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Aggregation and signature based comparisons of multi-state systems via decompositions of fuzzy measures

Jorge Navarro,[*,†] and Fabio Spizzichino[‡]

April 12, 2019

Abstract

In the reliability literature, several results have been presented to compare binary (two states) systems. Often, such results are based on *copula-based extensions of fuzzy measures*, where a fuzzy measure describes the structure of a system and a copula describes the stochastic dependence among the lifetimes of its components. Other similar results have been obtained in terms of the concept of *signature*. Here, we extend all those results to multi-state systems made up from binary components by suitably constructing corresponding *mixed binary systems*. For such a construction, we show how any fuzzy measure can be decomposed as a convex combination of $\{0,1\}$-valued fuzzy measures and how such a decomposition extends to the corresponding aggregation function. For a mixed system we can furthermore consider its signature and so we can also define a signature for the multi-state system. For mixed systems associated to different multi-state systems, we can thus obtain different comparison results, which can be translated into the corresponding comparisons for the parent multi-state systems. Stochastic comparisons are obtained for the discrete random variables which represent the states of two systems at time t, as well. The arguments in the paper will be illustrated by means of examples and related remarks.

Keywords: Aggregation functions · Fuzzy measures · Mixed systems · Decompositions of fuzzy measures · Copula.

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***Declaration of Interest Statement**

Declaration of interests

☒ The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

☐The authors declare the following financial interests/personal relationships which may be considered as potential competing interests:

*Revision Note

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Dear Editors,

I send you the second revised version of the paper entitled,

Aggregation and signature based comparisons of multi-state systems via decompositions of fuzzy measures;
By J. Navarro and F. Spizzichino;
Ref.: Ms. No. FSS-D-19-00181-R1;

that we have rewritten following the minor suggestions made by the reviewers.

We also enclose in the following pages point-by-point explanations on how and where the reviewers' comments have been incorporated in the text.

The main changes in the new version are printed in blue.

Yours sincerely,

Jorge Navarro

ANSWERS TO REVIEWER 2

First of all, we want to thank you for your positive report. We have incorporated your suggestion.

We write below your comment (in blue) and our answer (in black) explaining how they have been incorporated into the paper.

The main changes in the paper are written in blue.

The paper has been improved satisfactorily, so that the present version is suitable for publication.

Just a correction in Reference [23]: discrte $\rightarrow$ discrete

Fixed. Thanks!

ANSWERS TO REVIEWER 3

First of all, we want to thank you for your positive report. We have incorporated all your suggestions.

We write below your comments (in blue) and our answers (in black) explaining how they have been incorporated into the paper.

The main changes in the paper are written in blue.

You took all the comments into account and took this opportunity to improve your (already) nice paper. In my opinion, the paper is now suitable for publication.

I have two minor comments :

1) On page 9/l26 Replace "T_1 is not apt" by "Random variable T_1 is not apt", in order not to start a sentence with a mathematical symbol.

Changed.

2) I mentioned what is now in Remark 3.13 and Remark 3.14 in order to simplify the computations. For instance, in Example 5.2, the Boolean function $\max(y_1, ..., y_n)$ can be written $1 - (1 - y_1)...(1 - y_n)$. It is not difficult to expand this product as $\sum_{A\subset[n],A\neq\emptyset}(-1)^{|A|+1}\prod_{i\in A} y_i)$. Introducing random states and taking expectation, we obtain the formula that appears on page 21/l46. Of course this formula is known. But this computation can be done in the same way for Boolean functions and also produces in the same way the formula on page 22/l4. This computation seemed easier to me, but I understand now that simplicity depends on the reader/author and his favorite way of handling problems. Mine is clearly on the side of Boolean functions. So you can leave it as it is now.

We have decided to include both options in Example 5.2 (see page 21). Thus every reader can choose the preferred method.

Thank you for your useful suggestions!

Aggregation and signature based comparisons of multi-state systems via decompositions of fuzzy measures

Jorge Navarro,[*,†] and Fabio Spizzichino[‡]

October 25, 2019

Abstract

In the reliability literature, several results have been presented to compare binary (two states) systems. Often, such results are based on *copula-based extensions of fuzzy measures*, where a fuzzy measure describes the structure of a system and a copula describes the stochastic dependence among the lifetimes of its components. Other similar results have been obtained in terms of the concept of *signature*. Here, we extend all those results to multi-state systems made up from binary components by suitably constructing corresponding *mixed binary systems*. For such a construction, we show how any fuzzy measure can be decomposed as a convex combination of $\{0,1\}$-valued fuzzy measures and how such a decomposition extends to the corresponding aggregation function. For a mixed system we can furthermore consider its signature and so we can also define a signature for the multi-state system. For mixed systems associated to different multi-state systems, we can thus obtain different comparison results, which can be translated into the corresponding comparisons for the parent multi-state systems. Stochastic comparisons are obtained for the discrete random variables which represent the states of two systems at time t, as well. The arguments in the paper will be illustrated by means of examples and related remarks.

Keywords: Aggregation functions · Fuzzy measures · Mixed systems · Decompositions of fuzzy measures · Copula.

1 Introduction

The notion of aggregation function is important in Mathematics and in several other disciplines. It can have a unifying role in connecting studies of different types. Some recent papers have studied the connections between aggregation functions, coherent (engineering) systems and fuzzy measures, see [9, 13, 14, 28], the references therein and the basic monograph in [24]. These relationships are very interesting and can be used to simplify and improve expressions and proofs in these different research areas. In particular, it was proved in [9, 20] that the *reliability function*

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$R(t)$ of a coherent binary system can be obtained as an aggregation function applied to the respective reliability functions of the single components. This aggregation function is determined by the *survival copula* of the components' lifetimes and the *structure function* φ of the system.

Furthermore, it has been shown in [28] that also the expected value $\mathbb{E}(\mathcal{H}(t))$ of $\mathcal{H}(t)$, the state at time $t > 0$ of a multi-state coherent system, can be given in the form of an aggregation function applied to the respective reliability functions of the single components, as well. The difference between the two cases of multi-state and binary systems consists in the obvious circumstance that, in the latter case, the possible values for φ can only be 0 and 1, and that the function $\mathbb{E}(\mathcal{H}(t))$ reduces to the system's reliability function $R(t)$. Specifically, the aggregations representing $\mathbb{E}(\mathcal{H}(t))$ have in any case the form (3.5) below. By using a language used in the theory of monotone measures, we can say that the aggregation functions appearing in such a formula are *copula-based extensions* of the structure function φ, and they are the objects of main interest in our paper. The general formula (3.5) in particular encompasses the case when φ is binary and when, correspondingly, $\mathbb{E}(\mathcal{H}(t))$ reduces to $R(t)$.

Generally, the function φ appearing in the expression (3.5), can also take other remarkable interpretations beside the field of reliability. Thus it can be convenient to look at a coherent structure function φ from a more general viewpoint, just as a normalized *fuzzy measure* over the set $\mathcal{S} = \{0, 1/m, 2/m, \dots, 1\}$, for a fixed natural number m. Actually we will maintain here, for the function φ, the meaning of structure function of a coherent reliability system. This is not, however, a really restrictive choice. In fact we shall see that, in any case, aggregations of the form (3.5), even if emerging from different applied contexts, can be given corresponding interpretations in the field of reliability; see in particular Theorem 3.12 below, where the role of mixed binary systems is emphasized. Thus our results, inspired by the reliability context, can assume a rather general interest.

One of the main topics in reliability theory is the stochastic comparison of binary coherent systems. Many results have been obtained both for systems with independent and dependent components. Recently, some results have been obtained for systems with dependent components by using the theory of copulas to represent the dependence structure. A recent survey can be seen in [15].

In the treatment of these problems the representation (3.4), based on the concept of aggregation, reveals to be a useful tool. It can be used in fact to obtain distribution-free comparisons of systems just by comparing their aggregation functions (see [16, 17] and the references therein). Specific results can be obtained for homogeneous (identically distributed) components (see, e.g., [15]).

In the special cases when the components' lifetimes are independent and identically distributed (IID) or exchangeable an alternative, still very useful, representation can be obtained in terms of the concept of *signature*. The latter representation also emerges as a convenient one in the problem of comparing different binary systems, see e.g. [11, 18, 19].

The main purpose of the present paper is to extend the comparison results obtained for binary systems to multi-state systems composed with binary components. Comparing the performance of two different binary systems amounts to establishing some stochastic order between their respective lifetimes. In the case of a binary system with binary components, its lifetime is a non-negative random variable directly defined as a deterministic function of the components' lifetimes (see in particular [8] for details about the specific analytic form). Also for multi-state systems (still with binary components) one can consider a similar concept of system's lifetime, but the latter is not actually an appropriate object for the purpose of establishing significant stochastic comparisons.

To a multi-state system we will rather associate a lifetime defined by a probabilistic relation with the lifetimes of components. More precisely, we associate to a multi-state system a suitably defined *mixed system* (i.e. a mixture of a given set of binary systems). The lifetime of the mixed system corresponds to the lifetime of a system, chosen at random out of the given set of binary systems. For a fixed survival copula of components' lifetimes, such a mixed system shares the same aggregation function with the "parent" multi-state system and can be used to represent the performance of it. This representation, in particular, will be used to also define a specific concept of signature associated to the multi-state system. Both aggregation function and signature can be used to compute the reliability function of the corresponding mixed system and thus to get comparison results for different multi-state systems. Stochastic comparisons are also obtained for the discrete random variables which represent the states of the systems at time t. In order to suitably define the afore-mentioned mixed system, we will preliminarily point out that any arbitrary multi-state system admits a natural decomposition in terms of binary systems. We will then obtain different types of comparison results, which can also be translated into conditions for the respective copula-based aggregation functions.

The rest of the paper is organized as follows. The notation, basic definitions and preliminary results are introduced in Section 2. This section is actually divided into two different parts. The first one is devoted to the basic properties of binary systems, including the formal definitions of binary coherent systems, mixed systems and signatures. The second part presents a brief review of existing results concerning stochastic comparisons of binary systems. In Section 3 we obtain, for multi-state systems, different representation results based on the concepts of signature and of aggregation function. Such results will be employed in Section 4 to obtain comparison results for multi-state systems. Some illustrative examples are presented in Section 5. Section 6 will be devoted to a brief discussion and some final remarks.

2 Notation and preliminary results

2.1 A brief review about binary systems

The structure of a binary system with n components can be represented as a Boolean function

$$\varphi : \{0,1\}^n \to \{0,1\}$$

where y_j represents the state of the j-th component and $\varphi(y_1,\ldots,y_n)$ the state of the system which only depends on the components' states. Here $y_j = 1$ means that the j-th component is working and $y_j = 0$ means that it has failed. Then the binary system only has two possible states $\varphi(y_1,\ldots,y_n) = 1$ (it is working) and $\varphi(y_1,\ldots,y_n) = 0$ (it has failed). The multi-state systems defined in the next section will extend this definition allowing other possible intermediate states for the system.

According to the classic reliability theory (see, e.g., [1]), a binary system is *semi-coherent* if its structure function φ is increasing and satisfies $\varphi(0,\ldots,0) = 0$ and $\varphi(1,\ldots,1) = 1$. It is *coherent* if φ is increasing and it is strictly increasing in each variable in at least a point, that is, for each $j \in [n] := \{1,\ldots,n\}$, there exist values $y_1,\ldots,y_n \in \{0,1\}$ such that

$$0 = \varphi(y_1,\ldots,y_{j-1},0,y_{j+1},\ldots,y_n) < \varphi(y_1,\ldots,y_{j-1},1,y_{j+1},\ldots,y_n) = 1.$$

This last property means that all the components are relevant for the system. In particular, it implies that $\varphi(0,\ldots,0) = 0$ and $\varphi(1,\ldots,1) = 1$ (since φ is increasing), that is, all the coherent systems are in particular semi-coherent. The reverse is not true since a semi-coherent system may have irrelevant components. For example, $\varphi(y_1, \dots, y_n) = \min(y_1, y_2)$ is a semi-coherent system of order n (representing the series system formed with the two first components) but it is not coherent when $n > 2$ since then it has $n - 2$ irrelevant components.

In reliability theory one typically assumes that, for $j \in [n]$, the state of component j at time $t \geq 0$ is a (binary) random variable that we denote by $Y_j(t)$. Obviously, we assume $Y_j(0) = 1$ for $j = 1, \dots, n$. The lifetime of component j is then defined as

$$X_j = \sup\{t : Y_j(t) = 1\}.$$

We denote by T the corresponding random lifetime of the system. Then it is well known that T can be obtained from $X_1, \dots, X_n$ (see [1], p. 12; see also [8]). The components' lifetimes can be dependent or independent. This possible dependence can be modelled by the joint reliability (or survival) function of the random vector $(X_1, \dots, X_n)$ defined as

$$\bar{\mathbf{F}}(x_1, \dots, x_n) = \Pr(X_1 > x_1, \dots, X_n > x_n). \tag{2.1}$$

We denote by $\bar{F}_j(x_j) = \Pr(X_j > x_j)$ the marginal reliability function of X_j for $j = 1, \dots, n$ and by $\hat{C}$ the survival copula of $(X_1, \dots, X_n)$. As well-known, a copula is a multivariate distribution function with uniform marginals over the interval $(0, 1)$. The basic properties of copulas can be seen e.g. in [7, 22]. The survival copula $\hat{C}$ contains all the information about the dependence structure between the component's lifetimes and it is defined as the copula such that the joint reliability function in (2.1) can be written as

$$\bar{\mathbf{F}}(x_1, \dots, x_n) = \hat{C}(\bar{F}_1(x_1), \dots, \bar{F}_n(x_n)).$$

From Sklar's theorem, the survival copula is unique when $\bar{F}_1, \dots, \bar{F}_n$ are continuous.

Analogously, the reliability function $R(t) = \Pr(T > t)$ of the lifetime of a semi-coherent binary system can be written (see, e.g., [15, 20]) as

$$R(t) = \bar{Q}(\bar{F}_1(t), \dots, \bar{F}_n(t)) \tag{2.2}$$

for all $t > 0$, where $\bar{Q}$ is a continuous aggregation function (or distortion function), that is, $\bar{Q} : [0, 1]^n \to [0, 1]$ is a continuous increasing function such that $\bar{Q}(0, \dots, 0) = 0$ and $\bar{Q}(1, \dots, 1) = 1$.

The function $\bar{Q}$ depends both on the structure of the system φ and on the survival copula $\hat{C}$ of the component lifetimes (dependence structure). However, it does not depend on $\bar{F}_1, \dots, \bar{F}_n$. Hence (2.2) is a very useful representation since $\bar{Q}$ contains all the information about the system structure and the dependence and the different components can be represented by just different reliability functions (univariate probability models). More precisely, the dependence on the function φ and on the copula $\hat{C}$ of the function $\bar{Q}$ will be detailed in the next section, where can be obtained as a special case of the formula (3.4) below.

We also remind that the mixed (binary) systems were defined as finite mixtures of (binary) systems in [5]. Thus, given m semi-coherent systems with structures $\varphi_1, \dots, \varphi_m$ with n components and m positive numbers $w_1, \dots, w_m$ such that $w_1 + \cdots + w_m = 1$, the associated (random) mixed system structure φ is defined as φ_i with probability w_i, for $i = 1, \dots, m$. Note that $\varphi(0, \dots, 0) = 0$ and $\varphi(1, \dots, 1) = 1$. However, we cannot assure that φ is increasing (since it is random). What we get is that

$$\mathbb{E}(\varphi(u_1, \dots, u_n)) = \sum_{i=1}^{m} w_i \varphi_i(u_1, \dots, u_n) \leq \sum_{i=1}^{m} w_i \varphi_i(v_1, \dots, v_n) = \mathbb{E}(\varphi(v_1, \dots, v_n))$$

for all $u_i \leq v_i$, $i = 1, \ldots, n$. Mixed systems can be used in practice to represent systems that should fulfil different requirements at different times. For example, an online shop with n servers may work with just $k < n$ servers from Monday to Friday but it may need the n servers during the weekend. In other situations, some components in a system may be turned off during some hours (to save money) by changing in this way the structure of the system. In all these cases, the structure of the system can be modelled as a random structure.

2.2 Preliminaries about stochastic comparisons of binary systems

In order to compare the lifetimes of two different binary systems we shall consider the following well known stochastic orders. Their main properties can be seen in [27]. Given two random variables X and Y with reliability functions $\bar{F}$ and $\bar{G}$ and probability density (or probability mass) functions f and g, we say that X *is less than* Y:

- in the *usual stochastic order* (shortly written as $X \leq_{st} Y$) if $\bar{F} \leq \bar{G}$.
- in the *hazard order* (shortly written as $X \leq_{hr} Y$) if $\bar{G}/\bar{F}$ is increasing.
- in the *likelihood ratio order* (shortly written as $X \leq_{lr} Y$) if g/f is increasing (in the union of their supports).

It is well known that

$$X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{st} Y \Rightarrow \mathbb{E}(X) \leq \mathbb{E}(Y) \tag{2.3}$$

whenever these expectations exist.

Distribution-free comparison results for binary systems based on (2.2) were obtained in [16, 17, 20]. For completeness, we include some of them in the following proposition.

Proposition 2.1. *Let T_1 and T_2 be the lifetimes of two semi-coherent binary systems with aggregation functions $\bar{Q}_1$ and $\bar{Q}_2$ and common component reliability functions $\bar{F}_1, \ldots, \bar{F}_n$. Then:*

(i) $T_1 \leq_{st} T_2$ for all $\bar{F}_1, \ldots, \bar{F}_n$ if and only if $\bar{Q}_1 \leq \bar{Q}_2$.

(ii) $T_1 \leq_{hr} T_2$ for all $\bar{F}_1, \ldots, \bar{F}_n$ if and only if $\bar{Q}_2/\bar{Q}_1$ is decreasing in $(0,1)^n$.

Additional comparisons results can be obtained if we assume that the component lifetimes are independent and identically distributed (IID) or exchangeable (EXC), that is, $\bar{\mathbf{F}}$ is permutation invariant. In these cases, the system reliability function can be written (see, e.g., [15, 19, 26]) as

$$R(t) = s_1 \bar{F}_{1:n}(t) + \cdots + s_n \bar{F}_{n:n}(t), \tag{2.4}$$

where $\bar{F}_{1:n}, \ldots, \bar{F}_{n:n}$ are the reliability functions of the ordered component lifetimes $X_{1:n} \leq \cdots \leq X_{n:n}$ obtained from $X_1, \ldots, X_n$.

The vector with the coefficients $\mathbf{s} = (s_1, \ldots, s_n)$ in (2.4) is called the *signature of order n* of the system. If the joint reliability $\bar{\mathbf{F}}$ is absolutely continuous, then $s_k = \Pr(T = X_{k:n})$ for $k = 1, \ldots, n$. The signature can also be computed from the structure function as

$$s_k = \frac{1}{\binom{n}{k-1}} \sum_{\sum_{j=1}^n x_j = n-k+1} \varphi(y_1, \ldots y_n) - \frac{1}{\binom{n}{k}} \sum_{\sum_{j=1}^n x_j = n-k} \varphi(y_1, \ldots y_n), \quad k = 1, \ldots, n, \tag{2.5}$$

see Boland [4] (or [25]).

If the component lifetimes are IID or EXC, then representations similar to (2.4) can be obtained in terms of series or parallel systems (see, e.g., [15]). For example, for the series systems we have

$$R(t) = a_1\bar{F}_{1:1}(t) + \cdots + a_n\bar{F}_{1:n}(t), \tag{2.6}$$

where $\bar{F}_{1:j}(t) = \Pr(X_{1:j} > t)$ and $X_{1:j} = \min(X_1, \ldots, X_j)$ for $j = 1, \ldots, n$. The vector $\mathbf{a} = (a_1, \ldots, a_n)$ with the coefficients in (2.6) is called the *minimal signature of order* n of the system. Some of these coefficients can be negative (so they cannot be looked at as probabilities).

Note that $s_1 + \cdots + s_n = 1$. Thus the signature can be seen as a discrete distribution over $1, \ldots, n$. Hence, from (2.4) we can obtain the following comparison results extracted from [11] and [19], respectively.

Proposition 2.2. *Let T_1 and T_2 be the lifetimes of two semi-coherent binary systems based on n IID components with a common reliability function $\bar{F}$. Let $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ be the respective signatures of order n.*

(i) If $\boldsymbol{s}_1 \leq_{st} \boldsymbol{s}_2$, then $T_1 \leq_{st} T_2$ for all $\bar{F}$.

(ii) If $\boldsymbol{s}_1 \leq_{hr} \boldsymbol{s}_2$, then $T_1 \leq_{hr} T_2$ for all $\bar{F}$.

(iii) If $\boldsymbol{s}_1 \leq_{lr} \boldsymbol{s}_2$, then $T_1 \leq_{lr} T_2$ for all absolutely continuous $\bar{F}$.

Proposition 2.3. *Let T_1 and T_2 be the lifetimes of two semi-coherent binary systems based on n EXC components with a common joint reliability function $\bar{\mathbf{F}}$. Let $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ be the respective signatures of order n.*

(i) If $\boldsymbol{s}_1 \leq_{st} \boldsymbol{s}_2$, then $T_1 \leq_{st} T_2$ for all $\bar{\mathbf{F}}$.

(ii) If $\boldsymbol{s}_1 \leq_{hr} \boldsymbol{s}_2$, then $T_1 \leq_{hr} T_2$ for all $\bar{\mathbf{F}}$ such that

$$X_{1:n} \leq_{hr} \cdots \leq_{hr} X_{n:n}.$$

(iii) If $\boldsymbol{s}_1 \leq_{lr} \boldsymbol{s}_2$, then $T_1 \leq_{lr} T_2$ for all absolutely continuous $\bar{\mathbf{F}}$ such that

$$X_{1:n} \leq_{lr} \cdots \leq_{lr} X_{n:n}.$$

Note that in the two preceding propositions we just have sufficient conditions, based on signatures, for some stochastic orders to hold. Actually, these orders might hold even if the signatures are not ordered (see [15, 25]).

3 Representation of multi-state systems and related aggregation functions

As mentioned above, one main purpose of the present paper is to extend to multi-state systems the comparisons results reviewed in the preceding Subsection 2.2 for the case of binary systems. On this purpose we will devote the present section to point out relations existing between multi-state and binary systems. On such a basis we will suitably associate an aggregation function and a signature vector to any given multi-state system. For simplicity's sake and for readers' convenience we will first concentrate our attention to the special case of *multi-level* systems, defined by the condition that, for a fixed $m \in \mathbb{N}$, the set of possible states is given by $\mathcal{S} := \left\{0, \frac{1}{m}, \frac{2}{m}, \ldots, \frac{m-1}{m}, 1\right\}$. The general case when the structure function of the system is an arbitrary, regular, fuzzy measure will be separately analysed in Subsection 3.2.

In any case, we consider systems made up with n binary components. Recalling the notation introduced in the previous section, for $j = 1, \ldots, n$, the binary random variable $Y_j(t)$ is the random state of component j at time $t > 0$ and $X_j = \sup\{t : Y_j(t) = 1\}$ denotes the component's lifetime. $\hat{C}$ denotes the survival copula of $(X_1, \ldots, X_n)$ and $\bar{F}_1(t), \ldots, \bar{F}_n(t)$ are the marginal reliability functions.

3.1 The case of multi-level systems

Here we assume that, for a fixed $m \in \mathbb{N}$, the set of possible states of a system is

$$\mathcal{S} := \left\{0, \frac{1}{m}, \frac{2}{m}, \ldots, \frac{m-1}{m}, 1\right\}$$

where, as above, 1 represents the *perfect functioning state* and 0 the *state of failure*. In the middle, we have $m-1$ intermediate states. The evolution in time of the performance of the system can then be seen as a stochastic process starting from 1 (perfect functioning) and eventually going to 0 (failure) as $t \to \infty$. This representation is clearly equivalent to the classical representation using the levels $\{0, 1, \ldots, m\}$ for a given integer number m. We could of course consider systems with more general levels $\ell_0 = 0 < \ell_1 < \cdots < \ell_m$ by using the set

$$\mathcal{S} := \left\{h_0 = 0, h_1 = \frac{\ell_1}{\ell_m}, h_2 = \frac{\ell_2}{\ell_m}, \ldots, h_{m-1} = \frac{\ell_{m-1}}{\ell_m}, h_m = 1\right\}.$$

This general case is studied in the next Subsection 3.2.

We assume that the state of the system at time $t \geq 0$ is completely determined by the respective states at time t of the system's components. Thus we define the structure of a multi-level system with binary components as follows.

Definition 3.1. *The structure function of a multi-level system with binary components is a (component-wise) increasing function*

$$\varphi : \{0,1\}^n \to \mathcal{S}$$

such that $\varphi(0, \ldots, 0) = 0$ *and* $\varphi(1, \ldots, 1) = 1$.

Then we notice that φ has the properties of a normalized (or regular) *fuzzy measure*. It is convenient, in this respect, to keep in mind that φ can be considered both as a function defined on the space $\{0,1\}^n$ and as a *set function* defined over the family $2^{[n]}$ of all the subsets of $[n]$ where for $J \subseteq [n]$,

$$\varphi(J) := \varphi(\mathbf{1}_J)$$

and $\mathbf{1}_J := (y_1, \ldots, y_n)$ with $y_j = 1$ for $j \in J$ and $y_j = 0$ for $i \notin J$.

In this sense (see, e.g., [10]), the *Möbius transform* $\widehat{\varphi}$ *of* φ *is*

$$\hat{\varphi}(I) := \sum_{J \subseteq I} (-1)^{|I|-|J|} \varphi(J)$$

and it is such that the inverse relation

$$\varphi(J) = \sum_{I \subseteq J} \hat{\varphi}(I) \tag{3.1}$$

holds. It is also useful for our purposes below to rewrite the previous equation (3.1) *in a slightly different form. For* $\mathbf{y} \in \{0,1\}^n$ *and* $I \subseteq [n]$ *such that* $\mathbf{y} = \mathbf{1}_I$*, we can write*

$$\varphi(\mathbf{y}) = \sum_{J \subseteq I} \hat{\varphi}(J) = \sum_{J \subseteq [n]} \hat{\varphi}(J) \prod_{j \in J} y_j. \tag{3.2}$$

Remind from the previous section that, for $j \in [n]$, we denote by $Y_j(t)$ the (binary) random state of component j at time $t \geq 0$ and, obviously, we assume $Y_j(0) = 1$ for $j = 1, \ldots, n$. Then we can also consider that the *random state* of the system at time t is given by

$$\mathcal{H}(t) := \varphi(Y_1(t) \ldots, Y_n(t)).$$

For a single instant $t > 0$, $\mathcal{H}(t)$ is a discrete random variable, taking values in the set $\mathcal{S}$, with probability distribution given by

$$p_i(t) := \Pr\left(\mathcal{H}(t) = \frac{i}{m}\right) = \Pr\left(\varphi(Y_1(t) \ldots, Y_n(t)) = \frac{i}{m}\right), \quad i = 0, \ldots, m \tag{3.3}$$

and with the expected value

$$\mathbb{E}(\mathcal{H}(t)) = \frac{1}{m} \sum_{i=1}^{m} i \cdot \Pr\left(\varphi(Y_1(t) \ldots, Y_n(t)) = \frac{i}{m}\right).$$

Note that $\{\mathcal{H}(t)\}_{t \geq 0}$ is a stochastic process with $\mathcal{H}(0) = 1$ and $\mathcal{H}(\infty) = 0$.

By applying the above relation (3.2), one can write

$$\mathcal{H}(t) = \sum_{J \subseteq [n]} \hat{\varphi}(J) \prod_{j \in J} Y_j(t)$$

and the expected value $\mathbb{E}(\mathcal{H}(t))$ can be given in an alternative expression as follows. First we write

$$\mathbb{E}(\mathcal{H}(t)) = \sum_{J \subseteq [n]} \hat{\varphi}(J) \mathbb{E}\left(\prod_{j \in J} Y_j(t)\right).$$

Furthermore,

$$\mathbb{E}\left(\prod_{j \in J} Y_j(t)\right) = \Pr\left(\bigcap_{j \in J} \{Y_j = 1\}\right) = \Pr\left(\bigcap_{j \in J} \{T_j > t\}\right).$$

Thus we obtain the alternative form

$$\mathbb{E}(\mathcal{H}(t)) = M_\varphi\left(\bar{F}_1(t), \ldots, \bar{F}_n(t)\right), \tag{3.4}$$

where, for $\mathbf{u} = (u_1, \ldots, u_n) \in [0,1]^n$,

$$M_\varphi(\mathbf{u}) := \sum_{J \subseteq [n]} \widehat{\varphi}(J) \hat{C}(\mathbf{u}_J, \mathbf{1}), \tag{3.5}$$

with $\widehat{\varphi}$ denoting the Möbius transform of φ, $\hat{C}$ the survival copula and $(\mathbf{u}_J, \mathbf{1}) := (x_1, \ldots, x_n)$ with $x_j = u_j$ for $j \in J$ and $x_j = 1$ for $j \notin J$ (for more details see [28]).

The function M_φ in (3.4) and (3.5) is a continuous aggregation (distortion) function, that is, it is increasing and satisfies $M_\varphi(0, \ldots, 0) = 0$ and $M_\varphi(1, \ldots, 1) = 1$. Moreover, for $\mathbf{u} = (u_1, \ldots, u_n) \in \{0, 1\}^n$, one has

$$M_\varphi(\mathbf{u}) = \varphi(\mathbf{u})$$

and M_φ is told to be a *copula-based extension* of φ (see [12, 28]).

Of course, the binary semi-coherent systems considered in the preceding section are just multi-level systems with $m = 1$ (i.e., with two levels) and in this case we have

$$\mathbb{E}(\mathcal{H}(t)) = 0 \cdot \Pr(\mathcal{H}(t) = 0) + 1 \cdot \Pr(\mathcal{H}(t) = 1) = \Pr(\mathcal{H}(t) = 1) = M_\varphi\left(\bar{F}_1(t), \ldots, \bar{F}_n(t)\right),$$

where, if $T = \sup\{t : \mathcal{H}(t) = 1\}$ is the system lifetime, then

$$\Pr(\mathcal{H}(t) = 1) = \Pr(T > t) = M_\varphi\left(\bar{F}_1(t), \ldots, \bar{F}_n(t)\right), \tag{3.6}$$

that is, we obtain the representation (2.2) with $\bar{Q} = M_\varphi$. When we study the multi-level case, the lifetime of the system might be defined as

$$T_1 := \inf\{t : \mathcal{H}(t) = 0\} = \sup\{t : \mathcal{H}(t) > 0\}. \tag{3.7}$$

The random variable T_1 is not apt, however, to describe the performance of the system adequately for our purposes. For this reason, given a multi-level structure φ, we consider the associated binary systems with the following structures

$$\varphi_i(y_1, \ldots, y_n) = \begin{cases} 1, & \text{if} \quad \varphi(y_1, \ldots, y_n) \geq \frac{i}{m} \\ 0, & \text{if} \quad \varphi(y_1, \ldots, y_n) < \frac{i}{m} \end{cases} \tag{3.8}$$

for $i = 1, \ldots, m$. The binary structures $\varphi_1, \ldots, \varphi_m$ are semi-coherent and satisfy $\varphi_1 \geq \cdots \geq \varphi_m$. Moreover, we have

$$\varphi(y_1, \ldots, y_n) = \frac{1}{m} \sum_{i=1}^{m} \varphi_i(y_1, \ldots, y_n). \tag{3.9}$$

This representation is equivalent to that obtained in (30) of [13] and it can be traced back to [3].

Hence the state of the system at time t can also be written as

$$\mathcal{H}(t) = \varphi(Y_1(t), \ldots, Y_n(t)) = \frac{1}{m} \sum_{i=1}^{m} \varphi_i(Y_1(t), \ldots, Y_n(t)) = \frac{1}{m} \sum_{i=1}^{m} \mathcal{H}_i(t), \tag{3.10}$$

where $\mathcal{H}_i(t)$ is the state at time t of the binary system with structure φ_i for $i = 1, \ldots, m$, that is, $\mathcal{H}(t)$ is the average of the states of the binary systems. We denote by

$$T_1, \ldots, T_m$$

the respective lifetimes of the systems with structure φ_i for $i = 1, \ldots, m$. Note that T_1 coincides with the lifetime defined in (3.7). As a consequence of (3.9) and (3.10), we have the following property.

Proposition 3.2. *Let φ be a multi-level structure. Let $\varphi_1, \ldots, \varphi_m$ be the structures of the associated binary systems and let $T_1, \ldots, T_m$ be their respective system lifetimes that have the survival copula $\hat{C}$. Then*

$$\mathbb{E}\left(\mathcal{H}\left(t\right)\right) = \frac{1}{m} \sum_{i=1}^{m} R_i(t), \tag{3.11}$$

where $R_i(t) = \Pr(T_i > t)$ is the reliability function of T_i for $i = 1, \ldots, m$,

$$R_i(t) = M_i\left(\bar{F}_1\left(t\right), \ldots, \bar{F}_n\left(t\right)\right) = \sum_{I \subseteq [n]} \widehat{\varphi}_i(I) \hat{C}\left(\mathbf{u}_I, \mathbf{1}\right) \tag{3.12}$$

and $\widehat{\varphi}_i(I) = \sum_{J \subseteq I} (-1)^{|I|-|J|} \varphi_i(J)$ is the Möbius transform of φ_i.

Proof. From (3.5) and (3.6), the reliability function of T_i can be written as

$$R_i(t) = \Pr(T_i > t) = \Pr(\mathcal{H}_i\left(t\right) = 1) = M_i\left(\bar{F}_1\left(t\right), \ldots, \bar{F}_n\left(t\right)\right)$$

for $i = 1, \ldots, m$. Then, from (3.10), we get

$$\mathbb{E}\left(\mathcal{H}\left(t\right)\right) = \mathbb{E}\left(\frac{1}{m} \sum_{i=1}^{m} \mathcal{H}_i\left(t\right)\right) = \frac{1}{m} \sum_{i=1}^{m} \mathbb{E}(\mathcal{H}_i\left(t\right)),$$

where $\mathbb{E}(\mathcal{H}_i\left(t\right)) = \Pr(\mathcal{H}_i\left(t\right) = 1) = R_i(t)$. This proves (3.11). Expression (3.12) is then obtained by applying (3.4) to φ_i. $\square$

Remark 3.3. *Note that, from* (3.4)*, we also get*

$$\mathbb{E}\left(\mathcal{H}\left(t\right)\right) = M_\varphi\left(\bar{F}_1\left(t\right), \ldots, \bar{F}_n\left(t\right)\right),$$

where

$$M_\varphi(\mathbf{u}) = \sum_{J \subseteq [n]} \widehat{\varphi}(J) \hat{C}\left(\mathbf{u}_J, \mathbf{1}\right)$$

and

$$\widehat{\varphi}(J) = \sum_{I \subseteq J} (-1)^{|J|-|I|} \varphi(I).$$

By combining equations (3.4) *and* (3.5) *with* (3.9)*, we obtain*

$$\widehat{\varphi}(J) = \frac{1}{m} \sum_{J \subseteq J} (-1)^{|J|-|I|} \sum_{i=1}^{m} \varphi_i(I) = \frac{1}{m} \sum_{i=1}^{m} \sum_{J \subseteq I} (-1)^{|J|-|I|} \varphi_i(I) = \frac{1}{m} \sum_{i=1}^{m} \widehat{\varphi}_i(J) \tag{3.13}$$

and

$$M_\varphi(\mathbf{u}) = \sum_{J \subseteq [n]} \widehat{\varphi}(J) \hat{C}\left(\mathbf{u}_J, \mathbf{1}\right) = \frac{1}{m} \sum_{i=1}^{m} \sum_{J \subseteq [n]} \widehat{\varphi}_i(J) \hat{C}\left(\mathbf{u}_J, \mathbf{1}\right) = \frac{1}{m} \sum_{i=1}^{m} M_i(\mathbf{u}) \tag{3.14}$$

for all $J \subseteq [n]$ and all $\mathbf{u} \in [0,1]^n$. Therefore,

$$\mathbb{E}\left(\mathcal{H}\left(t\right)\right) = M_\varphi\left(\bar{F}_1\left(t\right), \ldots, \bar{F}_n\left(t\right)\right) = \frac{1}{m} \sum_{i=1}^{m} M_i(\bar{F}_1\left(t\right), \ldots, \bar{F}_n(t)) = \frac{1}{m} \sum_{i=1}^{m} R_i(t)$$

as stated in the preceding proposition. Thus, in (3.13) *and* (3.14)*, we have directly proven that the Möbius transform $\hat{\varphi}$ and the aggregation function M_φ of the multi-level system φ are just the average of the respective functions associated to the binary systems $\varphi_1, \ldots, \varphi_m$.*

Remark 3.4. *Another alternative (and simpler) proof for the preceding proposition can be obtained as follows. We remind that, for any fixed* t, $\mathcal{H}(t)$ *is a discrete random variable which can take the values* $0, 1/m, \ldots, m/m$. *Hence*

$$\mathbb{E}(\mathcal{H}(t)) = \sum_{i=1}^{m} \frac{i}{m}\, p_i(t)$$

with $p_i(t)$ *defined by (3.3). Moreover,*

$$p_i(t) = \Pr\left(\mathcal{H}(t) \geq \frac{i}{m}\right) - \Pr\left(\mathcal{H}(t) \geq \frac{i+1}{m}\right) = \Pr(T_i > t) - \Pr(T_{i+1} > t) = R_i(t) - R_{i+1}(t)$$

for $i = 1, \ldots, m-1$ *and*

$$p_m(t) = \Pr\left(\mathcal{H}(t) \geq \frac{m}{m}\right) = \Pr(T_m > t) = R_m(t).$$

Therefore,

$$\mathbb{E}(\mathcal{H}(t)) = R_m(t) + \sum_{i=1}^{m-1} \frac{i}{m}\,(R_i(t) - R_{i+1}(t)) = \frac{1}{m}\sum_{i=1}^{m} R_i(t).$$

The expression (3.11) proves that the function $\mathbb{E}(\mathcal{H}(t))$ is equal to the reliability function of a uniform mixture of the binary systems. Thus any multi-level system can be associated to a mixed system (see the definition in the preceding section) as follows.

Definition 3.5. *The mixed system associated to a multi-level system with structure function* φ *is that which is equal to the binary system* φ_i *with probability* $1/m$, *for* $i = 1, \ldots, m$.

If T is the lifetime of the mixed system associated to φ, then the expression (3.11) can be rewritten as

$$\Pr(T > t) = \mathbb{E}(\mathcal{H}(t)) = \frac{1}{m}\sum_{i=0}^{m} R_i(t).$$

In particular, we have proved that $\mathbb{E}(\mathcal{H}(t))$ can be seen as a reliability function. Thus, for our purposes, the lifetime T can be used to represent the performance of the multi-state system. Hence, the expression (3.4) can also be written as

$$\Pr(T > t) = \mathbb{E}(\mathcal{H}(t)) = \bar{Q}\left(\bar{F}_1(t), \ldots, \bar{F}_n(t)\right), \tag{3.15}$$

where

$$\bar{Q} = \frac{\bar{Q}_1 + \cdots + \bar{Q}_m}{m}$$

and $\bar{Q}_i$ is the aggregation function of the i-th binary system obtained from (2.2) (or from (3.12)). Note that $\bar{Q} = M_\varphi$ where M_φ can also be computed from (3.5). The preceding expression (3.15) will be used in the next section to compare stochastically two different multi-level systems.

We can go further if we assume that the components are exchangeable (or IID). In this case we can define the signature of a multi-level system as follows.

Definition 3.6. *The signature vector* $\mathbf{r}$ *of a multi-level system is defined as*

$$\mathbf{r} = \frac{\mathbf{s}_1 + \cdots + \mathbf{s}_m}{m}$$

where $\mathbf{s}_i$ *is the signature vector of the* i*-th binary system* T_i *for* $i = 1, \ldots, m$.

Note that $\mathbf{r}$ is the signature of the mixed system associated to φ. The minimal and maximal signatures can be defined in a similar way (as the average of the respective signatures of the binary systems). If the components are exchangeable (or IID), then we can obtain the following representation.

Proposition 3.7. *If $\mathbf{r} = (r_1, \ldots, r_n)$ is the signature vector of a multi-level system with exchangeable components, then*

$$\mathbb{E}\left(\mathcal{H}\left(t\right)\right) = r_1 \bar{F}_{1:n}(t) + \cdots + r_n \bar{F}_{n:n}(t), \tag{3.16}$$

where $\bar{F}_{i:n}$ is the reliability function of the i-th ordered lifetime obtained from $(X_1, \ldots, X_n)$.

The proof is immediate from (2.4) and (3.11).

Note that the signature vector $\mathbf{r} = (r_1, \ldots, r_n)$ of a multi-level system can also be defined as in (2.5) by

$$r_k = \frac{1}{\binom{n}{k-1}} \sum_{\sum_{j=1}^{n} y_j = n-k+1} \varphi(y_1, \ldots y_n) \; - \frac{1}{\binom{n}{k}} \sum_{\sum_{j=1}^{n} y_j = n-k} \varphi(y_1, \ldots y_n), \quad k = 1, \ldots, n.$$

From (3.9) the two definitions are equivalent. In Section 5 we show that the first definition is better in order to compute the signature of a multi-state system.

It is important to note here that to get (3.16), we need to assume that the components are exchangeable. If they are not, this definition of signature might be not useful (since (3.16) does not hold). The same happens for binary systems (see [19]).

3.2 The general case of multi-state systems

In the present subsection we extend and, up to some extent, we formalize the arguments presented in the previous subsection. Actually, all the properties studied therein can be extended to general multi-state systems with binary components having increasing structure functions

$$\varphi : \{0, 1\}^n \to [0, 1]$$

such that $\varphi\left(0, \ldots, 0\right) = 0$ and $\varphi\left(1, \ldots, 1\right) = 1$ (i.e. regular fuzzy measures).

Since the space $\{0, 1\}^n$ is finite, the set $\mathcal{S}_\varphi$ of the states of such a system is also finite and can be written as

$$\mathcal{S}_\varphi = \{0 = h_0 < h_1 < \cdots < h_{q-1} < h_q = 1\}$$

for an integer number q. Therefore, the structure function can be written as

$$\varphi : \{0, 1\}^n \to \mathcal{S}_\varphi$$

which is the definition given in [28].

Remark 3.8. *Note that the level set $\mathcal{S}_\varphi$ depends on the system φ. If we want to compare two systems φ_1 and φ_2 with different level sets $\mathcal{S}_{\varphi_1}$ and $\mathcal{S}_{\varphi_2}$, we just consider them as systems over the (finite) set $\mathcal{S} := \mathcal{S}_{\varphi_1} \cup \mathcal{S}_{\varphi_2}$. So, without loss of generality, we can assume that they have the same level set $\mathcal{S}$.*

The space Φ_n of all such structure functions is convex. Also, in these more general cases, we can consider, for any $t > 0$, the discrete random variable

$$\mathcal{H}(t) = \varphi\left(Y_1(t), \ldots, Y_n(t)\right),$$

indicating the state of the system at time t. By taking again into account (3.2) (see also [28]), we can easily extend the arguments in the previous subsection and we can still write

$$\mathbb{E}\left(\mathcal{H}(t)\right) = M_{\varphi}\left(\bar{F}_1\left(t\right), \ldots, \bar{F}_n\left(t\right)\right),$$

where M_φ is the aggregation function defined by

$$M_{\varphi}\left(\mathbf{u}\right) := \sum_{I \subseteq [n]} \widehat{\varphi}(I)\hat{C}\left(\mathbf{u}_I, \mathbf{1}\right). \tag{3.17}$$

Let now $\psi_1, \ldots, \psi_m \in \Phi_n$, be given coherent structure functions and set, for $i = 1, \ldots, m$,

$$\mathcal{H}_i(t) := \psi_i\left(Y_1(t), \ldots, Y_n(t)\right).$$

Then we can obtain the following result.

Proposition 3.9. *Suppose that m non-negative coefficients $\alpha_1, \ldots, \alpha_m$ exist such that $\alpha_1 + \cdots + \alpha_m = 1$ and*

$$\varphi\left(\mathbf{y}\right) = \sum_{i=1}^{m} \alpha_i \psi_i\left(\mathbf{y}\right)$$

for all $\mathbf{y} \in \{0, 1\}^n$. Then

$$\widehat{\varphi}\left(I\right) = \sum_{i=1}^{m} \alpha_i \widehat{\psi}_i\left(\mathbf{I}\right)$$

for all $I \subseteq [n]$ and

$$M_{\varphi}\left(\mathbf{u}\right) = \sum_{i=1}^{m} \alpha_i M_{\psi_i}\left(\mathbf{u}\right)$$

for all $\mathbf{u} \in [0, 1]^n$.

As the following result shows, any multi-state structure function $\varphi \in \Phi_n$ can be written as a convex combination of binary structure functions, namely the binary structures are the extreme points of the convex space Φ_n.

Fix a coherent structure function $\varphi : \{0, 1\}^n \to \mathcal{S}_\varphi$ and consider the q binary systems with structures $\varphi_1, \ldots, \varphi_q$ defined as

$$\varphi_i\left(y_1, \ldots, y_n\right) = \begin{cases} 1, & \text{if} \quad \varphi\left(\mathbf{y}\right) \geq h_i \\ 0, & \text{if} \quad \varphi\left(\mathbf{y}\right) < h_i \end{cases}. \tag{3.18}$$

Notice that such binary structures are obviously ordered. Consider also the functions $R_0(t), \ldots, R_q(t)$ defined by

$$R_i(t) := \Pr\left(\mathcal{H}\left(t\right) \geq h_i\right),$$

for $i = 0, \ldots, q$, so that

$$\Pr\left(\mathcal{H}\left(t\right) = h_i\right) = \Pr\left(\mathcal{H}\left(t\right) \geq h_i\right) - \Pr\left(\mathcal{H}\left(t\right) \geq h_{i+1}\right) = R_i(t) - R_{i+1}(t).$$

for $i = 0, \ldots, q-1$. Set

$$p_i(t) := \Pr\left(\mathcal{H}(t) = h_i\right) = R_i(t) - R_{i+1}(t).$$

for $i = 0, \ldots, q-1$ and

$$p_q(t) := \Pr\left(\mathcal{H}(t) = h_q\right) = \Pr\left(\mathcal{H}(t) = 1\right) = R_q(t).$$

Notice that, by definition, $p_i(t) \geq 0$ and

$$\sum_{i=0}^{q} p_i(t) = R_0(t) = 1.$$

As a consequence, we get the following result, which was also proven by Radojevic in different contexts (see, e.g., [23]).

Proposition 3.10. *If φ is a system and $\varphi_1, \ldots, \varphi_q$ are the associated binary systems defined by* (3.18)*, then we can write*

$$\varphi(\mathbf{y}) = \sum_{i=1}^{q} \rho_i \varphi_i(\mathbf{y}),$$

for all $\mathbf{y} \in \{0,1\}^n$, where $\rho_i = h_i - h_{i-1} \geq 0$ for $i = 1, \ldots, q$ and $\rho_1 + \cdots + \rho_q = 1$.

Proof. Suppose $\mathbf{y} \in \{0,1\}^n$ to be such that $\varphi(\mathbf{y}) = h_i$, for some $i = 1, \ldots, q$. Then

$$\varphi_1(\mathbf{y}) = \cdots = \varphi_{i-1}(\mathbf{y}) = \varphi_i(\mathbf{y}) = 1$$

and if $i < q$,

$$\varphi_{i+1}(\mathbf{y}) = \cdots = \varphi_q(\mathbf{y}) = 0.$$

Thus

$$\sum_{j=1}^{q} \rho_j \varphi_j(\mathbf{y}) = \sum_{j=1}^{i} \rho_j = \sum_{j=1}^{i} (h_j - h_{j-1}) = h_i = \varphi(\mathbf{y}).$$

If $\varphi(\mathbf{y}) = h_0 = 0$, then

$$\varphi_1(\mathbf{y}) = \cdots = \varphi_q(\mathbf{y}) = 0$$

and

$$\sum_{j=1}^{q} \rho_j \varphi_j(\mathbf{y}) = 0 = \varphi(\mathbf{y}).$$

□

Note that φ is a linear convex combination of the ordered binary structures $\varphi_1 \geq \cdots \geq \varphi_q$. Let us denote by $T_1, \ldots, T_q$ the respective lifetimes of the binary semi-coherent systems with structure φ_i for $i = 1, \ldots, q$, that is,

$$T_i := \sup\{t : \varphi_i(Y_1(t), \ldots, Y_n(t)) = 1\} = \sup\{t : \mathcal{H}(t) \geq h_i\}.$$

Then the functions $R_1, \ldots, R_q$ defined above satisfy $R_i(t) = \Pr(T_i > t)$, that is, they are the reliability functions of T_i for $i = 1, \ldots, q$. Analogously to the above Proposition 3.2 and as a direct consequence of above arguments we can state the following proposition.

Proposition 3.11. *For any $t > 0$, one has*

$$\mathbb{E}\left(\mathcal{H}\left(t\right)\right) = \sum_{i=1}^{q} \rho_i R_i(t),$$

where $\rho_i = h_i - h_{i-1} \geq 0$ for $i = 1, \ldots, q$ and $\rho_1 + \cdots + \rho_q = 1$.

Furthermore, we can also write

$$\mathbb{E}\left(\mathcal{H}\left(t\right)\right) = \sum_{i=1}^{q} h_i \Pr\left(\mathcal{H}\left(t\right) = h_i\right) = \sum_{i=1}^{q} h_i p_i(t) = \sum_{i=1}^{q} h_i \left[R_i(t) - R_{i+1}(t)\right].$$

Also in this case, $\mathbb{E}\left(\mathcal{H}\left(t\right)\right)$ is equal to the reliability function of a mixed system, namely the one obtained by taking the associated binary system φ_i with probabilities ρ_i, for $i = 1, \ldots, q$.

The converse property is not true, that is, we cannot define a multi-state system for any mixed system. This property holds just for mixed systems based on ordered binary systems. Thus if we consider the mixed system based on ordered semi-coherent systems $\varphi_1 \geq \cdots \geq \varphi_q$ with probabilities $\rho_1, \ldots, \rho_q$, then we can define a multi-state system with states

$$\mathcal{S} = \{0 = h_0 < h_1 := \rho_1 < h_2 := \rho_1 + \rho_2 < \cdots < h_q := \rho_1 + \cdots + \rho_q = 1\}$$

whose associated mixed system coincides with the parent one.

As already noticed the binary systems defined in (3.8) are ordered, that is, they satisfy the relation $\varphi_1 \geq \cdots \geq \varphi_q$. Hence the associated reliability functions satisfy $R_1(t) \geq \cdots \geq R_q(t)$ for all t, that is, $T_1 \geq_{st} \cdots \geq_{st} T_q$. Hence, the family of multi-state systems can be seen as a relevant subset of the family of general mixed systems (that based on ordered coherent systems).

We can, at this point, conclude with the following result concerning with aggregation functions of the form $M_{\varphi,\widehat{C}}\left(\mathbf{u}\right) := \sum_{I \subseteq [n]} \widehat{\varphi}(I) \hat{C}\left(\mathbf{u}_I, \mathbf{1}\right)$.

Theorem 3.12. *Let $\varphi \in \Phi_n$ and let $\hat{C}$ be an arbitrarily fixed n-dimensional copula. Then, for any choice of one-dimensional reliability functions $\bar{F}_1, \ldots, \bar{F}_n$, there exists a mixed binary system whose reliability function $R(t)$ is given by*

$$R(t) = M_{\varphi,\widehat{C}}\left(\bar{F}_1(t), \ldots, \bar{F}_n(t)\right),$$

for any $t > 0$.

Remark 3.13. *Let $\hat{C}$ be a fixed n-dimensional copula and consider the transformations*

$$\varphi \to \hat{\varphi} \text{ and } \varphi \to M_{\varphi},$$

where $\hat{\varphi}$ represents the Möbius transform and M_{φ} is the aggregation function defined in (3.5) as a copula-based extension of φ. Both of these transformations preserve convex combinations as pointed out in Proposition 3.9. Of course we cannot, stricto sensu, claim that such transformations are linear since the space Φ_n of all the multi-state systems is not a vector space. One can obtain a linear space by extending to $\mathbb{R}$ the target space of the functions φ. Thus the above considered transformations are indeed linear. In this case $\varphi : \{0,1\}^n \to \mathbb{R}$ admits a unique expression as a multilinear polynomial

$$\varphi(x_1, \ldots, x_n) = \sum_{J \subseteq [n]} a_J \prod_{j \in J} x_j.$$

The coefficients a_J in the canonical base are nothing but the values of the Möbius transform, that is. $a_J = \hat{\varphi}(J)$.

Remark 3.14. *As an immediate consequence of the relation* (3.2)*, also the random variable* $\mathcal{H}(t)$ *can be expressed as a multilinear polynomial*

$$\mathcal{H}(t) = \varphi(Y_1(t), \ldots, Y_n(t)) = \sum_{J \subseteq [n]} \hat{\varphi}(J) \prod_{j \in J} Y_j(t) \tag{3.19}$$

in the binary variables $Y_1(t), \ldots, Y_n(t)$*. This expression plays a key role for our results in view of linearity of expectations. In particular it point outs a more direct way to obtain the representation shown in the preceding theorem. Notice that* $\prod_{i \in I} Y_i(t)$ *represents the series system with components in* I *and that the values of the Möbius transform of the multi-state structure* φ *can be null, positive or negative. The polynomial form* (3.2) *for* φ *and the preceding one for* $\mathcal{H}(t)$ *are sometimes simple to obtain, remembering that the min (series structure) corresponds to a product and the max (parallel structure) to a coproduct, and that for Boolean variables* $x_i^2 = x_i$ *(see, e.g., p. 12 in [1]). Some examples are given in the following remark and in Section 5.*

Remark 3.15. *A very special and "extreme" class of fuzzy measures is obtained by imposing the condition of additivity:*

$$\varphi\left(y_1, \ldots, y_n\right) = \sum_{i=1}^{n} \alpha_i y_i,$$

for given non-negative quantities $\alpha_1, \ldots, \alpha_n$ *such that* $\sum_{i=1}^{n} \alpha_i = 1$*. In such a case the Möbius transform is given by*

$$\widehat{\varphi}\left(I\right) = \begin{cases} 0 & for\ |I| > 1, \\ \alpha_i & for\ I = \{i\}, \end{cases}$$

see, e.g., [10], p. 52. As an immediate consequence of (3.17), the aggregation function M_φ *becomes*

$$M_\varphi\left(\mathbf{u}\right) = \sum_{i=1}^{n} \alpha_i u_i$$

and thus it does not depend on the survival copula $\widehat{C}$*. This circumstance finds an immediate heuristic interpretation in the fact that additivity describes a lack of interaction, among the different components, for what concerns the level of the multi-state system. When the condition of additivity is combined with symmetry among the components, one obtains the very special case where the system level just coincides with the proportion of the working components in the system. This case is discussed in Example 5.1 below.*

In the above remark we have considered the extreme case of lack of interaction among the different components in contributing to the level of the system. An opposite extreme case is represented, in a sense, by the binary series system where $\varphi : \{0,1\}^n \to \{0,1\}$ is the supermodular function

$$\varphi\left(y_1, \ldots, y_n\right) = \min(y_1, \ldots, y_n). \tag{3.20}$$

Recalling the notation used in (2.2) for the case of binary systems, we write this time $M_\varphi = \overline{Q}$. The condition (3.20) gives rise to the case of maximum interaction among components (*unanimity game*) and the aggregation function $\overline{Q}$ does even coincide with the survival copula $\widehat{C}$. This can be seen as an immediate consequence of (3.17) (see also [20]). Moreover we can state a simple but remarkable characterization result. Remind that a function $g : \mathbb{R}^n \to \mathbb{R}$ is *supermodular* if the following inequality

$$g(\min(x_1, y_1), \dots, \min(x_n, y_n)) + g(\max(x_1, y_1), \dots, \max(x_n, y_n)) \geq g(x_1, \dots, x_n) + g(y_1, \dots, y_n)$$

holds for all $x_i, y_i \in \mathbb{R}$. Then the following properties are equivalent:

(i) $\overline{Q} = \overline{Q}_{\varphi,\widehat{C}}$ is supermodular.

(ii) φ has the form (3.20).

(iii) $\overline{Q}_{\varphi,\widehat{C}} = \widehat{C}$.

(iv) $\overline{Q}$ is a copula.

In view of the implication (ii) $\Rightarrow$ (iii), the above equivalences can be easily proven by taking into account the following facts:

- Since the fuzzy measure φ coincides with the restriction of the function $\overline{Q}$ over $\{0,1\}^n$, (i) implies that φ is supermodular.
- If a binary fuzzy measure φ is supermodular then it necessarily has the form (3.20) (see [2]). Hence (i) $\Rightarrow$ (ii).
- Any copula is supermodular and so (iv) $\Rightarrow$ (i).

4 Stochastic comparisons of multi-state systems

Different types of stochastic comparisons between two binary systems have been considered in the reliability literature, including those which have been reviewed in Section 2. Some of such results compare two systems with a common vector of marginal reliability functions and possibly different structure functions and/or survival copulas among component lifetimes. Other results compare two systems with different vectors of marginal distributions, but with same survival copula for them and same systems' structures. Results based on the concept of signature give finally rise, in the case of exchangeable lifetimes, to compare binary systems with same joint distribution for the component lifetimes (i.e., the same exchangeable copula and the same common marginal reliability function) and different structures.

The representations obtained in the preceding section can be used to extend those results to multi-state systems. The following two approaches emerge in a natural way and can be combined for this purpose:

a) Comparing the lifetimes of the associated (binary) mixed systems.

b) Comparing the discrete random variables $\mathcal{H}(t)$ and $\mathcal{H}^*(t)$ representing the states of the two systems at any time $t > 0$.

In this respect we have proven above (see Proposition 3.11) that the expected value $\mathbb{E}\left(\mathcal{H}(t)\right)$ coincides with a convex combination of the reliability functions of the binary system lifetimes $T_1, \dots, T_q$ at time t, and similarly for $\mathcal{H}^*(t)$. So the comparisons in both options are connected. Note that these convex combinations can be replaced by the reliability functions of the associated mixed systems. Thus the first result can be stated as follows.

Proposition 4.1. *Let T and T^* be the lifetimes of the mixed systems associated to two multi-state systems based on n components with reliability functions $\bar{F}_1, \ldots, \bar{F}_n$. Let $\bar{Q}$ and $\bar{Q}^*$ be the respective aggregation functions obtained from* (3.15)*. Then:*

(i) $T \leq_{st} T^*$ *for all* $\bar{F}_1, \ldots, \bar{F}_n$ *if and only if* $\bar{Q} \leq \bar{Q}^*$.

(ii) $T^* \leq_{hr} T^*$ *for all* $\bar{F}_1, \ldots, \bar{F}_n$ *if and only if* $\bar{Q}^*/\bar{Q}$ *is decreasing in* $(0,1)^n$.

The proof is immediate from (3.15) and Proposition 2.1. Some examples in the following section show how to use the preceding proposition to compare multi-state systems. More comparison properties can be obtained if we assume that the components are ordered or identically distributed by using the results given in [16] or [15], respectively.

Analogously, for multi-state systems with IID or EXC components, respectively, we have the following ordering properties.

Proposition 4.2. *Let T and T^* be the lifetimes of the mixed systems associated to two multi-state systems based on n IID components with a common reliability function $\bar{F}$. Let $\boldsymbol{s}$ and $\boldsymbol{s}^*$ be the respective signature vectors.*

(i) If $\boldsymbol{s} \leq_{st} \boldsymbol{s}^*$*, then* $T \leq_{st} T^*$ *for all* $\bar{F}$.

(ii) If $\boldsymbol{s} \leq_{hr} \boldsymbol{s}^*$*, then* $T \leq_{hr} T^*$ *for all* $\bar{F}$.

(iii) If $\boldsymbol{s} \leq_{lr} \boldsymbol{s}^*$*, then* $T \leq_{lr} T^*$ *for all absolutely continuous* $\bar{F}$.

The proof is immediate from Proposition 2.2, (3.15) and (3.16).

Proposition 4.3. *Let T and T^* be the lifetimes of the mixed systems associated to two multi-state systems based on n EXC components with lifetimes $X_1, \ldots, X_n$ having common joint reliability function $\bar{\mathbf{F}}$. Let $\boldsymbol{s}$ and $\boldsymbol{s}^*$ be the respective signatures and $X_{1:n}, \ldots, X_{n:n}$ the ordered component lifetimes.*

(i) If $\boldsymbol{s} \leq_{st} \boldsymbol{s}^*$*, then* $T \leq_{st} T^*$ *for all* $\bar{\mathbf{F}}$.

(ii) If $\boldsymbol{s} \leq_{hr} \boldsymbol{s}^*$*, then* $T \leq_{hr} T^*$ *for all* $\bar{\mathbf{F}}$ *such that* $X_{1:n} \leq_{hr} \cdots \leq_{hr} X_{n:n}$.

(iii) If $\boldsymbol{s} \leq_{lr} \boldsymbol{s}^*$*, then* $T \leq_{lr} T^*$ *for all absolutely continuous* $\bar{\mathbf{F}}$ *such that* $X_{1:n} \leq_{lr} \cdots \leq_{lr} X_{n:n}$.

The proof is immediate from Proposition 2.3, (3.15) and (3.16).

Notice that the results obtained so far refer to comparisons between two systems having different structures and the same components. Thus, in Propositions 4.2 and 4.3, we compare the mixed systems associated to two multi-state systems with different structures and same joint probability law for the components' lifetimes. However, in Proposition 4.1, we allow the lifetimes to also have different survival copulas, that is, only the vector of marginal reliability functions is required to be the same in the two systems. In fact we directly compare the corresponding aggregation functions.

In the following proposition we compare the discrete random variables which represent the states of two different system. Here we do not need to assume neither a common copula nor a common vector of marginals. Note that the ordering $T \leq_{st} T^*$ obtained above is equivalent to $\mathbb{E}\left(\mathcal{H}\left(t\right)\right) \leq \mathbb{E}\left(\mathcal{H}^*\left(t\right)\right)$ for all $t > 0$. We can obtain stronger orderings properties for the systems' states at time $t > 0$, $\mathcal{H}\left(t\right)$ and $\mathcal{H}^*\left(t\right)$, as follows. Note that from Remark 3.8 we can assume a common level set.

Proposition 4.4. *Let $\mathcal{H}(t)$ and $\mathcal{H}^*(t)$ be the state functions (processes) of two multi-state systems with the same level set $\mathcal{S} = \{0 = h_0 < h_1 < \cdots < h_q = 1\}$ and based on n components. Let R_i and R_i^* be the reliability functions of the respective associated binary systems for $i = 1, \ldots, q$. Let fix a $t > 0$. Then:*

(i) $\mathcal{H}(t) \leq_{st} \mathcal{H}^*(t)$ if and only if $R_i(t) \leq R_i^*(t)$ for all $i = 1, \ldots, q$.

(ii) $\mathcal{H}(t) \leq_{hr} \mathcal{H}^*(t)$ if and only if $R_i^*(t)/R_i(t)$ is increasing in i for all $i = 1, \ldots, q$.

(iii) $\mathcal{H}(t) \leq_{lr} \mathcal{H}^*(t)$ if and only if $(R_i^*(t) - R_{i+1}^*(t))/(R_i(t) - R_{i+1}(t))$ is increasing in i, for all $i = 0, \ldots, q$ where, by convention $R_0(t) := 1$ and $R_{m+1}(t) := 0$.

Note that the conditions of the preceding proposition can also be applied for all $t > 0$. Thus, for example, $\mathcal{H}(t) \leq_{st} \mathcal{H}^*(t)$ for all $t > 0$ if and only if $R_i(t) \leq R_i^*(t)$ for all $i = 1, \ldots, q$ and all $t > 0$. Then both conditions imply $\mathbb{E}(\mathcal{H}(t)) \leq \mathbb{E}(\mathcal{H}^*(t))$ for all $t > 0$. Recall that this last property can also be written as $T \leq_{st} T^*$ where T and T^* represent the lifetimes of the respective associated mixed systems. To summarize, from (2.3), we have

$$\mathcal{H}(t) \leq_{lr} \mathcal{H}^*(t)\ \forall t > 0 \Rightarrow \mathcal{H}(t) \leq_{hr} \mathcal{H}^*(t)\ \forall t > 0 \Rightarrow \mathcal{H}(t) \leq_{st} \mathcal{H}^*(t)\ \forall t > 0 \Rightarrow T \leq_{st} T^*.$$

Also note that the condition $R_i(t) \leq R_i^*(t)$ for all $i = 1, \ldots, m$ and all $t > 0$ is equivalent to get $T_i \leq_{st} T_i^*$ for all $i = 1, \ldots, m$, that is, the associated binary systems are st-ordered. This property does not necessarily hold for the other orders. A counterexample is given in Example 5.3 where $T_i \leq_{hr} T_i^*$ holds for $i = 1, \ldots, m$ but $\mathcal{H}(t) \leq_{hr} \mathcal{H}^*(t)$ does not hold.

The results given in [20] to compare systems with homogeneous (ID) components and heterogeneous components can also be extended to multi-state systems. The extensions are straightforward from representation (3.15). First we need the following definition.

Definition 4.5. *Let $g : \mathbb{R}^n \to \mathbb{R}$ be a real-valued function. We say that g is weakly Schur-concave (weakly Schur-convex) if*

$$g(u_1, \ldots, u_n) \leq g(\bar{u}, \ldots, \bar{u}) \quad (\geq)$$

for all $(u_1, \ldots, u_n)$, where $\bar{u} = (u_1 + \cdots + u_n)/n$.

Now we can state the following result for multi-state systems which extends Theorem 5.4 in [20]. The other results in [20] can be extended in a similar way. Again we compare two multi-state systems just by comparing their associated mixed systems. Remember again that we have proven that the reliability functions of the mixed system lifetimes at time $t \geq 0$ coincide with the expectations of the respective states at time t.

Proposition 4.6. *Let T and T^* be the lifetimes of the mixed systems associated to two multi-state systems based on the same structure function φ and on n components with a common survival copula $\hat{C}$ and reliability functions $\bar{F}_j$ and $\bar{F}_j^* = (\bar{F}_1 + \cdots + \bar{F}_n)/n$ for $j = 1, \ldots, n$. Let M_φ be the common aggregation function of both systems in representation* (3.15). *If M_φ is weakly-Schur concave (weakly-Schur convex) then $T \leq_{st} T^*$ ($\geq_{st}$).*

The proof is immediate from (3.15). Note that the preceding result can be used to compare multi-state systems with heterogeneous components with systems with homogeneous systems (or with randomly chosen components). It proves that if M_φ is weakly-Schur concave, then the system with the randomly chosen components is st better than that with heterogeneous components. This is called the Parrondo's paradox in the reliability theory (see [6]).

5 Examples

Let us see now how to apply the preceding theoretical results to some multi-state systems. In the first example, we study the most natural multi-state system.

Example 5.1. *Let us consider the multi-state structure defined by*

$$\varphi(y_1,\dots,y_n)=\frac{y_1+\cdots+y_n}{n}$$

that is, the system level is just the proportion of working components in the system. Then $m=n$ and $\mathcal{S}=\{0<1/n<2/n<\cdots<1\}$. Hence the binary systems defined by (3.8) *coincide with the k-out-of-n systems. For example, $\varphi_1(y_1,\dots,y_n)=1$ if and only if*

$$\varphi(y_1,\dots,y_n)=\frac{y_1+\cdots+y_n}{n}\geq\frac{1}{n},$$

that is, $y_1+\cdots+y_n\geq 1$ (at least a component works). Hence

$$T_1=\max(X_1,\dots,X_n)=X_{n:n}$$

is the lifetime of the parallel system. Analogously, we get $T_j=X_{n-j+1:n}$ for $j=1,\dots,n$. Note that $T_1\geq\cdots\geq T_n$. Therefore, from Proposition 3.2, we have

$$\mathbb{E}\left(\mathcal{H}\left(t\right)\right)=\frac{1}{n}\bar{F}_{1:n}(t)+\cdots+\frac{1}{n}\bar{F}_{n:n}(t),$$

that is, the associated mixed system just amounts to choosing randomly a k-out-of-n system. Hence, the signature of the multi-state system is $\mathbf{r}=(1/n,\dots,1/n)$. It is well known that

$$\frac{1}{n}\bar{F}_{1:n}(t)+\cdots+\frac{1}{n}\bar{F}_{n:n}(t)=\frac{1}{n}\bar{F}_1(t)+\cdots+\frac{1}{n}\bar{F}_n(t)$$

since choosing randomly a component is equivalent to choosing randomly an ordered component. Hence,

$$\mathbb{E}\left(\mathcal{H}\left(t\right)\right)=\frac{1}{n}\bar{F}_1(t)+\cdots+\frac{1}{n}\bar{F}_n(t),$$

that is, the associated mixed system just amounts to choosing randomly a component. Hence, the aggregation function $\bar{Q}$ in (3.15)*, is*

$$\bar{Q}(u_1,\dots,u_n)=\frac{1}{n}u_1+\cdots+\frac{1}{n}u_n$$

for $0\leq u_j\leq 1$ and $j=1,\dots,n$. Note that, in this example, $\bar{Q}$ does not depend on the survival copula.

If we want to compare two multi-state systems with this structure and different components we have $T\leq_{st}T^*$ for the respective mixed systems if and only if the respective component reliability functions satisfy

$$\bar{F}_1(t)+\cdots+\bar{F}_n(t)\leq\bar{G}_1(t)+\cdots+\bar{G}_n(t)$$

for all t.

In the second example we study a multi-state system which can represent the computer system used in an online store website.

Example 5.2. *Let us consider the multi-state system $\varphi^* : \{0,1\}^n \to \mathcal{S}$ defined by*

$$\varphi^*(y_1, \ldots, y_n) = \frac{y_1 + \max(y_2, \ldots, y_n)}{2}.$$

Then $m = 2$ and $\mathcal{S} = \{0 < 1/2 < 1\}$. Here, the first component represents the computer server which redirects the customers to the online system placed in the computers represented by $X_2, \ldots, X_n$ (depending on the queue in each computer). This is the usual system scheme represented by state 1. In the emergency mode, represented by level 1/2, any computer can be at the same time the server and the store system, and the system works in this state if at least a computer works (but more slowly).

Hence the first binary system defined by (3.8) is given by

$$\varphi_1^*(y_1, \ldots, y_n) = 1 \text{ if and only if } \max(y_1, \ldots, y_n) = 1,$$

that is, $T_1 = \max(X_1, \ldots, X_n) = X_{n:n}$ is the lifetime of the parallel system (as in the preceding example). Its reliability function can be obtained as

$$R_1^*(t) = \bar{F}_{n:n}(t) = \sum_{i=1}^{n} \bar{F}_i(t) - \sum_{i<j} \Pr(X_i > t, X_j > t) + \cdots + (-1)^{n+1} \Pr(X_1 > t, \ldots, X_n > t). \quad (5.1)$$

Alternatively, this expression can be obtained from (3.15) and (3.19) as follows. First, note that the Boolean function φ_1^* can also be written as

$$\varphi_1^*(y_1, \ldots, y_n) = \max(y_1, \ldots, y_n) = 1 - (1 - y_1) \ldots (1 - y_n) = \sum_{\emptyset \neq A \subseteq [n]} (-1)^{|A|+1} \prod_{i \in A} y_i$$

for $y_1, \ldots, y_n \in \{0,1\}$. Therefore, from (3.15), its reliability function at time t is

$$R_1^*(t) = \Pr(T_1 > t) = \mathbb{E}\left(\varphi_1^*(Y_1(t), \ldots, Y_n(t))\right),$$

where $Y_1(t), \ldots, Y_n(t)$ represent the component states at time t. Hence

$$R_1^*(t) = \mathbb{E}\left(\sum_{\emptyset \neq A \subseteq [n]} (-1)^{|A|+1} \prod_{i \in A} Y_i(t)\right) = \sum_{\emptyset \neq A \subseteq [n]} (-1)^{|A|+1} \mathbb{E}\left(\prod_{i \in A} Y_i(t)\right),$$

where $\mathbb{E}\left(\prod_{i \in A} Y_i(t)\right) = \Pr\left(\cap_{i \in A} X_i\right)$. Thus we obtain (5.1).

Analogously, the second binary system is defined by

$$\varphi_2^*(y_1, \ldots, y_n) = 1 \text{ if and only if } y_1 = 1 \text{ and } \max(y_2, \ldots, y_n) = 1,$$

that is,

$$\varphi_2^*(y_1, \ldots, y_n) = \min(y_1, \max(y_2, \ldots, y_n)).$$

The minimal path sets of this system are $P_i = \{1, i\}$ for $i = 2, \ldots, n$ and so its reliability function is

$$R_2^*(t) = \sum_{i=2}^{n} \Pr(X_1 > t, X_i > t) - \sum_{1<i<j} \Pr(X_1 > t, X_i > t, X_j > t) + \cdots + (-1)^n \Pr(X_1 > t, \ldots, X_n > t)$$

(see, e.g., [15]). *Proceeding as above, this expression can also be obtained from* (3.15) *and* (3.19) *by writing the Boolean function* φ_2^* *as*

$$\varphi_2^*(y_1,\dots,y_n)=\min(y_1,\max(y_2,\dots,y_n))=y_1(1-(1-y_2)\dots(1-y_n)).$$

Hence,

$$\mathbb{E}\left(\mathcal{H}^*\left(t\right)\right)=\frac{1}{2}R_1^*(t)+\frac{1}{2}R_2^*(t),$$

that is,

$$\mathbb{E}\left(\mathcal{H}^*\left(t\right)\right)=\frac{1}{2}\sum_{i=1}^{n}\bar{F}_i(t)-\frac{1}{2}\sum_{1<i<j}\Pr(X_i>t,X_j>t)+\cdots+\frac{1}{2}(-1)^n\Pr(X_2>t,\dots,X_n>t).$$

Note that if $\hat{C}$ *is the survival copula and* $J\subseteq[n]$*, then*

$$\Pr(X_j>t \textit{ for all } j\in J)=\hat{C}_J(\bar{F}_1(t),\dots,\bar{F}_n(t)),$$

where $\hat{C}_J(\mathbf{u}):=\hat{C}(\mathbf{u}_J,1)$ *is the marginal survival copula of the vector of random variables* X_j *for* $j\in J$*. Therefore*

$$\mathbb{E}\left(\mathcal{H}^*\left(t\right)\right)=\hat{Q}^*(\bar{G}_1(t),\dots,\bar{G}_n(t))$$

with

$$\hat{Q}^*(\mathbf{u})=\frac{1}{2}\sum_{i=1}^{n}u_i-\frac{1}{2}\sum_{1<i<j}\hat{C}_{\{i,j\}}(\mathbf{u})+\cdots+(-1)^n\frac{1}{2}\hat{C}_{\{2,\dots,n\}}(\mathbf{u}).$$

If we assume that the component lifetimes are exchangeable (or IID), then the signatures of the binary systems are $(0,\dots,0,1)$ *and* $(1/n,\dots,1/n,2/n,0)$*, respectively. Therefore, the signature of the multi-state system is*

$$\mathbf{r}^*=\frac{1}{2}(0,\dots,0,1)+\frac{1}{2}\left(\frac{1}{n},\dots,\frac{1}{n},\frac{2}{n},0\right)=\left(\frac{1}{2n},\dots,\frac{1}{2n},\frac{1}{n},\frac{1}{2}\right).$$

Hence, if we want to compare this system with the system studied in the preceding example with signature $\mathbf{r}=(1/n,\dots,1/n)$*, we obtain*

$$\frac{1/(2n)}{1/n}=\frac{1}{2}=\cdots=\frac{1/(2n)}{1/n}=\frac{1}{2}<\frac{1/n}{1/n}=1\leq\frac{1/2}{1/n}=\frac{n}{2}$$

which is true for $n\geq 2$*. In this case, from Proposition 4.2, the mixed system associated to the multi-state system of the present example is lr-greater than that of the preceding example for all absolutely continuous EXC joint reliability functions* $\bar{\mathbf{F}}$ *such that* $X_{1:n}\leq_{lr}\cdots\leq_{lr}X_{n:n}$ *holds. Analogously, it can be proved from Proposition 4.2 that they are st ordered for any EXC* $\bar{\mathbf{F}}$ *and that they are hr ordered for any EXC* $\bar{\mathbf{F}}$ *such that* $X_{1:n}\leq_{hr}\cdots\leq_{hr}X_{n:n}$ *holds.*

For $n=2$*, the signatures are* $(1/2,1/2)$ *and* $(1/2,1/2)$ *and so the mixed systems are equal in law when the components are EXC. This last property is true for general components (since when* $n=2$*,* T_2^* *is the series system).*

However, for general components these two systems are not st-ordered. For example, if $n=3$*, then the respective aggregation functions are*

$$\bar{Q}(u_1,u_2,u_3)=\frac{u_1+u_2+u_3}{3}$$

(for the system in the preceding example) and

$$\bar{Q}^*(u_1,u_2,u_3)=\frac{u_1+u_2+u_3}{2}-\frac{1}{2}\,\hat{C}(1,u_2,u_3)$$

(for the system in the present example). These two functions are not ordered because

$$\bar{Q}(0,0,1)=\frac{1}{3}<\frac{1}{2}=\bar{Q}^*(0,0,1)$$

and

$$\bar{Q}(0,1,1)=\frac{2}{3}>\frac{1}{2}=\frac{2}{2}-\frac{1}{2}=\bar{Q}^*(0,1,1).$$

So, from Proposition 4.1, the st order does not hold for general components. This example proves that the signature representation (3.16) *does not hold for general components (if it would hold, then these systems will be st-ordered since their signatures are st-ordered). Note that if the components are independent (i.e.* $\hat{C}(u_1,u_2,u_3)=u_1u_2u_3$*) then the first system is less than the second in the st order if and only if* $(\bar{F}_1(t),\bar{F}_3(t),\bar{F}_3(t))$ *belongs to the set*

$$D=\{(u_1,u_2,u_3)\in[0,1]^3:u_1+u_2+u_3-3u_2u_3\geq 0\}$$

for all t*. This set contains the diagonal since, for* (u,u,u)*, we obtain* $3u-3u^2\geq 0$ *for all* $u\in[0,1]$*. So they are st-ordered in the case of IID components (actually, they are lr-ordered as proved above by using signatures).*

In Figure 1 we plot $\mathbb{E}\left(\mathcal{H}\left(t\right)\right)$ *for both systems when* $n=3$ *and the components are IID with a common exponential distribution with mean* 1 *(left) or they are just independent with exponential distributions with means* $1,5,5$*, respectively (right). In the first case the reliability curves are ordered but in the second they cross.*

In the last example we compare stochastically the lifetimes T and T^* and the random variables $\mathcal{H}\left(t\right)$ and $\mathcal{H}^*\left(t\right)$ with the states at time t of two multi-state systems.

Example 5.3. *Let us consider the multi-state system* $\varphi^*:\{0,1\}^n\to\mathcal{S}$ *defined by*

$$\varphi^*(y_1,\dots,y_n)=j/n \text{ if } y_1=\cdots=y_j=1 \text{ and } y_{j+1}=0.$$

Then $m=n$ *and* $\mathcal{S}=\{0<1/n<2/n<\cdots<1\}$*. In this system the first component is the most important (if it does not work, the system does not work). Then level* $1/n$ *is attained when just the first component is working. To get level* $2/n$ *we need the first and the second components and so on. To get the perfect functioning (level* 1*) we need all the components to be working. Hence the binary systems defined by* (3.8) *coincide with the series systems with lifetimes* $T_j^*=X_{1:j}=\min(X_1,\dots,X_j)$ *for* $j=1,\dots,n$*. Their reliability functions are*

$$R_j^*(t)=\bar{F}_{1:j}(t)=\Pr(X_1>t,\dots,X_j>t)=\hat{C}(\bar{F}_1(t),\dots,\bar{F}_j(t),1,\dots,1)$$

for $j=1,\dots,n$ *where* $\hat{C}$ *represents the survival copula. In particular,* $R_1=\bar{F}_{1:1}=\bar{F}_1$*. Hence, from* (3.11)*, we have*

$$\mathbb{E}\left(\mathcal{H}^*\left(t\right)\right)=\frac{1}{n}\sum_{j=1}^{n}\bar{F}_{1:j}(t)=\frac{1}{n}\sum_{i=j}^{n}\hat{C}(\bar{F}_1(t),\dots,\bar{F}_j(t),1,\dots,1)=\bar{Q}(\bar{F}_1(t),\dots,\bar{F}_n(t)),$$

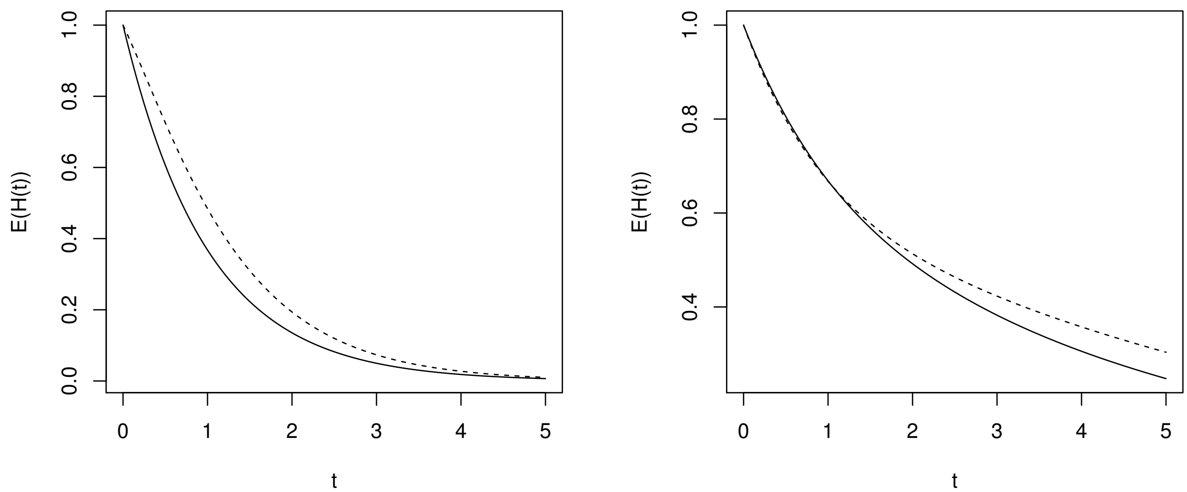

Figure 1: Plots of $\mathbb{E}\left(\mathcal{H}\left(t\right)\right)$ for the systems in Examples 5.1 and 5.2 when $n = 3$ and the components are IID with a common exponential distribution with mean 1 (left) or they are just independent with exponential distributions with means $1, 5, 5$ (right).

where

$$\bar{Q}^*(u_1, \ldots, u_n) = \frac{1}{n}\sum_{j=1}^{n} \hat{C}(u_1, \ldots, u_j, 1, \ldots, 1).$$

Hence, if the components are EXC, its minimal signature is $\mathbf{a} = (1/n, \ldots, 1/n)$. *If the components are independent, then*

$$\bar{Q}^*(u_1, \ldots, u_n) = \frac{1}{n}\sum_{j=1}^{n} u_1 \ldots u_j.$$

If we want to compare this system with the system defined in the first example, from Proposition 4.1, (i), we obtain

$$\mathbb{E}\left(\mathcal{H}^*\left(t\right)\right) \le \mathbb{E}\left(\mathcal{H}\left(t\right)\right)$$

for all $\bar{F}_1, \ldots, \bar{F}_n$ *and all* $\hat{C}$ *since*

$$\bar{Q}^*(\mathbf{u}) = \frac{1}{n}\sum_{j=1}^{n} \hat{C}(u_1, \ldots, u_j, 1, \ldots, 1) \le \frac{1}{n}\sum_{j=1}^{n} \hat{C}(1, \ldots, 1, u_j, 1, \ldots, 1) = \frac{1}{n}\sum_{j=1}^{n} u_j = \bar{Q}(\mathbf{u}).$$

Equivalently, if T *and* T^* *represent the lifetimes of the associated mixed systems, we have proved that* $T^* \le_{st} T$. *Note that the binary systems are also st ordered, that is,* $T_j^* = X_{1:j} \le_{st} X_{n-j+1:n} = T_j$ *for* $j = 1, \ldots, n$. *However,* T *and* T^* *are not always hr-ordered. For example, if* $n = 3$ *and the components are independent then, from Proposition 4.1, (ii),* $T^* \le_{hr} T$ *holds if and only if*

$$\frac{\bar{Q}(u_1, u_2, u_3)}{\bar{Q}^*(u_1, u_2, u_3)} = \frac{u_1 + u_2 + u_3}{u_1 + u_1u_2 + u_1u_2u_3}$$

is decreasing in $(0,1)^3$*. It is easy to see that it is decreasing in* u_1 *but that it is not always decreasing in* u_2, u_3*. For example, we have*

$$\frac{\bar{Q}(0.1,0,0.1)}{\bar{Q}^*(0.1,0,0.1)} = 2 < 2.702703 = \frac{\bar{Q}(0.1,0.1,0.1)}{\bar{Q}^*(0.1,0.1,0.1)}.$$

We can prove that they are hr-ordered in the case of IID components by using the results obtained for distorted distributions (see, e.g., Theorem 4 in [15]). In this case, the system reliability functions can be written as $\bar{F}_T(t) = \bar{q}(\bar{F}(t))$ *and* $\bar{F}_{T^*}(t) = \bar{q}^*(\bar{F}(t))$ *where* $\bar{F}$ *represents the common reliability,* $\bar{q}(u) = \bar{Q}(u,\dots,u) = u$ *and* $\bar{q}^*(u) = \bar{Q}^*(u,\dots,u) = (u+\dots+u^n)/n$*. Hence* $T^* \le_{hr} T$ *holds for all* $\bar{F}$ *since*

$$\frac{\bar{q}(u)}{\bar{q}^*(u)} = \frac{n}{1+u+\dots+u^{n-1}}$$

is decreasing in $(0,1)$*. Even more, from Theorem 4, (iv), in [15],* $T^* \le_{lr} T$ *holds for all* $\bar{F}$ *since*

$$\frac{\bar{q}'(u)}{(\bar{q}^*)'(u)} = \frac{n}{1+2u+\dots+nu^{n-1}}$$

is decreasing in $(0,1)$*.*

The same results can be proved by using signatures but the calculations are more complex. As we have seen in Example 5.1, the signature of T *is* $\mathbf{r} = (1/n,\dots,1/n)$*. To get the signature of the present system we need the signatures of order* n *of the series systems. It is well known (see, e.g. [19]) that the signature of order* n *of* $T_1 = X_{1:1}$ *is* $(1/n,\dots,1/n)$*. Of course, the signature of* $T_n = X_{1:n}$ *is* $(1,0,\dots,0)$*. A straightforward calculation shows that the signature of order* n *of* $T_k = X_{1:k}$ *for* $k=2,\dots,n$ *is* $(s_1^{(k)},\dots,s_n^{(k)})$ *with*

$$s_i^{(k)} = \frac{k(n-i)(n-i-1)\dots(n-i+2-k)}{n(n-1)\dots(n-k+1)}$$

for $i=1,\dots,n$ *(see (2.7) in [19]). Note in passing that there is a typo in expression (2.5) in [19] (i should be replaced with j at some positions). Hence, the signature of the present system is* $\mathbf{r}^* = (r_1^*,\dots,r_1^*)$ *with*

$$r_i^* = \frac{1}{n}\sum_{k=1}^{n} s_i^{(k)} = \frac{1}{n}\left(1+\sum_{k=2}^{n}\frac{k(n-i)(n-i-1)\dots(n-i+2-k)}{n(n-1)\dots(n-k+1)}\right)$$

for $i=1,\dots,n$*. For example, for* $n=4$*, the signatures of order* 4 *of the binary systems* $T_1 = X_1$*,* $T_2 = X_{1:2}$*,* $T_3 = X_{1:3}$*, and* $T_4 = X_{1:4}$ *are* $(1/4,1/4,1/4,1/4)$*,* $(1/2,1/3,1/6,0)$*,* $(3/4,1/4,0,0)$ *and* $(1,0,0,0)$*, respectively. Therefore* $\mathbf{r}^* = (30/48, 10/48, 5/48, 3/48)$*. Hence* $\mathbf{r}^* \le_{lr} \mathbf{r}$ *since*

$$\frac{1/4}{30/48} = \frac{2}{5} < \frac{1/4}{10/48} = \frac{6}{5} < \frac{1/4}{5/48} = \frac{12}{5} < \frac{1/4}{32/48} = 4.$$

Therefore, if the components are IID$\sim\bar{F}$*, from Proposition 4.2, (iii), we have* $T^* \le_{lr} T$ *holds for all absolutely continuous* $\bar{F}$*. Analogously, from Proposition 4.2, (ii), we obtain* $T^* \le_{hr} T$ *for all* $\bar{F}$*. Even more, if the components are EXC with a common joint reliability function* $\bar{\mathbf{F}}$*, then* $T \le_{lr} T^*$ *for all absolutely continuous* $\bar{\mathbf{F}}$ *such that* $X_{1:n} \le_{lr} \dots \le_{lr} X_{n:n}$*,* $T \le_{hr} T^*$ *for all* $\bar{\mathbf{F}}$ *such* *that* $X_{1:n} \leq_{hr} \cdots \leq_{hr} X_{n:n}$ *and* $T \leq_{st} T^*$ *for all* $\bar{\mathbf{F}}$. *Note that the last property (the st order) can be extended to all kind of components as proved above by using aggregations.*

Let us use now Proposition 4.4 to compare the random variables representing the states of both systems at time t. *First, we note that* $T_j^* = X_{1:j} \leq X_{n-j+1} = T_j$ *for* $j = 1, \ldots, n$. *Hence* $R_j^*(t) \leq R_j(t)$ *for all* t *and all* $j = 1, \ldots, n$ *and so* $\mathcal{H}^*(t) \leq_{st} \mathcal{H}(t)$ *for all* t. *In particular, by taking expectations, we obtain again* $T \leq_{st} T^*$ *for all* $\bar{\mathbf{F}}$. *However, the hazard rate order* $\mathcal{H}^*(t) \leq_{hr} \mathcal{H}(t)$ *does not necessarily hold. For example, if* $n = 2$, *from Proposition 4.4, (ii) it holds if and only if*

$$\frac{R_0(t)}{R_0^*(t)} = 1 \leq \frac{R_1(t)}{R_1^*(t)} \leq \frac{R_2(t)}{R_2^*(t)}.$$

If we assume that the components are IID$\sim \bar{F}$, *then this expression reduces to*

$$1 \leq \frac{2\bar{F}(t) - \bar{F}^2(t)}{\bar{F}(t)} \leq 1$$

which leads to $\bar{F}(t) = 0, 1$. *However, it does not hold when* $0 < \bar{F}(t) < 1$. *Hence* $\mathcal{H}^*(t) \leq_{hr} \mathcal{H}(t)$ *does not hold for all* t *and all non-atom reliability functions* $\bar{F}$. *For general components we get*

$$1 \leq \frac{\bar{F}_1(t) + \bar{F}_2(t) - \hat{C}(\bar{F}_1(t), \bar{F}_2(t))}{\bar{F}_1(t)} \leq 1$$

which leads to $\bar{F}_2(t) = \hat{C}(\bar{F}_1(t), \bar{F}_2(t))$. *This property holds for all* t *when the components are ID and* $\hat{C}(u, v) = \min(u, v)$. *This copula represents the extreme positive dependence and implies* $\Pr(X_1 = X_2) = 1$ *and* $X_1 =_{st} X_{2:2}$.

However, if the components are IID, then the mixed systems are hr-ordered since

$$T_1^* = X_1 \leq_{hr} X_{2:2} = T_1$$

and

$$T_2^* = X_{1:2} =_{hr} X_{1:2} = T_2.$$

Hence $T_i^* \leq_{hr} T_i$ *for* $i = 1, 2$ *does not imply* $\mathcal{H}^*(t) \leq_{hr} \mathcal{H}(t)$ *for all* t.

Obviously, the second condition above ($T_2^* = X_{1:2} =_{hr} X_{1:2} = T_2$*) holds for general components. However, if the components are ID, from Theorem 4.1 in [21], the first condition* $X_1 \leq_{hr} X_{2:2}$ *holds if and only if* $\hat{C}(u, u)/u$ *is increasing in* $(0, 1)$. *This last property is not always true as proved by Examples 4.1 and 4.2 in [21]. From Proposition 2.1, (ii),* $X_1 \leq_{hr} X_{2:2}$ *holds for all* $\bar{F}_1, \bar{F}_2$ *if and only if* $(v - \hat{C}(u, v))/u$ *is decreasing in* $(0, 1)^2$. *This last property is not always true. For example, it is not true when the components are just independent (see Figure 1 in [21]). However, note that, in this example,* $\mathcal{H}^*(t) \leq_{hr} \mathcal{H}(t)$ *for all* t, *implies* $X_1 =_{st} X_{2:2}$ *and so* $T_i^* \leq_{hr} T_i$ *holds for all* $i = 1, 2$. *It is easy to prove that this property is not true in general. For example, if* $m = 1$ *(binary systems), then* $\mathcal{H}^*(t) \leq_{hr} \mathcal{H}(t)$ *for all* t *is equivalent to* $T^* \leq_{st} T$ *which does not imply* $T^* \leq_{hr} T$ *(see, e.g., the examples given in [15]).*

6 Discussion and concluding remarks

We have provided several tools to stochastically compare the performance of multi-state systems with binary components. First, we have noted that their structure functions can be seen as fuzzy measures and we have defined the associated binary systems. The main property is that the multi-state system can be represented by convex combinations of these binary systems. As a consequence of this property we have obtained the representations based on signatures and aggregation functions for multi-state systems. The aggregation function can be also obtained from the Möbius transform. These representations are used to compare multi-state systems under different stochastic criteria. Specifically, we have obtained conditions for the usual stochastic order (reliability), the hazard rate order and the likelihood ratio order. Some examples and counterexamples show how to apply these theoretical results to specific multi-state systems.

The topic of performance of multi-state systems is then strictly related with aggregation functions of the form

$$M_{\varphi,C}(\mathbf{u}) := \sum_{I\subseteq[n]} \widehat{\varphi}(I)\ C(\mathbf{u}_I, \mathbf{1}),$$

where $\varphi : \{0,1\}^n \to [0,1]$ is a regular fuzzy measure and $C : [0,1]^n \to [0,1]$ is a copula. As well-known, such functions are of general interest also beyond the topic of the present paper. In the special case when C is the *comonotonicity* or *maximal* copula

$$C(u_1, ..., u_n) = \min(u_1, \ldots, u_n)$$

for instance, $M_{\varphi,C}(\mathbf{u})$ coincides with the *discrete Choquet integral*.

General properties of analytical type for such functions are directly obtained from results presented in [12]. Actually, that paper analyses the larger class of aggregation functions of the form

$$M_{\varphi,A}(\mathbf{u}) := \sum_{I\subseteq[n]} \widehat{\varphi}(I) A(\mathbf{u}_I, \mathbf{1})$$

where the copula C is replaced by a more general aggregation function A. In particular one obtains that the function $M_{\varphi,C}(\mathbf{u})$ can be seen as an extension to $[0,1]^n$ of the fuzzy measure φ defined over $\{0,1\}^n$.

Some further aspects, concerning the structure of the aggregation functions $M_{\varphi,C}$, can be derived from the arguments presented in the above Section 3. By combining propositions 3.9 and 3.10, in particular, we see that any such aggregation function can be obtained as a linear combination of aggregation functions extending binary fuzzy measures. Such a property might also be extended to aggregation functions of the form $M_{\varphi,A}$.

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References

[1] Barlow RE, Proschan F. Statistical theory of reliability and life testing. International Series in Decision Processes. Holt, Rinehart and Winston, Inc., NewYork, 1975.

[2] Block HW, Griffith WS, Savits TH. L-superadditive structure functions. Advances in Applied Probability 21 (1989), 919–929.

[3] Block HW, Savits, TH. A decomposition for multistate monotone systems. Journal of Applied Probability 19 (1982), 391–402.

[4] Boland P. Signatures of indirect majority systems. Journal of Applied Probability 38 (2001), 597–603.

[5] Boland P, Samaniego FJ. The signature of a coherent system and its applications in reliability. In: *Mathematical Reliability: An Expository Perspective* (R. Soyer, T. A. Mazzuchi, N. D. Singpurwalla, eds.), Internat. Ser. Oper. Res. Managament Sci., vol. **67**, Kluwer, Boston, 2004, pp. 1–29.

[6] Di Crescenzo A. A Parrondo paradox in reliability theory. The Mathematical Scientist 32 (2007), 17–22.

[7] Durante F, Sempi C. Principles of Copula Theory. CRC/Chapman & Hall, London, 2016.

[8] Duchovny P, Marichal J-L. Reliability of systems with dependent components based on lattice polynomial description. Stochastic Models 28 (2012), 167–184.

[9] Fantozzi F, Spizzichino F. Multi-attribute target-based utilities and extensions of fuzzy measures. Fuzzy Sets and Systems 259 (2015), 29–43.

[10] Grabisch M. Set Functions, Games and Capacities in Decision Making. Springer, 2016.

[11] Kochar S, Mukerjee H, Samaniego FJ. The 'signature' of a coherent system and its application to comparisons among systems. Naval Research Logistics 46 (1999), 507–523.

[12] Kolesárová A, Stupňanová A, Beganová J. Aggregation-based extensions of fuzzy measures. Fuzzy Sets and Systems 194 (2012), 1–14.

[13] Marichal JL, Mathonet P, Navarro J, Paroissin C. Joint signature of two or more systems with applications to multi-state systems made up of two-state components. European Journal of Operational Research 263 (2017), 559–570.

[14] Marichal J-L, Mathonet P, Tousset E. Characterization of some aggregation functions stable for positive linear transformations. Fuzzy Sets and Systems 102 (1999), 293–314.

[15] Navarro J. Stochastic comparisons of coherent systems. Metrika 81 (2018), 465–482.

[16] Navarro J, del Aguila Y. Stochastic comparisons of distorted distributions, coherent systems and mixtures with ordered components. Metrika 80 (2017), 627–648.

[17] Navarro J, del Águila Y, Sordo MA, Suárez-Llorens A. Preservation of stochastic orders under the formation of generalized distorted distributions. Applications to coherent systems. Methodology and Computing in Applied Probability 18 (2016), 529–545.

[18] Navarro J, Ruiz J., Sandoval C. A note on comparisons among coherent systems with dependent components using signatures. Statistics and Probability Letters, 72 (2008), 179–185

[19] Navarro J, Samaniego FJ, Balakrishnan N, Bhattacharya D. On the application and extension of system signatures to problems in engineering reliability. Naval Research Logistics 55 (2008), 313–327.

[20] Navarro J, Spizzichino F. Comparisons of series and parallel systems with components sharing the same copula. Applied Stochastic Models in Business and Industry 26 (2010), 775–791.

[21] Navarro J, Torrado N, Águila, Y. Comparisons between largest order statistics from multiple-outlier models with dependence. Methodology and Computing in Applied Probability 20 (2018), 411–433.

[22] Nelsen RB. An introduction to copulas. Second edition. Springer, New York, 2006.

[23] Radojevic D. Logical interpretation of discrete Choquet integral defined by general measure. International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 7 (1999), 577–588.

[24] Ramamurthy KG. Coherent structures and simple games. Kluwer Academic Publishers Group, Dordrecht, 1990.

[25] Rychlik T, Navarro J, Rubio R. Effective procedure of verifying stochastic ordering of system lifetimes. Journal of Applied Probability 55 (2018), 1261–1271.

[26] Samaniego FJ, Navarro J. On comparing coherent systems with heterogeneous components. Advances in Applied Probability 48 (2016), 88–111.

[27] Shaked M., Shanthikumar JG. Stochastic orders. Springer Series in Statistics. Springer, New York, 2007.

[28] Spizzichino, FL. On the probabilistic meaning of copula-based extensions of fuzzy measures. Applications to target-based utilities and multi-state reliability systems. Fuzzy Sets and Systems 354 (2019), 1–19.

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