\bar{x}_{ij}$,$\bar{y}_{ijk}$) is a binary solution. Otherwise, there must exist multiple variables y_{ilk} with $\bar{y}_{ilk} > 0$ for some k, all of them with the same s_{ilk} value. By taking $\hat{y}_{ilk} = 1$ for someone l for which $s_{ilk} = \min\{s_{ijk} : \bar{x}_{ij} = 1\}$ and $\hat{y}_{ijk} = 0$ for $j \neq l$, a new feasible solution ($\bar{x}_{ij}$,$\hat{y}_{ijk}$) to the *MILP* problem is obtained, which is a binary solution. Since both solutions ($\bar{x}_{ij}$,$\bar{y}_{ijk}$) and ($\bar{x}_{ij}$,$\hat{y}_{ijk}$) have the same objective value, it follows that ($\bar{x}_{ij}$,$\hat{y}_{ijk}$) is also an optimal solution to the *MILP* problem. □

From Property 2, it follows that the *MILP* formulation can also be used to solve problem $P_i(X_{-i})$. Note that the *BILP* formulation contains $n(m + 1)$ binary variables, while the *MILP* formulation only contains n binary variables. In subsection 4.1, we will show that run times to solve $P_i(X_{-i})$ with the *BILP* formulation are much more higher than run times to solve it with the *MILP* formulation.

3.3 Finding a NE

Once we have shown that problem $P_i(X_{-i})$ can be solved by Integer Linear Programming, finding a NE can be done by using the best response procedure, which is as follows:

Algorithm MFNE (Multi-facility Nash Equilibrium)

1: Select an initial set V^0 of nodes.
$V^0 = (V_1^0, V_2^0, ..., V_r^0)$, $|V_i^0| = f_i$, $i = 1, ..., r$. Set $\nu = 0$.

2: **For** $i = 1, ..., r$ **do**
Find an optimal solution $V_i^{\nu+1}$ to problem $P_i(V_1^{\nu+1}, ..., V_{i-1}^{\nu+1}, V_{i+1}^{\nu}, ..., V_r^{\nu})$.
Set $V_i^{\nu+1} = V_i^{\nu}$ if $\Pi_i(V_1^{\nu+1}, ..., V_i^{\nu+1}, V_{i+1}^{\nu}, ..., V_r^{\nu}) = \Pi_i(V_1^{\nu+1}, ..., V_{i-1}^{\nu+1}, V_i^{\nu}, ..., V_r^{\nu})$.
end for

3: If $V_i^{\nu+1} = V_i^{\nu}$, $i = 1, ..., r$, a NE is found, **STOP**.
Otherwise, set $\nu = \nu + 1$ and **go to** step 2.

In each iteration, problems $P_i(V_1^{\nu+1},...,V_{i-1}^{\nu+1},V_{i+1}^{\nu},...,V_r^{\nu})$, can be solved with either the *BILP* or the *MILP* formulation by using any standard *ILP* optimizer. Since the *BILP* formulation has too many binary variables, which would make large/medium size problems unable to solve in a short time, we will use the *MILP* formulation to find NE in the test problems shown in subsections 4.2 and 4.3.

4 An illustrative example

We have considered the transportation network in Spain to test our model. We have taken the municipalities over 5,000 inhabitants as demand nodes to have a real size example with more than 1000 markets. These municipalities have been numbered from 1 to 1.049 in decreasing population size, thus $M = \{1, 2, ..., 1.049\}$ (see Fig. 1- *Left*). Since firms may not be interested in locating in small municipalities, we have considered the municipalities over 40,000 inhabitants as location candidates, thus $L = \{1, 2, ..., 142\}$ (see Fig. 1-*Right*). It is assumed that a maximum of one out of a thousand of inhabitants in each municipality is able to buy one unit of product. We have then taken the following inverse demand function at municipality k:

$$p_k(q) = 1400 - \frac{1400}{m_k} \quad q \quad , \qquad 0 \leq q \leq m_k$$

where m_k is given by:

$$m_k = \frac{1}{1000} \times \text{size of municipality } k.$$

The population size and geographical coordinates of the Spanish municipalities can be seen on the web: *http://www.um.es/geloca/gio/datos-espana-2015.txt*. Distances $d(j, k)$ between any pair of municipalities j and k have been approximated by using the Harversine formula, which measure the distance between two geographical points from their longitudes and latitudes (see [24]).

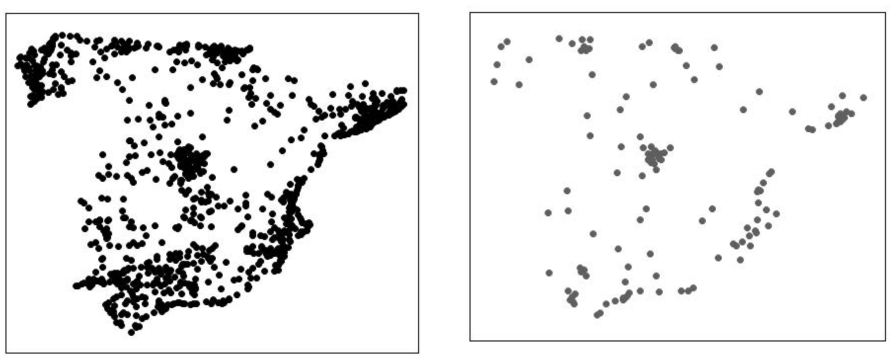

Figure 1: *Left*: Demand points , *Right*: Location candidates.

We have studied the performance of the proposed approach for the case of three competitors. For simplicity, the production cost of each firm i, $i = 1, 2, 3$, is the same in all locations and such costs are 200, 220, and 240 euros, respectively. The marginal transportation cost is taken proportional to the distance between municipalities j and k, $tc_i(j, k) = \mu d(j, k)$,

$\mu > 0$. The follower problem and the location game for the three firms have been ana-lyzed. In all test problems the software FICO Xpress Mosel [9], 64 bits v.3.10.0 for Linux, has been used on a computer with a processor Intel Core i7-6700 3.40 Ghzx8, RAM 8GB and OS Linux Ubuntu 15.10 64 bits.

4.1 The follower problem

Let us consider that firms 1 and 2 have their facilities already located. Firm 1 has 2 facilities located at the two most populated municipalities, $X_1 = \{1, 2\}$, and firm 2 has 3 facilities located at the three following most populated municipalities, $X_2 = \{3, 4, 5\}$. The number of facilities to be located by firm 3 (the follower) is $f_3 = 2, 3, 4, 5$. Transportation costs are proportional to distance with $\mu = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6$. We have used both the *MILP* and the *BILP* formulations to solve 24 location problems, which correspond to the different combinations of values for f_3 and μ. The same optimal locations were obtained by the two formulations, but with very different running times.

f_3	μ	X_3	$\Pi_1(X)$	$\Pi_2(X)$	$\Pi_3(X)$	Rt(M)	Rt(B)
2	0.1	31, 129	2323	2045	1761	0.93	68.37
	0.2	31, 129	2240	1985	1690	0.91	68.57
	0.3	1, 129	2163	1938	1625	0.86	68.31
	0.4	1, 129	2093	1900	1566	1.01	68.49
	0.5	1, 2	2028	1875	1513	1.61	68.43
	0.6	1, 2	1970	1860	1466	0.98	68.32
3	0.1	1, 129, 135	2301	2023	1819	0.94	68.18
	0.2	1, 129, 135	2200	1941	1804	0.97	68.03
	0.3	1, 129, 135	2108	1869	1792	0.86	68.08
	0.4	1, 129, 135	2026	1806	1784	1.02	68.11
	0.5	1, 2, 135	1952	1755	1778	1.64	68.80
	0.6	1, 2, 135	1887	1713	1776	0.78	67.97
4	0.1	2, 114, 125, 140	2287	2008	1859	0.83	68.06
	0.2	2, 114, 125, 140	2173	1911	1882	0.85	68.17
	0.3	2, 114, 125, 140	2071	1822	1907	0.82	68.06
	0.4	2, 114, 125, 140	1979	1743	1933	1.14	68.34
	0.5	1, 2, 29, 135	1896	1713	1965	1.04	68.66
	0.6	1, 2, 29, 135	1823	1668	2002	0.90	67.88
5	0.1	1, 2, 114, 125, 132	2274	1996	1894	0.93	67.60
	0.2	1, 2, 29, 114, 125	2148	1889	1954	0.89	67.56
	0.3	1, 2, 29, 114, 125	2035	1793	2016	0.87	67.66
	0.4	1, 2, 29, 114, 125	1933	1708	2081	0.83	67.64
	0.5	1, 2, 29, 114, 125	1843	1633	2149	0.81	67.61
	0.6	1, 2, 29, 114, 125	1765	1570	2219	0.81	67.65

Table 1: Optimal locations for firm 3 when $X_1 = \{1, 2\}$ and $X_2 = \{3, 4, 5\}$.

The results are shown in Table 1, where columns 1 and 2 correspond to the values of f_3 and μ. Column 3 gives the optimal location X_3 to the corresponding problem $P_3(X_1,\ X_2)$. Column 4, 5 and 6 give the profits of the three firms in thousands of euros for locations $X = (X_1,\ X_2,\ X_3)$. Column 6 and 7 show the running times in minutes to solve each problem by using the *MILP* and the *BILP* formulation, respectively.

The locations in X_1 and X_2 are shown in Figure 2-*Left* and the optimal locations in X_3 are shown in Figure 2-*Right*. Note that, for any pair of values of f_3 and μ, there is partial agglomeration of locations in X_3 around the most populated municipalities 1 and 2. See in Figure 2-*Right* that municipalities 31 and 140 are close to municipality 1, municipality 129 is close to municipality 2, and municipalities 1 and 2 are optimal facility locations in most of the problems, as it is shown in Table 1. Note also that for each value of f_3 the same optimal locations are obtained for most of values of parameter μ. Finally, note that it takes about one minute to solve every problem with the *MILP* formulation while it takes more than one hour with the *BILP* formulation.

In Figure 3 the profits per facility of the three firms are shown. Note that, for each value of the parameter μ, the profit per facility of each firm decreases as long as the number of entering facilities (f_3) increases. While the decrease in profit per facility for the pre-stablished firms (firms 1 and 2) is very low, the decrease in profit per facility for the entering firm (firm 3) when the number of entering facilities increases is high.

Once the facility locations are determined for all competitors, it seems that the profit of any firm will decrease if μ increases. However, the profit of one of the competing firms may increase if μ increases as it happens for firm 3 when $f_3 = 4$ and $f_3 = 5$ (see Table 1). This surprising result is explained by the fact that for fixed locations and $dc_i(x,k) = pc_i + \mu d(x,k)$, $i = 1,2,3$, where pc_i is a constant production cost, it is verified that if $\mu < \mu'$ then the equilibrium quantity q^*_{ik} for μ is smaller than the equilibrium quantity q^*_{ik} for μ' when $d_k(X_i)$ is less than $\frac{1}{3}(\sum_{h \neq i} d_k(X_h))$, being $d_k(A) = \min\{d(x,k) : x \in A\}$, for $A = X_i, X_h$. Thus, if the number of facilities of one firm i is much higher than the number of facilities of its competitors, the distance of many markets to their closest facility of firm i could be much smaller than the sum of distances to the closest facility of its competitors, and then firm i could obtain a greater profit if μ increases.

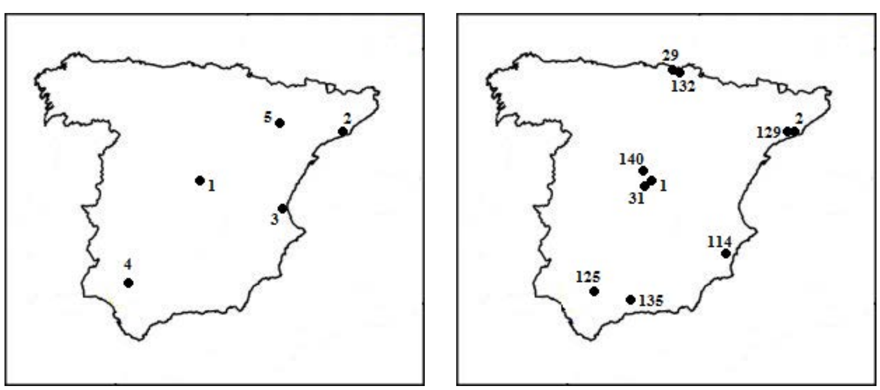

Figure 2: *Left*: Facility locations for firms 1 and 2, *Right*: Optimal locations for firm 3.

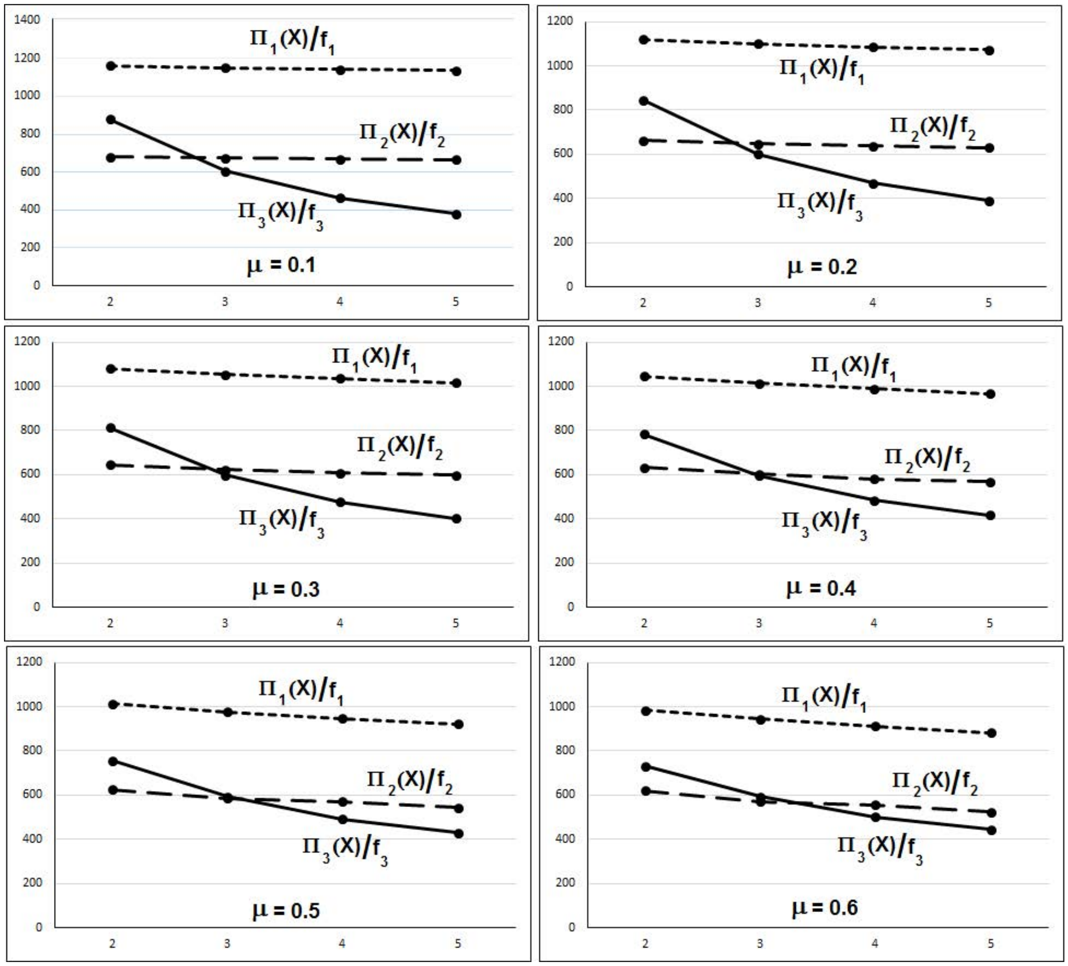

Figure 3: Profit per facility for firms 1,2, and 3.

4.2 The location game for three firms

Let us consider that the three firms locate f_1, f_2 and f_3 facilities, where $f_1 \leq f_2 \leq f_3$, $2 \leq f_i \leq 5$, $i = 1, 2, 3$. As in the previous section, transportation cost is proportional to distance with $\mu = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6$. We have solved 120 NE problems by using the *MILP* formulation which has been implemented in the best response procedure. These problems correspond to the different combinations of values for f_1, f_2, f_3 and μ. The results are shown for the six values of parameter μ in Tables 2, 3 and 4. For each value of μ, column 1 shows the values of (f_1, f_2, f_3). Columns 2, 3 and 4 give the locations X_1, X_2 and X_3 which are a NE to the corresponding location game. Column 5, 6 and 7 show the profits of the three firms, $X = (X_1, X_2, X_3)$. Column 8 gives the running time in minutes to find each NE. Column 9 shows the number of iterations (loops) of Algorithm *MFNE* to find a NE.

For most of triplets (f_1, f_2, f_3), there is partial collocation of firms 1,2 and 3 at equilibrium, being municipalities 1 and 2 location equilibria in many cases, as it is shown in Tables 2, 3 and 4. In particular, if $f_1 = f_2 = f_3$, the three firms co-locate their facilities at the same municipalities in most of cases. The location equilibrium for any triplet (f_1, f_2, f_3) is the same for most of values of the parameter μ, which means that NE are partially stable when the transportation cost changes. The running time ranges between 4.96 and 14.36 minutes to find a NE, and the number of iterations of Algorithm *MFNE* ranges between 2 and 5.

$\mu = 0.1$								
f_i	X_1	X_2	X_3	$\Pi_1(X)$	$\Pi_2(X)$	$\Pi_3(X)$	R. time	It.
(2,2,2)	31, 129	31, 129	31, 129	2324	2033	1762	7.14	2
(2,2,3)	31, 129	31, 129	1, 129, 135	2302	2013	1821	6.40	2
(2,2,4)	31, 129	31, 129	2, 114, 125, 140	2287	1999	1862	5.94	2
(2,2,5)	31, 129	31, 129	1, 2, 114, 125, 132	2275	1987	1897	6.78	2
(2,3,3)	31, 129	1, 129, 135	1, 129, 135	2281	2075	1801	5.79	2
(2,3,4)	31, 129	1, 129, 135	2, 114, 125, 140	2266	2061	1841	5.82	2
(2,3,5)	31, 129	1, 129, 135	1, 2, 114, 125, 132	2253	2049	1876	5.65	2
(2,4,4)	31, 129	2, 114, 125, 140	2, 114, 125, 140	2252	2104	1828	5.55	2
(2,4,5)	31, 129	2, 114, 125, 140	1, 2, 114, 125, 132	2239	2092	1863	5.40	2
(2,5,5)	31, 129	1, 2, 114, 125, 132	1, 2, 114, 125, 132	2226	2129	1851	5.77	2
(3,3,3)	129, 140, 141	129, 140, 141	1, 129, 135	2346	2054	1781	11.31	4
(3,3,4)	129, 140, 141	1, 129, 135	2, 114, 125, 140	2331	2040	1821	8.34	3
(3,3,5)	1, 129, 135	1, 129, 135	1, 2, 114, 125, 132	2319	2028	1857	5.29	2
(3,4,4)	1, 129, 135	2, 114, 125, 140	2, 114, 125, 140	2317	2082	1808	5.24	2
(3,4,5)	1, 129, 135	2, 114, 125, 140	1, 2, 114, 125, 132	2304	2070	1843	5.09	2
(3,5,5)	1, 129, 135	1, 2, 114, 125, 132	1, 2, 114, 125, 132	2292	2107	1831	4.99	2
(4,4,4)	2, 114, 125, 140	2, 114, 125, 140	2, 114, 125, 140	2361	2068	1794	5.06	2
(4,4,5)	2, 114, 125, 140	2, 114, 125, 140	1, 2, 114, 125, 132	2348	2056	1829	5.24	2
(4,5,5)	2, 114, 125, 140	1, 2, 114, 125, 132	1, 2, 114, 125, 132	2336	2093	1817	5.25	2
(5,5,5)	1, 2, 114, 125, 132	1, 2, 114, 125, 132	1, 2, 114, 125, 132	2374	2080	1806	5.47	2
$\mu = 0.2$								
f_i	X_1	X_2	X_3	$\Pi_1(X)$	$\Pi_2(X)$	$\Pi_3(X)$	R. time	It.
(2,2,2)	31, 129	31, 129	31, 129	2239	1954	1688	9.22	3
(2,2,3)	31, 129	31, 129	1, 129, 135	2197	1915	1807	8.65	3
(2,2,4)	31, 129	31, 129	2, 114, 125, 140	2169	1889	1889	5.47	2
(2,2,5)	31, 129	31, 129	1, 2, 114, 125, 132	2144	1866	1959	8.58	3
(2,3,3)	31, 129	1, 129, 135	1, 129, 135	2158	2037	1765	5.84	2
(2,3,4)	31, 129	1, 129, 135	2, 114, 125, 140	2129	2011	1847	8.32	3
(2,3,5)	31, 129	1, 129, 135	1, 2, 114, 125, 132	2105	1988	1917	5.79	2
(2,4,4)	31, 129	2, 114, 125, 140	2, 114, 125, 140	2102	2094	1818	4.96	2
(2,4,5)	31, 129	2, 114, 125, 140	1, 2, 114, 125, 132	2078	2071	1889	5.11	2
(2,5,5)	31, 129	1, 2, 114, 125, 132	1, 2, 114, 125, 132	2055	2143	1864	5.33	2
(3,3,3)	1, 129, 135	1, 129, 135	1, 129, 135	2282	1994	1726	5.96	2
(3,3,4)	1, 129, 135	1, 129, 135	2, 114, 125, 140	2254	1968	1807	5.74	2
(3,3,5)	1, 129, 135	1, 129, 135	1, 2, 114, 125, 132	2230	1945	1878	6.19	2
(3,4,4)	1, 129, 135	2, 114, 125, 140	2, 114, 125, 140	2227	2051	1779	5.36	2
(3,4,5)	1, 129, 135	2, 114, 125, 140	1, 2, 114, 125, 132	2203	2028	1849	5.48	2
(3,5,5)	1, 129, 135	1, 2, 114, 125, 132	1, 2, 114, 125, 132	2179	2100	1824	8.14	3
(4,4,4)	2, 114, 125, 140	2, 114, 125, 140	2, 114, 125, 140	2312	2022	1751	5.17	2
(4,4,5)	2, 114, 125, 140	2, 114, 125, 140	1, 2, 114, 125, 132	2287	1999	1821	5.20	2
(4,5,5)	2, 114, 125, 140	1, 2, 114, 125, 132	1, 2, 114, 125, 132	2264	2071	1797	7.74	3
(5,5,5)	1, 2, 114, 125, 132	1, 2, 114, 125, 132	1, 2, 114, 125, 132	2338	2046	1774	5.26	2

Table 2: NE for three firms with $\mu = 0.1$ and $\mu = 0.2$.

5 Conclusions

Multi-facility location choice under delivered quantity competition on a transportation network has been analyzed. If firms compete with the Cournot quantities, no procedure has been proposed to find a NE of the resulting location game. Under quite general conditions, it is proved that optimal locations for one firm, assuming that the facility locations of its competitors have been fixed, can be found at the nodes of the network. Then both a binary and a mixed integer linear programming formulations are proposed to solve the follower problem. This allows to apply the best response procedure to find a NE of the location game. Although both formulations can be used, only the mixed formulation allows to solve large size problems in a short running time, as it is shown by solving the follower problem for an illustrative example.

$\mu = 0.3$								
f_i	X_1	X_2	X_3	$\Pi_1(X)$	$\Pi_2(X)$	$\Pi_3(X)$	R. time	It.
(2,2,2)	31, 129	31, 129	31, 129	2156	1877	1617	11.07	3
(2,2,3)	31, 129	31, 129	1, 129, 135	2097	1821	1796	9.61	3
(2,2,4)	31, 129	31, 129	2, 114, 125, 140	2056	1784	1919	5.82	2
(2,2,5)	31, 129	31, 129	1, 2, 29, 114, 125	2021	1751	2025	5.29	2
(2,3,3)	1, 129	1, 129, 135	1, 129, 135	2042	2001	1732	8.84	3
(2,3,4)	1, 129	1, 129, 135	2, 114, 125, 140	2001	1963	1855	5.23	2
(2,3,5)	1, 129	1, 129, 135	1, 2, 29, 114, 125	1966	1931	1961	8.01	3
(2,4,4)	1, 129	2, 114, 125, 140	2, 114, 125, 140	1964	2086	1811	7.79	3
(2,4,5)	1, 129	2, 114, 125, 140	1, 2, 29, 114, 125	1929	2053	1917	5.26	2
(2,5,5)	1, 129	1, 2, 114, 125, 132	1, 2, 114, 125, 132	1896	2159	1879	8.18	3
(3,3,3)	1, 129, 135	1, 129, 135	1, 129, 135	2220	1936	1672	6.37	2
(3,3,4)	1, 129, 135	1, 129, 135	2, 114, 125, 140	2180	1899	1795	5.74	2
(3,3,5)	1, 129, 135	1, 129, 135	1, 2, 29, 114, 125	2145	1866	1901	5.81	2
(3,4,4)	1, 129, 135	2, 114, 125, 140	2, 114, 125, 140	2142	2021	1751	4.96	2
(3,4,5)	1, 129, 135	2, 114, 125, 140	1, 2, 29, 114, 125	2107	1988	1856	5.17	2
(3,5,5)	1, 129, 135	1, 2, 114, 125, 132	1, 2, 114, 125, 132	2074	2094	1819	8.23	3
(4,4,4)	2, 114, 125, 140	2, 114, 125, 140	1, 29, 129, 135	2275	1988	1712	7.65	3
(4,4,5)	2, 114, 125, 140	2, 114, 125, 140	1, 2, 29, 114, 125	2229	1944	1815	5.11	2
(4,5,5)	2, 114, 125, 140	1, 2, 114, 125, 132	1, 2, 114, 125, 132	2195	2050	1778	5.10	2
(5,5,5)	1, 2, 114, 125, 132	1, 2, 114, 125, 132	1, 2, 114, 125, 132	2302	2013	1743	5.37	2
$\mu = 0.4$								
f_i	X_1	X_2	X_3	$\Pi_1(X)$	$\Pi_2(X)$	$\Pi_3(X)$	R. time	It.
(2,2,2)	1, 129	1, 129	78, 135	2123	1847	1556	6.97	2
(2,2,3)	1, 129	3, 73	1, 129, 135	2023	1733	1809	9.41	3
(2,2,4)	1, 129	1, 129	2, 114, 125, 140	1949	1685	1955	8.67	3
(2,2,5)	31, 129	1, 129	1, 2, 29, 114, 125	1905	1644	2096	10.07	3
(2,3,3)	1, 129	1, 129, 135	1, 129, 135	1934	1967	1701	5.56	2
(2,3,4)	1, 129	1, 129, 135	2, 114, 125, 140	1882	1919	1865	8.52	3
(2,3,5)	1, 129	1, 129, 135	1, 2, 29, 114, 125	1838	1877	2007	5.61	2
(2,4,4)	1, 129	2, 114, 125, 140	2, 114, 125, 140	1836	2079	1805	8.48	3
(2,4,5)	1, 129	2, 114, 125, 140	1, 2, 29, 114, 125	1792	2037	1946	7.82	3
(2,5,5)	1, 129	1, 2, 114, 125, 132	1, 2, 29, 114, 125	1752	2176	1895	8.81	3
(3,3,3)	1, 129, 135	1, 129, 135	1, 129, 135	2160	1880	1620	10.00	3
(3,3,4)	1, 129, 135	1, 129, 135	2, 114, 125, 140	2108	1832	1784	5.71	2
(3,3,5)	1, 129, 135	1, 129, 135	1, 2, 29, 114, 125	2063	1791	1926	5.57	2
(3,4,4)	1, 129, 135	2, 114, 125, 140	2, 114, 125, 140	2061	1992	1724	8.71	3
(3,4,5)	1, 129, 135	2, 114, 125, 140	1, 2, 29, 114, 125	2017	1950	1865	8.28	3
(3,5,5)	1, 129, 135	1, 2, 114, 125, 132	1, 2, 29, 114, 125	1977	2089	1814	8.48	3
(4,4,4)	2, 114, 125, 140	2, 114, 125, 140	1, 29, 129, 135	2234	1950	1678	7.57	3
(4,4,5)	2, 114, 125, 140	2, 114, 125, 140	1, 2, 29, 114, 125	2172	1892	1809	5.51	2
(4,5,5)	1, 2, 114, 125	1, 2, 114, 125, 132	1, 2, 29, 114, 125	2131	2031	1760	8.94	3
(5,5,5)	1, 2, 114, 125, 132	1, 2, 114, 125, 132	1, 2, 114, 125, 132	2266	1979	1712	7.83	3

Table 3: NE for three firms with $\mu = 0.3$ and $\mu = 0.4$.

The location game has been solved for the illustrative example varying the number of competing firms, the number of facilities to be located, and the transportation cost. The same delivered costs have been taken for each firm. The results show that at equilibrium the firms partially co-locate their facilities. In particular, when the firms locate the same number of facilities, they locate all their facilities at the same municipalities in almost all cases. In most of the equilibria, some of the locations are the most populated municipalities, or municipalities close to them. For any fixed number of facilities of each firm, the same NE is obtained for most of values of the transportation cost. For three firms, the profit of each firm always decreases if the transportation cost increases. For two firms, it is shown that the profit of one firm may even increase if the transportation costs of the two firms increase. This is a surprising result which does not hold under Cournot competition if the

$\mu = 0.5$								
f_i	X_1	X_2	X_3	$\Pi_1(X)$	$\Pi_2(X)$	$\Pi_3(X)$	R. time	It.
(2,2,2)	1, 129	78, 135	1, 129	2064	1757	1543	10.83	3
(2,2,3)	1, 129	1, 129	3, 125, 140	1944	1682	1791	6.67	2
(2,2,4)	1, 129	1, 129	2, 114, 125, 140	1848	1592	1993	6.47	2
(2,2,5)	1, 129	1, 129	1, 2, 29, 114, 125	1795	1543	2171	10.33	3
(2,3,3)	1, 129	1, 129, 135	3, 125, 140	1869	1968	1677	8.62	3
(2,3,4)	1, 129	1, 129, 135	2, 114, 125, 140	1772	1877	1878	8.27	3
(2,3,5)	1, 129	1, 129, 135	1, 2, 29, 114, 125	1719	1828	2056	8.79	3
(2,4,4)	1, 129	2, 114, 125, 140	2, 114, 125, 140	1719	2073	1800	9.02	3
(2,4,5)	1, 129	1, 2, 114, 125	1, 2, 29, 114, 125	1667	2023	1979	8.17	3
(2,5,5)	1, 129	1, 2, 29, 114, 125	1, 2, 29, 114, 125	1621	2193	1911	8.04	3
(3,3,3)	1, 129, 135	3, 125, 140	1, 129, 135	2136	1828	1602	11.27	3
(3,3,4)	1, 129, 135	1, 129, 135	2, 114, 125, 140	2039	1768	1776	5.56	2
(3,3,5)	1, 129, 135	1, 129, 135	1, 2, 29, 114, 125	1986	1720	1954	8.62	3
(3,4,4)	1, 129, 135	2, 114, 125, 140	2, 114, 125, 140	1986	1964	1698	8.60	3
(3,4,5)	1, 129, 135	1, 2, 114, 125	1, 2, 29, 114, 125	1933	1914	1877	8.37	3
(3,5,5)	1, 129, 135	1, 2, 29, 114, 125	1, 2, 29, 114, 125	1887	2084	1810	7.88	3
(4,4,4)	1, 29, 129, 135	1, 2, 114, 125	1, 2, 114, 125	2192	1914	1653	5.71	2
(4,4,5)	2, 114, 125, 140	1, 2, 114, 125	1, 2, 29, 114, 125	2117	1841	1807	9.02	3
(4,5,5)	1, 2, 114, 125	1, 2, 29, 114, 125	1, 2, 29, 114, 125	2071	2011	1741	5.11	2
(5,5,5)	1, 2, 114, 125, 132	1, 2, 114, 125, 132	1, 2, 114, 125, 132	2232	1947	1682	7.68	3

$\mu = 0.6$								
f_i	X_1	X_2	X_3	$\Pi_1(X)$	$\Pi_2(X)$	$\Pi_3(X)$	R. time	It.
(2,2,2)	1, 129	1, 2	1, 125	2029	1763	1484	12.67	3
(2,2,3)	1, 129	1, 2	3, 125, 140	1869	1614	1803	11.84	4
(2,2,4)	1, 2	89, 135	2, 114, 125, 140	1870	1511	2069	11.55	4
(2,2,5)	1, 2	89, 135	1, 2, 29, 114, 125	1810	1455	2283	9.23	3
(2,3,3)	1, 129	1, 2, 135	3, 125, 140	1787	1948	1663	12.03	4
(2,3,4)	1, 129	1, 129, 135	2, 114, 125, 140	1671	1838	1892	8.39	3
(2,3,5)	1, 129	1, 129, 135	1, 2, 29, 114, 125	1611	1783	2106	7.64	3
(2,4,4)	1, 129	1, 2, 114, 125	1, 29, 129, 135	1611	2113	1799	7.93	3
(2,4,5)	1, 129	1, 2, 114, 125	1, 2, 29, 114, 125	1553	2013	2012	8.29	3
(2,5,5)	1, 129	1, 2, 29, 114, 125	1, 2, 29, 114, 125	1504	2211	1929	5.52	2
(3,3,3)	1, 129, 135	1, 2, 135	3, 125, 140	2090	1817	1541	14.36	5
(3,3,4)	1, 129, 135	1, 129, 135	2, 114, 125, 140	1973	1707	1769	13.72	5
(3,3,5)	1, 129, 135	1, 129, 135	1, 2, 29, 114, 125	1913	1652	1984	8.16	3
(3,4,4)	1, 129, 135	1, 2, 114, 125	1, 29, 129, 135	1913	1982	1677	8.30	3
(3,4,5)	1, 129, 135	1, 2, 114, 125	1, 2, 29, 114, 125	1854	1882	1889	7.82	3
(3,5,5)	1, 129, 135	1, 2, 29, 114, 125	1, 2, 29, 114, 125	1805	2080	1806	7.90	3
(4,4,4)	1, 2, 114, 125	1, 29, 129, 135	1, 2, 114, 125	2159	1886	1623	5.46	2
(4,4,5)	1, 2, 114, 125	1, 29, 129, 135	1, 2, 29, 114, 125	2110	1798	1806	5.75	2
(4,5,5)	1, 2, 114, 125	1, 2, 29, 114, 125	1, 2, 29, 114, 125	2016	1992	1724	5.30	2
(5,5,5)	1, 2, 114, 125, 132	1, 2, 114, 125, 132	1, 2, 114, 125, 132	2197	1915	1652	8.20	3

Table 4: NE for three firms with $\mu = 0.5$ and $\mu = 0.6$.

marginal delivered cost of one of the firms increases and the marginal delivered costs of its competitors remain fixed.

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