Solving the interference problem for ellipses and ellipsoids: new formulae

J. Caravantes[1], G.M. Diaz–Toca[2], M. Fioravanti[3], and L. Gonzalez-Vega[4]

[1]Universidad de Alcala (Spain)
[2]Universidad de Murcia (Spain)
[3]Universidad de Cantabria (Spain)
[4]CUNEF Universidad (Spain)

Abstract

The problem of detecting when two moving ellipses or ellipsoids overlap is of interest to robotics, CAD/CAM, computer animation, etc., where ellipses and ellipsoids are often used for modelling (and/or enclosing) the shape of the objects under consideration. By analysing symbolically the sign of the real roots of the characteristic polynomial of the pencil defined by two ellipses/ellipsoids $\mathcal{A}$ and $\mathcal{B}$ given by $X^TAX = 0$ and $X^TBX = 0$, we derive new formulae characterising when $\mathcal{A}$ and $\mathcal{B}$ overlap, are separate, or touch each other externally. This characterisation is defined by a minimal set of polynomial inequalities depending only on the entries of A and B so that we need only compute the characteristic polynomial of the pencil defined by A and B, $\det(TA + B)$, and not the intersection points between them. Compared with the best available approach dealing with this problem, the new formulae involve a smaller set of polynomials and less sign conditions. As an application, this characterisation provides also a new approach for exact collision detection of two moving ellipses or ellipsoids since the analysis of the univariate polynomials (depending on the time) in the previously mentioned formulae provides the collision events between them.

Keywords: Ellipses; Ellipsoids; Subresultants; Events detection

1 Introduction

The problem of detecting the collisions or overlap of two ellipses in the plane or two ellipsoids in the 3D space is of interest to robotics, CAD/CAM, computer animation, etc., where ellipses and ellipsoids are often used for modelling (or enclosing) the shape of the objects under consideration (see for example [8, 14, 15, 16, 18, 19, 24, 25, 29]).

In this paper, we propose new formulae to characterise the separation by a line of two ellipses in the plane, and by a plane of two ellipsoids in the three dimensional real affine space, by using subresultants to control the multiplicity structure of a univariate polynomial and Descartes' law of signs to control the number of positive real roots of a univariate polynomial. Our approach is based on the characterisation presented in [21] and [28] where the separation of two ellipses/ellipsoids is determined by the sign of the real roots of the characteristic polynomial of the matrix pencil defined by the two considered ellipses/ellipsoids.

Note that our goal is not the computation of the intersection points between the two considered ellipses or ellipsoids. In fact, this intersection problem can be solved by any numerical nonlinear solver or by "ad–hoc" methods. Nevertheless, the results later described can be used as a preprocessing step since any intersection problem is highly simplified if the structure of the intersection set is known in advance. Also observe that the problem considered here can be presented as a quantifier elimination problem over the reals (see [2]), since we are looking for conditions on the coefficients of the equations defining the considered ellipses/ellipsoids in order they do not share an interior common point.

The main result of this paper is the introduction of new formulae characterising when two ellipses/ellipsoids are separate, overlapping or externally tangent, improving, for the ellipsoids case, the best existing solution introduced in [22]. Compared with this solution, we present, at least, three improvements:

1. Less polynomials are involved: the solution in [22] requires to deal with 5 polynomials and our solution does require only 3 of them.

2. The way the relative positions are characterised is simpler than in [22] (less sign conditions are involved).

3. Our general approach is the same for ellipses and ellipsoids and the proofs are simpler since they do not require to use Sturm–Habicht or signed subresultant sequences and the formulae obtained are very easy to interpret.

For ellipses we obtain (in a different and very direct way) and improve slightly the formulae introduced in [1] for deciding when two ellipses are separate, overlapping or externally tangent. Moreover, the approach presented here is specially well suited for analysing the relative position of two ellipses/ellipsoids depending on one parameter (see for example [5, 6, 7, 10, 12, 22, 27]).

This paper is organised as follows. In Section 2, we introduce subresultants, their main properties and Descartes' law of signs. Section 3 deals with the ellipses separation problem introducing formulae characterising when two ellipses are separate, overlapping or externally tangent and using them to solve the interference problem for two moving ellipses. Section 4 deals with the ellipsoids separation problem introducing formulae characterising when two ellipsoids are separate, overlapping or externally tangent and using them to solve the interference problem for two moving ellipsoids. This section includes five examples highlighting how our approach works in practice when computing the collision free intervals for two moving ellipsoids. Last section is devoted to presenting some conclusions and further work.

2 Subresultants and Descartes' law of signs

Subresultants will be the algebraic tool to use to determine, in a very easy and compact way, the greatest common divisor of two univariate polynomials, or the number of different real roots of a univariate polynomial, when they involve parameters or algebraic numbers as coefficients.

Definition 1.
Let

$$P(T)=\sum_{i=0}^{p} a_i T^i \qquad \text{and} \qquad Q(T)=\sum_{i=0}^{q} b_i T^i$$

be two polynomials with coefficients in a field with $p \geq q$ and $j \in \{0,1,\ldots,q-1\}$. We define the j–th subresultant polynomial of $P(T)$ and $Q(T)$ with respect to T in the following way:

$$\mathbf{Sres}_j(P,Q)=\begin{vmatrix} a_p & a_{p-1} & a_{p-2} & \cdots & \cdots & a_0 & & & & \\ & \ddots & \ddots & \ddots & & & \ddots & & & \\ & & a_p & a_{p-1} & a_{p-2} & \cdots & \cdots & a_0 & & \\ b_q & b_{q-1} & b_{q-2} & \cdots & \cdots & \cdots & b_0 & & & \\ & \ddots & \ddots & \ddots & & & & \ddots & & \\ & & b_q & b_{q-1} & b_{q-2} & \cdots & \cdots & \cdots & b_0 & \\ & & & & & 1 & -T & & & \\ & & & & & & \ddots & \ddots & & \\ & & & & & & & 1 & -T & \end{vmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} q-j \\ \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} p-j \\ \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} j \end{matrix}$$

The resultant of $P(T)$ and $Q(T)$ with respect to T is:

$$\mathbf{Resultant}(P,Q)=\mathbf{Sres}_0(P,Q) \ .$$

The subresultants of $P(T)$ and $Q(T)$ always belong to the ideal generated by $P(T)$ and $Q(T)$ and allow to characterise easily the degree of the greatest common divisor of two univariate polynomials whose coefficients depend on one or several parameters. The determinants $\mathbf{sres}_j(P,Q)$, which are the formal leading coefficients of the polynomials in the subresultant sequence for $P(T)$ and $Q(T)$, can be used to compute the greatest common divisor of $P(T)$ and $Q(T)$ in the following way:

$$\mathbf{Sres}_i(P,Q)=\gcd(P,Q) \quad \Longleftrightarrow \quad \begin{cases} \mathbf{sres}_0(P,Q)=\ldots=\mathbf{sres}_{i-1}(P,Q)=0 \\ \mathbf{sres}_i(P,Q)\neq 0 \end{cases} \tag{1}$$

Proofs of this result can be found in [2, 11]. Next definition introduces the subresultant sequence associated to $P(T)$ as the subresultant sequence for $P(T)$ and $P'(T)$, the main tool, together with Descartes' law of signs, we will use to count the number of positive real roots of an univariate polynomial.

Definition 2.
Let $P(T)$ be a polynomial in $\mathbb{R}[T]$ with $p = \deg(P)$. We define the subresultant sequence of $P(T)$ as $\mathbf{Sres}_p(P) = P(T)$, $\mathbf{Sres}_{p-1}(P) = P'(T)$ and for every $j \in \{0, \ldots, p-2\}$:

$$\mathbf{Sres}_j(P) = \mathbf{Sres}_j(P, P').$$

There are many different ways of defining and computing subresultants: see [2, 11, 13, 20] for a short introduction and for a pointer to several references. The definition of the polynomials in the subresultant sequence through determinants allows to perform computations dealing with greatest common divisors in a generic way: if $P(T)$ is a polynomial with parameters or algebraic numbers as coefficients whose degree does not change after specialisation then the subresultant sequence for $P(T)$ can be computed without specialising the parameters, and the result is always correct after specialisation (modulo the condition over the degree of $P(T)$). This is not true when using Euclidean algorithm: the computation of the Euclidean remainders makes denominators to appear that may vanish after specialisation and, even fixing the degree of $P(T)$, the sequence has not always the same number of elements (see [2, 11] for a more detailed explanation).

Notation 3.
If $P(T)$ is a polynomial in $\mathbb{R}[T]$ with $p = \deg(P)$ and $0 \le k \le p-2$ then the coefficients of the subresultants of P of index k will be denoted in the following way:

$$\mathbf{Sres}_k(P) \stackrel{\text{def}}{=} s_k(P)T^k + s_{k,k-1}(P)T^{k-1} + \ldots + s_{k,1}(P)T + s_{k,0}(P)\ . \tag{2}$$

When $P(T)$ is clearly determined, we will write s_k and $s_{k,j}$ instead of $s_k(P)$ and $s_{k,j}(P)$.

We close this subsection by reviewing the relation between the discriminant of a univariate polynomial $P(T)$ and the resultant of $P(T)$ and $P'(T)$. Let $P(T)$ be a monic polynomial of degree p and $x_1, \ldots, x_p$ its roots (repeated according to their multiplicities). The discriminant of $P(T)$ (see for example [2, 17]) is defined to be

$$\mathrm{Disc}(P) = \prod_{1 \le i < j \le p} (x_i - x_j)^2.$$

In order to compute $\mathrm{Disc}(P)$ we have

$$\mathrm{Disc}(P) = (-1)^{\frac{p(p-1)}{2}} \mathbf{Resultant}(P, P') = (-1)^{\frac{p(p-1)}{2}} \mathbf{Sres}_0(P) = (-1)^{\frac{p(p-1)}{2}} s_0(P).$$

When $P(T)$ is not monic, $P(T) = a_p T^p + \ldots + a_1 T + a_0$, we have

$$\mathrm{Disc}(P) = a_p^{2p-2}\, \mathrm{Disc}\left(\frac{P}{a_p}\right) = \frac{(-1)^{\frac{p(p-1)}{2}}}{a_p} \mathbf{Sres}_0(P) = \frac{(-1)^{\frac{p(p-1)}{2}}}{a_p} s_0(P). \tag{3}$$

Typically the use of the discriminant is linked to characterise when a polynomial has multiple roots but, for low degree polynomials, it is also very useful to determine easily the number of different real roots (see [17] for example).

2.1 Descartes' law of signs

Descartes' law of signs (see [2]) provides an upper bound for the number of positive real roots (taking into account multiplicities) of a univariate polynomial by determining the number of sign changes in the coefficients sequence of the considered polynomial.

For a sequence of real numbers $b_0, b_1, \ldots, b_n$, $\mathbf{Var}(b_0, b_1, \ldots, b_n)$ will denote the number of sign changes in $b_0, b_1, \ldots, b_n$ after dropping the zeros in the sequence.

Proposition 4. *(Descartes' law of signs)*
Let $P(T)$ be the polynomial in $\mathbb{R}[T]$,

$$P(T) = \sum_{k=0}^{n} a_k T^k,$$

where a_n and a_0 are nonzero. The number of positive real roots of $P(T) = 0$, counted with multiplicity, is equal to

$$\mathbf{Var}(a_n, a_{n-1}, \ldots, a_0) - 2m,$$

for some non–negative integer m.

We will use here the following refined version of Descartes' law of signs computing exactly the number of positive real roots (taking into account multiplicities) of a univariate polynomial when all the roots are known to be real.

Proposition 5.
Let $P(T)$ be the polynomial in $\mathbb{R}[T]$,

$$P(T) = \sum_{k=0}^{n} a_k T^k,$$

where a_n and a_0 are nonzero. If all the roots of $P(T)$ are real then the number of positive real roots of $P(T) = 0$, counted with multiplicity, is equal to

$$\mathbf{Var}(a_n, a_{n-1}, \ldots, a_0).$$

See Remark 2.42 in [2] for a proof of this version of Descartes' law of signs.

3 On the ellipses interference problem

Let $\mathcal{A}$ be a conic in $\mathbb{R}^2$. Let $\mathcal{X} = (X, Y, 1)$. The implicit equation of $\mathcal{A}$ in $\mathbb{R}^2$ can be written as

$$a_{11}X^2 + a_{22}Y^2 + 2a_{12}XY + 2a_{13}X + 2a_{23}Y + a_{33} = 0,$$

or in matricial form $\mathcal{A} : \mathcal{X} A \mathcal{X}^t = 0$, where A is the symmetric matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}.$$

In what follows we assume that the matrix A is presented so that the interior of the ellipse $\mathcal{A}$ is defined by $\mathcal{X}A\mathcal{X}^t < 0$. If the conic is an ellipse (i.e., a non-degenerate conic with real affine points and no real points at infinity) then $\det(A) < 0$.

Definition 6.
Given two conics $\mathcal{A} : \mathcal{X}A\mathcal{X}^t = 0$ and $\mathcal{B} : \mathcal{X}B\mathcal{X}^t = 0$, their characteristic equation is defined as

$$P(T) = \det(TA + B) = \det(A)T^3 + \ldots + \det(B),$$

which is, if $\det(A) \neq 0$, a cubic polynomial in T with real coefficients.

Next theorem was introduced in [21] and connects some relative positions (separation, external tangency and overlapping) of two ellipses with the sign of the real roots of their characteristic equation. This is the starting point for the approach we introduce in this section when characterising the separation or the external tangency of two ellipses (see also [1] and [10]).

Theorem 7.
Let $\mathcal{A}$ and $\mathcal{B}$ be two ellipses with characteristic equation $P(T)$. Then:

1. *The leading coefficient and the constant term of $P(T)$ are strictly negative.*
2. *The characteristic equation $P(T)$ has at least one negative real root.*
3. *$\mathcal{A}$ and $\mathcal{B}$ are separated by a line if and only if $P(T)$ has two positive different real roots.*
4. *$\mathcal{A}$ and $\mathcal{B}$ touch each other externally if and only if $P(T)$ has a positive double root.*

This (very) well known lemma shows that, for cubic polynomials, the positivity of the discriminant characterises the case of having three different real roots (see [17] for example).

Lemma 8.
Let $P(T) = a_3T^3 + a_2T^2 + a_1T + a_0$ be a polynomial in $\mathbb{R}[T]$. Then $P(T)$ has three different real roots if and only if $\mathrm{Disc(P)} > 0$.

The following lemma shows how the roots and the coefficients of the characteristic polynomial are related when the two considered ellipses are separate (i.e., when there are two positive different real roots and one negative).

Lemma 9.
Let $\beta_1 > 0$, $\beta_2 > 0$ and $\gamma < 0$ be three different real numbers, and

$$P(T) = a_3T^3 + a_2T^2 + a_1T + a_0 = a_3(T - \beta_1)(T - \beta_2)(T - \gamma),$$

with $a_3 < 0$ and $a_0 < 0$. Then $s_0 > 0, a_1 > 0$ or $s_0 > 0, a_2 > 0$.

Proof. Previous lemma implies $\mathrm{Disc(P)} > 0$. By Equation (3), we have

$$\mathrm{Disc(P)} = -\frac{\mathrm{s}_0}{\mathrm{a}_3}.$$

Moreover, the coefficient a_3 is negative, which implies $s_0 > 0$.

On the other hand, according to Proposition 5, we have $\mathbf{Var}(a_3, a_2, a_1, a_0) = \mathbf{Var}(-, a_2, a_1, -) = 2$. There are only five possibilities (out of nine) for the signs of a_2 and a_1 compatible with having two sign variations: $a_2 > 0, a_1 > 0$ or $a_2 < 0, a_1 > 0$ or $a_2 = 0, a_1 > 0$ or $a_2 > 0, a_1 < 0$ or $a_2 > 0, a_1 = 0$. Since the first three possibilities are equivalent to $a_1 > 0$, and the last two ones to $a_1 \leq 0, a_2 > 0$, we conclude that the only possibilities are $s_0 > 0, a_1 > 0$ or $s_0 > 0, a_1 \leq 0, a_2 > 0$. Since $a_1 > 0$ or $a_2 > 0$ is the same set as $a_1 > 0$ or $a_1 < 0, a_2 > 0$ or $a_1 = 0, a_2 > 0$ (the complementary of $a_1 \leq 0$ and $a_2 \leq 0$ in both cases), we get the desired conclusion. □

Next lemma shows the formulae connecting subresultants, roots and coefficients of a cubic polynomial with a real and double root. These equalities will be the main tools for allowing us to characterise when two ellipses are externally tangent in terms of the coefficients of their characteristic equation.

Lemma 10.
Let β and γ be real numbers, $\beta \neq \gamma$, and

$$P(T) = a_3T^3 + a_2T^2 + a_1T + a_0 = a_3(T - \beta)^2(T - \gamma).$$

Then (see Equation (2)):

$$s_0 = 0, \quad s_1 = 2a_3^3(\gamma - \beta)^2, \quad s_{1,0} = -2a_3^3\beta(\gamma - \beta)^2, \quad \beta = -\frac{s_{1,0}}{s_1}, \quad \gamma = -\frac{a_0 s_1^2}{a_3 s_{1,0}^2}\,.$$

Proof. Equalities for s_0, s_1 and $s_{1,0}$ follow directly from Definition 1 applied to our concrete polynomial $P(T)$. Since $\gcd(P, P') = T - \beta$, using the equivalence in (1), we have

$$\gcd(P, P') = \mathbf{Sres}_1(P) = s_1T + s_{1,0}\,,$$

and the corresponding equality for β (since it is a root of $\mathbf{Sres}_1(P)$). Since

$$T - \gamma = \frac{P(T)}{a_3(T - \beta)^2}\,,$$

we get (making $T = 0$),

$$\gamma = -\frac{a_0}{a_3\beta^2}\,,$$

and the corresponding equality for γ (replacing β by its value in terms of s_1 and $s_{1,0}$). □

Remark 11.
If $P(T) = a_3T^3 + a_2T^2 + a_1T + a_0$ then

- $s_0 = a_3\,(27a_0^2a_3^2 - 18a_0a_1a_2a_3 + 4a_0a_2^3 + 4a_1^3a_3 - a_1^2a_2^2)$,
- $s_1 = 2a_3\,(a_2^2 - 3a_1a_3)$ and
- $s_{1,0} = a_3\,(a_1a_2 - 9a_0a_3)$.

In what follows, a_3 will refer to the determinant of a matrix defining an ellipse and it will be always negative. Thus, in the formulae to be introduced in the next subsections where the signs of s_0, s_1 and $s_{1,0}$ are required, we can replace a_3 by -1.

3.1 Characterising two separated ellipses

The best solution for identifying the relative position of two coplanar ellipses can be found in [1]. There, in Figures 1 and 4, the authors describe all the positional relationships between two ellipses and they present a binary decision tree to identify every position appearing in Figure 1 (in terms of the coefficients of the considered ellipses).

In this section we are interested in characterizing when two ellipses are separate (Type 3 in [1, Section 6, Figures 1 and 4]). If $P(T) = a_3T^3 + a_2T^2 + a_1T + a_0$ is the characteristic equation of the two considered ellipses, they prove that they are separate if and only if $s_0 > 0, a_1 > 0$ or $s_0 > 0, a_2 > 0$. Next theorem introduces a new proof of this characterisation mainly based on the Descartes' law of signs applied to $P(T)$. Recall that $a_3 < 0$ and $a_0 < 0$.

Theorem 12. *(see Type 3 in [1, Figures 1 and 4])*
Ellipses $\mathcal{A}$ and $\mathcal{B}$ are separate if and only if $s_0 > 0, a_1 > 0$ or $s_0 > 0, a_2 > 0$.

Proof. If the ellipses $\mathcal{A}$ and $\mathcal{B}$ are separate then, according to Theorem 7 (2 and 3), their characteristic equation has the structure required by Lemma 9, and we can conclude that $s_0 > 0, a_1 > 0$ or $s_0 > 0, a_2 > 0$.

Now we assume $s_0 > 0, a_1 > 0$ or $s_0 > 0, a_2 > 0$. Lemma 8 and $s_0 > 0$ implies that the characteristic equation of $\mathcal{A}$ and $\mathcal{B}$, $P(T) = a_3T^3 + a_2T^2 + a_1T + a_0$, has three different real roots being, at least, one of them negative (Theorem 7 (2)). This implies that Descartes' law of signs applied to $P(T)$ is exact when counting the number of positive real roots (Proposition 5). Since $a_1 > 0$ or $a_2 > 0$ (the complementary of $a_1 \leq 0$ and $a_2 \leq 0$) we have only the following five possibilities:

- $a_1 > 0$ and $a_2 > 0$: $\mathbf{Var}(a_3, a_2, a_1, a_0) = \mathbf{Var}(-, +, +, -) = 2$,
- $a_1 > 0$ and $a_2 < 0$: $\mathbf{Var}(a_3, a_2, a_1, a_0) = \mathbf{Var}(-, -, +, -) = 2$,
- $a_1 > 0$ and $a_2 = 0$: $\mathbf{Var}(a_3, a_2, a_1, a_0) = \mathbf{Var}(-, 0, +, -) = 2$,
- $a_1 < 0$ and $a_2 > 0$: $\mathbf{Var}(a_3, a_2, a_1, a_0) = \mathbf{Var}(-, +, -, -) = 2$,
- $a_1 = 0$ and $a_2 > 0$: $\mathbf{Var}(a_3, a_2, a_1, a_0) = \mathbf{Var}(-, +, 0, -) = 2$,

and so we can conclude that $P(T)$ has two different positive real roots. According to Theorem 7 (3), the ellipses $\mathcal{A}$ and $\mathcal{B}$ are separate. □

As a consequence of the proof of the previous theorem, we obtain immediately the following corollary.

Corollary 13.
Ellipses $\mathcal{A}$ and $\mathcal{B}$ are separate if and only if $\mathbf{Var}(-, a_2, a_1, -) = 2$ and $s_0 > 0$.

3.2 Characterising two externally tangent ellipses

Next theorem shows that we can characterise when two ellipses are externally tangent in terms of the signs of s_0 and $s_{1,0}$.

Theorem 14.
Ellipses $\mathcal{A}$ and $\mathcal{B}$ are externally tangent if and only if $s_0 = 0$ and $s_{1,0} > 0$.

Proof. If the ellipses touch each other externally then, according to Theorem 7, we have that $P(T)$ has one positive double real root β and one negative real root γ,

$$P(T) = a_3(T-\beta)^2(T-\gamma).$$

Using the notation introduced in Equation (2), Lemma 10 implies $s_0 = 0$, $s_1 < 0$ and $s_{1,0} > 0$, as desired, since $a_3 < 0$. In particular, we have proven that $s_0 = 0$ and $s_{1,0} > 0$.

If $s_0 = 0$ and $s_{1,0} > 0$, then $s_1 \neq 0$: if $s_1 = 0$ then $\gcd(P, P') = P'$, P' divides P, and this implies $P(T) = a_3(T-\beta)^3$, $\mathbf{Sres}_1(P) \equiv 0$ and $s_{1,0} = 0$. Hence $s_0 = 0$, $s_1 \neq 0$ and $\mathbf{Sres}_1(P) = \gcd(P, P')$, and $P(T) = a_3(T-\beta)^2(T-\gamma)$ is the unique possible factorisation for $P(T)$, with $\beta \neq \gamma$ real numbers. Lemma 10 implies $s_1 = 2a_3^3(\gamma-\beta)^2 < 0$, and since $s_{1,0} > 0$ and Lemma 10,

$$\beta = -\frac{s_{1,0}}{s_1},$$

we conclude $\beta > 0$. The equality

$$\gamma = -\frac{a_0}{a_3\beta^2}$$

from Lemma 10 implies $\gamma < 0$, since $a_3 < 0$ and $a_0 < 0$. This proves that $P(T)$ has a positive double real root β and a negative real root γ and, according to Theorem 7, the ellipses $\mathcal{A}$ and $\mathcal{B}$ are externally tangent, as desired. □

Since $a_3 < 0$, previous condition characterising when two ellipses are externally tangent reduces to

$$27a_0^2a_3^2 - 18a_0a_1a_2a_3 + 4a_0a_2^3 + 4a_1^3a_3 - a_1^2a_2^2 = 0, \quad 9a_0a_3 - a_1a_2 > 0.$$

Next corollary shows how the same strategy used before, together with Descartes' law of signs, produce the characterisation for externally tangent ellipses introduced in [1].

Corollary 15. *(see Type 7 in [1, Figures 1 and 4])*
Ellipses $\mathcal{A}$ and $\mathcal{B}$ are externally tangent if and only if $s_0 = 0$ and $a_1 > 0$ or $s_0 = 0$ and $a_2 > 0$.

Proof. If the ellipses touch each other externally then, according to Theorem 7, we have that $P(T)$ has one positive double real root β and one negative real root γ:

$$P(T) = a_3(T-\beta)^2(T-\gamma).$$

Lemma 10 implies $s_0 = 0$, and Descartes' law of signs (Proposition 5) implies $\mathbf{Var}(a_3, a_2, a_1, a_0) = 2$. Since $a_3 < 0$ and $a_0 < 0$, the unique possibilities are $a_1 > 0$ or $a_2 > 0$, as desired.

If $s_0 = 0$ then $P(T)$ has a multiple root and $P(T) = a_3(T-\beta)^3$ or $P(T) = a_3(T-\beta)^2(T-\gamma)$, with β and γ real numbers, and $\beta \neq \gamma$ are the only two possibilities for the factorisation of $P(T)$. Since $a_1 > 0$ or $a_2 > 0$, together with $a_3 < 0$ and $a_0 < 0$, we have $\mathbf{Var}(a_3, a_2, a_1, a_0) = 2$. Descartes' law of signs (Proposition 5) implies that $P(T)$ has two positive real roots counted with multiplicity. This excludes the first possibility and implies $\beta > 0$ (note that $\gamma < 0$ since $P(T)$, being the characteristic equation of two ellipses, has always one negative real root, (Theorem 7 (2)). This implies, by using Theorem 7, that $\mathcal{A}$ and $\mathcal{B}$ are externally tangent. □

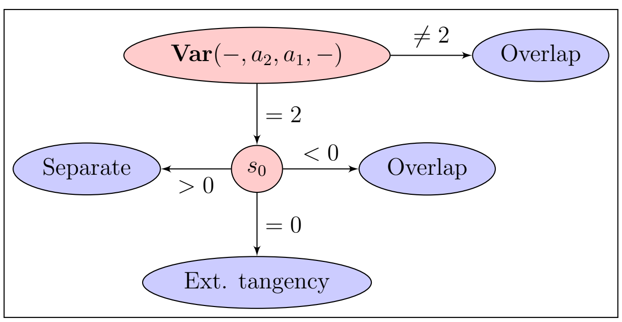
Observe that previous conditions characterising when two ellipses are externally tangent reduce to

$$27a_0^2a_3^2 - 18a_0a_1a_2a_3 + 4a_0a_2^3 + 4a_1^3a_3 - a_1^2a_2^2 = 0 \quad \text{and} \quad \big(a_1 > 0 \quad \text{or} \quad a_2 > 0\big).$$

Corollary 16.
Ellipses $\mathcal{A}$ and $\mathcal{B}$ are externally tangent if and only if $s_0 = 0$ and $\mathbf{Var}(-, a_2, a_1, -) = 2$.

3.3 Solving the interference problem for two ellipses

Next diagram summarizes the approach described in this section for deciding the relative position of two given ellipses:



The way we propose to decide the relative position of two given ellipses computes the value of s_0 only when it is really needed. This diagram can be specially useful when dealing with two ellipses continuously depending on one parameter t, the interval where t lives is discretised and it is required to decide the relative position of the two considered ellipses at every t value. A different way of approaching this problem is introduced in the next subsection.

3.4 Interference analysis for two ellipses continuously depending on one parameter

Results in this section allow to deal with the interference problem with two continuously changing ellipses. Let $\mathcal{A}(t)$ and $\mathcal{B}(t)$ be two ellipses depending continuously on t in a non–empty interval $I \subset \mathbb{R}$. Determining when they overlap or when they are separate requires only to compute when they are externally tangent since they depend on t continuously: if $t_0 < t_1$, $\mathcal{A}(t_0)$ and $\mathcal{B}(t_0)$ overlap (resp. are separate) and $\mathcal{A}(t_1)$ and $\mathcal{B}(t_1)$ are separate (resp. overlap) then there exists t' in (t_0, t_1) such that $\mathcal{A}(t')$ and $\mathcal{B}(t')$ are externally tangent. According to

Theorem 14, determining the points where they are externally tangent is the same as solving the system

$$s_0(t) = 0, \quad s_{1,0}(t) > 0,$$

or the system

$$s_0(t) = 0, \quad \mathbf{Var}(-, a_2(t), a_1(t), -) = 2,$$

for t in I .

If the entries of the matrices providing $\mathcal{A}(t)$ (resp. $\mathcal{B}(t)$) are rational functions of t then the ellipse and its interior definition do not change if we multiply them by plus or minus times the common denominator of their entries that has constant sign in I: observe that, since $\mathcal{A}(t)$ is always an ellipse for every $t \in I$, no denominator in the entries of $\mathcal{A}(t)$ can vanish in I so, by Bolzano's Theorem, the sign of the denominators is constant throughout I. The sign is chosen according to the sign of the common denominator on I: plus if it is positive, minus otherwise. Therefore, we can assume that the entries of the matrices providing $\mathcal{A}(t)$ and $\mathcal{B}(t)$ are polynomials in t. Then, $s_0(t)$ and $s_{1,0}(t)$, as subresultants of the characteristic equation

$$P(t,T) = a_3(t)T^3 + a_2(t)T^2 + a_1(t)T + a_0(t) \ ,$$

are also polynomials and the only possibilities arising are the following:

1. If the polynomial $s_0(t)$ is not identically zero then we compute the real roots of $s_0(t)$ in I such that $s_{1,0}(t) > 0$ (or such that $a_1(t) > 0$ or $a_2(t) > 0$): $\alpha_1 < \ldots < \alpha_m$. These t-values show where $\mathcal{A}(t)$ and $\mathcal{B}(t)$ are externally tangent, and we need only to decide what happens in each interval (α_i, α_{i+1}) (with $\alpha_0 = \inf(I)$ and $\alpha_{m+1} = \sup(I)$ being possibly infinite). This is solved by taking any value t_i in each interval and using Theorem 12: if $s_0(t_i) > 0$ and $a_1(t_i) > 0$ or $a_2(t_i) > 0$ then they are separate on (α_i, α_{i+1}), otherwise they overlap. If $s_0(t)$ has no real roots in I then the ellipses overlap or are separate for all t in I: in order to determine their relative position we evaluate $a_2(t)$, $a_1(t)$ and $s_0(t)$ at any $\alpha \in I$ (if $\mathbf{Var}(-, a_2(\alpha), a_1(\alpha), -) = 2$ and $s_0(\alpha) > 0$ then the ellipses are separate, otherwise, they overlap).

2. If the polynomial $s_0(t)$ is identically zero then $\mathcal{A}(t)$ and $\mathcal{B}(t)$ are externally tangent or they overlap on the whole interval I. Take any $t' \in I$: if $s_{1,0}(t') > 0$ then they are externally tangent for every $t \in I$; otherwise, they overlap for every $t \in I$.

While the first case is self explanatory, the second one requires to consider several cases in order to conclude that the sign of $s_{1,0}(t)$ evaluated at any $t' \in I$ characterises the relative position of $\mathcal{A}(t)$ and $\mathcal{B}(t)$.

First case: $s_0(t) \equiv 0$ and $s_1(t) \equiv 0$.
Observe that, when $s_0(t)$ and $s_1(t)$ are identically zero, $\mathcal{A}(t)$ and $\mathcal{B}(t)$ overlap for all $t \in I$: in this case, $P(t,T)$ has a triple negative root for every $t \in I$.

Second case: $s_0(t) \equiv 0$ and $s_1(t) \not\equiv 0$.
Now we consider the case where $s_0(t)$ vanishes identically but not $s_1(t)$. First we need the following lemma (under these conditions).

Lemma 17.
Polynomials $s_1(t)$ and $s_{1,0}(t)$ have exactly the same roots in I.

Proof. If $t_0 \in I$ is a root of $s_{1,0}(t)$ and $s_1(t_0) \neq 0$ then $s_1(t_0)T = \gcd(P(t_0,T), P'(t_0,T))$. This implies that $T = 0$ is a root of $P(t_0,T)$ and $a_0(t_0) = 0$, which is not possible since $a_0(t_0) < 0$. Thus $s_1(t_0) = 0$.

Let t_0 be a root of $s_1(t)$. If $s_{1,0}(t_0) \neq 0$ then $\mathbf{Sres}_1(P(t_0,T)) = s_{1,0}(t_0)$ is in the ideal generated by $P(t_0,T)$ and $P'(t_0,T)$, and so $\gcd(P(t_0,T), P'(t_0,T)) = 1$ and $s_0(t_0) \neq 0$, which is not possible since $s_0(t)$ is identically zero. Thus $s_{1,0}(t_0) = 0$. □

The fact that $s_0(t)$ is identically zero implies, when $s_1(t') \neq 0$ ($t' \in I$), that

$$P(t',T) = a_3(t')T^3 + a_2(t')T^2 + a_1(t')T + a_0(t') = a_3(t')(T-\beta(t'))^2(T-\gamma(t'))$$

with $\beta(t)$ and $\gamma(t)$ the rational functions (see Lemma 10)

$$\beta(t) = -\frac{s_{1,0}(t)}{s_1(t)}, \quad \gamma(t) = -\frac{a_0(t)s_1(t)^2}{a_3(t)s_{1,0}(t)^2} = -\frac{a_0(t)}{a_1(t)}\frac{1}{\beta^2(t)}$$

and $\gamma(t') < 0$.

Second case. First possibility: $[s_0(t) \equiv 0$ and $s_1(t) \not\equiv 0]$ $s_1(t)$ has no real roots.
If $s_1(t)$ has no real roots then $\beta(t)$ has no (real) poles and no (real) roots and constant sign on I. According to Lemma 10, we have for any $t' \in I$

$$s_{1,0}(t') = -2a_3(t')^3\beta(t')(\gamma(t') - \beta(t'))^2$$

and $\operatorname{sign}(s_{1,0}(t')) = \operatorname{sign}(\beta(t'))$. We conclude in this case that, when $s_{1,0}(t') > 0$, $\mathcal{A}(t)$ and $\mathcal{B}(t)$ are externally tangent over I and, when $s_{1,0}(t') < 0$, $\mathcal{A}(t)$ and $\mathcal{B}(t)$ overlap (note that $s_{1,0}(t)$ has no real roots as a consequence of Lemma 17).

Second case. Second possibility: $[s_0(t) \equiv 0$ and $s_1(t) \not\equiv 0]$ $s_1(t)$ has real roots.
Let $t_0 \in I$ be a real root of $s_1(t)$ (and a real root of $s_{1,0}(t)$ according to Lemma 17). Then there exists $\alpha < 0$ such that

$$P(t_0,T) = a_3(t_0)T^3 + a_2(t_0)T^2 + a_1(t_0)T + a_0(t_0) = a_3(t_0)(T-\alpha)^3 \ .$$

Being $P(t,T)$ a polynomial we have

$$\lim_{t\to t_0} P(t,T) = P(t_0,T) = a_3(t_0)(T-\alpha)^3$$

and

$$\begin{aligned}\lim_{t\to t_0}(T-\beta(t))^2(T-\gamma(t)) &= \lim_{t\to t_0}\left(T^3 - (2\beta(t)+\gamma(t))T^2 + \beta(t)(\beta(t)+2\gamma(t))T - \beta(t)^2\gamma(t)\right) = \\ &= (T-\alpha)^3 = T^3 - 3\alpha T^2 + 3\alpha^2 T - \alpha^3 \ .\end{aligned}$$

Equating coefficients allows us to write

$$\lim_{t\to t_0}(2\beta(t)+\gamma(t)) = \lim_{t\to t_0}\left(2\beta(t)-\frac{a_0(t)}{a_1(t)}\frac{1}{\beta^2(t)}\right) = 3\alpha$$

and

$$\lim_{t\to t_0}\beta(t)\,(\beta(t)+2\gamma(t)) = \lim_{t\to t_0}\left(\beta(t)^2-2\frac{a_0(t)}{a_1(t)}\frac{1}{\beta(t)}\right) = 3\alpha^2\ .$$

This implies automatically that the limit of the rational function $\beta(t)$ when t goes to t_0 is finite and different from 0. Denoting $L = \lim_{t\to t_0}\beta(t) \neq 0$ and using that $a_0(t_0)/a_1(t_0) = -\alpha^3$, we have then the equalities

$$2L - \frac{a_0(t_0)}{a_1(t_0)}\frac{1}{L^2} = 2L + \frac{\alpha^3}{L^2} = 3\alpha, \qquad L^2 - 2\frac{a_0(t_0)}{a_1(t_0)}\frac{1}{L} = L^2 + \frac{2\alpha^3}{L} = 3\alpha^2\ .$$

Clearing denominators, we have that L must satisfy simultaneously

$$2L^3 - 3\alpha L^2 + \alpha^3 = (2L+\alpha)(L-\alpha)^2 = 0, \qquad L^3 - 3\alpha^2 L + 2\alpha^3 = (L+2\alpha)(L-\alpha)^2 = 0$$

allowing us to conclude that

$$L = \lim_{t\to t_0}\beta(t) = \lim_{t\to t_0}\gamma(t) = \alpha < 0.$$

If there exists t' such that $\beta(t') > 0$ then there exist two consecutive real roots $t_1 < t_2$ of $s_1(t)$ such that $t_1 < t' < t_2$ ($t_1 = -\infty$ or $t_2 = +\infty$ if needed when t' is smaller or bigger, respectively, than all real roots of $s_1(t)$). But $\beta(t) = -s_{1,0}(t)/s_1(t)$ is continuous in $[t_1, t_2]$ and $\beta(t_1) < 0$ and $\beta(t_2) < 0$: therefore the exists $\tau \in (t_1, t_2)$ such that $s_{1,0}(\tau) = 0$. And this can not happen since, after Lemma 17, $s_1(\tau) = 0$ and in (t_1, t_2) there is no real roots of $s_1(t)$. This implies that in this possibility $\mathcal{A}(t)$ and $\mathcal{B}(t)$ overlap over I.

Summarizing: we can conclude analyzing all these cases and possibilities that the sign of $s_{1,0}(t)$ evaluated at any $t' \in I$ characterizes the relative position of $\mathcal{A}(t)$ and $\mathcal{B}(t)$:

- if $s_{1,0}(t') > 0$ then $\mathcal{A}(t)$ and $\mathcal{B}(t)$ are externally tangent for every $t \in I$.
- if $s_{1,0}(t') \le 0$ then $\mathcal{A}(t)$ and $\mathcal{B}(t)$ overlap for every $t \in I$.

4 On the ellipsoids interference problem

The equation of a quadric $\mathcal{A}$ in $\mathbb{R}^3$ can be written as

$$a_{11}X^2 + a_{22}Y^2 + a_{33}Z^2 + 2a_{12}XY + 2a_{13}XZ + 2a_{23}YZ + 2a_{14}X + 2a_{24}Y + 2a_{34}Z + a_{44} = 0$$

or in matricial form $\mathcal{A} : \mathcal{X}A\mathcal{X}^t = 0$, where $\mathcal{X} = (X,\, Y,\, Z,\, 1)$, and A is the symmetric matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{pmatrix}.$$

In what follows we assume that the matrix A is presented in such a way that the interior of the ellipsoid $\mathcal{A}$ is defined by $\mathcal{X}^T A \mathcal{X} < 0$. If the quadric is an ellipsoid (i.e., a non-degenerate quadric with real affine points and no real points at infinity) then $\det(A) < 0$.

Definition 18.
Given two quadrics $\mathcal{A} : \mathcal{X} A \mathcal{X}^t = 0$ and $\mathcal{B} : \mathcal{X} B \mathcal{X}^t = 0$, their characteristic equation is defined as

$$P(T) = \det(TA + B) = \det(A)T^4 + \ldots + \det(B)$$

which is, if $\det(A) \neq 0$, a quartic polynomial in T with real coefficients.

Next theorem was introduced in [28] and connects some relative positions (separation, external tangency and overlapping) of two ellipsoids with the sign of the real roots of their characteristic equation. This is the starting point for the new approach we introduce in this section when characterising the separation or the external tangency of two ellipsoids.

Theorem 19.
Let $\mathcal{A}$ and $\mathcal{B}$ be two real ellipsoids with the characteristic equation $P(T)$. Then:

1. *The leading coefficient and the constant term of $P(T)$ are strictly negative.*
2. *The characteristic equation $P(T)$ has at least two negative real roots.*
3. *$\mathcal{A}$ and $\mathcal{B}$ are separated by a plane if and only if $P(T)$ has two positive different real roots.*
4. *$\mathcal{A}$ and $\mathcal{B}$ touch each other externally if and only if $P(T)$ has a positive double root.*

The following lemmas show the formulae relating subresultants, roots and coefficients of a degree 4 polynomial with a real and multiple root. These equalities will be the main tools for allowing us to characterise when two ellipsoids are separate or externally tangent in terms of the coefficients of their characteristic equation.

Lemma 20.
Let β and γ be real numbers, $\beta \neq \gamma$, and

$$P(T) = a_4T^4 + a_3T^3 + a_2T^2 + a_1T + a_0 = a_4(T - \beta)^3(T - \gamma).$$

Then (see Equation (2)):

$$s_0 = s_1 = 0, \quad s_2 = -3a_4^3(\gamma - \beta)^2, \quad s_{2,1}^2 - 4s_2s_{2,0} = 0, \quad \beta = -\frac{s_{2,1}}{2s_2}, \quad \gamma = \frac{a_0}{a_4\beta^3}.$$

Proof. Since $P(T)$ has a triple root, s_0 and s_1 are equal to zero. Moreover, the polynomial $\mathbf{Sres}_2(P)$ is a multiple of $(T - \beta)^2$; actually, according to Definition 1, it is equal to

$$\mathbf{Sres}_2(P) = -3a_4^3(\gamma - \beta)^2(T - \beta)^2.$$

Thus, $s_2 = -3a_4^3(\gamma - \beta)^2$, the discriminant of $\mathbf{Sres}_2(P)$ must vanish, that is, $s_{2,1}^2 - 4s_2s_{2,0} = 0$, and the roots formulae can be deduced easily. □

Lemma 21.
Let β and γ be real numbers, $\beta \neq \gamma$, and

$$P(T) = a_4T^4 + a_3T^3 + a_2T^2 + a_1T + a_0 = a_4(T-\beta)^2(T-\gamma)^2.$$

Then (see Equation (2)):

$$s_0 = s_1 = 0, \quad s_2 = -4a_4^3(\gamma-\beta)^2, \quad s_{2,1}^2 - 4s_2s_{2,0} > 0, \quad \beta, \gamma = \frac{-s_{2,1} \pm \sqrt{s_{2,1}^2 - 4s_2s_{2,0}}}{2s_2}.$$

Proof. In this case, since $P(T)$ has two different double roots, s_0 and s_1 are equal to zero. Moreover, the polynomial $\mathbf{Sres}_2(P)$ is a multiple of $(T-\beta)(T-\gamma)$,

$$\mathbf{Sres}_2(P) = -4a_4^3(\gamma-\beta)^2(T-\beta)(T-\gamma).$$

Thus, $s_2 = -4a_4^3(\gamma-\beta)^2$, the discriminant of $\mathbf{Sres}_2(P)$ must be positive, $s_{2,1}^2 - 4s_2s_{2,0} > 0$, and the roots formulae are obtained easily. □

Lemma 22.
Let $\beta \neq 0$ be a real number, γ_1 and γ_2 be two different numbers (if they are real numbers, different from β and different from 0) and

$$P(T) = a_4T^4 + a_3T^3 + a_2T^2 + a_1T + a_0 = a_4(T-\beta)^2(T-\gamma_1)(T-\gamma_2).$$

Then (see Equation (2)):

$$s_0 = 0, \quad s_1 = 2a_4^5(\gamma_1-\gamma_2)^2(\beta-\gamma_2)^2(\beta-\gamma_1)^2, \quad \beta = -\frac{s_{1,0}}{s_1}, \quad (T-\gamma_1)(T-\gamma_2) = \frac{P(T)}{a_4(T-\beta)^2}.$$

Proof. Since $P(T)$ has only one double root, s_0 is equal to zero but not s_1. Formulae for β, γ_1 and γ_2 are a consequence of equality (1). Using Definition 1 we get

$$\mathbf{Sres}_1(P) = 2a_4^5(\gamma_1-\gamma_2)^2(\beta-\gamma_2)^2(\beta-\gamma_1)^2(T-\beta)$$

and $s_1 = 2a_4^5(\gamma_1-\gamma_2)^2(\beta-\gamma_2)^2(\beta-\gamma_1)^2$. □

We will use freely in Subsection 4.4 that, when dealing with the factorization of $P(T)$ in the previous lemma, the sign of s_1 will separate the case of γ_1 and γ_2 being different real numbers from the case of γ_1 and γ_2 being not real complex numbers and conjugate. Moreover, when γ_1 and γ_2 are real numbers, the sign of $\gamma_1\gamma_2$ agrees with the sign of a_0a_4.

Remark 23.
Given $P(T) = a_4T^4 + a_3T^3 + a_2T^2 + a_1T + a_0$, we have

$$\begin{aligned}
s_0 &= a_4(256a_0^3a_4^3 - 192a_0^2a_1a_3a_4^2 - 128a_0^2a_2^2a_4^2 + 144a_0^2a_2a_3^2a_4 - 27a_0^2a_3^4 + 144a_0a_1^2a_2a_4^2 \\
&\quad -6a_0a_1^2a_3^2a_4 - 80a_0a_1a_2^2a_3a_4 + 18a_0a_1a_2a_3^3 + 16a_0a_2^4a_4 - 4a_0a_2^3a_3^2 - 27a_1^4a_4^2 \\
&\quad +18a_1^3a_2a_3a_4 - 4a_1^3a_3^3 - 4a_1^2a_2^3a_4 + a_1^2a_2^2a_3^2), \\
s_1 &= 2a_4\left(16a_0a_2a_4^2 - 6a_0a_3^2a_4 - 18a_1^2a_4^2 + 14a_1a_2a_3a_4 - 3a_1a_3^3 - 4a_2^3a_4 + a_2^2a_3^2\right) \text{ and} \\
s_{1,0} &= -a_4\left(48a_0a_1a_4^2 - 32a_0a_2a_3a_4 + 9a_0a_3^3 - 3a_1^2a_3a_4 + 4a_1a_2^2a_4 - a_1a_2a_3^2\right).
\end{aligned}$$

In what follows a_4 and a_0 will refer to the determinants of the matrices defining the two considered ellipsoids and they will be always negative. Thus, in the formulae to be introduced in the next subsections where the signs of s_0, s_1 and $s_{1,0}$ are required, we can replace, when possible, both a_4 and a_0 by -1.

4.1 Characterising two externally tangent ellipsoids

Next theorem characterises when two ellipsoids are externally tangent in terms of the signs of only three subresultants coefficients s_0, s_1 and $s_{1,0}$ (instead of five as in [22]).

Theorem 24.
Ellipsoids $\mathcal{A}$ and $\mathcal{B}$ are externally tangent if and only if one of the two following conditions is verified

1. $s_0 = 0$, $s_1 = 0$ *and* $\mathbf{Var}(a_4, a_3, a_2, a_1, a_0) = 2$.
2. $s_0 = 0$, $s_1 < 0$ *and* $s_{1,0} > 0$.

Proof. If the ellipsoids touch each other externally then, according to Theorem 19, $P(T)$ has one positive double real root β and two negative real roots (equal or different). This implies, for $P(T)$, two possible factorisations:

$$P(T) = a_4(T - \beta)^2(T - \gamma)^2, \qquad P(T) = a_4(T - \beta)^2(T - \gamma_1)(T - \gamma_2),$$

with $\gamma < 0$ in the first case, and $\gamma_i < 0$ and $\gamma_1 \neq \gamma_2$ in the second one.

In the first possibility, Lemma 21 implies $s_0 = s_1 = 0$ and Descartes' law of signs (Proposition 5) implies $\mathbf{Var}(a_4, a_3, a_2, a_1, a_0) = 2$, which corresponds to the first case. In the second possibility, Lemma 22 implies $s_0 = 0$ and $s_1 < 0$ since $a_4 < 0$. Formula relating β, s_1 and $s_{1,0}$ in Lemma 22 implies $s_{1,0} > 0$.

We now prove the other implication. If $s_0 = s_1 = 0$ then $\gcd(P, P')$ has degree two or three. If $\deg(\gcd(P, P')) = 3$ (i.e., $s_2 = 0$), then P' divides P, and $P(T) = a_4(T - \gamma)^4$ with $\gamma \in \mathbb{R}$. According to Theorem 19, γ must be negative and by Descartes' law of signs (Proposition 5), we should have $\mathbf{Var}(a_4, a_3, a_2, a_1, a_0) = 0$. This leads to a contradiction if we assume initially $\mathbf{Var}(a_4, a_3, a_2, a_1, a_0) = 2$.

As a consequence of the previous discussion we have $s_2 \neq 0$ and $\deg(\gcd(P, P')) = 2$. There are only two possibilities under these conditions: either $P(T) = a_4(T - \beta)^3(T - \gamma)$ or $P(T) = a_4(T - \beta)^2(T - \gamma)^2$, with β and γ different real numbers since, by Theorem 19, $P(T)$ has, at least, two negative real roots counted with multiplicity. But we have $\mathbf{Var}(a_4, a_3, a_2, a_1, a_0) = 2$ and this happens only if P has a double negative root and a double positive root, according to Descartes' law of signs (Proposition 5). This implies, applying Theorem 19, that $\mathcal{A}$ and $\mathcal{B}$ are externally tangent as desired.

If $s_0 = 0$ and $s_1 < 0$ then $\deg(\gcd(P, P')) = 1$ and the unique multiple root of P is $\beta = -s_{1,0}/s_1$ (by Lemma 22). Since $s_{1,0} > 0$ we have that β is a positive double root of $P(T)$. This implies, according to Theorem 19, that $\mathcal{A}$ and $\mathcal{B}$ are externally tangent as desired. □

Note that the second condition in the previous theorem implies also $\mathbf{Var}(a_4, a_3, a_2, a_1, a_0) = 2$: this is again a consequence of Descartes' law of signs together with the fact that $-s_{1,0}/s_1$ is a positive double root of $P(T)$, and the other two roots of this polynomial must be real and negative (according to Theorem 19 (2)).

The first condition in the previous theorem can be further simplified as shown in the lemma and corollaries that follow.

Lemma 25.
Let $\mathcal{A}$ and $\mathcal{B}$ be two ellipsoids such that $s_0 = s_1 = 0$. The following three conditions are equivalent:

1. $\mathbf{Var}(a_4, a_3, a_2, a_1, a_0) = 2$.
2. $a_3a_1 < 0$ *or* $a_3 = 0, a_2 > 0, a_1 = 0$.
3. $a_3a_1 < 0$ *or* $a_2 > 0$.

Proof. If $s_0 = s_1 = 0$ and $\mathbf{Var}(a_4, a_3, a_2, a_1, a_0) = 2$, then there exist real numbers $\beta > 0$ and $\gamma < 0$ such that

$$\begin{aligned} P(T) &= a_4T^4 + a_3T^3 + a_2T^2 + a_1T + a_0 = a_4(T-\beta)^2(T-\gamma)^2 = \\ &= a_4T^4 - 2a_4(\beta+\gamma)T^3 + a_4((\beta+\gamma)^2 + 2\beta\gamma)T^2 - 2a_4\beta\gamma(\beta+\gamma)T + a_4\beta^2\gamma^2. \end{aligned}$$

This implies that

$$a_3 = 0 \iff a_1 = 0,$$

and under any of these conditions,

$$a_2 = 2a_4\beta\gamma > 0.$$

Moreover, if $a_3 \neq 0$ then

$$a_1a_3 = 4a_4^2\beta\gamma(\beta+\gamma)^2 < 0.$$

These two restrictions imply that the eighteen sign conditions for a_3, a_2 and a_1 such that $\mathbf{Var}(-, a_3, a_2, a_1, -) = 2$, namely

$$[+,+,+], [-,+,-], [-,+,+], [-,-,+], [+,-,-], [+,+,-], [+,0,-], [0,+,-], [+,-,0],$$

$$[-,0,+], [-,+,0], [0,-,+], [+,0,0], [0,+,0], [0,0,+], [+,0,+], [+,+,0], [0,+,+],$$

reduce to only seven: those marked in gray. Since

- $[-,+,+], [-,-,+], [-,0,+]$ is equivalent to $a_3 < 0, a_1 > 0$, and
- $[+,+,-], [+,-,-], [+,0,-]$ is equivalent to $a_3 > 0, a_1 < 0$

we get the desired sign conditions when $\mathbf{Var}(a_4, a_3, a_2, a_1, a_0) = 2$ (and $s_0 = s_1 = 0$). This shows that the equality in (1) implies the conditions in (2).

Conditions in (2) are included in the conditions in (3) and, taking into account all the possibilities when $a_3a_1 < 0$ or $a_2 > 0$, allows us to conclude that the conditions in (3) imply the equality in (1). ☐

Corollary 26.
Ellipsoids $\mathcal{A}$ and $\mathcal{B}$ are externally tangent if and only if one of the following conditions is verified

1. $a_3a_1 < 0, s_0 = 0, s_1 = 0$ *or*
2. $a_3 = 0, a_2 = +2\sqrt{a_0a_4}, a_1 = 0$ *or*

3. $s_0 = 0,\ s_1 < 0,\ s_{1,0} > 0$.

Proof. It is a direct consequence of Theorem 24, Lemma 25 (2) and, taking into account that, if $a_3 = 0$ and $a_1 = 0$ then

$$s_0 = 16a_0a_4^2(4a_0a_4 - a_2^2)^2, \qquad s_1 = 8a_2a_4^2(4a_0a_4 - a_2^2),$$

(see Remark 23). □

Corollary 27.
Ellipsoids $\mathcal{A}$ and $\mathcal{B}$ are externally tangent if and only if one of the following conditions is verified

1. $a_3a_1 < 0, s_0 = 0, s_1 = 0$ *or*

2. $a_2 > 0, s_0 = 0, s_1 = 0$ *or*

3. $s_0 = 0,\ s_1 < 0,\ s_{1,0} > 0$.

Proof. It is a direct consequence of Theorem 24 and Lemma 25 (3). □

In [23], the authors characterise externally touching ellipsoids by computing the Jordan form of $A^{-1}B$. The two ellipsoids are externally tangent when such Jordan form is

$$\begin{pmatrix} \alpha & 1 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \gamma \end{pmatrix}, \tag{4}$$

with $\alpha < 0 < \beta \leq \gamma$. Since the eigenvalues of $A^{-1}B$ are the roots of $P(-T)$, one can see that case (1) (resp. (2)) in Theorem 24 corresponds to the case $\beta = \gamma$ (resp. $\beta < \gamma$) in (4).

4.2 Characterising two separated ellipsoids

Next theorem characterises when two ellipsoids are externally tangent in terms of the signs of only three subresultants coefficients s_0, s_1 and $s_{1,0}$ (instead of five as in [22]). By Theorem 19, recall that given two ellipsoids $\mathcal{A}$ and $\mathcal{B}$, they are separated by a plane if and only if $P(T)$ has two positive different real roots.

Theorem 28.
Ellipsoids $\mathcal{A}$ and $\mathcal{B}$ are separated if and only if $\mathbf{Var}(a_4, a_3, a_2, a_1, a_0) = 2$ *and one of the two following conditions is verified*

1. $s_0 = 0,\ s_1 < 0$ *and* $s_{1,0} < 0$.

2. $s_0 < 0$.

Proof. The ellipsoids are separated by a plane if and only if $P(T)$ has two different positive real roots β_1 and β_2, and two negative real roots (equal or different). This implies, for $P(T)$, two possible factorisations:

$$P(T) = a_4(T-\beta_1)(T-\beta_2)(T-\gamma)^2, \qquad P(T) = a_4(T-\beta_1)(T-\beta_2)(T-\gamma_1)(T-\gamma_2),$$

with $\gamma < 0$ in the first case and $\gamma_1 \neq \gamma_2$ ($\gamma_1 < 0, \gamma_2 < 0$) in the second one.

We will prove that the first factorisation is equivalent to $\mathbf{Var}(a_4, a_3, a_2, a_1, a_0) = 2$, $s_0 = 0$, $s_1 < 0$ and $s_{1,0} < 0$ and that the second factorisation is equivalent to $\mathbf{Var}(a_4, a_3, a_2, a_1, a_0) = 2$ and $s_0 < 0$.

If $P(T) = a_4(T-\beta_1)(T-\beta_2)(T-\gamma)^2$ then, by Lemma 22, $s_0 = 0$ and $s_1 < 0$ and $s_{1,0} < 0$ must be also negative because the root of $\mathbf{Sres}_1(P)$ is γ. Moreover, Proposition 5 implies $\mathbf{Var}(a_4, a_3, a_2, a_1, a_0) = 2$.

Suppose now that $s_0 = 0$, $s_1 < 0$ and $s_{1,0} < 0$, and $\mathbf{Var}(a_4, a_3, a_2, a_1, a_0) = 2$. On the one hand, the fact that $s_0 = 0$, $s_1 < 0$ and $s_{1,0} < 0$ implies that $P(T)$ has a double negative root by Lemma 22 and this implies

$$P(T) = a_4(T-\beta_1)(T-\beta_2)(T-\gamma)^2,$$

with $\gamma < 0$. By Lemma 22, $s_1 = 2a_4^5(\beta_1-\beta_2)^2(\gamma-\beta_2)^2(\gamma-\beta_1)^2$, and by hypothesis, $s_1 < 0$ and $a_4 < 0$, so that, all roots must be real (otherwise, β_1 and β_2 being conjugate would render a_4 and s_1 of different signs). Since $\mathbf{Var}(a_4, a_3, a_2, a_1, a_0) = 2$, we have two different positive roots, that is, $\beta_1 > 0$ and $\beta_2 > 0$.

If $P(T) = a_4(T-\beta_1)(T-\beta_2)(T-\gamma_1)(T-\gamma_2)$ then the discriminant is positive and s_0 is negative by Equation (3). Obviously $\mathbf{Var}(a_4, a_3, a_2, a_1, a_0) = 2$. On the other hand, suppose $\mathbf{Var}(a_4, a_3, a_2, a_1, a_0) = 2$ and $s_0 < 0$. Then there are not multiple roots, at least two of them are negative, and the discriminant is positive. Since the discriminant is positive, the other two roots must be real. Since $\mathbf{Var}(a_4, a_3, a_2, a_1, a_0) = 2$, Proposition 5 implies that these other two roots are positive. □

In [23], the authors characterize ellipsoids separated by a plane by computing the Jordan form of $A^{-1}B$. The two ellipsoids are separated when such Jordan form is

$$\begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix}, \tag{5}$$

with $\alpha < \beta < 0 < \gamma \leq \delta$. Since the eigenvalues of $A^{-1}B$ are the roots of $P(-T)$, one can see that case (1) (resp. (2)) in Theorem 28 corresponds to the case $\gamma = \delta$ (resp. $\beta < \gamma$) in (5).

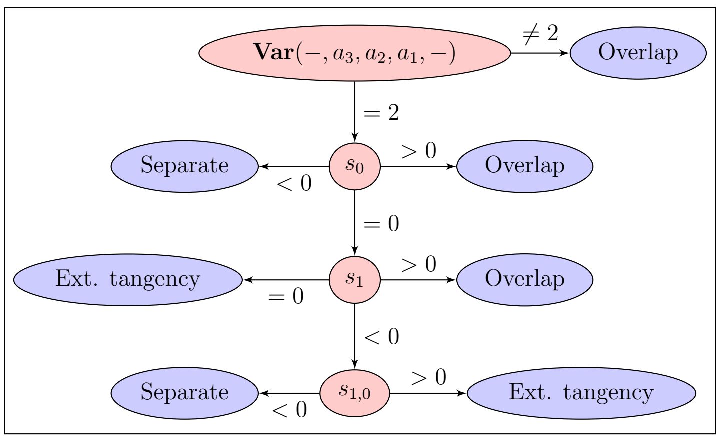
4.3 Solving the interference problem for two ellipsoids

Compared with the best solution dealing with this problem (see [22]), our approach presents, at least, two clear improvements:

1. Less polynomials are involved: we need to deal only with s_0, s_1 and $s_{1,0}$, while the solution in [22] requires to use two additional polynomials (s_2 and $s_{2,0}$).

2. The way the relative position is determined is simpler than in [22]. We start by checking if $\mathbf{Var}(-1, a_3, a_2, a_1, -1) \neq 2$ concluding overlapping. Otherwise, we compute s_0: if $s_0 < 0$ then the ellipsoids are separate, if $s_0 > 0$ then they overlap, and if $s_0 = 0$ then we compute s_1. When $s_0 = 0$, if $s_1 > 0$ then they overlap, if $s_1 = 0$ then they are externally tangent, and if $s_1 < 0$ then we compute $s_{1,0}$. When $s_0 = 0$ and $s_1 < 0$, the case $s_{1,0} = 0$ can not happen (if this is the case then 0 is a root $P(T)$ and $P(0) \neq 0$: see first item in Theorem 19) and if $s_{1,0} > 0$ then they are externally tangent, otherwise ($s_{1,0} < 0$) they are separate.

Next diagram summarises the approach here described:



The way we propose to decide the relative position of two given ellipsoids computes the values of s_0, s_1 and $s_{1,0}$ only when they are really needed. Moreover, as we will show next, our solution requires less multiplications and additions than the solution introduced in [22] since it uses less polynomials. According to Theorem 24 and Theorem 28 and once the characteristic polynomial is known, we only need to compute s_0, s_1 and $s_{1,0}$. Following [9] and [22] we proceed in the following way by denoting first:

$$\overline{a_4} = a_4,\ \overline{a_3} = -\frac{a_3}{4},\ \overline{a_2} = \frac{a_2}{6},\ \overline{a_1} = -\frac{a_1}{4},\ \overline{a_0} = a_0,$$

and determining:

$$d_2 = \overline{a_3}^2 - \overline{a_2}\ \overline{a_4}, \qquad w_1 = \overline{a_1}\ \overline{a_4} - \overline{a_3}\ \overline{a_2},$$

$$d_3 = \overline{a_2}^2 - \overline{a_3}\ \overline{a_1}, \qquad w_3 = \overline{a_0}\ \overline{a_4} - \overline{a_3}\ \overline{a_1},$$

$$\begin{aligned} &\star\ s_1 \to w_3\ d_2 + 3\ w_1^2 - 9\ d_2\ d_3,\\ &a = w_3 + 3\ d_3,\\ &b = \overline{a_1}\ w_1 + \overline{a_0}\ d_2 + \overline{a_2}\ d_3,\\ &\star\ s_{1,0} \to -(3\ b\ \overline{a_3} + a\ w_1),\\ &\star\ s_0 \to 27\ b^2 - a^3. \end{aligned}$$

The searched polynomials s_0, s_1 and $s_{1,0}$ are given up to a positive constant multiple. The above expressions require 18 multiplications and 11 additions (against the 28 multiplications and the 12 additions in [22]).

This approach can be specially useful when dealing with two ellipsoids continuously depending on one parameter t, the interval where t lives is discretised and it is required to decide the relative position of the two considered ellipsoids at every t value. A different way of approaching this problem is introduced in the next subsection.

4.4 Interference analysis for two ellipsoids continuously depending on one parameter

Results in this section allow to deal with the interference problem with two continuously changing ellipsoids. Let $\mathcal{A}(t)$ and $\mathcal{B}(t)$ be two ellipsoids depending continuously on t in a non–empty interval $I \subset \mathbb{R}$. Determining when they overlap or when they are separate requires only to compute when they are externally tangent, since they depend on t continuously: if $t_0 < t_1$, $\mathcal{A}(t_0)$ and $\mathcal{B}(t_0)$ overlap (resp. are separate) and $\mathcal{A}(t_1)$ and $\mathcal{B}(t_1)$ are separate (resp. overlap) then there exists t' in (t_0, t_1) such that $\mathcal{A}(t')$ and $\mathcal{B}(t')$ are externally tangent (see Theorem 5.1 in [22]). According to Theorem 24 this is the same as solving the two systems

$$s_0(t) = 0,\ s_1(t) = 0,\ \mathbf{Var}(-, a_3(t), a_2(t), a_1(t), -) = 2,$$

and

$$s_0(t) = 0,\ s_1(t) < 0,\ s_{1,0}(t) > 0,$$

for t in I.

As in the case of ellipses (see Subsection 3.4), if the entries of the matrices providing $\mathcal{A}(t)$ and $\mathcal{B}(t)$ are rational functions of t then we can assume that the entries of these matrices are polynomials in t. Then, $s_0(t)$, $s_1(t)$ and $s_{1,0}(t)$, as subresultants of the characteristic equation

$$P(t, T) = a_4(t)T^4 + a_3(t)T^3 + a_2(t)T^2 + a_1(t)T + a_0(t)\ ,$$

are also polynomials.

The interference analysis next described depends on the nature of the polynomials $s_0(t)$ and $s_1(t)$: if they are identically zero or not.

4.4.1 $s_0(t) \not\equiv 0$ (compare with Theorem 5.2 in [22])

If $s_0(t)$ has no real roots in I then its sign on I is constant. Taking any value $t' \in I$:

- If $s_0(t') > 0$ then $\mathcal{A}(t')$ and $\mathcal{B}(t')$ overlap since, according to the Example 5.4 in [17], $P(t', T)$ has has two real roots and two conjugate complex roots: these two different real roots must be negative (see Theorem 19) and the considered ellipsoids overlap on I.

- If $s_0(t') < 0$ and $\mathbf{Var}(-, a_3(t'), a_2(t'), a_1(t'), -) = 2$ then, according to the Example 5.4 in [17], $P(t', T)$ has four different real roots, two of them negative and two of them positive: $\mathcal{A}(t')$ and $\mathcal{B}(t')$ are separate. If, for any $t_0 \in I$, $\mathcal{A}(t_0)$ and $\mathcal{B}(t_0)$ overlap or they are externally tangent then there exists $t_1 \in I$ such that $s_0(t_1) = 0$. But this is not possible since $s_0(t)$ has no real roots in I. This implies that $\mathcal{A}(t)$ and $\mathcal{B}(t)$ are separate on I.

- If $s_0(t') < 0$ and $\mathbf{Var}(-, a_3(t'), a_2(t'), a_1(t'), -) \neq 2$ then $\mathcal{A}(t')$ and $\mathcal{B}(t')$ overlap (see Theorem 28). In this case $P(t', T)$ has four different real roots with, at most, only one of them positive. If, for any $t_0 \in I$, $\mathcal{A}(t_0)$ and $\mathcal{B}(t_0)$ are separate or they are externally tangent then there exists $t_1 \in I$ such that $s_0(t_1) = 0$. But this is not possible since $s_0(t)$ has no real roots in I. This implies that $\mathcal{A}(t)$ and $\mathcal{B}(t)$ overlap on I.

Let $\alpha_1 < \ldots < \alpha_m$ be the real roots of $s_0(t)$ in I such that $s_1(t) = 0$, $\mathbf{Var}(-, a_3, a_2, a_1, -) = 2$ or $s_1(t) < 0, s_{1,0}(t) > 0$: these are the t–values in I where $\mathcal{A}(t)$ and $\mathcal{B}(t)$ are externally tangent. Let $\alpha_0 = \inf(I)$, $\alpha_{m+1} = \sup(I)$ and t_i any number in (α_i, α_{i+1}): $\mathcal{A}(t)$ and $\mathcal{B}(t)$ are separate in (α_i, α_{i+1}) if and only if $\mathbf{Var}(-, a_3(t_i), a_2(t_i), a_1(t_i), -) = 2$, and $s_0(t_i) < 0$, or $s_0(t_i) = 0$, $s_1(t_i) < 0$ and $s_{1,0}(t_i) < 0$ $(0 \le i \le m)$.

4.4.2 $s_0(t) \equiv 0$ and $s_1(t) \not\equiv 0$ (compare with Theorem 5.5 in [22])

We start by proceeding like in the ellipses case. We consider the rational function

$$\beta(t) = -s_{1,0}(t)/s_1(t)$$

(after removing common factors). Observe that, when $s_1(t_0) \neq 0$, $T_0 = \beta(t_0)$ is a double root of $P(t_0, T)$. By continuity of $P(t, T)$, $\beta(t_0)$ is a triple root of $P(t_0, T)$ when $s_1(t_0) = 0$ (see the limit computation at Subsection 3.4 when dealing with the so called "Second case. Second possibility"). Now, since $\mathcal{B}(t)$ is always an ellipsoid, $a_0(t)$ never vanishes in I, so $T = 0$ is never a root of $P(t, T)$ and, then, $\beta(t)$ is never zero. Moreover, since $\mathcal{A}(t)$ is always an ellipsoid, $a_4(t)$ never vanishes in I, so $1/\beta(t)$, which, by continuity again, is a root of $T^4 P(t, 1/T)$, never vanishes. Therefore $\beta(t)$ is a rational continuous function whose range for $t \in I$ is either contained in $\mathbb{R}_{>0}$ or in $\mathbb{R}_{<0}$, because it has neither zeroes nor poles.

Since $s_1(t) \not\equiv 0$, there exists $\alpha \in I$ such that $s_1(\alpha) \neq 0$ and $s_{1,0}(\alpha) \neq 0$ (α exists since both polynomials have exactly the same real roots: Lemma 17 applies here too). Next we analyse the three possibilities arising when considering the signs of $s_1(\alpha)$ and $s_{1,0}(\alpha)$:

1. If $s_1(\alpha) > 0$ then $\beta(\alpha)$ is a double root of $P(\alpha, T)$ and, according to Lemma 22, the other two roots of $P(\alpha, T)$ are in $\mathbb{C} - \mathbb{R}$ ($\mathcal{A}(\alpha)$ and $\mathcal{B}(\alpha)$ overlap). By Theorem 19 (2), $\beta(\alpha) < 0$ and $\beta(t)$ is negative on I and a multiple root of $P(t, T)$. In this case we proceed by computing the real roots of $s_1(t)$: those roots such that $\mathbf{Var}(-, a_3(t), a_2(t), a_1(t), -) = 2$ give the values of t where they are externally tangent and the interval end points for separation and overlapping. If $s_1(t)$ has no real roots in I then the ellipsoids overlap for all t in I.

2. If $s_1(\alpha) < 0$ and $s_{1,0}(\alpha) > 0$ then $\beta(\alpha)$ is a double root of $P(\alpha, T)$ and, according to Lemma 22, $\beta(\alpha) > 0$ and the other two roots of $P(\alpha, T)$ are negative (by Theorem 19 (2)). This implies that $\beta(t)$ is positive on I and a positive double root of $P(t, T)$ for almost any $t \in I$. In this case, by continuity, $\mathcal{A}(t)$ and $\mathcal{B}(t)$ are externally tangent on I.

3. If $s_1(\alpha) < 0$ and $s_{1,0}(\alpha) < 0$ then $\beta(\alpha)$ is a double root of $P(\alpha, T)$ and, according to Lemma 22, $\beta(\alpha) < 0$ and the other two roots of $P(\alpha, T)$ are real and have the same sign. This implies that $\beta(t)$ is negative on I and a multiple root of $P(t, T)$.

In this case we proceed by computing the real roots of $s_1(t)$: those roots such that $\mathbf{Var}(-, a_3(t), a_2(t), a_1(t), -) = 2$ give the values of t where they are externally tangent and the interval end points for separation and overlapping. If $s_1(t)$ has no real roots in I then, when $\mathbf{Var}(-, a_3(\alpha), a_2(\alpha), a_1(\alpha), -) = 2$, the ellipsoids are separate for all t in I, otherwise they overlap for all t in I.

4.4.3 $s_0(t) \equiv 0$ and $s_1(t) \equiv 0$ (compare with Theorem 5.6 in [22])

Theorem 28 implies that $\mathcal{A}(t)$ and $\mathcal{B}(t)$ are not separate for any $t \in I$. Let α be in I. Then we analyse the two possibilities arising by considering the value of $\mathbf{Var}(-, a_3(\alpha), a_2(\alpha), a_1(\alpha), -)$:

1. If $\mathbf{Var}(-, a_3(\alpha), a_2(\alpha), a_1(\alpha), -) = 2$ then $P(\alpha, T) = a_4(T-\beta)^2(T-\gamma)^2$, with $\beta > 0$ and $\gamma < 0$. Repeating the argument used in Subsection 4.4.2, the roots must keep the sign throughout all I, so this factorisation is kept for the whole interval I, and $\mathcal{A}(t)$ and $\mathcal{B}(t)$ are externally tangent on I.
2. If $\mathbf{Var}(-, a_3(\alpha), a_2(\alpha), a_1(\alpha), -) \neq 2$ then $P(\alpha, T) = a_4(T-\beta)^2(T-\gamma)^2$, with $\gamma < 0$ and $\beta < 0$, or $P(\alpha, T) = a_4(T-\beta)(T-\gamma)^3$, with $\gamma < 0$ and $\beta < 0$, or $P(\alpha, T) = a_4(T-\gamma)^4$, with $\gamma < 0$. Repeating the argument used in Subsection 4.4.2, the roots must keep the sign throughout all I, so these factorisations are kept for the whole interval I, and $\mathcal{A}(t)$ and $\mathcal{B}(t)$ overlap on I.

4.5 Examples

The approach based on the formulae presented in the previous subsections allows to consider the problem of determining the non interference intervals for two moving ellipsoids. First experiments in `Maple` show a very good practical behaviour according to the performed experimentation: we include here five concrete examples showing different situations that can arise when dealing with this problem.

Example 29.
This is Example 5.1 from [22]. Let $\mathcal{A}(t)$ and $\mathcal{B}(t)$ be two moving ellipsoids defined by

$$\frac{(x+12t-11)^2}{4} + y^2 + z^2 = 1 \qquad \text{and} \qquad \frac{(x-3)^2}{4} + (y-4t+2)^2 + (z-4t+4)^2 = 1,$$

respectively, where $t \in [0, 1]$. In this particular case,

$$P(t, T) = -\frac{1}{4}(T+1)^2(T^2 + (-68t^2 + 96t - 34)T + 1),$$

$s_0(t) \equiv 0$ and $a_3(t) = a_1(t)$. Moreover,

$$s_1(t) = s_{1,0}(t) = -\frac{1}{2}(17t^2 - 24t + 8)(17t^2 - 24t + 9)^3$$

and, as a consequence, case 4.4.2(2) does not happen and we compute the real roots of $s_1(t)$:

$$\alpha_1 = \frac{12 - 2\sqrt{2}}{17} \approx 0.5395042868 \quad \text{and} \quad \alpha_2 = \frac{12 + 2\sqrt{2}}{17} \approx 0.8722604191 \;.$$

For both

$$\mathbf{Var}(-, a_3(\alpha_i), a_2(\alpha_i), a_1(\alpha_i), -) = \mathbf{Var}(-, 0, 1/2, 0, -) = 2.$$

This implies, by Theorem 24 (1), that the ellipsoids touch each other externally at α_1 and α_2. Next, in order to determine the relative position of $\mathcal{A}(t)$ and $\mathcal{B}(t)$ for $t \in [0, 1]$, we choose

$$\delta_1 = 0 \in [0, \alpha_1), \quad \delta_2 = 3/4 \in (\alpha_1, \alpha_2), \quad \delta_3 = 1 \in (\alpha_2, 1],$$

and check the conditions of Theorems 24 and 28 at δ_1, δ_2 and δ_3:

- Since $s_1(0) = s_{1,0}(0) < 0$ and $\mathbf{Var}(-, a_3(0), a_2(0), a_1(0), -) = \mathbf{Var}(-, 8, 33/2, 8, -) = 2$, $\mathcal{A}(t)$ and $\mathcal{B}(t)$ are collision free in $[0, \alpha_1)$.
- Since $s_1(3/4) = s_{1,0}(3/4) > 0$, $\mathcal{A}(t)$ and $\mathcal{B}(t)$ overlap in (α_1, α_2).
- Since $s_1(1) = s_{1,0}(1) < 0$ and $\mathbf{Var}(-, a_3(1), a_2(1), a_1(1), -) = \mathbf{Var}(-, 1, 5/2, 1, -) = 2$, $\mathcal{A}(t)$ and $\mathcal{B}(t)$ are collision free in $(\alpha_2, 1]$.

We remark here that using Theorem 3.10 in [22] to determine the relative position of $\mathcal{A}(t)$ and $\mathcal{B}(t)$ requires additionally to compute the signs of $s_2(t)$ and $s_{2,0}(t)$ evaluated at α_1 and α_2.

Example 30.
This is Example 1 from [5]. Consider the two moving ellipsoids

$$\mathcal{A} : \frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{4} = 1, \quad \mathcal{B} : x^2 + \frac{y^2}{9} + \frac{z^2}{16} = 1$$

under the rigid motions defined by the following degree two rotations

$$(R_{\mathcal{A}}(t), R_{\mathcal{B}}(t)) =$$

$$= \left(\begin{pmatrix} \frac{-8t^2+8t-1}{8t^2-8t+3} & \frac{-4t+2}{8t^2-8t+3} & \frac{4t-2}{8t^2-8t+3} \\ \frac{4t-2}{8t^2-8t+3} & \frac{1}{8t^2-8t+3} & \frac{2(2t-1)^2}{8t^2-8t+3} \\ \frac{-4t+2}{8t^2-8t+3} & \frac{2(2t-1)^2}{8t^2-8t+3} & \frac{1}{8t^2-8t+3} \end{pmatrix}, \begin{pmatrix} \frac{\sqrt{2}(t-1)(3t-1)}{-6t^2+6t-2} & \frac{2t(2t-1)}{-6t^2+6t-2} & \frac{\sqrt{2}(t-1)^2}{-6t^2+6t-2} \\ \frac{\sqrt{2}(2t-1)}{-6t^2+6t-2} & \frac{-2t(t-1)}{-6t^2+6t-2} & \frac{\sqrt{2}(2t-1)^2}{-6t^2+6t-2} \\ -\frac{\sqrt{2}t(3t-2)}{-6t^2+6t-2} & \frac{2(2t-1)(t-1)}{-6t^2+6t-2} & \frac{\sqrt{2}t^2}{-6t^2+6t-2} \end{pmatrix} \right)$$

and the degree three translations

$$(T_{\mathcal{A}}(t), T_{\mathcal{B}}(t)) =$$

$$= \left(\begin{pmatrix} -8t^3 + 24t^2 - 6t - 2 \\ -24t^3 + 24t^2 + 6t - 6 \\ -32t^3 + 48t^2 - 12t - 2 \end{pmatrix}, \begin{pmatrix} \left(72 - 24\sqrt{2}\right)t^3 + \left(-156 + 72\sqrt{2}\right)t^2 + \left(114 - 72\sqrt{2}\right)t - 27 + 24\sqrt{2} \\ 12t - 6 \\ \left(88 - 24\sqrt{2}\right)t^3 + \left(-168 + 72\sqrt{2}\right)t^2 + \left(114 - 72\sqrt{2}\right)t - 26 + 24\sqrt{2} \end{pmatrix} \right).$$

Note that the denominators have no real roots.

If $P(t, T)$ denotes the numerator of $\det(TA(t) + B(t))$ then its degree with respect to t is 14. In order to compute s_0, s_1 and $s_{1,0}$ we use the formulae in the Remark 23 and we will work with $\widehat{s_0} = -s_0/a_4$, $\widehat{s_1} = -s_1/a_4$ and $\widehat{s_{1,0}} = -s_{1,0}/a_4$. The corresponding degrees are $\deg_t(\widehat{s_0}) = 84$, $\deg_t(\widehat{s_1}) = 56$ and $\deg_t(\widehat{s_{1,0}}) = 56$. Since s_0 is not identically zero we proceed as described in the Subsection 4.4.1.

In order to determine the external tangency instants we compute first the real roots of $\widehat{s_0}$. It has six real roots, two of them are negative and four positive:

$$\alpha_1 \approx -0.2439303678,\ \alpha_2 \approx -0.1859966702,\ \alpha_3 = 0.5,$$
$$\alpha_4 \approx 0.6329555326,\ \alpha_5 \approx 0.7719039167,\ \alpha_6 \approx 0.9007790307.$$

Since $\gcd(\widehat{s_0}, \widehat{s_1}) = 1$, according to Theorem 24, $\mathcal{A}(\alpha_i)$ and $\mathcal{B}(\alpha_i)$ are externally tangent if and only if $\widehat{s_1}(\alpha_i) < 0$ and $\widehat{s_{1,0}}(\alpha_i) > 0$ and this happens only for α_1, α_2, α_3 and α_6. Moreover, using Theorem 28, we have

- $\widehat{s_1}(\alpha_4) < 0$, $\widehat{s_{1,0}}(\alpha_4) < 0$, $\mathbf{Var}(-, a_3(\alpha_4), a_2(\alpha_4), a_1(\alpha_4), -) = 0$: $\mathcal{A}(\alpha_4)$ and $\mathcal{B}(\alpha_4)$ overlap.
- $\widehat{s_1}(\alpha_5) < 0$, $\widehat{s_{1,0}}(\alpha_5) < 0$, $\mathbf{Var}(-, a_3(\alpha_5), a_2(\alpha_5), a_1(\alpha_5), -) = 0$: $\mathcal{A}(\alpha_5)$ and $\mathcal{B}(\alpha_5)$ overlap.

This implies that $\mathcal{A}(t)$ and $\mathcal{B}(t)$ overlap when $t \in (\alpha_3, \alpha_6)$. In order to check what happens in the other intervals determined by α_1, α_2, α_3 and α_6, we define

$$\delta_1 = -0.3 < \alpha_1 < \delta_2 = -0.2 < \alpha_2 < \delta_3 = 0 < \alpha_3 < \alpha_6 < \delta_4 = 1$$

and compute the relative position of $\mathcal{A}(\delta_i)$ and $\mathcal{B}(\delta_i)$:

- $\widehat{s_0}(\delta_1) < 0$ and $\mathbf{Var}(-, a_3(\delta_1), a_2(\delta_1), a_1(\delta_1), -) = 2$: $\mathcal{A}(\delta_1)$ and $\mathcal{B}(\delta_1)$ are separate.
- $\widehat{s_0}(\delta_2) > 0$: $\mathcal{A}(\delta_2)$ and $\mathcal{B}(\delta_2)$ overlap.
- $\widehat{s_0}(\delta_3) < 0$ and $\mathbf{Var}(-, a_3(\delta_3), a_2(\delta_3), a_1(\delta_3), -) = 2$: $\mathcal{A}(\delta_3)$ and $\mathcal{B}(\delta_3)$ are separate.
- $\widehat{s_0}(\delta_4) < 0$ and $\mathbf{Var}(-, a_3(\delta_4), a_2(\delta_4), a_1(\delta_4), -) = 2$: $\mathcal{A}(\delta_4)$ and $\mathcal{B}(\delta_4)$ are separate.

This allows us to conclude that:

- $\mathcal{A}(t)$ and $\mathcal{B}(t)$ overlap when $t \in (\alpha_1, \alpha_2)$ and $t \in (\alpha_3, \alpha_6)$.
- $\mathcal{A}(t)$ and $\mathcal{B}(t)$ are separate when $t \in (-\infty, \alpha_1)$, $t \in (\alpha_2, \alpha_3)$ and $t \in (\alpha_6, +\infty)$.
- $\mathcal{A}(t)$ and $\mathcal{B}(t)$ are externally tangent when $t = \alpha_1$, $t = \alpha_2$, $t = \alpha_3$ and $t = \alpha_6$.

Figure 1 shows the position of the two ellipsoids for different values of t.

Example 31.
This example shows external tangency without overlapping. Let

$$\mathcal{A} : \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1, \quad \mathcal{B} : \frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{4} = 1,$$

under the translations

$$T_{\mathcal{A}}(t) = \begin{pmatrix} 0 \\ 7 \\ 0 \end{pmatrix}, \quad T_{\mathcal{B}}(t) = \begin{pmatrix} 0 \\ 0 \\ \frac{t-4}{2} \end{pmatrix}.$$

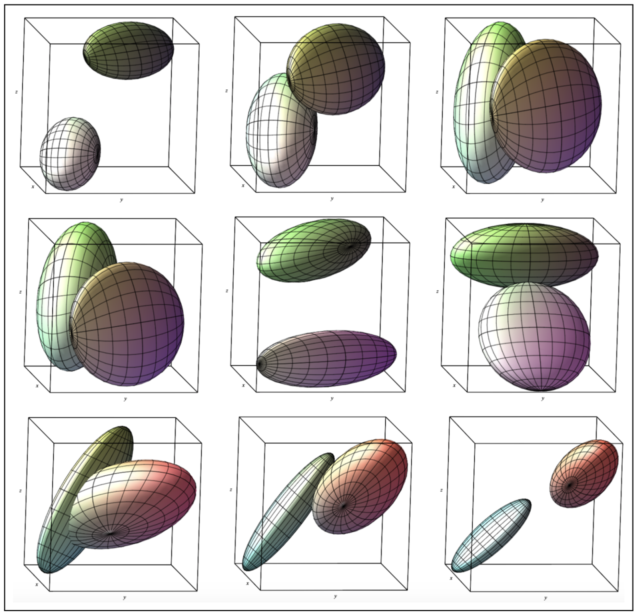


Figure 1: $\mathcal{A}(t)$ and $\mathcal{B}(t)$ when $t = \delta_1 < \alpha_1 < \delta_2 < \alpha_2 < \delta_3 < \alpha_3 < \alpha_5 < \alpha_6 < \delta_4$.

In this example, coefficients $a_4(t)$ and $a_0(t)$ of $P(t, T)$ are constants, and $a_3(t)$, $a_2(t)$ and $a_1(t)$ are different polynomials of degree 2. Moreover, $s_0(t)$ is a polynomial of degree 12 and we apply 4.4.1(1). The real roots of $s_0(t)$ are

$$\alpha_1 \approx -14.3303027798, \quad \alpha_2 = 4 \quad \text{and} \quad \alpha_3 \approx 22.3303027798,$$

and we have

$$s_1(\alpha_1) < 0, s_{10}(\alpha_1) < 0; \quad s_1(\alpha_2) < 0, s_{10}(\alpha_2) > 0; \quad s_1(\alpha_3) < 0, s_{10}(\alpha_3) < 0.$$

This implies, by Theorem 24 (2), that the ellipsoids touch each other externally only at $\alpha_2 = 4$. Next we consider values less and bigger than α_2, for example, 0 and 5. Since $s_0(0) < 0$, $s_0(5) < 0$, and $\mathbf{Var}(-, a_3(0), a_2(0), a_1(0), -) = \mathbf{Var}(-, a_3(5), a_2(5), a_1(5), -) = 2$, we conclude that the ellipsoids are externally tangent for $t = \alpha_2$ and they are separate for $t \neq \alpha_2$.

Example 32.
Consider the following ellipsoids

$$\mathcal{A} : \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1, \quad \mathcal{B} : \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{4} = 1,$$

moving following the translations

$$T_{\mathcal{A}}(t) = \begin{pmatrix} \frac{2(t^2-1)}{t^2+1} \\ \frac{-6t}{t^2+1} \\ 0 \end{pmatrix}, \quad T_{\mathcal{B}}(t) = \begin{pmatrix} -\frac{2(t^2-2)}{t^2+1} \\ \frac{6t}{t^2+1} \\ 0 \end{pmatrix}.$$

If $P(t,T)$ denotes the numerator of $\det(TA(t)+B(t))$ then we have $\deg_t(a_i) = 4$ (for $0 \le i \le 4$),

$$\deg_t(s_0) = 26, \quad \deg_t(s_1) = 20, \quad \deg_t(s_{1,0}) = 20,$$

and we follow 4.4.1(1).

The polynomial $s_0(t)$ has two real roots,

$$\alpha_1 \approx -1.118033989, \quad \alpha_2 \approx 1.118033989,$$

with $s_1(\alpha_1) < 0, s_{10}(\alpha_1) > 0$ and $s_1(\alpha_2) < 0, s_{10}(\alpha_2) > 0$. This implies, by Theorem 24 (2), that the ellipsoids touch each other externally at α_1 and α_2. Finally, in order to determine the relative position of $\mathcal{A}(t)$ and $\mathcal{B}(t)$ for t in $\mathbb{R}$, we choose

$$\delta_1 = -2 < \alpha_1, \quad \delta_2 = 0 \in (\alpha_1, \alpha_2), \quad \delta_3 = 2 > \alpha_2,$$

and check the conditions of Theorems 24 and 28 at δ_1, δ_2 and δ_3 in order to conclude:

- Since $s_0(\delta_1) > 0$, $\mathcal{A}(t)$ and $\mathcal{B}(t)$ overlap for $t < \alpha_1$.
- Since $s_0(\delta_2) < 0$ and $\mathbf{Var}(-, a_3(\delta_2), a_2(\delta_2), a_1(\delta_2), -) = 2$, $\mathcal{A}(t)$ and $\mathcal{B}(t)$ are separate in (α_1, α_2).
- Since $s_0(\delta_2) > 0$, $\mathcal{A}(t)$ and $\mathcal{B}(t)$ overlap for $t > \alpha_2$.

Example 33.
We consider again the ellipsoids in Example 32 but now moving them in a way such that they are always externally tangent. This is done by using the translations

$$T_{\mathcal{A}}(t) = \begin{pmatrix} \frac{2(t^2-1)}{t^2+1} \\ \frac{-6t}{t^2+1} \\ 0 \end{pmatrix}, \quad T_{\mathcal{B}}(t) = \begin{pmatrix} -\frac{2(t^2-1)}{t^2+1} \\ \frac{6t}{t^2+1} \\ 0 \end{pmatrix}.$$

In this case we have

$$P(t,T) = \det(TA(t) + B(t)) = -\frac{1}{576}(T+1)(T+4)(T-1)^2,$$

with $s_0 = 0$, $s_1 < 0$ and $s_{1,0} > 0$ for all $t \in \mathbb{R}$. Second condition in Theorem 24 is verified for all t confirming that they are always externally tangent.

Remark 34.
The main difficulty of this approach when applied to two moving ellipsoids relies on the (high) degree of s_0 since its real roots must be determined. But there are very efficient software packages for computing very fast and accurately real roots of univariate polynomials: for all examples in this section it was enough to use the `fsolve` function in `Maple` with a precision of 10 digits except for Example 30. For this example, since $\widehat{s_0}$ has degree 84 and its coefficients involve $\sqrt{2}$ and huge integer numbers it was needed instead to use the `Isolate` function from the `RootFinding` package with the option ABND activated, in order to compute correctly the six real roots of $\widehat{s_0}$. Moreover, in order to guarantee the correct evaluation of $\widehat{s_1}$ and $\widehat{s_{10}}$ at the real roots of $\widehat{s_0}$ (when required), it was needed to increase the precision from 10 to 35 digits.

5 Conclusions

We have characterised when two ellipses/ellipsoids are separate, overlapping or externally tangent. Our derivation is based on the use of subresultants and Descartes' law of signs in the algebraic conditions provided in [21, 28], translating the configurations of the two considered ellipses/ellipsoids to the root patterns of their characteristic polynomial. The explicit formulae here introduced involve only three subresultants coefficients of the characteristic polynomial and its derivative for ellipsoids and only one for ellipses. Moreover we have applied these formulae for the collision detection for two moving ellipses/ellipsoids.

From our point of view, three are our main contributions. The first one is mathematical since proofs from [21, 22] are greatly simplified (less subresultants are used and the proofs avoid, for example, using Puisseux expansions as in [22] for the ellipsoids case). The second one is related with the applications of the derived formulae: existing formulae for ellipses and ellipsoids were explicitly used in [8, 14, 15, 18, 19, 24, 25, 29]. And the last one relates to the computational cost since the new formulae requires less multiplications than the previous ones: we cannot compete with the strategy solving the ellipsoids interference problem by including the given ellipsoids inside spheres or boxes, but our more expensive test can be used to guarantee the ellipsoids to be separated in cases where the cheaper approaches before mentioned fail.

In the future we will study if our approach can also be adapted to develop explicit formulae to solve the same problems considered here for two general conics or quadrics with a fixed type (see for example [3, 4]).

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