Direct transformation from Cartesian into geodetic coordinates on a triaxial ellipsoid

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Abstract

This paper[1] presents two new direct symbolic-numerical algorithms for the transformation of Cartesian coordinates into geodetic coordinates considering the general case of a triaxial reference ellipsoid. The problem in both algorithms is reduced to finding a real positive root of a sixth degree polynomial. The first approach consists of algebraic manipulations of the equations describing the geometry of the problem and the second one uses Gröbner bases. In order to perform numerical tests and accurately compare efficiency and reliability, our algorithms together with the iterative methods presented by M. Ligas (2012) and J. Feltens (2009) have been implemented in C++. The numerical tests have been accomplished by considering 10 celestial bodies, referenced in the available literature. The obtained results clearly show that our algorithms improve the aforementioned iterative methods, in terms of both efficiency and accuracy.

Keywords: Coordinate transformation, Cartesian coordinates, Geodetic coordinates, Triaxial ellipsoid, Symbolic-numerical computation

1 Introduction

Transformation between Cartesian and Geodetic coordinates is an important, basic problem frequently encountered in Astronomy, Geodesy and Geoinformatics. Both coordinates are defined with respect to a Cartesian reference system and, in the case of geodetic coordinates, an ellipsoid with the center at the origin of the Cartesian reference system is also considered. Although computing Cartesian coordinates from geodetic coordinates can be easily performed, the inverse transformation is a non-trivial, challenging problem.

In our opinion, efficient innovative solutions of this problem, as well as another actual challenges faced in Geodesy and Geoinformatics reside in the application of algebraic computational techniques combined, if necessary, with numerical methods (see, for instance, [1]).

[1]First and third authors conceived the presented ideas, developed the theory and the algorithms, carried out the initial implementation in Maple and performed the initial computations and numerical tests. Second author carried out the final implementation in C++ and performed the final computations and numerical tests.

In the particular case of a reference biaxial ellipsoid, numerous solutions have been proposed (see, for instance, [7], [10] and [11] for iterative solutions, [22] for perturbation techniques based solutions and [3], [12] and [23] for closed form solutions). Interesting solutions have been recently developed in [20], [21] and [4].

Using as geometric model of the Earth a biaxial ellipsoid is barely justified by the computational simplicity of the approach, the existing standard reference systems (such as WGS 84) and the small difference between the axes in the equatorial plane (which rounds up to 69 m). Nevertheless, the triaxiality of the Earth has been studied in many papers during the last decades (see for instance [?], [?], [?] and [?]). Moreover, in [?], the authors explicitly state (on page 862), refering to the Earth's shape parameter: "Actually, with respect to the biaxial ellipsoid, fitting the triaxial ellipsoid is 65% better."

Therefore, the Earth and other celestial bodies (some of them listed in Table 1) can be much more appropriately (in terms of accuracy of the geometric model) approximated by triaxial ellipsoids. Furthermore, nowadays computational tools allow us to overcome the difficulty of working with three different semiaxes.

Historically, the Earth and celestial bodies with rather small diferences between semiaxes, had initially been modelled by spheres, afterwards by biaxial ellipsoids and nowadays the triaxial ellipsoid modelling is emerging. In our opinion, it might be just a matter of time until standard reference systems have based on triaxial ellipsoid.

At our best knowledge, the general case of triaxial reference ellipsoid has been considered up to the moment only in [8] and [14], both approaches giving iterative solutions. We present in this paper two new direct symbolic-numerical algorithms giving closed form solutions, which can be applied also to a biaxial reference ellipsoid.

Therefore, the novelty of our approaches resides in tackling the issue from the *symbolic* perspective, accompanied by better efficiency and accuracy results in comparison with the iterative methods developed in [8] and [14], and in using a *triaxial* reference ellipsoid. The symbolic perspective consists in generating some sixth degree polynomials, prove that they have only one positive root and afterwards compute them. In the proof of the uniqueness of the positive roots, the coefficients of these polynomials are not numerical values, but symbolic, generical expressions depending on the semiaxes of the reference ellipsoid and the cartesian coordinates of the considered point.

More concretely, in the algorithm called `Cartesian into Geodetic I`, described in Section 3, our closed form solution consists of finding the real positive root of a sixth degree polynomial in a variable t. This variable t serves to describe the cartesian coordinates of the given point. On the other hand, the algorithm called `Cartesian into Geodetic II`, described in Section 4, also consists of finding the real positive root of a sixth degree polynomial but in the variable z, which represents the third coordinate of the three-dimensional coordinate system.

The structure of the paper is as follows: Section 2 introduces some preliminaries and definitions. Sections 3 and 4 introduce the results that lead us to the algorithms materialized at the end of each section. Each algorithm is based on the numeric computation of the unique real positive root of a sixth degree polynomial. Both polynomials are symbolically generated: in the first approach by algebraic manipulations of the equations describing the geometry of the problem and in the second approach by computing a Gröbner basis. The uniqueness of the real positive roots is proven symbolically, by applying Descartes' rule of signs and studying the relative positions of several ellipsoids. The algorithm presented in Section 3 computes firstly the parametric coordinate (a parameter which serves to describe the cartesian coordinates) of the given point and secondly the Cartesian coordinates of the corresponding footpoint (the intersection point of the ellipsoidal normal vector passing through the given point and the ellipsoid). The algorithm presented in Section 4 computes firstly the z coordinate of the corresponding footpoint and secondly its x and y coordinates. The numerical tests performed with the celestial bodies listed in Table 1, together with the obtained results, are presented in Section 5. In Section 6 we present the main conclusions and further work.

2 Preliminaries

Given a point P_E on a triaxial ellipsoid, its Cartesian coordinates (X_E, Y_E, Z_E) satisfy the ellipsoid equation

$$f(X,Y,Z) = \frac{X^2}{a_x^2} + \frac{Y^2}{a_y^2} + \frac{Z^2}{a_z^2} - 1 = 0$$

and its geodetic and Cartesian coordinates are related as follows (see [16]):

$$X_E = \nu \cos\varphi \cos\lambda, \quad Y_E = \nu\,(1-e_e^2)\cos\varphi\sin\lambda, \quad Z_E = \nu\,(1-e_x^2)\sin\varphi,$$

where ν is equal to the radius of the prime vertical, $\nu = \dfrac{a_x}{\sqrt{1-e_x^2\sin^2\varphi - e_e^2\cos^2\varphi\sin^2\lambda}}$, and the first eccentricities squared are

$$e_x^2 = \frac{a_x^2 - a_z^2}{a_x^2},\ e_y^2 = \frac{a_y^2 - a_z^2}{a_y^2},\ e_e^2 = \frac{a_x^2 - a_y^2}{a_x^2}.$$

Obviously, if latitude φ and longitude λ are given, one obtains (X_E, Y_E, Z_E) by substitutions. Viceversa, if the coordinates (X_E, Y_E, Z_E) are given, then

$$\lambda = \begin{cases} \arctan\left(\dfrac{1}{(1-e_e^2)}\dfrac{Y_E}{X_E}\right), & \text{if } X_E > 0 \\ \arctan\left(\dfrac{1}{(1-e_e^2)}\dfrac{Y_E}{X_E}\right) + \pi, & \text{if } X_E < 0 \\ \operatorname{sign}(Y_E)\,\dfrac{\pi}{2}, & \text{if } X_E = 0 \text{ and } Y_E \neq 0 \\ \text{undefined}, & \text{if } X_E = Y_E = 0 \end{cases}$$

$$\varphi = \begin{cases} \arctan\left(\dfrac{(1-e_e^2)}{(1-e_x^2)}\dfrac{Z_E}{\sqrt{(1-e_e^2)^2X_E^2 + Y_E^2}}\right), & \text{if } X_E \neq 0 \text{ or } Y_E \neq 0 \\ \operatorname{sign}(Z_E)\,\frac{\pi}{2}, & \text{if } X_E = Y_E = 0 \end{cases} \tag{1}$$

However, suppose now that we have the cartesian coordinates of a point P_G and we want to compute its geodetic coordinates. In this case, there exists an ellipsoidal height h (see Figure 1) such that

$$X_G = (\nu + h)\cos\varphi\cos\lambda, \quad Y_G = (\nu\,(1-e_e^2) + h)\cos\varphi\sin\lambda, \quad Z_G = (\nu\,(1-e_x^2) + h)\sin\varphi, \tag{2}$$

and the point P_G will have the same latitude and longitude as the intersection point of the ellipsoidal normal vector passing through P_G and the ellipsoid. This point will be named the footpoint of P_G. Hence, obtaining the geodetic coordinate (φ, λ, h) from the Cartesian ones involves first to compute (X_E, Y_E, Z_E), the footpoint of P_G, and secondly to apply formulas (1).

The problem of computing the footpoint can be considered as the study of the distance from a point to an ellipsoid, a classical issue in Geometry, and it is tackled for example in [2],[13] and [6] from a less algebraic point of view than ours. Concretely, in [2] the formula (4) appears (on pages 112-113), but with practically no considerations about its resolution. [13] is interesting as a basic, seminal approach but it seems that the conclusions are drawn without much mathematical rigor. [6] is a much more interesting work, Eberly considered a function defined by formula (4) in our paper and analitically proved, by a Bolzano type theorem, that it had only one root in certain interval.

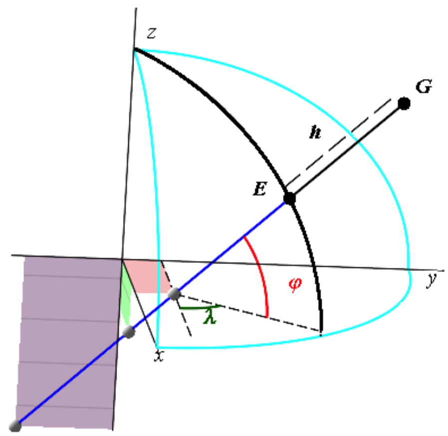

Figure 1: Geometry of the problem

3 Computing the footpoint. First approach

In our computations, we will apply Descartes' rule of signs, which determines the number of positive real roots of a univariate polynomial, and is based on the number of sign changes of its real coefficients.

Theorem 1. *[[15] Descartes' rule] Let $f(X) = a_n X^n + a_{n-1}X^{n-1} + \cdots + a_0$ be a polynomial in $\mathbb{R}[x]$, where a_n and a_0 are nonzero. Let v be the number of changes of signs in the sequence $[a_n, \ldots, a_0]$ of its coefficients and let r be the number of its real positive roots, counted with their orders of multiplicity. Then there exists some nonnegative integer m such that $r = v - 2m$.*

We will apply Descartes' rule several times across the paper, for polynomials whose number of sign changes in its lists of coefficients is equal to 0 or 1, therefore they have no or one positive real root, respectively. Analyzing the sign of the coefficients of these polynomials will be reduced to studying the relative positions of several ellipsoids. These ellipsoids have the same center and each ellipsoid will turn out to be placed inside or outside the others, having no intersection points.

The unique positive real roots of these polynomials will be used to determine the footpoint of a given point (see Equations (3) and (8)).

We assume throughout the paper, for simplicity, that our point $P_G \neq (0,0,0)$ is situated in the first octant and also that $a_x > a_y > a_z$. We define $P = (a_x - a_z)(a_x + a_z) > 0, Q = (a_y - a_z)(a_y + a_z) > 0$ and $R = (a_x - a_y)(a_x + a_y) > 0$.

Following [2], [8] and [14], the gradient of $f(X, Y, Z)$ evaluated in the footpoint P_E provides a normal vector to the ellipsoid, $\vec{n} = 2\left(\frac{X_E}{a_x^2}, \frac{Y_E}{a_y^2}, \frac{Z_E}{a_z^2}\right)$, and a vector connecting point P_G and P_E is

$$\vec{h} = (X_G - X_E, Y_G - Y_E, Z_G - Z_E) = h(\cos\varphi\cos\lambda, \cos\varphi\sin\lambda, \sin\varphi)$$

with $P_G = \vec{h} + P_E$. Both vectors $\vec{h}$ and $\vec{n}$ must be proportional and so, in the general case $|h| > 0$, there is a real value t with

$$t = \frac{X_G - X_E}{X_E/a_x^2} = \frac{Y_G - Y_E}{Y_E/a_y^2} = \frac{Z_G - Z_E}{Z_E/a_z^2},$$

and thus

$$X_E = \frac{a_x^2\, X_G}{t + a_x^2},\ Y_E = \frac{a_y^2\, Y_G}{t + a_y^2},\ Z_E = \frac{a_z^2\, Z_G}{t + a_z^2} \tag{3}$$

Since $\frac{X_E^2}{a_x^2} + \frac{Y_E^2}{a_y^2} + \frac{Z_E^2}{a_z^2} = 1$, we have

$$\frac{(a_x\, X_G)^2}{(t + a_x^2)^2} + \frac{(a_y\, Y_G)^2}{(t + a_y^2)^2} + \frac{(a_z\, Z_G)^2}{(t + a_z^2)^2} - 1 = 0. \tag{4}$$

The numerator of Equation (4) is the polynomial $A(t) = t^6 + A_5 t^5 + A_4 t^4 + A_3 t^3 + A_2 t^2 + A_1 t + A_0$, where

$$\begin{aligned}
A_5 &= 2\,(a_x^2 + a_y^2 + a_z^2) > 0,\\
A_4 &= -a_x^2 X_G^2 - a_y^2 Y_G^2 - a_z^2 Z_G^2 + (a_x^2 + a_y^2 + a_z^2)^2 + 2(a_x^2 a_y^2 + a_x^2 a_z^2 + a_y^2 a_z^2),\\
A_3 &= -2\left(\, a_x^2(a_y^2 + a_z^2) X_G^2 + a_y^2(a_x^2 + a_z^2) Y_G^2 + a_z^2(a_x^2 + a_y^2) Z_G^2 - \right.\\
&\quad \left. -(a_x^2 + a_y^2 + a_z^2)(a_y^2 a_z^2 + a_x^2 a_y^2 + a_x^2 a_z^2) - a_x^2 a_y^2 a_z^2 \,\right),\\
A_2 &= -a_x^2(a_y^4 + 4a_y^2 a_z^2 + a_z^4) X_G^2 - a_y^2(a_x^4 + 4a_x^2 a_z^2 + a_z^4) Y_G^2 - a_z^2(a_x^4 + 4a_x^2 a_y^2 + a_y^4) Z_G^2 + \\
&\quad +(a_x^2 a_y^2 + a_x^2 a_z^2 + a_z^2 a_y^2)^2 + 2a_x^2 a_y^2 a_z^2 (a_x^2 + a_y^2 + a_z^2),\\
A_1 &= -2a_x^2 a_y^2 a_z^2 \left(\, (a_y^2 + a_z^2) X_G^2 + (a_x^2 + a_z^2) Y_G^2 + (a_x^2 + a_y^2) Z_G^2 - a_x^2 a_y^2 - a_x^2 a_z^2 - a_y^2 a_z^2 \,\right),\\
A_0 &= -a_x^2 a_y^2 a_z^2 \left(\, a_x^2 a_y^2 Z_G^2 + a_x^2 a_z^2 Y_G^2 + a_y^2 a_z^2 X_G^2 - a_x^2 a_y^2 a_z^2 \,\right).
\end{aligned}$$

The variable t can be considered as a parametric coordinate of P_G and is positive if the point is situated outside the reference ellipsoid, negative if it is situated inside or 0 if it is situated on the reference ellipsoid. Obviously, the ellipsoidal heigh h is equal to 0 iff $A_0 = 0$.

Remark 1. *In the particular case of a biaxial reference ellipsoid, when $a_x = a_y$, the Equation (4) becomes*

$$\frac{(a_x\, X_G)^2 + (a_x\, Y_G)^2}{(t + a_x^2)^2} + \frac{(a_z\, Z_G)^2}{(t + a_z^2)^2} - 1 = 0 \tag{5}$$

and leads to the fourth degree polynomial $\alpha(t) = t^4 + \alpha_3 t^3 + \alpha_2 t^2 + \alpha_1 t + \alpha_0$ where

$$\begin{aligned}
\alpha_3 &= 2\left({a_x}^2 + {a_z}^2\right),\\
\alpha_2 &= -{a_x}^2({X_G}^2 + {Y_G}^2) - {a_z}^2 {Z_G}^2 + ({a_x}^2 + {a_z}^2)^2 + 2\,{a_x}^2 {a_z}^2,\\
\alpha_1 &= -2\,{a_x}^2 {a_z}^2 \left({X_G}^2 + {Y_G}^2 + {Z_G}^2 - {a_x}^2 - {a_z}^2\right),\\
\alpha_0 &= -{a_x}^2 {a_z}^2 \left({a_z}^2 {X_G}^2 + {a_z}^2 {Y_G}^2 + {a_x}^2 {Z_G}^2 - {a_x}^2 {a_z}^2\right).
\end{aligned}$$

The results obtained in this paper can be established also for the biaxial case. Nevertheless, we do not consider of any relevance this particular case: the aforementioned fourth degree polynomial has been studied in [12] completely symbolically, by using Sturm–Habicht coefficients and subresultants, having led to a close form solution.

Proposition 3.1. *The number of sign changes in $[A_5, A_4, A_3, A_2, A_1, A_0]$ is equal to 1 if the point P_G is situated outside the reference ellipsoid, or 0 if the point P_G is situated inside or on the reference ellipsoid.*

Proof. The sign of A_0 depends on the sign of the factor

$$a_x^2 a_y^2 Z_G^2 + a_x^2 a_z^2 Y_G^2 + a_y^2 a_z^2 X_G^2 - a_x^2 a_y^2 a_z^2,$$

which is the numerator of $f(X_G, Y_G, Z_G) - 1$. The sign of A_1 depends on the sign of the factor

$$(a_y^2 + a_z^2) X_G^2 + (a_x^2 + a_z^2) Y_G^2 + (a_x^2 + a_y^2) Z_G^2 - a_x^2 a_y^2 - a_x^2 a_z^2 - a_y^2 a_z^2,$$

which defines the ellipsoid of equation

$$e_1: \quad X^2 \frac{a_y^2 + a_z^2}{a_x^2 a_y^2 + a_x^2 a_z^2 + a_y^2 a_z^2} + Y^2 \frac{a_x^2 + a_z^2}{a_x^2 a_y^2 + a_x^2 a_z^2 + a_y^2 a_z^2} + Z^2 \frac{a_x^2 + a_y^2}{a_x^2 a_y^2 + a_x^2 a_z^2 + a_y^2 a_z^2} = 1.$$

Since

$$\frac{a_x^2 a_y^2 + a_x^2 a_z^2 + a_y^2 a_z^2}{a_y^2 + a_z^2} > a_x^2, \quad \frac{a_x^2 a_y^2 + a_x^2 a_z^2 + a_y^2 a_z^2}{a_x^2 + a_z^2} > a_y^2, \quad \frac{a_x^2 a_y^2 + a_x^2 a_z^2 + a_y^2 a_z^2}{a_x^2 + a_y^2} > a_z^2,$$

the original, reference ellipsoid $e_{original}$ is situated inside the ellipsoid e_1.
The coefficient A_2 defines the ellipsoid of equation

$$\begin{aligned} e_2: \quad & X^2 \frac{a_x^2 (a_y^4 + 4 a_y^2 a_z^2 + a_z^4)}{(a_x^2 a_y^2 + a_x^2 a_z^2 + a_z^2 a_y^2)^2 + 2 a_x^2 a_y^2 a_z^2 (a_x^2 + a_y^2 + a_z^2)} + \\ & + Y^2 \frac{a_y^2 (a_x^4 + 4 a_x^2 a_z^2 + a_z^4)}{(a_x^2 a_y^2 + a_x^2 a_z^2 + a_z^2 a_y^2)^2 + 2 a_x^2 a_y^2 a_z^2 (a_x^2 + a_y^2 + a_z^2)} + \\ & + Z^2 \frac{a_z^2 (a_x^4 + 4 a_x^2 a_y^2 + a_y^4)}{(a_x^2 a_y^2 + a_x^2 a_z^2 + a_z^2 a_y^2)^2 + 2 a_x^2 a_y^2 a_z^2 (a_x^2 + a_y^2 + a_z^2)} = 1. \end{aligned}$$

The semiaxes of the ellipsoid e_2 are bigger than the corresponding semiaxes of the ellipsoid e_1, and in consequence

$$e_{original} \subset e_1 \subset e_2.$$

The sign of the coefficient A_3 depends on a negative factor and on the factor

$$a_x^2 (a_y^2 + a_z^2) X_G^2 + a_y^2 (a_x^2 + a_z^2) Y_G^2 + a_z^2 (a_x^2 + a_y^2) Z_G^2 - (a_x^2 + a_y^2 + a_z^2)(a_y^2 a_z^2 + a_x^2 a_y^2 + a_x^2 a_z^2) - a_x^2 a_y^2 a_z^2.$$

This factor defines the ellipsoid of equation

$$\begin{aligned} e_3: \quad & X^2 \frac{a_x^2 (a_y^2 + a_z^2)}{(a_x^2 + a_y^2 + a_z^2)(a_y^2 a_z^2 + a_x^2 a_y^2 + a_x^2 a_z^2) + a_x^2 a_y^2 a_z^2} + \\ & + Y^2 \frac{a_y^2 (a_x^2 + a_z^2)}{(a_x^2 + a_y^2 + a_z^2)(a_y^2 a_z^2 + a_x^2 a_y^2 + a_x^2 a_z^2) + a_x^2 a_y^2 a_z^2} + \\ & + Z^2 \frac{a_z^2 (a_x^2 + a_y^2)}{(a_x^2 + a_y^2 + a_z^2)(a_y^2 a_z^2 + a_x^2 a_y^2 + a_x^2 a_z^2) + a_x^2 a_y^2 a_z^2} = 1. \end{aligned}$$

The semiaxes of the ellipsoid e_3 are also bigger than the corresponding semiaxes of the ellipsoid e_2, and in consequence

$$e_{original} \subset e_1 \subset e_2 \subset e_3.$$

Finally, the coefficient A_4 defines the ellipsoid of equation

$$\begin{aligned} e_4 : \quad & X^2 \frac{a_x^2}{(a_x^2+a_y^2+a_z^2)^2 + 2(a_x^2 a_y^2 + a_x^2 a_z^2 + a_y^2 a_z^2)} + \\ & + Y^2 \frac{a_y^2}{(a_x^2+a_y^2+a_z^2)^2 + 2(a_x^2 a_y^2 + a_x^2 a_z^{+} a_y^2 a_z^2)} + \\ & + Z^2 \frac{a_z^2}{(a_x^2+a_y^2+a_z^2)^2 + 2(a_x^2 a_y^2 + a_x^2 a_z^2 + a_y^2 a_z^2)} \quad = 1 \end{aligned}$$

The semiaxes of the ellipsoid e_4 are also bigger than the corresponding semiaxes of the ellipsoid e_3, and in consequence

$$e_{original} \subset e_1 \subset e_2 \subset e_3 \subset e_4.$$

Therefore, the signs of the list $[A_5, A_4, A_3, A_2, A_1, A_0]$ must be one of the following (being the number of sign changes equal to 1 for an outside point P_G and 0 otherwise):

- $[+, +, +, +, +, +]$ if P_G is inside the reference ellipsoid,
- $[+, +, +, +, +, 0]$ if P_G is on the reference ellipsoid,
- $[+, +, +, +, +, -]$ if P_G is outside the reference ellipsoid and inside e_1,
- $[+, +, +, +, 0, -]$ if P_G is on e_1,
- $[+, +, +, +, -, -]$ if P_G is outside e_1 and inside e_2,
- $[+, +, +, 0, -, -]$ if P_G is on e_2,
- $[+, +, +, -, -, -]$ if P_G is outside e_2 and inside e_3,
- $[+, +, 0, -, -, -]$ if P_G is on e_3,
- $[+, +, -, -, -, -]$ if P_G is outside e_3 and inside e_4,
- $[+, 0, -, -, -, -]$ if P_G is on e_4,
- $[+, -, -, -, -, -]$ if P_G is outside e_4.

□

Consequently if P_G is outside the reference ellipsoid, then the polynomial $A(t)$ has a unique real positive root. If P_G is inside the reference ellipsoid, then the polynomial $A(t)$ has no positive real roots. If P_G is on the reference ellipsoid, then it has no positive real roots and furthermore $A(0) = 0$.

3.1 P_G situated inside the ellipsoid

We will analyze in the following the case of P_G being situated inside the ellipsoid. Suppose first that $Z_G > 0$. Then $Z_E > 0$ and because of (3), we should have $t > -a_z^2$. Therefore, there exists $k > 0$ with $t = -a_z^2 + k$. That leads us to consider the polynomial $\bar{A}(k) = A(-a_z^2 + k)$, whose number of positive real roots is equal to the number of real (negative, since $A(t)$ has no positive real roots in this case) roots of $A(t)$ satisfying $t > -a_z^2$.
By applying Descartes' rule, we will see that $\bar{A}(k)$ has only one positive root. We obtain that $\bar{A}(k) = k^6 + \bar{A}_5 k^5 + \bar{A}_4 k^4 + \bar{A}_3 k^3 + \bar{A}_2 k^2 + \bar{A}_1 k + \bar{A}_0$, where

$$
\begin{aligned}
\bar{A}_5 &= 2(P+Q) > 0, \\
\bar{A}_4 &= -a_x^2 X_G^2 - a_y^2 Y_G^2 - a_z^2 Z_G^2 + P^2 + Q^2 + 4PQ, \\
\bar{A}_3 &= 2\left(-a_x^2 Q X_G^2 - a_y^2 P Y_G^2 - a_z^2 (P+Q) Z_G^2 + PQ(P+Q)\right), \\
\bar{A}_2 &= -a_x^2 Q^2 X_G^2 - a_y^2 P^2 Y_G^2 - a_z^2 (P^2 + Q^2 + 4PQ) Z_G^2 + P^2 Q^2, \\
\bar{A}_1 &= -2 a_z^2 PQ(P+Q) Z_G^2 \leq 0\,, \\
\bar{A}_0 &= -a_z^2 P^2 Q^2 Z_G^2 \leq 0\,.
\end{aligned}
$$

Proposition 3.2. *If $Z_G > 0$, the number of sign changes in the list $[\bar{A}_5, \bar{A}_4, \bar{A}_3, \bar{A}_2, \bar{A}_1, \bar{A}_0]$ is equal to 1.*

Proof. The coefficient $\bar{A}_2$ defines the ellipsoid $\bar{e}_2$,

$$\bar{e}_2 :\ X^2 \frac{a_x^2}{P^2} + Y^2 \frac{a_y^2}{Q^2} + Z^2 \frac{a_z^2 (P^2 + Q^2 + 4PQ)}{P^2 Q^2} = 1.$$

The coefficient $\bar{A}_3$ defines the ellipsoid of equation

$$\bar{e}_3 : X^2 \frac{a_x^2}{P(P+Q)} + Y^2 \frac{a_y^2}{Q(P+Q)} + Z^2 \frac{a_z^2}{PQ} = 1.$$

The coefficient $\bar{A}_4$ defines the ellipsoid of equation

$$\bar{e}_4 : X^2 \frac{a_x^2}{P^2 + Q^2 + 4PQ} + Y^2 \frac{a_y^2}{P^2 + Q^2 + 4PQ} + Z^2 \frac{a_z^2}{P^2 + Q^2 + 4PQ} = 1.$$

Since

$$P^2 < P(P+Q) < P^2+Q^2+4\,PQ, \quad Q^2 < Q(P+Q) < P^2+Q^2+4\,PQ, \quad \frac{P^2 Q^2}{P^2 + Q^2 + 4\,PQ} < PQ < P^2+Q^2+4\,P$$

we have $\bar{e}_2 \subset \bar{e}_3 \subset \bar{e}_4$. Therefore, the signs of the list $[\bar{A}_5, \bar{A}_4, \bar{A}_3, \bar{A}_2, \bar{A}_1, \bar{A}_0]$ must be one of the following:

- $[+,+,+,+,-,-]$ if the point P_G is inside $\bar{e}_2$,
- $[+,+,+,0,-,-]$ if the point P_G is on $\bar{e}_2$,
- $[+,+,+,-,-,-]$ if the point P_G is outside $\bar{e}_2$ and inside $\bar{e}_3$,
- $[+,+,0,-,-,-]$ if the point P_G is on $\bar{e}_3$,

- $[+,+,-,-,-,-]$ if the point P_G is outside $\bar{e}_3$ and inside $\bar{e}_4$,
- $[+,0,-,-,-,-]$ if the point P_G is on $\bar{e}_4$,
- $[+,-,-,-,-,-]$ if the point P_G is outside $\bar{e}_4$.

□

Consequently if P_G is situated inside the reference ellipsoid with $Z_G > 0$ then the polynomial $A(t)$ has a unique real root satisfying $-a_z^2 < t < 0$.

Suppose now that $Z_G = 0$. Then, $\varphi = 0$ and the footpoint P_E is on the ellipse

$$\frac{X^2}{a_x^2} + \frac{Y^2}{a_y^2} = 1. \tag{6}$$

Observe that if $Y_G = 0$, then $\lambda = 0$ and if $X_G = 0$ then $\lambda = \frac{\pi}{2}$. Suppose that $X_G > 0$ and $Y_G > 0$. Thus, following the same reasoning as before, we will have

$$\frac{(a_x\, X_G)^2}{(t + a_x^2)^2} + \frac{(a_y\, Y_G)^2}{(t + a_y^2)^2} - 1 = 0,$$

with the numerator equal to $\Delta(t) = t^4 + \Delta_3 t^3 + \Delta_2 t^2 + \Delta_1 t + \Delta_0$, where

$$\begin{aligned}
\Delta_3 &= 2\left(a_x^2 + a_y^2\right) > 0,\\
\Delta_2 &= \left(a_x^4 + 4\, a_x^2 a_y^2 + a_y^4 - a_x^2 X_G^2 - a_y^2 Y_G^2\right),\\
\Delta_1 &= 2\, a_x^2 a_y^2 \left(a_x^2 + a_y^2 - X_G^2 - Y_G^2\right),\\
\Delta_0 &= a_x^2 a_y^2 \left(a_x^2 a_y^2 - a_x^2 Y_G^2 - a_y^2 X_G^2\right).
\end{aligned}$$

In this case, Δ_0 is zero iff the point P_G is situated on the ellipse (6), and the number of sign changes in the list $[\Delta_3, \Delta_2, \Delta_1, \Delta_0]$ is zero for a point P_G inside or on the ellipse (6). However, by the same reasoning as before, t must be bigger than $-a_y^2$ and if we substitute $k - a_y^2$ for t in $\Delta(t)$, we obtain

$$\bar{\Delta}(k) = k^4 + \bar{\Delta}_3 k^3 + \bar{\Delta}_2 k^2 + \bar{\Delta}_1 k + \bar{\Delta}_0,$$

with

$$\bar{\Delta}_3 = 2R > 0, \quad \bar{\Delta}_2 = R^2 - a_x^2 X_G^2 - a_y^2 Y_G^2, \quad \bar{\Delta}_1 = -2a_y^2 Y_G^2 R < 0, \quad \bar{\Delta}_0 = -a_y^2 Y_G^2 R^2 < 0, \tag{7}$$

therefore the number of sign changes in the list $[\bar{\Delta}_3, \bar{\Delta}_2, \bar{\Delta}_1, \bar{\Delta}_0]$ is equal to 1.

Consequently if P_G is situated inside the reference ellipsoid with $Z_G = 0$, $X_G > 0$ and $Y_G > 0$, then the polynomial $\Delta(t)$ has a unique real root satisfying $-a_y^2 < t < 0$.

3.2 The algorithm

All these results lead to the following algorithm.

Algorithm `Cartesian into Geodetic I`

Require: The semiaxes of the triaxial reference ellipsoid.
The Cartesian coordinates $(X_G, Y_G, Z_G) \neq (0,0,0)$.

Ensure: The geodetic coordinates (φ, λ, h).

if $f(X_G, Y_G, Z_G) = 1$ **then**
  $(X_G, Y_G, Z_G) = (X_E, Y_E, Z_E)$, (φ, λ) are computed from Equalities (1) and $h = 0$;
else
  if $f(X_G, Y_G, Z_G) > 1$ **then**
    evaluate coefficients A_i, $i = 0, \ldots, 5$; {see Proposition 3.1}
    compute T the unique positive root of $A(t)$;
    substitute $t = \mathrm{T}$ in Equalities (3) for computing (X_E, Y_E, Z_E);
    $h = |(X_G, Y_G, Z_G) - (X_E, Y_E, Z_E)|$
  else
    if $Z_G > 0$ **then**
      evaluate coefficients $\bar{A}_i$, $i = 0, \ldots, 5$; {see Proposition 3.2}
      compute K the unique positive root of $\bar{A}(k)$;
      substitute $t = -a_z^2 + \mathrm{K}$ in Equalities (3) for computing (X_E, Y_E, Z_E);
      $h = -|(X_G, Y_G, Z_G) - (X_E, Y_E, Z_E)|$;
      compute (φ, λ) from Equalities (1)
    else
      $Z_E = 0$; $\varphi = 0$;
      if $X_G > 0$, $Y_G > 0$ **then**
        evaluate coefficients $\bar{\Delta}_i$, $i = 0, \ldots, 3$; {see Equations (7)}
        compute K the unique positive root of $\bar{\Delta}(k)$;
        substitute $t = -a_y^2 + \mathrm{K}$ in Equalities (3) for computing X_E and Y_E;
        $h = -|(X_G, Y_G) - (X_E, Y_E)|$;
        compute λ from Equalities (1)
      end if
      if $X_G = 0$ **then**
        $X_E = 0$; $Y_E = a_y$; $\lambda = \dfrac{\pi}{2}$; $h = Y_G - Y_E$
      end if
      if $Y_G = 0$ **then**
        $X_E = a_x$; $Y_E = 0$; $\lambda = 0$; $h = X_G - X_E$
      end if
    end if
  end if
end if

4 Computing the footpoint. Second approach

The ideal generated by a family of polynomials is defined to be the set of linear combinations, with polynomial coefficients, of these polynomials (see [?] pg.30 for details). If we have a system of equations with finitely many solutions, it is well known that a Gröbner basis (see [1] and [?] for details) of the ideal generated by the equations of such a system provides another equivalent system but in triangular form, which is much easier to solve. We will explore this idea in this section.

According to Section 3, the cartesian coordinates of the footpoint must satisfy the system of equations in three unknowns given by:

$$\frac{x^2}{a_x^2}+\frac{y^2}{a_y^2}+\frac{z^2}{a_z^2}=1,\quad \frac{X_G-x}{x/a_x^2}-\frac{Y_G-y}{y/a_y^2}=0,\quad \frac{X_G-x}{x/a_x^2}-\frac{Z_G-z}{z/a_z^2}=0,\quad \frac{Y_G-y}{y/a_y^2}-\frac{Z_G-z}{z/a_z^2}=0.$$

By assuming first that none of three variables is zero, this system is equivalent to the following one:

$$S:\left\{\begin{array}{lcl} a_y^2a_z^2x^2+a_x^2a_z^2y^2+a_x^2a_y^2z^2-a_x^2a_y^2a_z^2 &=& 0,\\ a_x^2xy-a_x^2X_Gy-a_y^2xy+a_y^2Y_Gx &=& 0,\\ a_x^2xz-a_x^2X_Gz-a_z^2xz+a_z^2Z_Gx &=& 0,\\ a_z^2yz+a_y^2Y_Gz-a_z^2Z_Gy-a_y^2yz &=& 0. \end{array}\right.$$

The system S has finitely many solutions, and so, as mentioned previously, a Gröbner basis of the ideal generated by the equations of S provides another equivalent system but in triangular form in the variables x, y, z. The univariate equation in z in the Gröbner basis[2] is given by $B(z) = B_6z^6 + B_5z^5 + B_4z^4 + B_3z^3 + B_2z^2 + B_1z + B_0$, where

$$\begin{array}{rcl} B_6 &=& P^2Q^2>0,\\ B_5 &=& 2\,a_z^2Z_G\,PQ\,(P+Q)\geq 0,\\ B_4 &=& a_z^2\left(a_x^2Q^2X_G^2+a_y^2P^2Y_G^2+a_z^2\left(P^2+Q^2+4\,PQ\right)Z_G^2-P^2Q^2\right),\\ B_3 &=& 2\,a_z^4Z_G\,\left(a_x^2QX_G^2+a_y^2P\,Y_G^2+a_z^2\,(P+Q)\,Z_G^2-PQ\,(P+Q)\right),\\ B_2 &=& a_z^6Z_G^2\left(a_x^2X_G^2+a_y^2Y_G^2+a_z^2Z_G^2-P^2-Q^2-4\,PQ\right),\\ B_1 &=& -2\,a_z^8\,Z_G^3\,(P+Q)\leq 0,\\ B_0 &=& -a_z^{10}Z_G^4\leq 0\,. \end{array}$$

Therefore, the positive root of $B(z)$ will be the coordinate Z_E required.

Proposition 4.1. *The number of sign changes in the list* $[B_6, B_5, B_4, B_3, B_2, B_1, B_0]$ *is equal to 1 if* $Z_G > 0$.

Proof. The signs of B_2, B_3 and B_4 are determined by the ellipsoids $\bar{e}_4$, $\bar{e}_3$ and $\bar{e}_2$, respectively, introduced in the proof of Proposition 3.2. Since $\bar{e}_2 \subset \bar{e}_3 \subset \bar{e}_4$, if $Z_G > 0$ the signs of the list $[B_6, B_5, B_4, B_3, B_2, B_1, B_0]$ must be one of the following:

- $[+,+,-,-,-,-,-]$ if P_G is inside $\bar{e}_2$,
- $[+,+,0,-,-,-,-]$ if P_G is on $\bar{e}_2$,
- $[+,+,+,-,-,-,-]$ if P_G is outside $\bar{e}_2$ and inside $\bar{e}_3$,
- $[+,+,+,0,-,-,-]$ if P_G is on $\bar{e}_3$,
- $[+,+,+,+,-,-,-]$ if P_G is outside $\bar{e}_3$ and inside $\bar{e}_4$,
- $[+,+,+,+,0,-,-]$ if P_G is on $\bar{e}_4$,

[2] The Gröbner basis using the lexicographical order with $y > x > z$ (see [?] pg.56 for details), computed with `Maple 2017` is available at `http://dx.doi.org/10.17632/xw5ws5gz8x.1`.

- $[+,+,+,+,+,-,-]$ if P_G is outside $\bar{e}_4$.

□

Consequently, if $Z_G > 0$, $B(z)$ has only one real positive root, which is equal to Z_E. Moreover, the polynomials

$$B_2(x,z) = \left(Pz + a_z^2 Z_G\right) x - a_x^2 X_G z, \quad B_3(y,z) = \left(Qz + a_z^2 Z_G\right) y - a_y^2 Y_G z,$$

part of the Gröbner basis, provide the coordinates X_E and Y_E:

$$X_E = \frac{a_x^2 X_G Z_E}{(P Z_E + a_z^2 Z_G)}, \quad Y_E = \frac{a_y^2 Y_G Z_E}{(Q Z_E + a_z^2 Z_G)}. \tag{8}$$

On the other hand, if $Z_G = 0$ then $Z_E = 0$ and we obtain a new system

$$a_x^2 y^2 + a_y^2 x^2 - a_x^2 a_y^2 = 0, \quad (a_x^2 - a_y^2) xy - a_x^2 X_G y + a_y^2 Y_G x = 0,$$

whose Gröbner basis[3] contains the polynomials

$$G_1(y) = R^2 y^4 + 2a_y^2 R Y_G y^3 - a_y^2 \left(R^2 - a_x^2 X_G^2 - a_y^2 Y_G^2\right) y^2 - 2a_y^4 R Y_G y - a_y^6 Y_G^2, \tag{9}$$

$$G_2(x,y) = \left(Ry + a_y^2 Y_G\right) x - a_x^2 X_G y,$$

which provide the coordinates Y_E and X_E. As the coefficients in y^4 and y^3 of $G_1(y)$ are positive and the coefficient in y and the independent one are negative, the number of changes of signs in the list of coefficients of $G_1(y)$ is equal to 1. Consequently, $G_1(y)$ has a unique real positive root.

Finally, if both $Z_G = 0$ and $Y_G = 0$ (unusual in practice) then $\varphi = \lambda = 0$.

[3] Available at `http://dx.doi.org/10.17632/xw5ws5gz8x.1`

Algorithm `Cartesian into Geodetic II`

Require: The semiaxes of the triaxial reference ellipsoid.
The Cartesian coordinates $(X_G, Y_G, Z_G) \neq (0, 0, 0)$.

Ensure: The geodetic coordinates (φ, λ, h).

if $Z_G \neq 0$ **then**
    evaluate the coefficients B_i, $i = 0, \ldots, 6$; {see Proposition 4.1}
    compute Z_E the unique positive root of $B(z)$;
    compute X_E and Y_E from Equalities (8);
    compute (φ, λ) from Equalities (1)
else
    $Z_E = 0$; $\varphi = 0$;
    if $Y_G \neq 0$, **then**
        evaluate the coefficients of the polynomial $G_1(y)$; {see Equations (9)}
        compute Y_E the unique positive root of $G_1(y)$;
        compute X_E the unique real root of $G_2(x, Y_E)$;
        compute λ from Equalities (1)
    else
        $Y_E = 0$; $X_E = a_x$; $\lambda = 0$
    end if
end if
if $f(X_G, Y_G, Z_G) \geq 1$ **then**
    $h = |(X_G, Y_G, Z_G) - (X_E, Y_E, Z_E)|$
else
    $h = -|(X_G, Y_G, Z_G) - (X_E, Y_E, Z_E)|$
end if

5 Numerical tests

Our algorithms have been initially implemented in the Scientific Computing System `Maple 2017`. We have implemented also the methods presented in [8] and [14], in order to accurately compare the results (maximum errors and running times). This initial study showed that the best running times and the best mean values of the maximum deviations were obtained with the algorithms `Cartesian into Geodetic I` and `Cartesian into Geodetic II`. Nevertheless, the CPU times obtained in `Maple` were high (as other formula processing systems, `Maple` runs in the interpreter mode, and therefore, it runs slow).

For this reason, the definitive implementation of the aforementioned algorithms has been performed in a compiler-type programing language, specifically in C++. The definitive CPU running times, in C++, differ in an order of magnitude 3 from the initial ones, in `Maple`. The results have been obtained working with double precision, on an Intel(R) Core(TM) i7-7700K CPU @ 4.20 GHz x 8 processor with 62,8GB of RAM.

The considered celestial bodies, together with their shape parameters (a_x, a_y and a_z respectively) (see [14], [17], [18], [19], [24]) are as follows:

Celestial body	a_x	a_y	a_z
Ariel	581.1	577.9	577.7
Earth	6378.173435	6378.1039	6356.7544
Enceladus	256.6	251.4	248.3
Europa	1564.13	1561.23	1560.93
Io	1829.4	1819.3	1815.7
Mars	3394.6	3393.3	3376.3
Mimas	207.4	196.8	190.6
Miranda	240.4	234.2	232.9
Moon	1735.55	1735.324	1734.898
Tethys	535.6	528.2	525.8

Table 1: Semiaxes (in km) of the considered celestial bodies

Following [14], we consider the points in the first octant defined by the geodetic coordinates $(\varphi_i, \lambda_j, h_k)$, where $\varphi_i = \dfrac{i\pi}{720}$ radians, $i = 1 \dots 359$, $\lambda_j = \dfrac{j\pi}{720}$ radians, $j = 1 \dots 359$, $h_k = k a_z$ km, $k \in \{0, \pm\dfrac{1}{50}, \pm\dfrac{1}{25}, \pm\dfrac{1}{15}, \pm\dfrac{1}{10}\}$. For each point, we compute its Cartesian coordinates from (2) and apply the corresponding algorithm for computing its geodetic coordinates, comparing the obtained values with the initial ones. We have excluded from the points considered for the numerical tests the following cases: $\varphi_0 = 0$, in which case $Z_G = 0$ and $X_G Y_G > 0$ and Case 3 of Ligas' method can't be applied, as the Jacobian is singular; $\varphi_{360} = \frac{\pi}{2}$, in which case $X_G = Y_G = 0$ and the longitude is undefined (see [16]); $\lambda_0 = 0$, in which case $Y_G = 0$ and $X_G > 0$ and Case 2 of Ligas' method can't be applied; and $\lambda_{360} = \frac{\pi}{2}$, in which case $X_G = 0$ and $Y_G > 0$ and Case 1 of Ligas' method can't be applied. Therefore, we considered, for each algorithm and each celestial body, 359 latitudes, 359 longitudes and 9 heights along the normal, i.e. a total of 1159929 different points. The averaged CPU times are computed by applying the corresponding algorithm once to all these points, and including the computation of the maximum errors.

The following tables present the maximum differences in absolute value between the real, known geodetic coordinates and the computed ones, on a base-10 log scale, together with the mean CPU running times in seconds. A logaritmic scale is a nonlinear scale often used when analyzing a very wide or narrow range of positive quantities. In the following tables, in the second, third and fourth columns, instead of displaying the maximum errors as $\varepsilon = 10^a$, where a is some negative real number, we display $\log_{10}(\varepsilon) = a$.

Celestial body	Max. err. λ	Max. err. φ	Max. err. h	Time
Ariel	**-18.789** / -18.664	-18.664 / -18.664	**-15.634** / -15.400	1.179343 / **1.093714**
Earth	-18.664 / -18.664	-18.664 / -18.664	**-14.700** / -14.500	1.191914 / **1.093113**
Enceladus	**-18.420** / -15.940	**-18.311** / -17.885	**-14.512** / -11.873	1.296452 / **1.190336**
Europa	-18.664 / -18.664	-18.567 / **-18.664**	**-15.244** / -12.768	1.182543 / **1.092273**
Io	**-18.789** / -18.664	**-18.664** / **-18.664**	**-15.277** / -14.767	1.183251 / **1.092522**
Mars	-18.664 / -18.567	**-18.664** / **-18.664**	**-15.000** / -14.816	1.228557 / **1.103634**
Mimas	-17.698 / **-18.664**	-17.550 / **-18.664**	-14.142 / **-15.559**	1.166641 / **1.152859**
Miranda	**-18.266** / -15.793	**-18.186** / -17.862	**-14.426** / -11.873	1.166864 / **1.107854**
Moon	**-18.789** / -18.664	**-18.664** / **-18.664**	**-15.244** / -15.045	1.180139 / **1.093541**
Tethys	**-18.664** / -17.311	-18.664 / **-18.664**	**-15.371** / -12.331	**1.175097** / 1.196335
Mean values	***-18.540*** / *-17.959*	*-18.460* / ***-18.506***	***-14.955*** / *-13.893*	*1.1950801* / ***1.1216181***

Table 2: Results obtained by applying the algorithms `Cartesian into Geodetic I` / `Cartesian into Geodetic II` implemented in C++

Celestial body	Max. err. λ	Max. err. φ	Max. err. h	Time
Ariel	-17.775 / -17.664 / -18.488	-18.337 / **-18.789** / **-18.664**	-13.664 / -13.662 / -13.663	1.270501 / 1.281323 / 1.270980
Earth	**-18.789** / **-18.789** / **-18.789**	-18.664 / -18.664 / **-18.789**	-14.552 / -14.627 / -14.612	1.275257 / 1.274793 / 1.270144
Enceladus	-14.804 / -15.169 / -17.580	-17.145 / -17.139 / -17.146	-13.305 / -13.304 / -13.299	1.279295 / 1.271683 / 1.270610
Europa	**-18.789** / **-18.789** / **-18.789**	**-18.664** / **-18.664** / **-18.664**	-14.084 / -14.084 / -14.083	1.268811 / 1.298762 / 1.322419
Io	-17.446 / -17.488 / -18.789	**-18.664** / **-18.664** / **-18.664**	-14.148 / -14.148 / -14.151	1.271437 / 1.271444 / 1.271251
Mars	**-18.789** / **-18.789** / **-18.789**	**-18.664** / **-18.664** / **-18.664**	-14.372 / -14.366 / -14.372	1.269966 / 1.277228 / 1.273551
Mimas	-16.583 / -14.260 / -16.780	-16.851 / -16.786 / -16.851	-13.185 / -13.186 / -13.183	1.348541 / 1.340201 / 1.340244
Miranda	-14.625 / -15.225 / -17.534	-17.257 / -17.257 / -17.257	-13.274 / -13.272 / -13.270	1.270835 / 1.273983 / 1.272070
Moon	**-18.789** / **-18.789** / **-18.789**	**-18.664** / **-18.664** / **-18.664**	-14.123 / -14.122 /-14.122	1.238008 / 1.240675 /1.245733
Tethys	-15.733 / -16.062 / -17.886	-17.972 / -18.187 / **-18.664**	-13.627 / -13.625 /-13.624	1.273102 / 1.277854 / 1.273062
Mean values	*-17.212 / -17.102 / -18.221*	*-18.088 / -18.148 / -18.203*	*-13.833 / -13.840 / -13.838*	*1.276575 / 1.280795 / 1.281006*

Table 3: Results obtained by applying Case 1 / Case 2 / Case 3 of Ligas' method implemented in C++

Celestial body	Max. err. λ	Max. err. φ	Max. err. h	Time
Ariel	-9.148 / -9.151 / -9.193	-11.515 /-11.533 /-11.470	-8.757 /-8.784 /-8.760	1.397315 /1.391594 / 1.359541
Earth	-12.760 / -12.902 /-12.898	-12.768 /-12.788 /-12.782	-9.011 /-9.034 /-9.029	1.381216 /1.363576 /1.366670
Enceladus	-8.062 /-8.243 /-8.216	-10.414 /-10.609 /-10.333	-8.020 /-8.221 /-7.992	1.623761 /1.531733 /1.587326
Europa	-10.044 /-10.112 /-10.062	-12.478 /-12.479 /-12.378	-9.214 /-9.256 /-9.242	1.161976 /1.158426 /1.170673
Io	-9.555 /-9.697 /-9.637	-11.882 /-12.027 /-11.868	-8.650 /-8.782 /-8.723	1.429542 /1.403343 /1.415738
Mars	-10.673 /-11.691 /-11.723	-12.302 /-12.346 /-12.325	-8.793 /-8.820 /-8.976	1.413745 /1.376237 /1.392895
Mimas	-7.592 /-7.812 /-7.709	-9.965 /-10.162 /-9.758	-7.633 /-7.826 /-7.560	1.951112 /1.740943 /1.858730
Miranda	-8.031 /-8.128 /-8.070	-10.394 /-10.498 /-10.325	-8.036 /-8.117 /-8.008	1.600380 /1.574738 /1.638333
Moon	-12.961 /-11.250 /-11.259	-13.227 /-13.864 /-13.228	-10.212 /-10.412 /-10.213	1.222829 /1.220730 /1.229268
Tethys	-8.616 /-8.721 /-8.738	-10.961 /-11.110 /-10.921	-8.265 /-8.386 /-8.675	1.533092 /1.498667 /1.534345
Mean values	*-9.744 / -9.771 / -9.751*	*-11.591 / -11.742 / -11.539*	*-8.659 / -8.764 / -7.113*	*1.471497 / 1.425999 / 1.455352*

Table 4: Results obtained by applying Case 1 / Case 2 / Case 3 of Feltens' method implemented in C++

In all the considered case studies, the best and second best running times are obtained with the algorithms `Cartesian into Geodetic II` and `Cartesian into Geodetic I`. Moreover, the best and second best mean values of the maximum deviations obtained in the 10 case studies correspond to our algorithms, except for the second best mean value of the maximum deviation of the longitude (which corresponds to the Case 3 of Feltens' method). The three best results are presented in the following table:

Position	Max. err. λ	Max. err. φ	Max. err. h	Time
Best result	`Cartesian into Geodetic I`	`Cartesian into Geodetic II`	`Cartesian into Geodetic I`	`Cartesian into Geodetic II`
Second best result	Case 3 of Feltens' method	`Cartesian into Geodetic I`	`Cartesian into Geodetic II`	`Cartesian into Geodetic I`
Third best result	`Cartesian into Geodetic II`	Case 3 of Feltens' method	Case 2 of Feltens' method	Case 2 of Feltens' method

Table 5: Ranking of the three best results in computing the mean values of the maximum deviations and CPU running times

These results show that our approaches improve the methods presented in [8] and [14], in terms of both efficiency and accuracy.

6 Conclusions and further work

We have presented two efficient algorithms for the transformation of Cartesian coordinates into geodetic coordinates, for a triaxial reference ellipsoid. Each algorithm is based on the numeric computation of the unique real positive root of a degree 6 polynomial, symbolically generated.

One of the main topics of our further work consists in studying the case of the hyperboloidal coordinates considered for triaxial reference hyperboloids and providing a similar approach for the transformation of the cartesian coordinates. From the geometric and algebraic points of view, both problems are closely related. This problem hasn't been tackled before and furthermore there are very few approaches for the biaxial case (see [5] for a closed form solution and [9] for a iterative solution).

7 Acknowledgments

The first and third authors are partially supported by FEDER/Ministerio de Ciencia, Innovación y Universidades - Agencia Estatal de Investigación/MTM2017-88796-P (Symbolic Computation: new challenges in Algebra and Geometry together with its applications). The second author is partially funded by the project TIN2017-86885-R cofinanced by the EU Feder program. The third author wish to kindly thank Prof. Enrique D. Fernández-Nieto for his support.

The authors wish to thank the anonymous referees for their valuable remarks which allowed to substantially improve the presentation of this paper.

8 Computer Code Availability

The implementations of our two algorithms are open source code and can be obtained at the link `http://dx.doi.org/10.17632/s5f6sww86x.2` (C++) and `http://dx.doi.org/10.17632/vf9r367m6d.3` (`Maple`).

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