Conditions on marginals and copula of component lifetimes for signature representation of system lifetime

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Abstract

The signature of a system is a probability vector that depends only on the system structure. Under the classic IID (independent and identically distributed) assumption on the component lifetimes, the system lifetime distribution is the convex combination of consecutive component failure times, and the signature coordinates constitute the mixture coefficients. In this case the signature representations are very useful in determining the system lifetime distributions and for stochastic comparisons of them. This first representation was obtained in 1985 by Samaniego. Then it was extended to the more general case of exchangeable component lifetimes. In 2011 Marichal, Mathonet and Waldhauser presented necessary and sufficient conditions assuring the Samaniego representation. There were expressed in terms of distributional properties of families of auxiliary indicator random vectors parametrized by positive numbers. In the paper we obtain other necessary and sufficient conditions represented in terms of the marginal distributions of component lifetimes and the dependence copula of them. Moreover, we study symmetry conditions for the equality of structural and probabilistic signatures.

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1 Introduction

The general interest of the concept of signature in the field of reliability is a well-known fact. The study of such a concept leads to analyze interesting and specific aspects of stochastic dependence among non-negative random variables. More precisely, such a study leads to compare different properties of partial symmetry, which can be respectively seen as suitable generalizations of the exchangeability condition.

The first signature representation was obtained by Samaniego [22] for coherent systems with independent and identically (IID) distributed components having a common continuous distribution function F. This representation proves that the system distribution is a mixture (linear combination with non-negative weights) of the distributions of the ordered component lifetimes. The vector formed with the coefficients in that representation does not depend on F and was called the *signature* of the system. It is composed of the probabilities that the system fails with the consecutive component failures (for the formal definitions of probabilistic and structural signatures see Section 2). These distributions coincide with that of the order statistics and so this representation can be used to compute the system reliability from its signature. It can also be used to compare two systems with different structures just by comparing their signature vectors [7, 10, 15, 20].

This representation was extended in [14] to systems with component lifetimes having a joint absolutely continuous exchangeable (EXC) distribution. Exchangeable means that the distribution (law) is invariant under permutations. Then it was extended to general EXC distributions in [15] but in this case the signature values should be computed from the system structure and cannot be interpreted as probabilities. A review on the properties and applications of signature representations for systems and networks can be seen in [23]. Extensions to multi-state systems and connections with fuzzy measures were given in [17, 30].

Example 5.1 in [15] proves that the signature representations do not hold for systems with independent non-identically distributed components. We show here that the ID assumption cannot be relaxed as far as one needs conditions for the validity of the signature representation for any coherent system. Note that for k-out-of-n no conditions are required. However, two recent results prove that the

EXC assumption can be relaxed.

The first one was obtained in Theorem 4 of [9]. There it is proved that a necessary and sufficient condition to get the signature representations of all the coherent systems is that the random variables with the components' states at time t are EXC for any $t \geq 0$.

The second one, obtained in Theorem 1 of [12], was based on the copula representation of the joint distribution of the component lifetimes. It is well know that this joint distribution is EXC if and only if the component (marginal) distributions are equal (ID) and the copula is EXC. It was proved in [12] that the EXC property for the copula can be relaxed. It is enough to merely assume that the copula is diagonal dependent (DD) (the formal definition is given in Section 2). Moreover, it was shown there that the set of DD copulas is much bigger than the set of EXC copulas and that it is dense in the set of copulas while the set of EXC copulas is not. Diagonal dependence of the copula is a sufficient condition on the dependence structure of component lifetimes for the signature representation of the system lifetime distribution, and this assumption is relaxed here so to obtain the necessary and sufficient condition.

In the present paper we obtain two new necessary and sufficient conditions for getting the signature representations for arbitrary systems. Other symmetry conditions are studied as well to get the equality between structural and probabilistic signatures. The rest of the paper is organized as follows. The main results are included in the following section. Examples and counterexamples are placed in Section 3. In Section 4, we analyze reliability models constructed with the use of multivariate conditional hazard rate functions, and investigate conditions on these functions which assure that the probabilistic and structural signatures of all the coherent systems are identical. A summary and some conclusions are given in Section 5.

Throughout the paper the terms 'increasing' and 'decreasing' are used in a wide sense, that is, they mean 'non-decreasing' and 'non-increasing', respectively. We use the notation $[n] := \{1, \ldots, n\}$ and $\mathbf{u}_I := (u_1, \ldots, u_n)$, $I \subseteq [n]$, for a vector with coordinates $u_i = u$ when $i \in I$ and $u_i = 1$ otherwise.

2 Main results

A (binary) *system* is a Boolean (structure) function $\psi : \{0,1\}^n \to \{0,1\}$. Here $x_i = 0$ means that the ith component does not work and $x_i = 1$ that it works. Then the system state $\psi(x_1, \ldots, x_n) \in \{0,1\}$ is completely determined by the structure function ψ and the component states $x_1, \ldots, x_n \in \{0, 1\}$. A system ψ is *semi-coherent* if it is increasing, $\psi(0, \ldots, 0) = 0$ and $\psi(1, \ldots, 1) = 1$. A system is *coherent* if it is increasing and all the components are relevant. The ith component is relevant if ψ is strictly increasing in at least a point in the ith variable (i.e., at this point, the system works if and only if the ith component works). In particular, if ψ is coherent, then $\psi(0, \ldots, 0) = 0$ and $\psi(1, \ldots, 1) = 1$ (it is also semi-coherent). The basic properties of systems can be seen in the classic book [1].

If the state of the ith component at a time $t \geq 0$ is represented by the random variable $X_i(t)$ for $i \in [n]$, where $X_i(t) = 1$ (resp. 0) means that the ith component is working (has failed) at time t, then the system state at time t is

$$X(t) = \psi(X_1(t), \ldots, X_n(t)).$$

We assume $X_i(0) = 1$ and $\lim_{t\to\infty} X_i(t) = 0$ for $i \in [n]$. Hence, for semi-coherent systems, we have $X(0) = 1$ and $\lim_{t\to\infty} X(t) = 0$.

Analogously we can define the component lifetimes

$$T_i := \sup\{t \geq 0 : X_i(t) = 1\}, \quad i \in [n]$$

and the system lifetime

$$T := \sup\{t \geq 0 : X(t) = 1\}.$$

They are non-negative random variables. Then we can consider the ordered component lifetimes $T_{1:n} \leq \cdots \leq T_{n:n}$ obtained from $T_1, \ldots, T_n$. Observe that $T_{1:n}, \ldots, T_{n:n}$ represent the lifetimes of k-out-of-n systems, i.e. ones that work when at least k of their n components work.

We know that the system fails with a component failure, that is, $T = T_{i:n}$ for an $i \in [n]$. Hence we can define the *probabilistic signature* $\mathbf{p} = (p_1, \ldots, p_n)$ of the system as

$$p_i := \Pr(T = T_{i:n}), \quad i \in [n]. \tag{2.1}$$

The meaning of $\mathbf{p}$ is clear. Note that $\mathbf{p}$ depends on both ψ and the joint distribution of $(T_1, \ldots, T_n)$. Also note that $p_i \geq 0$ but that $p_1 + \cdots + p_n = 1$ does not necessarily hold. For example, if T_1, T_2 are IID and $\Pr(X_i = 0) = \Pr(X_i = 1) = 1/2$ for $i = 1, 2$ (Bernoulli distributions), then the probabilistic signature of the series system $\psi_1(x_1, x_2) = \min(x_1, x_2)$ is $\mathbf{p} = (1, 1/2)$.

Samaniego [22] proved that if the component lifetimes $T_1, \dots, T_n$ are IID with a common continuous distribution function F, then the system lifetime distribution function $F_T(t) := \Pr(T \leq t)$ can be written as

$$F_T(t) = p_1 F_{1:n}(t) + \cdots + p_n F_{n:n}(t) \tag{2.2}$$

for all t. Even more, he proved that in this case $p_1, \dots, p_n$ do not depend on F (i.e. they only depend on the structure ψ). So expression (2.2) is very useful to compute the system distribution (or its reliability function $\bar{F}_T = 1 - F_T$) and to compare systems having different structures just by comparing their signatures, see [7]. This representation was extended to systems with components having a joint absolutely continuous EXC distribution in [14].

In the IID continous case, Boland [2] obtained an explicit formula to compute the signature from the structure function. This expression can be used to define the *structural signature* $\mathbf{s} = (s_1, \dots, s_n)$ of ψ as

$$s_i = \frac{1}{\binom{n}{n-i+1}} \sum_{\sum_{j=1}^n x_j = n-i+1} \psi(x_1, \dots x_n) \; - \frac{1}{\binom{n}{n-i}} \sum_{\sum_{j=1}^n x_j = n-i} \psi(x_1, \dots x_n) \tag{2.3}$$

for $i \in [n]$. Note that $\mathbf{s}$ only depends on ψ and that it satisfies $s_1, \dots, s_n \geq 0$ and $s_1 + \cdots + s_n = 1$. A symmetry condition for the components' joint distribution to get $\mathbf{p} = \mathbf{s}$ was given in Proposition 7 of [9].

The structural signature was used in [15] to extend the Samaniego representation (2.2) to systems with component lifetimes having a joint EXC distribution by using the following representation

$$F_T(t) = s_1 F_{1:n}(t) + \cdots + s_n F_{n:n}(t) \tag{2.4}$$

for all t. Note that this case includes the general IID case and it proves that (2.4) might hold when (2.2) does not hold. For example, this happens when the joint distribution of component lifetimes has ties (i.e. $\Pr(T_i = T_j) > 0$ for some $i \neq j$) and then $p_i = \Pr(T = T_{i:n})$, $i = 1, \dots, n$, sum up to a number greater than 1. Problems of validity of representations (2.2) and (2.4) are illustrated in Examples 3.2 and 3.3. The right-hand sides of (2.2) (when $p_1 + \cdots + p_n = 1$) and (2.4) can be used to define mixed systems as randomly chosen k-out-of-n systems with arbitrary choice probabilities p_i and s_i for $i \in [n]$, see [18].

The extensions of representations (2.2) and (2.4) were studied in the excellent paper by Marichal *et al.* [9]. In particular they proved in Theorem 4 that (2.4) holds for all the coherent systems of a fixed dimension $n > 2$ if and only if the random vector $(X_1(t),\dots,X_n(t))$ with the components' states is EXC for all $t \geq 0$. The case $n=2$ is trivial since there are only two coherent systems, the series system $\psi_1(x_1,x_2)=\min(x_1,x_2)$ and the parallel system $\psi_2(x_1,x_2)=\max(x_1,x_2)$ with structural signatures $(1,0)$ and $(0,1)$. Hence (2.4) is always true then.

Representation (2.4) was extended recently in [12] by using a condition based on the copula representation for the joint distribution function $\mathbf{F}$ of $(T_1,\dots,T_n)$. It is well known from Sklar's theorem (see, e.g., [4, 19]) that $\mathbf{F}$ can be written as

$$\mathbf{F}(t_1,\dots,t_n)=\Pr(T_1\leq t_1,\dots,T_n\leq t_n)=C(F_1(t_1),\dots,F_n(t_n)), \tag{2.5}$$

where C is a copula function. Even more, if the marginal distribution functions $F_1,\dots,F_n$ are continuous, the copula C is unique. From [15] we know that (2.4) holds when $(T_1,\dots,T_n)$ is EXC, that is,

$$(T_1,\dots,T_n)=_{st}(T_{\sigma(1)},\dots,T_{\sigma(n)})$$

for any permutation σ, where $=_{st}$ means equality in law. Moreover, it is easy to see that $(T_1,\dots,T_n)$ is EXC if and only if $F_1=\cdots=F_n$ (the components are ID) and C is EXC (i.e. permutation invariant). The ID assumption cannot be dropped out (see Example 5.1 in [15]). However, it was proved in [12] that the second condition (C is EXC) can be relaxed. In order to describe the relaxation we introduce the notion of diagonal dependent copula.

Definition 2.1. *Let S be a given subset of the interval $[0,1]$. We say that the n-dimensional copula function $C:[0,1]^n\to[0,1]$ is S-diagonal dependent (S-DD, for short) if*

$$C(\mathbf{u}_P)=C(\mathbf{u}_Q) \tag{2.6}$$

for every $u\in S$ and all subsets P and Q of $[n]$ with the same cardinality. We write that a copula is diagonal dependent (DD) if it is $[0,1]$-diagonal dependent.

The relaxation obtained in [12] consisted in replacing exchangeable copulas by diagonal dependent ones. A large exemplary family of DD copulas is given in Proposition 1 of [12]. Note that this concept is not the same as a d–copula which have all the lower dimensional k–marginal copulas equal each other for all $k\leq n$ since in the DD copulas these marginal copulas just coincide in their diagonal sections. Of course, all d–copulas are DD copulas but the reverse is not true (see Example 3.2 below).

In the following theorem we present three equivalent necessary and sufficient conditions for obtaining representation (2.4). Given an event (or a set) S, we use the notation S^c for the complementary event and $|S|$ for the cardinality of S.

Theorem 2.1. *If $n > 2$, the following conditions are equivalent:*

(i) Representation (2.4) holds for any coherent system with n components.

(ii) If $A_i = \{T_i \leq t\}$ and $A_i^c = \{T_i > t\}$, then

$$\Pr\left([\cap_{i\in I} A_i] \cap \left[\cap_{j\in I^c} A_j^c\right]\right) = \Pr\left([\cap_{i\in J} A_i] \cap \left[\cap_{j\in J^c} A_j^c\right]\right) \tag{2.7}$$

for all $I, J \subseteq \{1, \dots, n\}$ with $|I| = |J|$.

(iii) $(X_1(t), \dots, X_n(t))$ are EXC for all t.

(iv) $T_1, \dots, T_n$ are ID with a common distribution function F and their copula C is $\mathcal{R}(F)$-DD for $\mathcal{R}(F) = \{u = F(t) : t > 0\}$ denoting the image of the common marginal distribution function F of all component lifetimes.

Proof. The equivalence between conditions (i) and (iii) was given in Theorem 4 of [9].

Let us show that (ii) implies (iii). To this end we note that if $x_i = 0$ for $i \in I$ and $x_i = 1$ for $i \notin I$, then

$$\begin{aligned}
\mathbf{p}_t(x_1, \dots, x_n) :&= \Pr(X_1(t) = x_1, \dots, X_n(t) = x_n) \\
&= \Pr\left([\cap_{i\in I} A_i] \cap \left[\cap_{j\in I^c} A_j^c\right]\right) \\
&= \Pr\left([\cap_{i=1}^m A_i] \cap \left[\cap_{j=m+1}^n A_j^c\right]\right) \\
&= \Pr(X_1(t) = 0, \dots, X_m(t) = 0, X_{m+1}(t) = 1, \dots, X_n(t) = 1) \\
&= \mathbf{p}_t(\underbrace{0, \dots, 0}_{m}, \underbrace{1, \dots, 1}_{n-m}),
\end{aligned}$$

where $m = |I|$ is the cardinality of the set I and the third equality is obtained from (2.7). Hence the punctual probability function $\mathbf{p}_t$ of $(X_1(t), \dots, X_n(t))$ is permutation invariant and so $(X_1(t), \dots, X_n(t))$ is EXC for all t.

The reversed implication $(iii) \Rightarrow (ii)$ is easily obtained by changing the order in the above sequence of equations. Indeed, under (iii), for every $I \subset [n]$ of size m, if $x_i = 1$ for $i \in I$ and $x_i = 0$ for $i \notin I$, we have

$$\begin{aligned}
&\Pr\left([\cap_{i\in I} A_i] \cap \left[\cap_{j\in I^c} A_j^c\right]\right) = \Pr(X_1(t) = x_1, \dots, X_n(t) = x_n) \\
&\quad = \Pr(X_1(t) = 0, \dots, X_m(t) = 0, X_{m+1}(t) = 1, \dots, X_n(t) = 1) \\
&\quad = \Pr\left([\cap_{i=1}^m A_i] \cap \left[\cap_{j=m+1}^n A_j^c\right]\right).
\end{aligned}$$

The same happens for any other subset $J \subset [n]$ of size m, which gives the desired conclusion.

Finally, we show equivalence of (iii) and (iv). Under assumption (iii), the random variables $X_1(t), \dots, X_n(t)$ are identically distributed. Therefore for all $i \in [n]$, we have

$$\Pr(T_i \le t) = \Pr(X_i(t) = 0) = \Pr(X_1(t) = 0) = \Pr(T_1 \le t) = F(t),$$

say. Moreover, for every $u \in \mathcal{R}(F)$, $P \subset \{1, \dots, n\}$ with $|P| = m$, and some $t \ge 0$ satisfying $F(t) = u$, yields

$$\begin{aligned}
C(\mathbf{u}_P) &= \Pr(T_i \le t,\ i \in P) \\
&= \Pr(X_i(t) = 0,\ i \in P) \\
&= \Pr(X_i(t) = 0,\ i = 1, \dots, m) \\
&= \Pr(T_i \le t,\ i = 1, \dots, m) \\
&= C(\underbrace{F(t), \dots, F(t)}_{m}, 1, \dots, 1) \\
&= C(\underbrace{u, \dots, u}_{m}, 1, \dots, 1).
\end{aligned}$$

Conversely, if we assume now (iv), then identical distributions of $T_1, \dots, T_n$ implies identical distributions of $X_1(t), \dots, X_n(t)$ for all $t \ge 0$. If F is the common marginal and $u = F(t)$ for $t > 0$, the copula property assures exchangeability of $X_1(t), \dots, X_n(t)$ due to the following relations

$$\begin{aligned}
p_I :&= \Pr(X_i(t) = 0,\ i \in I,\ X_i(t) = 1,\ i \notin I) \\
&= \Pr(T_i \le t,\ i \in I,\ T_i > t,\ i \notin I) \\
&= \sum_{r=|I|}^{n} (-1)^{r-|I|} \sum_{B \supseteq I, |B| = r} C(\mathbf{u}_B) \\
&= \sum_{r=|I|}^{n} (-1)^{r-|I|} \binom{n - |I|}{r - |I|} C(\mathbf{u}_{\{1,\dots,r\}}) \\
&= \Pr(T_i \le t,\ i = 1, \dots, |I|,\ T_i > t,\ i = |I| + 1, \dots, n) \\
&= \Pr(X_i(t) = 0,\ i = 1, \dots, |I|,\ X_i(t) = 1,\ i = |I| + 1, \dots, n)
\end{aligned}$$

valid for all $I \subset [n]$. □

We must note that the equivalence between conditions (*ii*) and (*iii*) can be traced back to de Finetti [3] (see also Chapter 1 of [28]). We include here the proof for possible readers' convenience.

Remark 2.1. *Every copula is* $\{0,1\}$*-diagonal dependent. Therefore for every degenerate marginal component lifetime distribution function* $F(t) = \mathbf{1}_{[t_0,\infty)}(t)$*, the Samaniego formula* (2.4) *trivially holds*

$$\Pr(T \le t) = \mathbf{1}_{[t_0,\infty)}(t) = \sum_{i=1}^{n} s_i \mathbf{1}_{[t_0,\infty)}(t) = \sum_{i=1}^{n} s_i \Pr(T_{i:n} \le t).$$

Another extreme case is provided by any continuous common marginal lifetime distribution for which (2.4) *is assured by a DD-copula of mutual dependence. It follows that representation* (2.4) *is valid for all systems with identically distributed component lifetimes when their dependence copula is DD. This statement was proved recently in [12].*

Remark 2.2. *The first sufficient conditions for validity of formula* (2.4) *for all system structures, independence and common continuous marginal distribution of component lifetimes, were established by Samaniego [22]. It occurs that the identity of marginals is absolutely indispensable, but the independence assumption can be significantly relaxed. Navarro et al. [15] replaced independence by exchangeability of component lifetimes and removed restrictions on the common marginal distribution (the claim was proven earlier in [14] under continuity assumption). The necessary and sufficient conditions in (iii) of Theorem 2.1 were established by Marichal et al. [9]. The equivalent conditions (iv) seem to be more tractable because they refer directly to the distributional properties of component lifetimes. We show that the weakest possible dependence condition is described by a diagonal dependence of copula, with possible reduction to the diagonal arguments being values of the common marginal distribution in the discontinuous case.*

For better understanding of which properties of the dependence copula affect the lifetime distribution of the system composed of items with identically distributed lifetimes, we present the following theorem. We need the following concept. We say that a set $I \subseteq [n]$ is a cut set of a system ψ if $\psi(x_1, \dots, x_n) = 0$ when $x_i = 0$ for all $i \in I$.

Theorem 2.2. *Let* $\mathbf{T} = (T_1, \dots, T_n)$ *and* $\mathbf{S} = (S_1, \dots, S_n)$ *be vectors of identically distributed random variables with a common marginal distribution function* F *and* *copulas* $C_{\mathbf{T}}$ *and* $C_{\mathbf{S}}$*, respectively. Then for a given system structure* $\psi : \{0,1\}^n \to \{0,1\}$ *with the family of cut sets* $\mathcal{A} \subseteq 2^{[n]}$*, the system lifetimes* T_ψ *and* S_ψ *obtained from* $\mathbf{T}$ *and* $\mathbf{S}$*, respectively, have the same distribution if and only if*

$$C_{\mathbf{T}}(\mathbf{u}_I) = C_{\mathbf{S}}(\mathbf{u}_I) \qquad \forall\, I \in \mathcal{A} \quad \forall u \in \mathcal{R}(F). \tag{2.8}$$

Moreover, if

$$C_{\mathbf{T}}(\mathbf{u}_I) = C_{\mathbf{T}}(\mathbf{u}_{\{1,\ldots,|I|\}}) = C_{\mathbf{S}}(\mathbf{u}_I) = C_{\mathbf{S}}(\mathbf{u}_{\{1,\ldots,|I|\}}), \qquad \forall\, I \in \mathcal{A} \quad \forall u \in \mathcal{R}(F),$$

then representation (2.4) *holds and coincides for both systems.*

Proof. We have

$$\Pr(T \le t) = \sum_{I \in \mathcal{A}} \Pr(T_i \le t,\ i \in I,\ T_i > t,\ i \notin I),$$

and each particular summand has the form

$$\begin{aligned} \Pr(T_i \le t,\ i \in I,\ T_i > t,\ i \notin I) &= \sum_{k=0}^{n-|I|} (-1)^k \sum_{B \supseteq I, |B| = |I|+k} \Pr(T_i \le t,\ i \in B) \\ &= \sum_{k=0}^{n-|I|} (-1)^k \sum_{B \supseteq I, |B| = |I|+k} C_{\mathbf{T}}(\mathbf{u}_B) \end{aligned}$$

for $u = F(t)$. Note that every set containing a cut set is a cut set as well. It follows that the distribution function of T depends only on the values of the marginal distribution function F, and the diagonal values of the subcopulas generated by the cut sets of the system. If these are identical for $\mathbf{T}$ and $\mathbf{S}$, then T_ψ and S_ψ are identically distributed. The latter claim easily follows from the above establishments and Theorem 2.1. □

Remark 2.3. *If we want to check if condition (i) of Theorem 2.1 is fulfilled for a specific system, we need to check if the diagonal values of the subcopulas generated by the cut sets with identical sizes coincide at all arguments of the common marginal distribution function. For instance, for the parallel system with components having identically continuously distributed lifetimes, the system lifetime depends only on the diagonal of copula* C *itself and the distribution function* F*. For the system with the structure* $\psi(x_1, \ldots, x_n) = \max(\min(x_1, x_2), x_3, \ldots x_n)$ *we take into account merely three cuts sets* $\{2, 3, \ldots, n\}$*,* $\{1, 3, 4, \ldots, n\}$*, and* $[n]$*. On* *the other hand, if we admit different marginal distributions of component lifetimes in each system, but preserve the equality among distributions in the pairs* (T_i, S_i), $i = 1, \dots, n$, *we should modify the verifying condition* (2.8). *Namely we replace diagonal arguments in the formula by vectors* $(u_1, \dots, u_n) \in \{(F_1(t), \dots, F_n(t)) : t \geq 0\}$.

Remark 2.4. *In [12] it was also proved that the following representation based on the minimal signature vector* $\mathbf{a} = (a_1, \dots, a_n)$ *(see [13]) given by*

$$F_T(t) = a_1 F_{1:1}(t) + \cdots + a_n F_{1:n}(t) \tag{2.9}$$

holds for all t *when* $T_1, \dots, T_n$ *are ID and* C *is DD. Actually, the proof of* (2.4) *in [12] was based on this property. In a similar way it can be proved that* (2.9) *holds for a specific common marginal* F, *when the conditions in* (*iv*) *of Theorem 2.1 hold. Now we know from Theorem 2.1 that* (2.9) *holds for all the coherent systems with* n *components if and only if one of the properties in that theorem holds. In particular,* (2.4) *and* (2.9) *are equivalent for* $n > 2$. *The same happen for the representations based on the maximal signature* $\mathbf{b} = (b_1, \dots, b_n)$ *(see also [13]) given by*

$$F_T(t) = b_1 F_{1:1}(t) + \cdots + b_n F_{n:n}(t). \tag{2.10}$$

Some coefficients in representations (2.9) *and* (2.10) *can be negative. So they are not mixtures (they are called negative or generalized mixtures).*

Remark 2.5. *By using the preceding remark it is easy to see that* (2.4) *(or Theorem 2.1) can be extended to semi-coherent systems. Thus, if* $T = \psi(T_1, \dots, T_m)$ *is a coherent system,* $m < n$, *and* $(T_1, \dots, T_n)$ *satisfies one of the conditions in Theorem 2.1, then*

$$F_T(t) = s_1^{(n)} F_{1:n}(t) + \cdots + s_1^{(n)} F_{n:n}(t)$$

holds for all t, *where* $\mathbf{s}^{(n)} = (s_1^{(n)}, \dots, s_1^{(n)})$ *is the structural signature of dimension* n *obtained from* (2.3) *of the extension of* ψ *to* $\{0,1\}^n$. *This extension is a semi-coherent system of order* n. *The formulas to compute* $\mathbf{s}^{(n)}$ *from* $\mathbf{s}$ *were given in [15]. Note that representations* (2.9) *and* (2.10) *of dimension* n *also holds. In this case the minimal and maximal signatures of order* n *are trivial (we just add some zeros at the end of the minimal and maximal signatures of order* m*).*

However, note that if $n = 2$ *and* (T_1, T_2) *have an arbitrary joint distribution, then* (2.9) *and* (2.10) *are not necessarily true. They hold if and only if the components are ID. Note that all the bidimensional copulas are DD since*

$C(u,1) = C(1,u) = u$ for $u \in [0,1]$. For example, the representation (2.9) *for the series system $\psi_1(x_1,x_2) = \min(x_1,x_2)$ is always true since*

$$F_{1:2}(t) = 0F_{1:1}(t) + 1F_{1:2}(t).$$

However, the representation (2.9) *for the parallel system $\psi_2(x_1,x_2) = \max(x_1,x_2)$ is*

$$F_{2:2}(t) = 2F_{1:1}(t) - 1F_{1:2}(t),$$

where $F_{1:1} = F_1$. In the general case, $F_{2:2}$ can be computed as

$$\begin{aligned} F_{2:2}(t) &= \Pr(T_{2:2} \leq t) \\ &= 1 - \Pr(T_{2:2} > t) \\ &= 1 - \Pr(T_1 > t) - \Pr(T_2 > t) + \Pr(T_{1:2} > t) \\ &= F_1(t) + F_2(t) - F_{1:2}(t). \end{aligned}$$

Therefore, both expressions coincide if and only if $F_1 = F_2$. Note that the conditions (ii)-(iv) in Theorem 2.1 are equivalent to $F_1 = F_2$ when $n = 2$. With one of these conditions, representations (2.9) *and* (2.10) *hold.*

Note that (2.4) *is true for all the coherent systems of dimension $n = 2$ (i.e, the series and parallel ones) but it is not always true for the semi-coherent system $\psi(x_1,x_2) = x_1$. The structural signature of order* 2 *of ψ is* $(1/2, 1/2)$. *Then* (2.4) *holds if and only if*

$$F_T(t) = \frac{1}{2}F_{1:2}(t) + \frac{1}{2}F_{2:2}(t),$$

where $T = \psi(T_1,T_2) = T_1$ and $F_T = F_1$. By using the expression for $F_{2:2}$ given above, the preceding equality holds if and only if $F_1 = F_2$. Therefore, we can include the case $n = 2$ in Theorem 2.1 if we replace coherent systems with semi-coherent systems.

Of course, the ID assumption $F_1 = \cdots = F_n$ can be considered as the weakest exchangeability (symmetry) condition. Another condition is

$$\Pr(T_1 < \cdots < T_n) = \Pr(T_{\sigma(1)} < \cdots < T_{\sigma(n)}) \tag{2.11}$$

for any permutation σ of $[n]$. The examples included in the following sections show that they are not enough to get (2.4) and that (2.4) may hold when (2.11) fails. The extension of representation (2.2) was also studied in [15]. It is based on the concept of *weak exchangeability* for $(T_1, \ldots, T_n)$ defined by the following condition

$$\Pr(T_{i:n} \leq t) = \Pr(T_{i:n} \leq t | T_{\sigma(1)} < \cdots < T_{\sigma(n)}) \tag{2.12}$$

for every t, every $i = 1, \ldots, n$ and every permutation σ. Thus, it was proved that if $\Pr(T_{1:n} < \cdots < T_{n:n}) = 1$ (there are no ties) and (2.12) holds, then (2.2) holds. The (equivalent) conditions in Theorem 2.1 can also be used to define a new weak-exchangeability (or symmetry) condition. Theorem 8 in [9] provides conditions to get both (2.2) and (2.4) with $\mathbf{p} = \mathbf{s}$. Here the condition $\mathbf{p} = \mathbf{s}$ for all the coherent systems is also a symmetry condition. The relationships between these conditions are analyzed in the following examples.

3 Examples

The first example shows that the ID condition $F_1 = \cdots = F_n$ is not enough for getting either of useful formulae (2.2), (2.4), (2.9), (2.11) or $\mathbf{p} = \mathbf{s}$ when the component lifetimes dependence structure does not fulfil appropriate symmetry (e.g., EXC) conditions.

Example 3.1. *We treat the coherent system* $\psi(x_1, x_2, x_3) = \max(x_1, \min(x_2, x_3))$*. Its structure signature is* $\mathbf{s} = (0, 2/3, 1/3)$ *and its minimal and maximal signatures are* $\mathbf{a} = (1, 1, -1)$ *and* $\mathbf{b} = (0, 2, -1)$*, respectively. We assume that the component lifetimes are ID with a common continuous distribution function* F*. We also assume that the lifetime of the first component is independent of the lifetimes of other components which are dependent with the following Clayton copula*

$$C_\theta(u, v) = (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}$$

for $u, v \in [0, 1]$ *and* $\theta = 1$*. Hence, the copula* C *of* (T_1, T_2, T_3) *is*

$$C(u_1, u_2, u_3) = u_1 C_1(u_2, u_3) = \frac{u_1 u_2 u_3}{u_2 + u_3 - u_2 u_3}, \qquad u_1, u_2, u_3 \in [0, 1].$$

Note that C *is not a DD copula since* $C(u, u, 1) = u^2$ *and* $C(1, u, u) = u/(2 - u)$ *do not coincide for* $0 < u < 1$*. The probabilistic signature of* ψ *is then* $\mathbf{p} = (0, 2 - \ln 4, \ln 4 - 1)$ *which is approximately equal to* $(0, 0.6137056, 0.3862944)$, *and differs from* $\mathbf{s}$. *The system distribution function can be computed as*

$$\begin{aligned} F_T(t) &= \Pr(\max(T_1, \min(T_2, T_3)) \leq t) \\ &= \Pr(T_1 \leq t, \min(T_2, T_3) \leq t) \\ &= \Pr(T_1 \leq t) \Pr(\min(T_2, T_3) \leq t) \\ &= F(t) \left[\Pr(T_2 \leq t) + \Pr(T_3 \leq t) - \Pr(\max(T_2, T_3) \leq t)\right] \\ &= F(t) \left[2F(t) - C_1(F(t), F(t))\right] \\ &= 2F^2(t) - \frac{F^2(t)}{2 - F(t)}. \end{aligned}$$

We compare it with mixture distribution functions $F_{\mathbf{p}}$ *and* $F_{\mathbf{s}}$ *represented by formulae (2.2) and (2.4), respectively. To this end, we calculate the lifetime distribution functions of the* k*-out-of-3 systems*

$$\begin{aligned} F_{1:3}(t) &= 3F(t) - 2F^2(t) - C_1(F(t), F(t)) + F(t)C_1(F(t), F(t)) \\ &= 3F(t) - 2F^2(t) - \frac{F(t)}{2 - F(t)} + \frac{F^2(t)}{2 - F(t)}, \\ F_{2:3}(t) &= 2F^2(t) + C_1(F(t), F(t)) - 2F(t)C_1(F(t), F(t)) \\ &= 2F^2(t) + \frac{F(t)}{2 - F(t)} - \frac{2F^2(t)}{2 - F(t)}, \\ F_{3:3}(t) &= F(t)C_1(F(t), F(t)) = \frac{F^2(t)}{2 - F(t)}. \end{aligned}$$

Therefore

$$\begin{aligned} F_{\mathbf{s}}(t) &= \frac{2}{3}F_{2:3}(t) + \frac{1}{3}F_{3:3}(t) = \frac{4}{3}F^2(t) + \frac{2}{3}\frac{F(t)}{2 - F(t)} - \frac{F^2(t)}{2 - F(t)}, \\ F_{\mathbf{p}}(t) &= (2 - \ln 4)F_{2:3}(t) + (\ln 4 - 1)F_{3:3}(t) \\ &= (4 - 2\ln 4)F^2(t) + (2 - \ln 4)\frac{F(t)}{2 - F(t)} + (3\ln 4 - 5)\frac{F^2(t)}{2 - F(t)}. \end{aligned}$$

We can check that

$$\begin{aligned} F_{\mathbf{s}}(t) - F_T(t) &= \frac{2}{3}\,\frac{F(t)[1 - F(t)]^2}{2 - F(t)} \geq 0, \\ F_{\mathbf{s}}(t) - F_{\mathbf{p}}(t) &= \left(\ln 4 - \frac{4}{3}\right)\frac{F(t)[1 - F(t)][1 + 2F(t)]}{2 - F(t)} \geq 0, \end{aligned}$$

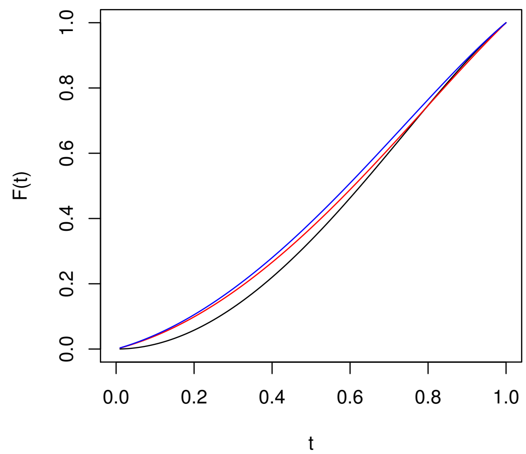

Figure 1: Distribution functions for the systems in Example 3.1.

and the inequalities are strict when $F(t)$ *differs from* 0 *and* 1*, whereas*

$$\begin{aligned} F_{\mathbf{p}}(t) - F_T(t) &= 2(\ln 4 - 2)\,\frac{F(t)[1-F(t)]}{2-F(t)}\left(\frac{1-\ln 2}{\ln 4 - 2} - F(t)\right) \\ & \begin{cases} \geq 0, \text{if} & F(t) \leq \frac{1-\ln 2}{\ln 4 - 1} \approx 0.7943497, \\ \leq 0, \text{if} & F(t) \geq \frac{1-\ln 2}{\ln 4 - 1}. \end{cases} \end{aligned}$$

The results of calculation are confirmed by Figure 1 where functions F_T*,* $F_{\mathbf{s}}$*, and* $F_{\mathbf{p}}$ *were plotted in black, blue and red, respectively, for the standard uniform distribution function* $F(t) = t$*,* $t \in [0,1]$*, of the single component lifetime. The picture shows that generally* $F_{\mathbf{p}}$ *better approximates* F_T *than* $F_{\mathbf{s}}$*, although we have*

$$|F_{\mathbf{s}}(t) - F_T(t)| < |F_{\mathbf{p}}(t) - F_T(t)|$$

if

$$F(t) > \frac{4 - 3\ln 2}{6\ln 2 - 2} \approx 0.8896084.$$

Finally we observe that

$$\begin{aligned} \Pr(T_1 > T_2 > T_3) &= \Pr(T_1 > T_3 > T_2) = \Pr(T_3 > T_2 > T_1) = \Pr(T_2 > T_3 > T_1) \\ &= \ln 2 - \frac{1}{2} \approx 0.1931472, \end{aligned}$$

are equal but they are different from

$$\Pr(T_2 > T_1 > T_3) = \Pr(T_3 > T_1 > T_2) = \frac{3}{2} - 2\ln 2 \approx 0.1137056.$$

This means that conditions (2.12) *guarantying the equality of the structural and probabilistic signatures do not hold, and actually* $\mathbf{p} \neq \mathbf{s}$*. Similar conclusions to all the above ones can be obtained for the Clayton copula of* T_2 *and* T_3 *with different* $\theta > 0$*, but the formulae were more complicated.*

The second example illustrates applicability of Theorems 2.1 and 2.2.

Example 3.2. *Let* (T_1, T_2, T_3) *be the ID lifetimes of three components with a common marginal distribution function* F *and copula* C *given by*

$$C(u_1, u_2, u_3) = u_3 C_2(u_1, u_2)$$

for $u_1, u_2, u_3 \in [0, 1]$*, where*

$$\begin{aligned} C_2(u_1, u_2) &= \min\left\{u_1, u_2, \frac{u_1^2 + u_2^2}{2}\right\} \\ &= \begin{cases} u_2, & \text{if } \ 0 \leq u_2 \leq 1 - \sqrt{1 - u_1^2}, \\ \frac{u_1^2+u_2^2}{2}, & \text{if } \ 1 - \sqrt{1 - u_1^2} \leq u_2 \leq \sqrt{2u_1 - u_1^2}, \\ u_1, & \text{if } \ \sqrt{2u_1 - u_1^2} \leq u_2 \leq 1, \end{cases} \end{aligned}$$

for $u_1, u_2 \in [0, 1]$ *(see Example 3.5 in [19, p. 62]). Copula* C_2 *is an exemplary singular copula with fixed diagonal* $C_2(u, u) = u^2$ *for* $0 \leq u \leq 1$ *constructed by Fredricks and Nelsen [5] (see also [4], p. 32). It is uniformly distributed on two quarter-circles* $u_2 = 1 - \sqrt{1 - u_1^2}$ *and* $u_2 = \sqrt{2u_1 - u_1^2}$*,* $0 \leq u_1 \leq 1$*, (see Figure 2), and its diagonal section satisfies*

$$C_2(u, u) = u^2, \qquad 0 \leq u \leq 1.$$

It follows that C *is diagonally dependent, and has all the diagonal sections identical with those of the product copula*

$$C(\mathbf{u}_P) = u^{|P|}, \qquad P \subseteq \{1, 2, 3\}.$$

However, note that C *is not a d–copula since the bivariate marginals*

$$C_{1,2}(u_1, u_2) = C(u_1, u_2, 1) = C_2(u_1, u_2)$$

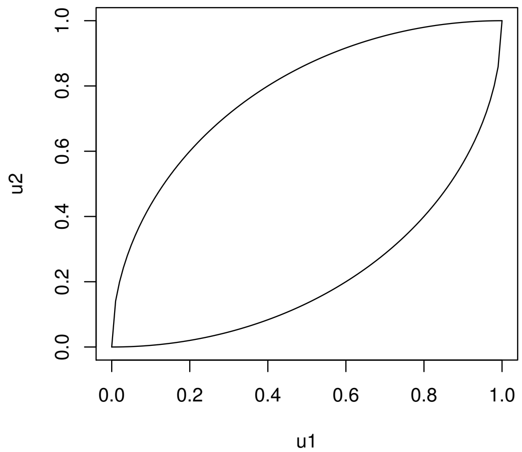

Figure 2: Support of the bidimensional copula C_2 considered in Example 3.2.

and

$$C_{1,3}(u_1, u_3) = C(u_1, 1, u_3) = u_3 C_2(u_1, 1) = u_1 u_3$$

do not coincide.

Theorems 2.1 and 2.2 imply that

$$\begin{aligned} F_{1:3}(t) &= 3F(t) - 3F^2(t) + F^3(t), \\ F_{2:3}(t) &= 3F^2(t) - F^3(t), \\ F_{3:3}(t) &= F^3(t), \end{aligned}$$

and

$$F_T(t) = F_{\mathbf{s}}(t) = \sum_{i=1}^{3} s_i F_{i:3}(t)$$

for every coherent system of size $n = 3$. We also have

$$\begin{aligned} \Pr(T_1 < T_2 < T_3) &= \Pr(T_2 < T_1 < T_3) = \Pr(T_3 < T_2 < T_1) = \Pr(T_2 < T_3 < T_1) \\ &= \frac{1}{2} - \frac{1}{\pi} \approx 0.1816901, \end{aligned}$$

but that they are different from

$$\Pr(T_1 > T_3 > T_2) = \Pr(T_2 > T_3 > T_1) = \frac{2}{\pi} - \frac{1}{2} \approx 0.1366198.$$

Accordingly (2.11) *does not hold, and equality* $\mathbf{p} = \mathbf{s}$ *is not assured.*

For example, if we consider the system $\psi(x_1, x_2, x_3) = \max(\min(x_1, x_2), x_3)$, *then* $\mathbf{s} = (0, 2/3, 1/3) \neq \mathbf{p} = (0, p_2, p_3)$ *because*

$$p_3 = \Pr(T_1 < T_2 < T_3) + \Pr(T_2 < T_1 < T_3) = 1 - \frac{2}{\pi} \approx 0.3633802$$

and $p_2 = 1 - p_3 = 2/\pi \approx 0.6366198$. *Moreover,*

$$F_T(t) = F_{\mathbf{s}}(t) = 2F^2(t) - F^3(t) \neq F_{\mathbf{p}}(t).$$

In the following example we analyze applicability of condition (2.12) for getting representation (2.2). This is a modification of Example 3.7 from [15].

Example 3.3. *Suppose that a random vector* $(T_1, \ldots, T_n)$ *takes values over all the permutations* $\Sigma([n])$ *of the set* $[n]$, *and not all of the respective probabilities*

$$\Pr(T_1 = \sigma(1), \ldots, T_n = \sigma(n)) = p_\sigma, \qquad \sigma \in \Sigma([n]),$$

are equal to $\frac{1}{n!}$ *which excludes exchangeability. We obviously have* $T_{i:n} = i$, $i = 1, \ldots, n$, *with probability one which implies* (2.12) *and, in consequence,* (2.2) *holds for any arbitrary coherent system of size* n. *However,* (2.11) *does not hold.*

In particular, we focus our attention now on our favorite system $\psi(x_1, x_2, x_3) = \max(x_1, \min(x_2, x_3))$ *(see Examples 3.1 and 3.2) with the structural signature* $(0, 2/3, 1/3)$. *Here we have* (2.2) *with*

$$\begin{aligned}
p_1 &= \Pr(T = T_{1:3}) = 0, \\
p_2 &= \Pr(T = T_{2:3}) = p_{123} + p_{132} + p_{213} + p_{312} = 1 - p_{231} - p_{321}, \\
p_3 &= \Pr(T = T_{3:3}) = p_{231} + p_{321}.
\end{aligned}$$

Note that if only $p_3 = p_{231} + p_{321} = \frac{1}{3}$, *we also obtain* (2.4) *even if not all of* p_σ, $\sigma \in \Sigma([3])$, *are equal to* $\frac{1}{6}$. *The explanation of the fact goes beyond the scope of acting of Theorem 2.1. It follows from the specific structure of the system: it fails with the last component failure if and only if* $T_{3:3} = T_1$, *and with the second one otherwise. So it suffices here that*

$$\Pr(T_1 = 3) = p_{231} + p_{321} = s_3 = \frac{1}{3},$$

and both the mixture representations (2.2) *and* (2.4) *hold even if* T_1, T_2, T_3 *have different marginal distributions. However, if* $p_3 \neq 1/3$*, then*

$$F_T(t) = F_{\mathbf{p}}(t) \neq F_{\mathbf{s}}(t)$$

for all t. *We finally recall an obvious observation: if there are no ties (i.e.* $\Pr(T_i = T_j) = 0$ *for all* $i \neq j$*), every* k*-out-of-*n *system satisfies both* (2.2) *and* (2.4) *whatever are the component lifetime distributions and interdependencies among them.*

Marichal *et al.* [9] proved that the structural and probabilistic signatures coincide if and only if the joint distribution of $(T_1, \ldots, T_n)$ has no ties and

$$q(A) := \Pr\left(\max_{i \in A^c} T_i < \min_{i \in A} T_i\right) = \frac{1}{\binom{n}{|A|}}, \qquad A \subset [n]. \tag{3.1}$$

The relation is obviously satisfied by exchangeable vectors $(T_1, \ldots, T_n)$. Next we show an example of the joint distribution of the vector so that $T_1, \ldots, T_n$ are not only differently distributed but they have different supports as well, and formula (3.1) still holds.

Example 3.4. *Let* $(J_1, \ldots, J_n)$ *denote a vector of random variables whose values are all permutations of the set* $[n]$*, and the respective probabilities are all identical and equal to* $\frac{1}{n!}$. *The joint distribution of the component lifetimes* $(T_1, \ldots, T_n)$ *is defined conditionally on the values of* $(J_1, \ldots, J_n)$ *as:* $T_i = n(J_i - 1) + i$, $i = 1, \ldots, n$. *It follows that each* T_i *is uniformly distributed on the discrete set* $\{i, n+i, \ldots, (n-1)n + i\}$, $i = 1, \ldots, n$. *Accordingly, all* T_i, $i = 1, \ldots, n$*, have different supports and they are obviously non-identically distributed. This together with Theorem 2.1 imply that formula* (2.4) *cannot hold for all* n*-component systems whose component lifetimes are* $T_1, \ldots, T_n$. *On the other hand, the orderings of the vectors* $(J_1, \ldots, J_n)$ *and* $(T_1, \ldots, T_n)$ *are identical. This gives*

$$\Pr\left(\max_{i \in A^c} T_i < \min_{i \in A} T_i\right) = \Pr\left(\max_{i \in A^c} J_i < \min_{i \in A} J_i\right) = \frac{1}{\binom{n}{|A|}}, \qquad A \subset [n],$$

which in turn implies $\mathbf{p} = \mathbf{s}$ *and finally non-validity of* (2.2). *We also notice that the* i*th order statistic is uniformly distributed on the points* $(i-1)n+1, (i-1)n+2, \ldots, in$ *for arbitrary* $i = 1, \ldots, n$.

In the special case of the system structure $\psi(x_1, x_2, x_3) = \max(x_1, \min(x_2, x_3))$*, an elementary algebra shows that the distribution of the system lifetime assigns* *probabilities* $1/3$ *to points* 4 *and* 7*, and* $1/6$ *to* 5 *and* 6*. However, representations* (2.2) *and* (2.4) *define a different distribution: the points* 4*,* 5*, and* 6 *have probabilities* $2/9$*, and the probabilities of* 7*,* 8 *and* 9 *are equal to* $\frac{1}{9}$*. Note that the support of* T *does not coincide with the common support of* $T_{\mathbf{s}}$ *and* $T_{\mathbf{p}}$*. So we have*

$$F_T(t) \neq F_{\mathbf{p}}(t) = F_{\mathbf{s}}(t)$$

for all t*.*

Clearly, spreading the massed of discrete points $i \in \mathbb{N}$ over the intervals $[i-1, i]$, we would obtain continuous distributions of component lifetimes without disturbing the properties of system signatures and lifetime distributions.

4 Signatures in the uniform frailty model

In this section we consider the uniform frailty reliability model which assumes particular symmetry properties of multivariate conditional hazard rates. Our purpose is to show that the uniform frailty property implies identity of the probabilistic and structural signatures (2.1) and (2.3), respectively. First we recall the basic notions. In what follows we assume that $(T_1, \ldots, T_n)$ has an absolutely continuous joint distribution.

Definition 4.1. *For every* $k = 1, \ldots, n-1$*, every* $k+1$ *elements* $i_1, ..., i_k, j$ *of* $[n]$*, and any ordered sequence* $0 < t_1 < \ldots < t_k < +\infty$*, the multivariate conditional hazard rate function* $\lambda_j(\cdot|i_1, \ldots, i_k; t_1, \ldots, t_k) : (t_k, +\infty) \rightarrow [0, +\infty)$ *is defined as*

$$\lambda_j(t|i_1, \ldots, i_k; t_1, \ldots, t_k) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \Pr(T_j \leq t + \Delta t | T_{i_1} = t_1, \ldots, T_{i_k} = t_k, T_{k+1:n} > t). \tag{4.1}$$

For $k = 0$ *and* $j = 1, \ldots, n$ *the multivariate conditional hazard rate function is defined by*

$$\lambda_j(t|\emptyset) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \Pr(T_j \leq t + \Delta t | T_{1:n} > t), \qquad t > 0. \tag{4.2}$$

The function (4.1) represents the failing tendency of the jth component if surviving at time t when we know that the components with labels $i_1, \ldots, i_k$ have failed at times $t_1 < \cdots < t_k < t$, respectively, whereas the remaining components are still operating at t. Similarly, (4.2) describes the failure inclination of the component number j when all the components are still working at time t. For further details see, e.g., [25, 26, 29], the review paper [27], and references cited therein.

Note that (4.1) are multivariate conditional versions of failure intensity functions. In particular, they satisfy

$$\int_{t_k}^{\infty} \lambda_j(t|i_1,\dots,i_k;t_1,\dots,t_k)\,dt = +\infty \tag{4.3}$$

for all $i_1,...,i_k,j \in [n]$, and almost all $0 < t_1 < \dots < t_k < +\infty$. Similarly, we have

$$\int_{0}^{\infty} \lambda_j(t|\emptyset)\,dt = +\infty, \qquad j = 1,\dots,n. \tag{4.4}$$

The uniform frailty model requires that the component failure tendencies (which depend on the labels of the already failed components and on their corresponding failure times) are however identical for all the working components. The formal definition is the following.

Definition 4.2. *We say that the joint probability distribution of* $(T_1,\dots,T_n)$ *satisfy the property of uniform frailty if there exist functions* $\lambda(\cdot|\emptyset) : (0,+\infty)\to[0,+\infty)$ *and* $\lambda(\cdot|i_1,\dots,i_k;t_1,\dots,t_k) : (t_k,+\infty)\to[0,+\infty)$ *for* $1 \le k \le n-1$, $i_1,\dots,i_k \in [n]$, *and* $0 < t_1 < \dots < t_k < +\infty$ *such that the following equalities hold:*

$$\begin{aligned}
\lambda_j(t|i_1,\dots,i_k;t_1,\dots,t_k) &= \lambda(t|i_1,\dots,i_k;t_1,\dots,t_k), && t > t_k,\, j \notin \{i_1,\dots,i_k\},\\
\lambda_j(t|\emptyset) &= \lambda(t|\emptyset), && t > 0,\ j=1,\dots,n.
\end{aligned} \tag{4.5}$$

Note that the above property for $k = n-1$ is always trivially true.

Theorem 4.1. *If the joint distribution of the component lifetimes* $(T_1,\dots,T_n)$ *of a coherent system possesses the uniform frailty property, then the probabilistic and structural signatures of the system are identical.*

Proof. We aim at proving that under the uniform frailty assumption the equality (3.1) holds which according to Proposition 7 in [9] is necessary and sufficient for the equality $\mathbf{p} = \mathbf{s}$.

We start by recalling that, as shown in [24], the joint density function f of $(T_1,\dots,T_n)$ can be expressed in terms of multivariate conditional hazard rate functions as follows

$$\begin{aligned} f(t_1,\dots,t_n) &= \lambda_{\pi(1)}(t_{\pi(1)}|\emptyset)\exp\left(-\int_0^{t_{\pi(1)}}\Lambda(u|\emptyset)\,du\right) \\ &\times \prod_{k=2}^{n}\Bigg[\lambda_{\pi(k)}(t_{\pi(k)}|\pi(1),\dots,\pi(k-1);t_{\pi(1)},\dots,t_{\pi(k-1)}) \\ &\qquad \exp\left(-\int_{t_{\pi(k-1)}}^{t_{\pi(k)}}\Lambda(u|\pi(1),\dots,\pi(k-1);t_{\pi(1)},\dots,t_{\pi(k-1)})\,du\right)\Bigg] \end{aligned}$$

for $t_1,\dots,t_n > 0$ with $t_{\pi(1)} < \dots < t_{\pi(n)}$ and π being the permutation of $[n]$ such that the ordered values $t_{1:n} < \dots < t_{n:n}$ obtained from $t_1,\dots,t_n$ satisfy $t_{i:n} = t_{\pi(i)}$ for $i = 1,\dots,n$, where

$$\Lambda(u|\emptyset) := \sum_{j=1}^{n}\lambda_j(u|\emptyset),$$

and

$$\begin{aligned} &\Lambda(u|\pi(1),\dots,\pi(k-1);t_{\pi(1)},\dots,t_{\pi(k-1)}) \\ &\qquad := \sum_{j=k}^{n}\lambda_{\pi(j)}(u|\pi(1),\dots,\pi(k-1);t_{\pi(1)},\dots,t_{\pi(k-1)}). \end{aligned}$$

Obviously, the above sums inherit the properties (4.4) and (4.3), respectively, of their summands.

Under the uniform frailty model assumption we have

$$\lambda_{\pi(1)}(t_{\pi(1)}|\emptyset) = \lambda(t_{\pi(1)}|\emptyset) = \frac{1}{n}\,\Lambda(t_{\pi(1)}|\emptyset),$$

and

$$\begin{aligned} &\lambda_{\pi(k)}(t_{\pi(k)}|\pi(1),\dots,\pi(k-1);t_{\pi(1)},\dots,t_{\pi(k-1)}) \\ &\qquad = \lambda(t_{\pi(k)}|\pi(1),\dots,\pi(k-1);t_{\pi(1)},\dots,t_{\pi(k-1)}) \\ &\qquad = \frac{1}{n+1-k}\Lambda(t_{\pi(k)}|\pi(1),\dots,\pi(k-1);t_{\pi(1)},\dots,t_{\pi(k-1)}) \end{aligned}$$

for $k = 2, \ldots, n$. Therefore we can write

$$\begin{aligned} f(t_1, \ldots, t_n) &= \frac{1}{n!} \Lambda(t_{\pi(1)}|\emptyset) \exp\left(-\int_0^{t_{\pi(1)}} \Lambda(u|\emptyset)\, du\right) \\ &\times \prod_{k=2}^{n} \Bigg[\Lambda(t_{\pi(k)}|\pi(1), \ldots, \pi(k-1); t_{\pi(1)}, \ldots, t_{\pi(k-1)}) \\ &\quad \exp\left(-\int_{t_{\pi(k-1)}}^{t_{\pi(k)}} \Lambda(u|\pi(1), \ldots, \pi(k-1); t_{\pi(1)}, \ldots, t_{\pi(k-1)})\, du\right)\Bigg]. \end{aligned}$$

Now we are in a position to prove

$$\Pr(T_{1:n} = T_{\pi(1)}, \ldots, T_{n:n} = T_{\pi(n)}) = \frac{1}{n!} \tag{4.6}$$

for every permutation π of $[n]$. We have

$$\begin{aligned} &\Pr(T_{1:n} = T_{\pi(1)}, \ldots, T_{n:n} = T_{\pi(n)}) \\ &\quad = \int \cdots \int_{\{0<t_{\pi(1)}<\ldots<t_{\pi(n)}<\infty\}} f(t_1, \ldots, t_n)\, dt_{\pi(1)} \ldots dt_{\pi(n)} \\ &\quad = \frac{1}{n!} \Bigg[\int_0^{\infty} \Lambda(t_{\pi(1)}|\emptyset) \exp\left(-\int_0^{t_{\pi(1)}} \Lambda(u|\emptyset)\, du\right) dt_{\pi(1)} \\ &\quad \Bigg[\int_{t_{\pi(1)}}^{\infty} \Lambda(t_{\pi(2)}|\pi(1); t_{\pi(1)}) \exp\left(-\int_{t_{\pi(1)}}^{t_{\pi(2)}} \Lambda(u|\pi(1); t_{\pi(1)})\, du\right) dt_{\pi(2)} \\ &\quad \ldots \\ &\quad \Bigg[\int_{t_{\pi(n)}}^{\infty} \Lambda(t_{\pi(n)}|\pi(1), \ldots, \pi(n-1); t_{\pi(1)}, \ldots, t_{\pi(n-1)}) \\ &\quad \exp\left(-\int_{t_{\pi(n-1)}}^{t_{\pi(n)}} \Lambda(u|\pi(1), \ldots, \pi(n-1); t_{\pi(1)}, \ldots, t_{\pi(n-1)})\, du\right) dt_{\pi(n)} \Bigg] \ldots \Bigg] \Bigg]. \end{aligned}$$

Observe that

$$\int_0^{\infty} \Lambda(t_{\pi(1)}|\emptyset) \exp\left(-\int_0^{t_{\pi(1)}} \Lambda(u|\emptyset)\, du\right) dt_{\pi(1)} = \left[-\exp\left(-\int_0^{t_{\pi(1)}} \Lambda(u|\emptyset)\, du\right)\right]_{t_{\pi(1)=0}}^{t_{\pi(1)=\infty}}$$

amounts to 1, and so do

$$\begin{aligned}
&\int_{t_{\pi(k-1)}}^{\infty} \Lambda(t_{\pi(k)}|\pi(1),\ldots,\pi(k-1);t_{\pi(1)},\ldots,t_{\pi(k-1)}) \\
&\qquad \exp\left(-\int_{t_{\pi(k-1)}}^{t_{\pi(k)}} \Lambda(u|\pi(1),\ldots,\pi(k-1);t_{\pi(1)},\ldots,t_{\pi(k-1)})\,du\right) dt_{\pi(k)} \\
= &\left[-\exp\left(-\int_{t_{\pi(k-1)}}^{t_{\pi(k)}} \Lambda(u|\pi(1),\ldots,\pi(k-1);t_{\pi(1)},\ldots,t_{\pi(k-1)})\,du\right)\right]_{t_{\pi(k)}=t_{\pi(k-1)}}^{t_{\pi(k)}=\infty}
\end{aligned}$$

for $k = 2,\ldots,n$. We calculate the multiple iterated integral in the following way: first we determine the last definite integral, and then we plug its value 1 into the previous one. This operation does not modify the form of the penultimate integral, and allows us to check that its value equals 1. Repeating the procedure n times we deduce that the iterated integral is equal to 1, and, in consequence, equality (4.6) holds.

Finally, we observe that (4.6) does imply

$$\Pr\left(\max_{i\in A^c} T_i < \min_{i\in A} T_i\right) = \sum_{\{\pi:\pi(i)>n-|A|,\, i\in A\}} \Pr(T_{1:n} = T_{\pi(1)},\ldots,T_{n:n} = T_{\pi(n)}) = \frac{1}{\binom{n}{|A|}},$$

as desired. □

The following example shows that the uniform frailty is not a necessary condition for the identity between the probabilistic and structural signatures.

Example 4.1. *We assume the restrictions of the time-homogenous load-sharing model (see [30] and the references therein) where the multivariate conditional hazard rate functions depend merely on the labels of failed components and are independent of the running time t and of the failure times $t_1,\ldots,t_k$ of the failed components, as well. This allows us to simplify the notation*

$$\begin{aligned}
\lambda_j(t|\emptyset) &= \lambda_j(\emptyset), \\
\lambda_j(t|i_1,\ldots,i_k;t_1,\ldots,t_k) &= \lambda_j(\{i_1,\ldots,i_k\}).
\end{aligned} \tag{4.7}$$

We further assume that $n = 3$,

$$\lambda_j(\emptyset) = \lambda(\emptyset), \qquad j = 1,2,3, \tag{4.8}$$

are all identical, and

$$\lambda_1(\{2\}) = \lambda_2(\{3\}) = \lambda_3(\{1\}) = a \neq \lambda_2(\{1\}) = \lambda_3(\{2\}) = \lambda_1(\{3\}) = b \qquad (4.9)$$

for some different positive numbers a and b. The conditions in (4.9) *violate the uniform frailty property. The values $\lambda_{j_1}(\{j_2, j_3\})$ are immaterial here. We have*

$$\begin{aligned} q(\{1\}) &= \Pr(T_{3:3} = T_1) = \Pr(T_{1:3} = T_2, T_{2:3} = T_3) + \Pr(T_{1:3} = T_3, T_{2:3} = T_2) \\ &= \frac{1}{3}\left[\frac{\lambda_3(\{2\})}{\lambda_1(\{2\}) + \lambda_3(\{2\})} + \frac{\lambda_2(\{3\})}{\lambda_1(\{3\}) + \lambda_2(\{3\})}\right]. \end{aligned}$$

Similarly, we calculate

$$\begin{aligned} q(\{2\}) &= \frac{1}{3}\left[\frac{\lambda_3(\{1\})}{\lambda_2(\{1\}) + \lambda_3(\{1\})} + \frac{\lambda_1(\{3\})}{\lambda_1(\{3\}) + \lambda_2(\{3\})}\right], \\ q(\{3\}) &= \frac{1}{3}\left[\frac{\lambda_1(\{2\})}{\lambda_1(\{2\}) + \lambda_3(\{2\})} + \frac{\lambda_2(\{1\})}{\lambda_2(\{1\}) + \lambda_3(\{1\})}\right]. \end{aligned}$$

Plugging (4.9) *into the above formulae, we obtain*

$$q(\{j\}) = \frac{1}{3}, \qquad j = 1, 2, 3.$$

Using (4.8)*, we also get*

$$q(\{j_1, j_2\}) = \Pr(T_{1:3} = T_{j_3}) = \frac{1}{3}, \qquad \{j_1, j_2, j_3\} = \{1, 2, 3\}.$$

The two above equalities guarantee (3.1)*, which in view of Proposition 7 in [9] implies* $\mathbf{p} = \mathbf{s}$.

Example 4.2. *Here we combine the assumptions of a* 3*-component time-homogeneous load-sharing model stated in the preceding example with the uniform frailty assumption. This enables us to drop the subscripts at λ's in the notation of* (4.7)*. We additionally assume that $\lambda(\{1\}) \neq \lambda(\{2\})$. Writing $\mathcal{L}(X) = \mathcal{L}(Y)$ below we mean that the distributions of random variables X and Y coincide. We also extend the notation to conditional distributions. By Corollary 3 of [21], we have*

$$\begin{aligned} \mathcal{L}(T_{2:3}|T_{1:3} = T_1, T_{2:3} = T_2, T_{3:3} = T_3) &= \mathcal{L}(T_{2:3}|T_{1:3} = T_1, T_{2:3} = T_2) \\ &= \mathcal{L}\left(\frac{V_1}{3\lambda(\emptyset)} + \frac{V_2}{2\lambda(\{1\})}\right), \qquad (4.10) \end{aligned}$$

where V_1 and V_2 denote two independent standard exponential random variables. Due to Proposition 1 in [21], the unconditional distribution of $T_{2:3}$ is the uniform mixture of three distributions

$$\mathcal{L}(T_{2:3}) = \frac{1}{3}\sum_{i=1}^{3}\mathcal{L}\left(\frac{V_1}{3\lambda(\emptyset)} + \frac{V_2}{2\lambda(\{i\})}\right). \tag{4.11}$$

If either $\lambda(\{1\}) \neq \lambda(\{2\})$ or $\lambda(\{1\}) \neq \lambda(\{3\})$, then distributions (4.10) *and* (4.11) *are different, and the weak exchangeability condition* (2.12) *assuring* (2.2) *is not satisfied.*

By Proposition 3 in [21],

$$\begin{aligned}\mathcal{L}(T_1) = {}& \frac{1}{3}\mathcal{L}\left(\frac{V_1}{3\lambda(\emptyset)}\right) + \frac{1}{6}\mathcal{L}\left(\frac{V_1}{3\lambda(\emptyset)} + \frac{V_2}{2\lambda(\{2\})}\right) + \frac{1}{6}\mathcal{L}\left(\frac{V_1}{3\lambda(\emptyset)} + \frac{V_2}{2\lambda(\{3\})}\right) \\ & + \frac{1}{6}\mathcal{L}\left(\frac{V_1}{3\lambda(\emptyset)} + \frac{V_2}{2\lambda(\{2\})} + \frac{V_3}{2\lambda(\{2,3\})}\right) \\ & + \frac{1}{6}\mathcal{L}\left(\frac{V_1}{3\lambda(\emptyset)} + \frac{V_2}{2\lambda(\{3\})} + \frac{V_3}{2\lambda(\{2,3\})}\right),\end{aligned}$$

where V_3 is another independent standard exponential random variable. We obtain the analogous representation for the distribution of T_2 if we replace in the right-hand side of the above formula $\lambda(\{2\})$ and $\lambda(\{2,3\})$ by $\lambda(\{1\})$ and $\lambda(\{1,3\})$, respectively. It follows that if either $\lambda(\{1\}) \neq \lambda(\{2\})$ or $\lambda(\{1,3\}) \neq \lambda(\{2,3\})$, then T_1 and T_2 have different marginal distributions. This contradicts the necessary condition of identical component lifetime distributions which assures representation (2.4) *for the system lifetime.*

5 Summary and conclusion

This section is devoted to comment on main results of the paper and to put them in an historical perspective. Several interesting studies have been devoted, in the last decade, to conceptual aspects of the general concept of signature. However, only in the cases when the *signature representation* (2.4) holds, one gets a really efficient tool for reliability analysis.

Actually (2.4) can be seen as a condition of symmetry on the distribution of the components' lifetimes $T_1, \ldots, T_n$, which is implied by exchangeability, but is actually more general than it. It becomes then interesting to characterize the multivariate distributions for which the property (2.4) holds. As recalled above, one such result had been first given in [9]. It had been shown therein that, for whatever coherent structure ψ with $n > 2$, the signature representation (2.4) holds if and only if, for any $t > 0$, the binary variables $X_1(t), \ldots, X_n(t)$, that represent the states of the components at time t, are exchangeable.

In the present paper we come back to considering the same characterization problem, aiming however to use concepts of copulas in the description of the stochastic dependence between the components. We also want to connect them with the results given in [9]. The main result of the paper consists in full characterization of the component lifetimes joint distributions which admit the signature representation (2.4). This result extends the representation obtained in [12] for identically distributed (ID) component lifetimes and diagonal dependent (DD) copulas by considering the weaker concept of S-DD copulas proposed in Definition 2.1. This condition preserves the diagonal dependence property on the range of the common marginal distribution. Our characterization result provides a useful tool for extending various stochastic ordering relations known for systems with exchangeable components to the systems with components satisfying much weaker dependence relations.

We remind that many papers had been devoted to analyzing structural properties of the system reliability function $\bar{F}_T$, when looked at as a functional of ψ and of the joint distribution of $T_1, \ldots, T_n$. Such papers have emphasized the role of the copulas among components' lifetimes. In particular it has been pointed out that, for any $t \geq 0$, the system reliability value $\bar{F}_T(t)$ can be written as a function of the component's reliability values $\bar{F}_1(t), \ldots, \bar{F}_n(t)$. It is natural that this function is parametrized by ψ and by the survival copula, namely that it is a *functional* of them. Moreover, it does not depend on t.

Actually, such a functional can be presented in a number of different forms, so that different types of notation and terminologies have been employed by the various authors, in the frame of distinct contexts. We just notice that, even if presented under different forms, the system reliability $\bar{F}_T(t)$ at time t can anyway be seen as the probability of a union of different basic events, depending on the structure of the system. Consequently, the computation of $\overline{F}_T(t)$ can be carried out, in any case, by appropriately resorting to the use of an *inclusion-exclusion* formula. Thus, in the paper [16] (see also [10, 11]), the probability of $(T > t)$ is written as a function of the probabilities $(T_i > t)$ for $i = 1, \ldots, n$. In this setting, the terms *generalized distortion function* and *dual distortion function* have been used to describe the form of the afore-mentioned functional.

On the other hand, using the language common in other contexts, it is natural to think in terms of *aggregation functions* (see, e.g., [6]). In fact, for any $t > 0$, $\bar{F}_T(t)$ can be seen as an aggregation of the components' marginal reliability functions. From the point of view of mathematical formalization, basic steps in this direction amount to looking at ψ as a (binary) *fuzzy measure* and to rely on the concept of *Möbius transform* in handling the inclusion-exclusion formula. By following this path, the afore-mentioned aggregation function can be described (see in particular [17, 30] and references therein) as a *copula-based extension of a fuzzy measure*, namely as a special case in the class of *aggregation-based extensions of fuzzy measures* as studied in [8] (see also references therein). The fuzzy measure is given by the system's structure ψ, while the copula appearing in the above description turns out to be the survival copula of the lifetimes. Any result obtained in the reliability context for the function $\bar{F}_T(t)$ can then be seen as a result for a continuous aggregation function obtained as a copula-based extension of a binary fuzzy measure.

Coming back to commenting on our main results in Section 2, we notice however that they are obtained by restricting our attention on the case when $T_1, \ldots, T_n$ are identically distributed (which is a necessary condition for (2.4) to hold). This entails that the function connecting $\bar{F}_T$ with $\bar{F}_1, \ldots, \bar{F}_n$, namely the aggregation function of our interest, is just analyzed on the diagonals, that is, in the points of type $\mathbf{u}_I$ (see Definition 2.1 and Theorem 2.2), when $\bar{F}_1 = \cdots = \bar{F}_n$.

A different type of problem, actually not involving the form of $\bar{F}_T$ directly, is studied in Section 4. Regarding the joint distribution of $T_1, ..., T_n$, we give a sufficient condition under which the probability signature $\mathbf{p}$ coincides, for any arbitrary coherent system, with the structural signature $\mathbf{s}$. A necessary and sufficient condition for the equality $\mathbf{p} = \mathbf{s}$ was already given in [9]. Also this equality amounts to a condition of symmetry, more general than exchangeability. The sufficient condition given here is *uniform frailty*, a notion that has been formalized in [21] and that is expressed in terms of the *multivariate conditional hazard rate functions*. An open problem for the future research is representing the necessary and sufficient conditions for the signature representation in terms of the multivariate conditional hazard rates.

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