Applied Probability Trust (9 December 2019)

ON THE EXTENSION OF SIGNATURE-BASED REPRESENTATIONS FOR COHERENT SYSTEMS WITH DEPENDENT NON-EXCHANGEABLE COMPONENTS

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Abstract

The signature representation shows that the reliability of the system is a mixture of the reliability functions of the k-out-of-n systems. The first representation was obtained for systems with independent and identically distributed (IID) components and after it was extended to exchangeable (EXC) components. The purpose of the present paper is to extended it to the class of systems with ID components which have a diagonal-dependent copula. We prove that this class is much larger than the class with EXC components. This extension is used to compare systems with non-EXC components.

Keywords: Signature; copula; stochastic orders; k-out-of-n systems

2010 Mathematics Subject Classification: Primary 60E15; 62K10

Secondary 90B25

1. Introduction

The coherent systems are basic concepts in the reliability theory, for their main properties we refer the reader to the classic book [1] (for completeness some of them are presented in Section 2). The *signature representation* obtained by Samaniego [22] (see also [23]) is a useful tool to study coherent systems. It proves that the reliability function of a coherent system with IID components having a continuous reliability

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function $\bar{F}$ can be written as a mixture (linear combination) of the reliability functions of the k-out-of-n systems (i.e., systems that fail when k of their n components fail). The vector with the coefficients in that representation is called the *signature* of the system. The lifetimes of the k-out-of-n systems coincide with the ordered component failure times and they are equivalent to the order statistics obtained from a sample of IID-$\bar{F}$ random variables. So this representation can be used to compute the system reliability (see [23]) and to compare (under different stochastic criteria) two systems with IID-$\bar{F}$ components just by comparing their signatures (see [7]).

The signature representation was extended to the case of *exchangeable* component lifetimes in [13, 16]. The coefficients are the same as that obtained in the IID case. The component lifetimes are EXC if and only if they are ID and their copula is EXC (i.e. invariant under permutations). Example 5.1 in [16] proved that this representation does not necessarily hold for systems with independent non-ID components.

In the present paper, this representation is extended to coherent systems whose component lifetimes are ID and have a dependence represented by a wide family of copulas called *diagonal-dependent* copulas. This family contains all the EXC copulas and also a lot of non-EXC copulas. This extension is used to compare systems (under different stochastic criteria) having this kind of dependent components.

The rest of the paper is organized as follows. The notation and some preliminary results are given in Section 2. The main results are in Section 3 where we prove the new representation and we obtain the corresponding comparison results. There we also prove that the class of diagonal-dependent copulas is a really big class when it is compared (from a topological viewpoint) with the class of EXC copulas. Section 4 contains some illustrative examples. The conclusions and some open problems for future research are placed in Section 5.

Throughout the paper, we say that a function $G : \mathbb{R}^n \to \mathbb{R}$ is increasing (resp. decreasing) if $G(x_1, \dots, x_n) \le G(y_1, \dots, y_n)$ ($\ge$) when $x_j \le y_j$ for all j.

2. Notation and preliminary results

From [1] a (two-states) *system* is a Boolean function $\psi : \{0,1\}^n \to \{0,1\}$, where $\psi(x_1, \dots, x_n)$ represents the state of the system (1 means that it works and 0 that it does not work) which is completely determined by the component states $x_1, \dots, x_n \in \{0,1\}$. A system ψ is *semi-coherent* if it is increasing, $\psi(0,\dots,0) = 0$ and $\psi(1,\dots,1) = 1$. We say that the jth component is *relevant* for the system ψ if there exist $x_1, \dots, x_{j-1}$, $x_{j+1}, \dots, x_n \in \{0,1\}$ such that

$$0 = \psi(x_1,\dots,x_{j-1},0,x_{j+1},\dots,x_n) < \psi(x_1,\dots,x_{j-1},1,x_{j+1},\dots,x_n) = 1.$$

A system ψ is *coherent* if it is increasing and all the components are relevant. Clearly, if ψ is coherent, then ψ is semi-coherent. However, for example, $\psi(x_1, x_2) = x_1$ is semi-coherent but it is not coherent (since the second component is irrelevant for the system).

Let T be the lifetime of a coherent system with component lifetimes $T_1, \dots, T_n$. Let $T_{1:n}, \dots, T_{n:n}$ be the associated ordered component lifetimes. Here $T_{k:n}$ represents the lifetime of the k-out-of-n system. It is well known that the system lifetime T is equal to one of these component lifetimes. Moreover, Samaniego [22] proved that, if $T_1, \dots, T_n$ are IID with a continuous reliability (survival) function $\bar{F}(t) = \mathbb{P}(T_j > t)$, then the system reliability function $\bar{F}_T(t) = \mathbb{P}(T > t)$ can be represented as

$$\bar{F}_T(t) = s_1 \bar{F}_{1:n}(t) + \cdots + s_n \bar{F}_{n:n}(t) \tag{1}$$

for any time t, where $\bar{F}_{i:n}(t) = \mathbb{P}(T_{i:n} > t)$ and $s_i = \mathbb{P}(T = T_{i:n})$ for $i = 1, \dots, n$. The vector with these coefficients $\mathbf{s} = (s_1, \dots, s_n)$ only depends on the structure ψ of the system and it is called the *signature* of the system (see [22, 23]). From the theory of order statistics, $\bar{F}_{1:n}, \dots, \bar{F}_{n:n}$ can be calculated from $\bar{F}$. So the signature-based mixture representation (1) is a useful tool to compute the system reliability (see [23]). It can also be used to compare stochastically systems with different structures (see [7, 16, 21, 23]).

Representation (1) (with the same coefficients) was extended in [16] (see also [13]) to the case in which the random vector $(T_1, \dots, T_n)$ with the component lifetimes has an EXC joint reliability function $\bar{\mathbf{F}}(t_1, \dots, t_n) = \mathbb{P}(T_1 > t_1, \dots, T_n > t_n)$, that is,

$$\bar{\mathbf{F}}(t_1, \dots, t_n) = \bar{\mathbf{F}}(t_{\sigma(1)}, \dots, t_{\sigma(n)})$$

holds for any permutation σ. In this case, the same property holds for the joint distribution function and $(T_1, \dots, T_n)$ is invariant in law under permutations.

From the copula theory (see, e.g., [6, p. 33] or [17, p. 32]), we know that $\bar{\mathbf{F}}$ can be written as

$$\bar{\mathbf{F}}(t_1,\dots,t_n) = \hat{C}(\bar{F}_1(t_1),\dots,\bar{F}_n(t_n))$$

for an n-dimensional copula function $\hat{C}$ (called *survival copula*) and for the marginal (component) reliability functions $\bar{F}_i(t_i) = \mathbb{P}(T_i > t_i)$, $i = 1,\dots,n$. A *copula* is an n-dimensional distribution function with uniform marginals over the interval $(0,1)$.

It is easy to prove that $\bar{\mathbf{F}}$ is EXC if and only if $\hat{C}$ is EXC and the component lifetimes are ID, that is, $\bar{F}_1 = \cdots = \bar{F}_n$. As above, in the ID case, the common component reliability function will be represented simply as $\bar{F}$.

Example 5.1 in [16] proves that representation (1) does not necessarily hold when the component lifetimes are independent but not ID. Therefore, the ID assumption (included in the EXC case) cannot be relaxed if we want to get (1). However, in the following section, we will prove that the other assumption: "$\hat{C}$ is EXC" can be relaxed.

For this purpose we need the following representation for the system reliability which is valid in the general case, that is, for any joint reliability function $\bar{\mathbf{F}}$. From [1, p. 12], we know that the system lifetime T can be written as

$$T = \max_{i=1,\dots,r} T_{P_i},$$

where $T_{P_i} = \min_{j\in P_i} T_j$ is the lifetime of the series system with the components in P_i for $i = 1,\dots,r$ and $P_1,\dots,P_r$ are the minimal path sets of the system. A *path set* is a set $P \subseteq \{1,\dots,n\}$ of components such that the system works when all the components in P work (i.e., $\psi(x_1,\dots,x_n) = 1$ when $x_i = 1$ for all $i \in P$). A path set is a *minimal path set* if it does not contain other path sets. Then, by using the inclusion-exclusion formula, the system reliability can be written as

$$\begin{aligned}\bar{F}_T(t) &= \mathbb{P}(T > t) = \mathbb{P}\left(\max_{i=1,\dots,r} T_{P_i} > t\right) = \mathbb{P}\left(\bigcup_{i=1}^{r}\{T_{P_i} > t\}\right)\\ &= \sum_{i=1}^{r}\bar{F}_{P_i}(t) - \sum_{i=1}^{r-1}\sum_{j=i+1}^{r}\bar{F}_{P_i\cup P_j}(t) + \cdots + (-1)^{r+1}\bar{F}_{P_1\cup\cdots\cup P_r}(t) \qquad (2)\end{aligned}$$

for all t, where we use the notation $\bar{F}_P(t) = \mathbb{P}(T_P > t)$ for the reliability function of the series system with components in the set P. Expression (2) proves that the reliability function of the system $\bar{F}_T$ is a linear combination of the reliability functions of series systems. However it is not a mixture representation since it may contain some negative coefficients. The preceding expression can also be used to compute the system reliability and to compare systems (see the review in [11]).

3. Main results

Let us start with a definition extracted from [18]. From now on we will use the following notation. For any set $I \subseteq \{1, \dots, n\}$, $\mathbf{u}_I := (u_1, \dots, u_n)$ denotes the vector with $u_i = u$ for $i \in I$ and $u_i = 1$ if $i \notin I$. The cardinality of the set I is denoted by $|I|$.

Definition 1. An n-dimensional copula C is said to be **diagonal-dependent** (denoted by DD) if

$$C(\mathbf{u}_P) = C(\mathbf{u}_Q) \text{ for all } P, Q \subseteq \{1, \dots, n\} \text{ with } |P| = |Q|. \tag{3}$$

The function $\delta(u) = C(u, \dots, u)$ is called the *diagonal section* of the copula C. Hence note that C is DD if and only if

$$C(\mathbf{u}_P) = \delta_m(u) \text{ for all } P \subseteq \{1, \dots, n\} \text{ with } |P| = m \tag{4}$$

for $m = 1, \dots, n$, where

$$\delta_m(u) := C(\underbrace{u, \dots, u}_{m-times}, \underbrace{1, \dots, 1}_{(n-m)-times})$$

is the diagonal section for the copula of the marginal distribution of the first m-variables. Clearly, $\delta_n(u) = C(u, \dots, u) = \delta(u)$ and $\delta_1(u) = u$ for all $u \in [0, 1]$ (since all the univariate marginals have a uniform distribution over the interval $(0, 1)$). So we just need to check (4) for $m = 2, \dots, n-1$.

In particular, a copula C is DD when all the marginals of dimension m have the same copula for all $1 < m < n$. Of course, all the EXC copulas are, in particular, DD. The reverse is not true (see Proposition 1 and Example 1 below).

Now we are ready to state the main result of the paper.

Theorem 1. *If T is the lifetime of a coherent system and the component lifetimes are ID and have a DD survival copula, then* (1) *holds for the same coefficients $s_1, \dots, s_n$ obtained in the IID continuous case.*

Proof. From (2) we know that the system reliability function $\bar{F}_T$ can be written as a linear combination of the reliability functions of the series systems. If the component lifetimes are ID with a reliability function $\bar{F}$ and a DD survival copula $\hat{C}$, then

$$\bar{F}_P(t) = \mathbb{P}\left(\min_{j\in P} T_j > t\right) = \hat{C}_P(\bar{F}(t),\dots,\bar{F}(t)) = \hat{\delta}_m(\bar{F}(t)) \tag{5}$$

holds for all t and all $P \subseteq \{1,\dots,n\}$, where $\hat{C}_P(u_1,\dots,u_n) := \hat{C}(u_1^P,\dots,u_n^P)$ and $u_i^P = u_i$ if $i \in P$ and $u_i^P = 1$ if $i \notin P$, $m = |P|$ and $\hat{\delta}_m$ is defined as

$$\hat{\delta}_m(u) := \hat{C}(\underbrace{u,\dots,u}_{m-times}, \underbrace{1,\dots,1}_{(n-m)-times})$$

for all $u \in [0,1]$ and $m = 1,\dots,n$. Hence, all the series systems with the same number of components m do have the same reliability function given by (5). Therefore, the general representation (2) can be reduced to

$$\bar{F}_T(t) = a_1\hat{\delta}_1(\bar{F}(t)) + \cdots + a_n\hat{\delta}_n(\bar{F}(t)), \tag{6}$$

where $a_1,\dots,a_n$ are some coefficients that only depend on the system structure (i.e., the minimal path sets).

The preceding representation (6) holds for any system structure (with the appropriate coefficients $a_1,\dots,a_n$). For example, the series system with n components has just one minimal path set $P_1 = \{1,\dots,n\}$ and lifetime $T_{1:n} = \min(T_1,\dots,T_n)$. Hence

$$\bar{F}_{1:n}(t) = \mathbb{P}(T_1 > t,\dots,T_n > t) = \hat{C}(\bar{F}(t),\dots,\bar{F}(t)) = \hat{\delta}_n(\bar{F}(t)) \tag{7}$$

for all t.

Analogously, the minimal path sets of $T_{2:n}$ are all the subsets with $n-1$ elements. So there are $n = \binom{n}{n-1}$ minimal path sets and, from (2),

$$\bar{F}_{2:n}(t) = n\hat{\delta}_{n-1}(\bar{F}(t)) - (n-1)\hat{\delta}_n(\bar{F}(t)) \tag{8}$$

holds for all t. The last coefficient in the preceding expression is $n-1$ because the coefficients in (6) sum up to 1 (take $t \to -\infty$).

In general, $T_{i:n}$ has $\binom{n}{n-i+1}$ minimal path sets and, from (2), its reliability function can be written as

$$\bar{F}_{i:n}(t) = a_{i,n-i+1}\hat{\delta}_{n-i+1}(\bar{F}(t)) + \cdots + a_{i,n}\hat{\delta}_n(\bar{F}(t)) \tag{9}$$

for some coefficients $a_{i,n-i+1},\dots,a_{i,n}$ such that $a_{i,n-i+1}+\dots+a_{i,n}=1$ and $a_{i,n-i+1}=\binom{n}{n-i+1}$ for $i=1,\dots,n$.

Thus, if we define the column vectors $\mathbf{r}(t)=(\bar{F}_{1:n}(t),\dots,\bar{F}_{n:n}(t))'$ and $\mathbf{d}(t)=(\hat{\delta}_1(\bar{F}(t)),\dots,\hat{\delta}_n(\bar{F}(t)))'$, (9) proves that $\mathbf{r}(t)=A\mathbf{d}(t)$ for a triangular real-valued matrix $A=(a_{i,j})$ such that $a_{i,n-i+1}=\binom{n}{n-i+1}$ and $a_{i,j}=0$ for $i=1,\dots,n$ and $j=1,\dots,n-i$. Hence A is not singular and so we can write $\mathbf{d}(t)=A^{-1}\mathbf{r}(t)$, where A^{-1} is the inverse matrix of A. Moreover, note that (6) can be rewritten as

$$\bar{F}_T(t)=(a_1,\dots,a_n)\mathbf{d}(t).$$

Then

$$\bar{F}_T(t)=(a_1,\dots,a_n)A^{-1}\mathbf{r}(t)=(c_1,\dots,c_n)\mathbf{r}(t)=c_1\bar{F}_{1:n}(t)+\dots+c_n\bar{F}_{n:n}(t)$$

for all t, where $(c_1,\dots,c_n)=(a_1,\dots,a_n)A^{-1}$ are some coefficients which only depend on the structure of the system. Therefore, these coefficients should be the same as that obtained in the IID continuous case, that is, $c_i=s_i$ for $i=1,\dots,n$. So (1) holds with the same coefficients for systems with ID component lifetimes and DD survival copulas. □

Remark 1. The usual copula representation for the joint distribution function of $(T_1,\dots,T_n)$ is

$$\mathbf{F}(t_1,\dots,t_n):=\mathbb{P}(T_1\le t_1,\dots,T_n\le t_n)=C(F_1(t_1),\dots,F_n(t_n)),$$

where $F_i(t_i)=\mathbb{P}(T_i\le t_i)$, $i=1,\dots,n$, are the univariate marginal distribution functions and C is the *distributional copula*. Both copulas C and $\hat{C}$ determine the dependence structure of $(T_1,\dots,T_n)$. So C determines $\hat{C}$ and vice versa. Moreover, it is easy to see that $\hat{C}$ is DD if and only if C is DD. Hence, we can obtain an alternative proof of Theorem 1 by using copula C and the representation of the system lifetime in terms of its minimal cut sets (see [1, p. 12]).

Remark 2. If $T_{1:j}=\min(T_1,\dots,T_j)$, then $\bar{F}_{1:j}(t):=\mathbb{P}(T_{1:j}>t)=\hat{\delta}_j(\bar{F}(t))$ for all t and $j=1,\dots,n$. Hence, under the assumptions of the preceding theorem, expression (6) can also be written as

$$\bar{F}_T(t)=a_1\bar{F}_{1:1}(t)+\dots+a_n\bar{F}_{1:n}(t).$$

The vector $(a_1, \ldots, a_n)$ with these coefficients is called the *minimal signature* (or the domination coefficients) of the system (see, e.g., [15] and [23, p. 77]). Hence, the representation based on minimal signatures obtained in [15] for systems with EXC component lifetimes can also be extended to systems with ID component lifetimes and DD survival copulas. From the comments given in the preceding remark, the same can be applied to the representation based on parallel systems and the *maximal signature* (see [15]).

As an immediate consequence of the main theorem we obtain several properties that can be used to compare stochastically systems with different structures. In the following theorem we state the results for the (usual) stochastic order ($\leq_{st}$), the hazard rate order ($\leq_{hr}$), the mean residual life order ($\leq_{mrl}$), and the likelihood ratio order ($\leq_{lr}$), extending the comparison results obtained in [7] (IID case) and [16] (EXC case). Similar results can be obtained for the reversed hazard rate and mean inactivity time orders extending that given in [14]. For the formal definitions of these orders, their basic properties and their main applications we refer the reader to [8, 10, 24]. We just note here that the relationships between these orders are as follows:

$$\begin{array}{ccccc} X \leq_{lr} Y & \Rightarrow & X \leq_{hr} Y & \Rightarrow & X \leq_{mrl} Y \\ & & \Downarrow & & \Downarrow \\ & & X \leq_{st} Y & \Rightarrow & \mathbb{E}(X) \leq \mathbb{E}(Y). \end{array}$$

Because the signature vector $\mathbf{s} = (s_1, \ldots, s_n)$ of a system can be seen as the probability mass function of a discrete random variable with support contained in the set $\{1, \ldots, n\}$, these stochastic orders can also be applied to compare two signature vectors (as discrete distributions). Thus we can state the following theorem.

Theorem 2. *Let T and T^* be the lifetimes of two coherent systems with respective signatures $\mathbf{s} = (s_1, \ldots, s_n)$ and $\mathbf{s}^* = (s_1^*, \ldots, s_n^*)$. Let us assume that the component lifetimes of both systems are ID with a reliability function $\bar{F}$ and a common DD survival copula $\hat{C}$.*

(i) If $\mathbf{s} \leq_{st} \mathbf{s}^*$, then $T \leq_{st} T^*$ for all $\bar{F}$.

(ii) If $\mathbf{s} \leq_{hr} \mathbf{s}^*$ and

$$T_{1:n} \leq_{hr} \cdots \leq_{hr} T_{n:n}, \tag{10}$$

then $T \leq_{hr} T^*$ *for all* $\bar{F}$.

(iii) If $\mathbf{s} \leq_{hr} \mathbf{s}^*$ *and*

$$T_{1:n} \leq_{mrl} \cdots \leq_{mrl} T_{n:n}, \tag{11}$$

then $T \leq_{mrl} T^*$ *for all* $\bar{F}$.

(iv) If $\mathbf{s} \leq_{lr} \mathbf{s}^*$ *and*

$$T_{1:n} \leq_{lr} \cdots \leq_{lr} T_{n:n}, \tag{12}$$

then $T \leq_{lr} T^*$ *for all abs. cont.* $\bar{F}$.

The proof is immediate from (1), Theorem 1 and Theorems 1.A.6, 1.B.14, 2.A.15 and 1.C.17 in [24], respectively. Note that in *(iii)* we need the hr ordering between the signatures and the mrl ordering in (11), to get the mrl ordering between the system lifetimes. The mrl ordering between the signatures is not enough. In the IID case, if we just assume $\mathbf{s} \leq_{mrl} \mathbf{s}^*$, then we need some extra-conditions (see [9]).

Example 2 shows how to use the preceding theorem to obtain (distribution-free) comparisons results for systems with different structures. The signature vectors and the ordering relationships between all the systems (signatures) with 1-4 components are given in Table 1 and Figures 1–3 of [16] (see also Figures 1–3 in [11]). These orderings can be extended to systems satisfying the assumptions of Theorems 1 and 2. Note that Theorem 2 can also be applied to mixed systems (i.e., mixtures of coherent systems) and, in particular, to semi-coherent systems since they can be written as mixed systems (from Theorem 1).

Remark 3. Expression (9) can be used jointly with the results for distorted distributions obtained in [12] (see also Theorem 4 in [11]) to check if (10), (11) and (12) hold for a given DD survival copula $\hat{C}$. We show this procedure in Example 2. These properties depend on the survival copula. For example, from (7) and (8), $T_{1:n} \leq_{hr} T_{2:n}$ holds for all $\bar{F}$ if and only if the function

$$\frac{n\hat{\delta}_{n-1}(u) - (n-1)\hat{\delta}_n(u)}{\hat{\delta}_n(u)}$$

is decreasing in $(0,1)$, that is, if and only if the function

$$\frac{\hat{\delta}_n(u)}{\hat{\delta}_{n-1}(u)} = \frac{\hat{C}(u,\ldots,u,u)}{\hat{C}(u,\ldots,u,1)}$$

is increasing in $(0,1)$.

Now we introduce a new family of copulas which are DD but not EXC.

Proposition 1. *Let D be the absolutely continuous n-dimensional copula which has the following probability density function*

$$d(\mathbf{u}) = \begin{cases} 2^{n-1}-\alpha, & \text{for } \mathbf{u} \in I_0 \times \cdots \times I_0, \\ 2^{n-1}-\alpha, & \text{for } \mathbf{u} \in I_1 \times \cdots \times I_1, \\ \frac{\alpha}{2^{n-1}-1}, & \text{for } \mathbf{u} \in I_{i_1} \times \cdots \times I_{i_n}, i_j = 0,1, \text{but } i_1 = \cdots = i_n \text{ does not hold,} \end{cases}$$

where $\mathbf{u} = (u_1, \dots, u_n)$, $I_0 = [0, 1/2)$, $I_1 = [1/2, 1]$ and $1/2 < \alpha < 2^{n-1} - 1$. Let $g_1, \dots, g_n : [0,1] \to [0,1]$ be different absolutely continuous functions such that $g_i(0) = g_i(1) = 0$ and $-1 \le 2g_i'(u) \le 1$ for all $i = 1, \dots, n$ and all $u \in [0,1]$ such that this derivative exists. Then

$$C(u_1, \dots, u_n) = D(u_1, \dots, u_n) + g_1(u_1) \dots g_n(u_n) \tag{13}$$

is a non-EXC DD copula with $\delta_m(u) \neq u^m$ for all $m = 2, \dots, n-1$. Moreover, if for an $i \in \{1, \dots, n\}$, $g_i(1/2) = 0$, then $\delta_n(u) \neq u^n$.

Proof. First we note that D is an EXC n-dimensional absolutely continuous copula since $2^{n-1} - \alpha > 0$ and

$$2\frac{2^{n-1}-\alpha}{2^n} + (2^n - 2)\frac{\alpha}{2^n(2^{n-1}-1)} = \frac{2^n - 2\alpha}{2^n} + \frac{2\alpha}{2^n} = 1.$$

Moreover, its diagonal section satisfies

$$\delta_D(1/2) = D(1/2, \dots, 1/2) = \frac{2^{n-1}-\alpha}{2^n} > \frac{1}{2^n} \tag{14}$$

(since $\alpha < 2^{n-1} - 1$).

Secondly, let us prove that the function C defined by (13) is a copula. Its probability density function c is given by

$$c(u_1, \dots, u_n) = d(u_1, \dots, u_n) + g_1'(u_1) \dots g_n'(u_n).$$

As $1/2 < \alpha < 2^{n-1} - 1$, then

$$2^{n-1} - \alpha > 1 > \frac{\alpha}{2^{n-1}-1}$$

and

$$d(u_1, \dots, u_n) \ge \frac{\alpha}{2^{n-1}-1} > \frac{1}{2^n - 2}.$$

Hence, as $-1 \le 2g_i'(u) \le 1$ for $i = 1, \dots, n$, then

$$c(u_1, \dots, u_n) \ge \frac{\alpha}{2^{n-1}-1} - \frac{1}{2^n} > \frac{1}{2^n - 2} - \frac{1}{2^n} > 0 \tag{15}$$

for all $u_1, \dots, u_n \in (0,1)$. Therefore

$$C(x_1, \dots, x_n) = \int_{[0,x_1]\times\cdots\times[0,x_n]} c(u_1, \dots, u_n) du_1 \dots du_n$$

is a copula since $g_i(0) = g_i(1) = 0$ for $i = 1, \dots, n$ and so

$$C(1, \dots, 1) = D(1, \dots, 1) + g_1(1) \dots g_n(1) = 1$$

and the other border conditions hold.

Clearly, C is not EXC (since we assume that the functions $g_1, \dots, g_n$ are different continuous functions). However, it is DD because D is EXC and

$$C_P(u, \dots, u) = D_P(u, \dots, u)$$

for all $P \subseteq \{1, \dots, n\}$ with $|P| < n$ (since $g_i(1) = 0$).

Moreover, for $n = 3, \dots$ and $m = 2, \dots, n-1$, $\delta_m(u) \neq u^m$ since

$$\begin{aligned}
\delta_m(1/2) &= C(\underbrace{1/2, \dots, 1/2}_{m-times}, \underbrace{1, \dots, 1}_{(n-m)-times}) \\
&= \frac{2^{n-1} - \alpha}{2^n} + \frac{2^{n-m} - 1}{2^n} \frac{\alpha}{2^{n-1} - 1} \\
&= \frac{1}{2} - \alpha \frac{2^{-1} - 2^{-m}}{2^{n-1} - 1} \\
&> \frac{1}{2^m}
\end{aligned}$$

because $\alpha < 2^{n-1} - 1$. Even more, if for an i, $g_i(1/2) = 0$, then

$$\delta_n(1/2) = C(1/2, \dots, 1/2) = D(1/2, \dots, 1/2) = \frac{2^{n-1} - \alpha}{2^n} > \frac{1}{2^n}$$

from (14). This concludes the proof. □

In order to characterize the relative size of the class $\mathcal{C}_{EXC}$ of EXC copulas in the class $\mathcal{C}_{DD}$ of DD copulas, we will use a topological approach similar to that suggested in [20] working with Baire's categories (see, e.g., [4, 5, 19]). Let us recall some basic topological definitions. A subset N of a metric space (Ω, d) is called *nowhere dense* if its closure has an empty interior. A subset $A \subseteq \Omega$ is of *first category* in (Ω, d) if it can be expressed as (or covered by) a countable union of nowhere dense sets. The subset A is called of *second category* if it is not of first category. Following [2], in complete metric spaces, first category sets are "small sets" and nowhere dense sets are "very small sets".

The next property shows that, from a topological viewpoint, the set $\mathcal{C}_{EXC}$ is very small (i.e. it is nowhere dense) into the set $\mathcal{C}_{DD}$. Therefore, Theorem 1 provides a relevant extension of signature-based representations from EXC copulas to DD copulas. It is easy to see that $\mathcal{C}_{DD}$ is a closed set in the set of all the copulas $\mathcal{C}$. Hence it is compact and complete (see [3]). Then, from Baire's theorem, it is a second category set in itself.

Proposition 2. *$\mathcal{C}_{EXC}$ is nowhere dense in $\mathcal{C}_{DD}$.*

Proof. As $\mathcal{C}_{EXC}$ is a closed subset of $\mathcal{C}$, then so is in $\mathcal{C}_{DD}$. Hence we need to prove that $\mathcal{C}_{EXC}$ does not have interior points. Let us see that for any $C_* \in \mathcal{C}_{EXC}$ and any $\varepsilon > 0$, we have

$$B\left(C_*, \varepsilon\right) = \left\{C \in \mathcal{C}_{DD} : d_\infty\left(C, C_*\right) < \varepsilon\right\} \subsetneqq \mathcal{C}_{EXC},$$

where $d_\infty\left(C, C_*\right) := \sup_{\mathbf{u}\in[0,1]^n} |C(\mathbf{u}) - C_*(\mathbf{u})|$. If $C_{**} \in \mathcal{C}_{DD} - \mathcal{C}_{EXC}$ and we define $C_n = \frac{1}{n}C_{**} + \frac{n-1}{n}C_*$, then the sequence $\{C_n\}$ converges to C_* with $C_n \in \mathcal{C}_{DD} - \mathcal{C}_{EXC}$ for all n. So, for any $\varepsilon > 0$, there exists n_0 (depending on ε) such that $C_n \in B\left(C_*, \varepsilon\right)$ for all $n \geq n_0$. Therefore $B\left(C_*, \varepsilon\right) \subsetneqq \mathcal{C}_{EXC}$ and so $\mathcal{C}_{EXC}$ is nowhere dense in $\mathcal{C}_{DD}$. $\square$

4. Examples

The first example shows how to use Theorem 1 in a coherent system with ID component lifetimes having a fixed non-EXC DD survival copula.

Example 1. Let us consider the system with lifetime $T = \min(T_1, \max(T_2, T_3))$. The signature of this coherent system is $(1/3, 2/3, 0)$ (see, e.g., [23, p. 24]). Let us assume that the component lifetimes T_1, T_2, T_3 are ID with a reliability function $\bar{F}$ and with the following survival copula

$$\hat{C}(u_1, u_2, u_3) = u_1 C_{FN}(u_2, u_3), \tag{16}$$

where $C_{FN}(u_2, u_3) = \min\left(u_2, u_3, \frac{u_2^2+u_3^2}{2}\right)$ is the Fredricks-Nelsen copula (see, e.g., [6, p. 32]). Clearly, the first component is independent from the other components but components 2 and 3 are dependent. So $\hat{C}$ is not EXC. Therefore the signature-based representations obtained in [22] (IID case) and [16] (EXC case) cannot be applied to this system. However, $\hat{C}$ is DD since $\hat{\delta}_2(u) := \hat{C}(u,u,1) = \hat{C}(u,1,u) = \hat{C}(1,u,u) = u^2$ for all $u \in [0,1]$. Note that $\hat{\delta}_1(u) := \hat{C}(u,1,1) = \hat{C}(1,u,1) = \hat{C}(1,1,u) = u$ (as expected since the univariate marginal distributions of copulas are uniform) and $\hat{\delta}_3(u) := \hat{C}(u,u,u) = u^3$ for all $u \in [0,1]$.

Therefore, we can apply Theorem 1, obtaining the following representation for the system reliability function $\bar{F}_T(t) = \frac{1}{3}\bar{F}_{1:3}(t) + \frac{2}{3}\bar{F}_{2:3}(t)$, where, from (7) and (8), $\bar{F}_{1:3}(t) = \hat{\delta}_3(\bar{F}(t)) = (\bar{F}(t))^3$ and $\bar{F}_{2:3}(t) = 3\hat{\delta}_2(\bar{F}(t)) - 2\hat{\delta}_3(\bar{F}(t)) = 3(\bar{F}(t))^2 - 2(\bar{F}(t))^3$ for all t. Hence this system (with two dependent components) has the same law (reliability) as the system with the same structure and three IID-$\bar{F}$ components.

The second example shows that Theorem 2 allows us to compare two coherent systems with different structures (signatures).

Example 2. Let us consider the systems with lifetimes $T = \min(T_1, \max(T_2, T_3))$ (studied in the preceding example) and $T^* = \max(T_1, \min(T_2, T_3))$. We assume that the component lifetimes are ID with a reliability function $\bar{F}$ and a survival copula $\hat{C}$. The signatures of these coherent systems are $\mathbf{s} = (1/3, 2/3, 0)$ and $\mathbf{s}^* = (0, 2/3, 1/3)$, respectively (see, e.g., [23, p. 24]). Therefore, as

$$\frac{0}{1/3} = 0 < \frac{2/3}{2/3} = 1 < \frac{1/3}{0} = \infty,$$

we have $\mathbf{s} \leq_{lr} \mathbf{s}^*$. As the likelihood ratio order is the strongest one, all the signature orderings in Theorem 2 hold. Then $T \leq_{st} T^*$ holds for any $\bar{F}$ and any DD copula $\hat{C}$.

Analogously, to get $T \leq_{hr} T^*$, we need to check if (10) holds. If we choose the survival copula of the preceding example, given in (16), then the distributions of $T_{1:3}, T_{2:3}, T_{3:3}$ coincide with that obtained in the IID case. Hence, (10) holds and we get $T \leq_{hr} T^*$ for all $\bar{F}$. Actually, (12) also holds and we have $T \leq_{lr} T^*$ for all abs.

cont. $\bar{F}$. However, if we select the survival copula from the family (13) given by

$$\hat{C}(u_1, u_2, u_3) = D(u_1, u_2, u_3) + g_1(u_1)g_2(u_2)g_3(u_3),$$

where $\alpha \in (1/2, 3)$ and $2g_1(u) = 3g_2(u) = 3g_3(u) = u(1-u)$. Then

$$\hat{\delta}_2(u) = \hat{C}(u, u, 1) = D(u, u, 1) = \begin{cases} \frac{6-\alpha}{3}\, u^2, & \text{for } u \in [0, 1/2], \\ h(u), & \text{for } u \in (1/2, 1], \end{cases}$$

and $\hat{\delta}_3(u) = D(u, u, u) + g_1(u)g_2(u)g_3(u)$, where

$$h(u) = \frac{6-\alpha}{12} + \frac{\alpha}{3}\left(u - \frac{1}{2}\right) + \left(2 - \frac{\alpha}{3}\right)\left(u - \frac{1}{2}\right)^2 = 1 + \frac{6-\alpha}{3}u^2 - \frac{6-2\alpha}{3}u - \frac{\alpha}{3},$$

$$D(u, u, u) = \begin{cases} (4-\alpha)u^3, & \text{for } u \in [0, 1/2], \\ \frac{4-\alpha}{8} + (4-\alpha)\left(u - \frac{1}{2}\right)^3 + \frac{\alpha}{3}\left[u^3 - \frac{1}{8} - \left(u - \frac{1}{2}\right)^3\right], & \text{for } u \in (1/2, 1], \end{cases}$$

and $g_1(u)g_2(u)g_3(u) = u^3(1-u)^3/18$. Therefore $\hat{\delta}_3(u) \neq u^3$.

Note that the reliability functions of $T_{1:3}, T_{2:3}, T_{3:3}$ are $\bar{q}_{1:3}(\bar{F}(t)), \bar{q}_{2:3}(\bar{F}(t)), \bar{q}_{3:3}(\bar{F}(t))$, respectively, where $\bar{q}_{1:3}(u) = \hat{\delta}_3(u)$, $\bar{q}_{2:3}(u) = 3\hat{\delta}_2(u) - 2\hat{\delta}_3(u)$, and $\bar{q}_{3:3}(u) = 3u - 3\hat{\delta}_2(u) + \hat{\delta}_3(u)$ for $u \in [0, 1]$.

Therefore, as per the results given in [12] (or in Theorem 4, (ii), of [11]), $T_{1:3} \leq_{hr} T_{2:3}$ holds for all $\bar{F}$ and that survival copula, if and only if $\bar{q}_{2:3}/\bar{q}_{1:3}$ is decreasing in $[0, 1]$, that is, the ratio $r = \hat{\delta}_2/\hat{\delta}_3$ is decreasing in $(0, 1)$. By plotting this ratio we see that this property is not always true. For example, when $\alpha = 1$, we obtain the plot given in Figure 1, left, which is not decreasing. However, when $\alpha = 2$, we obtain the plot given in Figure 1, right, which is decreasing in $(0, 1)$. Hence, $T_{1:3} \leq_{hr} T_{2:3}$ holds for all $\bar{F}$ when $\alpha = 2$. It can be proved analogously that $T_{2:3} \leq_{hr} T_{3:3}$ holds for all $\bar{F}$ when $\alpha = 2$. Therefore, (10) holds and, from Theorem 2, (ii), we get $T \leq_{hr} T^*$ for all $\bar{F}$ and that survival copula with $\alpha = 2$.

5. Conclusions

The signature-based representations can be extended to systems with ID non-EXC component lifetimes. Specifically, they are extended here for the dependence models determined by a DD copula. This extension is relevant since we have proved that the family of DD copulas is much larger (from a topological viewpoint) than the family of

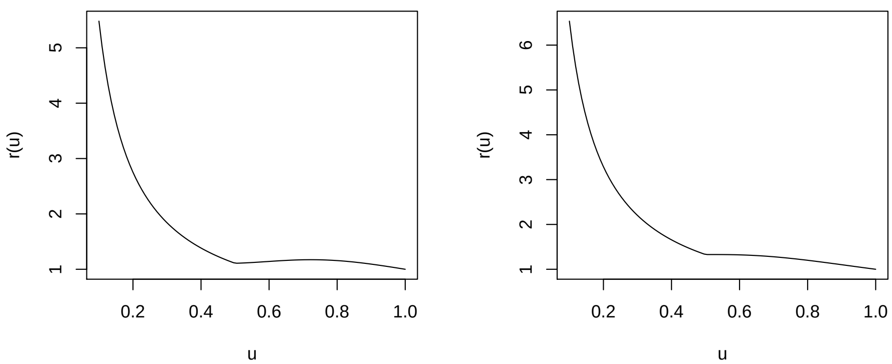

FIGURE 1: Plots of the ratio $r = \hat{\delta}_2/\hat{\delta}_3$ for the survival copula studied in Example 2 when $\alpha = 1$ (left) and $\alpha = 2$ (right).

EXC copulas. Moreover, some examples show that the new representation can be used to compare systems with these dependence models.

The main problem for future research could be to determine if these representations can be extended for other (bigger) families of copulas. We think that this extension is not possible by using the approach used in the present paper. However, it could be possible if we use a different technique. Other relevant problem is to determine when (10), (11) and (12) hold. These conditions are needed in order to get the comparison results given in Theorem 1. The distorted distributions may help in this task (as showed in Example 2).

Acknowledgements

We would like to thank the anonymous reviewers for several helpful suggestions that have served to add clarity and breadth to the earlier version of this paper.

JN was supported in part by Ministerio de Economía, Industria y Competitividad of Spain under grant MTM2016-79943-P (AEI/FEDER, UE) and JFS by Ministerio de Economía y Competitividad under grant MTM2014-60594-P (AEI/FEDER, UE).

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