Minimal repair of failed components in coherent systems

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Abstract

The minimal repair replacement is a reasonable assumption in many practical systems. Under this assumption a failed component is replaced by another one whose reliability is the same as that of the component just before the failure, i.e., a used component with the same age. In this paper we study the minimal repair in coherent systems. We consider both the cases of independent and dependent components. Three replacement policies are studied. In the first one, the first failed component in the system is minimally repaired while, in the second one, we repair the component which causes the system failure. A new technique based on the relevation transform is used to compute the reliability of the systems obtained under these replacement policies. In the third case, we consider the replacement policy which assigns the minimal repair to a fixed component in the system. We compare these three options under different stochastic criteria and for different system structures. In particular, we provide the optimal strategy for all the coherent systems with 1-4 independent and identically distributed components.

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1. Introduction

There exists an extensive literature on redundancy, repair and replacement policies for coherent systems subject to failure. The basic concepts can be seen in [8, 12, 29]. Roughly speaking, maintenance actions can be performed in different ways as replacement strategies, minimal repairs, imperfect repairs, redundancies, etc. On the one hand, it is addressed in literature the concept of active (hot) redundancy, where some additional components are included in the system by using parallel structures, see [45, 50, 51, 52] and [14, 48] for systems having independent and dependent components, respectively. On the other hand, it is addressed the concept of standby (cold) redundancy, where a component is replaced or repaired when it fails. A nice summary of those cases is described in Aven [5]. Among the standby policies, many papers study the case when the broken unit is physically replaced by a new and identical unit (perfect repair), see [25, 43, 46].

In this paper we will focus on minimal repairs. The basic minimal repair model was introduced in Barlow and Hunter [11]. Since then, many works have been published attempting to extend the minimal repair concept, see, among others, Aven [2, 3], Aven and Castro [6], Aven and Jensen [7, 8], Phelps [40], Bergman [16], Block et al. [17], Balakrishnan et al. [10], Stadje and Zuckerman

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[44], Shaked and Shanthikumar [41], Beichelt [13], Zhang and Jardine [49], Finkelstein [23] and Zequeira and Berenguer [47]. In short, under the minimal repair concept it is generally assumed that the maintenance action is such that the system is restored to the condition it had immediately before the failure occurred. This concept allows us to describe many repairs in real cases where it is not unrealistic to think that repairs basically bring the system to the same condition it was just before the failure. To formalize this idea, the basic model assumes that the repair time is negligible and the number of failures that occur in the interval $(0, t]$ follows a nonhomogeneous Poisson process (NHPP) with an intensity function, $\lambda(t)$.

At this point it is worth to mention that minimal repairs are usually split into two cases: physical minimal repairs and statistical minimal repairs. The former is used when a component of the system is repaired and the latter, also called black box minimal repair, when the system is repaired. Of course, if we look at a component as a subsystem, a physical minimal repair is just a statistical minimal repair at component level. The previous cases clearly depends on if we have information about the state of the components or just about failures of the whole system, respectively, see Aven [5] and Aven and Jensen [7, 8] for further details and illustrative examples. As it is nicely discussed in Aven [5], the need to be precise with the level of information leads to some author to frame their works in the theory of point processes by taking into account the history of the system. Some valuables contributions in this sense are given by Aven [2, 3, 4], Aven and Jensen [7, 8], Bergman [16], Arjas and Norros [1], Natvig [27, 28] and Gåsemyr and Natvig [24].

In this paper we will address the problem of comparing different repair policies by using minimal repairs. From the above discussion, we will assume we have information about the condition of each component of the system. In other words, all components can be observed at any time. Under this assumption, we will focus on comparing three different repair policies. The first policy, denoted by case I, consists in a statistical minimal repair of the component that fails first. The second one, denoted by case II, consists in a statistical minimal repair of the component that causes the system failure. The last one, denoted by case III, consists of assigning a statistical minimal repair to a fixed component. In all these cases we will consider only one repair although the same technique can be applied to k replacements as it will be commented later on. For our purpose we will consider a representation of the system's reliability based on the dependence structure (copula) of the components, the underlying distributions and the relevation transform as a tool to deal with the different policies. Some studies that share a similar approach have been recently published. Some stochastic comparisons of repairable coherent systems with independent components can be seen in [15, 19, 22]. Different replacement policies, under minimal repairs of fixed components and number of repairs, are compared in [9]. Some preservation results and aging properties of repairable systems under minimal repair are established in [18]. Our approach satisfies some advantages. First, the representation of the reliability function in terms of distortion functions leads to simplify the complex algebraic expressions derived from the computation of the system's reliability. Secondly, results can be applied to systems with dependent components. Furthermore, this approach can be used to deal with heterogeneous components. Finally, our approach allows us to make distribution-free comparisons. Additionally, we give a procedure to compare these three policies under different stochastic criteria (orders). In particular, we prove that the replacement policy of case II is better than that of case I under the assumption of independent and identically distributed (IID) components. However, some examples prove that they are not ordered with case III. We also apply this procedure to determine the best replacement policy in terms of the usual stochastic order for all the systems with 1-4 IID components.

The rest of the paper is organized as follows. In Section 2 we introduce the notation and the basic tools needed in the paper. The main results are given in Section 3, where we give a procedure to determine the reliability functions of the systems obtained with the replacement policies of cases I and II. These functions are used in Section 4 to compare the different replacement policies, obtaining some general results for the IID case. The conclusions are placed in Section 5.

Throughout the paper, we say that a function g is increasing (resp. decreasing) if $g(x) \leq g(y)$ ($\geq$) for all $x \leq y$. If $G : [0,1]^n \to [0,1]$, then $\partial_i G$ represents the partial derivative of G with respect to the ith variable.

2. Notation and preliminary results

2.1. Relevation transform

Let X and Y be two nonnegative independent random variables with absolutely continuous reliability (survival) functions $\bar{F}$ and $\bar{G}$. Then the reliability function of $X+Y$ (convolution) is

$$\bar{F} * \bar{G}(t) = \Pr(X+Y>t) = \int_t^\infty f(x)dx + \int_0^t \int_{t-x}^\infty g(y)f(x)dydx = \bar{F}(t) + \int_0^t \bar{G}(t-x)f(x)dx,$$

where f and g are the respective probability density functions. Under a *perfect repair* in a cold standby procedure, the unit X is replaced when failed by an independent unit Y having the same distribution as X (when new). Then the resulting reliability is

$$\bar{F} * \bar{F}(t) = \bar{F}(t) + \int_0^t \bar{F}(t-x)f(x)dx.$$

If X and Y are dependent, we obtain the expression included in the following definition.

Definition 2.1. *If X and Y are two nonnegative dependent random variables with reliability functions $\bar{F}$ and $\bar{G}$, then the relevation transform (or conditional convolution) $\bar{F}\#\bar{G}$ is the reliability of $X+Y$ given by*

$$\bar{F}\#\bar{G}(t) = \bar{F}(t) + \int_0^t \bar{G}_x(t-x)f(x)dx, \tag{2.1}$$

where f is the probability density function of X and $\bar{G}_x$ is the reliability function of $(Y|X=x)$.

Under a *classic relevation transform*, the unit X is replaced when it fails at a time x by a unit having reliability $\bar{G}$ but with the same age as X, that is, by $Y_x = (Y-x|Y>x)$ with reliability

$$\bar{G}_x(y) = \Pr(Y-x>y|Y>x) = \frac{\bar{G}(x+y)}{\bar{G}(x)}$$

for $y \geq 0$. Hence,

$$\bar{F}\#\bar{G}(t) = \Pr(X+Y_X>t) = \bar{F}(t) + \int_0^t \frac{\bar{G}(t)}{\bar{G}(x)} f(x)dx. \tag{2.2}$$

Under a *minimal repair*, the failed unit X is replaced by a unit having the same reliability as X and with the same age (that is, it is repaired to be as it was just before its failure). Then, from (2.2), the resulting reliability is

$$\bar{F}\#\bar{F}(t) = \bar{F}(t) + \int_0^t \frac{\bar{F}(t)}{\bar{F}(x)} f(x)dx = \bar{F}(t) - \bar{F}(t)\ln\bar{F}(t).$$

After k replacements, the resulting reliability is

$$\bar{F}\#^k\bar{F}(t) = \bar{F}(t)\sum_{i=0}^{k}\frac{1}{i!}[-\ln\bar{F}(t)]^i,$$

where $\bar{F}\#^0\bar{F} = \bar{F}$, $\bar{F}\#^1\bar{F} = \bar{F}\#\bar{F}$, $\bar{F}\#^2\bar{F} = (\bar{F}\#\bar{F})\#\bar{F}$ and so on. Note that $(\bar{F}\#\bar{F})\#\bar{F} \neq \bar{F}\#(\bar{F}\#\bar{F})$. We shall write it as $\bar{F}\#^k\bar{F}(t) = \bar{q}_k(\bar{F}(t))$ with

$$\bar{q}_k(u) = u\sum_{i=0}^{k}\frac{1}{i!}(-\ln u)^i. \tag{2.3}$$

The distributions that can be written in this way are called *distorted distributions* (see, e.g., [32, 37] and the references therein). Thus, we say that a distribution function F_q is a distortion of another distribution F if $F_q(t) = q(F(t))$ for a *distortion function* $q : [0,1] \to [0,1]$ increasing, continuous and such that $q(0) = 0$ and $q(1) = 1$. A similar representation holds for the respective reliability functions, that is, $\bar{F}_q(t) = \bar{q}(\bar{F}(t))$, where $\bar{q}(u) = 1 - q(1-u)$ for $u \in [0,1]$. It is also a distortion function, that is, it is an increasing continuous function in $[0,1]$ such that $\bar{q}(0) = 0$ and $\bar{q}(1) = 1$ (see, e.g., (2.3)). It is called the *dual distortion function* associated to q.

2.2. Coherent systems

Let T be the lifetime of a coherent system with component lifetimes $X_1,\dots,X_n$. In the general case, the components can be dependent and this possible dependency will be represented by the joint reliability of the components lifetimes which can be written as

$$\bar{\boldsymbol{F}}(x_1,\dots,x_n) = \Pr(X_1 > x_1,\dots,X_n > x_n) = K(\bar{F}_1(x_1),\dots,\bar{F}_n(x_n)),$$

where K is the survival copula and $\bar{F}_i$ is the reliability function of the ith component for $i = 1,\dots,n$ (see, e.g., [21, p. 33]). Note that the case of independent components is included here and that it is represented by the product copula $K = \Pi$, where $\Pi(u_1,\dots,u_n) = u_1\dots u_n$ for $u_1,\dots,u_n \in [0,1]$. From now on we assume that $\bar{\boldsymbol{F}}$ is absolutely continuous with joint probability density function

$$\boldsymbol{f}(x_1,\dots,x_n) = k(\bar{F}_1(x_1),\dots,\bar{F}_n(x_n))f_1(x_1)\dots f_n(x_n),$$

where f_i is the probability density function of X_i and

$$k(u_1,\dots,u_n) = \partial_1\dots\partial_n K(u_1,\dots,u_n) = \frac{\partial^n}{\partial u_1\dots\partial u_n}K(u_1,\dots,u_n)$$

is the probability density function associated to K.

Then it is well known (see, e.g., [26, 35]) that the system reliability can be written as

$$\bar{F}_T(t) = \bar{Q}(\bar{F}_1(t),\dots,\bar{F}_n(t)), \tag{2.4}$$

where $\bar{Q}$ is a *distortion function*, that is, a continuous increasing function $\bar{Q} : [0,1]^n \to [0,1]$ such that $\bar{Q}(0,\dots,0) = 0$ and $\bar{Q}(1,\dots,1) = 1$ which depends on the system structure and on K (the dependence structure). In particular, if the components are identically distributed (ID), then (2.4) reduces to $\bar{F}_T(t) = \bar{q}(\bar{F}(t))$ (see, e.g., [37]) where $\bar{F}$ is the common reliability function of the components and $\bar{q}(u) = \bar{Q}(u,\dots,u)$ (i.e., T has a distorted distribution from the common distribution of the components). If the components are just independent, then $\bar{Q}$ is a multinomial expression (see [12, p. 21]). Finally, if they are independent and identically distributed (IID), then $\bar{q}(u) = \sum_{i=1}^{n} a_i u^i$, where $(a_1, \ldots, a_n)$ is called the *minimal signature* of the system (see, e.g., [36]).

For example, if $n = 2$, the reliability function of the parallel system $X_{2:2} = \max(X_1, X_2)$ is

$$\bar{F}_{2:2}(t) = \Pr(\{X_1 > t\} \cup \{X_2 > t\}) = \bar{F}_1(t) + \bar{F}_2(t) - \Pr(X_1 > t, X_2 > t) = \bar{Q}_{2:2}(\bar{F}_1(t), \bar{F}_2(t)),$$

where $\bar{Q}_{2:2}(u, v) = u + v - K(u, v)$ and, in the IID case, $\bar{q}_{2:2}(u) = \bar{Q}_{2:2}(u, u) = 2u - u^2$.

2.3. Reliability of systems using the relevation transform

Let us see how the relevation transform can also be used to compute the system's reliability. This new technique will be used in the following sections to compute the reliability of systems with minimal repairs on failed components. As in the preceding section we consider the simple case of a two-component parallel system.

Example 2.2. *Let us consider $X_{2:2} = \max(X_1, X_2)$. If the component lifetimes X_1, X_2 are IID with a common reliability $\bar{F}$, then*

$$\bar{F}_{2:2}(t) = \bar{F}_{1:2} \# \bar{F}(t) = \bar{F}_{1:2}(t) + \int_0^t \frac{\bar{F}(t)}{\bar{F}(x)} f_{1:2}(x) dx$$

and, as $\bar{F}_{1:2}(t) = \bar{F}^2(t)$ and $f_{1:2}(t) = 2\bar{F}(t)f(t)$, we have

$$\bar{F}_{2:2}(t) = \bar{F}^2(t) + \int_0^t \frac{\bar{F}(t)}{\bar{F}(x)} 2\bar{F}(x)f(x)dx = \bar{F}^2(t) + 2\bar{F}(t)F(t) = 2\bar{F}(t) - \bar{F}^2(t).$$

Let us assume now that both components can be dependent with a survival copula K. Then

$$\begin{aligned} \bar{F}_{2:2}(t) &= \Pr(X_1 < X_2)\Pr(X_{2:2} > t | X_1 < X_2) + \Pr(X_2 < X_1)\Pr(X_{2:2} > t | X_2 < X_1) \\ &= \Pr(X_1 < X_2)\bar{F}_1^{(X_1<X_2)} \# \bar{G}_1(t) + \Pr(X_2 < X_1)\bar{F}_2^{(X_2<X_1)} \# \bar{G}_2(t), \end{aligned}$$

where $\bar{F}_1^{(X_1<X_2)}(t) = \Pr(X_1 > t | X_1 < X_2)$, $\bar{F}_2^{(X_2<X_1)}(t) = \Pr(X_2 > t | X_2 < X_1)$, $\bar{G}_{1,x}(y) = \Pr(X_2 - x > y | X_1 = x, X_2 > x)$ and $\bar{G}_{2,x}(y) = \Pr(X_1 - x > y | X_2 = x, X_1 > x)$. Note that

$$p_1 = \Pr(X_1 < X_2) = \int_0^\infty \int_x^\infty f_1(x)f_2(y)\partial_{1,2}K(\bar{F}_1(x), \bar{F}_2(y))dydx = \int_0^\infty f_1(x)\partial_1 K(\bar{F}_1(x), \bar{F}_2(x))dx$$

when $\lim_{u\to 0^+} \partial_1 K(\bar{F}_1(x), u) = 0$ (see [39]). Analogously,

$$p_2 = \Pr(X_2 < X_1) = 1 - p_1 = \int_0^\infty f_2(x)\partial_2 K(\bar{F}_1(x), \bar{F}_2(x))dx$$

when $\lim_{u\to 0^+} \partial_2 K(u, \bar{F}_2(y)) = 0$. The joint density of $(X_1, X_2 | X_1 < X_2)$ is $\boldsymbol{h}(x, y) = \boldsymbol{f}(x, y)/p_1$ for all $x \le y$ (0 otherwise). Then the marginal density of $(X_1 | X_1 < X_2)$ is

$$h_1(x) = \frac{1}{p_1}\int_x^\infty \boldsymbol{f}(x, y)dy = \frac{1}{p_1}\int_x^\infty f_1(x)f_2(y)\partial_{1,2}K(\bar{F}_1(x), \bar{F}_2(y))dy = \frac{1}{p_1} f_1(x)\partial_1 K(\bar{F}_1(x), \bar{F}_2(x)).$$

Hence, the conditional density of $(X_2 | X_1 = x, X_2 > x)$ is

$$h_{2|1}(y|x) = \frac{\boldsymbol{h}(x, y)}{h_1(x)} = \frac{f_2(y)\partial_{1,2}K(\bar{F}_1(x), \bar{F}_2(y))}{\partial_1 K(\bar{F}_1(x), \bar{F}_2(x))}$$

and then the reliability function $\bar{G}_{1,x}$ *is given by*

$$\bar{G}_{1,x}(y) = \int_{x+y}^{\infty} h_{2|1}(z|x)dz = \int_{x+y}^{\infty} \frac{f_2(z)\partial_{1,2}K(\bar{F}_1(x), \bar{F}_2(z))}{\partial_1 K(\bar{F}_1(x), \bar{F}_2(x))}dz = \frac{\partial_1 K(\bar{F}_1(x), \bar{F}_2(x+y))}{\partial_1 K(\bar{F}_1(x), \bar{F}_2(x))}. \quad (2.5)$$

In a similar way (by the symmetry), we get

$$\bar{G}_{2,x}(y) = \frac{\partial_2 K(\bar{F}_1(x+y), \bar{F}_2(x))}{\partial_2 K(\bar{F}_1(x), \bar{F}_2(x))}. \quad (2.6)$$

Therefore, from (2.1)*, we obtain*

$$\begin{aligned}
\bar{F}_1^{(X_1<X_2)} \# \bar{G}_1(t) &= \bar{F}_1^{(X_1<X_2)}(t) + \int_0^t \bar{G}_{1,x}(t-x)h_1(x)dx \\
&= \bar{F}_1^{(X_1<X_2)}(t) + \frac{1}{p_1}\int_0^t f_1(x)\partial_1 K(\bar{F}_1(x), \bar{F}_2(t))dx \\
&= \bar{F}_1^{(X_1<X_2)}(t) + \frac{1}{p_1}\left[\bar{F}_2(t) - K(\bar{F}_1(t), \bar{F}_2(t))\right].
\end{aligned}$$

Analogously, we have $\bar{F}_2^{(X_2<X_1)} \# \bar{G}_2(t) = \bar{F}_2^{(X_2<X_1)}(t) + \frac{1}{p_2}\left[\bar{F}_1(t) - K(\bar{F}_1(t), \bar{F}_2(t))\right]$. *Then*

$$\begin{aligned}
\bar{F}_{2:2}(t) &= p_1 \bar{F}_1^{(X_1<X_2)} \# \bar{G}_1(t) + p_2 \bar{F}_2^{(X_2<X_1)} \# \bar{G}_2(t) \\
&= p_1 \bar{F}_1^{(X_1<X_2)}(t) + p_2 \bar{F}_2^{(X_2<X_1)}(t) + \bar{F}_1(t) + \bar{F}_2(t) - 2K(\bar{F}_1(t), \bar{F}_2(t)) \\
&= p_1 \Pr(X_{1:2} > t | X_1 < X_2) + p_2 \Pr(X_{1:2} > t | X_2 < X_1) + \bar{F}_1(t) + \bar{F}_2(t) - 2K(\bar{F}_1(t), \bar{F}_2(t)) \\
&= \Pr(X_{1:2} > t) + \bar{F}_1(t) + \bar{F}_2(t) - 2K(\bar{F}_1(t), \bar{F}_2(t)) \\
&= \bar{F}_1(t) + \bar{F}_2(t) - K(\bar{F}_1(t), \bar{F}_2(t)).
\end{aligned}$$

These expressions can be simplified if $\boldsymbol{F}$ *is exchangeable (EXC), that is,* K *is permutation invariant and the components are ID. In this case we have* $\bar{F}_{2:2} = \bar{F}_{1:2} \# \bar{G}$*, where*

$$\bar{G}_x(y) = \Pr(X_2 - x > y | X_1 = x, X_2 > x) = \frac{\Pr(X_2 > x+y | X_1 = x)}{\Pr(X_2 > x | X_1 = x)}.$$

Then, from (2.5)*, we get* $\bar{G}_x(y) = \dfrac{\partial_1 K(\bar{F}(x), \bar{F}(x+y))}{\partial_1 K(\bar{F}(x), \bar{F}(x))}$*. Hence, from (2.1), we have*

$$\bar{F}_{1:2} \# \bar{G}(t) = \bar{F}_{1:2}(t) + \int_0^t \bar{G}_x(t-x) f_{1:2}(x)dx = \bar{F}_{1:2}(t) + \int_0^t \frac{\partial_1 K(\bar{F}(x), \bar{F}(t))}{\partial_1 K(\bar{F}(x), \bar{F}(x))} f_{1:2}(x)dx,$$

where $\bar{F}_{1:2}(x) = K(\bar{F}(x), \bar{F}(x))$ *and* $f_{1:2}(x) = 2f(x)\partial_1 K(\bar{F}(x), \bar{F}(x))$*. Therefore,*

$$\begin{aligned}
\bar{F}_{1:2} \# \bar{G}(t) &= \bar{F}_{1:2}(t) + 2\int_0^t \partial_1 K(\bar{F}(x), \bar{F}(t)) f(x)dx \\
&= K(\bar{F}(t), \bar{F}(t)) - 2K(\bar{F}(t), \bar{F}(t)) + 2K(1, \bar{F}(t)) \\
&= 2\bar{F}(t) - K(\bar{F}(t), \bar{F}(t)).
\end{aligned}$$

Another approach for the general case is

$$\bar{F}_{2:2} = \bar{F}_{1:2} \# \bar{G}, \tag{2.7}$$

where

$$\begin{aligned}\bar{G}_x(y) &= p_1(x)\Pr(X_2 - x > y | X_1 = x, X_2 > x) + p_2(x)\Pr(X_1 - x > y | X_2 = x, X_1 > x)\\ &= p_1(x)\frac{\Pr(X_2 > x + y | X_1 = x)}{\Pr(X_2 > x | X_1 = x)} + p_2(x)\frac{\Pr(X_1 > x + y | X_2 = x)}{\Pr(X_1 > x | X_2 = x)},\end{aligned}$$

$p_1(x) = \Pr(X_1 < X_2 | X_{1:2} = x)$ *and* $p_2(x) = \Pr(X_2 < X_1 | X_{1:2} = x)$.

Similar expressions can be obtained for other order statistics (k-out-of-n systems), that is, for $X_{i:n}$, $i = 1, \ldots, n$. For example, in the IID case, the reliability of $X_{2:3}$ can be written as $\bar{F}_{2:3} = \bar{F}_{1:3} \# \bar{F}_{1:2}$ or that of $X_{3:3}$ as $\bar{F}_{3:3} = (\bar{F}_{1:3} \# \bar{F}_{1:2}) \# \bar{F}$. Analogous (but more complicated) expressions hold for general coherent systems.

3. Main results

With the notation introduced in the preceding section, let us assume that we have a coherent system with lifetime T based on n components with lifetimes $X_1, \ldots, X_n$. If we apply a single minimal repair to the system then the main options are:

Case I: To repair the component which fails first.

Case II: To repair the component which leads to the system failure.

Case III: To repair a fixed component (e.g., to repair the ith component).

Other options will be considered later. If we can choose among these options (this is not always the case in practice), we need to determine which one is the best one under some stochastic criteria. To do this, we need to obtain the reliability of the resulting systems after these replacement policies.

From now on, we will denote by T_I and T_{II} the lifetimes associated to the resulting system under the policy I and II, respectively. In the third option, if we repair the ith component, the resulting system lifetime will be represented by $T_{III}^{(i)}$. If the dependence structure does not change after the replacement, then the reliability of $T_{III}^{(i)}$ is

$$\bar{F}_{T_{III}^{(i)}}(t) = \bar{Q}(\bar{F}_1(t), \ldots, \bar{F}_{i-1}(t), \bar{q}_1(\bar{F}_i(t)), \bar{F}_{i+1}(t), \ldots, \bar{F}_n(t)),$$

where $\bar{q}_1$ is given in (2.3). If the components are ID, then $\bar{F}_{T_{III}^{(i)}}(t) = \bar{q}_{III}^{(i)}(\bar{F}(t))$, where

$$\bar{q}_{III}^{(i)}(u) = \bar{Q}(u, \ldots, u, \bar{q}_1(u), u, \ldots, u) \tag{3.1}$$

and $\bar{q}_1$ is placed at the ith position. Comparison results for these kinds of replacements were given in [9]. Let us study the other two cases.

3.1. Case I

In this case we repair the component which fails first. Its lifetime is $X = X_{1:n}$. Then the broken component is minimally repaired and the resulting system has the same structure as T but we know that all the components are working and have age X. Hence its reliability is

$$\bar{F}_{T_I}(t) = \bar{F}_{1:n} \# \bar{G}(t), \tag{3.2}$$

where

$$\bar{G}_x(y) = \Pr(T - x > y | X_1 > x, \dots, X_n > x) = \frac{\Pr(T > x + y, X_1 > x, \dots, X_n > x)}{\Pr(X_1 > x, \dots, X_n > x)}$$

when $X = x$. In Proposition 3 of [30] is proved that this reliability can be written as $\bar{G}_x(t) = \bar{Q}_x(\bar{F}_{1,x}(t), \dots, \bar{F}_{n,x}(t))$, where $\bar{F}_{i,x}(t) = \Pr(X_i - x > t | X_i > x) = \bar{F}_i(t + x)/\bar{F}_i(x)$ for $i = 1, \dots, n$ and $\bar{Q}_x$ is a distortion function (see Section 2). Hence, from (2.1), we have,

$$\bar{F}_{T_I}(t) = \bar{F}_{1:n}(t) + \int_0^t \bar{G}_x(t - x) f_{1:n}(x) dx = \bar{F}_{1:n}(t) + \int_0^t \bar{Q}_x(\bar{F}_{1,x}(t - x), \dots, \bar{F}_{n,x}(t - x)) f_{1:n}(x) dx. \tag{3.3}$$

Let us see an example.

Example 3.1. *If $T = X_{2:2}$ (a parallel system with two components), then*

$$\begin{aligned}
\bar{G}_x(y) &= \Pr(T - x > y | X_1 > x, X_2 > x) \\
&= \frac{\Pr(T > x + y, X_1 > x, X_2 > x)}{\Pr(X_1 > x, X_2 > x)} \\
&= \frac{\Pr(X_1 > x + y, X_2 > x) + \Pr(X_2 > x + y, X_1 > x) - \Pr(X_1 > x + y, X_2 > x + y)}{\Pr(X_1 > x, X_2 > x)} \\
&= \frac{K(\bar{F}_1(x + y), \bar{F}_2(x)) + K(\bar{F}_1(x), \bar{F}_2(x + y)) - K(\bar{F}_1(x + y), \bar{F}_2(x + y))}{K(\bar{F}_1(x), \bar{F}_2(x))} \\
&= \bar{Q}_x(\bar{F}_{1,x}(y), \bar{F}_{2,x}(y))
\end{aligned}$$

for $y \geq 0$, with $\bar{F}_{1,x}(y) = \bar{F}_1(x + y)/\bar{F}_1(x)$, $\bar{F}_{2,x}(y) = \bar{F}_2(x + y)/\bar{F}_2(x)$ and

$$\bar{Q}_x(u_1, u_2) = \frac{K(u_1 \bar{F}_1(x), \bar{F}_2(x)) + K(\bar{F}_1(x), u_2 \bar{F}_2(x)) - K(u_1 \bar{F}_1(x), u_2 \bar{F}_2(x))}{K(\bar{F}_1(x), \bar{F}_2(x))}$$

whenever $K(\bar{F}_1(x), \bar{F}_2(x)) > 0$. Hence, from (2.1) *and* (3.2)*,*

$$\begin{aligned}
\bar{F}_{T_I}(t) &= \bar{F}_{1:2}(t) + \int_0^t \bar{Q}_x(\bar{F}_{1,x}(t - x), \bar{F}_{2,x}(t - x)) f_{1:2}(x) dx \\
&= \bar{F}_{1:2}(t) + \bar{F}_{1:2}(t) \ln(\bar{F}_{1:2}(t)) + \int_0^t \frac{K(\bar{F}_1(t), \bar{F}_2(x)) + K(\bar{F}_1(x), \bar{F}_2(t))}{\bar{F}_{1:2}(x)} f_{1:2}(x) dx
\end{aligned} \tag{3.4}$$

holds. In particular, if the components are IID, then

$$\begin{aligned}
\bar{F}_{T_I}(t) &= \bar{F}^2(t) + 2\bar{F}^2(t) \ln(\bar{F}(t)) + \int_0^t \frac{\bar{F}(t)\bar{F}(x) + \bar{F}(x)\bar{F}(t)}{\bar{F}^2(x)} 2f(x)\bar{F}(x) dx \\
&= \bar{F}^2(t) + 2\bar{F}^2(t) \ln(\bar{F}(t)) + 4\bar{F}(t)F(t).
\end{aligned}$$

Therefore, $\bar{F}_{T_I}(t) = \bar{q}_I(\bar{F}(t))$ with $\bar{q}_I(u) = 4u - 3u^2 + 2u^2 \ln(u)$. A straightforward calculation shows that $\bar{q}^{(i)}_{III}(u) = 2u - u^2 - u \ln u + u^2 \ln u$ and $\bar{q}_I \leq \bar{q}^{(i)}_{III}$ for $i = 1, 2$. So, $T_I \leq_{ST} T^{(i)}_{III}$ holds for all F, that is, in this system, it is better to replace a fixed component than to replace the first failure.

If the components are just ID, from (3.4), we get

$$\begin{aligned}\bar{F}_{T_I}(t) &= \bar{F}_{1:2}(t) + \bar{F}_{1:2}(t)\ln(\bar{F}_{1:2}(t)) + \int_0^t \frac{K(\bar{F}(t),\bar{F}(x)) + K(\bar{F}(x),\bar{F}(t))}{\bar{F}_{1:2}(x)} f_{1:2}(x)dx \\ &= \bar{F}_{1:2}(t) + \bar{F}_{1:2}(t)\ln(\bar{F}_{1:2}(t)) \\ &\quad + \int_0^t \frac{K(\bar{F}(t),\bar{F}(x)) + K(\bar{F}(x),\bar{F}(t))}{K(\bar{F}(x),\bar{F}(x))}[\partial_1 K(\bar{F}(x),\bar{F}(x)) + \partial_2 K(\bar{F}(x),\bar{F}(x))] f(x)dx,\end{aligned}$$

where $\bar{F}_{1:2}(t) = K(\bar{F}(t),\bar{F}(t))$. Now, if we do the change $v = \bar{F}(x)$, then

$$\bar{F}_{T_I}(t) = \delta_K(\bar{F}(t)) + \delta_K(\bar{F}(t))\ln(\delta_K(\bar{F}(t))) + \int_{\bar{F}(t)}^1 \frac{K(\bar{F}(t),v) + K(v,\bar{F}(t))}{\delta_K(v,v)}\delta'_K(v)dv,$$

where $\delta_K(v) = K(v,v)$ is the diagonal section of the copula K and $\delta'_K(v) = \partial_1 K(v,v) + \partial_2 K(v,v)$ for $v \in (0,1)$. Therefore $\bar{F}_{T_I}(t) = \bar{q}_I(\bar{F}(t))$ with

$$\bar{q}_I(u) = \delta_K(u) + \delta_K(u)\ln(\delta_K(u)) + \int_u^1 \frac{K(u,v) + K(v,u)}{\delta_K(v)}\delta'_K(v)dv.$$

A similar representation is obtained in the following theorem for an arbitrary coherent system.

Theorem 3.2. *Let T be the lifetime of a coherent system with ID components having a common reliability $\bar{F}$. Then the reliability function of T_I can be written as*

$$\bar{F}_{T_I}(t) = \bar{q}_I(\bar{F}(t)) \tag{3.5}$$

for all $t \geq 0$ and a distortion function $\bar{q}_I$ which does not depend on $\bar{F}$.

Proof. In the ID case, the general representation obtained in (3.3), can be written as

$$\bar{F}_{T_I}(t) = \bar{F}_{1:n}(t) + \int_0^t \bar{G}_x(t-x) f_{1:n}(x)dx = \bar{F}_{1:n}(t) + \int_0^t \bar{q}_x(\bar{F}_x(t-x)) f_{1:n}(x)dx, \tag{3.6}$$

where $\bar{q}_x(u) = \bar{Q}_x(u,\dots,u)$ and $\bar{F}_x(t) = \Pr(X_i - x > t | X_i > x) = \bar{F}(t+x)/\bar{F}(x)$ for $i = 1,\dots,n$. Even more, in this case, $\bar{G}_x$ can be written as $\bar{G}_x(y) = \bar{q}(\bar{F}(x+y);\bar{F}(x))$, see [30]. Hence

$$\bar{F}_{T_I}(t) = \bar{F}_{1:n}(t) + \int_0^t \bar{q}(\bar{F}(t);\bar{F}(x)) f_{1:n}(x)dx$$

where $\bar{F}_{1:n}(t) = \delta_K(\bar{F}(t))$, $\delta_K(u) = K(u,\dots,u)$ and $f_{1:n}(t) = f(t)\delta'_K(\bar{F}(t))$. Then

$$\bar{F}_{T_I}(t) = \delta_K(\bar{F}(t)) + \int_0^t \bar{q}(\bar{F}(t);\bar{F}(x))\delta'_K(\bar{F}(x)) f(x)dx.$$

Finally, if we do the change $u = \bar{F}(x)$, then

$$\bar{F}_{T_I}(t) = \delta_K(\bar{F}(t)) + \int_{\bar{F}(t)}^1 \bar{q}(\bar{F}(t);u)\delta'_K(u)du \tag{3.7}$$

and therefore (3.5) holds. □

The dual distortion function $\bar{q}_I$ in (3.5) depends on the structure of the system and on the underlying survival copula K. In the next sections we will show how to compute it. However, we must say that, sometimes, it is not easy to get an explicit expression for it (since we have to solve the integral in (3.7)). In the IID case, the preceding theorem can be simplified as follows.

Theorem 3.3. *Let T be the lifetime of a coherent system with IID components having a common reliability $\bar{F}$. Then the reliability function of T_I can be written as $\bar{F}_{T_I}(t) = \bar{q}_I(\bar{F}(t))$ where*

$$\bar{q}_I(u) = n \sum_{i=1}^{n-1} \frac{a_i}{n-i} u^i + \left(1 - n \sum_{i=1}^{n-1} \frac{a_i}{n-i}\right) u^n - n a_n u^n \ln u \tag{3.8}$$

and $(a_1, \ldots, a_n)$ is the minimal signature of the system.

Proof. If the components are independent, then $\bar{G}_x(t) = \bar{Q}(\bar{F}_{1,x}(t), \ldots, \bar{F}_{n,x}(t))$ holds from Proposition 5 in [30], that is, $\bar{Q}_x = \bar{Q}$, where $\bar{Q}$ is the distortion function in (2.4). Then, if they are IID, we have $\bar{G}_x(t) = \bar{q}(\bar{F}_x(t))$, where $\bar{F}_x(t) = \bar{F}(t+x)/\bar{F}(x)$ and $\bar{q}(u) = \sum_{i=1}^n a_i u^i$ (see Section 2). Hence, from (3.6), we have

$$\begin{aligned}
\bar{F}_{T_I}(t) &= \bar{F}_{1:n}(t) + \int_0^t \bar{q}(\bar{F}_x(t-x)) f_{1:n}(x) dx \\
&= \bar{F}^n(t) + \int_0^t \bar{q}\left(\frac{\bar{F}(t)}{\bar{F}(x)}\right) n \bar{F}^{n-1}(x) f(x) dx \\
&= \bar{F}^n(t) + n \sum_{i=1}^{n} a_i \bar{F}^i(t) \int_0^t \bar{F}^{n-i-1}(x) f(x) dx \\
&= \bar{F}^n(t) + n \sum_{i=1}^{n-1} \frac{a_i}{n-i} \bar{F}^i(t)(1 - \bar{F}^{n-i}(t)) + n a_n \bar{F}^n(t)(-\ln \bar{F}(t))
\end{aligned}$$

which concludes the proof. □

The minimal signatures of all the coherent systems with 1-5 IID components were obtained in [36]. Hence, from the preceding theorem, we have explicit expressions for $\bar{q}_I$ for all these systems.

3.2. Case II

Let us assume now that we repair the component which is critical for the system. We may expect that this option leads to a better performance since the most relevant components for the system have higher probabilities of being repaired. Note that, in case I, we just repair the first failure and so, for example, if the components are exchangeable, then all the components have the same probability of being repaired. However, we must note that case II is not always available in practice for all systems.

In this case it is not easy to obtain the reliability $\bar{F}_{T_{II}}$ of the resulting system lifetime T_{II}. Let us see a simple example. If the system is a series system, then cases I and II coincide since the first failure is always critical for the system. So let us consider again a parallel system.

Example 3.4. *If $T = X_{2:2}$ and the components are IID, then, from* (2.1)*, we have*

$$\bar{F}_{T_{II}}(t) = \bar{F}_T \# \bar{F}(t) = \bar{F}_T(t) + \int_0^t \frac{\bar{F}(t)}{\bar{F}(x)} f_T(x) dx,$$

where $\bar F_T(t)=2\bar F(t)-\bar F^2(t)$ *and* $f_T(t)=2(1-\bar F(t))f(t)$. *Hence*

$$\bar F_{T_{II}}(t)=2\bar F(t)-\bar F^2(t)+2\bar F(t)\int_0^t\frac{1-\bar F(x)}{\bar F(x)}f(x)dx=\bar F^2(t)-2\bar F(t)\ln\bar F(t)=\bar q_{II}(\bar F(t))$$

with $\bar q_{II}(u)=u^2-2u\ln u$. *So* T_{II} *also has a distorted distribution from* F. *Hence it is easy to compare the three replacement policies for this system just by comparing the three distortion functions. Thus a straightforward calculation leads to* $\bar q\le\bar q_I\le\bar q^{(i)}_{III}\le\bar q_{II}$ *and so* $T\le_{ST}T_I\le_{ST}T^{(i)}_{III}\le_{ST}T_{II}$ *for all* $\bar F$ *and* $i=1,2$, *that is, the best option in this system is to repair the component which is critical for the system. The second best option is to replace a fixed component and, of course, the three options are better than the original system* T. *They are also better than a parallel system with three components (active redundancy) with* $\bar q_{3:3}(u)=3u-3u^2+u^3$.

Let us assume now that the component lifetimes are just exchangeable. Then, proceeding as in Section 2, we have $\bar F_{T_{II}}(t)=\bar F_T\#\bar G(t)$, *where*

$$\begin{aligned}\bar G_x(y)&=\Pr(X_2-x>y|X_1\le x,X_2>x)\\&=\frac{\Pr(X_1\le x,X_2>x+y)}{\Pr(X_1\le x,X_2>x)}\\&=\frac{\Pr(X_2>x+y)-\Pr(X_1>x,X_2>x+y)}{\Pr(X_2>x)-\Pr(X_1>x,X_2>x)}\\&=\frac{\bar F(x+y)-K(\bar F(x),\bar F(x+y))}{\bar F(x)-K(\bar F(x),\bar F(x))}\end{aligned}$$

for $x,y\ge0$. *Hence, from* (2.1), *we have*

$$\bar F_{T_{II}}(t)=\bar F_T(t)+\int_0^t\bar G_x(t-x)f_T(x)dx=\bar F_T(t)+\int_0^t\frac{\bar F(t)-K(\bar F(x),\bar F(t))}{\bar F(x)-K(\bar F(x),\bar F(x))}f_T(x)dx,$$

where $\bar F_T(t)=2\bar F(t)-K(\bar F(t),\bar F(t))$ *and* $f_T(t)=2(1-\partial_1K(\bar F(t),\bar F(t)))f(t)$. *Therefore*

$$\begin{aligned}\bar F_{T_{II}}(t)&=2\bar F(t)-K(\bar F(t),\bar F(t))+2\int_0^t\frac{\bar F(t)-K(\bar F(x),\bar F(t))}{\bar F(x)-K(\bar F(x),\bar F(x))}(1-\partial_1K(\bar F(x),\bar F(x)))f(x)dx\\&=2\bar F(t)-K(\bar F(t),\bar F(t))+2\int_{\bar F(t)}^1\frac{\bar F(t)-K(v,\bar F(t))}{v-K(v,v)}(1-\partial_1K(v,v))dv=\bar q_{II}(\bar F(t))\end{aligned}$$

with

$$\bar q_{II}(u)=2u-K(u,u)+2\int_u^1\frac{u-K(v,u)}{v-K(v,v)}(1-\partial_1K(v,v))dv. \tag{3.9}$$

Note that we need K *(and to solve this integral) to get an explicit expression for* $\bar q_{II}$. *Of course, if* $K(u,v)=uv$, *then we obtain the expression obtained above for the IID case.*

Finally, in the general case, proceeding as in (2.7)*, we get* $\bar{F}_{T_{II}}(t) = \bar{F}_T \# \bar{G}(t)$*, where*

$$\begin{aligned}
\bar{G}_x(y) &= p_1(x)\Pr(X_2 - x > y | X_1 \le x, X_2 > x) + p_2(x)\Pr(X_1 - x > y | X_2 \le x, X_1 > x) \\
&= p_1(x)\frac{\Pr(X_1 \le x, X_2 > x+y)}{\Pr(X_1 \le x, X_2 > x)} + p_2(x)\frac{\Pr(X_2 \le x, X_1 > x+y)}{\Pr(X_2 \le x, X_1 > x)} \\
&= p_1(x)\frac{\Pr(X_2 > x+y) - \Pr(X_1 > x, X_2 > x+y)}{\Pr(X_2 > x) - \Pr(X_1 > x, X_2 > x)} \\
&\quad + p_2(x)\frac{\Pr(X_1 > x+y) - \Pr(X_1 > x+y, X_2 > x)}{\Pr(X_1 > x) - \Pr(X_1 > x, X_2 > x)} \\
&= p_1(x)\frac{\bar{F}_2(x+y) - K(\bar{F}_1(x), \bar{F}_2(x+y))}{\bar{F}_2(x) - K(\bar{F}_1(x), \bar{F}_2(x))} + p_2(x)\frac{\bar{F}_1(x+y) - K(\bar{F}_1(x+y), \bar{F}_2(x))}{\bar{F}_1(x) - K(\bar{F}_1(x), \bar{F}_2(x))}, \quad (3.10)
\end{aligned}$$

$p_1(x) := \Pr(X_1 < X_2 | T = x)$ *and* $p_2(x) := \Pr(X_2 < X_1 | T = x) = 1 - p_1(x)$ *for* $x, y \ge 0$*. To compute* $p_1(x)$*, we need the joint reliability of* $(X_1, X_{2:2})$ *given by*

$$\begin{aligned}
\bar{\boldsymbol{H}}(x,y) &= \Pr(X_1 > x, X_{2:2} > y) \\
&= \Pr(X_1 > x, X_1 > y) + \Pr(X_1 > x, X_2 > y) - \Pr(X_1 > x, X_1 > y, X_2 > y) \\
&= \bar{F}_1(y) + K(\bar{F}_1(x), \bar{F}_2(y)) - K(\bar{F}_1(y), \bar{F}_2(y))
\end{aligned}$$

for all $x \le y$*. Hence, its joint density is* $\boldsymbol{h}(x,y) = f_1(x)f_2(y)\partial_{1,2}K(\bar{F}_1(x), \bar{F}_2(y))$ *for all* $x \le y$ *(*0 *otherwise) and the conditional density function of* $(X_1 | X_{2:2} = y)$ *is*

$$\boldsymbol{h}_{1|2}(x|y) = \frac{f_1(x)f_2(y)\partial_{1,2}K(\bar{F}_1(x), \bar{F}_2(y))}{f_T(y)}$$

for $0 \le x \le y$*. Therefore*

$$\begin{aligned}
p_1(y) &= \Pr(X_1 < X_2 | T = y) = \Pr(X_1 < X_{2:2} | T = y) = \int_0^y \boldsymbol{h}_{1|2}(x|y)dx \\
&= \int_0^y \frac{f_1(x)f_2(y)\partial_{1,2}K(\bar{F}_1(x), \bar{F}_2(y))}{f_T(y)}dx = \frac{f_2(y) - f_2(y)\partial_2 K(\bar{F}_1(y), \bar{F}_2(y))}{f_T(y)} \qquad (3.11)
\end{aligned}$$

when $\lim_{u \to 1^-} \partial_2 K(u, \bar{F}_2(y)) = 1$ *(see [39]). Analogously, we get*

$$p_2(y) = \Pr(X_2 < X_1 | T = y) = \frac{f_1(y) - f_1(y)\partial_1 K(\bar{F}_1(y), \bar{F}_2(y))}{f_T(y)}. \qquad (3.12)$$

Hence, from (2.1), (3.10), (3.11) *and* (3.12)*, we have*

$$\begin{aligned}
\bar{F}_{T_{II}}(t) &= \bar{F}_T(t) + \int_0^t \bar{G}_x(t-x)f_T(x)dx \\
&= \bar{F}_1(t) + \bar{F}_2(t) - K(\bar{F}_1(t), \bar{F}_2(t)) \\
&\quad + \int_0^t [1 - \partial_2 K(\bar{F}_1(x), \bar{F}_2(x))]\frac{\bar{F}_2(t) - K(\bar{F}_1(x), \bar{F}_2(t))}{\bar{F}_2(x) - K(\bar{F}_1(x), \bar{F}_2(x))}f_2(x)dx \\
&\quad + \int_0^t [1 - \partial_1 K(\bar{F}_1(x), \bar{F}_2(x))]\frac{\bar{F}_1(t) - K(\bar{F}_1(t), \bar{F}_2(x))}{\bar{F}_1(x) - K(\bar{F}_1(x), \bar{F}_2(x))}f_1(x)dx.
\end{aligned}$$

In the exchangeable case, we have $\Pr(X_1 < X_2 | T = y) = \Pr(X_2 < X_1 | T = y) = 1/2$ *and* (3.9).

The preceding example shows that it is not easy to get an expression for the reliability in the general case. So, we are going to try to solve the case of exchangeable components. In this case, we know that the system's reliability can be written as

$$\bar{F}_T(t) = \sum_{i=1}^{n} s_i \bar{F}_{i:n}(t), \tag{3.13}$$

where $\boldsymbol{s} = (s_1, \dots, s_n)$ is the *signature* of the system and $s_i = \Pr(T = X_{i:n})$ for $i = 1, \dots, n$. We can use this representation to obtain the following result.

Theorem 3.5. *Let T be the lifetime of a coherent system with components having an absolutely continuous exchangeable joint reliability. Then the reliability function of T_{II} can be written as*

$$\bar{F}_{T_{II}}(t) = \bar{q}_{II}(\bar{F}(t)) \tag{3.14}$$

for all $t \geq 0$ and for a distortion function $\bar{q}_{II}$ which does not depend on $\bar{F}$.

Proof. Let us consider the events $E_\sigma = \{X_{\sigma(1)} < \cdots < X_{\sigma(n)}\}$ for σ in the set P_n of all the permutations of order n. If the components are exchangeable, then $\Pr(E_\sigma) = 1/n!$. Let us divide the set P_n in the disjoint subsets $A_1, \dots, A_k$ where A_j contains all the permutations which lead to $T = X_{i_j:n}$ and to a fixed repaired system T_j. Let $H_j = \cup_{\sigma \in A_j} E_\sigma$. Then $p_j := \Pr(H_j) = |A_j|/n!$, where $|A_j|$ is the cardinal of the set A_j for $j = 1, \dots, k$. Hence

$$\bar{F}_{T_{II}}(t) = \Pr(T_{II} > t) = \sum_{j=1}^{k} p_j \Pr(T_{II} > t | H_j). \tag{3.15}$$

Note that under H_j, we know which component failure causes the system failure. Moreover $(T|H_j) =_{ST} (X_{i_j:n}|H_j)$. Also note that $X_{i_j:n} =_{ST} (X_{i_j:n}|H_j)$ due to the assumption about exchangeable components. Proceeding as in Section 2, we get $\Pr(T_{II} > t|H_j) = \bar{F}_{i_j:n} \# \bar{G}_j(t)$, where

$$\bar{G}_{j,x}(y) = \Pr(T_j - x > y | X_{i_j:n} = x, H_j) \tag{3.16}$$

and T_j is the system obtained after a minimal repair of the component broken in the i_jth position and at a given time x under H_j. Note that the structure of this system is completely determined by H_j. This event also determines which components are working and which have failed at time x. Hence, from (2.1),

$$\bar{F}_{T_{II}}(t) = \sum_{j=1}^{k} p_j \left[\bar{F}_{i_j:n}(t) + \int_0^t \bar{G}_{j,x}(t-x) f_{i_j:n}(x) dx \right] \tag{3.17}$$

holds. Note that the semi-coherent system T_j has $n - i_j + 1$ working components (some of them can be irrelevant for the system). These components are exchangeable and the corresponding joint reliability function $\bar{\boldsymbol{H}}(y_1, \dots, y_{n-i_j+1})$ is given by

$$\Pr(X_{i_j} - x > y_1, \dots, X_n - x > y_{n-i_j+1} | X_1 \leq x, \dots, X_{i_j-1} \leq x, X_{i_j} > x, \dots, X_n > x).$$

Proceeding as in case I, this joint reliability can be written as

$$\bar{\boldsymbol{H}}(y_1, \dots, y_{n-i_j+1}) = \bar{Q}_x(\bar{F}_x(y_1), \dots, \bar{F}_x(y_{n-i_j+1})) \tag{3.18}$$

for a distortion function $\bar{Q}_x$ which depends on $\bar{F}(x)$, where $\bar{F}_x(y) = \bar{F}(x+y)/\bar{F}(x)$. Let $\bar{H}^j_{1:n}, \dots, \bar{H}^j_{n-i_j+1:n}$ be the reliability functions of the order statistics obtained from these exchangeable components and let $(s^j_1, \dots, s^j_{n-i_j+1})$ be the signature (of order $n-i_j+1$) of T_j. Then

$$\bar{F}_{T_{II}}(t) = \sum_{j=1}^{k} p_j \left[\bar{F}_{i_j:n}(t) + \sum_{\ell=1}^{n-i_j+1} s^j_\ell \int_0^t \bar{H}^j_{\ell:n}(t-x) f_{i_j:n}(x)dx \right].$$

It is well known (see, e.g., [32]) that $\bar{F}_{i_j:n}(t)$ can be written as $\bar{F}_{i_j:n}(t) = \bar{q}_{i_j:n}(\bar{F}(t))$ where $\bar{q}_{i_j:n}$ depends on K. Analogously, from (3.18), we know that $\bar{H}^j_{\ell:n}$ can be written as $\bar{H}^j_{\ell:n}(y) = \bar{q}^j_{i:n}(\bar{F}(x+y); \bar{F}(x))$ where $\bar{q}^j_{i:n}$ depends on K. Therefore

$$\bar{F}_{T_{II}}(t) = \sum_{j=1}^{k} p_j \left[\bar{q}_{i_j:n}(\bar{F}(t)) + \sum_{\ell=1}^{n-i_j+1} s^j_\ell \int_0^t \bar{q}^j_{i:n}(\bar{F}(t); \bar{F}(x)) \bar{q}'_{i_j:n}(\bar{F}(x)) f(x)dx \right] \quad (3.19)$$

and by doing the change $v = \bar{F}(x)$ we get (3.14). □

The coefficients in the signature used in (3.13) can also be computed as $s_k = |B_k|/n!$, where B_k is the subset of P_n with the permutations which lead to $T = X_{k:n}$, that is, $B_k = \cup_{j:i_j=k} A_j$. Hence (3.19) can also be written as

$$\bar{F}_{T_{II}}(t) = \bar{F}_T(t) + \sum_{j=1}^{k} p_j \sum_{\ell=1}^{n-i_j+1} s^j_\ell \int_0^t \bar{q}^j_{i:n}(\bar{F}(t); \bar{F}(x)) \bar{q}'_{i_j:n}(\bar{F}(x)) f(x)dx, \quad (3.20)$$

where $\bar{F}_T(t) = \bar{q}_T(\bar{F}(t))$. These general expressions can be simplified in the IID case as follows.

Theorem 3.6. *Let T be the lifetime of a coherent system with IID components having a common absolutely continuous reliability $\bar{F}$. Then the reliability function of T_{II} can be expressed as $\bar{F}_{T_{II}}(t) = \bar{q}_{II}(\bar{F}(t))$ for all $t \geq 0$, where*

$$\bar{q}_{II}(u) = \sum_{i=1}^{n} c_i u^i + \sum_{i=1}^{n} d_i u^i \ln u \quad (3.21)$$

for some coefficients c_i, d_i, $i = 1, \dots, n$ which only depend on the structure of the system.

Proof. Let $a^j = (a^j_1, \dots, a^j_{n-i_j+1})$ be the minimal signature the system T_j considered in the proof of the preceding theorem for $j = 1, \dots, k$. In the IID case, this semi-coherent system has $n-i_j+1$ IID components with the common reliability $\bar{F}_x(y) = \bar{F}(x+y)/\bar{F}(x)$. Hence the reliability in (3.16) is

$$\bar{G}_{j,x}(y) = \sum_{\ell=1}^{n-i_j+1} a^j_\ell \left(\frac{\bar{F}(x+y)}{\bar{F}(x)} \right)^\ell.$$

Therefore, from (3.17) and (3.20), we have

$$\bar{F}_{T_{II}}(t) = \bar{F}_T(t) + \sum_{j=1}^{k} p_j \sum_{\ell=1}^{n-i_j+1} a^j_\ell \int_0^t \frac{\bar{F}^\ell(t)}{\bar{F}^\ell(x)} \bar{q}'_{i_j:n}(\bar{F}(x)) f(x)dx$$

where $\bar{F}_{i:n}(t) = \bar{q}_{i:n}(\bar{F}(t))$ for a polynomial

$$\bar{q}_{i:n}(u) = \sum_{r=n-i+1}^{n} (-1)^{r-n+i-1} \binom{n}{r} \binom{r-1}{n-i} u^r$$

(see, e.g., [20, p. 46]). So

$$f_{i:n}(t) = f(t)\bar{q}'_{i:n}(\bar{F}(t)) = f(t) \sum_{r=n-i+1}^{n} (-1)^{r-n+i-1} r \binom{n}{r} \binom{r-1}{n-i} \bar{F}^{r-1}(t).$$

Therefore, if $(a_1, \dots, a_n)$ is the minimal signature of T, then

$$\begin{aligned} \bar{F}_{T_{II}}(t) &= \bar{F}_T(t) + \sum_{j=1}^{k} p_j \sum_{\ell=1}^{n-i_j+1} a_\ell^j \bar{F}^\ell(t) \sum_{r=n-i_j+1}^{n} r(-1)^{r-n+i_j-1} \binom{n}{r} \binom{r-1}{n-i_j} \int_0^t \bar{F}^{r-\ell-1}(x) f(x) dx \\ &= \sum_{j=1}^{n} a_j \bar{F}^j(t) + \sum_{j=1}^{k} p_j \sum_{\ell=1}^{n-i_j+1} a_\ell^j \bar{F}^\ell(t) \sum_{r=n-i_j+1}^{n} r(-1)^{r-n+i_j-1} \binom{n}{r} \binom{r-1}{n-i_j} \phi_{r-\ell}(t), \end{aligned}$$

where $\phi_s(t) = (1 - \bar{F}^s(t))/s$ if $s > 0$ and $\phi_s(t) = -\ln \bar{F}(t)$ if $s = 0$. This concludes the proof. □

Example 4.4 shows how to apply the preceding theorem. In [31] we provide an R-script to compute the coefficients c_i and d_i for a given coherent system with IID components.

In general it is not easy to compute the reliability function associated to the case II of a coherent system with dependent components. However, the reliability function of k-out-of-n systems can be obtained by assuming exchangeable components. Thus, if $T = X_{i:n}$ for a fixed $i \in \{2, \dots, n\}$ and the components are exchangeable, then $\bar{F}_{T_{II}}(t) = \bar{F}_{i:n} \# \bar{G}(t)$, where

$$\begin{aligned} \bar{G}_x(y) &= \Pr(X_i > x+y, \dots, X_n > x+y | X_1 \le x, \dots, X_{i-1} \le x, X_i > x, \dots, X_n > x) \\ &= \frac{\Pr(X_1 \le x, \dots, X_{i-1} \le x, X_i > x+y, \dots, X_n > x+y)}{\Pr(X_1 \le x, \dots, X_{i-1} \le x, X_i > x, \dots, X_n > x)} = \frac{H_i(\bar{F}(x), \bar{F}(x+y))}{H_i(\bar{F}(x), \bar{F}(x))}, \end{aligned}$$

with a function H_i such that $\Pr(X_1 \le x, \dots, X_{i-1} \le x, X_i > t, \dots, X_n > t) = H_i(\bar{F}(x), \bar{F}(t))$ for all $0 \le x \le t$. Note that H_i only depends on K. Therefore, from (2.1), we have

$$\Pr(T_{II} > t) = \bar{F}_{i:n}(t) + \int_0^t \frac{H_i(\bar{F}(x), \bar{F}(t))}{\bar{H}_i(\bar{F}(x), \bar{F}(x))} f_{i:n}(x) dx. \tag{3.22}$$

If the components are IID, then the following result provide an explicit expression for (3.22).

Proposition 3.7. *Given an i-out-of-n system with IID components and lifetime $T = X_{i:n}$ for a fixed $i \in \{2, \dots, n\}$, then $\bar{F}_{T_{II}}(t) = \bar{q}_{II}(\bar{F}(t))$, where*

$$\begin{aligned} \bar{q}_{II}(u) = &\binom{n}{n-i+1} u^{n-i+1} + u^{n-i+1} \sum_{k=n-i+2}^{n} (-1)^{k-n+i-1} \frac{k}{k-n+i-1} \binom{n}{k} \binom{k-1}{n-i} \\ &+ \sum_{k=n-i+2}^{n} (-1)^{k-n+i} \frac{n-i+1}{k-n+i-1} \binom{n}{k} \binom{k-1}{n-i} u^k - i \binom{n}{i} u^{n-i+1} \ln u. \end{aligned}$$

Proof. If the components are IID, then

$$\bar{G}_x(y) = \Pr(X_i > x+y, \dots, X_n > x+y | X_1 \le x, \dots, X_{i-1} \le x, X_i > x, \dots, X_n > x)$$
$$= \Pr(X_i > x+y | X_i > x) \dots \Pr(X_n > x+y | X_n > x) = \frac{\bar{F}^{n-i+1}(x+y)}{\bar{F}^{n-i+1}(x)}.$$

Moreover, as $\bar{F}_{i:n}(t) = \sum_{k=n-i+1}^{n} (-1)^{k-n+i-1} \binom{n}{k} \binom{k-1}{n-i} \bar{F}^k(t)$ (see, e.g., [20, p. 46]), we have

$$\begin{aligned}
\Pr(T_{II} > t) &= \bar{F}_{i:n}(t) + \int_0^t \frac{\bar{F}^{n-i+1}(t)}{\bar{F}^{n-i+1}(x)} f_{i:n}(x)dx \\
&= \bar{F}_{i:n}(t) + \bar{F}^{n-i+1}(t) \sum_{k=n-i+1}^{n} (-1)^{k-n+i-1} k \binom{n}{k} \binom{k-1}{n-i} \int_0^t \bar{F}^{k-n+i-2}(x) f(x) dx \\
&= \bar{F}_{i:n}(t) + \sum_{k=n-i+2}^{n} (-1)^{k-n+i-1} k \binom{n}{k} \binom{k-1}{n-i} \frac{\bar{F}^{n-i+1}(t) - \bar{F}^k(t)}{k-n+i-1} - i \binom{n}{i} \bar{F}^{n-i+1}(t) \ln \bar{F}(t) \\
&= \binom{n}{n-i+1} \bar{F}^{n-i+1}(t) - \sum_{k=n-i+2}^{n} (-1)^{k-n+i-1} \frac{n-i+1}{k-n+i-1} \binom{n}{k} \binom{k-1}{n-i} \bar{F}^k(t) \\
&\quad + \bar{F}^{n-i+1}(t) \sum_{k=n-i+2}^{n} (-1)^{k-n+i-1} \frac{k}{k-n+i-1} \binom{n}{k} \binom{k-1}{n-i} - i \binom{n}{i} \bar{F}^{n-i+1}(t) \ln \bar{F}(t)
\end{aligned}$$

which concludes the proof. □

3.3. Other cases

The purpose of this section is to show that we can study other cases following the procedures used above in cases I and II. For example, if we know that the system does not fail with the first component failure, we can consider to repair the system at the second component failure with a minimal repair of the broken component at this point. Then, if the components are exchangeable, the reliability function of the repaired system is $\bar{F}_{(2)}(t) = \bar{F}_{2:n} \# \bar{G}(t)$, where

$$\bar{G}_x(y) = \frac{1}{n} \sum_{i=1}^{n} \Pr(T_i - x > y | X_i \le x, X_j > t \text{ for all } j \ne i)$$

and T_i is the lifetime of the semi-coherent system obtained from T when we know that the ith component is broken. A similar expression can be obtained if the system is repaired at the jth failure for $j = 3, 4, \dots$.

In all the options studied above, we just repair one component. We can of course consider k replacements. For example, if $k = 2$ and, in case III, we repair components i and j (for fixed $i < j$), then the reliability of the repaired system is

$$\bar{F}_{T_{III}^{(i,j)}}(t) = \bar{Q}(\bar{F}_1(t), \dots, \bar{F}_{i-1}(t), \bar{q}_1(\bar{F}_i(t)), \bar{F}_{i+1}(t), \dots, \bar{F}_{j-1}(t), \bar{q}_1(\bar{F}_j(t)), \bar{F}_{j+1}(t), \dots, \bar{F}_n(t)),$$

where $\bar{q}_1$ is given in (2.3). If the components are ID, then this representation can be reduced to $\bar{F}_{T_{III}^{(i,j)}}(t) = \bar{q}_{III}^{(i,j)}(\bar{F}(t))$, where $\bar{q}_{III}^{(i,j)}(u) = \bar{Q}(u, \dots, u, \bar{q}_1(u), u, \dots, u, \bar{q}_1(u), u, \dots, u)$ and $\bar{q}_1(u)$ is placed at the ith and jth positions. Analogously, if we repair the ith component twice, then

$$\bar{F}_{T_{III}^{(i,i)}}(t) = \bar{Q}(\bar{F}_1(t), \dots, \bar{F}_{i-1}(t), \bar{q}_2(\bar{F}_i(t)), \bar{F}_{i+1}(t), \dots, \bar{F}_n(t)),$$

where $\bar q_2$ is given in (2.3). If the components are ID, we get $\bar F_{T^{(i,i)}_{III}}(t) = \bar q^{(i,i)}_{III}(\bar F(t))$, where

$$\bar q^{(i,i)}_{III}(u) = \bar Q(u,\dots,u,\bar q_2(u),u,\dots,u)$$

and $\bar q_2$ is placed at the ith position. Other options with fixed repairs were studied in [9].

We could consider other options with $k=2$ minimal repairs. For example, we can repair the two first broken components. In this case, if $X_1,\dots,X_n$ are IID, the resulting reliability is

$$\bar F^{(2)}_I(t) = (\bar F_{1:n}\#\bar G_{1:n})\#\bar G(t),$$

where $\bar F_{1:n}(t) = \bar F^n(t)$ is the reliability function of $X_{1:n} = \min(X_1,\dots,X_n)$,

$$(\bar G_{1:n})_x(y) = \bar F^n_x(y) = \frac{\bar F^n(x+y)}{\bar F^n(x)}$$

is the reliability function of $Y_{1:n} = \min(Y_1,\dots,Y_n)$ (a series system with n IID components and a common reliability $\bar F_x(y) = \bar F(x+y)/\bar F(x)$) and $\bar G_y(z) = \bar q_T(\bar F_y(z))$ is the reliability of a system with the same structure as T, having n IID components with reliability $\bar F_y$ when $Y_{1:n}=y$. The reliability $\bar H = \bar F_{1:n}\#\bar G_{1:n}$ can be computed from (2.1) as

$$\bar H(t) = \bar F^n(t) + \int_0^t \frac{\bar F^n(t)}{\bar F^n(x)} n\bar F^{n-1}(x)f(x)dx = \bar F^n(t) - n\bar F^n(t)\ln \bar F(t).$$

Its density is $h(t) = -n^2\bar F^{n-1}(t)f(t)\ln\bar F(t)$. Then, by using (2.1) again, the system's reliability is

$$\begin{aligned}\bar F^{(2)}_I(t) &= \bar H(t) + \int_0^t \bar G_y(t-y)h(y)dy\\ &= \bar H(t) - n^2\int_0^t \bar q_T\left(\frac{\bar F(t)}{\bar F(y)}\right)\bar F^{n-1}(y)f(y)\ln\bar F(y)dy\\ &= \bar H(t) - n^2\sum_{i=1}^n a_i\bar F^i(t)\int_0^t \bar F^{n-i-1}(y)\ln\bar F(y)f(y)dy,\end{aligned}$$

where $(a_1,\dots,a_n)$ is the minimal signature of the system T. Then

$$\begin{aligned}\bar F^{(2)}_I(t) = \bar H(t) - n^2a_n\bar F^n(t)\int_0^t \bar F^{-1}(y)\ln\bar F(y)f(y)dy - n^2\sum_{i=1}^{n-1} a_i\bar F^i(t)\int_0^t \bar F^{n-i-1}(y)\ln\bar F(y)f(y)dy\\ = \bar H(t) + n^2\frac{a_n}{2}\bar F^n(t)\ln^2\bar F(t) - n^2\sum_{i=1}^{n-1} a_i\bar F^i(t)\int_0^t \bar F^{n-i-1}(y)\ln\bar F(y)f(y)dy.\end{aligned}$$

Finally, by doing the change $x = -\ln\bar F(y)$, in $I_i(t) = \int_0^t \bar F^{n-i-1}(y)\ln\bar F(y)f(y)dy$, we get

$$I_i(t) = \int_0^{-\ln\bar F(t)} xe^{-(n-i)x}dx = \frac{\bar F^{n-i}(t)\ln\bar F(t)}{n-i} + \frac{1-\bar F^{n-i}(t)}{(n-i)^2}.$$

Therefore

$$\bar F^{(2)}_I(t) = \bar q_1(F^n(t)) + \frac{n^2a_n}{2}\bar F^n(t)\ln^2\bar F(t) + n^2\sum_{i=1}^{n-1} a_i\frac{\bar F^n(t)\ln\bar F(t)}{n-i} + n^2\sum_{i=1}^{n-1} a_i\frac{\bar F^i(t)-\bar F^n(t)}{(n-i)^2}.$$

Note that the reliability can be written as $\bar{F}_I^{(2)}(t) = \bar{q}_I^{(2)}(\bar{F}(t))$ for a distortion function $\bar{q}_I^{(2)}$. For example, for $T = X_{1:n}$, we obtain $\bar{q}_I^{(2)}(u) = u^n - nu^n \ln u + (n^2/2)u^n(\ln u)^2$. For this system, if we repair the first k broken components, then we get $\bar{q}_I^{(k)}(u) = \sum_{i=0}^{k} n^i u^n(-\ln u)^i/i!$.

Other similar replacement policies can be studied in a similar way. However, in the following section we restrict ourselves to the cases with $k = 1$ to develop fair comparisons, that is comparisons of replacement policies with the same number of repairs (i.e. with the same cost).

4. Comparison results

The representations obtained in the preceding section can be used jointly with the ordering results for distorted distributions given in [32, 34] to compare the different replacement policies. For sake of completeness we include some of these ordering results in the following theorem. We shall consider the following (well known) stochastic orders.

The main order is the *usual stochastic order*, denoted by $X \leq_{ST} Y$, that compares the respective reliability functions $\bar{F}_X(t) \leq \bar{F}_Y(t)$ for any time t. This ordering implies that $E(X) \leq E(Y)$ (if these expectations exist). An alternative (stronger) order is the *hazard rate order*, denoted by $X \leq_{HR} Y$, that compares the respective residual lifetimes $(X - t|X > t) \leq_{ST} (Y - t|Y > t)$ for any time t. While the ST order compares new units, the HR order compares (in the ST order) used units with the same age t. Analogously, the *mean residual life order*, denoted by $X \leq_{MRL} Y$, compares the respective mean (expected) residual lifetimes $E(X - t|X > t) \leq E(Y - t|Y > t)$ for any time t. The HR order implies the MRL order. An order similar to the HR order is the *reversed hazard rate order*, denoted by $X \leq_{RHR} Y$, that compares the inactivity times $(t - X|X < t) \geq_{ST} (t - Y|Y < t)$ for any time t. Finally, the *likelihood ratio order*, denoted by $X \leq_{LR} Y$, holds if the ratio of their densities f_Y/f_X is increasing in the union of their supports. This order implies all the preceding orders. For basic properties and applications of these orders we refer the reader to [12, 42].

Theorem 4.1. *Let X_1 and X_2 be two random variables with distribution functions $F_{q_1} = q_1(F)$ and $F_{q_2} = q_2(F)$ obtained as distorted distributions from the same distribution function F and from the distortion functions q_1 and q_2, respectively. Let $\bar{q}_1$ and $\bar{q}_2$ be the respective dual distortion functions. Then:*

(i) $X_1 \leq_{ST} X_2$ *for all* $F \iff \bar{q}_1(u) \leq \bar{q}_2(u)$ [*or* $q_1(u) \geq q_2(u)$] *for all* $u \in (0, 1)$.

(ii) $X_1 \leq_{HR} X_2$ *for all* $F \iff \bar{q}_2(u)/\bar{q}_1(u)$ *is decreasing in* $(0, 1)$.

(iii) $X_1 \leq_{RHR} X_2$ *for all* $F \iff q_2(u)/q_1(u)$ *is increasing in* $(0, 1)$.

(iv) $X_1 \leq_{LR} X_2$ *for all* $F \iff \bar{q}_2'(u)/\bar{q}_1'(u)$ *is decreasing in* $(0, 1)$.

(v) $X_1 \leq_{MRL} X_2$ *for all* $F \Longleftarrow \bar{q}_2(u)/\bar{q}_1(u)$ *is bathtub in* $(0, 1)$ *and* $E(X_1) \leq E(X_2)$.

We apply these ordering results in the following theorems and examples comparing the different replacement policies. In the first main result we prove that, for any system with IID components, the replacement policy of case II is always ST-better than that of case I.

Theorem 4.2. *Let T be the lifetime of a coherent system with IID components having a common absolutely continuous reliability $\bar{F}$. Let T_I and T_{II} be the system lifetimes obtained with the replacement policies of cases I and II, respectively. Then $T_I \leq_{ST} T_{II}$ for all $\bar{F}$.*

Proof. If we assume that the component lifetimes $X_1, \ldots, X_n$ are IID, then the system's reliability can be written as $\bar{F}_T(t) = \bar{q}(\bar{F}(t))$ for a polynomial $\bar{q}(u)$. From Theorems 3.3 and 3.6, we also know that the reliability functions of T_I and T_{II} can be written as $\Pr(T_I > t) = \bar{q}_I(\bar{F}(t))$ and $\Pr(T_{II} > t) = \bar{q}_{II}(\bar{F}(t))$. So we just need to prove that $\bar{q}_I(u) \le \bar{q}_{II}(u)$ for all $u \in [0,1]$.

From the proof of Theorem 3.3, we know that $T_I = X_{1:n} + Y^I$, where $X_{1:n} = \min(X_1, \ldots, X_n)$,

$$\Pr(Y^I - x > y | X_{1:n} = x) = \Pr(T^* > y)$$

and T^* is the lifetime of a system with the same structure as T and having IID components with the common reliability function $\bar{F}_x(y) = \bar{F}(x+y)/\bar{F}(x)$ for $y \ge 0$. Hence

$$\Pr(Y^I - x > y | X_{1:n} = x) = \Pr(T^* > y) = \bar{q}(\bar{F}_x(y)).$$

On the other hand, from the proof of Theorem 3.6, we know that $T_{II} = T + Y^{II}$, where

$$\Pr(Y^{II} - x > y | T = x) = \Pr(T^{**} > y)$$

and T^{**} is a mixture of different semi-coherent systems with n (or less) IID components with the common reliability function $\bar{F}_x$.

Now let assume that the IID components are exponential with mean 1, that is, $\bar{F}(t) = e^{-t}$ for $t \ge 0$. This model has the lack of memory property and so $\bar{F}_x(y) = \bar{F}(y)$ for all $y \ge 0$. Hence

$$\Pr(Y^I - x > y | X_{1:n} = x) = \bar{q}(\bar{F}(y)) = \Pr(T > y)$$

for all $x, y \ge 0$, that is, $(Y^I - x | X_{1:n} = x) =_{ST} T$. So $X_{1:n}$ and Y^I are independent. Analogously, T^{**} is a mixture of different semi-coherent systems with n (or less) components and having IID components with the common reliability function $\bar{F}$. Hence T and Y^{II} are independent. Moreover, as all these semi-coherent systems are ST-better than $X_{1:n}$ (because they have n or less components), then $X_{1:n} \le_{ST} T^{**}$. Finally, from Theorem 1.A.3, b, in [42, p. 6], we get

$$T_I =_{ST} X_{1:n} + T^* \le_{ST} T + T^{**} =_{ST} T_{II}$$

for $\bar{F}(t) = e^{-t}$, where $T^* =_{ST} T$. Hence $\bar{q}_I(e^{-t}) \le \bar{q}_{II}(e^{-t})$ for all $t \ge 0$. So $\bar{q}_I(u) \le \bar{q}_{II}(u)$ for all $u \in [0,1]$ and the proof is completed. □

In the second theorem we prove that this property can be extended to the hazard rate order for the systems which preserve the IFR (increasing failure rate) aging property. A similar result can be stated for the likelihood ratio order from Theorem 1.C.9 in [42, p. 46] and the preservation results for the ILR class of logconcave densities given in Proposition 2.2 of [33].

Theorem 4.3. *Let T be the lifetime of a coherent system with IID components having a common absolutely continuous reliability $\bar{F}$. Let T_I and T_{II} be the system lifetimes obtained with the replacement policies of cases I and II, respectively. Let $\bar{q}$ be the dual distortion function of T. If $\alpha(u) = u\bar{q}'(u)/\bar{q}(u)$ is decreasing in $(0,1)$, then $T_I \le_{HR} T_{II}$ for all $\bar{F}$.*

Proof. As in the preceding theorem, we have $\Pr(T_I > t) = \bar{q}_I(\bar{F}(t))$ and $\Pr(T_{II} > t) = \bar{q}_{II}(\bar{F}(t))$. So, from Theorem 4.1, (ii), we need to prove that $\bar{q}_{II}/\bar{q}_I$ is decreasing in $(0,1)$. With the notation used in the proof of the preceding theorem, if we assume that $\bar{F}(t) = e^{-t}$ for $t \ge 0$ (exponential components), we have $T_I =_{ST} X_{1:n} + T^*$ and $T_{II} =_{ST} T + T^{**}$, where $T^* =_{ST} T$ and T^{**} is a mixture of semi-coherent systems of order n. Then its reliability can be written as

$$\Pr(T^{**} > t) = s_1^{**}\bar{F}_{1:n}(t) + \cdots + s_n^{**}\bar{F}_{n:n}(t)$$

j	A_j	H_j	$\|A_j\|$	T	i_j	T_j
1	$(1, i_2, i_3)$	$X_1 < X_{i_2} < X_{i_3}$	2	$T = X_{i_2}$	2	$\min(X_2, X_3)$
2	$(i_1, 1, i_3)$	$X_{i_1} < X_1 < X_{i_3}$	2	$T = X_1$	2	X_1
3	$(i_1, i_2, 1)$	$X_{i_1} < X_{i_2} < X_1$	2	$T = X_1$	3	X_1

Table 1: Repairing options for the system in Example 4.4.

for all $t \geq 0$. The vector $(s_1^{**}, \ldots, s_n^{**})$ is called the signature (of order n) of T^{**} (see, e.g., [38]). The signature of $X_{1:n}$ is $(1, 0, \ldots, 0)$. Hence, as $(1, 0, \ldots, 0) \leq_{HR} (s_1^{**}, \ldots, s_n^{**})$, from Theorem 4.4 in [38], we get $X_{1:n} \leq_{HR} T^{**}$ for $\bar{F}(t) = e^{-t}$. Moreover, we know that T^* is independent of $X_{1:n}$ and T^{**} is independent of T. Then we can apply Lemma 1.B.3 in [42, p. 18] obtaining

$$T_I =_{ST} X_{1:n} + T^* \leq_{HR} T + T^{**} =_{ST} T_{II}$$

for $\bar{F}(t) = e^{-t}$ whenever T is IFR. Now we note that, from the results given in [33, p. 447], if the function α defined above is decreasing, then the system preserves the IFR property. So, as the exponential distribution is IFR, then T is also IFR and $T_I \leq_{HR} T_{II}$ holds for $\bar{F}(t) = e^{-t}$, that is,

$$\frac{\Pr(T_{II} > t)}{\Pr(T_I > t)} = \frac{\bar{q}_{II}(\bar{F}(t))}{\bar{q}_I(\bar{F}(t))} = \frac{\bar{q}_{II}(e^{-t})}{\bar{q}_I(e^{-t})}$$

is increasing for $t \geq 0$. Therefore, $\bar{q}_{II}(u)/\bar{q}_I(u)$ is decreasing in $(0, 1)$ and the proof is completed. □

The following example shows that, sometimes, to repair a fixed component (case III) is better than to repair the critical component of the system (case II).

Example 4.4. *Let us consider a coherent system with three IID components and lifetime* $T = \max(X_1, \min(X_2, X_3))$. *Then the distortion functions of the system are* $\bar{Q}(u_1, u_2, u_3) = u_1 + u_2u_3 - u_1u_2u_3$ *and* $\bar{q}(u) = \bar{Q}(u, u, u) = u + u^2 - u^3$. *Furthermore, the dual distortion functions associated to the lifetimes obtained after the minimal repair of the components* 1, 2 *and* 3 *are given by*

$$\bar{q}_{III}^{(1)}(u) = \bar{Q}(\bar{q}_1(u), u, u) = u + u^2 - u^3 - (u - u^3)\ln u$$

and

$$\bar{q}_{III}^{(2)}(u) = \bar{q}_{III}^{(3)}(u) = \bar{Q}(u, \bar{q}_1(u), u) = u + u^2 - u^3 - (u^2 - u^3)\ln u.$$

On the other hand, the distortion function for case I can be obtained from (3.8) *as*

$$\bar{q}_I(u) = \frac{3}{2}u + 3u^2 - \frac{7}{2}u^3 + 3u^3 \ln u.$$

Finally, we compute $\bar{q}_{II}$ *from* (3.21). *The signature of the system is* $(0, 2/3, 1/3)$. *It can be computed from the permutations given in Table 1. This table also contains the numbers* i_j *of component failures which cause the system failure and the expressions of the repaired system lifetimes* T_j *for each* $j = 1, 2, 3$. *Hence, from* (3.15), *we get*

$$\Pr(T_{II} > t) = \frac{1}{3}\sum_{j=1}^{3} \Pr(T_{II} > t | H_j)$$

for the events H_j given in Table 1. The first probability can be computed as

$$\Pr(T_{II} > t|H_1) = \bar{F}_{i_1:3}\#\bar{G}_1(t) = \bar{F}_{2:3}\#\bar{G}_1(t),$$

where if $X_{2:3} = x$, then

$$\bar{G}_{1,x}(y) = \Pr(T_1 - x > y|X_{2:3} = x, H_1) = \Pr(\min(X_2, X_3) - x > y|X_1 < x < X_2 < X_3) = \frac{\bar{F}^2(x+y)}{\bar{F}^2(x)}$$

since the components are IID. Therefore, from (2.1), we have

$$\Pr(T_{II} > t|H_1) = \bar{F}_{2:3}(t) + \int_0^t \frac{\bar{F}^2(t)}{\bar{F}^2(x)} f_{2:3}(x)dx,$$

where $\bar{F}_{2:3}(t) = 3\bar{F}^2(t) - 2\bar{F}^3(t)$ and $f_{2:3}(t) = 6(\bar{F}(t) - \bar{F}^2(t))f(t)$. Hence

$$\begin{aligned}
\Pr(T_{II} > t|H_1) &= \bar{F}_{2:3}(t) + 6\bar{F}^2(t)\int_0^t \frac{\bar{F}(x) - \bar{F}^2(x)}{\bar{F}^2(x)} f(x)dx \\
&= \bar{F}_{2:3}(t) + 6\bar{F}^2(t)\int_0^t \left(\frac{1}{\bar{F}(x)} - 1\right) f(x)dx \\
&= \bar{F}_{2:3}(t) + 6\bar{F}^2(t)\left(-\log\bar{F}(t) - F(t)\right) \\
&= -3\bar{F}^2(t) + 4\bar{F}^3(t) - 6\bar{F}^2(t)\log\bar{F}(t).
\end{aligned}$$

A straightforward (analogous) calculation for H_2 and H_3 leads us to

$$\Pr(T_{II} > t|H_2) = 3\bar{F}(t) - 3\bar{F}^2(t) + \bar{F}^3(t)$$

and

$$\Pr(T_{II} > t|H_3) = -\frac{3}{2}\bar{F}(t) + 3\bar{F}^2(t) - \frac{1}{2}\bar{F}^3(t) - 3\bar{F}(t)\log\bar{F}(t).$$

Hence

$$\begin{aligned}
\Pr(T_{II} > t) &= \frac{1}{3}\Pr(T_{II} > t|H_1) + \frac{1}{3}\Pr(T_{II} > t|H_2) + \frac{1}{3}\Pr(T_{II} > t|H_3) \\
&= \frac{1}{2}\bar{F}(t) - \bar{F}^2(t) + \frac{3}{2}\bar{F}^3(t) - \bar{F}(t)\log\bar{F}(t) - 2\bar{F}^2(t)\log\bar{F}(t) \\
&= \bar{q}_{II}(\bar{F}(t)),
\end{aligned}$$

where $\bar{q}_{II}(u) = u/2 - u^2 + (3/2)u^3 - u\log u - 2u^2\log u$ for $u \in (0,1)$.

In Figure 1 (left) we compare the distortion functions of the three cases. From these plots we conclude that $T \le_{ST} T_{III}^{(2)} \le_{ST} T_I \le_{ST} T_{II} \le_{ST} T_{III}^{(1)}$. In order to clarify the last inequality, we plot the ratio $\bar{q}_{III}^{(1)}/\bar{q}_{II}$ in the interval (0,1) (see Figure 1, right). This quotient is always above the line $y = 1$. However it is not decreasing and therefore T_{II} and $T_{III}^{(1)}$ are not HR-ordered. Hence, we can state that against the expected, the replacement policy of case II is not always the best strategy in the case of IID components.

The following example shows that Theorem 4.2 is not true when the components are dependent.

Figure 1: Plots of the dual distortion functions for the cases: I, II, III ((1) and (2)) and for the system given in Example 4.4 (left). Ratio $\bar{q}_{III}^{(1)}/\bar{q}_{II}$ in the interval (0,1) (right).

Example 4.5. *Let us consider a parallel system with* 2 *exchangeable components having a common absolutely continuous reliability function* $\bar{F}$. *Let us assume that both components are dependent and have the following Clayton-Oakes survival copula*

$$K(u,v) = \frac{u\,v}{u+v-u\,v}.$$

Taking into account that both components are ID and have survival copula K, *we get*

$$\bar{F}_{1:2}(t) = K(\bar{F}(t),\bar{F}(t)) = \frac{\bar{F}(t)}{2-\bar{F}(t)} \quad \text{and} \quad f_{1:2}(t) = \frac{2\,f(t)}{(2-\bar{F}(t))^2},$$

where f *represents the common density function of both components. Hence, the reliability function associated to* T_I *can be obtained from* (3.4) *as follows*

$$\begin{aligned}\bar{F}_{T_I}(t) &= \bar{F}_{1:2}(t) + \bar{F}_{1:2}(t)\ln(\bar{F}_{1:2}(t)) + 2\int_0^t \frac{K(\bar{F}(t),\bar{F}(x))}{\bar{F}_{1:2}(x)} f_{1:2}(x)dx \\ &= \frac{\bar{F}(t)}{2-\bar{F}(t)}\left(1-3\ln\bar{F}(t)-\ln(2-\bar{F}(t))\right) = \bar{q}_I(\bar{F}(t)),\end{aligned}$$

where $\bar{q}_I(u) = (u-3u\ln u - u\ln(2-u))/(2-u)$ *represents the dual distortion associated to* T_I.

On the other hand, we can obtain immediately the expression for the dual distortion associated to T_{II} *just by replacing* $K(u,v)$ *in* (3.9) *as follows*

$$\begin{aligned}\bar{q}_{II}(u) &= 2u - K(u,u) + 2\int_u^1 \frac{u-K(v,u)}{v-K(v,v)}(1-\partial_1 K(v,v))dv \\ &= \frac{u(3-2u)}{2-u} + \frac{u\,(3-u)}{1-u}\ln(2-u) + \frac{u^2\,(5-3u)}{(2-u)(1-u)}\ln u.\end{aligned}$$

Finally, we obtain the dual distortion functions for the case III. Firstly, we note that both distortions must be the same because we are considering exchangeable components. Moreover, $\bar{Q}(u,v) = u+v-K(u,v)$. *Hence, the dual distortion function of* $T_{III}^{(1)}$ *can be obtained as follows*

$$\bar{F}_{T_{III}^{(1)}}(t) = \bar{Q}(\bar{q}_1(\bar{F}(t)),\bar{F}(t)) = \bar{q}_1(\bar{F}(t)) + \bar{F}(t) - \frac{\bar{q}_1(\bar{F}(t))\,\bar{F}(t)}{\bar{q}_1(\bar{F}(t)) + \bar{F}(t) - \bar{q}_1(\bar{F}(t))\,\bar{F}(t)} = \bar{q}_{III}^{(1)}(\bar{F}(t)),$$

where

$$\bar{q}_{III}^{(1)}(u) = 2u - u\ln u - \frac{u-u\ln u}{1+(1-u)(1-\ln u)}.$$

We compare $\bar{q}_I$, $\bar{q}_{II}$ *and* $\bar{q}_{III}^{(1)}$ *in Figure 2 (left) along with the dual distortion function associated to the system without repairs. We observe that* $T \leq_{ST} T_{II} \leq_{ST} T_I$ *and* $T_{II} \leq_{ST} T_{III}^{(1)}$. *In Figure*

Figure 2: Plots of the dual distortion functions for the system in Example 4.5 for cases I, II, III and without repairs (left) and plots of the ratios $\bar{q}^{(1)}_{III}/\bar{q}_{II}$, $\bar{q}_I/\bar{q}_{II}$ and $\bar{q}_I/\bar{q}^{(1)}_{III}$ in the interval (0.3,1) (right).

2 (right) we represent the quotients $\bar{q}^{(1)}_{III}/\bar{q}_{II}$, $\bar{q}_I/\bar{q}_{II}$ and $\bar{q}_I/\bar{q}^{(1)}_{III}$. The first and second ratios are above the line $y = 1$ and they are decreasing, therefore $T_{II} \leq_{HR} T^{(1)}_{III}$ and $T_{II} \leq_{HR} T_I$. However, $\bar{q}_I/\bar{q}^{(1)}_{III}$ crosses the line $y = 1$ at the value $u_0 = 0.5862$ and thereby T_I and $T^{(1)}_{III}$ are not comparable in the ST order. As the ratio is bathtub, we have $T^{(1)}_{III} \leq_{MRL} T_I$ whenever $E(T^{(1)}_{III}) \leq E(T_I)$.

Proceeding as in the examples above, we can obtain the stochastic comparisons among the three policies considered in this paper for any coherent system. In particular, Table 2 provides the best replacement policy in terms of the usual stochastic order for all the coherent systems with 1-4 IID components. The coefficients c_i and d_j, associated to the distortion function $\bar{q}_{II}$ are given for each system as well. As one would expect in the case of IID components, the policy II induces a more reliable system in most of cases (see Theorem 4.2). However, there exist some systems where repairing a fix component is better than repairing the component which causes the failure of the system. In particular, the systems 7 and 24 in Table 2 satisfy that the system's reliability is improved in a higher level if we apply the policy III rather than the policies I or II. For both systems the first component is the most important component and its functioning implies the system functioning. Furthermore, the policies II and III are better than policy I for the systems 25 and 26 and both policies are not ordered. In this case, the optimal policy depends on if the decision maker is interested in improving the reliability of the system in an advanced or early age.

5. Conclusions

In the present paper we give a procedure to determine the reliability functions of coherent systems under a minimal repair maintenance and three different replacement policies. The components can be dependent or independent. In the first replacement policy, the first broken component is repaired. In the second case, a minimal repair is applied to the component which produces the failure of the system. In the third one, a fixed component is repaired in case of failure. Note that in the two first cases we do not know a priori which component will be repaired. In this context, we have proved that if the components are ID, then the reliability function associated to the lifetime of the repaired system in case I can be expressed as a distortion of the common component reliability function (see Theorem 3.2). This distortion depends on the structure of the system and on the underlying survival copula. We provide an explicit expression of this distortion in Theorem 3.3 for IID components. Analogously, we have proved that the reliability function for the case II can also be expressed using a distortion function when the components are exchangeable. This distortion is simplified for the IID case in Theorem 3.6. The new technique developed here can also be used to study other replacement policies. As an example, we provide an explicit expression for the dual distortion functions associated to the case of repairing the two first broken components in a general system or the k first broken components in a series system.

These representation results are used to compare the three replacement policies using the main stochastic orders. In this sense, our first comparison result shows that, for any coherent system with IID components, the case II is always a better strategy of replacement than the case I in the stochastic order (see Theorem 4.2). We prove with an example that this property is not true when

Table 2: Coefficients c_i and d_i associated to the dual distortion function $\bar{q}_{II}$ (see Theorem 3.6) for all the coherent systems with 1-4 IID components and the best replacement policy in the stochastic order. Cases I, II and III (i) are denoted by C_I, C_{II} and $C_{III}^{(i)}$, respectively.

N	$T = \phi(X_1, X_2, X_3, X_4)$	**c**	**d**	Best ST-policy
1	$X_{1:1} = X_1$	(1)	(-1)	$C_I \equiv C_{II} \equiv C_{III}^{(1)}$
2	$X_{1:2} = \min(X_1, X_2)$	(0,1)	(0,-2)	$C_I \equiv C_{II}$
3	$X_{2:2} = \max(X_1, X_2)$	(0,1)	(-2,0)	C_{II}
4	$X_{1:3} = \min(X_1, X_2, X_3)$	(0,0,1)	(0,0,-3)	$C_I \equiv C_{II}$
5	$\min(X_1, \max(X_2, X_3))$	(0,0,1)	(0,-4,1)	C_{II}
6	$X_{2:3}$ (2-out-of-3:F)	(0,-3,4)	(0,-6,0)	C_{II}
7	$\max(X_1, \min(X_2, X_3))$	(1/2,-1,3/2)	(-1,-2,0)	$C_{III}^{(1)}$
8	$X_{3:3} = \max(X_1, X_2, X_3)$	(-3/2,3,-1/2)	(-3,0,0)	C_{II}
9	$X_{1:4} = \min(X_1, X_2, X_3, X_4)$	(0,0,0,1)	(0,0,0,-4)	$C_I \equiv C_{II}$
10	$\max(\min(X_1, X_2, X_3),$ $\min(X_2, X_3, X_4))$	(0,0,0,1)	(0,0,-6,2)	C_{II}
11	$\min(X_{2:3}, X_4)$	(0,0,-3,4)	(0,0,-9,2)	C_{II}
12	$\min(X_1, \max(X_2, X_3), \max(X_2, X_4))$	(0, 1/2, -1, 3/2)	(0, -2, -3, 1)	C_{II}
13	$\min(X_1, \max(X_2, X_3, X_4))$	(0,-3/2,3,-1/2)	(0,-6,3,-1)	C_{II}
14	$X_{2:4}$ (2-out-of-4:F)	(0,0,-8,9)	(0,0,-12,0)	C_{II}
15	$\max(\min(X_1, X_2), \min(X_1, X_3, X_4),$ $\min(X_2, X_3, X_4))$	(0,0,-4,5)	(0,-2,-6,0)	C_{II}
16	$\max(\min(X_1, X_2), \min(X_3, X_4))$	(0,0,0,1)	(0,-4,0,0)	C_{II}
17	$\max(\min(X_1, X_2), \min(X_1, X_3),$ $\min(X_2, X_3, X_4))$	(0,-1,0,2)	(0,-4,-2,0)	C_{II}
18	$\max(\min(X_1, X_2), \min(X_2, X_3),$ $\min(X_3, X_4))$	(0,-2,4,-1)	(0,-6,2,0)	C_{II}
19	$\max(\min(X_1, \max(X_2, X_3, X_4)),$ $\min(X_2, X_3, X_4))$	(0,-3,4,0)	(0,-6,0,0)	C_{II}
20	$\min(\max(X_1, X_2), \max(X_1, X_3),$ $\max(X_2, X_3, X_4))$	(0,-5,8,-2)	(0,-8,2,0)	C_{II}
21	$\min(\max(X_1, X_2), \max(X_3, X_4))$	(0,-4,8,-3)	(0,-8,4,0)	C_{II}
22	$\min(\max(X_1, X_2), \max(X_1, X_3, X_4),$ $\max(X_2, X_3, X_4))$	(0,-8,12,-3)	(0,-10,2,0)	C_{II}
23	$X_{3:4}$ (3-out-of-4:F)	(0,-12,16,-3)	(0,-12,0,0)	C_{II}
24	$\max(X_1, \min(X_2, X_3, X_4))$	(2/3,0,-2,7/3)	(-1,0,-3,0)	$C_{III}^{(1)}$
25	$\max(X_1, \min(X_2, X_3), \min(X_2, X_4))$	(1/3,-3,5,-4/3)	(-1,-4,1,0)	C_{II}, $C_{III}^{(1)}$
26	$\max(X_{2:3}, X_4)$	(5/6,-5,13/2,-4/3)	(-1,-4,0,0)	C_{II}, $C_{III}^{(4)}$
27	$\max(X_1, X_2, \min(X_3, X_4))$	(1/3,0,1,-1/3)	(-2,0,0,0)	C_{II}
28	$X_{4:4} = \max(X_1, X_2, X_3, X_4)$	(-10/3,6,-2,1/3)	(-4,0,0,0)	C_{II}

the components are dependent. Furthermore, the previous result holds for the hazard rate order when we consider systems which preserve the IFR property (see Theorem 4.3). Unfortunately, the case III is not ST-ordered with neither case I nor case II, even assuming IID components. We provide both counterexamples as well as some interesting examples including the comparisons of all the coherent systems with 1-4 IID components (see Table 2).

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