PERIODIC MODULES AND ACYCLIC COMPLEXES

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Abstract. We study the behaviour of modules M that fit into a short exact sequence $0 \to M \to C \to M \to 0$, where C belongs to a class of modules $\mathcal{C}$, the so-called $\mathcal{C}$-periodic modules. We find a rather general framework to improve and generalize some well-known results of Benson and Goodearl and Simson. In the second part we will combine techniques of hereditary cotorsion pairs and presentation of direct limits, to conclude, among other applications, that if M is any module and C is cotorsion, then M will be also cotorsion. This will lead to some meaningful consequences in the category $\mathrm{Ch}(R)$ of unbounded chain complexes and in Gorenstein homological algebra. For example we show that every acyclic complex of cotorsion modules has cotorsion cycles, and more generally, every map $F \to C$ where C is a complex of cotorsion modules and F is an acyclic complex of flat cycles, is null-homotopic. In other words, every complex of cotorsion modules is dg-cotorsion.

Introduction

Throughout this paper R is an associative ring with identity and all modules will be right R-modules.

The goal of this work is the study of periodic and pure periodic modules with respect to an arbitrary class of modules $\mathcal{C}$. More precisely, one of the main objectives we pursue is to know when $\mathcal{C}$-periodic modules (resp. pure $\mathcal{C}$-periodic modules) are trivial, where an R-module M is called *$\mathcal{C}$-periodic* (resp. *pure $\mathcal{C}$-periodic*) if it fits into an exact sequence (resp. into a pure exact sequence) of the form $0 \to M \to C \to M \to 0$, with $C \in \mathcal{C}$, and it is called *trivial* if it belongs to $\mathcal{C}$. The origin of this problem comes from the celebrated result by Benson and Goodearl [BG00, Theorem 2.5] in which they show that each flat Proj-periodic module is trivial (here Proj denotes the class of all projective modules). It is then easy to observe that Benson and Goodearl statement can be reformulated to saying that each pure Proj-periodic module is trivial. This is because M is always flat in each pure short exact sequence of the form $0 \to M \to P \to M \to 0$, with P projective.

This module-theoretic property has a remarkable consequence at the level of chain complexes of modules: every acyclic complex of projective modules with flat cycles is contractible. This connection between flat Proj-periodic modules and acyclic complexes of projective modules was firstly observed by Christensen and Holm [CH15] and entitled them to find the module-theoretic proof conjectured by

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Neeman in [Nee08, Remark 2.15] to get the aforementioned result on acyclic complexes of projective modules with flat cycles (Neeman already gives a proof of this fact by using homotopy techniques). In 2002 Simson [Sim02] realized that Benson and Goodearl's theorem can be established in the pure setting of a finitely accessible Grothendieck category, by showing that if M fits into a pure exact sequence $0 \to M \to T \to M \to 0$, with T pure projective (i.e. a direct summand of a direct sum of finitely presented objects), then M itself is pure projective. In other words, every pure PProj-periodic module is trivial (PProj is the class of all pure projective modules).

We devote the first part of this paper to show that both Benson and Goodearl and Simson results are encoded under the following rather more general statement (see Theorem 2.5, Corollary 2.6(1) and (2)):

Theorem 0.1. *Any short exact sequence* $0 \to M \hookrightarrow G \xrightarrow{g} M \to 0$ *in which* g *is locally split and* G *is a direct sum of countably generated modules is split.*

Aside from the preceding particular instances of this Theorem, we also get (Corollary 2.6) that each pure $\mathrm{Add}(\mathcal{P})$-periodic module is trivial (here $\mathcal{P}$ denotes a class of finitely presented modules and $\mathrm{Add}(\mathcal{P})$ is the class of direct summands of direct sums of modules in $\mathcal{P}$).

Now, since a module F is flat if and only if every short exact sequence $0 \to M \to N \to F \to 0$ is pure, we immediately realize that the dual notion of flat module is that of absolutely pure (=FP-injective) module (i.e. a module E such that each exact sequence of modules $0 \to E \to L \to T \to 0$ is pure). Thus it seems natural to wonder whether or not the dual version of Benson and Goodearl's theorem holds. Namely, is every Inj-periodic absolutely pure module injective? (Inj is the class of all injective modules). Or even more generally, is every pure PInj-periodic module trivial? (here PInj stands for the category of pure injective modules, i.e. modules L such that each pure short exact sequence $0 \to L \to A \to B \to 0$ splits). Recently, the answer to this question has been positively settled by Šťovíček [Šťo14, Corollary 5.5] by using complete cotorsion pairs in the category of unbounded chain complexes.

Thus, at this point, we know that every pure PProj-periodic module is trivial and that every pure PInj-periodic module is also trivial. For the global case we know that flat Proj-periodic modules are trivial and that Inj-periodic absolutely pure modules are also trivial. But there are rings for which there exist non trivial $\mathcal{C}$-periodic modules for the classes Proj, Inj and Flat. These non trivial $\mathcal{C}$-periodic modules can be constructed when $\mathcal{C}$ is a proper generating and cogenerating class, see Corollary 1.5.

However when considering the class Cot of cotorsion modules (i.e. the right Ext-orthogonal with respect to the flat modules) one confronts the major problem that this class is almost never generating (except in the trivial case in which the ring is right perfect, when all modules are cotorsion). This is due to a result of Guil Asensio and Herzog [GAH05, Corollary 10]. This suggests that Cot-periodic modules might be trivial, and indeed one of the main applications of the second part of the paper is to show that this is the case.

Hence we devote the second part of this paper to address, among others, this question. We will work directly with hereditary cotorsion pairs in the module category, rather than in the category of complexes. So we get a slightly more direct proof of the dual of Benson and Goodearl Theorem. Also our methods seem to be more far reaching as they allow to prove, as we announced before, that each Cot-periodic module is trivial and also allow to get significant consequences in Gorenstein homological algebra. We state below the main applications based on the general theorem (Theorem 3.7). The reader can find the proof of these applications in Proposition 3.8 and Corollaries 3.10 and 4.5.

Theorem 0.2. *The following hold true:*

1. *Every* Inj*-periodic absolutely pure module is trivial.*
2. *Let* $(\mathcal{A}, \mathcal{B})$ *be a hereditary cotorsion pair in* Mod-R*. Assume that* $\mathcal{A}$ *is closed under pure epimorphic images. Then every* $\mathcal{B}$*-periodic module is cotorsion. In particular, every* Cot*-periodic module is trivial (and so every* Inj*-periodic module is cotorsion).*
3. *Assume that each finitely generated right ideal has finite flat dimension. Then each Gorenstein injective module is injective. In particular if a ring has finite weak global dimension, every Gorenstein injective module is injective.*
4. *Assume that each finitely generated right ideal has finite injective dimension. Then each Gorenstein injective module is injective.*
5. *Every pure* PInj*-periodic module is trivial.*

The third and fourth statements are applications of Theorem 3.7 to the realm of Gorenstein homological algebra. The first statement shows the dual version of Benson and Goodearl Theorem. The second statement has very interesting, and perhaps surprising, consequences for chain complexes of modules as we will indicate now. The last part of the paper (Section 4) is therefore devoted to infer these and other applications of $\mathcal{C}$-periodic modules for chain complexes.

We have already mentioned the relationship observed by Holm and Christensen between flat Proj-periodic modules and acyclic chain complexes of projective modules with flat cycles. In a recent paper Estrada, Fu and Iacob [EFI17] show that Christensen's and Holm's argument can be easily extended to provide a nice correspondence between $\mathcal{C}$-periodic modules in a class $\mathcal{D}$ of modules, and acyclic complexes of modules in $\mathcal{C}$ with cycles in $\mathcal{D}$ (see Proposition 1.4 for a precise formulation of the statement). This bridge between periodic modules and acyclic complexes, is the key to find applications of our results in the category $\mathrm{Ch}(R)$ of unbounded chain complexes. The first application is a consequence of Theorem 0.2(2). We recall that a chain complex C is called *dg-cotorsion* if each C_n is a cotorsion module and each map $f\colon F \to C$ from an acyclic complex F of flat modules with flat cycles, is null-homotopic.

Theorem 0.3. *Every acyclic complex of cotorsion modules has cotorsion cycles. As a consequence, every complex of cotorsion modules is dg-cotorsion.*

The proof of this Theorem can be found in theorems 4.1 and 4.3. As a particular instance, we get that every acyclic complex of injectives has cotorsion cycles. This was already proved by Šťovíček [Šťo14, Corollary 5.9].

The second application (Corollary 4.6) also recovers a result of Šťovíček ([Šťo14, Theorem 5.5]):

Corollary 0.4. *Let* $\mathcal{G}$ *be a finitely accessible additive category, and let* $\widetilde{\mathrm{PurAc}}$ *be the class of pure acyclic complexes in* $\mathrm{Ch}(\mathcal{G})$ *(i.e. concatenation of pure short exact sequences in* $\mathcal{G}$*) and* $dw\mathrm{PInj}$ *be the class of chain complexes with pure injective components. The pair of classes* $(\widetilde{\mathrm{PurAc}}, dw\mathrm{PInj})$ *is a complete hereditary cotorsion pair in* $\mathrm{Ch}(\mathcal{G})_{dw\text{-}pur}$ *(this is the category* $\mathrm{Ch}(\mathcal{G})$ *of unbounded chain complexes with the degreewise pure exact structure induced from* $\mathcal{G}$*).*

1. Preliminaries

Given an ordinal κ, a family of sets $\{A_\alpha \mid \alpha < \kappa\}$ is called *continuous* if $\bigcup_{\alpha<\lambda} A_\alpha = A_\lambda$ for each limit ordinal λ smaller than κ. Given a map $f\colon A \to B$

and C a subset of A, we shall denote by $f \restriction C$ the restriction of f to C. The cardinality of a set A will be denoted by $|A|$; ω will be the first infinite ordinal number.

Let $\mathcal{X}$ be a class of modules containing all projective modules and n a natural number. We shall denote by $\mathcal{X}_n$ the class of all modules with $\mathcal{X}$-projective dimension less than or equal to n (a module M is said to have $\mathcal{X}$-projective dimension less than or equal to n if there exists a projective resolution of M such that its $(n-1)st$ syzygy belongs to $\mathcal{X}$). The right finitistic $\mathcal{X}$-projective dimension of R is

$$\mathrm{FPD}_{\mathcal{X}}(R) = \mathrm{Min}\{n < \omega : \mathcal{X}_n = \mathcal{X}_{n+1}\}$$

in case the set in the right is not empty, or ∞ otherwise. The right finitistic $\mathcal{Y}$-injective dimension, $\mathrm{FID}_{\mathcal{Y}}(R)$, for a class $\mathcal{Y}$ containing all injective modules is defined analogously. We will denote by Proj, Inj, Flat, Abs, Cot, PProj and PInj the classes Mod-R consisting of all projective, injective, flat, absolutely pure, cotorsion, pure projective and pure injective modules respectively.

1.1. **Locally split short exact sequences.** An epimorphism (resp. monomorphism) $f : M \to N$ in Mod-R is *locally split* if for each $x \in N$ (resp. $x \in M$) there exists $g\colon N \to M$ such that $fg(x) = x$ (resp. $gf(x) = x$).

By [Azu92, Corollary 2], the morphism f is a locally split epimorphism if and only if for each finite subset F of N, there exists $g\colon N \to M$ such that $fg(x) = x$ for each $x \in F$. Moreover, by [ZH92, Proposition 11], if f is a locally split epimorphism and N is countably generated, then f is actually split. We shall use the following property of locally split epimorphisms:

Lemma 1.1. *Consider the following commutative diagram in* Mod-R

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} \\ K & \xrightarrow{i} & L \end{array}$$

in which f and g are locally split epimorphisms. Then so is i.

Proof. Let $x \in L$ and $\overline{g}\colon\ L \to N$ be such that $g\overline{g}(x) = x$. Moreover, let $\overline{f}\colon\ N \to M$ be such that $f\overline{f}(\overline{g}(x)) = \overline{g}(x)$. Then the morphism $\overline{i} := h\overline{f}\overline{g}$ from L to K satisfies $i\overline{i}(x) = x$. Since x is arbitrary, this means that i is locally split. □

Given a submodule K of a module M, if the inclusion $K \to M$ (resp. the projection $M \to \frac{M}{K}$) is locally split, we shall say that K (resp. $\frac{M}{K}$) is a *locally split submodule* (resp. *quotient*) of M. Given $\mathcal{C}$ a class of modules we shall denote by:

- $\mathrm{Add}(\mathcal{C})$ the class of all modules that are isomorphic to a direct summand of a direct sum of modules in $\mathcal{C}$;
- $\mathrm{add}(\mathcal{C})$ the class of all modules that are isomorphic to a direct summand of a finite direct sum of modules in $\mathcal{C}$;
- $\mathrm{G}(\mathcal{C})$ the class of all modules M for which there exists a locally split epimorphism $A \to M$ with $A \in \mathrm{Add}(\mathcal{C})$;
- and $\mathrm{S}(\mathcal{C})$ the class of all modules M for which there exists a locally split monomorphism $M \to A$ with $A \in \mathrm{Add}(\mathcal{C})$.

The following lemma recalls properties of the classes $\mathrm{G}(\mathcal{C})$ and $\mathrm{S}(\mathcal{C})$. The properties (1), (2) and (4) are well known. The property (3) is an extension of a classical result for free modules due to Villamayor (see [Cha60, Proposition 2.2]):

Lemma 1.2. *Let $\mathcal{C}$ be a class of modules.*

(1) The class $\mathrm{G}(\mathcal{C})$ is closed under locally split quotients and locally split submodules. Consequently, $\mathrm{S}(\mathcal{C}) \subseteq \mathrm{G}(\mathcal{C})$.

(2) *Countably generated modules belonging to* $\mathrm{G}(\mathcal{C})$ *or* $\mathrm{S}(\mathcal{C})$ *actually belong to* $\mathrm{Add}(\mathcal{C})$.
(3) *If* $\mathcal{C}$ *consists of finitely generated modules, then* $\mathrm{S}(\mathcal{C})$ *is closed under pure submodules.*
(4) *If* $\mathcal{C}$ *consists of finitely presented modules, then* $\mathrm{G}(\mathcal{C})$ *is closed under pure submodules.*

Proof. (1) The class $\mathrm{G}(\mathcal{C})$ is clearly closed under locally split quotients. It is closed under locally split submodules as a consequence of [AH03, Lemma 2.1].
(2) Since a locally split epimorphism onto a countably generated module is split, a countably generated module belonging to $\mathrm{G}(\mathcal{C})$ actually belongs to $\mathrm{Add}(\mathcal{C})$. As $\mathrm{S}(\mathcal{C}) \subseteq \mathrm{G}(\mathcal{C})$, countably generated modules in $\mathrm{S}(\mathcal{C})$ belong to $\mathrm{Add}(\mathcal{C})$ too.
(3) Let $N \in \mathrm{S}(\mathcal{C})$ and K a pure submodule of N. Since N is isomorphic to a locally split submodule of a module A which is a direct summand of a direct sum of finitely generated modules, we may assume that there exists a family $\{N_i \mid i \in I\}$ of finitely generated modules such that N is a locally split submodule of $\bigoplus_{i\in I} N_i$.
We prove that K is a locally split submodule of N which implies, by [Zim02, Proposition 1.3], that $K \in \mathrm{S}(\mathcal{C})$. Let U be a finitely generated submodule of K. There exists a finite set $J \subseteq I$ and a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \hookrightarrow & \bigoplus_{j\in J} N_j & \xrightarrow{p'} & C' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle q} & & \\ 0 & \longrightarrow & K & \hookrightarrow & \bigoplus_{i\in I} N_i & \xrightarrow{p} & C & \longrightarrow & 0. \end{array}$$

Since K is pure in $\bigoplus_{i\in I} N_i$ and C' is finitely presented, there exists a morphism $r\colon C' \to \bigoplus_{i\in I} N_i$ such that $pr = q$. By [Wis88, 7.16] there exists $\overline{g}\colon \bigoplus_{j\in J} N_j \to K$ such that $\overline{g}(u) = u$ for each $u \in U$. This $\overline{g}$ extends to a morphism $g'\colon \bigoplus_{i\in I} N_i \to K$ whose restriction to N, $g\colon N \to K$, trivially satisfies $g(u) = u$ for each $u \in U$. Then K is a locally split submodule of N.
(4). This is [AH03, Proposition 2.3]. □

A short exact sequence

$$0 \to K \xrightarrow{f} M \xrightarrow{g} N \to 0$$

is said to be locally split if f is a locally split monomorphism and g is a locally split epimorphism. Contrary to the case of pure exact sequences, there exist locally split epimorphisms and locally split monomorphisms such that the corresponding short exact sequence is not locally split (see [ZH92, Example 13]). However, if in a short exact sequence

$$0 \to K \xrightarrow{f} M \xrightarrow{g} N \to 0,$$

g is locally split and M is a direct sum of countably generated modules, then the sequence is locally split by [ZH92, Proposition 12]. This is the situation of our Theorem 0.1.

1.2. **Complete cotorsion pairs.** A pair of classes $(\mathcal{A}, \mathcal{B})$ in an abelian category $\mathcal{G}$ is called a *cotorsion pair* if $\mathcal{A}^\perp = \mathcal{B}$ and ${}^\perp\mathcal{B} = \mathcal{A}$, where, for a given class of objects $\mathcal{C}$, the right orthogonal class $\mathcal{C}^\perp$ is defined as the class of objects M such that $\mathrm{Ext}^1_{\mathcal{G}}(C, M) = 0$ for each object $C \in \mathcal{C}$. Similarly, the left orthogonal class ${}^\perp\mathcal{C}$ is defined. The cotorsion pair is called *hereditary* if $\mathrm{Ext}^i_{\mathcal{G}}(A, B) = 0$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$, and $i \geq 1$. Finally we say that the cotorsion pair is *complete* when it has *enough injectives* and *enough projectives*. This means that for each $M \in \mathcal{G}$ there exist exact sequences $0 \to M \to B \to A \to 0$ (enough injectives) and $0 \to B' \to A' \to M \to 0$ (enough projectives), where $A, A' \in \mathcal{A}$ and $B, B' \in \mathcal{B}$.

1.3. **Chain complexes of modules.** Let $\mathcal{G}$ be an abelian category, we denote by $\mathrm{Ch}(\mathcal{G})$ the category of unbounded chain complexes of objects in $\mathcal{G}$, i.e. complexes G of the form

$$\cdots \to G_{n+1} \xrightarrow{d^G_{n+1}} G_n \xrightarrow{d^G_n} G_{n-1} \to \cdots .$$

We will denote by Z_nG the n *cycle of* G, i.e. $Z_nG = \mathrm{Ker}(d^G_n)$. Given a chain complex G the n^{th}*-suspension of* G, $\Sigma^n G$, is the complex defined as $(\Sigma^n G)_k = G_{k-n}$ and $d^{\Sigma^n G}_k = (-1)^n d^G_{k-n}$. And for a given object $A \in \mathcal{G}$, the n*-disk* complex is the complex with the object A in the components n and $n-1$, d_n as the identity map, and 0 elsewhere. Given a covariant functor $F\colon \mathcal{G} \to \mathcal{H}$ between abelian categories, we shall denote by $F(G)$ the complex

$$\cdots \to F(G_{n+1}) \xrightarrow{F(d^G_{n+1})} F(G_n) \xrightarrow{F(d^G_n)} F(G_{n-1}) \to \cdots$$

If F is contravariant, we define $F(G)$ in a similar way.

In case $\mathcal{G} = \text{Mod-}R$, we will denote $\mathrm{Ch}(\mathcal{G})$ simply by $\mathrm{Ch}(R)$. An acyclic complex G in $\mathrm{Ch}(R)$ consisting of projective modules (resp. injective modules) is said to be *totally acyclic* if $\mathrm{Hom}_R(G, P)$ (resp. $\mathrm{Hom}_R(I, G)$) is acyclic for each projective module P (resp. injective module I).

If X, Y are complexes in $\mathrm{Ch}(R)$, we have the well known formula (see [Hap88, §1]):

$$(*) \quad \mathrm{Ext}^1_{dw}(X, Y) \cong \frac{\mathrm{Hom}_{\mathrm{Ch}(R)}(X, Y[1])}{\sim}$$

where $\mathrm{Ext}^1_{dw}(X, Y)$ is the subgroup of $\mathrm{Ext}^1_{\mathrm{Ch}(R)}(X, Y)$ consisting of the degreewise splitting short exact sequences and $\sim$ is the homotopy relation on $\mathrm{Ch}(R)$.

For every class $\mathcal{C}$ of modules, we shall consider the following classes of complexes of modules:

- $\mathrm{dw}\,\mathcal{C}$ is the class of all complexes $X \in \mathrm{Ch}(R)$ such that $X_n \in \mathcal{C}$ for all $n \in \mathbb{Z}$. $\mathrm{Ch}(\mathcal{C})$ will denote the full subcategory of $\mathrm{Ch}(R)$ with objects in $\mathrm{dw}\,\mathcal{C}$.
- $\mathrm{ex}\,\mathcal{C}$ is the class of all acyclic complexes in $\mathrm{dw}\,\mathcal{C}$.
- $\widetilde{\mathcal{C}}$ is the class class of all complexes $X \in \mathrm{ex}\,\mathcal{C}$ with the cycles Z_nX in $\mathcal{C}$ for all $n \in \mathbb{Z}$.
- If $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair in Mod-R, then $\mathrm{dg}\,\mathcal{A}$ is the class of all complexes $X \in \mathrm{dw}\,\mathcal{A}$ such that any morphism $f\colon X \to Y$ with $Y \in \widetilde{\mathcal{B}}$ is null homotopic. Since $\mathrm{Ext}^1_R(X_n, Y_n) = 0$ for every $n \in \mathbb{Z}$, formula $(*)$ shows that $\mathrm{dg}\,\mathcal{A} = {}^\perp\widetilde{\mathcal{B}}$. Similarly, $\mathrm{dg}\,\mathcal{B}$ is the class of all complexes $Y \in \mathrm{dw}\,\mathcal{B}$ such that any morphism $f\colon X \to Y$ with $X \in \tilde{\mathcal{A}}$ is null homotopic. Hence $\mathrm{dg}\,\mathcal{B} = \tilde{\mathcal{A}}^\perp$.

If $\mathcal{C}$ is a class of complexes of modules, we shall denote by $\mathrm{Z}(\mathcal{C})$ the class of all modules isomorphic to a cycle of some complex belonging to $\mathcal{C}$.

1.4. **$\mathcal{C}$-periodic modules.** We are interested in periodic modules with respect to a class of modules.

Definition 1.3. Let $\mathcal{C}$ be a class of modules and M a module. We say that M is $\mathcal{C}$-periodic (resp. pure $\mathcal{C}$-periodic) if there exists an exact sequence (resp. a pure exact sequence)

$$0 \to M \to C \to M \to 0$$

with $C \in \mathcal{C}$.

These modules are related with Gorenstein modules. Recall that a module M is *Gorenstein projective* (resp. *Gorenstein injective*) if there exists a totally acyclic complex of projective modules (resp. injective modules) such that $M \cong Z_0G$. And

M is said to be *strongly Gorenstein projective* (resp. *strongly Gorenstein injective*) if it is Gorenstein projective and Proj-periodic (resp. Gorenstein injective and Inj-periodic), see [BM07, Proposition 2.9]. By [BM07, Theorem 2.7], each Gorenstein projective (resp. Gorenstein injective) module is a direct summand of a strongly Gorenstein projective (resp. strongly Gorenstein injective) module.

Analogously, a module M is said to be *Gorenstein flat* if there exists an acyclic complex G consisting of flat modules such that $G \otimes_R I$ is exact for each injective left R-module I and $M \cong Z_0 G$. And M is *strongly Gorenstein flat* if it is Flat-periodic and Gorenstein flat. By [BM07, Theorem 3.5], each Gorenstein flat module is a direct summand of a strongly Gorenstein flat module.

The relationship between Gorenstein modules and strongly Gorenstein modules observed by Bennis and Mahdou can be extended to cycles of certain chain complexes and periodic modules, as we show in the following proposition. This approach was used by Christensen and Holm [CH15] and by Fu and Herzog [FH16] in the special case of flat Proj-periodic modules. A general formulation appears in [EFI17] which we present here including its pure version whose proof can be easily obtained.

Proposition 1.4. *(*[EFI17, Proposition 1 and Proposition 2]*) Let $\mathcal{C}$ be a class of modules closed under direct sums or under direct products and let $\mathcal{D}$ be a class closed under direct summands. The following are equivalent:*

(1) Every (pure) cycle of an acyclic complex with components in $\mathcal{C}$ belongs to $\mathcal{D}$, that is $\mathrm{Z}(\mathrm{ex}(\mathcal{C})) \subseteq \mathcal{D}$.
(2) Every (pure) $\mathcal{C}$-periodic module belongs to $\mathcal{D}$.

Let $\mathcal{C}$ be a class of modules. One of our main concerns is when $\mathcal{C}$-periodic modules are trivial, in the sense that they belong to the initial class $\mathcal{C}$. The preceding result gives us a procedure to construct non-trivial $\mathcal{C}$-periodic modules: we only have to find a complex in $\mathrm{ex}\,\mathcal{C}$ whose cycles do not lie in $\mathcal{C}$. This complex exists if $\mathcal{C}$ is a generating and cogenerating class in Mod–R.

Corollary 1.5. *Let $\mathcal{C}$ be a generating and cogenerating class of modules closed under direct sums or direct products. Let $\mathcal{D}$ be a class of modules closed under direct summands which is not equal to the whole category* Mod-R*. Then there exist $\mathcal{C}$-periodic modules not belonging to $\mathcal{D}$.*

Proof. Let M be a module not belonging to $\mathcal{D}$. Since $\mathcal{C}$ is generating and cogenerating, there exists a complex $C \in \mathrm{ex}\,\mathcal{C}$ such that $M \cong Z_0 C$. Then, if $\mathcal{C}$ is closed under direct sums (resp. direct products), $\bigoplus_{n\in\mathbb{Z}} Z_n C$ (resp. $\prod_{n\in\mathbb{Z}} Z_n C$) is a $\mathcal{C}$-periodic module not belonging to $\mathcal{D}$. □

2. Locally split monomorphisms and generalized periodic modules

This section is devoted to prove Theorem 0.1. We begin with a technical lemma. In general, if f_1 and f_2 are split monomorphisms in the commutative diagram

$$\begin{array}{ccc} K' & \xrightarrow{\;i\;} & M' \\ \big\downarrow{\scriptstyle f_1} & & \big\downarrow{\scriptstyle f_2} \\ K & \xrightarrow{\;k\;} & M \end{array}$$

then the splittings of f_1 and f_2 need not make the diagram commutative. The following lemma constructs a splitting which makes the diagram commutative in some particular cases.

Lemma 2.1. *Consider the following diagram in* Mod-R *with exact rows*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K' & \overset{i}{\hookrightarrow} & M' & \overset{j}{\longrightarrow} & L' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \\ 0 & \longrightarrow & K & \overset{k}{\longrightarrow} & M & \overset{l}{\longrightarrow} & L & \longrightarrow & 0 \end{array}$$

such that f_1, f_3 and k are splitting monomorphisms. Then for any splitting $\overline{f}_1$ of f_1, there exists a splitting $\overline{f}_2$ of f_2 such that $i\overline{f}_1 = \overline{f}_2 k$.

Proof. Let $\overline{f}_3$ be a splitting of f_3 and $\overline{k}$ a splitting of k. Then $\overline{i} := \overline{f}_1\overline{k}f_2$ is a splitting of i and, consequently, there exists $\overline{j}$ a splitting of j.
Now let l' be a splitting of l and note that $L = \operatorname{Im} f_3 \oplus N$ for some N. Denoting by $\overline{l}$ the direct sum of the morphisms $f_2\overline{j}(\overline{f}_3 \restriction \operatorname{Im} f_3)\colon \operatorname{Im} f_3 \to M$ and $l' \restriction N\colon N \to M$, we obtain a splitting of l satisfying $\overline{l}f_3 = f_2\overline{j}$.
Finally, notice that $M = \operatorname{Im} k \oplus \operatorname{Im} \overline{l}$. Define $\overline{f}_2$ as the direct sum of the morphisms $i\overline{f}_1\overline{k}\colon \operatorname{Im} k \to M'$ and $\overline{j}\overline{f}_3 l\colon \operatorname{Im} \overline{l} \to M'$. It is easy to see that $\overline{f}_2$ is a splitting of f_2 satisfying $i\overline{f}_1 = \overline{f}_2 k$. □

In general, the union of a chain of direct summands of a module is not a direct summand. The following lemma shows a situation in which the union of a continuous chain of direct summands is a direct summand.

Lemma 2.2. *Let G be a module and M a submodule of G. Suppose that there exist an ordinal κ, a continuous chain $\{M_\alpha \mid \alpha < \kappa\}$ of submodules of M, with $M = \bigcup_{\alpha<\kappa} M_\alpha$, and a continuous chain $\{G_\alpha \mid \alpha < \kappa\}$ of submodules of G, such that:*

(a) $\bigcup_{\alpha<\kappa} G_\alpha$ is a direct summand of G;
(b) G_α is a direct summand of $G_{\alpha+1}$ for each $\alpha < \kappa$;
(c) $M_\alpha = M_{\alpha+1} \cap G_\alpha$ for each $\alpha < \kappa$;
(d) M_0 is a direct summand of G_0 and $\frac{M_{\alpha+1}+G_\alpha}{G_\alpha}$ is a direct summand of $\frac{G_{\alpha+1}}{G_\alpha}$ for each $\alpha < \kappa$.

Then M is a direct summand of G.

Proof. Denote by $f_\alpha\colon M_\alpha \to G_\alpha$ the inclusion for each $\alpha < \kappa$. We are going to construct, for each $\alpha < \kappa$, a morphism $\overline{f}_\alpha\colon G_\alpha \to M_\alpha$ such that $\overline{f}_\alpha f_\alpha = 1_{M_\alpha}$ and $i_{\alpha\gamma}\overline{f}_\gamma = \overline{f}_\alpha j_{\alpha\gamma}$ for each $\gamma < \alpha$, where $i_{\alpha\gamma}\colon M_\gamma \to M_\alpha$ and $j_{\alpha\gamma}\colon G_\gamma \to G_\alpha$ are the inclusions. Then the direct limit map of the $\overline{f}_\alpha$'s is a splitting of the inclusion $M \hookrightarrow \bigcup_{\alpha<\kappa} G_\alpha$, which implies that M is a direct summand of G by (a).
We shall make the construction recursively on α. If $\alpha = 0$, take a splitting $\overline{f}_0\colon G_0 \to M_0$ of f_0. If α is limit, let $\overline{f}_\alpha$ be the direct limit of the system $\{\overline{f}_\gamma \mid \gamma < \alpha\}$. Finally, suppose that α is successor, say $\alpha = \mu+1$. Using the snake lemma we can construct the following commutative diagram,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_\mu & \overset{f_\mu}{\hookrightarrow} & G_\mu & \overset{g_\mu}{\longrightarrow} & \frac{G_\mu}{M_\mu} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle i_{\mu+1\mu}} & & \downarrow{\scriptstyle j_{\mu+1\mu}} & & \downarrow{\scriptstyle k_\mu} & & \\ 0 & \longrightarrow & M_{\mu+1} & \overset{f_{\mu+1}}{\hookrightarrow} & G_{\mu+1} & \overset{g_{\mu+1}}{\longrightarrow} & \frac{G_{\mu+1}}{M_{\mu+1}} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle p_\mu} & & \downarrow{\scriptstyle q_\mu} & & \downarrow{\scriptstyle r_\mu} & & \\ 0 & \longrightarrow & \frac{M_{\mu+1}}{M_\mu} & \overset{h_\mu}{\longrightarrow} & \frac{G_{\mu+1}}{G_\mu} & \overset{l_\mu}{\longrightarrow} & \frac{G_{\mu+1}}{M_{\mu+1}+G_\mu} & \longrightarrow & 0 \end{array}$$

in which k_μ is monic, since $M_{\mu+1} \cap G_\mu = M_\mu$ by (c) and, consequently, h_μ is monic too. But $h_\mu(x + M_\mu) = x + G_\mu$ for each $x \in M_{\mu+1}$, which means that $\operatorname{Im} h_\mu = \frac{M_{\mu+1}+G_\mu}{G_\mu}$. By (d), h_μ splits. Since $j_{\mu+1\mu}$ splits by (b) and f_μ splits by induction hypothesis, we can apply Lemma 2.1 to the splitting $\overline{f}_\mu$ of f_μ to construct a splitting $\overline{f}_{\mu+1}$ of $f_{\mu+1}$ such that $\overline{f}_{\mu+1} j_{\mu+1\mu} = i_{\mu+1\mu}\overline{f}_\mu$. This concludes the construction. □

The proof of Theorem 0.1 relies on the following lemmas.

Lemma 2.3. *Let λ be an infinite cardinal and $g\colon G \to M$ be a locally split epimorphism. Then for each $\leq \lambda$-generated submodule C of M, there exists a $\leq \lambda$-generated submodule D of G such that $C \subseteq g(D)$ and $g \restriction D\colon D \to g(D)$ satisfies the following property: for each $x \in g(D)$, there exists $f\colon M \to G$ such that $gf(x) = x$ and $fg(D) \leq D$.*

If, in addition, $G = \oplus_{i\in I} G_i$ for a family $\{G_i \mid i \in I\}$ of countably generated modules, then there exists a subset J of I of cardinality less than or equal to λ such that $D = \oplus_{j\in J} G_j$.

Proof. We are going to construct a chain $\{D_n \mid n < \omega\}$ of $\leq \lambda$-generated submodules of G with $C \leq g(D_0)$, and a chain $\{H_n \mid n < \omega\}$ of subsets of $\operatorname{Hom}_R(M, G)$ such that $|H_n| \leq \lambda$, verifying the following properties for every $n < \omega$:

(a) given $x \in g(D_n)$ there exists $f \in H_n$ with $gf(x) = x$,
(b) and $fg(D_n) \leq D_{n+1}$ for each $f \in H_n$.

We shall make the construction recursively on n. For $n = 0$ let D_0 be a λ-generated submodule of G such that $C \leq g(D_0)$ and take a generating set $\{x^0_\alpha \mid \alpha < \lambda\}$ of $g(D_0)$. Since g is locally split, for each finite subset Γ of λ there exists, by [Azu92, Corollary 2], a morphism $f^0_\Gamma\colon M \to G$ such that $gf^0_\Gamma(x^0_\gamma) = x^0_\gamma$ for each $\gamma \in \Gamma$. Then the set $H_0 := \{f^0_\Gamma \mid \Gamma \subseteq \lambda \text{ is finite}\}$ has cardinality less than or equal to λ and for each $x \in g(C)$, there exists $f \in H_0$ with $gf(x) = x$.
Assume that we have constructed D_n and H_n for some $n < \omega$. Let $D_{n+1} = D_n + \sum_{f\in H_n} f(g(D_n))$, which is a $\leq \lambda$-generated submodule (since $|H_n| \leq \lambda$ and D_n is $\leq \lambda$-generated), and satisfies $fg(D_n) \leq D_{n+1}$ for each $f \in H_n$. Let $\{x^{n+1}_\alpha \mid \alpha < \lambda\}$ be a generating system of $g(D_{n+1})$. Again by [Azu92, Corollary 2] since g is locally split, there exists, for each $\Gamma \subseteq \lambda$ finite, a morphism $f^{n+1}_\Gamma\colon M \to G$ such that $gf^{n+1}_\Gamma(x^{n+1}_\gamma) = x^{n+1}_\gamma$ for each $\gamma \in \Gamma$. Set $H_{n+1} = \{f^{n+1}_\Gamma \mid \Gamma \subseteq \lambda \text{ is finite}\} \cup H_n$. Then H_{n+1} has cardinality less than or equal to λ and satisfies that for each $x \in g(D_{n+1})$ there exists $f \in H_{n+1}$ with $gf(x) = x$. This concludes the construction.
Now take $D = \bigcup_{n<\omega} D_n$, which is $\leq \lambda$-generated. Given $x \in g(D)$, there exists $n < \omega$ such that $x \in g(D_n)$. By (a) there exists $f \in H_n$ such that $fg(x) = x$. Moreover, this f satisfies that $fg(D) \leq D$ since, given $k < \omega$ with $k \geq n$, $fg(D_k) \leq D_{k+1} \leq D$ by (b), because $f \in H_k$. And, if $k < n$, then $D_k \leq D_n$ which implies that $f(D_k) \leq D_{n+1}$ by (b) again. This finishes the proof of the first part of the lemma.
In order to prove the last statement simply note that, if G is a direct sum of countably generated modules, $G = \oplus_{i\in I} G_i$, then, in the previous construction, D_n can be taken of the form $\oplus_{i\in I_n} G_i$ for some subset I_n of I satisfying $I_n \subseteq I_{n+1}$ for each $n < \omega$. □

Lemma 2.4. *Let*

$$0 \longrightarrow M \xhookrightarrow{k} G \xrightarrow{g} M \longrightarrow 0$$

be a short exact sequence in Mod-R *such that k is the inclusion $M \leq G$, g is locally split and G is a direct sum of a family $\{G_i \mid i \in I\}$ of countably generated* *modules. Then, for each countably generated submodule K of M, there exists a countable subset J of I such that $K \leq \oplus_{j\in J}G_j$, $g(\oplus_{j\in J}G_j) = M \cap (\oplus_{j\in J}G_j)$ and the sequence*

$$0 \longrightarrow M \cap (\oplus_{j\in J}G_j) \hookrightarrow \oplus_{j\in J}G_j \xrightarrow{\overline{g}} g(\oplus_{j\in J}G_j) \longrightarrow 0$$

is split exact, where $\overline{g} = g \restriction \oplus_{j\in J}G_j$.

Proof. We are going to construct two chains of countable subsets of I, $\{I_n \mid n < \omega\}$ and $\{J_n \mid n < \omega\}$, such that $K \leq \oplus_{i\in I_0}G_i$, and a chain of subsets of $\mathrm{Hom}_R(M, G)$, $\{H_n \mid n < \omega\}$, satisfying, for each $n < \omega$:

(a) $I_n \leq J_n \leq I_{n+1}$;
(b) $M \cap (\oplus_{i\in I_n}G_i) \leq g(\oplus_{i\in J_n}G_i) \leq \oplus_{i\in I_{n+1}}G_i$,
(c) and for each $x \in g(\oplus_{i\in I_n}G_i)$ there exists $f \in H_n$ such that $gf(x) = x$ and $f(g(\oplus_{i\in I_n}G_i)) \leq \oplus_{i\in I_n}G_i$.

We shall make the construction recursively on n. For $n = 0$ let L_0 be a countable subset of I such that $K \leq \oplus_{i\in L_0}G_i$. Now apply Lemma 2.3 to get a countable subset I_0 of I containing L_0, and a subset H_0 of $\mathrm{Hom}_R(M, G)$ satisfying that for each $x \in g(\oplus_{i\in I_0}G_i)$ there exists $f \in H_0$ with $gf(x) = x$ and $f(g(\oplus_{i\in I_0}G_i)) \leq \oplus_{i\in I_0}G_i$. In particular, $g \restriction \oplus_{i\in I_0}G_i$ is locally split and, as $g(\oplus_{i\in I_0}G_i)$ is countably generated, $g \restriction \oplus_{i\in I_0}G_i$ is actually split. This means that $M \cap (\oplus_{i\in I_0}G_i)$ is a direct summand of $\oplus_{i\in I_0}G_i$ and, consequently, it is countably generated. Therefore, there exists a countable subset J_0 of I containing I_0 such that $M \cap (\oplus_{i\in I_0}G_i) \leq g(\oplus_{i\in J_0}G_i)$. This concludes case $n = 0$.

Now assume that we have constructed I_n and J_n for some $n < \omega$, and let us construct I_{n+1}, J_{n+1} and H_{n+1}. Since $g(\oplus_{i\in J_n}G_i)$ is countably generated there exists a countable subset L_{n+1} containing J_n such that $g(\oplus_{i\in J_n}G_i) \leq \oplus_{i\in L_{n+1}}G_i$. Then we can apply again Lemma 2.3 to find a countable subset I_{n+1} of I containing L_{n+1} and a subset H_{n+1} of $\mathrm{Hom}_R(M, G)$ satisfying that for each $x \in g(\oplus_{i\in I_{n+1}}G_i)$ there exists $f \in H_{n+1}$ with $gf(x) = x$ and $f(g(\oplus_{i\in I_{n+1}}G_i)) \leq \oplus_{i\in I_{n+1}}G_i$. In particular, $g \restriction \oplus_{i\in I_{n+1}}G_i$ is locally split and, as $g(\oplus_{i\in I_{n+1}}G_i)$ is countably generated, $g \restriction \oplus_{i\in I_{n+1}}G_i$ is actually split. This means that $M \cap (\oplus_{i\in I_{n+1}}G_i)$ is a direct summand of $\oplus_{i\in I_{n+1}}G_i$ and so it is countably generated. Consequently, there exists a countable subset J_{n+1} of I containing I_{n+1} such that $M \cap (\oplus_{i\in I_{n+1}}G_i) \leq g(\oplus_{i\in J_{n+1}}G_i)$. This concludes the construction.

Finally, let $J = \bigcup_{n<\omega} J_n = \bigcup_{n<\omega} I_n$. Then, by (b), $g(\oplus_{j\in J}G_j) = M \cap \oplus_{j\in J}G_j$ so that the sequence

$$0 \longrightarrow M \cap (\oplus_{j\in J}G_j) \hookrightarrow \oplus_{j\in J}G_j \xrightarrow{\overline{g}} g(\oplus_{j\in J}G_j) \longrightarrow 0$$

is exact, where $\overline{g} = g \restriction \oplus_{j\in J}G_j$. Moreover, as a consequence of (c), $g \restriction \oplus_{j\in J}G_j$ is locally split and, since $g(\oplus_{j\in J}G_j)$ is countably generated, it is actually split. This concludes the proof. □

We are now in position to prove Theorem 0.1.

Theorem 2.5. *Any short exact sequence*

$$0 \longrightarrow M \hookrightarrow G \xrightarrow{g} M \longrightarrow 0$$

in which g is locally split and G is a direct sum of countably generated modules is split.

Proof. Write $G = \oplus_{i\in I}C_i$ as a direct sum of countably generated modules and fix $\{x_\alpha \mid \alpha < \kappa\}$ a generating system of M for some cardinal κ. Our aim is to use Lemma 2.2 with the submodule M of G. In order to construct the chain of submodules $\{M_\alpha \mid \alpha < \kappa\}$ and $\{G_\alpha \mid \alpha < \kappa\}$ satisfying the hypotheses of this lemma, we are going to apply recursively Lemma 2.4. Actually, we are going to construct a continuous chain of subsets of I, $\{I_\alpha \mid \alpha < \kappa\}$, such that $G_\alpha = \oplus_{i\in I_\alpha} C_i$. Let us make the construction of the chain $\{M_\alpha \mid \alpha < \kappa\}$ of submodules of M, and of the chain $\{I_\alpha \mid \alpha < \kappa\}$ of subsets of I satisfying Lemma 2.2 and, for each $\alpha < \kappa$,

(a) $x_\alpha \in M_\alpha$;
(b) $M_\alpha = M \cap G_\alpha = g(G_\alpha)$,
(c) and M_α is a direct summand of G_α.

We shall proceed recursively on α.
For $\alpha = 0$, take I_0 the countable subset obtained in Lemma 2.4 for the countably generated submodule x_0R of M, and set $M_0 = M \cap (\oplus_{i\in I_0} C_i)$ and $G_0 = \oplus_{i\in I_0} C_i$. Notice that, as a consequence of Lemma 2.4, M_0 is a direct summand of G_0.
Let α be a nonzero ordinal such that we have made the construction for each ordinal smaller than α. If α is limit, simply take $M_\alpha = \bigcup_{\gamma<\alpha} M_\gamma$, $I_\alpha = \bigcup_{\gamma<\alpha} I_\gamma$ and $G_\alpha = \oplus_{i\in I_\alpha} C_i$.
Finally, assume that α is successor, say $\alpha = \mu + 1$. We can construct, using the snake lemma, the following commutative diagram with exact rows, in which f_μ is the inclusion, g_μ is the restriction of g, and p_μ and q_μ are projections:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_\mu & \overset{f_\mu}{\hookrightarrow} & G_\mu & \overset{g_\mu}{\longrightarrow} & M_\mu & \longrightarrow & 0 \\
 & & \downarrow i_\mu & & \downarrow j_\mu & & \downarrow i_\mu & & \\
0 & \longrightarrow & M & \hookrightarrow & G & \overset{g}{\longrightarrow} & M & \longrightarrow & 0 \\
 & & \downarrow p_\mu & & \downarrow q_\mu & & \downarrow p_\mu & & \\
0 & \longrightarrow & \frac{M}{M_\mu} & \overset{\hat{f}_\mu}{\longrightarrow} & \frac{G}{G_\mu} & \overset{\hat{g}_\mu}{\longrightarrow} & \frac{M}{M_\mu} & \longrightarrow & 0
\end{array}$$

Note that f_μ and j_μ are split, so that i_μ is split too. Moreover, by Lemma 1.1, $\hat{g}_\mu$ is locally split.
Now, since $\hat{f}_\mu(x+M_\mu) = x+G_\mu$ for each $x \in M$, $\operatorname{Im} \hat{f}_\mu = \frac{M+G_\mu}{G_\mu}$ and, consequently, we have a short exact sequence

$$0 \longrightarrow \frac{M+G_\mu}{G_\mu} \hookrightarrow \frac{G}{G_\mu} \xrightarrow{\hat{f}_\mu \hat{g}_\mu} \frac{M+G_\mu}{G_\mu} \longrightarrow 0$$

in which $\hat{f}_\mu \hat{g}_\mu$ is a locally split epimorphism and $\frac{G}{G_\mu}$ is the direct sum of the family of countably generated modules $\left\{\frac{C_i+G_\mu}{G_\mu} \mid i \in I - I_\mu\right\}$. Then we can apply Lemma 2.4 to the countably generated submodule $(x_{\mu+1} + G_\mu)R$ of $\frac{M+G_\mu}{G_\mu}$, to get a countable subset J_μ of $I - I_\mu$ such that $x_{\mu+1} + G_\mu \in \bigoplus_{i\in J_\mu} \frac{C_i+G_\mu}{G_\mu}$,

$$\frac{M+G_\mu}{G_\mu} \bigcap \left(\bigoplus_{i\in J_\mu} \frac{C_i+G_\mu}{G_\mu}\right) = \hat{f}_\mu \hat{g}_\mu \left(\bigoplus_{i\in J_\mu} \frac{C_i+G_\mu}{G_\mu}\right)$$

and the morphism $\hat{f}_\mu \hat{g}_\mu \restriction \left(\bigoplus_{i\in J_\mu} \frac{C_i+G_\mu}{G_\mu}\right)$ from $\bigoplus_{i\in J_\mu} \frac{C_i+G_\mu}{G_\mu}$ to $\hat{f}_\mu \hat{g}_\mu \left(\bigoplus_{i\in J_\mu} \frac{C_i+G_\mu}{G_\mu}\right)$ is split. Set $I_{\mu+1} = I_\mu \cup J_\mu$, $G_{\mu+1} = \oplus_{i\in I_{\mu+1}} C_i$ and $M_{\mu+1} = g(G_{\mu+1})$. It is easy to show that $g(G_{\mu+1}) = M \cap G_{\mu+1}$ and that $x_{\mu+1} \in \oplus_{i\in I_{\mu+1}} C_i$.
Now we see that $\frac{M_{\mu+1}+G_\mu}{G_\mu}$ is a direct summand of $\frac{G_{\mu+1}}{G_\mu}$ and $M_{\mu+1}$ is a direct summand of $G_{\mu+1}$ to finish the proof. Applying again the snake lemma, we get a

commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_\mu & \xhookrightarrow{f_\mu} & G_\mu & \xrightarrow{g_\mu} & M_\mu & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & M_{\mu+1} & \xhookrightarrow{f_{\mu+1}} & G_{\mu+1} & \xrightarrow{g_{\mu+1}} & M_{\mu+1} & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & \frac{M_{\mu+1}}{M_\mu} & \xrightarrow{\tilde{f}_\mu} & \frac{G_{\mu+1}}{G_\mu} & \xrightarrow{\tilde{g}_\mu} & \frac{M_{\mu+1}}{M_\mu} & \longrightarrow & 0
\end{array}$$

Since there exists a commutative diagram

$$\begin{array}{ccc}
\bigoplus_{i \in J_\mu} \frac{C_i + G_\mu}{G_\mu} & \xrightarrow{\hat{f}_\mu \hat{g}_\mu} & \hat{f}_\mu \hat{g}_\mu \left(\bigoplus_{i \in J_\mu} \frac{C_i + G_\mu}{G_\mu} \right) \\
\big\downarrow \cong & & \big\downarrow \cong \\
\frac{G_{\mu+1}}{G_\mu} & \xrightarrow{\tilde{g}_\mu} & \frac{M_{\mu+1}}{M_\mu}
\end{array}$$

we conclude that $\tilde{g}_\mu$ is split. Then, by Lemma 2.1, $f_{\mu+1}$ is split too and $M_{\mu+1}$ is a direct summand of $G_{\mu+1}$. Moreover, $\operatorname{Im} \tilde{f}_\mu$ is a direct summand of $\frac{G_{\mu+1}}{G_\mu}$. But this image is precisely $\frac{M_{\mu+1}+G_\mu}{G_\mu}$. This concludes the construction. □

We apply the previous result to get that some pure periodic modules are trivial.

Corollary 2.6. *Let $\mathcal{P}$ be a class of finitely presented modules. Then each pure* $\operatorname{Add}(\mathcal{P})$*-periodic module belongs to* $\operatorname{Add}(\mathcal{P})$*. In particular we get the following:*

(1) [Sim02, Theorem 1.3] *Each pure* PProj*-periodic module is pure projective.*
(2) [BG00, Theorem 2.5] *Each pure* Proj*-periodic module is projective (equivalently, each flat* Proj*-periodic module is projective).*

Proof. Let M be a pure $\operatorname{Add}(\mathcal{P})$-periodic module. Then there exists a pure exact sequence

$$0 \longrightarrow M \longrightarrow P \xrightarrow{g} M \longrightarrow 0$$

with $P \in \operatorname{Add}(\mathcal{P})$. By Lemma 1.2 (4), $M \in \operatorname{G}(\mathcal{P})$, so that there exists a locally split epimorphism $f \colon Q \to M$ with $Q \in \operatorname{Add}(\mathcal{P})$. Since Q is pure projective by [Wis88, 33.6], there exists $h \colon Q \to P$ such that $gh = f$. Applying Lemma 1.1 to the commutative diagram

$$\begin{array}{ccc}
Q & = & Q \\
\big\downarrow h & & \big\downarrow f \\
P & \xrightarrow{g} & M
\end{array}$$

we conclude that g is locally split. By Theorem 2.5 g is actually split and $M \in \operatorname{Add}(\mathcal{P})$.
Now, to get (1) we simply note that PProj is equal to $\operatorname{Add}(\mathcal{P})$ for the class $\mathcal{P}$ of all finitely presented modules. Finally to get (2), if M is a pure Proj-periodic module then it is pure projective by (1). Since it is flat too, we conclude that M is projective. □

It is easy to see that the class of projective modules is not closed under periodic modules. Actually, there exist Proj-periodic modules that are not pure projective.

Example 1. *Let R be a QF ring which is not right pure semisimple. Then we can apply Corollary 1.5 with $\mathcal{C}$ the class* Proj *and $\mathcal{D}$ the class* PProj *to construct a* Proj*-periodic module M which is not pure projective. In particular, M is not trivial. Since* Flat = Proj *in this case, this also gives us an example of a* Flat*-periodic module which is not trivial.*

Another consequence of Corollary 2.6 is that a flat and strongly Gorenstein projective module is projective.

Corollary 2.7. *Any flat and strongly Gorenstein projective module is projective.*

A natural question arises:

Question 2.8. *Is every flat Gorenstein projective module projective?*

Using Corollary 2.6 we can give a partial answer to this question.

Proposition 2.9. *Suppose that $\mathrm{FPD}_{\mathrm{Flat}}(R) < \infty$. If a cycle in an acyclic complex of projective modules is flat, then all cycles are projective modules. In particular, every flat Gorenstein projective module is projective.*

Proof. Let P be an acyclic complex of projectives and suppose that $M = Z_0(P)$ is flat. By [Wis88, 36.6], $Z_n(P)$ is flat for each $n > 0$. Now let $d = \mathrm{FPD}_{\mathrm{Flat}}(R)$ and $n < 0$. Since $Z_{n-d-1}(P)$ has finite flat dimension, it has flat dimension less than or equal to d. As $Z_n(P)$ is a syzygy of M, it has to be flat. The conclusion is that $Z_n(P)$ is flat for each $n \in \mathbb{Z}$. Now each $Z_n(P)$ is a direct summand of $\bigoplus_{n\in\mathbb{Z}} Z_n(P)$, which is a flat Proj-periodic module. By Corollary 2.6(2), the module $\bigoplus_{n\in\mathbb{Z}} Z_n(P)$ is projective, and so is $Z_n(P)$, $n \in \mathbb{Z}$. □

3. Periodic modules with respect to hereditary cotorsion pairs

In this section we study the dual notion of flat Proj-periodic modules: Inj-periodic absolutely pure modules. In order to do this, we consider periodic modules with respect to the right class of a hereditary cotorsion pair. Let us start with the following two lemmas, which are useful for computing the Ext functors with periodic modules.

Lemma 3.1. *Let $(\mathcal{A}, \mathcal{B})$ be a hereditary cotorsion pair in* Mod-R *and let M be a $\mathcal{B}$-periodic module. Then, for every module $L \in \mathcal{A}$ and nonzero natural number n, $\mathrm{Ext}^n_R(L, M) \cong \mathrm{Ext}^1_R(L, M)$.*

Proof. Let $L \in \mathcal{A}$. Since M is $\mathcal{B}$-periodic, there exists an exact sequence $0 \to M \to B \to M \to 0$ with $B \in \mathcal{B}$. The usual long exact sequence of cohomology attained to this short exact sequence, gives us an exact sequence

$$\cdots \to \mathrm{Ext}^n_R(L, B) \to \mathrm{Ext}^n_R(L, M) \to \mathrm{Ext}^{n+1}_R(L, M) \to \mathrm{Ext}^{n+1}_R(L, B) \to \cdots .$$

Since $(\mathcal{A}, \mathcal{B})$ is hereditary we have that $\mathrm{Ext}^i_R(L, B) = 0$, for every $i \geq 1$. Therefore, it follows that $\mathrm{Ext}^n_R(L, M) \cong \mathrm{Ext}^{n+1}_R(L, M)$, for every $n \geq 1$. So we get our claim. □

Symmetrically we have:

Lemma 3.2. *Let $(\mathcal{A}, \mathcal{B})$ be a hereditary cotorsion pair in* Mod-R *and let M be an $\mathcal{A}$-periodic module. Then, for every module $T \in \mathcal{B}$ and nonzero natural number n, $\mathrm{Ext}^n_R(M, T) \cong \mathrm{Ext}^1_R(M, T)$.*

We shall use the following relative version of the 2-out-of-3 property for a class of modules.

Definition 3.3. Let $\mathcal{C}$, $\mathcal{D}$ be two classes of modules. We say that $\mathcal{D}$ has the 2-out-of-3 property with respect to $\mathcal{C}$ if the following holds: for every exact sequence $0 \to C_1 \to C_2 \to C_3 \to 0$ in $\mathcal{C}$, if two of the C_i's are in $\mathcal{D}$ then the the third term is in $\mathcal{D}$ too.

Now we prove that the left orthogonal of a module which is periodic with respect to the right class of a hereditary cotorsion pair has the relative 2-out-of-3 property.

Lemma 3.4. *Let $(\mathcal{A}, \mathcal{B})$ be a hereditary cotorsion pair in* Mod-R *and let M be a $\mathcal{B}$-periodic module. Then ${}^{\perp}M$ has the* 2*-out-of-*3 *property with respect to $\mathcal{A}$.*

Proof. Let $0 \to X \to Y \to Z \to 0$ be an exact sequence in $\mathcal{A}$. We have the long exact sequence of cohomology

$$\begin{aligned} 0 \to \operatorname{Hom}_R(Z, M) &\to \operatorname{Hom}_R(Y, M) \to \operatorname{Hom}_R(X, M) \to \\ \to \operatorname{Ext}^1_R(Z, M) &\to \operatorname{Ext}^1_R(Y, M) \to \operatorname{Ext}^1_R(X, M) \to \\ \to \operatorname{Ext}^2_R(Z, M) &\to \operatorname{Ext}^2_R(Y, M) \to \operatorname{Ext}^2_R(X, M) \to \cdots . \end{aligned}$$

Now, if $X, Z \in {}^{\perp}M$ the exact sequence

$$\operatorname{Ext}^1_R(Z, M) \to \operatorname{Ext}^1_R(Y, M) \to \operatorname{Ext}^1_R(X, M)$$

gives us that $Y \in {}^{\perp}M$. If $X, Y \in {}^{\perp}M$ the sequence

$$\operatorname{Ext}^1_R(X, M) \to \operatorname{Ext}^2_R(Z, M) \to \operatorname{Ext}^2_R(Y, M)$$

together with Lemma 3.1, gives that $Z \in {}^{\perp}M$. Finally, if $Y, Z \in {}^{\perp}M$, the sequence

$$\operatorname{Ext}^1_R(Y, M) \to \operatorname{Ext}^1_R(X, M) \to \operatorname{Ext}^2_R(Z, M)$$

together with Lemma 3.1, gives that $X \in {}^{\perp}M$. □

We have also the symmetric statement.

Lemma 3.5. *Let $(\mathcal{A}, \mathcal{B})$ be a hereditary cotorsion pair in* Mod-R *and let M be an $\mathcal{A}$-periodic module. Then $M^{\perp}$ has the* 2*-out-of-*3 *property with respect to $\mathcal{B}$.*

We will now prove that under the hypothesis of Lemma 3.4 and assuming that $\mathcal{A}$ is also closed under pure epimorphic images, the class ${}^{\perp}M \cap \mathcal{A}$ is closed under direct limits. We start by showing that ${}^{\perp}M \cap \mathcal{A}$ is closed under well ordered direct unions of pure submodules.

Lemma 3.6. *Let $(\mathcal{A}, \mathcal{B})$ be a hereditary cotorsion pair in* Mod-R *and let M be a $\mathcal{B}$-periodic module. Assume that $\mathcal{A}$ is closed under pure epimorphic images. Let K be the direct union of a chain $\{K_\alpha \mid \alpha < \lambda\}$ of pure submodules with $K_\alpha \in {}^{\perp}M \cap \mathcal{A}$, for each $\alpha < \lambda$. Then $K \in {}^{\perp}M \cap \mathcal{A}$.*

Proof. If λ is successor, say $\lambda = \mu + 1$, then $K = K_\mu$ and there is nothing to prove. Suppose that λ is a limit ordinal. Consider the *continuous* chain $\{L_\alpha \mid \alpha < \lambda\}$ of submodules of K given by:

- $L_\alpha = K_\alpha$ if $\alpha < \omega$.
- $L_\omega = \bigcup_{n<\omega} L_n$.
- $L_{\alpha+1} := K_\alpha$ for each $\alpha \geq \omega$.
- $L_\alpha := \bigcup_{\gamma<\alpha} L_\gamma$ if α is a limit ordinal greater than ω.

Since $\mathcal{A}$ is closed under direct sums and pure epimorphic images, $\mathcal{A}$ is closed under direct limits by [Wis88, 33.9]. Therefore, $L_\alpha \in \mathcal{A}$, for each $\alpha < \lambda$. It is also clear that K is the direct union of the continuous chain $\{L_\alpha \mid \alpha < \lambda\}$. Since the class of pure submodules of a given module is closed under direct unions [Wis88, 33.8], we deduce that L_α is a pure submodule of K, for each $\alpha < \lambda$. Let us call $L_\lambda = K$. We prove now by induction that $L_\alpha \in {}^{\perp}M$, for every $\alpha \leq \lambda$. If α is successor, $K_\alpha = L_\alpha$

which belongs to ${}^{\perp}M$ by hypothesis. Assume that α is a limit ordinal. For each $\gamma < \alpha$ we have the exact sequence

$$0 \to L_\gamma \to L_{\gamma+1} \to \frac{L_{\gamma+1}}{L_\gamma} \to 0 \tag{1}$$

in which L_γ is a pure submodule of K so that, by [Wis88, 33.3], it is a pure submodule of $L_{\gamma+1}$. Since $\mathcal{A}$ is closed under pure epimorphic images, we infer that $\frac{L_{\gamma+1}}{L_\gamma} \in \mathcal{A}$. By our induction hypothesis L_γ and $L_{\gamma+1}$ belong to ${}^{\perp}M$. Thus, the exact sequence (1) has all its terms in $\mathcal{A}$ and therefore by Lemma 3.4, the quotient module $\frac{L_{\gamma+1}}{L_\gamma}$ belongs to ${}^{\perp}M$. Finally by Eklof Lemma ([GT12, Lemma 6.2]), we conclude that L_α lies in ${}^{\perp}M$. □

Theorem 3.7. *Let $(\mathcal{A}, \mathcal{B})$ be a hereditary cotorsion pair in* Mod-R *and let M be a $\mathcal{B}$-periodic module. Assume that $\mathcal{A}$ is closed under pure epimorphic images. If $\{X_i; f_{ji} \mid i \leq j \in I\}$ is a direct system of modules in ${}^{\perp}M \cap \mathcal{A}$, then $\varinjlim_{i\in I} X_i$ is in ${}^{\perp}M$.*

Proof. By [AR94, Corollary 1.7] we can assume that I is an ordinal λ so that the direct system is a λ-sequence of the form $\{X_\alpha; f_{\beta\alpha} \mid \alpha \leq \beta < \lambda\}$, that is a sequence indexed by the ordinal λ and such that for each limit ordinal $\mu < \lambda$, the subsystem $\{X_\alpha; f_{\beta\alpha} \mid \alpha \leq \beta < \mu\}$ has colimit X_μ.
If $\lambda = \omega$, then the well known presentation of a countable direct limit (e.g. [GT12, Lemma 2.12]) gives a short exact sequence

$$0 \to \bigoplus_{\alpha<\omega} X_\alpha \to \bigoplus_{\alpha<\omega} X_\alpha \to \varinjlim_{\alpha<\omega} X_\alpha \to 0.$$

This is an exact sequence with all its terms in $\mathcal{A}$ and where the first two terms belong to ${}^{\perp}M$. Hence by Lemma 3.4, we get that $\varinjlim_{\alpha<\omega} X_\alpha \in {}^{\perp}M$.
Now for an arbitrary limit ordinal λ, let

$$0 \to K \to \bigoplus_{\alpha<\lambda} X_\alpha \to \varinjlim_{\alpha<\lambda} X_\alpha \to 0 \tag{2}$$

be the canonical exact sequence associated to the direct limit. Since $(\mathcal{A}, \mathcal{B})$ is hereditary, we get that $K \in \mathcal{A}$. Then, the exact sequence (2) has all its terms in $\mathcal{A}$, and clearly $\bigoplus_{\alpha<\lambda} X_\alpha \in^{\perp} M$. So in view of Lemma 3.4, to get our claim we only need to show that $K \in {}^{\perp}M$. We use Lemma 3.6 to prove this.
As in the proof of [GPGA00, Lemma 2.1] we have that K is the direct union of a chain $\{K_\alpha \mid \alpha < \lambda\}$, where each K_α is a direct summand of $\bigoplus_{\alpha<\lambda} X_\alpha$. We need to check that the system $\{K_\alpha \mid \alpha < \lambda\}$ fulfills the requirements of Lemma 3.6, i.e.

- $K_\alpha \in {}^{\perp}M \cap \mathcal{A}$, for each $\alpha < \lambda$.
- K_α is pure in K, for each $\alpha < \lambda$.

Since both classes $\mathcal{A}$ and ${}^{\perp}M$ are closed under direct summands, we get that $K_\alpha \in {}^{\perp}M \cap \mathcal{A}$, for each $\alpha < \lambda$. For the second condition, note that K_α actually is a direct summand of K. □

We illustrate some consequences of the previous result.

Proposition 3.8. *The following hold true:*

(1) An Inj*-periodic absolutely pure module is injective.*
(2) Let $(\mathcal{A}, \mathcal{B})$ be a hereditary cotorsion pair in Mod-R. *Assume that $\mathcal{A}$ is closed under pure epimorphic images. Then every $\mathcal{B}$-periodic module is cotorsion. In particular, every* Cot*-periodic module is trivial, i.e. cotorsion (and so every* Inj*-periodic module is cotorsion).*

(3) *Assume that each finitely generated right ideal has finite flat dimension. Then each* Inj*-periodic module is trivial.*

Proof. (1) Let M be an Inj-periodic absolutely pure module. Then $\operatorname{Ext}^1_R(F, M) = 0$ for every finitely presented module. By Theorem 3.7, $\operatorname{Ext}^1_R(X, M) = 0$ for every module X, hence M is injective.
(2) Let M be a $\mathcal{B}$-periodic module. Since Proj $\subseteq \mathcal{A}$, Theorem 3.7 implies that Flat $\subseteq {}^{\perp}M$, hence M is cotorsion.
(3) Let M be an Inj-periodic module. We only have to show that $\operatorname{Ext}^1_R\left(\frac{R}{I}, M\right) = 0$ for each right ideal I of R. Let I be a right ideal and write $I = \bigcup_{\gamma\in\Gamma} I_\gamma$ as a direct union of finitely generated right ideals. By (2) M is cotorsion so, for each $\gamma \in \Gamma$, there exists a nonzero natural number n_γ such that $\operatorname{Ext}^{n_\gamma}_R(I_\gamma, M) = 0$. By Lemma 3.1 $I_\gamma \in {}^{\perp}M$, and by Theorem 3.7 we get that actually I belongs to ${}^{\perp}M$. But then, again by Lemma 3.1, we get that $\operatorname{Ext}^2_R\left(\frac{R}{I}, M\right) \cong \operatorname{Ext}^1_R\left(\frac{R}{I}, M\right) = 0$. □

The same argument used in Example 1 can be used to see that there exist Inj-periodic modules which are not pure injective.

Example 2. *Let R be a QF ring which is not right pure semisimple. We can apply Corollary 1.5 with $\mathcal{C}$ the class* Inj *and $\mathcal{D}$ the class* PInj *to construct an* Inj*-periodic module M which is not pure injective. In particular, M is a non-trivial* Inj*-periodic module.*

Remark 3.9. Note that Cot does not satisfy the hypothesis of Corollary 1.5, so that we cannot use that result to construct a non-trivial Cot-periodic module. This is because if Cot is generating, then R_R is Σ-cotorsion (that is, $R^{(I)}$ is a cotorsion right module for each set I) and, by [GAH05, Corollary 10], R is right perfect. This means that Cot = Mod-R.

Now, regarding Proposition 3.8(3), we show an example of a ring with infinite weak global dimension but such that each finitely generated right ideal has finite flat dimension.

Example 3. *Let k be a field and let $k[x_1, \ldots, x_n]$ be the polynomial ring in n variables over k. The ring $R = \varinjlim_{n<\omega} k[x_1, \ldots, x_n]$ has infinite weak global dimension but each finitely generated ideal of R has finite projective dimension (see Glaz,* [Gla89, p. 202]*).*

We can apply Proposition 3.8 to Gorenstein injective modules.

Corollary 3.10. (1) *Any absolutely pure strongly Gorenstein injective module is injective.*
(2) *Assume that each finitely generated right ideal has finite flat dimension. Then each Gorenstein injective module is injective. In particular, if a ring has finite weak global dimension, then each Gorenstein injective module is injective.*
(3) *Assume that each finitely generated right ideal has finite injective dimension. Then each Gorenstein injective module is injective.*

Proof. Note that every Gorenstein injective module is a summand of a strongly Gorenstein injective module.

(1) follows from Proposition 3.8 (1) and (2) follows by Proposition 3.8 (3) and the above remark.

(3) Let M be a strongly Gorenstein injective module. Since M is in particular Gorenstein injective, ${}^{\perp}M$ contains the modules of finite injective dimension. Then, arguing as in the proof of Proposition 3.8 (3) we conclude that M is injective, thus so is every Gorenstein injective module. □

We ask the following:

Question 3.11. *Is every absolutely pure Gorenstein injective module injective?*

As in the case of flat Gorenstein projective modules, we can give a partial answer to this question in the following result. The proof is similar to the proof of Proposition 2.9, but using Corollary 3.8(1).

Proposition 3.12. *Suppose that R is right coherent and $\mathrm{FID}_{\mathrm{Abs}}(R) < \infty$. If a cycle in an acyclic complex of injective modules is absolutely pure, then all cycles are injective modules. In particular, every absolutely pure Gorenstein injective module is injective.*

4. Acyclic complexes and $\mathcal{C}$-periodic modules

This section is devoted to exploit the power of periodic modules in shortening and simplifying recent proofs of some meaningful results in homotopy categories.

We apply the previous results to classes of complexes of R-modules. The following result has the rank of Theorem, because of its relevant statements and its subsequent consequences. But the proof is an easy and immediate application of propositions 1.4 and 3.8.

Theorem 4.1. *(1) Every acyclic complex of injective modules with absolutely pure cycles is contractible, i.e.* $\mathrm{dw}\,\mathrm{Inj} \cap \widetilde{\mathrm{Abs}} = \widetilde{\mathrm{Inj}}$.

(2) Every acyclic complex of cotorsion modules has cotorsion cycles, that is, $\mathrm{ex}\,\mathrm{Cot} = \widetilde{\mathrm{Cot}}$. *In particular, every acyclic complex of flat cotorsion modules with flat cycles is contractible, i.e.* $\widetilde{\mathrm{Flat}} \cap \mathrm{dw}\,\mathrm{Cot} = \widetilde{\mathrm{Flat} \cap \mathrm{Cot}}$.

(3) Every pure acyclic complex (i.e. a concatenation of pure short exact sequences) of pure projective modules is contractible. As a consequence, if there exists a pure exact sequence

$$0 \to M \xrightarrow{f} P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \xrightarrow{g} M \to 0$$

in Mod-R*, where the modules $P_0, \ldots, P_n$ are pure projective, then M is pure projective.*

Proof. The statements (1) and (2) are direct consequences of propositions 1.4 and 3.8. The assertion (3) follows from Corollary 2.6(1) and (the pure version of) Proposition 1.4. Finally, the second part in statement (3) follows because from that pure exact sequence, we get the following pure acyclic complex with pure projective components

$$\cdots \to P_0 \xrightarrow{fg} P_n \to \cdots \to P_0 \xrightarrow{fg} P_n \to \cdots$$

□

Remark 4.2. The assertions (1) and (3) were already proved by Šťovíček [Šťo14, Corollary 5.5] (see also Emmanouil [Emm16, Corollary 3.7] for (3)). The application stated in Theorem 4.1 (3) was firstly shown by Simson in [Sim02, Theorem 1.3]. We just want to emphasize how these statements easily follow from the corresponding properties of periodic modules. What seems to be unknown is the remarkable statement (2). Notice that, as a consequence of (2), we get that every acyclic complex of injectives has cotorsion cycles. This was also shown by Šťovíček [Šťo14, Corollary 5.9].

Theorem 4.3. *Let C be a complex of cotorsion modules. Then every chain map $f\colon F \to C$, where $F \in \widetilde{\mathrm{Flat}}$ is null-homotopic. That is, the classes* $\mathrm{dg}\,\mathrm{Cot}$ *and* $\mathrm{dw}\,\mathrm{Cot}$ *coincide.*

Proof. Let us consider the complete hereditary cotorsion pair $(\widetilde{\mathrm{Flat}}, \mathrm{dg\,Cot})$ in $\mathrm{Ch}(R)$ ([Gil04, Corollary 4.10]). Then, there is an exact sequence

$$0 \to C \to D \to G \to 0,$$

with $D \in \mathrm{dg\,Cot}$, $G \in \widetilde{\mathrm{Flat}}$ and the sequence splits on each degree. Since D_n is cotorsion, it follows that G_n is cotorsion for each $n \in \mathbb{Z}$. Therefore $G \in \widetilde{\mathrm{Flat}} \cap \mathrm{dw\,Cot}$. Now, by Theorem 4.1(2) we get that G is contractible. Therefore C and D are homotopically equivalent, and so $C \in \mathrm{dg\,Cot}$. □

4.1. **Application to finitely accessible additive categories.** Throughout this section $\mathcal{G}$ will denote a *finitely accessible* additive category. That is, $\mathcal{G}$ has all direct limits, the class of finitely presented objects is skeletally small and every object in $\mathcal{G}$ is a direct limit of finitely presented objects. A well-known Representation Theorem (see [MP89, Corollary 2.1.9], [CB94, Theorem 1.4] and [AR94, Theorem 2.26] and the remark that follow) states the following

Theorem 4.4. *Every finitely accessible additive category $\mathcal{G}$ is equivalent to the full subcategory* $\mathrm{Flat}(A)$ *of the category* Mod-A *of unital right A-modules consisting of flat right A-modules where A is the functor ring of $\mathcal{G}$ (that is, a ring with enough idempotents). This equivalence gives a 1-1 correspondence between pure exact sequences in $\mathcal{G}$ and exact sequences in* $\mathrm{Flat}(A)$.

In other words, $\mathcal{G}$ with its pure exact structure $\mathcal{E}$ is equivalent to $\mathrm{Flat}(A)$ with its canonical exact structure inherited from Mod-A. In particular, the equivalence takes injective objects in $(\mathcal{G}; \mathcal{E})$ (i.e. pure injectives) to injective objects in $\mathrm{Flat}(A)$ (cotorsion flat modules). Thus, from the preceding theorem and Proposition 3.8(2), we immediately get the following.

Corollary 4.5. *Every pure* PInj*-periodic object of $\mathcal{G}$ is trivial (i.e. pure injective).*

Proof. Let M be a pure PInj-periodic module and $0 \to M \to E \to M \to 0$ a pure exact sequence with E pure injective. By using the Representation Theorem, we get an exact sequence in Mod-A, $0 \to \overline{M} \to \overline{E} \to \overline{M} \to 0$, with $\overline{M}$ flat and $\overline{E}$ flat cotorsion. But then, by Theorem 4.1(2) (whose argument is still valid for a category of unital modules over a ring with enough idempotents) the sequence splits, so $\overline{M}$ is flat cotorsion and therefore M is pure injective. □

The equivalence between $(\mathcal{G}; \mathcal{E})$ and $\mathrm{Flat}(A)$ takes pure acyclic complexes in $\mathrm{Ch}(\mathcal{G})$ (i.e. concatenation of conflations in $(\mathcal{G}; \mathcal{E})$) to acyclic complexes in $\mathrm{Ch}(A)$ with flat cycles. We will denote by $\mathrm{Ch}(\mathcal{G})_{\text{dw-pur}}$ the exact category of unbounded chain complexes $\mathrm{Ch}(\mathcal{G})$ with the degreewise pure exact structure.

The following result (Šťovíček [Šťo14, Theorem 5.4]) can also be easily proved by using the Representation Theorem for finitely accessible additive categories and Theorem 4.3 (which still holds for unital modules over a ring with enough idempotents).

Corollary 4.6. *Let $\widetilde{\mathrm{PurAc}}$ be the class of pure acyclic chain complexes in* $\mathrm{Ch}(\mathcal{G})$. *The pair of classes* $(\widetilde{\mathrm{PurAc}}, \mathrm{dw\,PInj})$ *is a complete hereditary cotorsion pair in* $\mathrm{Ch}(\mathcal{G})_{\text{dw-pur}}$.

Proof. We have the complete hereditary cotorsion pair $(\widetilde{\mathrm{Flat}}, \mathrm{dg\,Cot})$ in $\mathrm{Ch}(A)$. Then we have the induced complete hereditary cotorsion pair $(\widetilde{\mathrm{Flat}}, \mathrm{dg\,Cot} \cap \mathrm{dw\,Flat})$ in $\mathrm{Ch}(\mathrm{Flat})$ (see for instance [Gil16, Corollary 7.5]). Now, by Theorem 4.3, $\mathrm{dg\,Cot} = \mathrm{dw\,Cot}$. Therefore the previous cotorsion pair is $(\widetilde{\mathrm{Flat}}, \mathrm{dw\,Cot} \cap \mathrm{dw\,Flat})$. Now we use the Representation Theorem to get the complete hereditary cotorsion pair $(\widetilde{\mathrm{PurAc}}, \mathrm{dw\,PInj})$ in $\mathrm{Ch}(\mathcal{G})_{\text{dw-pur}}$. □

Remark 4.7. For any complex E of pure injective objects in $\mathcal{G}$ and any complex A, we have that $\mathrm{Ext}^1_{\mathrm{Ch}(\mathcal{G})_{\text{dw-pur}}}(A, E) = 0$ if and only if every map $A \to \Sigma E$ is null-homotopic. Then it follows from Corollary 4.6 that a complex A is pure acyclic if and only if any map $A \to I$ is null-homotopic, where I is a complex of pure injective objects in $\mathcal{G}$.

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