On the existence and computation of Nash equilibrium in network competitive location under delivered pricing and price sensitive demand

Blas Pelegrín*and Pascual Fernández
Dept. of Statistics and Operatinal Research. University of Murcia, Spain
E-mail: pelegrin@um.es, pfdez@um.es

María Dolores García
Dept. of Business Administration. San Antonio Catholic University, Spain
E-mail: mdolores@ucam.edu

Abstract

We address the location-price decision problem for firms that offer the same type of product and compete on delivered pricing. If firms set equilibrium prices at demand points, the problem can be seen as a location game for which the Nash equilibrium (NE) is used as solution concept. For spatially separated markets, with inelastic demand, there exists a NE and it can be found by social cost minimization, as happens in network and planar location. However, with price sensitive demand, the existence of a NE has not been proven yet and socially optimal locations may not be a NE. In this paper we show that a NE can be found in discrete and network location when demand is price sensitive. A Mixed Integer Linear Programming formulation is implemented in the best response procedure which allow to find a NE for a variety of demand functions. An empirical study with data of Spanish municipalities is performed in which the procedure is applied to 200 large size test problems with linear, quadratic, exponential and hyperbolic demand functions.

Keywords: Facility location, Discrete optimization, Nash equilibrium, Spatial competition.

1 Introduction

Facility location choice is a strategic decision for firms that compete to provide goods to customers. If the competing firms simultaneously enter into the market two interesting question are, will there exist a location Nash equilibrium (NE)? , and, how to find a NE if it exists?. A NE is a set of locations for the firms so that no firm will increase its profit by changing its location while the locations of the other firms do not change. There are hundreds of papers which have studied the underlying location game for a variety of alternative pricing policies. Most of the papers consider the customers distributed on a linear segment, a circumference, or a circle, where firms will locate their facilities (see for

*Correspondence to: Dr. Blas Pelegrín, Facultad de Matemáticas, Universidad de Murcia, 30100 Espinardo, Murcia, Spain. E-mail: pelegrin@um.es

instance [2, 3, 9, 16, 25]). However, due to its complexity, the problem of existence and computation of a NE has been less studied for spatially separated markets when the location space is a network. When firm's decision is based on quantities, different network facility location models have been formulated for simultaneous decision on production levels (see for instance [8, 11, 21, 28]). Alternative formulations deal with Stackelberg location games in which the firms enter sequentially in the market and production and distribution decisions of the leader are determined simultaneously over a discrete network (see [20, 22]).

For a price-based modeling approach, most of models proposed in the literature consider a two stage game. In the first stage, the competing firms select their facility locations, in the second stage, they compete on price with the aim of profit maximization. The division into two stages is motivated by the fact that the choice of location is usually prior to the decision on price. The two-stage game is reduced to a single stage location game if there exists a price equilibrium in the second stage, which is determined by the set of facility locations chosen in the first stage. The existence of price equilibrium often occurs when firms compete on delivered prices (see [24, 29]). With this price policy, firms offer a price at each market area, which includes production and transportation cost, and customers buy from the firm that offers the lowest price. We will deal with the location game under this price policy for the case of non-cooperative firms.

The notion of NE is probably the most used solution concept in non-cooperative games ([6]), however such equilibrium may not exist. For completely inelastic demand and constant marginal production cost, Hoover [14] analyzed the location game considering each firm locates one facility, and determined that the equilibrium market price of a firm with the lowest delivery cost is equal to the next lowest delivery cost. This result was extended to a spatial duopoly on a compact subset of the plane by Lederer and Hurter[18] and to a network by Lederer and Thisse [19]. In oligopoly, the same result was obtained by Dorta et al. [5], who proposed a model where firms make location and delivery price decisions along a network of connected but spatially separated markets. Existence of NE for the location game has been proved in the above mentioned papers by showing that any global minimizer of the social cost is a NE. Then finding a NE reduces to minimize social cost. If each firm locates multiple facilities on a network, the problem of minimizing the social cost can be solved by using Mixed Integer Linear Programming (MILP) formulations as it was shown in [26, 27]. The problem of minimizing the social cost on a plane is harder to solve and it has been solved for two competing firms which locate single facilities (see [4, 7]). If marginal production costs are not constant, or demand is price-sensitive, the socially optimal locations may not be a NE of the location game, as it has been shown in a linear location space (see [12] and [13]).

In network and planar location, to our knowledge, the existence of a NE for the location game with price sensitive demand has not been proven yet and the computation of a NE for this game has not been studied. Our main contribution is to show that for price-sensitive demand a NE can be found when the set of location candidates is finite, or the location candidates are nodes and points on the edges on a network. First, we show that socially optimal locations may not be a NE on a network, which means that social cost minimization for price sensitive demand can not be used to find a NE. Then we use the best response procedure to find a NE, which requires to compute the optimal locations for one of the firms, assuming the locations of its competitors are fixed. This optimization problem can be solved for a finite set of location candidates by a MILP formulation proposed by the authors in [10]. On a network, it is proved that a NE can be found at the nodes, and the procedure can also be applied to network location taking the nodes as the set of location candidates (the points in the edges can be omitted). A computational study with 200 real size test problems is performed to show that such procedure can find a NE for different types of demand functions in a small number of iterations and short running time.

The paper is organized as follows. In Section 2, the location game is described. In Section 3, existence and computation of NE for inelastic demand is briefly discussed. An example is given to show that socially optimal locations may not be a NE on a network if demand is price sensitive. In Section 4, the best response procedure for the location game is described together with the MILP formulation. An empirical study is presented in Section 5 to show that the procedure is able to find NE in a variety of real size test problems with different types of demand functions. Finally, some conclusions are given in Section 6.

2 The location game

Let us consider a set of firms which compete for demand of an homogeneous product in spatially separated markets. The firms have to make decisions on facility location and price with the aim of profit maximization. First, firms select the location of their facilities, then firms set delivered prices at each market. Note that location decision is permanent (strategic decision) while decision on price can be easily changed (tactic decision). It is assumed that the profit of a firm in any market is independent of the profit obtained in any other market. The marginal delivered cost is supposed to be independent of the quantity delivered.

The following notation will be used:

Indices

- i index of firms; $i = 1, \ldots, r$.
- j index of location candidates (in discrete location space) ; $j = 1, \ldots, n$.
- k index of demand nodes; $k = 1, \ldots, m$.

Data

$F = \{1, 2, ..., r\}$	set of firms.
L	set of location candidates.
$M = \{1, 2, ..., m\}$	set of markets.
$q_k(p)$	demand in market k at price p; $k \in M$.
f_i	number of facilities of firm i.
$pc^i(x)$	unit production cost of firm i at location x; $x \in L$.
$d(x, k)$	distance between location x and market k; $x \in L$, $k \in M$.
$tc^i(x, k) = T_i(d(x, k))$	unit transportation cost of firm i from location x to market k; $x \in L$, $k \in M$.
$dc^i(x, k) = pc^i(x) + tc^i(x, k)$	unit delivered cost of firm i from location x to market k; $x \in L$, $k \in M$.

Decision variables

- X^i set of facility locations of firm i.

Miscellaneous

$X = (X^1, X^2, ..., X^r)$	location strategy for the facilities of the competing firms.
X^{-i}	vector of locations for the facilities of the firms other than i; $X = (X^i, X^{-i})$.
C_k(A)	minimum delivered cost from the facilities in A to market k.

Once facility locations are fixed, the firms will compete on price. Each firm offer a price at each market, which include production and transportation costs, then customers buy from the firm which offers the lowest price. It is assumed that the firms will not set a price below the marginal delivered cost. If two firms offer a minimum price at market k, the one with the minimum marginal delivered cost can lower its price and it obtains all the demand at market k. Then ties in price are broken in favour of the firm with the lowest marginal delivered cost. Therefore, we consider that each market is served by the firm with the min-imum marginal delivered cost. If more than one firm can price the minimum delivered cost at a given market, the competition process will make the profit obtained from that market to be zero.

Given the set of facility locations X, the prices at any market k are obtained as follows:

i) If $C_k(X^i) < C_k(X^{-i})$, firm i obtains a maximum profit from market k by offering a price equal to the optimal solution of the following problem:

$$Max \ \{\Pi_k^i(p) = q_k(p)(p - C_k(X^i)) : C_k(X^i) \leq p \leq C_k(X^{-i})\}$$

The optimal solution to this problem is :

$$\hat{p}_k^i(X) = \begin{cases} p_k^{mon}(C_k(X^i)) & if \quad p_k^{mon}(C_k(X^i)) \leq C_k(X^{-i}) \\ C_k(X^{-i}) & if \quad p_k^{mon}(C_k(X^i)) > C_k(X^{-i}) \end{cases}$$

where $p_k^{mon}(C_k(X^i))$ is the monopoly price of firm i at market k.

The monopoly price is the optimal solution to the following optimization problem:

$$Max \ \{\Pi_k(p) = q_k(p)(p - C_k(X)) : C_k(X) \leq p \leq p_k^{max}\}$$

where p_k^{max} is the maximum price that customers are willing to pay for the product in market k.

ii) If $C_k(X^i) \geq C_k(X^{-i})$, then firm i obtains zero profit from market k. In this case, firm i sets a price $\hat{p}_k^i(X) = C_k(X^i)$ to make its competitors obtain a minimum profit from market k.

It is easy to see that $\hat{p}_k^i(X)$, $i = 1, 2, \ldots, r$, are equilibrium prices at market k. We assume that for any strategy location X, the firms will set the equilibrium prices. Since the group of markets monopolized by each firm i is:

$$M^i(X) = \{k : C_k(X^i) < C_k(X^{-i})\}$$

the profit obtained by firm i is given by:

$$\Pi^i(X) = \sum_{k \in M^i(X)} q_k(\hat{p}^i_k(X))(\hat{p}^i_k(X) - C_k(X^i))$$

Therefore, the location-price decision problem for the competing firms can be seen as a location game. The firms are the players, the alternatives of player i are subsets X^i of the set of location candidates L, and the payoff that player i obtains is $\Pi^i(X)$, $i = 1, ..., r$.

3 The Nash equilibrium and social cost minimization

If the firms do not cooperate on location decision, the most used solution concept of the location game is the Nash equilibrium. A strategic profile of locations $\hat{X} = (\hat{X}_1, \hat{X}_2, \dots, \hat{X}_r)$ is a NE if for any i it is verified that:

$$\Pi^i(\hat{X}^i, \hat{X}^{-i}) \geq \Pi^i(X^i, \hat{X}^{-i}), \qquad \forall X^i \subset L$$

Most of the existing results on existence of NE for non-cooperative games assume a convex compact set in the Euclidean space as the set of alternatives for each player (see [1, 32]), which, unfortunately, can not be applied to a discrete location space. Other results avoid the convexity requirement, but their conditions on the payoffs are difficult to verify for the proposed location game (see [15, 30, 33]). The only known result that guarantees existence of a NE for the location game is for the case of inelastic demand. In such a case, existence of a NE is proved by showing that a minimizer of social cost is a NE (see [5]).

The social cost is defined as the total cost incurred to supply demand to customers if each customer would pay for the product the minimum delivered cost. For any fixed set of locations $X = (X^1, X^2, \dots, X^r)$, the social cost is:

$$S(X) = \sum_{k=1}^{m} q_k(C_k(X))\, C_k(X)$$

For inelastic demand, $q_k(C_k(X)) = \alpha_k$, where α_k is a fix demand at market k. Then, it is verified that (see [27]):

$$\Pi^i(X) \quad = \quad \sum_{k=1}^{m} \alpha_k C_k(X^{-i}) - S(X)$$

From the previous expression follows that any global minimizer of $S(X)$ is also a NE.

For price sensitive demand, socially optimal locations may not be a NE, as it is shown by the following example. Consider two firms which locate a single facility at one of the nodes of the network in Figure 1. Demand function at each node k is given by $q_k(p) = 3 - p$, $0 \leq p \leq 3$. Delivered costs are the same for the two firms, $dc^1(j,k) = dc^2(j,k)$. The number on each edge (j,k) is the marginal delivered cost between nodes j and k. Note that in this example the location game is symmetric.

Table 1 shows the social cost and the profits of the two firms for all possible location choices, which are represented by $(\mathbf{S(X)}; \Pi^1(X), \Pi^2(X))$. The pairs $(2,4)$ and $(4,2)$ are the minimizers of social cost, but they are not NE. Observe that for the pair $(2, 4)$, firm 2 can obtain a greater profit if it changes its location from node 4 to node 3. Similarly, for the pair $(4, 2)$, firm 1 can obtain a greater profit if it changes its location from node 4 to node 3. The pairs $(2, 3)$ and $(3, 2)$ are NE.

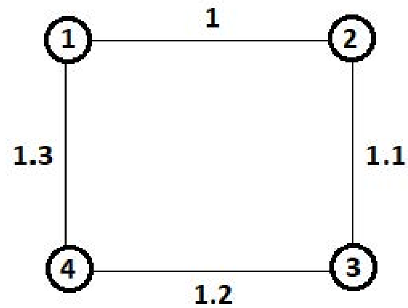

Figure 1:

Node	1	2	3	4
1	(**6.1**;0,0)	(**4.30**;2.72,2.90)	(**4.16**;2.44,2.42)	(**4.16**;3.21,2.21)
2	(**4.30**;2.90,2.72)	(**5.7**;0,0)	(**4.16**;3.09,2.90)	(**4.09**;2.94,2.25)
3	(**4.16**;2.42,2.44)	(**4.16**;2.90,3.09)	(**6.14**;0,0)	(**4.30**;3.06,2.88)
4	(**4.16**;2.21,3.21)	(**4.09**;2.25,2.94)	(**4.30**;2.88,3.06)	(**5.98**;0,0)

Table 1: Social cost and profits for each pair of locations at the nodes.

From the previous example follows that social cost minimization can not be used to find NE for price-sensitive demand. The following section is devoted to the best response procedure, which we will show that it is able to find a NE for the proposed location game.

4 Computation of a NE

Although existence of NE is not guaranteed for price sensitive demand, we will try to find NE by using a well known method in Game Theory, called the best response procedure (see [6]). This procedure involves solving a sequence of optimization problems. Each optimization problem consist of determining the best strategy for a player assuming that the strategies of the other players are known. In the following we will describe such a procedure for the location game.

4.1 The best response procedure

The best location strategy for firm i, assuming that facility location of its competitors are known, is obtained by solving the following problem:

$$BR^i(X^{-i}): \; Max\{\Pi^i(Y^i, X^{-i}) : |Y^i| = f_i, Y^i \subset L\}$$

where X^{-i} are the facility locations for the firms other than i, $i = 1, .., r$. Based on the previous problem, the best response procedure is as follows:

1: Select an initial set of facility locations X_0.
$X_0 = (X_0^1, X_0^2, ..., X_0^r)$, $|X_0^i| = f_i$, $i = 1, ..., r$. Set $\nu = 0$.

2: **For** $i = 1, ..., r$ **do**
Find an optimal solution $X_{\nu+1}^i$ to problem $BR^i(X_{\nu+1}^1, ..., X_{\nu+1}^{i-1}, X_\nu^{i+1}, ..., X_\nu^r)$.
Set $X_{\nu+1}^i = X_\nu^i$ if $\Pi^i(X_{\nu+1}^1, ..., X_{\nu+1}^i, X_\nu^{i+1}, ..., X_\nu^r) = \Pi^i(X_{\nu+1}^1, ..., X_{\nu+1}^{i-1}, X_\nu^i, ..., X_\nu^r)$.
end for

3: If $X_{\nu+1}^i = X_\nu^i$, $i = 1, ..., r$, then $X = (X_\nu^1, X_\nu^2, ..., X_\nu^r)$ is a NE , **STOP**.
Otherwise, set $\nu = \nu + 1$ and **go to** step 2.

Observe that the previous procedure finds a NE if it stop. Otherwise, the procedure might cycle and a NE is not found. The procedure can be used whenever problem $BR^i(X^{-i})$ can be solved in appropriated run time, since that problem has to be solved many times. We will use a $MILP$ formulation which allow to solve $BR^i(X^{-i})$ in a short run time when L is a finite set. If L is not finite, the problem seems to be very difficult to solve, particularly for multi-facility location. However, if L is the set of points on a network (nodes or points in the edges), the following property holds (see [10]):

Property 1 *There exists a set of nodes which is an optimal solution of $BR^i(X^{-i})$ if the following assumptions hold for all $i \in F$:*
i) The marginal production cost, $pc^i(x)$, is a positive concave function when x varies along any edge in the network, and it is independent of the quantity produced.
ii) The marginal transportation cost, $T^i(d(x,k))$, is a positive, concave and increasing function with respect to the distance from x to each node k.

Therefore, if the location space is a network with nodes and points on the edges as location candidates, $BR^i(X^{-i})$ can be solved taking L as the set of nodes if Property 1 holds. Concavity of marginal production cost and marginal transportation cost is realistic in certain situations as it has been remarkable by many authors (see for instance [17, 31]).

4.2 A MILP formulation of $BR^i(X^{-i})$

We will consider that L is any finite set of location candidates, for simplicity $L = \{1, 2, \ldots, n\}$. Then the problem $BR^i(X^{-i})$ can be formulated as follows.

Let us define the following sets and variables:

$$L_k^i = \{j \in L : dc^i(j,k) < C_k(X^{-i})\}$$

$$L^i = \cup\{L_k^i : k \in M\}$$

$$M^i = \{k : L_k^i \neq \emptyset\}$$

$$x_j^i = \begin{cases} 1 & \text{if firm } \ i \text{ locates a facility at } \ j \\ 0 & \text{otherwise} \end{cases}$$

$$w_{kj}^i = \begin{cases} 1 & \text{if maket } k \text{ is served by firm } i \text{ from location } j \\ 0 & \text{otherwise} \end{cases}$$

L_k^i is the set of locations at which firm i can price below its competitors at market k and get the full demand at k. L^i is the set of locations at which firm i can get a positive profit. M^i is the set of markets where firm i can get a positive profit. Variables x_j^i and w_{kj}^i are location and allocation variables, respectively.

If market k is served by firm i from location $j \in L_k^i$, the equilibrium price at k is:

$$\hat{p}_k^i(j) = \begin{cases} p_k^{mon}(dc^i(j,k)) & if \quad p_k^{mon}(dc^i(j,k)) \leq C_k(X^{-i}) \\ \\ C_k(X^{-i}) & if \quad p_k^{mon}(dc^i(j,k)) > (C_k(X^{-i})) \end{cases}$$

Then the profit maximization problem of firm i can be formulated as follows:

$$BR^i(X^{-i}): \qquad \max \sum_{k \in M^i} \sum_{j \in L_k^i} q_k(\hat{p}_k^i(j))(\hat{p}_k^i(j) - dc^i(j,k))w_{kj}^i$$

$$\text{s.t.} \qquad \sum_{j \in L^i} x_j^i = f_i \tag{1}$$

$$w_{kj}^i \leq x_j^i, \qquad j \in L_k^i, \quad k \in M^i \tag{2}$$

$$\sum_{j \in L_k^i} w_{kj}^i \leq 1, \;\; k \in M^i \tag{3}$$

$$x_j^i, w_{kj}^i \in \{0,1\}, \;\; j \in L_k^i, \quad k \in M^i \tag{4}$$

The objective function of problem $BR^i(X^{-i})$ represents the profit of firm i. Constraint (1) represents the number of facilities to be located by firm i. Constraints (2) guarantee that variable w_{kj}^i may be positive only if firm i locates a facility at location j. Constraints (3) mean that each market $k \in M^i$ can be served from at most one of the facilities of firm i (the facility with the minimum marginal delivered cost in the optimal solution). Constraints (4) require that the variables are binary. The above problem is a Binary Integer Linear Programming (*BILP*) problem which contains a lot of binary variables. Observe that the optimal value of problem $BR^i(X^{-i})$ does not change if variables w_{kj}^i are taken as non negative variables instead of a binary variables. Replacing constraints $w_{kj}^i \in \{0,1\}$ by $w_{kj}^i \geq 0$ in the above formulation a Mixed Integer Linear Programming (*MILP*) is obtained which allow to solve instances of larger size than the *BILP* formulation.

5 An empirical study

We present an example with some real data where both markets and location candidates are Spanish municipalities. Markets are municipalities with a population over 5000 inhabitants (1046 cities) which have been numbered from 1 to 1.049 in decreasing population size, thus $M = \{1, 2, ..., 1049\}$. We have taken five sets of municipalities as location candidates, which are the municipalities with a population over 5000 inhabitants ($L = M = \{1, 2, ..., 1049\}$), 10000 inhabitants ($L = \{1, 2, ..., 592\}$), 25000 inhabitants ($L = \{1, 2, ..., 250\}$), 50000 inhabitants ($L = \{1, 2, ..., 121\}$) and 100000 inhabitants ($L = \{1, 2, ..., 54\}$), respectively (see Fig. 2). The population size and geographical coordinates of the Spanish municipalities can be seen on the website: *http://www.um.es/geloca/gio/datos-espana-2015.txt*.

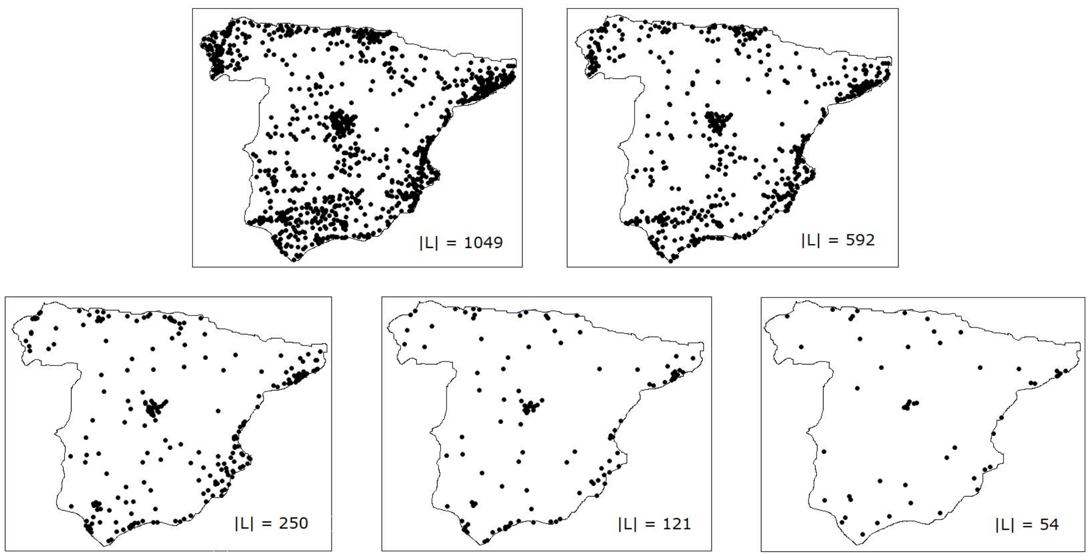

Figure 2: Location candidates.

We consider four types of demand functions which are shown in Table 2. The value of α_k represents the maximum demand at municipality k in the linear, quadratic and exponential cases. Demand is unlimited in the hyperbolic case. Note that the maximum price the customers in market k are willing to pay for the product, p_k^{max}, depends on α_k and β_k in the linear and quadratic cases, and it is unlimited in the exponential and hyperbolic cases.

We have studied the performance of the proposed approach to find Nash equilibria for the case of two competing firms. For simplicity, we assume that marginal delivered costs are given by $dc^i(j, k) = td(j, k) + pc^i$, where $d(j, k)$ is the distance between municipalities j and k, $i = 1, 2$; t is the transportation cost per unit of distance; and pc^i is the production cost of firm i at any municipality j (for each firm it is assumed equal production cost at different location candidates). Distances $d(j, k)$ between any pair of municipalities j and k have been approximated by using the Harversine formula, which measure the distance between two geographical points from their longitudes and latitudes (see [23]). The transportation cost per unit of distance and the production cost are randomly generated for each test problem in the intervals [0.1 , 0.3] and [50 , 100] , respectively.

Demand function	$q_k(p)$	p_k^{max}	$p_k^{mon}(c)$
Linear	$\alpha_k - \beta_k p$	$\frac{\alpha_k}{\beta_k}$	$\frac{1}{2}(c + \frac{\alpha_k}{\beta_k})$
Quadratic	$\alpha_k - \beta_k p^2$	$\sqrt{\frac{\alpha_k}{\beta_k}}$	$\frac{1}{3}(c + \sqrt{c^2 + 3\frac{\alpha_k}{\beta_k}})$
Exponential	$\alpha_k e^{-\beta_k p}$	∞	$c + \frac{1}{\beta_k}$
Hyperbolic	$\alpha_k p^{-\beta_k}$	∞	$\frac{c\ \beta_k}{\beta_k - 1}$, $\beta_k > 1$

Table 2: Some demand functions and their monopoly prices.

Taking into account the given delivered costs, and assuming that a maximum of one out of 1000 inhabitants buy the product, and that the maximum price the customers in market k are willing to pay is 1000, the following values of parameters α_k and β_k have been chosen:

$\alpha_k = \frac{\text{size of municipality } k}{1000}$ (linear, quadratic and exponential demand)

$\alpha_k =$ size of municipality k (hyperbolic demand)

$\beta_k = \frac{\alpha_k}{1000}$ (linear and exponential demand)

$\beta_k = \frac{\alpha_k}{1000000}$ (quadratic demand)

$\beta_k = \frac{\ln \alpha_k}{\ln 1000}$ (hyperbolic demand)

We have applied the best response procedure to 200 test problems. For each type of demand function, 50 test problems have been solved (10 problems for each set of location candidates). The number of iterations has been limited to 20 and the number of facilities of each firm has been randomly generated in the interval [1, 25]. The MILP problems have been solved with the software FICO Xpress Mosel [34], 64 bits v.3.10.0 for Linux, on a computer with a processor Intel Core i7-6700 3.40 Ghzx8, RAM 8GB and OS Linux Ubuntu 15.10 64 bits.

The results are shown in the Tables 3 to 6 which are given in the Appendix. Column 1 show the cardinality of the set of location candidates. Columns 2 and 3 give the number of facilities to be located by each firm, $[f_1, f_2]$, and the production costs, $[pc^1, pc^2]$, respectively. Column 4 shows the unit transportation cost, t. Columns 5 and 6 show the number of iterations of the best response procedure and the running time in minutes. Finally, column

7 indicates if a NE has been found (YES) or not (NO). Note that a Nash equilibrium has been obtained in 100 % of the test problems in the linear and quadratic cases, in 82 % of the test problems in the exponential case and 64 % of the test problems in the hyperbolic case. Observe that the all NE are found in a few number of iterations with an small running time.

6 Conclusions and future research

The existence and computation of a Nash equilibrium for competing firms under delivered pricing and price-sensitive demand has been analyzed. It is remarkable that the location-price decision problem for the firms reduces to a location game if firms set equilibrium prices. When demand is price sensitive, as opposed to fixed demand, existence of equilibrium is not guaranteed and social cost minimizing locations may not be an equilibrium. Then we have applied the best response procedure to find a Nash equilibrium, which requires to solve a sequence of optimization problems. Each optimization problem in the sequence is often hard to solve and the sequence may cycle without finding an equilibrium. We have shown that each of these optimization problems can be solved by a Mixed Integer Linear Programming formulation if the set of location candidates is finite, which allow to solve the sequence of optimization problems by using an standard optimizer. If the location candidates are the points on a network (nodes and points on the edges), under some concavity assumptions, a Nash equilibrium can be found taking the nodes as the set of location candidates.

We have performed an empirical study in which four types of demand functions have been considered. The procedure has been applied to check if a NE is found in 200 test problems with two firms which locate a high number of facilities on different sets of location candidates. This study shows that an equilibrium can often be computed for realistic size location games. In particular, an equilibrium has always been found for linear and quadratic demand function.

Although an equilibrium has not been found in some of the test problems, it could exist. To our knowledge, no proof of existence of Nash equilibrium has been given for the location game with price sensitive demand. The results presented in this paper seems to indicate existence of equilibrium, in particular for linear and quadratic demand functions. Finding such a proof of existence will be a subject for future research.

Acknowledgments
This research has been supported by the Fundación Séneca (The Agency of Science and Technology of the Region of Murcia) under the research project 19241/PI/14.

Competing interests
The authors do not have any conflict of interest.
Data are available on the web site http://www.um.es/geloca/gio/datos-espana-2015.txt

References

[1] Cheng S., Reeves D. M. , Vorobeychik Y., Wellman M.P. (2004) Notes on equilibria in symmetric games, In: Proceedings of the 6th International Workshop on Game

Theoretic and Decision Theoretic Agents (GTDT), 71–78.

[2] D´Aspremont C., Gabszewicz J. and Thisse J.F. (1979). On Hotelling´s Stability in Competition. *Econometrica* 47, 1145–1150.

[3] De Palma A., Ginsburgh V. and Thisse J.F. (1987). On existence of location equilibria in the 3-firm Hotelling problem. *The Journal of industrial Economics* 36, 245–252.

[4] Díaz-Bañez J.M., Heredia M. , Pelegrín B., Pérez-Lantero P. , Ventura I. (2011). Finding all pure strategy Nash Equilibria in a planar location game. *European Journal of Operational Research* 214, 91–98.

[5] Dorta-González P., Santos-Peñate D.R. and Suárez-Vega R. (2005). Spatial competition in networks under delivered pricing. *Papers Regional Science* 84, 271–280.

[6] Eichberger J. (1993). Game Theory for Economists. *Academic Press*.

[7] Fernández J., Salhi S. , Toth B. G. (2014). Location equilibria for a continuous competitive facility location problem under delivered pricing. *Computers and Operations Research* 4, 185–195.

[8] Friesz T.L., , Tobin R.L. and Miller T.C. (1989) Existence theory for spatially competitive network facility location models. *Annals of Operations Research* 18, 267-276

[9] Gabszewicz J.J. and Thisse J.F. (1992). Location. In: Aumann R and Hart S (Eds.), *Handbook of Game Theory with Economic Applications*. Elsevier: Amsterdam, pp 281–304.

[10] García M.D., Pelegrín B. and Fernández P. (2011). Location strategy for a firm under competitive delivered prices. *Annals of Regional Science* 47, 1–23.

[11] García M.D. and Pelegrín B (2020) Exact equilibrium quantities in spatially separated markets under production constraints. *Journal of Interdisciplinary Mathematics* 23 , 1309-1323.

[12] Gupta P. (1994). Competitive spatial price discrimination with strictly convex production costs. *Regional Science and Urban Economics* 24, 265–272.

[13] Hamilton J.H., Thisse J.F. and Weskamp A. (1989). Spatial discrimination, Bertran vs. Cournot in a model of location choice. *Regional Science and Urban Economics* 19, 87–102.

[14] Hoover E.M. (1936). Spatial price discrimination. *Review of Economic Studies* 4, 182–191.

[15] Iimura T. and Watanabe T. (2014). Existence of a pure strategy equilibrium in fi-nite symmetric games where payoff functions are integrally concave. *Discrete Applied Mathematics* 166, 26–33.

[16] Kilkenny M. and Thisse J.F. (1999). Economics of location: A selective survey. *Computers and Operations Research* 26, 1369–1394.

[17] Labbe M., Hakimi S.L. (1991). Market and location equilibrium for two competitors. *Operations Research* 39, 749–756.

[18] Lederer P.J. and Hurter A.P. (1986). Competition of chains, discriminatory pricing and location. *Econometrica* 54, 623–640.

[19] Lederer P.J. and Thisse J.F. (1990). Competitive location on networks under delivered pricing. *Operations Research Letters* 9, 147–153.

[20] Miller T.C., Tobin R.L. and Friesz T.L.(1991) Stackelberg games on a network with Cournot-Nash oligopolistic competitors. *Journal of Regional Science* 31, 435–454.

[21] Miller T.C., Friesz T.L. and Tobin R.L. (1992). Heuristic algorithms for delivered price spatially competitive network facility location problems. *Annals of Operations Research* 34, 177–202.

[22] Miller T.C., Tobin R.L. and Friesz T.L.(1995) Network facility-location models in Stackelberg-Nash-Cournot spatial competption. *Papers in Regional Science* 71, 277–291.

[23] Mwemezi J.J. anf Huang Y. (2011). Optimal Facility Location on Spherical Surfaces: Algorithm and Application. *New York Science Journal* 4, 21–28.

[24] Pacheco C. (2009). Location choice under delivered pricing: a reinterpretation. *The Annals of Regional Science* 43, 199–213.

[25] Osborne M.J. and Pitchik C. (1987). Equilibrium in Hotelling´s model of spatial competition. *Econometrica* 55, 911–922.

[26] Pelegrín B., Dorta P. , and Fernández P. (2011). Finding location equilibria for competing firms under delivered pricing. *Journal of the Operational Research Society* 62, 729–741.

[27] Pelegrín B. , Fernández P. and García M.D. (2017). Nash equilibria in network facility location under delivered prices, in *Spatial interaction models: Facility location using Game Theory*, Springer, pp. 273–292.

[28] Pelegrín B. , Fernández P. and García M.D. (2018) Computation of Multi-facility Location Nash Equilibria on a Network Under Quantity Competition, *Networks and Spatial Economics* 18, 999–1017.

[29] Phlips L. (1983). *The Economics of Price Discrimination.* Cambridge University Press.

[30] Reny P.J. (1999). On the Existence of Pure and Mixed Strategy Nash Equilibria in Discontinuous Games, *Econometrica* 67, 1029–1056.

[31] Sarkar J., Gupta B. and Pal D. (1997). Location equilibrium for Cournot oligopoly in spatially separated markets. *Journal of Regional Science* 2, 195–212.

[32] Scalzo V. (2019). Equilibrium existence in games: Slight single deviation property and Ky Fan minimax inequality. *Journal of Mathematical Economics* 82, 197–201.

[33] Tian G. (2015). On the existence of equilibria in games with arbitrary strategy spaces and preferences. *Journal of Mathematical Economics* 60, 9–16.

[34] FICO Xpress Mosel (2009), Fair Isaac Corporation.

Appendix

$\mid L \mid$	$[f_1, f_2]$	$[pc^1, pc^2]$	t	$\#iter$	$time(min)$	$NE?$
54	[12,7]	[79,55]	0.15	3	0.02	YES
	[6,24]	[65,69]	0.19	2	0.01	YES
	[15,2]	[83,69]	0.15	3	0.03	YES
	[2,25]	[51,99]	0.15	2	0.02	YES
	[14,2]	[70,61]	0.2	3	0.03	YES
	[5,6]	[87,93]	0.15	3	0.03	YES
	[10,25]	[56,85]	0.24	2	0.01	YES
	[3,10]	[67,54]	0.15	3	0.04	YES
	[3,6]	[80,86]	0.13	3	0.03	YES
	[15,16]	[74,73]	0.21	3	0.01	YES
121	[25,25]	[69,96]	0.2	2	0.03	YES
	[19,14]	[91,77]	0.29	3	0.03	YES
	[25,14]	[98,51]	0.1	2	0.13	YES
	[4,24]	[88,97]	0.2	3	0.03	YES
	[2,17]	[89,82]	0.28	3	0.08	YES
	[2,17]	[65,80]	0.19	2	0.04	YES
	[3,20]	[51,71]	0.19	2	0.03	YES
	[11,3]	[63,87]	0.18	2	0.09	YES
	[25,12]	[63,58]	0.21	3	0.03	YES
	[12,25]	[73,92]	0.18	2	0.03	YES
250	[8,18]	[59,58]	0.15	3	0.07	YES
	[1,5]	[67,82]	0.14	3	0.23	YES
	[9,2]	[97,72]	0.25	3	0.20	YES
	[12,11]	[58,77]	0.17	3	0.14	YES
	[9,21]	[69,93]	0.27	2	0.06	YES
	[5,21]	[91,65]	0.2	3	0.18	YES
	[23,5]	[72,64]	0.26	3	0.08	YES
	[15,6]	[57,89]	0.27	2	0.14	YES
	[6,25]	[92,69]	0.17	2	0.26	YES
	[19,16]	[63,90]	0.21	3	0.11	YES
592	[15,6]	[65,73]	0.13	3	0.35	YES
	[19,3]	[58,98]	0.11	2	1.60	YES
	[22,7]	[84,96]	0.21	3	0.58	YES
	[12,12]	[54,95]	0.12	2	1.39	YES
	[3,25]	[85,63]	0.21	3	1.12	YES
	[18,20]	[55,75]	0.19	3	0.34	YES
	[22,8]	[58,84]	0.13	2	0.77	YES
	[14,6]	[76,87]	0.14	3	0.44	YES
	[21,16]	[87,94]	0.26	3	0.20	YES
	[24,6]	[57,51]	0.19	3	0.24	YES
1049	[16,2]	[75,56]	0.24	3	1.35	YES
	[3,15]	[58,88]	0.25	3	1.03	YES
	[25,10]	[86,99]	0.28	3	0.56	YES
	[20,11]	[60,61]	0.12	3	0.45	YES
	[17,12]	[58,64]	0.3	3	0.47	YES
	[20,1]	[72,64]	0.23	3	2.82	YES
	[16,17]	[81,69]	0.25	3	0.58	YES
	[25,18]	[77,89]	0.15	2	0.37	YES
	[1,7]	[100,55]	0.29	3	9.03	YES
	[6,12]	[98,92]	0.18	4	0.88	YES

Table 3: Results for linear demand function.

$\mid L \mid$	$[f_1, f_2]$	$[pc^1, pc^2]$	t	$\#iter$	$time(min)$	$NE?$
54	[10,13]	[92,98]	0.27	3	0.01	YES
	[9,25]	[57,64]	0.26	3	0.01	YES
	[22,24]	[84,54]	0.28	3	0.01	YES
	[1,11]	[92,95]	0.1	3	0.05	YES
	[15,18]	[87,55]	0.13	2	0.02	YES
	[10,7]	[99,92]	0.13	3	0.02	YES
	[22,11]	[67,85]	0.11	2	0.01	YES
	[13,1]	[75,89]	0.17	2	0.03	YES
	[1,11]	[80,98]	0.21	2	0.02	YES
	[16,11]	[71,75]	0.28	2	0.01	YES
121	[4,8]	[56,65]	0.2	3	0.05	YES
	[24,18]	[76,53]	0.22	3	0.03	YES
	[3,18]	[91,62]	0.13	2	0.11	YES
	[6,5]	[86,86]	0.13	3	0.06	YES
	[14,11]	[77,92]	0.11	2	0.04	YES
	[22,4]	[56,93]	0.13	2	0.10	YES
	[24,5]	[57,67]	0.16	2	0.04	YES
	[23,20]	[51,62]	0.17	2	0.03	YES
	[22,7]	[66,75]	0.19	2	0.03	YES
	[21,11]	[64,75]	0.25	2	0.03	YES
250	[7,10]	[58,88]	0.19	3	0.21	YES
	[14,7]	[97,87]	0.26	3	0.07	YES
	[22,1]	[80,71]	0.14	2	0.12	YES
	[21,15]	[57,52]	0.24	3	0.06	YES
	[25,16]	[65,59]	0.22	3	0.07	YES
	[15,20]	[53,53]	0.24	2	0.04	YES
	[22,6]	[76,84]	0.19	2	0.12	YES
	[21,5]	[86,93]	0.26	3	0.12	YES
	[3,18]	[55,65]	0.12	3	0.12	YES
	[22,8]	[50,78]	0.12	2	0.26	YES
592	[11,20]	[96,70]	0.29	4	0.50	YES
	[22,12]	[56,92]	0.1	2	1.10	YES
	[24,23]	[61,77]	0.27	3	0.24	YES
	[18,23]	[84,75]	0.11	3	0.31	YES
	[21,4]	[85,85]	0.23	3	0.31	YES
	[23,8]	[92,97]	0.14	3	0.73	YES
	[13,4]	[56,76]	0.21	3	1.10	YES
	[17,25]	[90,94]	0.23	3	0.16	YES
	[2,23]	[54,62]	0.16	2	0.30	YES
	[7,2]	[60,60]	0.3	3	0.60	YES
1049	[22,14]	[74,65]	0.19	3	0.48	YES
	[8,21]	[92,58]	0.21	3	2.70	YES
	[25,24]	[53,64]	0.3	3	0.34	YES
	[14,22]	[90,99]	0.17	3	0.42	YES
	[6,9]	[93,73]	0.25	4	1.03	YES
	[7,5]	[61,79]	0.22	3	1.11	YES
	[12,18]	[65,77]	0.25	3	0.47	YES
	[10,1]	[96,51]	0.26	2	0.92	YES
	[12,22]	[87,78]	0.14	4	1.00	YES
	[13,4]	[63,71]	0.19	3	1.39	YES

Table 4: Results for quadratic demand function.

$\mid L \mid$	$[f_1, f_2]$	$[pc^1, pc^2]$	t	$\#iter$	$time(min)$	$NE?$
	[1,20]	[78,64]	0.22	20	0.28	NO
	[25,16]	[70,81]	0.2	3	0.01	YES
	[8,18]	[66,64]	0.22	3	0.02	YES
	[24,25]	[60,51]	0.24	3	0.01	YES
54	[24,6]	[75,82]	0.29	3	0.02	YES
	[2,17]	[83,80]	0.13	4	0.05	YES
	[6,3]	[73,92]	0.13	20	0.29	NO
	[8,2]	[79,91]	0.27	3	0.04	YES
	[8,10]	[81,66]	0.15	3	0.02	YES
	[10,17]	[81,92]	0.16	3	0.02	YES
	[5,22]	[72,75]	0.3	3	0.03	YES
	[2,10]	[61,80]	0.25	3	0.06	YES
	[5,10]	[91,62]	0.18	3	0.09	YES
	[8,15]	[68,75]	0.3	3	0.03	YES
121	[4,4]	[91,61]	0.23	3	0.10	YES
	[5,23]	[54,86]	0.17	20	0.43	NO
	[15,5]	[90,91]	0.29	3	0.05	YES
	[1,4]	[65,62]	0.12	3	0.14	YES
	[22,12]	[67,99]	0.28	3	0.04	YES
	[9,14]	[75,63]	0.3	20	0.18	NO
	[14,3]	[78,85]	0.17	4	0.21	YES
	[1,21]	[73,89]	0.2	3	0.36	YES
	[2,23]	[92,73]	0.22	20	1.80	NO
	[10,12]	[81,96]	0.26	3	0.08	YES
250	[11,3]	[92,91]	0.28	3	0.16	YES
	[14,8]	[83,76]	0.24	3	0.07	YES
	[11,22]	[93,95]	0.23	3	0.06	YES
	[1,11]	[90,93]	0.13	3	0.34	YES
	[24,19]	[68,55]	0.29	20	0.39	NO
	[13,21]	[88,56]	0.17	2	0.09	YES
	[7,20]	[54,79]	0.16	20	2.70	NO
	[2,19]	[72,72]	0.27	3	0.67	YES
	[12,4]	[89,53]	0.12	20	13.64	NO
	[25,13]	[65,71]	0.16	4	0.23	YES
592	[11,19]	[60,84]	0.29	20	1.48	NO
	[19,22]	[91,70]	0.21	4	0.39	YES
	[12,25]	[60,82]	0.26	20	1.52	NO
	[22,7]	[52,90]	0.22	2	0.43	YES
	[18,25]	[81,84]	0.1	3	0.14	YES
	[2,2]	[81,57]	0.16	3	1.19	YES
	[25,7]	[64,62]	0.23	4	0.46	YES
	[1,3]	[63,98]	0.13	4	2.80	YES
	[9,9]	[82,84]	0.25	4	0.55	YES
	[9,6]	[77,61]	0.19	3	0.54	YES
1049	[25,4]	[54,85]	0.14	2	2.26	YES
	[22,5]	[59,87]	0.21	20	20.62	NO
	[11,13]	[53,95]	0.25	20	13.56	NO
	[25,20]	[70,64]	0.16	5	0.53	YES
	[2,3]	[89,50]	0.24	5	4.89	YES
	[11,23]	[50,67]	0.13	3	0.89	YES

Table 5: Results for exponential demand function.

$\mid L \mid$	$[f_1, f_2]$	$[pc^1, pc^2]$	t	$\#iter$	$time(min)$	$NE?$
54	[2,20]	[72,55]	0.18	20	0.26	NO
	[21,15]	[92,73]	0.11	2	0.01	YES
	[9,6]	[68,83]	0.29	3	0.02	YES
	[9,18]	[96,98]	0.13	3	0.02	YES
	[25,4]	[96,84]	0.21	3	0.02	YES
	[24,7]	[84,91]	0.14	4	0.02	YES
	[18,7]	[65,71]	0.15	3	0.02	YES
	[1,12]	[70,89]	0.12	2	0.02	YES
	[25,21]	[54,73]	0.22	4	0.02	YES
	[6,16]	[78,96]	0.12	20	0.11	NO
121	[17,19]	[56,73]	0.17	20	0.20	NO
	[22,18]	[56,52]	0.19	4	0.03	YES
	[22,5]	[69,74]	0.15	20	0.28	NO
	[8,6]	[77,55]	0.11	20	0.58	NO
	[22,14]	[57,62]	0.14	3	0.03	YES
	[10,4]	[93,95]	0.19	4	0.07	YES
	[14,1]	[54,76]	0.3	20	0.99	NO
	[13,25]	[84,60]	0.23	20	0.22	NO
	[17,5]	[89,74]	0.25	3	0.03	YES
	[18,7]	[99,60]	0.2	4	0.08	YES
250	[16,11]	[72,96]	0.29	20	0.58	NO
	[11,9]	[58,52]	0.28	4	0.09	YES
	[7,11]	[60,90]	0.14	3	0.22	YES
	[10,18]	[54,56]	0.14	3	0.05	YES
	[10,16]	[93,72]	0.29	20	0.58	NO
	[11,14]	[81,69]	0.24	20	0.55	NO
	[21,7]	[87,51]	0.11	2	0.34	YES
	[24,1]	[53,78]	0.17	2	0.42	YES
	[1,13]	[81,63]	0.12	20	3.53	NO
	[13,18]	[88,58]	0.26	5	0.16	YES
592	[1,20]	[51,80]	0.18	3	0.57	YES
	[16,13]	[95,72]	0.17	20	4.58	NO
	[6,13]	[93,60]	0.18	2	0.54	YES
	[13,24]	[83,62]	0.15	2	0.78	YES
	[1,19]	[72,64]	0.21	20	12.92	NO
	[1,23]	[92,58]	0.12	2	1.93	YES
	[22,13]	[63,61]	0.13	6	0.27	YES
	[23,11]	[93,64]	0.15	2	0.64	YES
	[12,1]	[55,83]	0.25	20	10.16	NO
	[10,20]	[82,57]	0.2	20	2.63	NO
1049	[6,25]	[86,56]	0.22	3	2.38	YES
	[6,4]	[50,65]	0.13	4	2.85	YES
	[18,1]	[55,63]	0.11	20	30.94	NO
	[22,24]	[83,68]	0.18	20	3.20	NO
	[16,21]	[84,77]	0.12	4	0.53	YES
	[9,15]	[57,97]	0.19	3	2.91	YES
	[22,11]	[97,99]	0.19	3	0.41	YES
	[22,13]	[50,91]	0.15	2	1.95	YES
	[21,8]	[66,99]	0.27	20	18.76	NO
	[10,5]	[68,65]	0.13	7	1.04	YES

Table 6: Results for hyperbolic demand function.