# UNIVERSIDAD DE MURCIA

ESCUELA INTERNACIONAL DE DOCTORADO


Flag curvature in pseudo-Finsler manifolds
Curvatura bandera en variedades de pseudo-Finsler


**D. Matthieu Huber**

2023

# Resumen

Cuando Bernard Riemann introdujo la geometría diferencial en 1854 [2], permitiendo por primera vez la investigación sistemática de espacios no euclidianos, anticipó en su disertación inaugural la posibilidad de generalizar aún más la métrica que definía, a partir de productos interiores en los espacios tangentes a la variedad diferenciable. No fue hasta 1918 cuando esa generalización se concretó con la tesis de Paul Finsler [5] que extendió varios teoremas de geometría diferencial al contexto de métricas definidas a partir de una norma en cada espacio tangente. Posteriormente, Ludwig Berwald definió el tensor curvatura de una variedad de Finsler utilizando una conexión que desafortunadamente no preserva la métrica [17]. Preocupado por esta falta de elegancia, Elie Cartan presentó en 1933 una conexión que sí preservaba la métrica [6]. Sin embargo, la geometría de Finsler no fue tan estudiada como la geometría riemanniana debido en parte a la dificultad de realizar cálculos en ese caso más general. Cuando Albert Einstein sugirió en 1905 la teoría de la relatividad especial [3], ofreció otra motivación física para la forma bilineal que reemplaza el producto interior del espacio euclidiano en el espacio-tiempo de Minkowski. Esa relajación de la condición de la métrica riemanniana de ser definida positiva para ser tan solo no degenerada, dio lugar a la geometría pseudo-riemanniana que se empleó para enunciar la teoría de la relatividad general. Esa relevancia física de la geometría pseudo-riemanniana frente a la geometría pseudo-Finsler fue seguramente otra de las razones por las que hubo un interés desproporcionado por la primera sobre la segunda.

Sin embargo, Gunnar Randers introdujo en 1941 [9] en el contexto de la teoría de la relatividad general un caso sencillo de la métrica de Finsler: la suma de una métrica riemanniana con una 1-forma. Estas métricas bastante simples fueron investigadas esporádicamente hasta que Shiing-Shen Chern hizo revivir la geometría de Finsler reintroduciendo en 1992 [18] las nociones sobre las que Chern ya había escrito con anterioridad en 1943 [10] (see also [11]) con el formalismo moderno que utilizamos hoy en día. Propuso una conexión libre de torsión que preserva la métrica, de la manera más análoga posible a la conexión de Levi-Civita de una variedad pseudo-riemanniana. A pesar de ese renacimiento de la geometría de Finsler, la extensión de los resultados al caso no degenerado sigue teniendo mucho retraso comparado a la geometría pseudo-riemanniana utilizada para la teoría de la relatividad general.

Confrontados con la irreconciliabilidad de la teoría de la relatividad general con la descripción del mundo natural a escala cuántica, así como con las lagunas que esa teoría presenta, notablemente en lo que concierne al interior de los agujeros negros o a los primeros instantes del cosmos, se requerirá irremediablemente, tarde o temprano, una extensión a una nueva teoría de la gravitación. Cabe suponer que tal extensión se fundamente en una geometría pseudo-Finsler, la extensión más natural y conceptualmente más económica de la geometría pseudo-riemanniana utilizada en la teoría actual. En el contexto de determinar las ecuaciones de campo de Einstein y sus soluciones, es imprescindible disponer de la mayor clase posible de ejemplos de variedades de pseudo-Finsler. Las ecuaciones de Gauss y Codazzi permiten calcular el tensor curvatura de cualquier subvariedad no degenerada, y así investigar sobre una gran colección de variedades nuevas generadas a partir de variedades ambiente conocidas. Las ecuaciones fundamentales de una sumersión se ocupan de la configuración dual de

una aplicación diferenciable cuya diferencial es sobreyectiva, en vez de la inyectividad de la inclusión de una subvariedad como subconjunto de la variedad ambiente, y constituyen también una rica fuente de resultados sobre relaciones entre variedades de pseudo-Finsler. Aunque las ecuaciones de Gauss y Codazzi de una subvariedad de Finsler se investigaron tan pronto como 1936 [8], desde que Barrett O'Neill introduce las ecuaciones fundamentales de una sumersión riemanniana en 1966 [14], aún no parece haber ninguna generalización a ecuaciones fundamentales de una sumersión en geometría de Finsler en toda su generalidad (véase [32] y [47] para el caso de la curvatura bandera). Esta tesis doctoral tiene por propósito llenar ese vacío y contribuir así al conocimiento de las variedades pseudo-Finsler.

Las métricas de Finsler definen un funcional de Minkowski en cada espacio tangente, y una norma en el caso definido positivo. Se caracterizan por su tensor fundamental no degenerado (Ecuación (1.16) de la Definición 1.2.1), que define una forma bilineal, es decir un producto interior en el caso definido positivo, para cada elección de un vector del espacio tangente. El vector en el que se evalúa el tensor fundamental como una forma bilineal se llama informalmente la variable anisotrópica. En el caso pseudo-riemanniano, esa forma bilineal es constante en esa variable y define la métrica pseudo-riemanniana no degenerada. El ejemplo más sencillo de una métrica de Finsler es la métrica de Randers, a saber, la suma de una métrica riemanniana definida positiva con una 1-forma. En este caso, la indicatriz de la métrica es la traslación a lo largo de un campo de la indicatriz de una métrica riemanniana (véase la Sección 1.2.2), cuyo par es denominado datos de Zermelo en referencia al problema navegacional de encontrar el camino que minimiza el tiempo recorrido entre dos puntos en presencia de una corriente o de un viento. Los cálculos más sencillos tienen lugar con la métrica de Randers, y la generalización a cualquier métrica pseudo-Finsler complica necesariamente la computación, si no la imposibilita. El cálculo tensorial anisotrópico se efectúa más adecuadamente con respecto a la conexión de Chern (Definición 1.3.10), una conexión libre de torsión que preserva la métrica. Se trata de la conexión más análoga a la conexión de Levi-Civita del caso pseudo-riemanniano. Muchos cálculos clásicos se pueden mimetizar con un término adicional, el tensor de Cartan (Definición 1.2.6), cuando se deriva el tensor fundamental de la métrica (Ecuación (1.38) de la Sección 1.3.2). Se obtiene con facilidad una fórmula análoga a la fórmula de Koszul (Teorema 1.3.8), que por la propiedad del tensor de Cartan (Proposición 1.2.7) se simplifica al evaluarse en vectores idénticos a la variable anisotrópica en la que se evalúa el tensor fundamental, y así verifica la unicidad de la conexión de Chern. El mismo procedimiento permite mitigar la dificultad que presenta el estudio de las subvariedades pseudo-Finsler, para las que la conexión inducida por la conexión de Chern de la variedad ambiente no coincide con la conexión de Chern intrínseca a la subvariedad. El tensor diferencia $\widehat{Q}$ entre la conexión inducida y la conexión intrínseca (Definición 2.1.9, véase también la Definición 3.1.24 para su generalización), se puede expresar en términos del tensor de Cartan cuya propiedad permite simplificar la evaluación de ese tensor en un vector idéntico a la variable anisotrópica (Lema 2.1.11). Estas simplificaciones son claves para relacionar la curvatura bandera de la variedad ambiente con la curvatura bandera de la subvariedad (Corolario 2.2.5), ya que el tensor curvatura de la conexión inducida por la conexión de Chern de la variedad ambiente difiere del tensor curvatura de la conexión de Chern intrínseca a la subvariedad por términos en $\widehat{Q}$ en los que dos de los cuatro vectores en los que se evalúa la expresión total coinciden con la variable anisotrópica. Como paso intermedio a la relación entre las curvaturas bandera (Corolario 2.2.5), las ecuaciones de Gauss y Codazzi se obtienen en el contexto pseudo-Finsler sin poder hacer uso de estas propiedades de simplificación, resultando en largas expresiones que difieren de las ecuaciones de Gauss y Codazzi de una subvariedad pseudo-riemanniana por muchos términos más con dependencia en $\widehat{Q}$ y en la derivada vertical de la conexión de Chern (véase el Teorema 2.2.4). Incluso en las variedades de Randers, los cálculos necesarios para expresar la curvatura bandera de una subvariedad son un verdadero desafío. Los términos adicionales, dependientes del campo vectorial que representa el viento en los datos de Zermelo y que se cancelarían en el caso riemanniano, forman una expresión homogénea únicamente en la variable anisotrópica y con factores extremadamente sofisticados cuando se comparan a la definición sencilla de una métrica de Randers en términos de los datos de Zermelo. El capítulo 2 concluye con esta expresión de la curvatura bandera de una subvariedad de un espacio de Randers-Minkowski en función de los datos de Zermelo de su métrica de Randers y de la curvatura seccional con respecto a su métrica riemanniana (Teorema 2.3.5). En el caso de las hipersuperficies de un espacio de Randers-Minkowski, se obtiene una relación algo más simplificada de la que se deduce inmediatamente que las hipersuperficies llanas con respecto a la métrica riemanniana de los datos de Zermelo son hipersuperficies de curvatura bandera escalar con respecto a la métrica de Randers (Corolario 2.3.8).

En el capítulo siguiente, estudiamos la noción dual a las subvariedades: las sumersiones pseudo-Finsler, que preservan la métrica de los vectores horizontales (en el sentido de la Definición 3.1.1) y cuyas fibras son no-degeneradas (Definición 3.0.5). El objeto del capítulo es derivar en el caso pseudo-Finsler las ecuaciones fundamentales de una sumersión (Ecuaciones {0} a {4}), que completan las ecuaciones de Gauss y Codazzi (Ecuaciones {0} y {1}), válidas para las fibras de la sumersión en calidad de subvariedades. La primera dificultad con la que nos encontramos es la identificación del tensor fundamental $g$ de la variedad ambiente con el tensor fundamental $\tilde{g}$ de la variedad de base de la sumersión (Proposición 3.1.6), que en principio sólo se consigue probar cuando se evalúa en una variable anisotrópica $v$ horizontal en el sentido anisotrópico, y para un vector horizontal, es decir ortogonal a los vectores tangentes a la fibra de la sumersión, con respecto a la forma bilineal $g_v$. Se definen los tensores de O'Neill $T$ y $A$ (Definición 3.1.18), que generalizan la segunda forma fundamental de una subvariedad, de manera análoga a [14]. Para ello, se generalizan los operadores $^\top$ y $^\perp$ de tomar la parte tangente y ortogonal a las fibras con respecto a $g_v$ con una dependencia en la variable anisotrópica, lo cual significa que estos operadores producen a partir de un campo vectorial, un campo vectorial anisotrópico que depende de esa variable anisotrópica. Es importante destacar que para la definición rigurosa de los tensores de O'Neill se requiere que el espacio vertical sea no-degenerado con respecto a $g_v$ para cada $v$ en el dominio de la métrica de pseudo-Finsler, lo cual se puede garantizar restringiendo ese dominio con la consecuencia de que pueda no ser necesariamente conexo. Los objetos anisotrópicos, ya sean funciones, campos vectoriales o tensores, se pueden evaluar localmente en una extensión admisible $V$ del valor $v$ de la variable anisotrópica en un punto, para ser así manipulados en términos de objetos clásicos. Se definen entonces los tensores $T$ y $A$ de O'Neill como tensores anisotrópicos, consiguiendo generalizar muchas de las propiedades de estos tensores a propiedades similares en el caso pseudo-Finsler. Se destacan dos diferencias principales que complican los cálculos a la hora de obtener las ecuaciones fundamentales de una sumersión: en primer lugar, la parte $g_v$-horizontal de la derivada covariante con respecto a la conexión de Chern de un campo vectorial localmente $g_V$-horizontal para una extensión localmente admisible $V$ de $v$, difiere de los tensores de O'Neill por un término dependiente del tensor de Cartan que es cero en el caso pseudo-riemanniano (Lema 3.1.20); y en segundo lugar, para una extensión horizontal y proyectable $V$, la parte $g_v$-horizontal de la derivada covariante a lo largo de un vector $g_v$-horizontal de un campo vectorial proyectable $g_V$-horizontal no coincide con el levantamiento $g_v$-horizontal de la derivada covariante con respecto a la conexión de Chern de la variedad de base de la proyección en esa base de los mismos vectores y campo vectorial. En otras palabras, aparece otro tensor $\widetilde{Q}$ (Definición 3.1.26) que también se puede expresar en términos del tensor de Cartan y que relaciona entonces estos objetos de la misma manera que el tensor $\widehat{Q}$ relaciona la parte tangente a las fibras de la derivada covariante de campos vectoriales verticales, con la derivada covariante de estos mismos con respecto a la conexión de Chern intrínseca a estas fibras. Estos tensores $\widetilde{Q}$ y $\widehat{Q}$ permiten relacionar el tensor curvatura intrínseco a las fibras de la sumersión con un tensor curvatura $R^{\top}$ tangente a ellas que generaliza el tensor curvatura de la conexión inducida por la conexión de Chern ambiente, para ser evaluado en vectores no necesariamente tangentes, y relacionar el tensor curvatura $\widetilde{R}$ de la variedad de base de la sumersión con un tensor curvatura $R^{\perp}$ definido por analogía a este tensor curvatura tangente intercambiando los operadores anisotrópicos de tomar la parte tangente $^{\top}$ y la parte horizontal $^{\perp}$ con respecto a la variable anisotrópica (Definición 3.2.2). Las ecuaciones fundamentales se generalizan con estos tensores curvatura $R^{\top}$ y $R^{\perp}$ para ser evaluadas en valores de la variable anisotrópica no necesariamente tangentes u ortogonales a las fibras (Ecuación (3.83) y Corolario 3.2.11). De la misma manera que con las ecuaciones de Gauss y Codazzi, dos de las cinco ecuaciones fundamentales pueden modificarse para expresar el tensor curvatura $R$ de la conexión de Chern de la variedad ambiente en función de los tensores curvatura $\widehat{R}$ y $\widetilde{R}$ de las fibras y de la variedad de base, cuando la variable anisotrópica es tangente u ortogonal a las fibras respectivamente (Teorema 3.2.12). Al margen de los términos en $\widetilde{Q}$, $\widehat{Q}$ y en el tensor de Cartan, similares a los que aparecen en las ecuaciones de Gauss y Codazzi, la mayor diferencia con las ecuaciones clásicas de O'Neill es la ausencia de una fórmula cíclica que permita recombinar los términos que sólo son productos de tensores de O'Neill y sus derivadas covariantes. Esas ecuaciones fundamentales tienen como consecuencia inmediata una relación muy sencilla entre la curvatura bandera de la variedad ambiente y la curvatura bandera de la variedad de base, otra vez debido a las propiedades del tensor de Cartan que simplifica mucho las expresiones evaluadas en vectores idénticos a la variable anisotrópica. En el caso definido positivo, todo aquello permite establecer una desigualdad entre la curvatura bandera de la variedad ambiente y la curvatura bandera de la variedad de base de la sumersión (Teorema 3.2.17). Se concluye el capítulo con consideraciones sobre sumersiones cuyas fibras son totalmente geodésicas, lo cual quiere decir que las curvas que son geodésicas intrínsecas a las fibras también son geodésicas de la variedad ambiente. Gracias a la destacable y sencilla relación entre las geodésicas de la variedad de base y las geodésicas horizontales de la variedad ambiente, que coinciden exactamente con los levantamientos horizontales de las geodésicas de la variedad de base, cada arco geodésico de la variedad de base induce un difeomorfismo entre las fibras de cada uno de sus puntos. En el caso definido positivo, bajo ciertas condiciones de regularidad, estos difeomorfismos son isometrías si y solo si las fibras de la sumersión son totalmente geodésicas (Proposición 3.3.4). Por último, probamos que bajo estas condiciones de regularidad, si las fibras de una sumersión de Finsler son totalmente geodésicas, entonces esa sumersión es la proyección de un fibrado asociado con un fibrado principal cuyo grupo de estructura es el grupo de Lie de isometrías de la fibra (Teorema 3.3.6).

# Summary

When Bernard Riemann introduced differential geometry in 1854 [2], allowing for the first time the systematic study of non-Euclidean spaces, he anticipated in his inaugural lecture the possibility of generalising even more the metric he defined from inner products in the tangent spaces of the differentiable manifold. It was not until 1918 when this generalisation materialised with the doctoral thesis of Paul Finsler [5] who extended various theorems of differential geometry to the context of metrics defined from norms in the tangent spaces. Later, Ludwig Berwald defined the curvature tensor of a Finsler manifold using a connection that unfortunately does not preserve the metric [17]. Concerned by that lack of elegance, Elie Cartan presented in 1933 a connection that did preserve the metric [6]. However, Finsler geoemtry was not as thoroughly studied as Riemannian geometry, in part due to the difficulty of performing calculations in the more general setting. When Albert Einstein suggested in 1905 the theory of special relativity [3], he provided another physical motivation for the bilinear form which replaces the inner product of Euclidean space in the Minkowski space-time. The weakening of the positive definiteness condition of Riemannian metrics to a non-degeneracy condition gave rise to pseudo-Riemannian geometry, which was used to enunciate the theory of general relativity. This physical relevance of pseudo-Riemannian geometry over pseudo-Finsler geometry was certainly another reason for which there has been a disproportionate interest for the first one over the second.

Nevertheless, Gunnar Randers introduced in 1941 [9] in the context of the theory of general relativity a simple case of Finsler metrics: the sum of a Riemannian metric and a 1-form. This fairly simple metric was sporadically studied until Shiing-Shen Chern revived Finsler geometry by reintroducing in 1992 [18] the notions he had previously investigated in 1943 [10] and 1948 [11], with the modern formalism we still use nowadays. He proposed a torsion-free connection which preserves the metric, in the most analogous way to the Levi-Civita connection of a pseudo-Riemannian manifold. Despite of this renaissance of Finsler geometry, the extension of the results to the non-degenerate case is still far behind the results of pseudo-Riemannian geometry used in general relativity theory.

Confronted with the irreconcilability of general relativity theory with the description of the natural world at the quantum scale, as well as with the degeneracies of this theory, notably with regards to the interior of black holes or the first instants of the cosmos, sooner or later, an extension of a new theory of gravitation will irremediably be required. It is reasonable to assume that such an extension would be based in pseudo-Finsler geometry, the most natural and conceptually sparing extension of pseudo-Riemannian geometry used in the current theory. In the context of the search for the Einstein field equations and their solutions, it is indispensable to dispose of the greatest possible class of examples of pseudo-Finsler manifolds. The Gauss and Codazzi equations enable the computation of the curvature tensor of any non-degenerate submanifold, and thus studying a large collection of new manifolds generated from known ambient manifolds. The fundamental equations of a submersion deal with the dual configuration of a differentiable map whose differential is surjective, instead of the injectivity of the inclusion of a submanifold as a subset of the ambient manifold, and also constitute a rich source of results on relations between pseudo-Finsler manifolds. Though the

Gauss and Codazzi equations of a Finsler submersion were studied as early as 1936 [8], since Barrett O'Neill introduced the fundamental equations of a Riemannian submersion in 1966 [14], there does not appear to have been any generalisation to the fundamental equations of a Finsler submersion in all its generality (see [32] and [47] for a treatment of the case of flag curvature). This doctoral thesis has as prime purpose to fill this gap and contribute to the understanding of pseudo-Finsler manifolds.

Finsler metrics define a Minkowski functional in each tangent space, and a norm in the positive definite case. They are characterised by their non-degenerate fundamental tensor (Equation (1.16) of Definition 1.2.1), which defines a bilinear form, that is to say, an inner product in the positive definite case, for each choice of a vector in the tangent space. That vector in which the fundamental tensor is evaluated as a bilinear form is informally called the anisotropic variable. In the pseudo-Riemannian case, that bilinear form is constant in the anisotropic variable and defines the non-degenerate pseudo-Riemannian metric. The simplest example of a Finsler metric is that of Randers metrics, that is, the sum of a positive definite Riemannian metric and a 1-form. In this case, the indicatrix of the metric is the translation along a vector field of the indicatrix of a Riemannian metric (see Section 1.2.2), whose pair is referred to as the Zermelo data, in reference to the navigational problem of finding the path that minimises the time of travel between two points in the presence of a current or wind. The simplest calculations occur with Randers metrics, and the generalisation to arbitrary pseudo-Finsler metrics necessarily complicates the computations, when it does not impede them altogether. Anisotropic tensor calculus is best performed with respect to the Chern connection (Definition 1.3.10), a torsion-free connection that preserves the metric. It is the most analogous connection to the Levi-Civita connection of the pseudo-Riemannian case. Many classical computations can be mimicked with an additional term, the Cartan tensor (Definition 1.2.6), when differentiating the fundamental tensor of the metric (Equation (1.38) of Section 1.3.2). A formula analogous to the Koszul formula is easily obtained (Theorem 1.3.8), which by property of the Cartan tensor (Proposition 1.2.7) simplifies when evaluated in vectors that are identical to the anisotropic variable in which the fundamental tensor is evaluated, and thus verifies the uniqueness of the Chern connection. The same process mitigates the difficulty of studying pseudo-Finsler submanifolds, for which the connection induced by the Chern connection of the ambient manifold does not coincide with the Chern connection intrinsic to the submanifold. The difference tensor $\widehat{Q}$ between the induced connection and the intrinsic connection (Definition 2.1.9, see also Definition 3.1.24 for its generalisation) can be expressed in terms of the Cartan tensor whose property allows the simplification of the evaluation of that tensor in a vector identical to the anistropic variable (Lemma 2.1.11). These simplifications are key to relate the flag curvature of the ambient manifold with the flag curvature of the submanifold (Corollary 2.2.5), since the curvature tensor of the connection induced by the Chern connection of the ambient manifold differs from the curvature tensor of the Chern connection intrinsic to the submanifold by terms in $\widehat{Q}$ in which two of the four vectors in which the total expression is evaluated coincide with the anisotropic variable. As an intermediate step to the relating of the flag curvatures (Corollary 2.2.5), the Gauss and Codazzi equations are obtained in the pseudo-Finsler setting without the help of these simplifying properties, resulting in long expressions that differ from the Gauss and Codazzi equations of a pseudo-Riemannian submanifolds by many more terms which depend on $\widehat{Q}$ and on the vertical derivative of the Chern connection (see Theorem 2.2.4). Even in Randers manifolds, the calculations needed to express the flag curvature of a submanifold are truly challenging. The additional terms, with a dependence on the vector field that represents the wind in the Zermelo data and that would vanish in the Riemannian case, form an expression homogeneous only in the anisotropic variable and with extremely sophisticated factors when compared to the simple definition of a Randers metric in terms of its Zermelo data. Chapter 2 concludes with this expression of the flag curvature of a submanifold in a Randers-Minkowski space in terms of the Zermelo data of its Randers metric and the sectional curvature with respect to its Riemannian metric (Theorem 2.3.5). In the case of hypersurfaces of a Randers-Minkowski space, a more simplified relation is obtained from which to deduce immediately that hypersurfaces that are flat with respect to the Riemannian metric of the Zermelo data are hypersurfaces of scalar flag curvature with respect to the Randers metric (Corollary 2.3.8).

In the next chapter, we study the notion dual to submanifolds: the pseudo-Finsler submersions, which preserve the metric of horizontal vectors (in the sense of Definition 3.1.1) and whose fibres are non-degenerate (Definition 3.0.5). The objective of the chapter is to derive in the pseudo-Finsler case the fundamental equations of a submersion (Equations {0} to {4}), which complete the Gauss and Codazzi equations (Equations {0} and {1}) valid for the fibres of the submersion as submanifolds. The first difficulty we encounter is that of identifying the fundamental tensor $g$ of the ambient manifold with the fundamental tensor $\tilde{g}$ of the base manifold of the submersion (Proposition 3.1.6), which in principle can only be proved when evaluated in an anisotropic variable $v$ horizontal in the anisotropic sense, and for a horizontal vector, that is to say orthogonal to the vectors that are tangent to the submersion fibre, with respect to the bilinear form $g_v$. The O'Neill tensors $T$ and $A$ are defined (Definition 3.1.18), which generalise the second fundamental form of a submanifold, analogously to [14]. To this end, the operators $^\top$ and $^\perp$ of taking the parts tangent and orthogonal to the submersion fibres are defined with respect to $g_v$ with a dependence in the anisotropic variable $v$, which means that these operators produce from a vector field an anisotropic vector field that depends on this anisotropic variable. It is important to note that in order to define the O'Neill tensors rigorously the vertical space must be non-degenerate with respect to $g_v$ for each $v$ in the domain of definition of the pseudo-Finsler metric, which can be guaranteed up to a restriction of that domain with the possible consequence that it would no longer necessarily be connected. Anisotropic objects, be it a function, vector field or tensor, can be evaluated locally in an admissible extension $V$ of $v$ of the anisotropic variable at a point, in order to be manipulated in terms of classical objects. The O'Neill tensors $T$ and $A$ are thus defined as anisotropic tensors, and many of their classical properties are generalised to similar properties in the pseudo-Finsler case. Two main differences are to be noted that make the calculations difficult when obtaining the fundamental equations of a submersion: on the one hand, the $g_v$-horizontal part of the covariant derivative with respect to the Chern connection of a locally $g_V$-horizontal vector field for a locally admissible extension $V$ of $v$ differs from the O'Neill tensors by a term that depends on the Cartan tensor which is zero in the pseudo-Riemannian case (Lemma 3.1.20); and on the other hand, for a projectable horizontal extension $V$, the $g_v$-horizontal part of the covariant derivative along a $g_v$-horizontal vector of a projectable $g_V$-horizontal vector field does not coincide with the $g_v$-horizontal lift of the covariant derivative with respect to the Chern connection of the base manifold of the projection onto that base manifold of the same vectors and vector field. In other words, another $\widetilde{Q}$ tensor appears (Definition 3.1.26) which can also be expressed in terms of the Cartan tensor and relates these objects in the same way as the $\widehat{Q}$ tensor relates the part tangent to the fibres of the covariant derivative of vertical vector fields with the covariant derivative of these terms with respect to the Chern connection intrinsic to these fibres. These tensors $\widetilde{Q}$ and $\widehat{Q}$ allow relating the curvature tensor intrinsic to the fibres of the submersion with a curvature tensor $R^\top$ tangent to them, which generalises the curvature tensor of the connection induced by the ambient Chern connection in order to be evaluated in vectors that are not necessarily tangent to the fibres, and relating the curvature tensor $\widetilde{R}$ of the base manifold of the submersion with a curvature tensor $R^\perp$ defined by analogy to this tangent curvature tensor $R^\top$ interchanging only the anisotropic operators of taking the tangent $\top$ and horizontal $\perp$ parts with respect to the anisotropic variable (Definition 3.2.2). The fundamental equations are generalised with these curvature tensors $R^{\top}$ and $R^{\perp}$ to be evaluated in values of the anisotropic variable not necessarily tangent or orthogonal to the fibres (Equation (3.83) and Corollary 3.2.11). Similarly to the Gauss and Codazzi equations, two of the five fundamental equations can be modified to express the curvature tensor $R$ of the Chern connection of the ambient manifold in terms of the curvature tensors $\widehat{R}$ and $\widetilde{R}$ of the fibres and of the base manifold, when the anisotropic variable is tangent or orthogonal to the fibres respectively (Theorem 3.2.12). Notwithstanding the terms in $\widetilde{Q}$, $\widehat{Q}$ and in the Cartan tensor, similar to the ones appearing in the Gauss and Codazzi equations, the greatest difference with the classical equations of O'Neill is the absence of a cyclic formula that allowed him to recombine the terms that only feature the product of O'Neill tensors and their covariant derivatives. An immediate consequence of these fundamental equations is the very simple relation between the flag curvature of the ambient manifold and the flag curvature of the base manifold, due again to the properties of the Cartan tensor that simplifies greatly the expression evaluated in vectors that are identical to the anisotropic variable. In the definite positive case, all this allows to establish an inequality between the flag curvature of the ambient manifold and the flag curvature of the base manifold of the submersion (Theorem 3.2.17). The chapter concludes with considerations about submersions whose fibres are totally geodesics, which means that the curves that are geodesics intrinsic to the fibres are also geodesics of the ambient manifold. Thanks to the remarkable and simple relation between geodesics of the base manifold and the horizontal geodesics of the ambient manifold, which coincide exactly with the horizontal lift of the geodesics of the base manifold, each geodesic arc of the base manifold induces a diffeomorphism between the fibres of each of its points. In the definite positive case, under certain conditions of regularity, these diffeomorphisms are isometries if and only if the fibres of the submersion are totally geodesic (Proposition 3.3.4). Lastly, we prove under these conditions of regularity that if the fibres of a Finsler submersion are totally geodesic, then that submersion is the projection of a bundle associated with the principal fibre bundle whose structure group is the Lie group of isometries of the fibre (Theorem 3.3.6).

# Acknowledgements

I would like to express my gratitude to my supervisor Miguel Ángel Javaloyes Victoria for his patience and pedagogy, as well as for introducing me to the state of the art of cosmological models and to the mathematics of general relativity, the most natural extension of which I have so far concluded in agreement with him to be very likely a Finslerian modification.

# Contents

# Foreword

The recent history of all human knowledge can be thought of as a long process of sculpting: in the words of Michelangelo, *“beauty is the purgation of superfluities”*. Mathematics has undergone a similar enlightenment in the aftermath of the Renaissance; since the beginning of what has come to be called the Modern Period, thinkers have chiseled away superfluous assumptions that had prevented us from seeing the sculpture already complete within the marble block.

In the year before his death in May 1543, Nicolas Copernicus finally handed over his controversial manuscript in order to be published as *De revolutionibus orbium coelestium* and setting into motion, perhaps inadvertently, the gearworks of the Scientific Revolution. The preface famously reads *“if [some hypotheses] provide a calculus consistent with the observation, that alone is enough [. . . ] they are not put forward to convince anyone that they are true, but merely to provide a reliable basis for computation”*. The mathematical disciplines of astronomy and optics soon transmitted their highly contagious methods to other fields of study, and by the 17th century every self-respecting scientist was contrasting computational predictions from mathematical models of the world with observational data. Observations made by Galileo Galilei and Johannes Kepler inspired Isaac Newton’s laws of motion, which removed the first of three major assumptions: *there need not exist a preferred frame of rest*. Similarly, the Michelson-Morley experiment (later complemented by the Hughes-Drever experiments and the likes) inspired Albert Einstein’s equivalence principle, which removed the second major assumption: *there need not exist a preferred frame of simultaneity*. Exactly four centuries after the publication of *De revolutionibus orbium coelestium*, the British Tube Alloys nuclear weapons programme’s disclosure to the Manhattan Project at the First Quebec Conference would mark the end the Modern Period and the beginning of the Atomic Age.

During these 400 years, mathematical developments accompanied and enabled the shift in philosophy towards an ever more daring challenge to old beliefs; to quote once more the Italian artist, *“the more the marble wastes, the more the statue grows”*. Carl Friedrich Gauss lifted Euclidean geometry to curved surfaces, and his student Bernhard Riemann brilliantly extended these insights to higher dimensional submanifolds: they freed geometry from the bounds of flat space. Resting on the shoulders of Hendrik Lorentz and Henri Poincaré, Hermann Minkowski recognised the importance of relaxing the positive definiteness of Euclidean metric space in order to make room for a new kind of gauge invariance: they freed geometry from the bounds of isochrony and gave its actual meaning to the then Aristotelian notion of causality.

This brings us to the third and, thus far, most challenging of assumptions to remove: *there need not exist a preferred frame of ontology*. In the latest mathematical modelisation of the very small and of the very large, the foundational object of study (respectively the Hilbert space of quantum mechanical wave functions and the algebraic topological category of Lorentzian manifolds) is demonstrably insubstantial to our real world. In a nutshell, neither the wave function nor the space-time metric which both hold all the information of the physical system being described are ever quite literally purported to be any more real than the tangible reality they are used to describe. Albert Einstein, having witnessed and contributed to both macroscopic and microscopic physics, was personally struggling with the incompatibility of locality, causality and special relativity (the EPR paradox) as well as the incompatibility of determinism, general covariance and the materiality of a space-time metric (the hole argument). One of the three must be abandoned in order to remain logically consistent, but which one to choose is irrelevant to computations and to a theory's predictive power. Nevertheless, the many worlds interpretation (quantum measurements are due to relative abundance of parallel universes that differ only in the outcome of the measurements), pilot wave theory (they are due to an empirically inaccessible deterministic process), four dimensionalism (all general relativistic events coexist) and the neo-Lorentzian interpretation (they only exist briefly on an empirically inaccessible Cauchy surface of existence) are a testament of humanity's resistance to this last strike of chisel.

Since the generalisation of Euclidean geometry, in the century that followed Paul Finsler's own 1918 doctoral dissertation *Über Kurven und Flächen in allgemeinen Räumen*, progress on Finsler geometry has been slow in comparison to the work on pseudo-Riemannian manifolds (used in classical general relativity theory). The simple difference, but root of all the difficulty, is the dependence of the metric on the direction of travel, specifically, and more generally the asymmetry of its indicatrix (the set of unit vectors with respect to the metric). In Finsler geometry, the length of a curve is computed by measuring its velocity in a way that is sensitive to the reversal of the parametrisation of that curve, in addition there is a strong dependence on the direction of travel; this geometry is therefore referred to as *anisotropic* differential geometry. Informally, in an anisotropic universe, reproducing an experiment identically but in a different orientation might give a different result, as if under some background influence that violates Lorentz invariance. Though this influence is severely restricted by a so far fruitless search for such a violation, the possibility of imaging black holes and probing gravitational waves has raised the hopes of finding a violation in extremely energetic events such as the vicinity of a black hole or a black hole merger, and thus attracted interest to possible generalisations of general relativity theory such as the study of pseudo-Finsler manifolds in general and Finsler space-times in particular (see [29] for a presentation on the possibility of extending general relativity theory using Finsler manifolds, and [49] for an overview of the recent progress on the subject).

Chapter 1 is an introduction to pseudo-Finsler geometry that provides an elementary description that reproduces previous results. In particular, the anisotropic Koszul formula is derived following the classical proof, and defines a unique Chern connection analogous to the Levi-Civita connection. The symmetries of the classical curvature tensor are substituted by almost-symmetries of the anisotropic curvature tensor, with additional terms due to the anisotropic differentiation. Vectors as derivations lift to the vertical derivative of anisotropic objects; the vector they are evaluated in is therefore informally called the *vertical* or *anisotropic* variable. This variable is the flagpole of the so called *flag curvature*, analogous to the sectional curvature along a plane at one point which in our context must also depend on this variable chosen in that plane acting as a flag. For the sake of illustration, the simplest example of Randers metrics is studied in terms of their Zermelo data modeling the problem of navigating a manifold in the presence of an anisotropic influence.

In Chapter 2, the Gauss and Codazzi equations are derived in the context of pseudo-Finsler manifolds: they feature mainly the Cartan tensor and the $\widehat{Q}$ tensor (the difference tensor between the induced and intrinsic connections). The Gauss formula no longer identifies the submanifold's Chern connection as the restriction of the ambient one, unlike the Levi-Civita connection. The special case of Randers-Minkowski spaces illustrates the complex compuations needed to describe even the simplest Finsler manifolds. Their flag curvature can be expressed in terms of the Zermelo data mentioned above. One important consequence is the fact that the hypersurfaces of a Randers-Minkowski space are of scalar flag curvature if and only if they are flat with respect to the Riemannian metric of the Zermelo data.

Chapter 3 concerns itself with pseudo-Finsler submersions. The Gauss and Codazzi equations can be extended to the fundamental equations of a submersion. In this doctoral thesis, a single unified equation is provided that holds for arbitrary vectors, from which it is easier to derive the classical equations in the form obtained by Barrett O'Neill in 1966. The O'Neill tensors $T$ and $A$ are generalised to anisotropic tensors due to the dependence on the anisotropic variable $v$ of the decomposition into parts tangent and orthogonal to the fibres. The tensorial difference between some curvature tensors fails to vanish like the difference tensor $\widehat{Q}$ of the Gauss formula. This failure manifests as the intermediary curvature tensors $R^{\top}$ and $R^{\perp}$ that are related to the curvature tensors of the fibres and of the base manifold in terms of those difference tensors. The fundamental equations imply that the base and ambient flag curvatures are related in pseudo-Finsler submersions, with further applications to pseudo-Finsler submersions whose fibres are totally geodesic. Since the geodesics orthogonal to the fibres of the submersion and the lift of the geodesics of the base are in a one-to-one correspondence, every geodesic arc of the base defines a diffeomorphism between the fibres. In the positive-definite case, this diffeomorphism is an isometry if and only if the fibres of the submersion are totally geodesic; under the right conditions, a Finsler submersion with totally geodesic fibres is the projection of a bundle associated with a principal fibre bundle whose structure group is the Lie group of isometries of the fibre.

The majority of space-time models admits a pseudo-Riemannian submersion structure useful to calculate their curvature tensor. We hope that in the future the same will be done for Finsler space-times.

# Chapter 1

# Preliminary notions

Let us recall in this first chapter some elementary facts of anisotropic differential geometry. To do so is no luxury, when what is taken for granted in the Riemannian case and pseudo-Riemannian case must be rediscovered under a new light in the Finslerian case and peudo-Finslerian case. Some symmetries are lost with unforeseeable repercussions, either because:

(1) the intuition behind an earlier argument no longer proves an identity, or

(2) the identity in question simply fails altogether.

As a first example of (1), consider a pseudo-Finsler manifold with fundamental metric tensor $g$, and the unique Chern connection, analogous to the Levi-Civita connection of a pseudo-Riemannian manifold. In that context, the anisotropic Koszul formula expresses anisotropic Christoffel symbols, coefficients of the Chern connection, under the local form

$$2\Gamma^k_{ij} g_{kl} = \frac{\delta g_{lj}}{\delta x^i} + \frac{\delta g_{il}}{\delta x^j} - \frac{\delta g_{ij}}{\delta x^l}\,,$$

replacing by $\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - y^j \Gamma^k_{ij} \frac{\partial}{\partial y^k}$ the traditional $\frac{\partial}{\partial x^i}$ that appears in the isotropic case. Consequently, one must then express the contraction $y^j \Gamma^k_{ij}$ independently of $\Gamma^k_{ij}$. This additional step is not required when computing the coefficients of the Levi-Civita connection of a pseudo-Riemannian manifold due to the vanishing vertical derivative $\frac{\partial g_{ij}}{\partial y^k}$ along the anisotropic variable. Once this detail is addressed, the result and method are virtually identical.

As a second example of (1), consider identifying the anisotropic metric tensors of the base and ambient manifolds in a pseudo-Finsler submersion, as studied in Chapter 3. Contrary to pseudo-Riemannian submersions, the splitting of the tangent bundle along horizontal and vertical directions has a dependence on the choice of an anisotropy variable. We can still identify $g_v(w,\cdot)$ with its projection onto the base manifold for certain (horizontal) vectors $v$ and certain ($g_v$-horizontal) vectors $w$, though the proof is more involved than the pseudo-Riemannian case.

To illustrate (2), let us look at two properties that will fail. As a first example, while the metric-preserving condition $\nabla g = 0$ persists in its usual form, the idea of a compatibility condition is replaced by an *almost* compatibility condition (1.38), dependent on the generally non-vanishing vertical derivative of the metric tensor, called the Cartan tensor.

A second example is the Gauss formula (2.8) for pseudo-Finsler submanifolds, which no longer identifies the Chern connection of a submanifold with the connection induced on the submanifold by the ambient Chern connection, the so-called *induced connection*. The difference between the two is a generally non-vanishing tensor that will be the main concern of Chapter 2 and a prominent feature of the anisotropic Gauss and Codazzi equations.

Most of these phenomena, whether their belong to the kind (1) or the other kind (2), can be made manifest by concise manipulations reminiscent of the analogous proofs of Riemannian geometry. In order to ground things firmly, let us briefly recall the foundations of anisotropic tensor calculus.

## 1.1 The tangent bundle

For the purposes of anisotropic tensor calculus, the definition of the tangent bundle $TM$ of a manifold $M$ is to be accompanied by the manifold structure of $TM$ itself and its own tangent bundle $T(TM)$. Anisotropic objects ultimately differ from isotropic ones merely in their evaluation at a *tangent vector* rather than a *point*. Isotropic objects are trivially anisotropic, constant on each tangent space. That constance hides a substantial difficulty: that the paths along which to differentiate anisotropically must span the manifold $TM$, rather than $M$, with results acting on $T(TM)$.

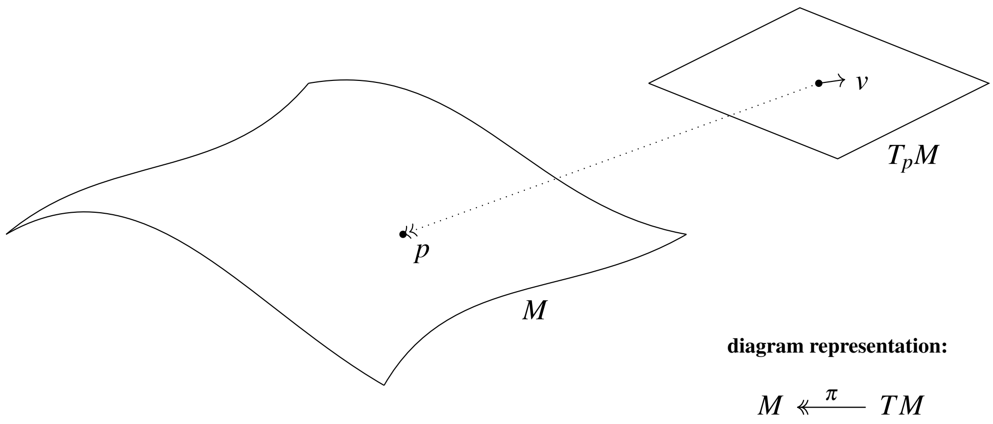


FIGURE 1.1: Visual representation of the tangent bundle.

A good intuitive illustration of this hidden difficulty is thermometry versus thermography. Thermometers provide a reading of temperature at each point of measurement, a scalar field with no dependence on an anisotropic variable. Thermographic cameras, by contrast, provide a reading of the infrared heat received from all around when measured at a point: an anisotropic scalar field. The information required to determine the change in temperature seen by a moving thermometer consists of the velocity $v \in TM$, whereas that information is insufficient for a moving thermographic camera, for which we must also know about the tilting and panning of the camera as an element of $T_v(TM)$.

### 1.1.1 Formal definition

Consider a set, which we shall denote by the letter $M$ for the German word *"Mannigfaltigkeit"*[1], and a fixed cover of $M$ by subsets. For a fixed non-negative integer $n \in \mathbb{N}$, let us call a *chart* any injective map from a subset of that cover into the $n$-dimensional Euclidean space $\mathbb{R}^n$. Furthermore, let us call an *atlas* the choice of at least one chart for each subset such that chart changes between two subsets define differentiable functions on their intersection.

In other words, if $\varphi$ and $\psi$ represent generic charts of an atlas, and their domains are $\mathbb{U}, \mathbb{V} \subset M$, then $\psi \circ \varphi^{-1}$ shall, by hypothesis, be differentiable as an $n$-tuple of real-valued functions

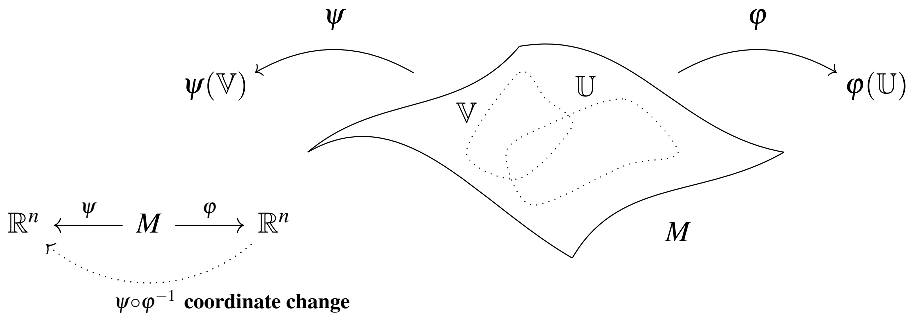


FIGURE 1.2: Compatible charts belonging to the same atlas.

of $n$ real variables at each point of the domain $\varphi(\mathbb{U}\cap\mathbb{V}) \subset \mathbb{R}^n$. When such is the case, $M$ is referred to as a *differentiable manifold* with respect to our choice of an atlas of charts.

Admittedly, differentiability can in essence be formulated in terms of the Euclidean topology of Euclidean space, rather than real analysis, with no reference to a choice of Cartesian coordinates. The image of each chart of our atlas is always an open subset of the Euclidean topology, up to reducing the domain to exclude all points mapped to its adherence, such that we can induce a topology on $M$ by defining the open neighbourhoods to be the preimages of open neighbourhoods of the Euclidean topology as well as their arbitrary unions. (Those points that are excluded from all maps may aggregate into the boundary of connected components, in the case of a manifold with boundary, which is not the object of this doctoral thesis.) The charts of the atlas are homeomorphisms for that choice of a topology on $M$ and the Euclidean topology on $\mathbb{R}^n$.

**Definition.** *An (n-dimensional)* differentiable manifold *is a topological space equipped with an atlas of local homeomorphisms into (n-dimensional) Euclidean topological spaces.*

The conditions that the space be Hausdorff and second countable has been omitted, but is tacitally implied in order to avoid pathological examples that are of no concern in our context.

Conceptual purity notwithstanding, we shall persist in interpreting elements of $n$-dimensional Euclidean space as real number $n$-tuples. Each chart of the atlas of an $n$-dimensional differentiable manifold is, equivalently, an $n$-tuple $(x^1,\cdots,x^n)$ of local Cartesian coordinate functions $x^i$ that each send points $p \in \mathbb{U}$ of a neighbourhood $\mathbb{U} \subset M$ to their $i^{\text{th}}$ coordinate $x^i(p) \in \mathbb{R}$.

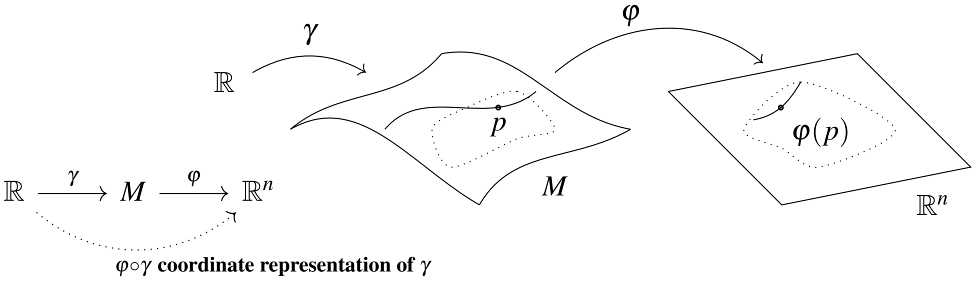


FIGURE 1.3: Local chart of a curve around one of its points.

Composing a curve $\gamma\colon \mathbb{R} \to M$ with a chart $\varphi\colon M \to \mathbb{R}^n$ around one of its points produces $\varphi\circ\gamma\colon \mathbb{R} \to \mathbb{R}^n$, which when differentiable can be differentiated as an $n$-tuple of *real-valued single real variable* functions to obtain the $n$-tuple of derivative functions $(\varphi\circ\gamma)'$. Such a curve whose coordinate expression is differentiable in this way is called *smooth*. This will serve to define the derivative of $\gamma$ itself, by coordinate independence.

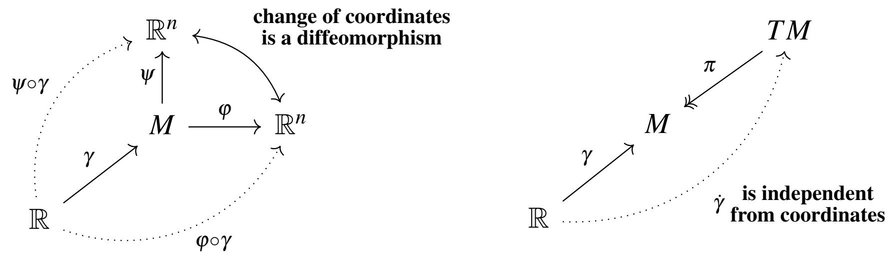


FIGURE 1.4: Diagrams illustrating the well-definedness.

Since chart changes are diffeomorphisms of $\mathbb{R}^n$, they can neither break apart coinciding derivatives of two paths at one point, nor merge differing derivatives. This makes the concept of derivative $\dot\gamma$ of $\gamma$ well-defined pointwise as the equivalence class of paths $\gamma$ through a point with matching $(\gamma\circ\varphi)'$ for some local chart $\varphi$ around $p$. This definition, independent of the choice of $\varphi$, gives rise to the vector space $T_pM$ of all possible derivatives a smooth path through $p \in M$ can have.

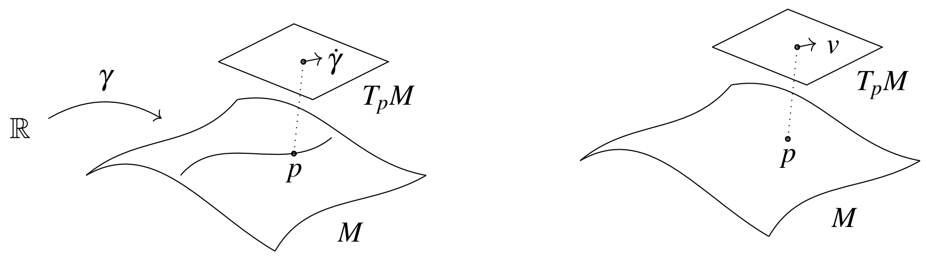


FIGURE 1.5: Well-definedness allows silencing $\gamma$.

What follows is a terser definition of $T_pM$, though far more esoteric, shaped around the product rule for smooth real-valued single real variable functions.

Indulging a brief moment's elementariness, recall that the product of smooth real-valued single real variable functions $a$ and $b$ has derivative $a'b+ab'$. In particular, for the composition of curves $\gamma\colon \mathbb{R}\to M$ with smooth functions $f,g\colon M\to\mathbb{R}$ and their pointwise product $fg\colon p\mapsto f(p)g(p)$,

$$(fg\circ\gamma)' = (f\circ\gamma)'g\circ\gamma + (g\circ\gamma)'f\circ\gamma$$

This fact leads to the following condition (1.1) for the reformulation of our construction of $T_pM$ in terms of derivations.

Let us call a *derivation* on $M$ at $p\in M$ any linear map $v$ that takes smooth scalar fields $f,g\colon M\to\mathbb{R}$ and their pointwise product $fg$ to real numbers $vf$, $vg$ and $v(fg)$ satisfying

$$v(fg) = vf\,g(p) + vg\,f(p)\,. \tag{1.1}$$

Let us furthermore define the addition of two derivations $v$ and $w$ at $p$ quite naturally by

$$(v+w)f = vf + wf\,,$$

and the scalar action of $\lambda\in\mathbb{R}$ on $v$ by

$$(\lambda v)f = \lambda vf\,.$$

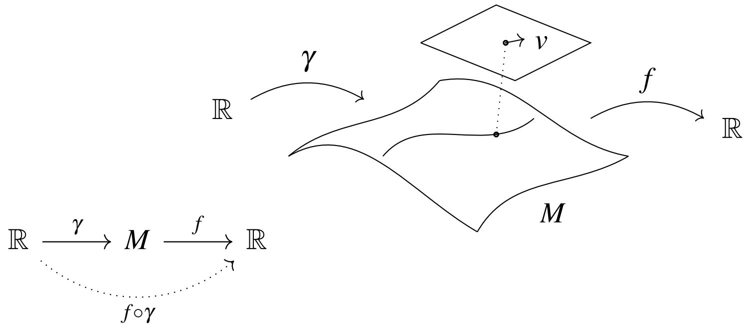


FIGURE 1.6: $v$ acts on $f$ as a directional derivative.

It is very straightforward to check that these operations give the space of derivations the structure of a vector space. Less obvious is the fact that it is precisely the space of tangent vectors we have constructed as the equivalence class of paths with coinciding derivatives, up to identifying tangent vectors with the directional derivative at the point where they are based along any smooth path of which they are the velocity.

**Definition.** *The* tangent space $T_pM$ *is the vector space of derivations on* $M$ *at* $p$*.*

The condition (1.1) captures an essential defining property; to construct $T_pM$ by equivalence classes of curves through $p \in M$ with the same speed in some (and therefore in all) local chart may fuel an inaccurate impression that tangent vectors represent information about paths, when any smooth curve $\gamma\colon \mathbb{R} \to M$ can also be interpreted as the collection of derivations

$$\dot{\gamma}(t)\colon f \mapsto (f \circ \gamma)'(t) \tag{1.2}$$

at $p = \gamma(t)$ for each $t \in \mathbb{R}$. The product rule emphasises that all local behaviour in the neighbourhood of the point belongs to the smooth functions being operated upon, while the derivations themselves are really defined by a condition (1.1) at a single point of the manifold.

**Definition.** *The* tangent bundle $TM$ *of a differentiable manifold* $M$ *is the disjoint union of its tangent spaces. Its* natural projection $\pi$ *maps each tangent space* $T_pM$ *to its point* $p$*.*

At the risk of notational inconsistencies, let $f$ temporarily denote a smooth function *between* differentiable manifolds $M$ and $N$; this choice of notation should hopefully not be a confusing one, since it generalises the previous case of $N = \mathbb{R}$. By smooth, we should understand that the composition with charts produces smooth local maps between Euclidean spaces.

As $f$ takes points of $M$ to points of $N$, the function drags $\gamma\colon \mathbb{R} \to M$ to $f \circ \gamma\colon \mathbb{R} \to N$, and in doing so unambiguously defines their derivative at each transported point. Given $v \in TM$, represented by $\gamma$ such that $\dot{\gamma} = v$ at $\pi(v)$, we can uniquely define $\mathrm{d}f \cdot v$ as the derivative of $f \circ \gamma$ at $f \circ \pi(v)$. In light of (1.2), we may condense this intuition into the following form.

**Definition.** *The* differential $\mathrm{d}f$ *of a smooth map* $f\colon M \to N$ *between differentiable manifolds maps* $v \in TM$ *to* $\mathrm{d}f \cdot v \in TN$ *satisfying for any scalar field* $g\colon N \to \mathbb{R}$ *the condition*

$$(\mathrm{d}f \cdot v)g = v(g \circ f)\,. \tag{1.3}$$

Note that if $f\colon M \to \mathbb{R}$ is a scalar field, then denoting by $\mathbb{1}_{\mathbb{R}}$ the identity map $t \mapsto t$ on the set of real numbers, we have

$$vf = v(\mathbb{1}_{\mathbb{R}} \circ f) = (\mathrm{d}f \cdot v)\mathbb{1}_{\mathbb{R}} = \mathrm{d}f \cdot v$$

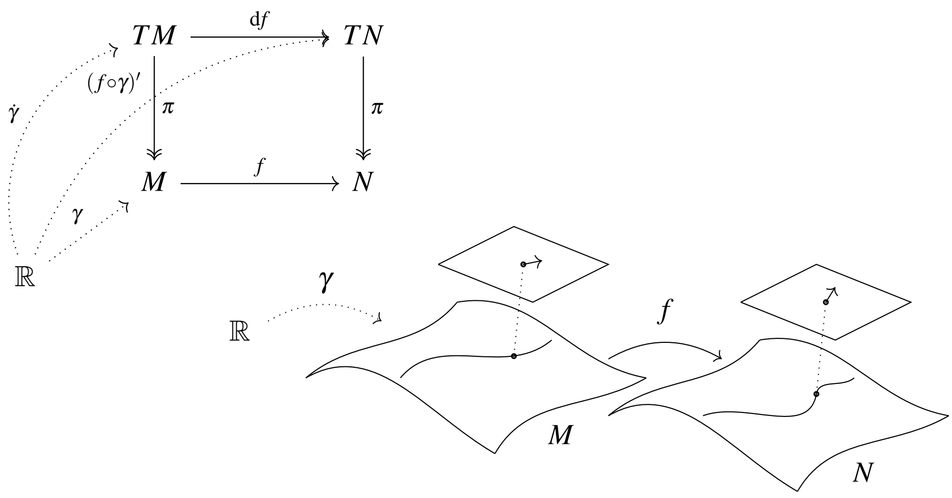


FIGURE 1.7: The differential d$f$ takes the derivative of $\gamma$ to that of $f \circ \gamma$, well-defined since both are, pointwise, elements of well-defined tangent bundles.

by setting $g$ to $\mathbb{1}_{\mathbb{R}}$ in the definition (1.3). Note that the expression $(\mathrm{d}f \cdot v)\mathbb{1}_{\mathbb{R}}$ denotes here the derivation of the identity function along the 1-dimensional vector $\mathrm{d}f \cdot v$, which confusingly consists of a real number.

It is worth illustrating visually that the chain rule for differential forms is trivial. Once more, just for the purposes of the next theorem, let us allow ourselves to momentarily label by $f, g$ differentiable maps rather than scalar fields.

**Theorem 1.1.1.** *For smooth maps $g\colon L \to M$ and $f\colon M \to N$ between differentiable manifolds,*

$$\mathrm{d}(f \circ g) = \mathrm{d}f \cdot \mathrm{d}g\,. \tag{1.4}$$

*Proof.* By definition (1.3) and for any scalar field $h\colon N \to \mathbb{R}$ and tangent vector $v \in TL$,

$$(\mathrm{d}(f \circ g) \cdot v)h = v(h \circ (f \circ g))\,,$$

while applying the definition (1.3) twice reveals that

$$(\mathrm{d}f \cdot \mathrm{d}g \cdot v)h = (\mathrm{d}g \cdot v)(h \circ f) = v((h \circ f) \circ g)\,.$$

Both $h \circ (f \circ g)$ and $(h \circ f) \circ g$ are ways of writing the same function $p \mapsto h(f(g(p)))$. □

We shall thus extend the derivation notation of $v \in TM$ to any smooth map $f\colon M \to N$ into an arbitrary differentiable manifold as

$$v \cdot f = \mathrm{d}f \cdot v\,,$$

producing an element of $TN$, generalised from the real number obtained in the case $N = \mathbb{R}$. This is where the differential-first approach pays off: many objects are particular cases of the general maps between differentiable manifolds. A map from $\mathbb{R}$ to a generic differentiable manifold is a curve; by definition, its differential is the derivative as a tangent vector, up to the identification (1.2) (strictly speaking it is the multiplication by the derivative of that curve, sending a scalar to an element of the tangent bundle). A map from a generic differentiable

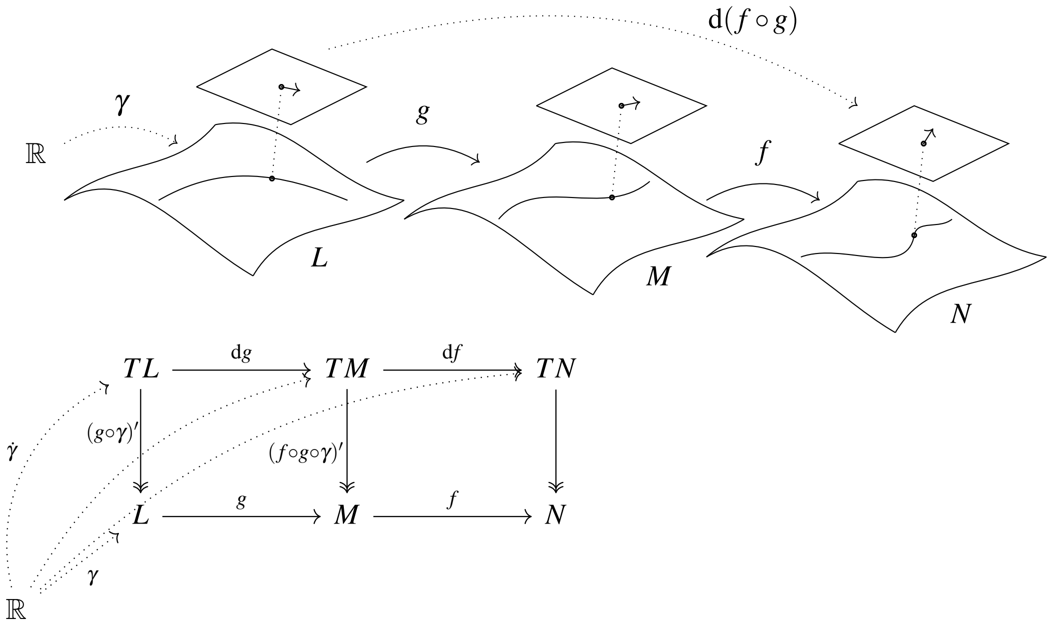


FIGURE 1.8: Visual illustration and commutative diagram of (1.4).

manifold into $\mathbb{R}$ is a scalar field, such as coordinate functions whose differential will form, pointwise, a basis of the dual vector space of linear forms acting on tangent vectors.

**Definition 1.1.2.** *Let $\mathfrak{X}(M)$ denote the set of smooth sections of $TM$, or vector fields $V: M \to TM$ satisfying $\pi \circ V = \mathbb{1}_M$ and whose local coordinate coefficients are differentiable.*

**Definition 1.1.3.** *Let $C^\infty(M)$ denote the ring of smooth real-valued functions on $M$, and $\mathfrak{X}(M)$ the $C^\infty(M)$-module of smooth sections of $TM$.*

For local coordinates $\varphi = (x^1, \cdots, x^n)$, we may define partial derivation along $x^i$ of a smooth scalar field $f: M \to \mathbb{R}$ as the partial derivative along the $i^{\text{th}}$ variable of the real-valued function of $n$ real variables $f \circ \varphi^{-1}: (t^1, t^2, \cdots, t^n) \mapsto f(\varphi^{-1}(t^1, t^2, \cdots, t^n))$, denoted as $\frac{\partial f}{\partial x^i}$. The partial derivations $\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}$ as actions on smooth scalar fields are elements of the tangent bundle up to the isomorphism (1.2), and define, locally, a frame of $\mathfrak{X}(M)$ and, pointwise, a basis of the tangent space as a space of derivations. Each tangent vector $v \in TM$ of the domain of the coordinate functions can be written under the form

$$v = \sum_i v^i \frac{\partial}{\partial x^i}|_{\pi(v)} \tag{1.5}$$

for some unique coefficients $v^1, \cdots, v^n \in \mathbb{R}$.

In the converse case of a map from the manifold to $\mathbb{R}$, our definition of differential produces 1-forms. Their restriction at a point form a vector space dual to the tangent space at that point. For local coordinates $(x^1, \cdots, x^n)$, the differentials $\mathrm{d}x^1, \cdots, \mathrm{d}x^n$ define, pointwise, a basis of that dual space. Any 1-form can be written under the form

$$\sum_i \omega_i \, \mathrm{d}x^i(q) \tag{1.6}$$

for some unique coefficients $\omega_1, \cdots, \omega_n \in \mathbb{R}$.

From (1.5), and by definition of the tangent space as a space of derivations, we can write for any scalar field $f$

$$vf = \sum_i (\mathrm{d}x^i \cdot v) \tfrac{\partial f}{\partial x^i}|_{\pi(v)} \,. \tag{1.7}$$

Furthermore,

$$\mathrm{d}f \cdot v = vf = \sum_i (\mathrm{d}x^i \cdot v) \tfrac{\partial f}{\partial x^i}|_{\pi(v)} = \left[\sum_i \tfrac{\partial f}{\partial x^i} \mathrm{d}x^i\right] \cdot v \,,$$

recovering the total derivative

$$\mathrm{d}f = \sum_i \tfrac{\partial f}{\partial x^i} \mathrm{d}x^i \tag{1.8}$$

as a differential form. In particular, setting $f$ in (1.8) to the coordinate functions $x^i$ themselves in a different coordinate system $(\bar{x}^1, \cdots, \bar{x}^n)$ yields

$$\mathrm{d}x^i = \sum_j \tfrac{\partial x^i}{\partial \bar{x}^j} \mathrm{d}\bar{x}^j \,, \tag{1.9}$$

while the associated differential operators $\frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial \bar{x}^i}$ appear in the decomposition

$$\sum_i \tfrac{\partial f}{\partial x^i} \mathrm{d}x^i = \mathrm{d}f = \sum_i \tfrac{\partial f}{\partial \bar{x}^i} \mathrm{d}\bar{x}^i \,,$$

producing

$$\tfrac{\partial}{\partial \bar{x}^j} = \sum_i \tfrac{\partial x^i}{\partial \bar{x}^j} \tfrac{\partial}{\partial x^i} \,. \tag{1.10}$$

More explicitly, this identity can be obtained, for instance, via the manipulation

$$\tfrac{\partial}{\partial \bar{x}^j} f = \mathrm{d}f \cdot \tfrac{\partial}{\partial \bar{x}^j} = \sum_i \tfrac{\partial f}{\partial x^i} \mathrm{d}x^i \cdot \tfrac{\partial}{\partial \bar{x}^j} = \sum_{i,k} \tfrac{\partial x^i}{\partial \bar{x}^k} \tfrac{\partial f}{\partial x^i} \mathrm{d}\bar{x}^k \cdot \tfrac{\partial}{\partial \bar{x}^j} = \left[\sum_i \tfrac{\partial x^i}{\partial \bar{x}^j} \tfrac{\partial}{\partial x^i}\right] f \,.$$

### 1.1.2 Einstein summation

Given local coordinates, we can induce as explained previously a basis for the tangent spaces of the domain in which to decompose tangent vectors into the weighted sum (1.5), whose coefficients will vary *against* any change of coordinates. That is to say the coefficients $v^1, \cdots, v^n$ of a tangent vector in a local coordinate system $(x^1, \cdots, x^n)$ are related to the coefficients $\bar{v}^1, \cdots, \bar{v}^n$ of the same tangent vector in another local coordinate system $(\bar{x}^1, \cdots, \bar{x}^n)$ by (1.9)

$$\bar{v}^i = \mathrm{d}\bar{x}^i \cdot v = \sum_j v^j \tfrac{\partial x^i}{\partial \bar{x}^j} \,.$$

Note the $\frac{\partial x^i}{\partial \bar{x}^j}$ factor, differentiating the old coordinate function $x^i$ with respect to the new coordinate function $\bar{x}^j$. This *contravariance* is indicated by the use of an upper index, to match the contravariance (1.9) of $\mathrm{d}x^i$.

Analogously, elements of the dual also decompose into the weighted sum (1.6), whose coefficients are *covariant* with changes of coordinates. Indeed by (1.10)

$$\bar{\omega}_i = \omega \cdot \tfrac{\partial}{\partial \bar{x}^i} = \sum_j \omega_j \tfrac{\partial \bar{x}^j}{\partial x^i} \,.$$

Note the $\frac{\partial \bar{x}^j}{\partial x^i}$ factor, differentiating the new coordinate function $x^j$ with respect to the old coordinate function $x^i$. This *covariance* is indicated by the use of a lower index matching the covariance (1.10) of $\frac{\partial}{\partial x^i}$ which, concordantly, shall be interpreted as a lower index (often rendered as $\partial_i$).

By virtue of the balance of upper and lower indices in each decomposition, we can unambiguously omit the summation sign $\Sigma$ over the silent index $i$. When presented with an expression that balances a lower index $i$ with an upper index $i$, that index is implicitly to be summed over its values from 1 to $n$. We shall write

$$\frac{\partial}{\partial x^i} = \frac{\partial \bar{x}^j}{\partial x^i}\frac{\partial}{\partial \bar{x}^j}$$

and

$$\mathrm{d}x^i = \frac{\partial x^i}{\partial \bar{x}^j}\mathrm{d}\bar{x}^j\,,$$

intuitively recalling them as anomalous cancellations. This implicit summation called the *Einstein summation convention* makes it much more straightforward to introduce *tensors*, the generalisation of scalar fields and 1-forms.

For example, *metric tensors* conventionally denoted by $g$ for *gravitation* (see [4, S. A38] and [13, p. 119] for the historical context) are linear operators whose variables are a pair of vector fields, smooth sections of the tangent bundle, and whose output defines a scalar field, pointwise the inner product of vectors. Using the Einstein summation convention, we can locally decompose two vector fields $V = V^i\frac{\partial}{\partial x^i}$ and $W = W^j\frac{\partial}{\partial x^j}$ for some local coordinates $(x^1,\cdots,x^n)$ and the coefficients $V^i = \mathrm{d}x^i \cdot V$ and $W^j = \mathrm{d}x^j \cdot W$. Factoring them out by linearity yields

$$g(V,W) = g(V^i\frac{\partial}{\partial x^i}, W^j\frac{\partial}{\partial x^j}) = V^iW^j\underbrace{g(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j})}_{g_{ij}}\,, \tag{1.11}$$

a *tensorial* expression for $g(V,W)$ in local coordinates, so called because of its manifest bilinearity in the coefficients of its input. That is to say,

$$g = g_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j$$

is a tensor field, where $\mathrm{d}x^i \otimes \mathrm{d}x^j$ denotes the bilinear function that sends an ordered pair of vector fields to the product of their respective $i^{\text{th}}$ and $j^{\text{th}}$ coefficients.

The coefficients $g_{ij}$ of $g$ feature two covariant indices, balancing the coefficients of the two vector field inputs. This fact is encoded into the type $(0,2)$ of the tensor. Vector fields and 1-forms are trivial examples of tensors, up to identifying $\frac{\partial}{\partial x^i}$ with $\omega \mapsto \omega(\frac{\partial}{\partial x^i})$. A vector field $V = V^i\frac{\partial}{\partial x^i}$ can take as input a 1-form $\omega = \omega_i\mathrm{d}x^i$ and is of type $(1,0)$, while the 1-form can take as input the vector field and is of type $(0,1)$, both producing the scalar field

$$V(\omega) = \omega(V) = V^i\omega_i\,.$$

Scalar fields themselves are, by extension, tensors of type $(0,0)$.

More generally, a tensor field of type $(a,b)$ is a pointwise multilinear function of $a$ vector inputs and $b$ covector inputs. Its smooth local coordinate coefficients have $a$ contr*a*-variant upper indices *a*bove at an *a*ltitude and $b$ co-variant lower indices *b*elow at the *b*ase. Scalar fields take no input, vector fields take one covector input, 1-forms take one vector input and metric tensors take two.

There is an alternative definition of tensor fields of type $(a,b)$: at each point $p$, they are elements of the tensor product of $a$ copies of $T_pM$ and $b$ copies of its dual. The two definitions are equivalent up to the identification of each multilinear scalar field-valued function (defined at $p$ on the Cartesian product of $a$ copies of $T_pM$ and $b$ copies of its dual) with the unique element of the tensor product provided by the universal characterisation of the tensor product.

Both definitions extend quite naturally to anisotropic tensor fields of type $(a,b)$: they are, at each point $p$ and each tangent vector $v$ at that point, elements of the tensor product of $a$ copies of $T_pM$ and $b$ copies of its dual, or equivalently multilinear scalar field-valued functions defined at $p$ and each tangent vector $v$ at that point on the Cartesian product of $a$ copies of $T_pM$ and $b$ copies of its dual. We shall define them more rigorously in the next section on pseudo-Finsler metrics.

From now on, let us use the word (anisotropic) *tensor* as a shorthand for (anisotropic) *tensor field*.

**Definition.** *A* pseudo-Riemannian manifold $(M,g)$ *is a differentiable manifold $M$ equipped with a symmetric nowhere-degenerate type* $(0,2)$ *tensor $g$.*

In this context, it is customary to identify each vector $v$ with the covector $g(v,\cdot)\colon w \mapsto g(v,w)$ at $\pi(v)$. If the former decomposes as $v^i \frac{\partial}{\partial x^i}|_{\pi(v)}$, the latter shall be written $v_i \, \mathrm{d}x^i|_{\pi(v)}$ with $v_i = v^j g_{ij}$ bearing the same coefficient symbol with a lowered index. Concretely, we want to allow ourselves to write

$$g(V,W) = V^i W^j g_{ij} = V^i W_i = V_i W^i \,. \tag{1.12}$$

This can be formalised by requiring that $g^{ij}$ be the coefficients of the inverse to the matrix of coefficients $g_{ij}$, and that any tensor coefficient with upper index $i$ contracts with $g_{ij}$ to the same symbol with a lower index $j$ in its place, and vice versa for a lower index $i$ contracting with $g^{ij}$ to an upper index $j$.

The notoriously confounding index notation of differential geometry can be pushed to the extreme as is the case in the context of Ricci calculus, which reduces, among other things, the covariant derivation along $\frac{\partial}{\partial x^i}$ to a mere "$_{;i}$" subscript. For our purposes, we will prefer to write tensorial expressions with the minimal use of local coefficients, and generally favour a more evocative notation for the sake of intelligibility in the tenuous enough anisotropic tensor calculus computations that we will encounter.

### 1.1.3 Manifold structure

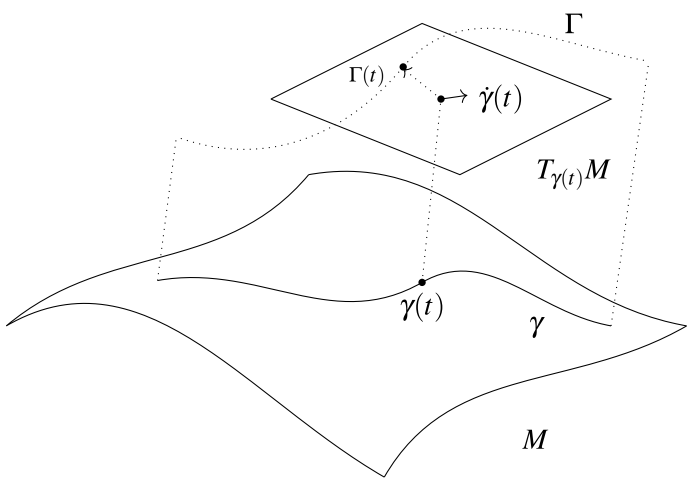


FIGURE 1.9: Tantalising illustration of a curve $\Gamma$ drawn though $TM$. The 2-dimensionality of $M$ suggested here should in reality call for 4 dimensions.

Anisotropy requires clarification, and one aspect we take to heart is the subtlety of deriving objects taking values not solely on $M$ but on the entire $TM$. In particular, if the metric tensor $g$ is to have a dependence on a vertical variable $v$, evaluating as a different inner product $g_v$ for each vector of one same tangent space, then its covariant derivative must involve partial derivation in the vertical variable. We are bound to treat $TM$ as a manifold in its own right.

The topology of the tangent bundle $TM$ as a differentiable manifold is locally that of the Cartesian product of a neighbourhood of the original manifold $M$ with Euclidean space, up to isomorphism with each tangent space. The charts of the tangent bundle is the Cartesian product of the natural projection with the identity of the tangent space. That is to say, if $\varphi$ is a local chart of $M$ on some open neighbourhood $U \subset M$, then it induces an isomorphism between $T_pM$ and $\mathbb{R}^n$ at each $p \in U$ and $(p,v) \mapsto (x^1(p), x^2(p), \cdots, x^n(p), \mathrm{d}x^1 \cdot v, \mathrm{d}x^2 \cdot v, \cdots, \mathrm{d}x^n \cdot v)$ is a local chart of $TM$. We may apply the previous construction to obtain the tangent bundle $T(TM)$ of the manifold $TM$.

The key to making sense of $T(TM)$ is to understand the curves of $TM$. The derivative of a curve of $M$ is an example of such a curve, we only need to widen our perspective to cover all possible curves of $TM$. To this end, some notation is helpful.

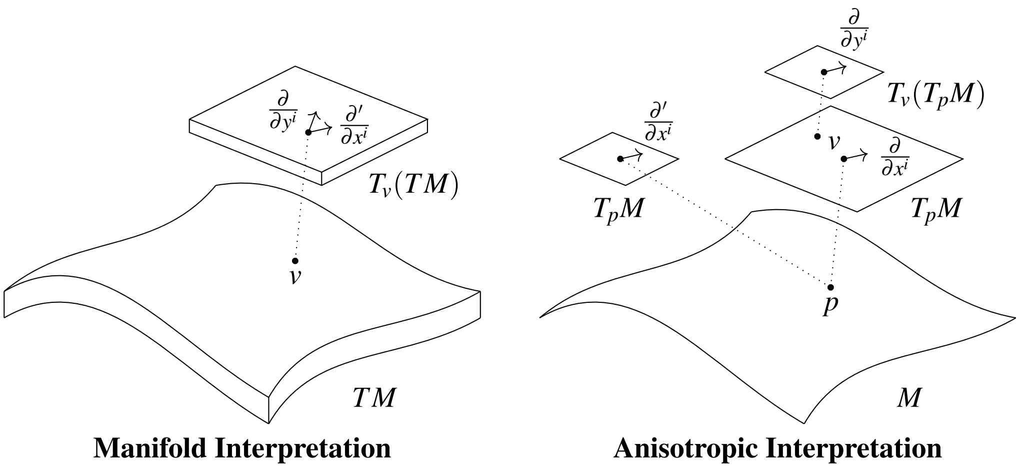


FIGURE 1.10: Two complementary perspectives on $T(TM)$.

Local coordinates $(x^1, \cdots, x^n)$ on a neighbourhood $\mathbb{U} \subset M$ can be completed by $y^i = \mathrm{d}x^i$ into local coordinates

$$(x^1, \cdots, x^n, y^1, \cdots, y^n)$$

on $TM$. The $2n$-tuple of partial derivations

$$\left(\tfrac{\partial'}{\partial x^1}, \cdots, \tfrac{\partial'}{\partial x^n}, \tfrac{\partial}{\partial y^1}, \cdots, \tfrac{\partial}{\partial y^n}\right)$$

forms a local frame of the $C^\infty(M)$-module $\mathfrak{X}(TM)$ of vector fields of $TM$, as opposed to $\mathfrak{X}(M)$. Similarly,

$$(\mathrm{d}'x^1, \cdots, \mathrm{d}'x^n, \mathrm{d}y^1, \cdots, \mathrm{d}y^n)$$

forms a basis of the space of 1-forms of $TM$. Note that we have used an apostrophe $'$ to avoid confusion with $\frac{\partial}{\partial x^i}$ and $\mathrm{d}x^i$ arising from $x^i$ as a coordinate function on $M$.

As an alternative to formally treating of the tangent bundle as a manifold in its own right, we can think of scalar fields of $TM$ as anisotropic scalar fields of $M$. Let $\mathrm{Im}(V) = \{V(p),\ p \in M\}$ represent the image of $V \in \mathfrak{X}(M)$. We can lay $\frac{\partial'}{\partial x^i}$ and $\frac{\partial}{\partial x^i}$ together side by side in the same commutative diagram. For simplicity, assimilate the sections $\mathrm{d}h \cdot \mathrm{Im}\frac{\partial'}{\partial x^i}$ and $\mathrm{d}f \cdot \mathrm{Im}\frac{\partial}{\partial x^i}$ of $T\mathbb{R}$ appearing below to $\mathbb{R}$ itself, representing the restriction of $\pi_{\mathbb{R}}$ to those sections as $\mathbb{R} \to \mathbb{R}$ by abuse of notation.

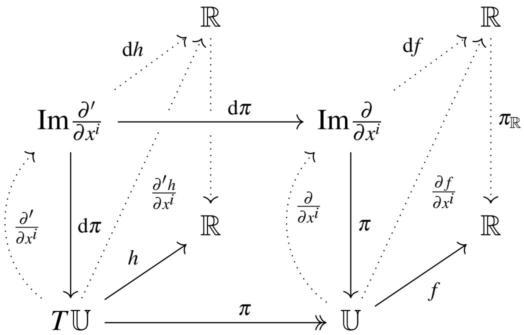


FIGURE 1.11: Commutative diagram of the actions of $\frac{\partial}{\partial x^i}$ and $\frac{\partial'}{\partial x^i}$.

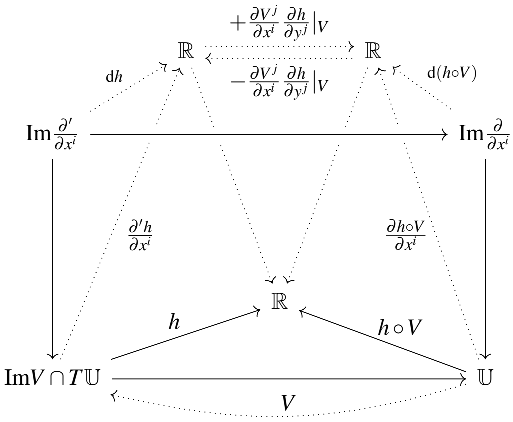


FIGURE 1.12: Commutative diagram illustrating equation (1.13).

A vector field $V \in \mathfrak{X}(M)$ takes each point $p$ of the domain with coordinates $(x^1(p), \cdots, x^n(p))$ to the vector $V(p) \in T_pM$ with coordinates $(x^1(p), \cdots, x^n(p), V^1(p), \cdots, V^n(p))$. By definition and in light of (1.3) and (1.8), its formal differential $\mathrm{d}V : TM \to T(TM)$ can be expressed in local coordinates for each $w = w^i \frac{\partial}{\partial x^i} \in T_pM$ as

$$\mathrm{d}V \cdot w = (w^1, \cdots, w^n, \frac{\partial V^1}{\partial x^i} w^i, \cdots, \frac{\partial V^n}{\partial x^i} w^i),$$

in other words

$$\mathrm{d}V = \frac{\partial'}{\partial x^i}\mathrm{d}x^i + \frac{\partial V^j}{\partial x^i}\frac{\partial}{\partial y^j}\mathrm{d}x^i .$$

Recalling that by the same definition

$$\mathrm{d}h = \frac{\partial' h}{\partial x^i}\mathrm{d}' x^i + \frac{\partial h}{\partial y^i}\mathrm{d}y^i ,$$

we can apply the chain rule (1.4) to compute

$$\mathrm{d}(h \circ V)|_p = \mathrm{d}h|_v \cdot \mathrm{d}V|_p = \frac{\partial' h}{\partial x^i}|_v \, \mathrm{d}x^i|_p + \frac{\partial V^j}{\partial x^i}|_p \, \frac{\partial h}{\partial y^j}|_v \, \mathrm{d}x^i|_p .$$

Up to an identification involving (1.3), $\frac{\partial'}{\partial x^i}$ and $\frac{\partial}{\partial x^i}$ are related by $\frac{\partial}{\partial y^i}$ terms as

$$\frac{\partial h(V)}{\partial x^i}|_p = \frac{\partial' h}{\partial x^i}|_v + \frac{\partial V^j}{\partial x^i}\frac{\partial h}{\partial y^j}|_v . \tag{1.13}$$

Expressed in a different coordinate system $(\bar{x}^1, \cdots, \bar{x}^n, \bar{y}^1, \cdots, \bar{y}^n)$, we would have

$$\frac{\partial' h}{\partial x^i}|_v = \frac{\partial h(V)}{\partial x^i}|_p - \frac{\partial V^j}{\partial x^i}\frac{\partial h}{\partial y^j}|_v = \frac{\partial \bar{x}^j}{\partial x^i}\left(\frac{\partial h(V)}{\partial \bar{x}^j}|_p - \frac{\partial \bar{y}^l}{\partial y^k}\frac{\partial V^k}{\partial \bar{x}^j}\frac{\partial h}{\partial \bar{y}^l}|_v\right)$$
$$= \frac{\partial \bar{x}^j}{\partial x^i}\left(\frac{\partial' h}{\partial \bar{x}^j}|_p + (\frac{\partial V^k}{\partial \bar{x}^j} - \frac{\partial \bar{y}^k}{\partial y^l}\frac{\partial V^l}{\partial \bar{x}^j})\frac{\partial h}{\partial \bar{y}^k}|_v\right),$$

manifesting the need to balance $\frac{\partial'}{\partial x^i}$ by terms in $\frac{\partial}{\partial y^i}$ in order to obtain an object that is independent from our choice of coordinates, as will be done in (1.34).

The same as (1.13) works if we replace the anisotropic scalar field $h$ by an anisotropic vector field $H$, or a smooth choice for each $v \in TM$ of some $H(v) \in T_{\pi(v)}M$, which we can decompose locally as $H^i(v)\frac{\partial}{\partial x^i}|_{\pi(v)}$. The coefficients $H^i$ are anisotropic scalar fields to which we can apply (1.13) and obtain the similar relation

$$\frac{\partial H^k \circ V}{\partial x^i}\frac{\partial}{\partial x^k} = \frac{\partial' H^k}{\partial x^i}|_V \frac{\partial}{\partial x^k} + \frac{\partial V^j}{\partial x^i}\frac{\partial H^k}{\partial y^j}|_V \frac{\partial}{\partial x^k}. \tag{1.14}$$

We shall later rewrite this identity independently of coordinates in equation (1.36).

In anticipation of this, so as not to refer to any choice of coordinates, let us represent by $\dot{\partial}$ the vertical derivation, such that the action $X^i\frac{\partial}{\partial y^i}$ of the vector field $X^i\frac{\partial}{\partial x^i}$ as a vertical derivative on anisotropic scalar fields may be rewritten in the following way independently of a choice of coordinates.

**Definition 1.1.4.** *Let $\dot{\partial}$ denote the* vertical derivative *of anisotropic scalar fields, defined for $h \in C^\infty(TM)$, $v \in TM$ and $w \in T_{\pi(v)}M$ by*

$$(\dot{\partial} h)_v(w) = \frac{\partial h(v+tw)}{\partial t}\Big|_{t=0}$$

*and, concordantly, $\dot{\partial}_i$ the vertical derivation $\frac{\partial}{\partial y^i}$ induced by local coordinates whenever $\partial_i$ is used to denote the derivation $\frac{\partial}{\partial x^i}$.*

Resting a vertical derivative from

$$X(h \circ V) = X^i \frac{\partial h(V)}{\partial x^i} = X^i \frac{\partial' h}{\partial x^i}|_v + (\dot{\partial} h)_v(X^i \frac{\partial V^j}{\partial x^i}\frac{\partial}{\partial x^j}) \tag{1.15}$$

can effectively isolate a part independent from $\frac{\partial V^j}{\partial x^i}$ and, consequently, independent of the choice of a vector field extension $V$ of the tangent vector $v$. Complementing this independence of a choice of extension with an independence of a choice of local coordinates will give rise to our definition of $\nabla h$ in (1.34), well-defined at each point independently of both the choice of coordinates and the choice of an extension $V$.

The same will make viable the condition on $\nabla$ that $\nabla g = 0$ for a fixed anisotropic metric tensor $g$, leading to the existence of an almost $g$-compatible anisotropic linear connection, while a connection verifying the more naive $g$-compatible condition need not exist at all, as explained in (1.29).

## 1.2 Pseudo-Finsler metrics

This section focuses on (squared) *pseudo-Finsler metrics* $L$ defined on a set of admissible tangent vectors. Informally and as rough summary, the associated fundamental tensor $g$, that is to say the anisotropic metric tensor, is a choice of an inner product $g_v$ on the tangent space at $\pi(v)$ for each admissible tangent vector $v$, which will satisfy inter alia $g_v(v,v) = L(v)$.

In the positive definite case, we call *Finsler metric* the well-defined positive square root $F = \sqrt{L}$. One natural deviation from Riemannian metrics, which define a norm $\|\cdot\|$ on each tangent space, are Randers metrics $F = \|\cdot\| + \omega$ for which that norm is displaced by a 1-form $\omega$ of norm strictly less than 1.The indicatrix $F^{-1}(\{1\})$ of $F$ is, in that case, the translation by some vector field $W$ of the indicatrix of some Riemannian metric tensor $\langle\cdot,\cdot\rangle$. The pair $(\langle\cdot,\cdot\rangle, W)$ fully determines $F$, and is called *Zermelo data*, named in reference to the navigation problem of finding the shortest path a powered vessel can take in the presence of wind.

For Randers metrics, it is spectacularly straightforward to compute both $g$ and its vertical derivative the Cartan tensor $C = \frac{1}{2}\dot{\partial} g$. The result gives an easily accessible indication for how to generalise to arbitrary pseudo-Finsler metrics.

### 1.2.1 Definition and properties

Being $\mathbf{0}$ the zero section of the tangent bundle, let $\mathbb{A} \subset TM \setminus \mathbf{0}$, for *admissible*, be a conic subset, open for the topology we have chosen on $TM$, not containing any zero tangent vectors of $M$, in the sense that $v \in \mathbb{A}$ implies $\lambda v \in \mathbb{A}$ for every positive real number $\lambda$. This set may be modified a posteriori to suit the conditions of a given problem of study, such as when restricting to the future-directed time-like directions delimited by the future light-cone in a pseudo-Finsler spacetime.

**Definition 1.2.1.** *A smooth* 2*-positive homogeneous function* $L\colon \mathbb{A} \to \mathbb{R}$ *is said to be a (squared)* pseudo-Finsler metric *when its fundamental tensor* $g$ *given for* $v \in \mathbb{A}$ *and* $x, y \in T_{\pi(v)}M$ *by*

$$g_v(x,y) = \frac{1}{2}\frac{\partial^2 L(v+sx+ty)}{\partial s \partial t}\bigg|_{s,t=0} \tag{1.16}$$

*defines a non-degenerate inner product* $g_v$ *on* $T_{\pi(v)}M$ *for all* $v \in \mathbb{A}$*. When* $g_v$ *is positive definite over* $v \in \mathbb{A}$*, the square root* $F = \sqrt{L}$ *will be called a* Finsler metric.

Given a set of local coordinate functions $x^i$, we can equivalently write the previous definition (1.16) in local coordinates as

$$g_{ij} = \tfrac{1}{2}\dot{\partial}_i\dot{\partial}_j L \left( = \tfrac{1}{2}\tfrac{\partial^2 L}{\partial y^i \partial y^j}\right) \tag{1.17}$$

where $g_{ij} \in C^\infty(TM)$ are the anisotropic coefficients of $g$ given by $g_{ij}(v) = g_v(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$.

**Proposition 1.2.2.** *Subadditivity of* $F = \sqrt{L}$ *(respectively superadditivity) is equivalent to inward (respectively outward) convexity of the indicatrix* $F^{-1}(\{1\})$.

*Proof.* Assuming subadditivity, we have for $v \in \mathbb{A}$, $w \in T_{\pi(v)}M$ and $t \in [0,1]$

$$F(tv + (1-t)w) \leq F(tv) + F((1-t)w) = tF(v) + (1-t)F(w)\,,$$

which means the left hand is less than 1 for $v, w \in \Sigma$. Conversely, since $F(\frac{v}{F(v)}) = F(\frac{w}{F(w)}) = 1$, assuming convexity of $\Sigma$ gives us

$$F(t\tfrac{v}{F(v)} + (1-t)\tfrac{w}{F(w)}) \leq 1\,.$$

Setting $t = \frac{F(v)}{F(v)+F(w)}$ and therefore $1-t = \frac{F(w)}{F(v)+F(w)}$, we obtain

$$\frac{F(v+w)}{F(v)+F(w)} = F\left(\tfrac{v+w}{F(v)+F(w)}\right) = F\left(\tfrac{v}{F(v)+F(w)} + \tfrac{w}{F(v)+F(w)}\right)$$
$$= F\left(\tfrac{F(v)}{F(v)+F(w)}\tfrac{v}{F(v)} + \tfrac{F(w)}{F(v)+F(w)}\tfrac{w}{F(w)}\right) \leq 1\,,$$

giving us the subadditivity inequality $F(v+w) \leq F(v)+F(w)$. Same goes for superadditivity and outward convexity of $\Sigma$, up to reversing all inequalities. □

**Proposition 1.2.3.** *For $v \in \mathbb{A}$, $g_v(v,\cdot) = \frac{1}{2}(\dot{\partial}L)_v$ and $g_v(v,v) = L(v)$.*

*Proof.* Let $w \in T_{\pi(v)}M$. $L(v+sv+tw) = (1+s)^2L(v+\frac{t}{1+s}w)$ differentiates to $2g_v(v,w) = (\dot{\partial}L)_v(w)$ since the right hand side first becomes $(1+s)^2(\dot{\partial}L)_v(\frac{w}{1+s}) = (1+s)(\dot{\partial}L)_v(w)$ at $t=0$. Similarly $L(v+sv+tv) = (1+s+t)^2L(v)$ differentiates to $g_v(v,v) = 2L(v)$. □

**Proposition 1.2.4.** *At each $p \in M$, the level set $L^{-1}(\{t\}) \cap T_pM$ of $L|_{T_pM}$ for $t \in \mathbb{R}\setminus\{0\}$ is a hypersurface of $T_pM$ to which each non-lightlike admissible $v$ with $L(v) = t$ is $g_v$-orthogonal.*

*Proof.* By the implicit function theorem, and since by the previous proposition the differential $(\dot{\partial}L)_v = 2g_v(v,\cdot)$ is non-degenerate for each admissible $v$, the level set is a hypersurface of $T_pM$. Again by the previous proposition, $(\dot{\partial}L)_v = 2g_v(v,\cdot)$ implies that the set of vectors $g_v$-orthogonal to $v$ is precisely the set of vectors tangent to the level set. Assuming that some ray $\{\lambda v,\ \lambda \in ]0,+\infty[\}$ intersects the level set in some $w$, we can check from $t = L(w) = L(\lambda v) = \lambda^2 L(v) = \lambda^2 t$ that necessarily $\lambda = 1$ and $w = v$, such that each such ray intersects the level set in exactly one point for each admissible $v$. Therefore at each point the 1-dimensional span$(\{v\})$ and the space tangent to the level set at $v$ are supplementary subsets of $T_pM$, such that the level set must have the dimensionality of a hypersurface. □

**Proposition 1.2.5.** *For $v \in \mathbb{A}$ and $\lambda > 0$, $g_{\lambda v} = g_v$.*

*Proof.* Let $x,y \in T_{\pi(v)}M$. $L(\lambda v+sx+ty) = \lambda^2 L(v+\frac{s}{\lambda}x+\frac{t}{\lambda}y)$ differentiates to $g_{\lambda v}(x,y) = \lambda^2 g_v(\frac{x}{\lambda},\frac{y}{\lambda})$, noting that (1.16) is manifestly bilinear in $x$ and $y$ concludes the proof. □

**Definition 1.2.6.** *Let the Cartan tensor $C = \frac{1}{2}\dot{\partial}g$ be given for $v \in \mathbb{A}$ and $x,y,z \in T_{\pi(v)}M$ as*

$$C_v(x,y,z) = \frac{1}{2}\frac{\partial g_{v+tz}(x,y)}{\partial t}\bigg|_{t=0} = \frac{1}{4}\frac{\partial^3 L(v+rx+sy+tz)}{\partial r\partial s\partial t}\bigg|_{r,s,t=0}\,. \tag{1.18}$$

Note that this definition is well defined by openness of $\mathbb{A}$, due to which $v+tz$ is necessarily admissible for $t \in \mathbb{R}$ close to 0.

**Proposition 1.2.7.** *For $v \in \mathbb{A}$, $C_v(v,\cdot,\cdot) = C_v(\cdot,v,\cdot) = C_v(\cdot,\cdot,v) = 0$.*

*Proof.* By Proposition 1.2.5, $g_{v+tv}(x,y) = g_v(x,y)$ differentiates to $C_v(x,y,v) = 0$, conclude the proof by noting that (1.18) is manifestly symmetric by commutativity of partial differentiation on smooth functions. □

**Proposition 1.2.8.** *For $v \in \mathbb{A}$ and $\lambda > 0$, $C_{\lambda v} = \frac{1}{\lambda}C_v$.*

*Proof.* Let $x,y,z \in T_{\pi(v)}M$. Then $L(\lambda v + rx + sy + tz) = \lambda^2 L(v + \frac{r}{\lambda}x + \frac{s}{\lambda}y + \frac{t}{\lambda}z)$ differentiates to $C_{\lambda v}(x,y,z) = \lambda^2 C_v(\frac{x}{\lambda}, \frac{y}{\lambda}, \frac{z}{\lambda}) = \frac{1}{\lambda} C_v(x,y,z)$ where we obtain the last equality from the manifest trilinearity of (1.18). □

**Definition 1.2.9.** *Let $C^\sharp$ be the unique tensor satisfying*

$$g_v(C^\sharp_v(x,y),z) = C_v(x,y,z) \tag{1.19}$$

*for every admissible $v \in \mathbb{A}$ and tangent vectors $x,y,z \in T_{\pi(v)}M$, obtained by juggling the indices of the Cartan tensor as explained in (1.12).*

**Proposition 1.2.10.** *For $v \in \mathbb{A}$, $C^\sharp_v$ is symmetric, its image in $T_{\pi(v)}M$ is $g_v$-orthogonal to $v$ and $C^\sharp_v(v,\cdot) = C^\sharp_v(\cdot,v) = 0$.*

*Proof.* From the definition of $C^\sharp$ and Proposition 1.2.7. □

### 1.2.2 Randers metrics and Zermelo data

The simplest non-Riemannian example of Finsler metrics is the disturbance $F = \|\cdot\| + \omega$ of a Riemannian metric $\|\cdot\|$ on $M$ by a 1-form $\omega \in \mathfrak{X}^*(M)$ satisfying $\|\omega\| < 1$. By the definition (1.16), for $v \in \mathbb{A}$ and $x,y \in T_{\pi(v)}M$ we have

$$\begin{aligned} g_v(x,y) &= \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\left(\|v+sx+ty\| + \omega(v+sx+ty)\right)^2\Big|_{s,t=0} \\ &= \frac{\partial}{\partial s}\left(\|v+sx\| + \omega(v+sx)\right)\left(\tfrac{\langle v+sx,y\rangle}{\|v+sx\|} + \omega(y)\right)\Big|_{s=0} \\ &= \left(\tfrac{\langle v,x\rangle}{\|v\|} + \omega(x)\right)\left(\tfrac{\langle v,y\rangle}{\|v\|} + \omega(y)\right) + \tfrac{F(v)}{\|v\|}\left(\langle x,y\rangle - \tfrac{\langle v,x\rangle\langle v,y\rangle}{\|v\|^2}\right) \end{aligned} \tag{1.20}$$

where $\langle\cdot,\cdot\rangle$ is the metric tensor associated to $\|\cdot\|$. As mentioned in Section 1.1, $\omega|_{T_{\pi(v)}M} = \langle w,\cdot\rangle$ for some $w \in T_{\pi(v)}M$, allowing us to write

$$g_v(x,y) = F\left(\tfrac{v}{\|v\|}\right)\left(\langle x,y\rangle - \left\langle \tfrac{v}{\|v\|},x\right\rangle\left\langle \tfrac{v}{\|v\|},y\right\rangle\right) + \left\langle \tfrac{v}{\|v\|}+w,x\right\rangle\left\langle \tfrac{v}{\|v\|}+w,y\right\rangle. \tag{1.21}$$

The data $(\langle\cdot,\cdot\rangle,\omega)$ is related to but generally different from the Zermelo data of $F$: for some other Euclidean product $\langle\cdot,\cdot\rangle_b$ on $T_{\pi(v)}M$ and $w_b \in T_{\pi(v)}M$ non-unitary for $\|\cdot\|_b$, we can arrange that

$$\|x\|^2 = \frac{\|x\|_b^2}{1-\|w_b\|_b^2} + \frac{\langle w_b,x\rangle_b^2}{(1-\|w_b\|_b^2)^2}$$

or equivalently

$$\langle\cdot,\cdot\rangle = \frac{\langle\cdot,\cdot\rangle_b}{1-\|w_b\|_b^2} + \frac{\langle w_b,\cdot\rangle_b\langle w_b,\cdot\rangle_b}{(1-\|w_b\|_b^2)^2}$$

by setting $\langle\cdot,\cdot\rangle_b^2$ to $\langle\cdot,\cdot\rangle^2 - \omega^2$ up to a factor of $1-\|w\|^2$ where $w \in T_{\pi(v)}M$ verifies $\omega = \langle w,\cdot\rangle$, and

$$\omega = \frac{-\langle w_b,\cdot\rangle_b}{1-\langle w_b,w_b\rangle_b}$$

by setting $w_b$ to $-w$ up to a factor of $\lambda = (1-\|w\|^2)^{-1}$, as indeed

$$\frac{\|x\|_b^2}{1-\|w_b\|_b^2} + \frac{\langle w_b, x\rangle_b^2}{(1-\|w_b\|_b^2)^2} = \frac{\|x\|^2 - \omega^2(x)}{\lambda(1-\frac{\|w_b\|^2-\omega^2(w_b)}{\lambda})} + \frac{(\langle w_b, x\rangle - \omega(w_b)\omega(x))^2}{\lambda^2(1-\frac{\|w_b\|^2-\omega^2(w_b)}{\lambda})^2}$$
$$= \frac{\|x\|^2-\omega^2(x)}{\lambda(1-\lambda(\|w\|^2-\omega^2(w)))} + \frac{(\langle w,x\rangle - \omega(w)\omega(x))^2}{(1-\lambda(\|w\|^2-\omega^2(w)))^2}$$
$$= \|x\|^2 - \omega^2(x) + \lambda^2(\omega(x) - \|w\|^2\omega(x))^2 = \|x\|^2$$

noting that by definition $\|w\|^2 - \omega^2(w) = \lambda^{-1}\|w\|^2$. The Randers metric then takes the form

$$F(v) = \sqrt{\frac{\|v\|_b^2}{1-\langle w_b, w_b\rangle_b} + \left(\frac{\langle w_b, v\rangle_b}{1-\langle w_b, w_b\rangle_b}\right)^2} - \frac{\langle w_b, v\rangle_b}{1-\langle w_b, w_b\rangle_b},$$

which incidentally solves the quadratic equation

$$F^2(v) - \langle v - F(v)w_b, v - F(v)w_b\rangle_b = 0$$

more revealingly written as

$$\left\|\tfrac{v}{F(v)} - w_b\right\|_b = 1\,. \tag{1.22}$$

That is to say Randers metrics are precisely those Finsler metrics whose indicatrix is the Euclidean sphere of some Riemannian metric translated along some vector field.

The pair $(\langle\cdot,\cdot\rangle_b, W_b)$ of the Riemannian metric tensor $\langle\cdot,\cdot\rangle_b$ associated to $\|\cdot\|_b$ and the vector field of that (smooth) choice of translations $w_b = W_b(\pi(v))$ for each $v \in TM$ are called the Zermelo data of $F = \|\cdot\| + \omega$, referencing the optimal control problem proposed in 1931 by Ernst Zermelo in which a ship navigates under external forces of wind or current.

For time independent forces, we can model this problem with precisely the Randers metric described here, in which the Euclidean spheres represent the isotropic reach of the ship in the absence of wind and the translation vector field the wind itself.

**Proposition 1.2.11.** *Let $F$ be a Randers metric on $M$ with Zermelo data $(\langle\cdot,\cdot\rangle, W)$. Then*

$$g_v(\tfrac{v}{F(v)}, \cdot) = \frac{\langle\frac{v}{F(v)} - W, \cdot\rangle}{\langle\frac{v}{F(v)} - W, \frac{v}{F(v)}\rangle}\,. \tag{1.23}$$

*Proof.* By property of the indicatrix, every $w \in T_{\pi(v)}M$ can be respresented as $w = \rho + \lambda\frac{v}{F(v)}$ for $\lambda \in \mathbb{R}$ and some $\rho \in T_{\pi(v)}M$ tangent to the indicatrix $F^{-1}(\{1\})$ at $\frac{v}{F(v)}$. Since the indicatrix is the translation by $W$ of that of $\langle\cdot,\cdot\rangle$, we have $g_v(\frac{v}{F(v)}, \rho) = \frac{1}{2}\dot{(\partial F)}_v(\rho) = 0$ and $= \langle\frac{v}{F(v)} - W, \rho\rangle = 0$. Then, by Proposition 1.2.3,

$$g_v(\tfrac{v}{F(v)}, w) = g_v(\tfrac{v}{F(v)}, \rho + \lambda\tfrac{v}{F(v)}) = g_v(\tfrac{v}{F(v)}, \lambda\tfrac{v}{F(v)}) = \lambda\frac{g_v(v,v)}{F^2(v)} = \lambda$$
$$= \lambda\frac{\langle\frac{v}{F(v)} - W, \frac{v}{F(v)}\rangle}{\langle\frac{v}{F(v)} - W, \frac{v}{F(v)}\rangle} = \frac{\langle\frac{v}{F(v)} - W, \lambda\frac{v}{F(v)}\rangle}{\langle\frac{v}{F(v)} - W, \frac{v}{F(v)}\rangle} = \frac{\langle\frac{v}{F(v)} - W, w\rangle}{\langle\frac{v}{F(v)} - W, \frac{v}{F(v)}\rangle}\,.$$

□

**Proposition 1.2.12.** *Let $F$ be a Randers metric on $M$ with Zermelo data $(\langle\cdot,\cdot\rangle, W)$ and $x,y \in T_{\pi(v)}M$ tangent to the indicatrix $F^{-1}(\{1\})$ at $\frac{v}{F(v)}$. Then*

$$g_v(x,y) = \frac{\langle x,y\rangle}{\langle \frac{v}{F(v)} - W, \frac{v}{F(v)}\rangle}\,. \tag{1.24}$$

*Proof.* The condition that $x,y$ be tangent to the indicatrix can be expressed as $\langle \frac{v}{F(v)} - W, x\rangle = \langle \frac{v}{F(v)} - W, y\rangle = 0$ allowing us substitute $\langle \frac{v}{F(v)}, x\rangle = \langle W, x\rangle$ and $\langle \frac{v}{F(v)}, y\rangle = \langle W, y\rangle$ in equation (1.21). Note that $\langle \frac{v}{F(v)} - W, \frac{v}{F(v)} - W\rangle = 1$ and thus $1 - \langle W,W\rangle + \langle \frac{v}{F(v)}, W\rangle = \langle \frac{v}{F(v)} - W, \frac{v}{F(v)}\rangle$, such that

$$\begin{aligned}
\frac{F(v)}{\sqrt{\frac{\|v\|^2}{1-\langle W,W\rangle} + \frac{\langle v,W\rangle^2}{(1-\langle W,W\rangle)^2}}} &= \frac{F(v)}{\sqrt{\frac{\|v\|^2}{1-\langle W,W\rangle} + \frac{\langle v,W\rangle^2}{(1-\langle W,W\rangle)^2}} - \frac{\langle v,W\rangle}{1-\langle W,W,\rangle} + \frac{\langle v,W\rangle}{1-\langle W,W\rangle}} \\
&= \frac{F(v)}{F(v) + \frac{\langle v,W\rangle}{1-\langle W,W\rangle}} = \frac{1}{1 + \frac{\langle \frac{v}{F(v)},W\rangle}{1-\langle W,W\rangle}} = \frac{1-\langle W,W\rangle}{1-\langle W,W\rangle + \langle \frac{v}{F(v)}, W\rangle} = \frac{1-\langle W,W\rangle}{\langle \frac{v}{F(v)} - W, \frac{v}{F(v)}\rangle}\,.
\end{aligned}$$

Similar identities such as $1 + \frac{\langle W, \frac{v}{F(v)}\rangle}{1-\langle W,W\rangle} = \frac{\langle \frac{v}{F(v)} - W, \frac{v}{F(v)}\rangle}{1-\langle W,W\rangle}$ and $1 + \frac{1-\langle W,W\rangle}{\langle \frac{v}{F(v)} - W, \frac{v}{F(v)}\rangle} = \frac{\langle W, \frac{v}{F(v)}\rangle}{\langle \frac{v}{F(v)} - W, \frac{v}{F(v)}\rangle}$ account for the remaining simplifications that allow us to rewrite equation (1.21) under the form

$$\begin{aligned}
g_v(x,y) &= \frac{1-\langle W,W\rangle}{\langle \frac{v}{F(v)} - W, \frac{v}{F(v)}\rangle}\left(\frac{\langle x,y\rangle}{1-\langle W,W\rangle} + \frac{\langle W,x\rangle\langle W,y\rangle}{(1-\langle W,W\rangle)^2}\right) \\
&\quad - \frac{\langle W, \frac{v}{F(v)}\rangle}{\langle \frac{v}{F(v)} - W, \frac{v}{F(v)}\rangle^3}\left(\langle x, \tfrac{v}{F(v)}\rangle + \frac{\langle W,x\rangle\langle W, \frac{v}{F(v)}\rangle}{1-\langle W,W\rangle}\right)\left(\langle y, \tfrac{v}{F(v)}\rangle + \frac{\langle W,y\rangle\langle W, \frac{v}{F(v)}\rangle}{1-\langle W,W\rangle}\right) + \frac{\langle W,x\rangle\langle W,y\rangle}{(1-\langle W,W\rangle)^2} \\
&= \frac{\langle x,y\rangle}{\langle \frac{v}{F(v)} - W, \frac{v}{F(v)}\rangle} - \frac{\langle W, \frac{v}{F(v)}\rangle\langle W,x\rangle\langle W,y\rangle}{\langle \frac{v}{F(v)} - W, \frac{v}{F(v)}\rangle(1-\langle W,W\rangle)^2} + \frac{\langle W, \frac{v}{F(v)}\rangle\langle W,x\rangle\langle W,y\rangle}{\langle \frac{v}{F(v)} - W, \frac{v}{F(v)}\rangle(1-\langle W,W\rangle)^2}\,.
\end{aligned}$$

□

**Proposition 1.2.13.** *Let $F$ be a Randers metric on $M$ with Zermelo data $(\langle\cdot,\cdot\rangle, W)$, $v \in \mathbb{A}$ and arbitrary vectors $\tilde{x} = x + \lambda_x v$, $\tilde{y} = y + \lambda_y v$, $\tilde{z} = z + \lambda_z v$ given in terms of $x,y,z \in T_{\frac{v}{F(v)}}F^{-1}(\{1\})$, $\lambda_x, \lambda_y, \lambda_z \in \mathbb{R}$. Then the Cartan tensor of $F$ is given by*

$$C_v(\tilde{x},\tilde{y},\tilde{z}) = -\frac{\langle x,y\rangle\langle z,v\rangle + \langle z,x\rangle\langle y,v\rangle + \langle y,z\rangle\langle x,v\rangle}{2F^2(v)\langle \frac{v}{F(v)} - W, \frac{v}{F(v)}\rangle^2}\,. \tag{1.25}$$

*Proof.* Note that by property of the Cartan tensor we have $C_v(\tilde{x},\tilde{y},\tilde{z}) = C_v(x,y,z)$. Let $x_t, y_t$ be for $t \in \mathbb{R}$ the parts of $x,y$ tangent to $F^{-1}(\{1\})$ at $\frac{v+tz}{F(v+tz)}$, explicitly

$$x = g_{v+tz}(x, \tfrac{v+tz}{F(v+tz)})\frac{v+tz}{F(v+tz)} + x_t\,, \tag{1.26}$$

$$y = g_{v+tz}(y, \tfrac{v+tz}{F(v+tz)})\frac{v+tz}{F(v+tz)} + y_t\,. \tag{1.27}$$

By the previous proposition, $g_{v+tz}(x_t,y_t) = \frac{\langle x_t,y_t\rangle}{\langle \frac{v+tz}{F(v+tz)} - W, \frac{v+tz}{F(v+tz)}\rangle}$. Since $\langle \frac{v+tz}{F(v+tz)} - W, \frac{v+tz}{F(v+tz)} - W\rangle = 1$ and $\mathrm{d}_v F = \frac{\mathrm{d}_v F^2}{2F(v)} = g_v(\frac{v}{F(v)}, \cdot)$, we may write

$$\begin{aligned}
\frac{\partial}{\partial t}\langle \tfrac{v+tz}{F(v+tz)} - W, \tfrac{v+tz}{F(v+tz)}\rangle^{-1}\Big|_{t=0} &= -\frac{1}{\langle \frac{v}{F(v)} - W, \frac{v}{F(v)}\rangle^2}\frac{\partial}{\partial t}\left(1 + \langle \tfrac{v+tz}{F(v+tz)} - W, W\rangle\right)\Big|_{t=0} \\
&= -\frac{\langle z - g_v(\frac{v}{F(v)}, z), W\rangle}{F(v)\langle \frac{v}{F(v)} - W, \frac{v}{F(v)}\rangle^2} = -\frac{\langle z,v\rangle}{F^2(v)\langle \frac{v}{F(v)} - W, \frac{v}{F(v)}\rangle^2}\,,
\end{aligned}$$

and compute

$$C_v(x,y,z) = \frac{1}{2}\frac{\partial g_{v+tz}(x,y)}{\partial t}\Big|_{t=0} = \frac{1}{2}\frac{\partial g_{v+tz}(x_t,y_t)}{\partial t}\Big|_{t=0}$$
$$= \frac{1}{2\langle\frac{v}{F(v)}-W,\frac{v}{F(v)}\rangle}\frac{\partial\langle x_t,y_t\rangle}{\partial t}\Big|_{t=0} - \frac{\langle z,v\rangle}{2F^2(v)\langle\frac{v}{F(v)}-W,\frac{v}{F(v)}\rangle^2}$$

where, recalling that $g_v(v,x)=g_v(v,y)=g_v(v,z)=0$,

$$\frac{\partial\langle x_t,y_t\rangle}{\partial t}\Big|_{t=0} = \frac{\partial}{\partial t}\Big(g_{v+tz}(x,\tfrac{v+tz}{F(v+tz)})\langle\tfrac{v+tz}{F(v+tz)},y\rangle + g_{v+tz}(y,\tfrac{v+tz}{F(v+tz)})\langle x,\tfrac{v+tz}{F(v+tz)}\rangle$$
$$+g_{v+tz}(x,\tfrac{v+tz}{F(v+tz)})g_{v+tz}(y,\tfrac{v+tz}{F(v+tz)})\langle\tfrac{v+tz}{F(v+tz)},\tfrac{v+tz}{F(v+tz)}\rangle\Big)\Big|_{t=0}$$
$$= -\frac{g_v(x,z)\langle y,v\rangle+g_v(y,z)\langle x,v\rangle}{F^2(v)}.$$

Conclude by $g_v(\cdot,z) = \frac{\langle\cdot,z\rangle}{\langle\frac{v}{F(v)}-W,\frac{v}{F(v)}\rangle}$ for $x,y$ using again the previous proposition. □

Though we will later prove in equation (2.5) a version of one of these propositions for general pseudo-Finsler metrics, the case of Randers metrics is worth noting for being computable directly with relatively few steps in terms of the Zermelo data.

The general case follows essentially the same outline for the Cartan tensor $C_v(x,y,z) = (\dot{\partial}g(x,y))_v(z) = \frac{\partial}{\partial t}g_{v+tz}(x,y)|_{t=0}$ of the fundamental metric tensor $g$ associated to $L$ by decomposing the entries $x$ and $y$ along the direction $v+tz$, by definition $g_{v+tz}$-orthogonal to the indicatrix of $g$ in the fibre $T_{\pi(v)}M$. There, we can use the previous expression of $g$ for vectors tangent to the indicatrix on one part of the expression, and vanish the rest when evaluating at $t=0$.

That of the fundamental metric tensor $g$ itself is somewhat more subtle. The particularity of the Randers metrics is that the indicatrix $\Sigma$ of $L=F^2$ is a Euclidean sphere for the inner product $\langle\cdot,\cdot\rangle$ of the Zermelo data. Along the vector $\frac{v}{F(v)}-W$ $\langle\cdot,\cdot\rangle$-orthogonal to $\Sigma$ at $\frac{v}{F(v)}$, this product is incidentally the second fundamental form of $\Sigma$, which as recalled in Proposition 2.3 is related by a proportionality factor of precisely $\langle\frac{v}{F(v)}-W,\frac{v}{F(v)}\rangle$ to its second fundamental form along $v$ itself. Equation (1.24) suggests $-g_v$ is that second fundamental form. The more general proof consists of a sort of shortcut, in which we show this fact directly, at the cost of ending up with a more obscure expression which does not follow from a naive, intuitive approach.

Naturally, expressing $g$ and $2C=\dot{\partial}g$ in terms of the second fundamental form of the indicatrix $\Sigma$ of $L$ is, in hindsight, the best we could do; if we allow $\Sigma$ to take any convex shape beside the Euclidean sphere of some specified inner product, then we must embrace the full generality of our somewhat less transparent equation (2.5). Hopefully, in light of equations (1.24) and (1.25), their incidental nature should feel less arbitrary.

## 1.3 Anisotropic tensor calculus

In this section, let us review elementary notions of anisotropic linear connections, anisotropic tensor derivation and anisotropic curvature tensor associated to a connection.

First recall that $\frac{\partial}{\partial x^i}$ as a differential operator does not generally define on its own the differentiation of a vector field $Y=Y^j\frac{\partial}{\partial x^j}$ along another $X=X^i\frac{\partial}{\partial x^i}$ by the mere differentiation of the coordinate coefficient $Y^j$, as indeed $Y^j = \mathrm{d}x^j\cdot Y = \frac{\partial x^j}{\partial\bar{x}^k}\mathrm{d}\bar{x}^k\cdot Y = \frac{\partial x^j}{\partial\bar{x}^k}\bar{Y}^k$ and similarly $X^i = \frac{\partial x^i}{\partial\bar{x}^l}\bar{X}^l$ for a generic alternative set of local coordinate functions $\bar{x}^i$ by the transformation rule (1.9), and thus by (1.10) we have

$$X(Y^j)\frac{\partial}{\partial x^j} = X^i\frac{\partial Y^j}{\partial x^i}\frac{\partial}{\partial x^j} = X^i\frac{\partial x^j}{\partial \bar{x}^k}\frac{\partial \bar{Y}^k}{\partial x^i}\frac{\partial}{\partial x^j} + X^i\bar{Y}^k\frac{\partial^2 x^j}{\partial x^i\partial \bar{x}^k}\frac{\partial}{\partial x^j} = \bar{X}^l\frac{\partial \bar{Y}^k}{\partial \bar{x}^l}\frac{\partial}{\partial \bar{x}^k} + \bar{X}^l\bar{Y}^k\frac{\partial^2 x^j}{\partial \bar{x}^l\partial \bar{x}^k}\frac{\partial \bar{x}^m}{\partial x^j}\frac{\partial}{\partial \bar{x}^m}\,.$$

A more robust differentiation formula would therefore require coefficients $\Gamma^m_{lk}$ capable of absorbing the $\frac{\partial^2 x^j}{\partial \bar{x}^l\partial \bar{x}^k}\frac{\partial \bar{x}^m}{\partial x^j}$ term into an expression satisfying the coordinate transformation rule

$$X^i\frac{\partial Y^j}{\partial x^i}\frac{\partial}{\partial x^j} + X^iY^j\Gamma^k_{ij}\frac{\partial}{\partial x^k} = \bar{X}^i\frac{\partial \bar{Y}^j}{\partial \bar{x}^i}\frac{\partial}{\partial \bar{x}^j} + \bar{X}^i\bar{Y}^j\bar{\Gamma}^k_{ij}\frac{\partial}{\partial \bar{x}^k}\,. \tag{1.28}$$

The map $(X,Y) \mapsto X^i\frac{\partial Y^j}{\partial x^i}\frac{\partial}{\partial x^j} + X^iY^j\Gamma^k_{ij}\frac{\partial}{\partial x^k}$ typically denoted by $\nabla\colon (X,Y)\mapsto \nabla_X Y$, or alternatively the 1-form $Y \mapsto (\frac{\partial Y^j}{\partial x^i}\frac{\partial}{\partial x^j} + Y^j\Gamma^k_{ij}\frac{\partial}{\partial x^k})\mathrm{d}x^i$ denoted by $Y\mapsto \nabla Y$, defines a connection, or covariant derivation, for every choice of such coefficients $\Gamma^k_{ij}$, and all connections can be defined in this way in terms of local coordinate coefficients.

The most sparing definition of anisotropic connections is to then replace the coefficients $\Gamma^k_{ij}\in C^\infty(M)$, also known as *Christoffel symbols*, with anisotropic coefficients $\Gamma^k_{ij}\in C^\infty(TM)$ such that $\nabla_X Y$ need not be constant on each tangent space. This provides us with the degree of freedom required for the almost $g$-compatible condition (equivalently the metric-preserving condition $\nabla g = 0$ defined later), which reads for each admissible $v$ with a (locally admissible) extension $V\in\mathfrak{X}(M)$ and in the neighbourhood of $\pi(v)$

$$Xg_V(Y,Z) = g_V(\nabla^V_X Y, Z) + g_V(Y,\nabla^V_X Z) + 2C_V(Y,Z,\nabla^V_X V)\,, \tag{1.29}$$

and would otherwise be doomed by the mere fact that, a priori, the sum of the first two terms $g_v(w_Y,Z)+g_v(Y,w_Z)$ of the right hand side is not constant when varying $v$ in its tangent space for a fixed choice of $w_Y,w_Z\in T_{\pi(v)}M$, while $Xg_V(Y,Z) - 2C_V(Y,Z,\nabla^V_X V)$ is, at $\pi(v)$, independent of the choice of extension $V$. Indeed, similarly to (1.13), we can compute

$$Xg_V(Y,Z) - 2C_V(Y,Z,\nabla^V_X V) = \underbrace{X^i\frac{\partial g_V(Y,Z)}{\partial x^i} - X^i\frac{\partial V^j}{\partial x^i}\frac{\partial g(Y,Z)}{\partial y^j}|_V}_{X^i\frac{\partial' g(Y,Z)}{\partial x^i}|_V} - X^iV^j\Gamma^k_{ij}(V)\frac{\partial g(Y,Z)}{\partial y^j}|_V\,. \tag{1.30}$$

For the same reason, no choice of torsion-free anisotropic coefficients $\Gamma^k_{ij}\in C^\infty(TM)$ can systematically overcome the dependence of $Xg_V(Y,Z)$ at $\pi(v)$ on the choice of extension $V$ to obtain a $g$-compatibility condition not featuring a Cartan tensor term in $\nabla^v_X V$. The best we can hope for is equation (1.29).

The Koszul formula resulting from (1.29) produces $2\Gamma^k_{ij}g_{kl} = \frac{\delta g_{lj}}{\delta x^i} + \frac{\delta g_{il}}{\delta x^j} - \frac{\delta g_{ij}}{\delta x^l}$ immediately, see (1.40), replacing $\frac{\partial}{\partial x^i}$ of the classical coordinate expression for the coefficients of the Levi-Civita connection, as promised, by $\frac{\delta}{\delta x^i} = \frac{\partial'}{\partial x^i} - y^j\Gamma^k_{ij}\frac{\partial}{\partial y^k}$ acting on $C^\infty(TM)$. Proposition 1.2.7 suffices to then solve for

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} + (2y^py^q\gamma^j_{pq}C^k_{ij} - y^j\gamma^k_{ij})\frac{\partial}{\partial y^k}$$

where $\gamma^k_{ij}$ stands for the *formal* classical Christoffel symbols $\frac{g^{kl}}{2}(\frac{\partial g_{kj}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k})$.

This fortunate conjunction allows for a sketch of proof essentially identical to the Riemannian case, throughout. Furthermore, the occurrence of $\frac{\delta}{\delta x^i}$ is far from a coincidence as it, rather unsurprisingly after all, also appears in the coefficients of the curvature tensor associated to the Chern connection, taking the shape of the familiar expression involving $\omega\wedge\omega$, see (1.52), where $\omega$ is the connection form with coefficient expansion $\omega^k_j = \Gamma^k_{ij}\mathrm{d}x^i$.

### 1.3.1 Anisotropic Linear connections

Let us define connections more rigorously by listing its defining properties axiomatically, making abstraction of local coordinates. Recall for the following definition that $\mathbb{A}$ (for *admissible*) designates an open set of the slit tangent bundle, but note that in order to define a connection it need not be a conic subset. Since we will later use a metric-preserving connection, we will however require that $\mathbb{A}$ be the domain of definition of the pseudo-Finsler metric when defining the Chern connection.

**Definition 1.3.1.** *Anisotropic linear connections, denoted here generically by* $\nabla$*, take as input* $X,Y\in\mathfrak{X}(M)$ *and produce the anisotropic vector field* $\nabla_X Y$*, evaluating at each admissible* $v\in\mathbb{A}$ *as* $\nabla^v_X Y\in T_{\pi(v)}M$*, and satisfying the following conditions: for* $Z\in\mathfrak{X}(M)$ *and* $f\in C^\infty(M)$*,*

1. $\nabla^v Y$ *is* $C^\infty(M)$*-linear:*

$$\nabla^v_{X+Z}Y=\nabla^v_X Y+\nabla^v_Z Y\,,$$
$$\nabla^v_{fX}Y=f(\pi(v))\nabla^v_X Y\,,$$

2. $\nabla^v_X$ *is a derivation:*

$$\nabla^v_X(Y+Z)=\nabla^v_X Y+\nabla^v_X Z\,,$$
$$\nabla^v_X(fY)=(Xf)Y(\pi(v))+f(\pi(v))\nabla^v_X Y\,. \tag{1.31}$$

For local representations $X=X^i\frac{\partial}{\partial x^i}$ and $Y=Y^i\frac{\partial}{\partial x^i}$, we can verify the decomposition (1.28)

$$\begin{aligned}\nabla^v_X Y&=\nabla^v_{X^i\frac{\partial}{\partial x^i}}Y=X^i(\pi(v))\nabla^v_{\frac{\partial}{\partial x^i}}Y=X^i(\pi(v))\nabla^v_{\frac{\partial}{\partial x^i}}(Y^j\tfrac{\partial}{\partial x^j})\\&=X^i(\pi(v))\tfrac{\partial Y^j}{\partial x^i}(\pi(v))\tfrac{\partial}{\partial x^j}(\pi(v))+X^i(\pi(v))Y^j(\pi(v))\Gamma^k_{ij}(v)\tfrac{\partial}{\partial x^k}(\pi(v))\,,\end{aligned} \tag{1.32}$$

where the *anisotropic Christoffel symbols* $\Gamma^k_{ij}\in C^\infty(TM)$ are given by:

$$\nabla^v_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}=\Gamma^k_{ij}(v)\frac{\partial}{\partial x^k}(\pi(v))\,.$$

As a result, concordantly to what was mentioned earlier, $\nabla$ is completely determined by the Christoffel symbols $\Gamma^k_{ij}$, and each choice of such coefficients locally defines a connection in the coordinate neighbourhood.

Note that the expression for (1.32) is independent of the values of $X$ away from $\pi(v)$, we shall write $\nabla^v_X Y$ and $\nabla^v_{X(\pi(v))}Y$ interchangeably to emphasize this fact accordingly.

**Definition 1.3.2.** *The* Lie brackets $[\cdot,\cdot]$ *stand for the commutator of* $\mathfrak{X}(M)$ *acting as a set of derivations on* $C^\infty(M)$*, in the sense that, given* $X,Y\in\mathfrak{X}(M)$ *and* $f\in C^\infty(M)$*, we have*

$$[X,Y](f)=X(Y(f))-Y(X(f))\,.$$

In local coordinates, $[X,Y]$ is the vector field

$$\left(X^i\frac{\partial Y^j}{\partial x^i}-Y^i\frac{\partial X^j}{\partial x^i}\right)\frac{\partial}{\partial x^j} \tag{1.33}$$

due to the commutativity of the differential operator $\frac{\partial^2}{\partial x^i\partial x^j}=\frac{\partial}{\partial x^i}\circ\frac{\partial}{\partial x^j}$ on $C^\infty(M)$. Reversing the indices of the coefficients $\Gamma^k_{ij}$ as $\Gamma^k_{ji}$ in the decomposition of $\nabla_X Y$ defines a connection, namely

$$X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial}{\partial x^j} + X^i Y^j \Gamma^k_{ji} \frac{\partial}{\partial x^k} = \underbrace{(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i}) \frac{\partial}{\partial x^j}}_{[X,Y]} + \underbrace{Y^j \frac{\partial X^i}{\partial x^j} \frac{\partial}{\partial x^i} + X^i Y^j \Gamma^k_{ji} \frac{\partial}{\partial x^k}}_{\nabla_Y X}.$$

Torsion is the difference between them, that is,

$$\nabla_X Y - \nabla_Y X - [X,Y] = X^i Y^j (\Gamma^k_{ij} - \Gamma^k_{ji}) \frac{\partial}{\partial x^k},$$

measuring the failure of the coefficients $\Gamma^k_{ij}$ of $\nabla$ to be symmetric in its lower indices, or equivalently the failure to be commutative for vector fields with vanishing Lie brackets, such as the vector fields $\frac{\partial}{\partial x^i}$ induced by local coordinate functions $x^i$. When the torsion is zero, the connection is said to be torsion-free.

Any arbitrary connection $\nabla$ with Christoffel symbols $\Gamma^k_{ij}$ can be written as the sum of the torsion-free connection whose coefficients are $\frac{1}{2}(\Gamma^k_{ij} + \Gamma^k_{ji})$ and half the torsion tensor of $\nabla$. In that sense, we can identify $\nabla$ with the pair of this torsion-free connection, unique for our given $\nabla$, and the torsion of $\nabla$. The set of anisotropic linear connections can therefore also be thought of as that of torsion-free anisotropic linear connections perturbed by specified torsion tensors, in a one-to-one correspondence.

Given an anisotropic linear connection $\nabla$ with local coefficients $\Gamma^k_{ij}$, we can rewrite the chain rule (1.13) as follows: for $X = \frac{\partial}{\partial x^i}$ and an extension $V \in \mathfrak{X}(M)$ of $v$, forcibly *nudge* $\frac{\partial V^j}{\partial x^i}$ by $V^j \Gamma^k_{ij}(V) \frac{\partial}{\partial x^k}$ so as to provoke (1.28), for the representation

$$\underbrace{\frac{\partial h(V)}{\partial x^i}(\pi(v))}_{X(h(V))(\pi(v))} = \underbrace{\frac{\partial' h}{\partial x^i}(v) - \left(v^j \Gamma^k_{ij}(v)\right) \frac{\partial h}{\partial y^k}(v)}_{\frac{\delta h}{\delta x^i}(v)} + \underbrace{\left(\frac{\partial V^j}{\partial x^i}(\pi(v)) + v^j \Gamma^k_{ij}(v)\right) \frac{\partial h}{\partial y^j}(v)}_{(\dot\partial h)_v(\nabla^v_X V)} \tag{1.34}$$

more robust across all choices of local coordinates. The first term $\frac{\delta h}{\delta x^i} = \frac{\partial' h}{\partial x^i} - y^j \Gamma^k_{ij} \frac{\partial h}{\partial y^k}$, manifestly independent from the choice of extension $V$, shall be written $\nabla_X h$ by convention and shall consist, as in (1.30), of $X(h(V))$ to which we substract the vertical derivative $\dot\partial h$ along $\nabla^V_X V = (\frac{\partial V^k}{\partial x^i} + V^j \Gamma^k_{ij}(V)) \frac{\partial}{\partial x^k}$.

**Definition 1.3.3.** *For $v \in TM$ with any extension $V \in \mathfrak{X}(M)$ such that $V(\pi(v)) = v$, $x \in T_{\pi(v)}M$ and $h\colon TM \to \mathbb{R}$ a smooth anisotropic scalar field of $M$, let*

$$\nabla^v_x h = x(h \circ V)(\pi(v)) - (\dot\partial h)_v(\nabla^v_x V) \tag{1.35}$$

Similarly, from (1.14) we can differentiate $H(V)$ for some choice of $V \in \mathfrak{X}(M)$ extending $V(\pi(v)) = v$ and obtain along $X = \frac{\partial}{\partial x^i}$

$$\begin{aligned}\nabla^v_X(H(V)) &= \left(\frac{\partial H^k(V)}{\partial x^i}(\pi(v)) + H^j(v)\Gamma^k_{ij}(v)\right) \frac{\partial}{\partial x^k}(\pi(v)) \\ &= \underbrace{\left(\frac{\delta H^k}{\delta x^i}(v) + H^j(v)\Gamma^k_{ij}(v)\right) \frac{\partial}{\partial x^k}(\pi(v))}_{\nabla^v_X H} + \underbrace{(\dot\partial H^j)_v(\nabla^v_X V) \frac{\partial}{\partial x^j}(\pi(v))}_{(\dot\partial H)_v(\nabla^v_X V)}.\end{aligned}$$

By the same convention, the expression highlighted as $\nabla^v_X H$, manifestly independent of the choice of extension, consists of $\nabla^v_X(H(V))$ to which we rest the vertical derivative term.

**Definition 1.3.4.** *For each admissible $v$ with any locally admissible vector field $V$ such that $V(\pi(v)) = v$, $x \in T_{\pi(v)}M$ and $H\colon TM \to TM$ an anisotropic vector field of $M$ such that $\pi \circ H =$*

*$\pi$, let*

$$\nabla^v_x H = \nabla^v_x (H \circ V) - (\dot{\partial} H)_v (\nabla^v_x V) \tag{1.36}$$

*where the vertical derivative $\dot{\partial} H$ evaluates for any $w \in T_{\pi(v)}M$ as*

$$(\dot{\partial} H)_v(w) = \left.\frac{\partial H(v+tw)}{\partial t}\right|_{t=0}$$

*well defined on $T_{\pi(v)}M$.*

Having defined the covariant derivative of anisotropic scalar fields and anisotropic vector fields, respectively anisotropic tensors of type $(0,0)$ and $(0,1)$, let us briefly extend the definition to cover anisotropic 1-forms $\omega$, of type $(1,0)$, which is to say the smooth choice of a 1-form $\omega_v$ on the tangent space $T_{\pi(v)}M$ for each (admissible) vector $v$.

**Definition 1.3.5.** *For each admissible $v$ with a locally admissible vector field $V$ such that $V(\pi(v)) = v$, $x \in T_{\pi(v)}M$ and $\omega\colon TM \to (TM)^*$ an anisotropic 1-form of $M$, let*

$$\nabla^v_x \omega = \nabla^v_x \omega_V - (\dot{\partial}\omega)_v(\nabla^v_x V)$$

*where the vertical derivative $\dot{\partial}\omega$ evaluates for any $w \in T_{\pi(v)}M$ as the 1-form on $T_{\pi(v)}M$*

$$(\dot{\partial}\omega)_v(w)\colon x \mapsto \left.\frac{\partial \omega_{v+tw}(x)}{\partial t}\right|_{t=0}$$

*whose output is well defined as the derivative of the real-valued single real variable function $t \mapsto \omega_{v+tw}(x)$.*

More generally, this construction applies to all anisotropic tensors, and in particular to the anisotropic metric tensor $g$. The covariant derivative $\nabla_X g$ along $X \in \mathfrak{X}(M)$ will be that of the evaluation $g_V$ for some $V \in \mathfrak{X}(M)$ as an isotropic tensor on $M$, to which we rest the vertical derivative $(\dot{\partial} g)_V(\cdot,\cdot,\nabla^V_X V) = 2C_V(\cdot,\cdot,\nabla^V_X V)$.

Specifically for the anisotropic metric tensor $g$, the product rule for computing the derivative of real-valued single real variable functions allows us to expand in local coordinates

$$\begin{aligned}
\frac{\partial g_V(X,Y)}{\partial x^k} &= \frac{\partial X^i Y^j g_{ij}(V)}{\partial x^k} = \frac{\partial X^i}{\partial x^k} Y^j g_{ij}(V) + X^i \frac{\partial Y^j}{\partial x^k} g_{ij}(V) + X^i Y^j \frac{\partial' g_{ij}}{\partial x^k}|_V + X^i Y^j \frac{\partial V^l}{\partial x^k}\frac{\partial g_{ij}}{\partial y^l}|_V \\
&= \Big(\frac{\partial X^i}{\partial x^k} + X^l \Gamma^i_{kl}(V)\Big) Y^j g_{ij}(V) + X^i \Big(\frac{\partial Y^j}{\partial x^k} + Y^l \Gamma^j_{kl}(V)\Big) g_{ij}(V) + 2X^i Y^j \Big(\frac{\partial V^l}{\partial x^k} + V^m \Gamma^l_{km}(V)\Big) C_{ijl}(V) \\
&\quad + X^i Y^j \Big(\frac{\partial' g_{ij}}{\partial x^k}|_V - 2V^l \Gamma^m_{kl}(V) C_{ijl}(V) - \Gamma^l_{ki}(V) g_{lj}(V) - \Gamma^l_{kj}(V) g_{il}(V)\Big) \\
&= g_V(\nabla^V_{\frac{\partial}{\partial x^k}} X, Y) + g_V(X, \nabla^V_{\frac{\partial}{\partial x^k}} Y) + 2C_V(X,Y,\nabla^V_{\frac{\partial}{\partial x^k}} V) \\
&\quad + X^i Y^j \left(\frac{\partial' g_{ij}}{\partial x^k} - 2y^l \Gamma^m_{kl} C_{ijl} - \Gamma^l_{ki} g_{lj} - \Gamma^l_{kj} g_{il}\right)(V),
\end{aligned} \tag{1.37}$$

using our definition of the Cartan tensor satisfying $2C = \dot{\partial} g$, such that

$$\begin{aligned}
Zg_V(X,Y) &= g_V(\nabla^V_Z X, Y) + g_V(X, \nabla^V_Z Y) + 2C_V(X,Y,\nabla^V_Z V) \\
&\quad + X^i Y^j Z^k \left(\frac{\partial' g_{ij}}{\partial x^k} - 2y^l \Gamma^m_{kl} C_{ijl} - \Gamma^l_{ki} g_{lj} - \Gamma^l_{kj} g_{il}\right)(V) \\
&= g_V(\nabla^V_Z X, Y) + g_V(X, \nabla^V_Z Y) + 2C_V(X,Y,\nabla^V_Z V) + (\nabla_Z g)_V(X,Y)
\end{aligned}$$

where $\nabla g$ is the tensor defined by its local coefficients

$$(\nabla_Z g)_V(X,Y) = X^i Y^j Z^k \left( \frac{\partial' g_{ij}}{\partial x^k} - 2y^l \Gamma^m_{kl} C_{ijl} - \Gamma^l_{ki} g_{lj} - \Gamma^l_{kj} g_{il} \right)(V) .$$

The same computation is possible for general anisotropic tensors of any type.

**Proposition 1.3.6.** *For every anisotropic tensor $s$ of type $(0,b)$, taking a (locally admissible) vector field $V$ and a $b$-tuple $(X,Y,\cdots,Z)$ of vector fields as input and producing a scalar field $s_V(X,Y,\cdots,Z)$ as output, the expression*

$$W s_V(X,Y,\cdots,Z) - s_V(\nabla^V_W X,Y,\cdots,Z) - s_V(X,\nabla^V_W Y,\cdots,Z) - \cdots - s_V(X,Y,\cdots,\nabla^V_W Z) \\ - (\dot\partial s(X,Y,\cdots,Z))_V(\nabla^V_W V) ,$$

*for every anisotropic tensor $T$ of type $(1,b)$, taking a (locally admissible) vector field $V$ and a $b$-tuple $(X,Y,\cdots,Z)$ of vector fields as input and producing a vector field $T_V(X,Y,\cdots,Z)$ as output, the expression*

$$\nabla^V_W T_V(X,Y,\cdots,Z) - T_V(\nabla^V_W X,Y,\cdots,Z) - T_V(X,\nabla^V_W Y,\cdots,Z) - \cdots - T_V(X,Y,\cdots,\nabla^V_W Z) \\ - (\dot\partial T(X,Y,\cdots,Z))_V(\nabla^V_W V)$$

*and more generally, when some or all of the inputs are 1-forms $\alpha,\beta,\cdots$ instead, the similar expression*

$$\nabla^V_W T_V(\alpha,\beta,\cdots,X,Y,\cdots,Z) - T_V(\nabla^V_W\alpha,\beta,\cdots,X,Y,\cdots,Z) - T_V(\alpha,\nabla^V_W\beta,\cdots,X,Y,\cdots,Z) \\ - \cdots - T_V(\alpha,\nabla^V_W\beta,\cdots\nabla^V_W X,Y,\cdots,Z) - T_V(\alpha,\nabla^V_W\beta,\cdots,X,\nabla^V_W Y,\cdots,Z) - \cdots \\ - T_V(\alpha,\nabla^V_W\beta,\cdots,X,Y,\cdots,\nabla^V_W Z) - (\dot\partial T(\alpha,\nabla^V_W\beta,\cdots,X,Y,\cdots,Z))_V(\nabla^V_W V)$$

*all define tensors.*

Note that in the case of a tensor of type $(0,0)$, that is to say an anisotropic scalar field $h$, this definition is consistent with our earlier definition (1.35) and justifies the notation $\nabla h$.

**Definition 1.3.7.** *Let $\nabla T\colon (V,W,X,Y,\cdots,Z) \mapsto (\nabla_W T)_V(X,Y,\cdots,Z)$ denote that tensor.*

By the definition (1.36) we may also write it as

$$\nabla^V_W T(X,Y,\cdots,Z) - T_V(\nabla^V_W X,Y,\cdots,Z) - T_V(X,\nabla^V_W Y,\cdots,Z) - \cdots - T_V(X,Y,\nabla^V_W Z) ,$$

taking on the appearance of an isotropic expression when considering the anisotropic function of $V$

$$\nabla_W T(X,Y,\cdots,Z) - T(\nabla^V_W X,Y,\cdots,Z) - T(X,\nabla^V_W Y,\cdots,Z) - \cdots - T(X,Y,\nabla^V_W Z) .$$

In relation with the anisotropic metric tensor $g$, defined on a set of admissible vectors, we need only require that the evaluation $T_v$ of an anistotropic tensor $T$ be defined for each admissible $v$, and the evaluation $T_V$ for a locally admissible $V$ on the neighbourhood where it is admissible. To make clear that this is not an issue, we shall evaluate expressions at a point whenever practical.

### 1.3.2 The Chern connection

Consider differentiating $g_V(X,Y) = g_V(X^i \frac{\partial}{\partial x^i}, Y^j \frac{\partial}{\partial x^j}) = X^i Y^j g_{ij}(V)$ as a smooth function on $M$ along $Z = Z^k \frac{\partial}{\partial x^k}$. By nudging as in (1.34) the terms featuring $\frac{\partial X^i}{\partial x^k}$, $\frac{\partial Y^j}{\partial x^k}$ and $\frac{\partial V^l}{\partial x^k}$ into the coordinate representations of $\nabla_Z X$, $\nabla_Z Y$ and $\nabla_Z V$. Let $v$ be an admissible vector, and $x,y,z \in T_{\pi(v)}M$. For a locally admissible vector field extension $V$ of $v$ and vector fields $X$ and $Y$ extending $x$ and $y$, recall from (1.37) that by definition of $\nabla g$

$$zg_V(X,Y) = g_v(\nabla^v_z X, y) + g_v(x, \nabla^v_z Y) + 2C_v(x,y,\nabla^v_z V) + (\nabla_z g)_v(x,y) \tag{1.38}$$

for every choice of an anisotropic linear connection $\nabla$. By substituting the variables, we also have for any vector field extension $Z$ of $z$

$$\begin{aligned} xg_V(Z,Y) &= g_v(\nabla^v_x Z, y) + g_v(z, \nabla^v_x Y) + 2C_v(z,y,\nabla^v_x V) + (\nabla_x g)_v(z,y) \\ yg_V(X,Z) &= g_v(\nabla^v_y X, z) + g_v(x, \nabla^v_y Z) + 2C_v(x,z,\nabla^v_y V) + (\nabla_y g)_v(x,z)\,, \end{aligned}$$

which means

$$\begin{aligned} & xg_V(Z,Y) + yg_V(X,Z) - zg_V(X,Y) - (\nabla_x g)_v(z,y) - (\nabla_y g)_v(x,z) \\ &\quad + (\nabla_z g)_v(x,y) - 2C_v(z,y,\nabla^v_x V) - 2C_v(x,z,\nabla^v_y V) + 2C_v(x,y,\nabla^v_z V) \\ &\qquad = g_v(\nabla^v_x Z - \nabla^v_z X, y) + g_v(\nabla^v_x Y + \nabla^v_y X, z) + g_v(X, \nabla^v_y Z - \nabla^v_z Y) \end{aligned}$$

where $\nabla^v_x Y - \nabla^v_y X = [X,Y]|_{\pi(v)} + \mathrm{Tor}_v(x,y)$ and $\nabla^v_x Y + \nabla^v_y X = 2\nabla^v_x Y + [Y,X]|_{\pi(v)} + \mathrm{Tor}_v(y,x)$ for Tor the torsion tensor associated to $\nabla$, therefore

$$\begin{aligned} & xg_V(Z,Y) + yg_V(X,Z) - zg_V(X,Y) - (\nabla_x g)_v(z,y) - (\nabla_y g)_v(x,z) \\ &\quad + (\nabla_z g)_v(x,y) + g_v([Z,X] + \mathrm{Tor}_v(z,x), y) + g_v(x, [Z,Y] + \mathrm{Tor}_v(z,y)) \\ &\qquad + g_v([X,Y] + \mathrm{Tor}_v(x,y), z) - 2C_v(z,y,\nabla^v_x V) \\ &\qquad\quad - 2C_v(x,z,\nabla^v_y V) + 2C_v(x,y,\nabla^v_z V) = 2g_v(\nabla^v_x Y, z)\,. \end{aligned}$$

For torsion-free connections $\nabla$ such that $\nabla g = 0$, all that remains is (1.39).

**Theorem 1.3.8** (Koszul Formula)**.** *Let $v$ be an admissible vector, and $V,X,Y,Z \in \mathfrak{X}(M)$ with $V$ locally admissible such that $V(\pi(v)) = v$ and $x,y,z$ their respective image at $\pi(v)$. Torsion-free anisotropic linear connections $\nabla$ satisfying $\nabla g = 0$ necessarily satisfy*

$$\begin{aligned} 2g_v(\nabla^v_x Y, z) &= xg_V(Z,Y) + yg_V(X,Z) - zg_V(X,Y) \\ &\quad + g_v([Z,X],y) + g_v(x,[Z,Y]) + g_v([X,Y],z) \\ &\qquad - 2C_v(z,y,\nabla^v_x V) - 2C_v(x,z,\nabla^v_y V) + 2C_v(x,y,\nabla^v_z V)\,. \end{aligned} \tag{1.39}$$

Note that the right hand side is $C^\infty(M)$-linear in $X$, noting that

$$g_v([Z,fX],y) = (\mathrm{d}f \cdot z) g_v(x,y) + g_v([Z,X],y)$$

cancels out the extra term from

$$-zg_V(fX,Y) = -(\mathrm{d}f \cdot z) g_v(x,y)\,,$$

and similarly for $g_v([fX,Y],Z)$ and $yg_V(X,Z)$, while setting $Y$ to $fY$ in (1.39) establishes by similar cancellations that any operator that satisfies the Koszul formula is a derivation for which the Leibniz rule (1.31) holds. Such a connection can be shown to be unique, and in that sense it is the anisotropic analogue to the Levi-Civita connection.

**Corollary 1.3.9.** *There exists a unique torsion-free anisotropic linear connection satisfying $\nabla g = 0$.*

*Proof.* As mentioned, it suffices that $\nabla$ satisfies the Koszul formula for it to necessarily be an anisotropic linear connection. By property of the Cartan tensor, from (1.39)

$$2g_v(\nabla^v_x V, z) = xg_V(Z,V) + vg_V(X,Z) - zg_V(X,V) \\ + g_v([Z,X],v) + g_v(x,[Z,V]) + g_v([X,V],z) - 2C_v(x,z,\nabla^v_v V),$$

and furthermore from (1.39)

$$2g_v(\nabla^v_v V, z) = 2vg_V(V,Z) - zg_V(V,V) + 2g_v([Z,V],v),$$

the right side of which is independent from $\nabla$, uniquely determining $\nabla^v_v V$ by non-degeneracy, which uniquely determines $\nabla^v V$ by reinserting into the previous equation, and finally $\nabla^v$ by reinserting into (1.39). □

Local coordinate computations can be performed in order to obtain the Christoffel symbols $\Gamma^k_{ij}$ of the Chern connection:

$$2g_V\left(\nabla^V_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right) = 2g_V\left(\Gamma^l_{ij}(V)\frac{\partial}{\partial x^l}, \frac{\partial}{\partial x^k}\right) = 2\Gamma^l_{ij}(V)g_{lk}(V),$$

and therefore

$$2\Gamma^l_{ij}(V)g_{lk}(V) = \frac{\delta g_{kj}}{\delta x^i}(V) + \frac{\delta g_{ik}}{\delta x^j}(V) - \frac{\delta g_{ij}}{\delta x^k}(V), \tag{1.40}$$

solving for $\Gamma^l_{ij}(V)$ in terms of the differential operators $\frac{\delta}{\delta x^k}|_V = \frac{\partial}{\partial x^k}|_V - V^l\Gamma^m_{kl}(V)\frac{\partial}{\partial y^m}$, up to applying the coefficients $g^{lk}$ of the matrix inverse to $g_{lk}$, such that

$$\Gamma^l_{ij} = \frac{g^{lk}}{2}\left(\frac{\delta g_{kj}}{\delta x^i} + \frac{\delta g_{ik}}{\delta x^j} - \frac{\delta g_{ij}}{\delta x^k}\right). \tag{1.41}$$

We can in turn solve for $\frac{\delta}{\delta x^i}$ by contracting the expression using from Proposition 1.2.7 the property $C_V(V,\cdot,\cdot) = C_V(\cdot,V,\cdot) = C_V(\cdot,\cdot,V) = 0$, under the form $V^iC_{ijk}(V) = V^jC_{ijk}(V) = V^kC_{ijk}(V) = 0$. We have

$$2V^j\Gamma^l_{ij}(V)g_{lk}(V) = V^j\left(\frac{\partial g_{kj}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k} - V^l\Gamma^m_{jl}(\dot{\partial} g_{ik})(\tfrac{\partial}{\partial x^m})\right)(V)$$

$$2V^iV^j\Gamma^l_{ij}(V)g_{lk}(V) = V^iV^j\left(\frac{\partial g_{kj}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right)(V), \tag{1.42}$$

solving to

$$2V^j\Gamma^l_{ij}(V)g_{lk}(V) = 2V^j\left(\frac{\partial g_{kj}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k} - V^lg^{mn}\left(\frac{\partial g_{nl}}{\partial x^j} + \frac{\partial g_{jn}}{\partial x^l} - \frac{\partial g_{jl}}{\partial x^n}\right)C_{ikm}\right)(V).$$

**Definition 1.3.10.** *Let us call the Chern connection $\nabla$ of a pseudo-Finsler manifold $(M,L)$ this unique torsion-free anisotropic linear connection satisfying the condition $\nabla g = 0$ for the anisotropic metric tensor $g$ associated to $L$.*

Note the similarity of (1.41) with the coefficients of the Levi-Civita connection of Riemannian manifolds. In hindsight, little effort is required to generalise to pseudo-Finsler manifolds, as the groundwork of establishing a Koszul formula is already laid and practically unveils its local coefficients up to the cancelling property of the Cartan tensor while contracting to (1.42). Working in reverse from these relatively simple expressions, we may for instance check the following homogeneity property of the Chern connection.

**Proposition 1.3.11.** *The Chern connection $\nabla$ of a pseudo-Finsler manifold satisfies $\nabla^v = \nabla^{\lambda v}$ for each admissible $v$ and positive scalar $\lambda$.*

*Proof.* For $\lambda$ positive, by homogeneity of $g$ and $C$ if $\nabla^v$ satisfies the Koszul formula (1.39) then so does $\nabla^{\lambda v}$ satisfy

$$\begin{aligned}2g_v(\nabla_x^{\lambda v}Y,z) = xg_V(Z,Y) + yg_V(X,Z) - zg_V(X,Y) \\ + g_v([Z,X],y) + g_v(x,[Z,Y]) + g_v([X,Y],z) \\ - 2C_v(z,y,\nabla_x^{\lambda v}V) - 2C_v(x,z,\nabla_y^{\lambda v}V) + 2C_v(x,y,\nabla_z^{\lambda v}V)\,,\end{aligned}$$

and as explained before we obtain that $\nabla^v = \nabla^{\lambda v}$. □

This trick will be used again to derive the Gauss formula (Theorem 2.1.10) and its dual (equation (3.49) of Lemma 3.1.27).

Evidently, the Chern connection is not the only connection of interest in the study of pseudo-Finsler manifolds. Particularly relevant is the fact that the contraction (1.42) of the Christoffel symbols of the Chern connection eliminates the Cartan tensor terms due to its cancelling property. Therefore the Euler-Lagrange equation (1.47) describing geodesics can work with the naive choice of formal Christoffel symbols inspired by those of the Levi-Civita connection blindly applied to the anisotropic metric tensor, as they appear on the right hand side of (1.42). Though these coefficients do not define a connection, one way to artifically force a torsion-free connection $\bar\nabla$ (called the Berwald connection) is to take its coefficients to be

$$\begin{aligned}\bar\Gamma_{ij}^k &= \tfrac12\dot\partial_i\dot\partial_j(y^my^n\Gamma_{mn}^k) = \tfrac12\Big((\dot\partial_jy^m)y^n\Gamma_{mn}^k + y^m(\dot\partial_jy^n)\Gamma_{mn}^k + y^my^n\dot\partial_j\Gamma_{mn}^k\Big) \\ &= \dot\partial_i(y^n\Gamma_{nj}^k) + \tfrac12\dot\partial_i(y^my^n\dot\partial_j\Gamma_{mn}^k) = \Gamma_{ij}^k + \tfrac12y^n(\dot\partial_i\Gamma_{nj}^k + \dot\partial_j\Gamma_{in}^k) + \tfrac12y^my^n\dot\partial_i\dot\partial_j\Gamma_{mn}^k\,,\end{aligned}$$

which can be easily proved to satisfy the same homogeneity property $\bar\Gamma_{ij}^k(\lambda v) = \bar\Gamma_{ij}^k(v)$. As was rather expected, however, $\bar\nabla$ does not satisfy the metric-preserving condition, and $\bar\nabla g$ is generally not the zero tensor.

The Berwald connection was not mentioned in vain: the homogeneity properties $\Gamma_{ij}^k(\lambda v) = \Gamma_{ij}^k(v)$ and $\bar\Gamma_{ij}^k(\lambda v) = \bar\Gamma_{ij}^k(v)$ differentiate with respect to $\lambda$ yielding $y^l\dot\partial_l\Gamma_{ij}^k = y^l\dot\partial_l\bar\Gamma_{ij}^k = 0$, such that the coefficients of the so called Berwald tensor

$$\dot\partial_l\bar\Gamma_{ij}^k = \dot\partial_l\Gamma_{ij}^k + \tfrac12(\dot\partial_i\Gamma_{lj}^k + \dot\partial_j\Gamma_{il}^k) + y^m\dot\partial_i\dot\partial_j\Gamma_{lm}^k + \tfrac12y^m\dot\partial_l(\dot\partial_i\Gamma_{mj}^k + \dot\partial_j\Gamma_{im}^k) + \tfrac12y^my^n\dot\partial_i\dot\partial_j\dot\partial_l\Gamma_{ij}^k$$

contract to

$$0 = y^l\dot\partial_l\bar\Gamma_{ij}^k = \tfrac12y^l(\dot\partial_i\Gamma_{lj}^k + \dot\partial_j\Gamma_{il}^k) + y^ly^m\dot\partial_i\dot\partial_j\Gamma_{lm}^k$$

and further to

$$0 = y^iy^j\dot\partial_l\Gamma_{ij}^k\,. \tag{1.43}$$

The coefficients $X^lV^iV^j\dot\partial_l\Gamma_{ij}^k(V)$ correspond to the derivative at 0 of $t \mapsto \nabla_V^{V+tX}V$. We may thus phrase the result in terms of the vertical derivative $P$ of the connection.

**Definition 1.3.12.** *Let $P$ denote the tensor defined for $v$ admissible and $x,y,z \in T_{\pi(v)}M$ by*

$$P_v(x,y,z) = x^iy^jz^l\dot\partial_l\Gamma_{ij}^k(v)\partial_k|_{\pi(v)}$$

*or, equivalently, for any choice of a vector field extension $Y$ of $y$*

$$P_v(x,y,z) = \frac{\partial}{\partial t}\nabla_x^{v+tz}Y\Big|_{t=0}\,.$$

We may rephrase (1.43) as

$$P_v(v,v,\cdot)=0\,. \tag{1.44}$$

This equation will provide a very useful simplification in Chapter 3, notably in the proof of Corollary 3.2.16. We will also make profuse use of the following identity when proving Proposition 3.2.6.

**Lemma 1.3.13.** *Let $H$ be an anisotropic vector field. For each admissible $v$ and $x\in T_{\pi(v)}M$,*

$$(\dot\partial(\nabla_x H))_v = P_v(x,H_v,\cdot)+(\nabla_x(\dot\partial H))_v-(\dot\partial H)_v(P_v(x,v,\cdot))\,. \tag{1.45}$$

*Proof.* For $y\in T_{\pi(v)}M$, a locally admissible extension $V$ of $v$ and vector field extensions $X$ and $Y$ of $x$ and $y$, by (1.36)

$$(\dot\partial(\nabla_x H))_v(y) = \tfrac{\partial}{\partial t}\nabla_x^{v+ty}H\Big|_{t=0} = \tfrac{\partial}{\partial t}\left(\nabla_x^{v+ty}H_{V+tY}-(\dot\partial H)_{v+ty}(\nabla_x^{v+ty}(V+tY))\right)\Big|_{t=0}$$
$$= P_v(x,H_v,y)+\nabla_x^v((\dot\partial H)_V(Y))-(\dot\partial(\dot\partial H))_v(\nabla_x^v V,y)-(\dot\partial H)_v(P_v(x,v,y))-(\dot\partial H)_v(\nabla_x^v Y)\,.$$

Conclude by symmetry of second order derivative from the definition of $\nabla(\dot\partial H)$ under the form

$$(\nabla_x(\dot\partial H))_v(y)=\nabla_x^v(\dot\partial H)_V(Y)-(\dot\partial H)_v(\nabla_x^v Y)-(\dot\partial(\dot\partial H))_v(y,\nabla_x^v V)\,.$$

□

The same computations performed with different building blocks that nevertheless produce the same geometry can be wildly more complicated than they need to be, and it is therefore of capital importance to carefully make the choices that will simplify our calculations the most. The Chern connection is such a building block. With a clear idea of which connection to define in the first place, and a good understanding of the properties of the Cartan tensor, the otherwise obscure relevance of the operator $\frac{\delta}{\delta x^i}$ and its intricate expression independently of $\Gamma^k_{ij}$ become straightforward.

### 1.3.3 Geodesics

Denoting by $\nabla$ the Chern connection of a pseudo-Finsler manifold $(M,L)$, the unique torsion-free and metric-preserving covariant differentiation, consider smooth curves $\gamma\colon\mathbb{R}\to M$, and its velocity $\dot\gamma$. Whether $\gamma$ is self-parallel with respect to $\nabla$, that is to say, whether

$$\nabla_{\dot\gamma(t)}^{\dot\gamma(t)}\dot\gamma=0 \tag{1.46}$$

holds is a property called *geodesicity*.

Note that, strictly speaking, this expression is only well defined as $\nabla_v^v V=0$ for a vector field extension $V$ of $\dot\gamma$ and $v=\dot\gamma(t)$; this is only a minor quibble. One could first define the covariant derivative along curves or use the well-defined local coordinate expression (1.47) differentiating along the curve's parameter, and reserve the $\nabla$ notation for regular curves that can be extended in this way. Thereafter, all things considered, we may extend $\dot\gamma(t)$ locally to a geodesic vector field (a vector field $V$ satisfying $\nabla_V^V V$, which exists uniquely as a solution to the differential equation (1.47)). Then $V$ would incidentally extend $\dot\gamma$, locally at least. For that reason, let us take the liberty to use the $\nabla$ notation directly, at the expense perhaps of implicitly defining it rigorously from (1.47) rather than the other way around.

A curve satisfying (1.46) shall be referred to as a *geodesic* of $M$ with respect to the metric $L$. Since the property is local, we typically only consider some arc of $\gamma$ at a time, let us therefore also take the liberty to leave real intervals out altogether, for simplicity, while a more rigorous approach would admittedly demand that one considers the fact that geodesics do not systematically extend to the whole number line in general (that they would extend in this way is a restrictive condition, referred to as *geodesic completeness*). For the time being, implicitly restrict $\mathbb{R} \to M$ to some generic interval to obtain the most general formulation.

**Proposition 1.3.14.** *For each non-zero $v \in TM$, there exists a geodesic $\gamma$, unique up to reparametrisation, verifying $\dot{\gamma}(t) = v$ for some $t \in \mathbb{R}$.*

*Proof.* A curve $\gamma\colon t \mapsto \gamma(t)$ is said to satisfy the geodesic equation when its real-valued single real variable local coordinate coefficients $\gamma^i$ satisfy

$$\frac{\mathrm{d}^2\gamma^k(t)}{\mathrm{d}t^2} + \frac{\mathrm{d}\gamma^i(t)}{\mathrm{d}t}\frac{\mathrm{d}\gamma^j(t)}{\mathrm{d}t}\Gamma^k_{ij}(\dot{\gamma}) = 0 \tag{1.47}$$

for $k \in [\![1,n]\!]$, which consists of a system of second order homogeneous ordinary differential equations in $\gamma$, with initial conditions $\gamma(t) = \pi(v)$ and $\dot{\gamma}^i(t) = \mathrm{d}x^i \cdot v$. After reducing it to a first order equation in $\dot{\gamma}$, by the Cauchy-Lipschitz existence and uniqueness theorem, the system of equations is satisfied by a set of tangent vectors along a unique path. □

Consider the exponential map $\exp\colon TM \to M$, extended to the set $\mathbf{0}$ of zero vectors as $\exp|_{\mathbf{0}} = \pi\colon TM \to M$ and otherwise sending $v \in TM$ to $\exp(v) = \gamma_v(1)$ along the unique geodesic $\gamma_v$ satisfying the initial conditions $\gamma_v(0) = \pi(v)$ and $\dot{\gamma}_v(0) = v$ and parametrised such that $g_{\dot{\gamma}}(\dot{\gamma},\dot{\gamma})$ is constant, which the previous proposition guarantees to exist. As mentioned before, in all rigour exp is to be implicitly restricted to a neighbourhood of $0 \in T_{\pi(v)}M$ at each point such that $\gamma_v(t+1)$ is defined, since geodesics need not in general extend indefinitely.

**Definition 1.3.15.** *Being $\gamma$ any geodesic of $(M,L)$ parametrised such that $g_{\dot{\gamma}}(\dot{\gamma},\dot{\gamma})$ is constant,* exp *maps $\dot{\gamma}(t)$ to $\gamma(t+1)$.*

The name comes from a collision with group theory, in particular the differentiable manifold $M = \mathbb{R}\setminus\{0\}$ which is a group with abelian group operation $(p,q) \mapsto pq$. Since this operation as well as the inverse map $p \mapsto \frac{1}{p}$ are smooth, this gives $M$ the structure of a Lie group. The trivial inner product $g_1\colon (v,w) \mapsto vw$ at the identity extends uniquely to $g_x = \frac{1}{x^2}g_1$ at each point $x$ of the connected component with action $f\colon p \mapsto xp$ by

$$g_1(v,w) = g_{f(1)}(\mathrm{d}f \cdot v, \mathrm{d}f \cdot w) = g_x(xv,xw) = x^2 g_x(v,w)\,,$$

inducing the Riemannian metric $v \mapsto \frac{|v|}{p}$ invariant under left translation by the group operation. The only path through 1 with prescribed derivative $v$ is $t \mapsto 1+tv$. The speed $t \mapsto \frac{|v|}{1+tv}$ integrates on $[0,\tau] \subset \mathbb{R}$ to $s = \frac{|v|}{v}\ln(1+\tau v)$, which solves to $1+\tau v = e^{s\frac{v}{|v|}}$ for the arclength parametrisation $s \mapsto e^{s\frac{v}{|v|}}$. The reparametrisation $\gamma\colon t \mapsto e^{tv}$ still satisfies that $g_{e^{tv}}(ve^{tv},ve^{tv}) = v^2$ is constant, such that $\exp(v) = \exp(\dot{\gamma}(0)) = \gamma(1) = e^v$.

**Proposition 1.3.16.** exp *is continuously differentiable, and smooth away from the zero section. The differential at the zero section is the identity map.*

*Proof.* For the second assertion, given a non-vector $v$ the function of a single real variable $t \mapsto \exp(tv)$ is the geodesic through $\pi(v)$ with velocity $v$ and therefore differentiates at 0 to $\mathrm{d}\exp \cdot v = v$. For the first assertion, proceeding as in section 5.3 of [21]. □

**Definition 1.3.17.** *A locally admissible vector field $V$ satisfying $\nabla^V_V V = 0$, or whose integral lines $\gamma\colon \mathbb{R} \to M$, such that $\dot{\gamma} = V \circ \gamma$, verify equivalently the geodesic equation (1.46), shall be referred to as a* geodesic vector field *with respect to the connection $\nabla$.*

**Proposition 1.3.18.** *For each admissible $v$ and being $\nabla$ an anisotropic connection on $M$. There exists a choice of extension $V \in \mathfrak{X}(M)$ such that every $w \in T_{\pi(v)}M$ satisfies*

$$\nabla^v_w V = 0 \,. \tag{1.48}$$

*Proof.* For local coordinates $(x^1, \cdots, x^i)$ around $p = \pi(v)$, we may decompose each $w = w^i \frac{\partial}{\partial x^i} = w^i \partial_i$ and obtain by linearity that $\nabla^v_w V$ will be zero for each of them provided

$$\nabla^v_{\partial_i} V = 0$$

for each $i \in [\![1,n]\!]$. For each index $i$, consider the path $\gamma_i$ sending $t$ from some arbitrarily small interval about 0 to the point with coordinates $(x^1(p), \cdots, x^{i-1}(p), x^i(p)+t, x^{i+1}(p), \cdots, x^n)$, essentially tracing the $i^{\text{th}}$ coordinate line through $\pi(v)$, and define the vector field $X : t \mapsto V \circ \gamma_i(t)$ along that path. Denote by $X^j$ its coefficients in the decomposition $X(t) = X^j(t) \frac{\partial}{\partial x^j}|_{\gamma_i(t)}$, and note how

$$\nabla^v_{\partial_i} V = \left( \tfrac{\partial V^k}{\partial x^i}|_p + v^j \Gamma^k_{ij}(v) \right) \partial_k = \left( \tfrac{\mathrm{d}X^j(t)}{\mathrm{d}t} + X^j(t) \Gamma^k_{ij}(v) \right) \partial_k$$

is zero if and only if for each $j \in [\![1,n]\!]$

$$\tfrac{\mathrm{d}X^j(t)}{\mathrm{d}t} = -X^j(t) \Gamma^k_{ij}(v) \,,$$

which constitutes a system of first order homogeneous ordinary differential equations in the coefficients $X^j$, with initial conditions $X^j(0) \frac{\partial}{\partial x^j}|_p = v$. Therefore, by the Cauchy-Lipschitz existence and uniqueness theorem, the system of equations is satisfied by a set of coordinate coefficients that define a unique vector field $X$ along the path $\gamma_i$. We may proceed similarly from each point of the image of $\gamma_i$ along the direction of another index $j$, then from each point of the rectangular shape completed in this way along the direction of yet another index $k$, and so on until exhausting the indices from 1 to $n$. We thus obtain coefficients of some vector field $V$ in the neighbourhood formed by the Cartesian product of the small intervals on which we defined the paths $\gamma_i$, $\gamma_j$, $\gamma_k$ and so on. Using a bump function, this vector field can be extended arbitrarily near the boundary and outside of this neighbourhood. □

That choice will allow us to eliminate vertical derivative terms at one point, while also eliminating derivatives of some expressions in $\nabla^V_V V$ when evaluated along the geodesic of $v$ in the neighbourhood of $\pi(v)$.

### 1.3.4 Curvature tensors

Consider for $V$ locally admissible on some neighbourhood and $X, Y \in \mathfrak{X}(M)$ the differential operator

$$\nabla^V_X \circ \nabla^V_Y \tag{1.49}$$

well-defined on that neighbourhood. For $Z \in \mathfrak{X}(M)$, we can expand classically

$$\begin{aligned}\nabla^V_X \nabla^V_Y Z = X^i \tfrac{\partial Y^j}{\partial x^i} \left( \tfrac{\partial Z^l}{\partial x^j} + Z^k \Gamma^l_{jk}(V) \right) \tfrac{\partial}{\partial x^l} + X^i Y^j \left( \tfrac{\partial^2 Z^l}{\partial x^i \partial x^j} + \tfrac{\partial Z^k}{\partial x^i} \Gamma^l_{jk}(V) + Z^k \tfrac{\partial \Gamma^l_{jk}(V)}{\partial x^i} \right) \tfrac{\partial}{\partial x^l} \\ + X^i Y^j \left( \tfrac{\partial Z^l}{\partial x^j} + Z^k \Gamma^l_{jk}(V) \right) \Gamma^m_{il}(V) \tfrac{\partial}{\partial x^m} \,,\end{aligned}$$

where, as in (1.34), we have

$$X^i \frac{\partial \Gamma^l_{jk}(V)}{\partial x^i} = X^i \left( \frac{\partial' \Gamma^l_{jk}}{\partial x^i}(V) + \frac{\partial V^m}{\partial x^i} \frac{\partial \Gamma^l_{jk}}{\partial y^m}(V) \right) = X^i \frac{\delta \Gamma^l_{jk}}{\delta x^i}(V) + (\dot{\partial} \Gamma^l_{jk})_V (\nabla^V_X V) \,,$$

and can then compute the Riemann curvature tensor associated to $\nabla^V : (X,Y) \mapsto \nabla^V_X Y$ as

$$\begin{aligned}\nabla^V_X \nabla^V_Y Z - \nabla^V_Y \nabla^V_X Z - \nabla^V_{[X,Y]} Z \\ = X^i Y^j Z^k \left( \frac{\partial \Gamma^m_{jk}(V)}{\partial x^i} - \frac{\partial \Gamma^m_{ik}(V)}{\partial x^j} + \Gamma^l_{jk}(V)\Gamma^m_{il}(V) - \Gamma^l_{ik}(V)\Gamma^m_{jl}(V) \right) \frac{\partial}{\partial x^m} \\ = Y^j Z^k \left( X^i \left( \frac{\delta \Gamma^m_{jk}}{\delta x^i} - \frac{\delta \Gamma^m_{ik}}{\delta x^j} + \Gamma^l_{jk}\Gamma^m_{il} - \Gamma^l_{ik}\Gamma^m_{jl} \right)(V) + ((\dot\partial \Gamma^m_{jk})_V - (\dot\partial \Gamma^m_{ik})_V)(\nabla^V_X V) \right) \frac{\partial}{\partial x^m}.\end{aligned}$$

Specifically, let us denote by $R^V$ the curvature tensor defined as follows.

**Definition 1.3.19.** *Being $(M,L)$ a pseudo-Finsler manifold and $\nabla$ and anisotropic linear connection on $(M,L)$, for each admissible $v$ with an extension $V$ locally admissible in the neighbourhood of $\pi(v)$ and $X,Y,Z \in \mathfrak{X}(M)$ let*

$$R^V(X,Y)Z = \nabla^V_X \nabla^V_Y Z - \nabla^V_Y \nabla^V_X Z - \nabla^V_{[X,Y]} Z$$

*on that neighbourhood of $\pi(v)$.*

Recall our definition of the anisotropic tensor $P$ as the vertical derivative of the connection $\nabla$ in the following sense: being $(M,L)$ a pseudo-Finsler manifold and $\nabla$ and anisotropic linear connection on $(M,L)$, for each admissible $v$ and $X,Y,Z \in \mathfrak{X}(M)$ with $z = Z(\pi(v))$

$$P_v(X,Y,Z) = (\dot\partial(\nabla_X Y))_v(Z) = \frac{\partial}{\partial t} \nabla^{v+tz}_X Y \Big|_{t=0} = X^i Y^j Z^l \dot\partial_l \Gamma^k_{ij}(v) \partial_k|_{\pi(v)}. \tag{1.50}$$

Our definition (1.36) extends $\nabla$ to differentiate anisotropic vector fields, relating the previous computation of (1.49) to that of the differential operator

$$\nabla_X \circ \nabla_Y \tag{1.51}$$

which, up to composition by $\pi$ to lift coefficients from $C^\infty(M)$ to $C^\infty(TM)$, expands into

$$\begin{aligned}\nabla_X \nabla_Y Z = X^i \frac{\partial Y^j}{\partial x^i} \left( \frac{\partial Z^l}{\partial x^j} + Z^k \Gamma^l_{jk} \right) \frac{\partial}{\partial x^l} + X^i Y^j \left( \frac{\partial^2 Z^l}{\partial x^i \partial x^j} + \frac{\partial Z^k}{\partial x^i} \Gamma^l_{jk} + Z^k \frac{\delta \Gamma^l_{jk}}{\delta x^i} \right) \frac{\partial}{\partial x^l} \\ + X^i Y^j \left( \frac{\partial Z^l}{\partial x^j} + Z^k \Gamma^l_{jk} \right) \Gamma^m_{il} \frac{\partial}{\partial x^m},\end{aligned}$$

giving us

$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z = X^i Y^j Z^k \underbrace{\left( \frac{\delta \Gamma^m_{jk}}{\delta x^i} - \frac{\delta \Gamma^m_{ik}}{\delta x^j} + \Gamma^l_{jk}\Gamma^m_{il} - \Gamma^l_{ik}\Gamma^m_{jl} \right)}_{R^m{}_{kij}} \frac{\partial}{\partial x^m}. \tag{1.52}$$

**Definition 1.3.20.** *Being $(M,L)$ a pseudo-Finsler manifold and $\nabla$ and anisotropic linear connection on $(M,L)$, for $X,Y,Z \in \mathfrak{X}(M)$ let*

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The curvature tensors $R$ and $R^V$ are related by

$$R^V(X,Y)Z = R_V(X,Y)Z + P_V(Y,Z,\nabla^V_X V) - P_V(X,Z,\nabla^V_Y V). \tag{1.53}$$

Since $P$ is tensorial, we can write for each admissible $v$ with any locally admissible extension $V$ and $x,y,z \in T_{\pi(v)}M$

$$R_v(x,y)z = R^V_{\pi(v)}(x,y)z - P_v(y,z,\nabla^v_x V) + P_v(x,z,\nabla^v_y V). \tag{1.54}$$

This identity will be used repeatedly when proving identities such as the Gauss and Codazzi equations (2.14) and (2.15). This permits the calculation of $g_v(R_v(x,y)z,w)$ in isotropic terms, dependent on the choice of a locally admissible extension $V$ of $v$. Thus, many computational steps can be imported straight from Riemannian proofs up to vertical derivative terms such as the $C$ and $P$ tensors.

**Theorem 1.3.21.** *The anisotropic curvature tensor $R$ associated with the Chern connection $\nabla$ of a pseudo-Finsler manifold $(M,L)$ exhibits for $v \in \mathbb{A}$ and $w,x,y,z \in T_{\pi(v)}M$ the following symmetries:*

1. *Antisymmetry of $R_v$,*
$$R_v(x,y) = R_v(y,x)\,, \tag{1.55}$$
2. *Almost skew-symmetry of $R_v(x,y)$,*
$$g_v(w,R_v(x,y)z) = -g_v(R_v(x,y)w,z) - 2C_v(R_v(x,y)v,w,z)\,, \tag{1.56}$$
3. *Almost symmetry of $((x,y),(z,w)) \mapsto g_v(w,R_v(x,y)z)$,*
$$\begin{aligned} g_v(R_v(x,y)z,w) &= g_v(R_v(z,w)x,y) - C_v(z,w,R_v(x,y)v) + C_v(R_v(z,w)v,x,y) \\ &\quad - C_v(R_v(w,x)v,y,z) + C_v(w,x,R_v(y,z)v) - C_v(w,R_v(x,z)v,y) + C_v(x,R_v(w,y)v,z)\,. \end{aligned} \tag{1.57}$$

*Proof.* The first antisymmetry identity (1.55) is trivial, by definition of $R$. Let $v^i = \mathrm{d}x^i \cdot v$ the coefficients of $v$ in some local coordinates. For simplicity, let us write $\frac{\partial}{\partial x^i}$ to denote that for which we previously reserved the notation $\frac{\partial'}{\partial x^i}$. Noting that $\frac{\delta y^j}{\delta x^i}\Gamma^l_{jk} = -y^m\Gamma^j_{im}\Gamma^l_{jk}$, the operator $\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - y^j\Gamma^k_{ij}\frac{\partial}{\partial y^k}$ iterates on anisotropic scalar fields $f$ as

$$\begin{aligned} \frac{\delta^2 f}{\delta x^i \delta x^j} &= \frac{\delta}{\delta x^i}\left[\frac{\partial f}{\partial x^j} - y^k\Gamma^l_{jk}\frac{\partial f}{\partial y^l}\right] \\ &= \frac{\partial^2 f}{\partial x^i\partial x^j} - y^m\Gamma^n_{im}\frac{\partial^2 f}{\partial y^n\partial x^j} - y^m\Gamma^k_{im}\Gamma^l_{jk}\frac{\partial f}{\partial y^l} - y^k\frac{\delta\Gamma^l_{jk}}{\delta x^i}\frac{\partial f}{\partial y^l} - y^k\Gamma^l_{jk}\frac{\partial^2 f}{\partial x^i\partial y^l} + y^k y^m\Gamma^l_{jk}\Gamma^n_{im}\frac{\partial^2 f}{\partial y^l\partial y^n}\,. \end{aligned}$$

Note that the right hand side commutes in $i$ and $j$ safe for

$$-y^m\Gamma^k_{im}\Gamma^l_{jk}\frac{\partial f}{\partial y^l} - y^k\frac{\delta\Gamma^l_{jk}}{\delta x^i}\frac{\partial f}{\partial y^l}\,.$$

Observe how, consequence of the commutativity of mixed partial derivatives on the manifold $TM$, we can commute the indices of the iterated operator to the expense of a Cartan tensor term featuring $R_v(\cdot,\cdot)v$, a fact that is easier to spot when ignoring all terms linear in $\Gamma$ which vanish when evaluated at $v$ for some choice of coordinates since it amounts to adding $-v^k\frac{\delta\Gamma^l_{ik}}{\delta x^j}(v)\frac{\partial f}{\partial y^l}(v) + v^k\frac{\delta\Gamma^l_{ik}}{\delta x^j}(v)\frac{\partial f}{\partial y^l}(v) = 0$ to $\frac{\delta^2 f}{\delta x^i\delta x^j}(v) = \frac{\partial^2 f}{\partial x^j\partial x^i}(v) - v^k\frac{\delta\Gamma^l_{jk}}{\delta x^i}(v)\frac{\partial f}{\partial y^l}(v)$. Let us, for completion's sake, to nevertheless spell out the commuted expression

$$\begin{aligned} \frac{\delta^2 f}{\delta x^i\delta x^j} &= \overbrace{\frac{\partial^2 f}{\partial x^j\partial x^i} - y^k\Gamma^l_{jk}\frac{\partial^2 f}{\partial y^l\partial x^i} - y^m\Gamma^k_{jm}\Gamma^l_{ik}\frac{\partial f}{\partial y^l} - y^k\frac{\delta\Gamma^l_{ik}}{\delta x^j}\frac{\partial f}{\partial y^l} - y^m\Gamma^n_{im}\frac{\partial^2 f}{\partial x^j\partial y^n} + y^m y^k\Gamma^n_{im}\Gamma^l_{jk}\frac{\partial^2 f}{\partial y^n\partial y^l}}^{\frac{\delta^2 f}{\delta x^j\delta x^i}} \\ &\quad + y^k\left(\frac{\delta\Gamma^l_{ik}}{\delta x^j} - \frac{\delta\Gamma^l_{jk}}{\delta x^i} + \Gamma^m_{ik}\Gamma^l_{jm} - \Gamma^m_{jk}\Gamma^l_{im}\right)\frac{\partial f}{\partial y^l}\,. \end{aligned}$$

For the coefficient functions $f = g_{ab}$ of the anisotropic metric tensor $g$, this amounts to

$$\frac{\delta^2 g_{ab}}{\delta x^i \delta x^j}(v) - \frac{\delta^2 g_{ab}}{\delta x^j \delta x^i}(v) = 2C_v\big(R_v(\tfrac{\partial}{\partial x^j}, \tfrac{\partial}{\partial x^i})v, \tfrac{\partial}{\partial x^a}, \tfrac{\partial}{\partial x^b}\big)\,. \tag{1.58}$$

Reinserting the coefficients (1.41) of the Chern connection into those of the associated curvature tensor yields

$$\begin{aligned}
2g_v(\tfrac{\partial}{\partial x^h}, R_v(\tfrac{\partial}{\partial x^i}, \tfrac{\partial}{\partial x^j})\tfrac{\partial}{\partial x^k}) &= 2(R^l{}_{kij} g_{hl})(v) \\
&= \Big(\frac{\delta(2\Gamma^l_{jk} g_{hl})}{\delta x^i} - \frac{\delta(2\Gamma^l_{ik} g_{hl})}{\delta x^j} - 2\Gamma^l_{ik}\frac{\delta g_{hl}}{\delta x^j} + 2\Gamma^l_{jk}\frac{\delta g_{hl}}{\delta x^i} + 2\Gamma^m_{jl} g_{hm}\Gamma^l_{ik} - 2\Gamma^m_{il} g_{hm}\Gamma^l_{jk}\Big)(v) \\
&= \Big(\frac{\delta^2 g_{hk}}{\delta x^i \delta x^j} + \frac{\delta^2 g_{jh}}{\delta x^i \delta x^k} - \frac{\delta^2 g_{jk}}{\delta x^i \delta x^h} - \frac{\delta^2 g_{hk}}{\delta x^j \delta x^i} - \frac{\delta^2 g_{ih}}{\delta x^j \delta x^k} + \frac{\delta^2 g_{ik}}{\delta x^j \delta x^h} \\
&\qquad - \Gamma^l_{ik}\frac{\delta g_{hl}}{\delta x^j} + \Gamma^l_{jk}\frac{\delta g_{hl}}{\delta x^i} + \Gamma^l_{ik}\frac{\delta g_{jh}}{\delta x^l} - \Gamma^l_{ik}\frac{\delta g_{jl}}{\delta x^h} - \Gamma^l_{jk}\frac{\delta g_{ih}}{\delta x^l} + \Gamma^l_{jk}\frac{\delta g_{il}}{\delta x^h}\Big)(v) \\
&= \Big(\frac{\delta^2 g_{jh}}{\delta x^i \delta x^k} - \frac{\delta^2 g_{jk}}{\delta x^i \delta x^h} - \frac{\delta^2 g_{ih}}{\delta x^j \delta x^k} + \frac{\delta^2 g_{ik}}{\delta x^j \delta x^h} + \Gamma^l_{jk}\big(\frac{\delta g_{hl}}{\delta x^i} + \frac{\delta g_{il}}{\delta x^h} - \frac{\delta g_{ih}}{\delta x^l}\big) \\
&\qquad + \Gamma^l_{ik}\big(-\frac{\delta g_{hl}}{\delta x^j} - \frac{\delta g_{jl}}{\delta x^h} + \frac{\delta g_{jh}}{\delta x^l}\big)\Big)(v) + \frac{\delta^2 g_{hk}}{\delta x^i \delta x^j}(v) - \frac{\delta^2 g_{hk}}{\delta x^j \delta x^i}(v) \\
&= \Big(\frac{\delta^2 g_{jh}}{\delta x^i \delta x^k} - \frac{\delta^2 g_{jk}}{\delta x^i \delta x^h} - \frac{\delta^2 g_{ih}}{\delta x^j \delta x^k} + \frac{\delta^2 g_{ik}}{\delta x^j \delta x^h} + 2(\Gamma^m_{ih}\Gamma^l_{jk} - 2\Gamma^m_{jh}\Gamma^l_{ik}) g_{lm}\Big)(v) \\
&\qquad + 2C_v\big(R_v(\tfrac{\partial}{\partial x^j}, \tfrac{\partial}{\partial x^i})v, \tfrac{\partial}{\partial x^h}, \tfrac{\partial}{\partial x^k}\big)
\end{aligned}$$

by (1.58), making the difference

$$\begin{aligned}
g_v(\tfrac{\partial}{\partial x^h}, R_v(\tfrac{\partial}{\partial x^i}, \tfrac{\partial}{\partial x^j})\tfrac{\partial}{\partial x^k}) - C_v\big(R_v(\tfrac{\partial}{\partial x^j}, \tfrac{\partial}{\partial x^i})v, \tfrac{\partial}{\partial x^h}, \tfrac{\partial}{\partial x^k}\big) \\
= \tfrac{1}{2}\Big(\frac{\delta^2 g_{jh}}{\delta x^i \delta x^k} - \frac{\delta^2 g_{jk}}{\delta x^i \delta x^h} - \frac{\delta^2 g_{ih}}{\delta x^j \delta x^k} + \frac{\delta^2 g_{ik}}{\delta x^j \delta x^h} + 2(\Gamma^l_{ih}\Gamma^m_{jk} - 2\Gamma^l_{jh}\Gamma^m_{ik}) g_{lm}\Big)(v)
\end{aligned}$$

anti-symmetric in $h \leftrightarrow k$, and by symmetry of the Cartan tensor adding to the previous equality the expression commuted in $h$ and $k$ yields

$$g_v(\tfrac{\partial}{\partial x^h}, R_v(\tfrac{\partial}{\partial x^i}, \tfrac{\partial}{\partial x^j})\tfrac{\partial}{\partial x^k}) + g_v(\tfrac{\partial}{\partial x^k}, R_v(\tfrac{\partial}{\partial x^i}, \tfrac{\partial}{\partial x^j})\tfrac{\partial}{\partial x^h}) + 2C_v\big(R_v(\tfrac{\partial}{\partial x^i}, \tfrac{\partial}{\partial x^j})v, \tfrac{\partial}{\partial x^h}, \tfrac{\partial}{\partial x^k}\big) = 0\,,$$

proving (1.56). For (1.57), note how we also have

$$\begin{aligned}
g_v(\tfrac{\partial}{\partial x^h}, R_v(\tfrac{\partial}{\partial x^i}, \tfrac{\partial}{\partial x^j})\tfrac{\partial}{\partial x^k}) + g_v(\tfrac{\partial}{\partial x^j}, R_v(\tfrac{\partial}{\partial x^h}, \tfrac{\partial}{\partial x^k})\tfrac{\partial}{\partial x^i}) \\
- C_v\big(R_v(\tfrac{\partial}{\partial x^j}, \tfrac{\partial}{\partial x^i})v, \tfrac{\partial}{\partial x^h}, \tfrac{\partial}{\partial x^k}\big) - C_v\big(R_v(\tfrac{\partial}{\partial x^k}, \tfrac{\partial}{\partial x^h})v, \tfrac{\partial}{\partial x^j}, \tfrac{\partial}{\partial x^i}\big) \\
= \tfrac{1}{2}\Big(\frac{\delta^2 g_{jh}}{\delta x^i \delta x^k} - \frac{\delta^2 g_{jk}}{\delta x^i \delta x^h} - \frac{\delta^2 g_{ih}}{\delta x^j \delta x^k} + \frac{\delta^2 g_{ik}}{\delta x^j \delta x^h} + \frac{\delta^2 g_{kj}}{\delta x^h \delta x^i} - \frac{\delta^2 g_{ki}}{\delta x^h \delta x^j} - \frac{\delta^2 g_{hj}}{\delta x^k \delta x^i} + \frac{\delta^2 g_{hi}}{\delta x^k \delta x^j}\Big)(v) \\
= C_v\big(R_v(\tfrac{\partial}{\partial x^k}, \tfrac{\partial}{\partial x^i})v, \tfrac{\partial}{\partial x^j}, \tfrac{\partial}{\partial x^h}\big) - C_v\big(R_v(\tfrac{\partial}{\partial x^h}, \tfrac{\partial}{\partial x^i})v, \tfrac{\partial}{\partial x^j}, \tfrac{\partial}{\partial x^k}\big) \\
- C_v\big(R_v(\tfrac{\partial}{\partial x^k}, \tfrac{\partial}{\partial x^j})v, \tfrac{\partial}{\partial x^i}, \tfrac{\partial}{\partial x^h}\big) + C_v\big(R_v(\tfrac{\partial}{\partial x^h}, \tfrac{\partial}{\partial x^j})v, \tfrac{\partial}{\partial x^i}, \tfrac{\partial}{\partial x^k}\big)\,,
\end{aligned}$$

producing again by (1.58) the desired Cartan tensor terms. □

One of the most important geometric invariants of a pseudo-Finsler manifold is the flag curvature: the sectional curvature along a plane that contains the vertical variable. The plane is typically visualised intuitively as a *flag* and the vertical variable as a *flagpole*. The flag curvature, dependent on both flag and flagpole, measures the amount by which geodesics deviate from the flagpole geodesic.

**Definition 1.3.22.** *Let the flag curvature $K$ of a pseudo-Finsler manifold $(M,L)$ be given for $v \in \mathbb{A}$ and $u \in T_{\pi(v)}M$ by*

$$K_v(u) = \frac{g_v(R_v(v,u)u,v)}{g_v(u,u)g_v(v,v) - g_v(u,v)g_v(u,v)}\,, \tag{1.59}$$

*where $g$ is the anisotropic metric tensor associated with $L$, as defined in* (1.16)*, and $R$ the anisotropic curvature tensor associated to its Chern connection, as defined in* (1.52)*.*

For pseudo-Riemannian manifolds, this invariant reduces to the traditional sectional curvature, and by Shur's Lemma a Riemannian manifold with scalar sectional curvature has constant sectional curvature. Similar properties are extensively explored in [26] and [43]. The next chapter will conclude with the expression of the flag curvature of the submanifold of a Randers-Minkowski space in terms of its Zermelo data.

# Chapter 2

# Pseudo-Finsler submanifolds

Progress in the study of pseudo-Finsler submanifolds has been slow in comparison to Riemannian geometry due to the much more demanding complexity of computation. Understanding submanifolds is a prerequisite to establishing a classification of pseudo-Finsler manifolds, as it naturally allows the plentiful generation of new manifolds. To overcome the daunting calculations of the Finslerian setting, a judicious choice of the objects used when calculating is essential: indeed the classification of Randers manifolds was only feasible once the Randers metric (the sum of a Riemannian metric with a 1-form of norm strictly less than 1) was expressed in terms of its Zermelo data (another Riemannian metric together with a vector field with respect to which the Randers indicatrix of unit vectors is the translation along this vector field of that Riemannian metric's indicatrix) as done in [27].

The content of this chapter is the object of an article we have recently published [51], in which we successfully derive the Gauss and Codazzi equations (Theorem 3.1 and Theorem 3.2 of [31], to cite one earlier instance of the equations in previous research papers) under the forms (2.14) and (2.15) in an index-free formulation of anisotropic tensor calculus (see [46]) more appropriate for our purposes, and furthermore obtain the flag curvature of Randers-Minkowski spacetimes in terms of Zermelo data.

**Definition 2.0.1.** *A smooth map $\iota : \widehat{M} \to M$ of a manifold $\widehat{M}$ into a higher dimensional ambient manifold $M$ is an* immersion *when the restriction of its differential to the tangent space at each point $p$ is injective into $T_{\iota(p)}M$.*

Recall from elementary differential geometry that immersions are local embeddings, and the distinction between an immersed manifold and a proper submanifold only occurs globally.

**Theorem 2.0.2.** *There exist* adapted *local coordinates $(\hat{x}_1, \cdots, \hat{x}_k)$ around any $p \in \widehat{M}$ and $(x_1, \cdots, x_n)$ around $\iota(p)$ for which $\iota$ takes the form*

$$(\hat{x}_1, \cdots, \hat{x}_k) \mapsto (x_1, \cdots, x_k, 0, \cdots, 0)\,.$$

When $\iota$ is the inclusion map of a subset $\widehat{M} \subset M$, up to a topological homeomorphism between $\widehat{M}$ and $\iota(\widehat{M})$, we call $\widehat{M}$ a submanifold of $M$ and its topology is obtained by intersection with the ambient topology. If $(M, L)$ is a pseudo-Finsler manifold, and $\widehat{M} \subset M$ is a submanifold of $M$, then the restriction $L|_{\widehat{M}}$ of the ambient pseudo-Finsler metric will define a pseudo-Finsler metric on $\widehat{M}$ provided the associated anisotropic metric tensor is non-degenerate, in which case the submanifold is said to be a non-degenerate submanifold.

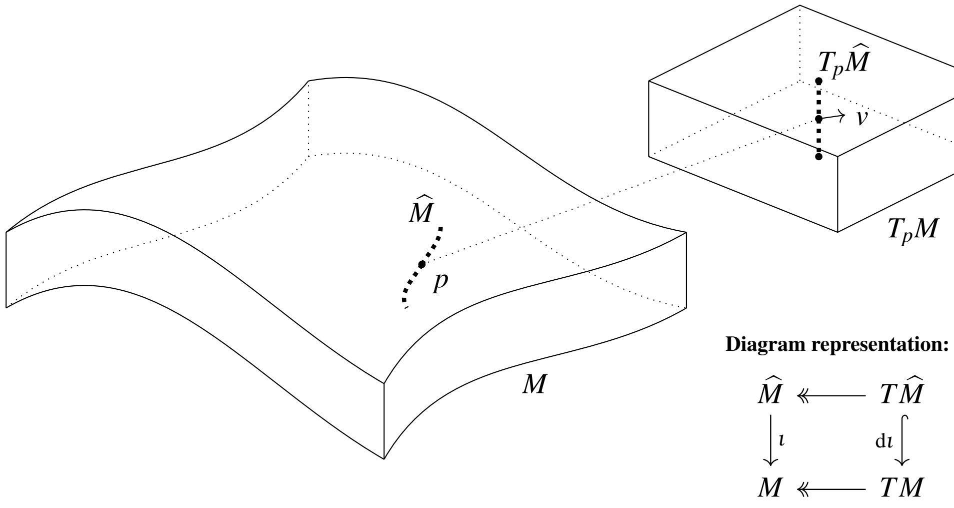


FIGURE 2.1: Visual illustration of an immersion.

## 2.1 Preliminaries

This section reviews the definitions of anisotropic second fundamental forms $\mathrm{I\!I}$ and induced connection $\nabla^\perp$ resulting from decomposing an anisotropic connection $\nabla$ on $(M,L)$ into parts tangent and $g$-orthogonal to a non-degenerate submanifold $\widehat{M} \subset M$, namely such that the restriction $\hat{L} = L|_{T\widehat{M}}$ defined on $\hat{\mathbb{A}} = \mathbb{A} \cap T\widehat{M}$ is a pseudo-Finsler metric on $\widehat{M}$.

### 2.1.1 Second fundamental form

Let $v \in \mathbb{A}$, $x \in T_{\pi(v)}\widehat{M}$ and $Y \in \mathfrak{X}(M)$ such that $Y \circ \gamma$ is tangent to $\widehat{M}$ along some curve $\gamma$ such that $\dot\gamma(\pi(v)) = x$. Consider for any anisotropic linear connection $\nabla$ of $M$ the local coordinate decompositions $x = x^i \frac{\partial}{\partial x^i}|_{\pi(v)}$, $Y = Y^j \frac{\partial}{\partial x^j}$ and

$$\nabla^v_x Y = x^i \left( \frac{\partial Y^k}{\partial x^i}|_{\pi(v)} + Y^j(\pi(v))\Gamma^k_{ij}(v) \right) \frac{\partial}{\partial x^k}|_{\pi(v)} .$$

Note that $x^i \frac{\partial Y^j}{\partial x^i}|_{\pi(v)}$ only depends on the values of $Y$ along $\widehat{M}$ since $x \in T\widehat{M}$. That is, given $\hat{Y} \in \mathfrak{X}(\widehat{M})$, we can choose any local extension $Y \in \mathfrak{X}(M)$ such that $Y|_{\widehat{M}} = \hat{Y}$ and compute the well-defined $\nabla^v_x \hat{Y} = \nabla^v_x Y$. In that precise sense, $\nabla$ also inadvertedly defines an operator within $\widehat{M}$.

**Definition 2.1.1.** *Assuming $\widehat{M}$ is a non-degenerate submanifold of $(M,L)$, for $v \in \mathbb{A}$ and $w \in T_{\pi(v)}M$, let $w_v^\top \in T_{\pi(v)}\widehat{M}$ and $w_v^\perp \in T_{\pi(v)}M$ be the parts of $w$ respectively tangent and $g_v$-orthogonal to $T_{\pi(v)}\widehat{M}$, such that $w = w_v^\top + w_v^\perp$.*

When applying this $g_v$-orthogonal decomposition to both $\hat{Y}$ and $f\hat{Y}$ for $f \in C^\infty(\widehat{M})$, we have

$$\nabla^v_x f\hat{Y} = (\nabla^v_x f\hat{Y})^\top_v + (\nabla^v_x f\hat{Y})^\perp_v .$$

When combined to the product rule

$$\nabla^v_x f\hat{Y} = (xf)\hat{Y}(\pi(v)) + f(\pi(v))\nabla^v_x \hat{Y} ,$$

this provides us with the properties

$$(\nabla^v_x f\hat{Y})^\top_v = (xf)\hat{Y}(\pi(v)) + f(\pi(v))(\nabla^v_x\hat{Y})^\top_v\,, \tag{2.1}$$
$$(\nabla^v_x f\hat{Y})^\perp_v = f(\pi(v))(\nabla^v_x\hat{Y})^\perp_v\,. \tag{2.2}$$

**Definition 2.1.2.** *Let*

$$\nabla^\top\colon (v,x,\hat{Y}) \mapsto (\nabla^v_x\hat{Y})^\top_v$$

*be the connection induced on* $\widehat{M}$ *by* $\nabla$*, satisfying for every* $z \in T_{\pi(v)}\widehat{M}$

$$g_v((\nabla^\top)^v_x\hat{Y}, z) = g_v(\nabla^v_x\hat{Y}, z)\,,$$

*and*

$$\mathrm{I\!I}\colon (v,x,\hat{Y}) \mapsto (\nabla^v_x\hat{Y})^\perp_v$$

*be the second fundamental form of* $\widehat{M}$ *as a pseudo-Finsler submanifold of* $(M,L)$*, satisfying for every* $w \in T_{\pi(v)}M$ $g_v$*-orthogonal to* $\widehat{M}$

$$g_v(\mathrm{I\!I}_v(x,\hat{Y}), w) = g_v(\nabla^v_x\hat{Y}, w)\,.$$

**Proposition 2.1.3.** *The maps* $X \mapsto X^\top$ *and* $X \mapsto X^\perp$ *sending a vector field* $X$ *to the anisotropic vector fields* $v \mapsto X^\top_v$ *and* $v \mapsto X^\perp_v$ *are smooth.*

*Proof.* Consider at $p \in \widehat{M}$ and for some admissible $v$ tangent to $\widehat{M}$ at $p$ a non-degenerate basis $(e_1,\cdots,e_k,\cdots,e_n)$ of $T_pM$ such that the first vectors form a basis $(e_1,\cdots,e_k)$ of $T_p\widehat{M}$ and the last vectors form a basis $(e_{k+1},\cdots,e_n)$ of the non-degenerate subspace supplementary to $T_p\widehat{M}$ with respect to $g_v$. Such a basis can be locally obtain explicitly by some constant combination of the vector fields $\frac{\partial}{\partial x^1},\cdots,\frac{\partial}{\partial x^n}$ induced by local coordinates $(x^1,\cdots,x^n)$ adapted to the submanifold. By applying the Gram-Schmidt process, we may generate a $g_{v'}$-orthonormal basis $(e_1(v'),\cdots,e_k(v'),\cdots,e_n(v'))$ for each admissible $v'$ in the neighbourhood of $v$ such that the first vectors again form a basis of the space of vertical vectors. Locally,

$$X^\top_{v'} = \sum_{i\in[\![1,k]\!]} \frac{g_{v'}(e_i(v'),X)}{g_{v'}(e_i(v'),e_i(v'))} e_i(v')$$

depends smoothly on $v'$. □

**Proposition 2.1.4.** *The induced connection* $\nabla^\top$ *is a well-defined connection on* $\widehat{M}$*, and the second fundamental form* $\mathrm{I\!I}$ *of* $\widehat{M}$ *a well-defined anisotropic tensor.*

*Proof.* $\nabla^\top$ satisfies equation (2.1), making it a well-defined connection on $\widehat{M}$ by Definition 1.3.1. Meanwhile, $\mathrm{I\!I}_v(x,\hat{Y})$ satisfies (2.2) and is therefore bilinear in $x$ and $\hat{Y}$. Consequently, $\mathrm{I\!I}$ is an anisotropic tensor and depends only on the value $\hat{Y}(\pi(v))$, and not on the choice of extension $\hat{Y}$. □

**Proposition 2.1.5.** *If* $\nabla$ *is the Chern connection of* $(M,L)$*, then so* $\nabla^\top$ *is torsion-free and* $\mathrm{I\!I}$ *is a symmetric anisotropic tensor.*

*Proof.* We have for each admissible $v$ and $X,Y \in \mathfrak{X}(M)$ tangent to $\widehat{M}$

$$[X,Y] = \nabla^v_X Y - \nabla^v_Y X = (\nabla^v_X Y)^\top_v - (\nabla^v_Y X)^\top_v + \mathrm{I\!I}_v(X,Y) - \mathrm{I\!I}_v(X,Y)\,.$$

Recognising $[X,Y]=(X^i\frac{\partial Y^j}{\partial x^i}-Y^i\frac{\partial X^j}{\partial x^i})\frac{\partial}{\partial x^j}$ as tangent along $\widehat{M}$, one identifies the tangent part $(\nabla^v_X Y)^\top_v-(\nabla^v_Y X)^\top_v=[X,Y]$ and the $g_v$-orthogonal $\mathrm{II}_v(X,Y)-\mathrm{II}_v(X,Y)=0$. □

When $\widehat{M}$ is a hypersurface, the second fundamental form $\mathrm{II}$ may be expressed for each admissible $v$ and vectors $x$ and $y$ all tangent to $\widehat{M}$ at some point as $\mathrm{II}_v(x,y)=\sigma_v(x,y)w$ for $w$ $g_v$-orthogonal to $\widehat{M}$ and $\sigma_v(x,y)\in\mathbb{R}$.

**Proposition 2.1.6.** *Consider (isotropic) second fundamental forms $\sigma w,\tilde{\sigma}\tilde{w}$ at $p$ of a generic hypersurface $S\subset T_pM$ of the tangent space at that point along arbitrary vectors $w\in T_{\pi(v)}M\setminus T_vS$ and $\tilde{w}=\lambda w+s$ with $\lambda\in\mathbb{R}\setminus\{0\}$ and $s\in TS$. Then*

$$\sigma=\lambda\tilde{\sigma}\,. \tag{2.3}$$

*Proof.* The part of the Levi-Civita connection transverse along $w$ is $\sigma w$, let $\rho$ be the remaining part tangent to $S$. The part of the connection transverse to $S$ along $\lambda w+s$ is $\lambda\tilde{\sigma}w+\tilde{\sigma}s$, let $\tilde{\rho}$ be the remaining part tangent to $S$. Since $s$ is also tangent to $S$, we have $\rho=\tilde{\rho}+\tilde{\sigma}s$ by identifying the tangent part of either decomposition of the connection. We obtain that the remaining parts transverse to $S$ must be equal. □

### 2.1.2 Implications for pseudo-Finsler metrics

The restriction of the evaluation $g_v$ in a unitary admissible vector $v$ of the anisotropic metric tensor $g$ associated to an arbitrary pseudo-Finsler metric $L$ to the space tangent to its indicatrix at $v$ coincides with the second fundamental form of the indicatrix $L^{-1}(\{1\})\cap T_{\pi(v)}M$ as a submanifold of the Euclidean space $T_{\pi(v)}M$.

**Proposition 2.1.7.** *For $v\in\mathbb{A}$, let $\bar{\nabla}^v$ be a flat linear connection of the vector space $T_{\pi(v)}M$, and $x,y\in T_{\pi(v)}M$. Then, for any extension $Y\in\mathfrak{X}(T_{\pi(v)}M)$*

$$g_v(x,y)=\tfrac{1}{2}\left(x(Y(L|_{T_{\pi(v)}M}))-(\bar{\nabla}^v_xY)(L|_{T_{\pi(v)}M})\right)(v)\,. \tag{2.4}$$

*Proof.* Locally, $x=\alpha^i\frac{\partial}{\partial x^i}|_{\pi(v)}$, $y=\beta^i\frac{\partial}{\partial x^i}|_{\pi(v)}$ for $\alpha^i,\beta^i\in\mathbb{R}$ and $Y=Y^i\frac{\partial}{\partial x^i}\circ\pi$ for $Y^i\in C^\infty(T_{\pi(v)}M)$. Write as $\bar{\Gamma}^k_{ij}(v)\in C^\infty(T_{\pi(v)}M)$ the coefficients of $\bar{\nabla}^v$. Then we can compute the right hand side of (2.4)

$$\begin{aligned}\tfrac{1}{2}&\left(x(Y(L|_{T_{\pi(v)}M}))-(\bar{\nabla}^v_xY)(L|_{T_{\pi(v)}M})\right)(v)\\ &=\frac{\alpha^i}{2}\left(\frac{\partial Y^j}{\partial y^i}\frac{\partial L}{\partial y^j}+\beta^j\frac{\partial^2 L}{\partial y^i\partial y^j}-\left(\frac{\partial Y^k}{\partial y^i}+\beta^j\bar{\Gamma}^k_{ij}\right)\frac{\partial L}{\partial y^k}\right)(v)\\ &=\frac{\alpha^i\beta^j}{2}\left(\frac{\partial^2 L}{\partial y^i\partial y^j}-\bar{\Gamma}^k_{ij}\frac{\partial L}{\partial y^k}\right)(v)\,.\end{aligned}$$

The connection $\bar{\nabla}$ admits natural coordinates on $T_pM$ as a manifold, for which the coefficients $\bar{\Gamma}^k_{ij}(v)$ are zero at $v$. In those coordinates, we have obtained

$$\tfrac{1}{2}\left(x(Y(L|_{T_{\pi(v)}M}))-(\bar{\nabla}^v_xY)(L|_{T_{\pi(v)}M})\right)(v)=\frac{\alpha^i\beta^j}{2}\frac{\partial^2 L}{\partial y^i\partial y^j}|_v\,,$$

whose right hand side is $g_v(x,y)$ by definition (1.17) independently of a choice of coordinates. □

**Proposition 2.1.8.** *For each unitary admissible vector $v \in \mathbb{A} \cap L^{-1}(\{1\})$ and $\Sigma = L^{-1}(\{1\}) \cap T_{\pi(v)}M$ the indicatrix of $L$ in its tangent plane,*

$$g_v|_{T_{\pi(v)}\Sigma \times T_{\pi(v)}\Sigma} = -\sigma \tag{2.5}$$

*where $\sigma\colon T_v\Sigma \times T_v\Sigma \to \mathbb{R}$ such that $\sigma v$ is the second fundamental form of $\Sigma$ along $v$ with respect to the Riemannian (squared) metric $h\colon w \mapsto g_v(w,w)$.*

*Proof.* Let $\bar{\nabla}$ be trivial connection of $T_{\pi(v)}M$, that is to say the Levi-Civita connection with respect to any Euclidean metric. By (2.4), we have for $x, y \in T_v\Sigma \subset T_{\pi(v)}M$

$$g_v(x,y) = -\tfrac{1}{2}(\dot{\partial}L)_v(\sigma(x,y)v) = -\sigma(v,v)g_v(v,v) = -\sigma(x,y)$$

after eliminating the action, null on $L$, of vectors tangent to $\Sigma$, being $\sigma(x,y)v$ by definition the part of $\bar{\nabla}^v_x Y$ $g_v$-orthogonal to $v$. □

This proposition generalises (1.23), up to (2.3) accounting for the factor $\langle \frac{v}{F(v)} - W, \frac{v}{F(v)} \rangle^{-1}$ coming from

$$\tfrac{v}{F(v)} = \rho_{\frac{v}{F(v)}} + \langle \tfrac{v}{F(v)} - W, \tfrac{v}{F(v)} \rangle (\tfrac{v}{F(v)} - W),$$

recalling that $\langle \frac{v}{F(v)} - W, \frac{v}{F(v)} - W \rangle = 1$ and where $\rho_{\frac{v}{F(v)}}$ is tangent to $L^{-1}(\{1\})$ at $\frac{v}{F(v)}$.

### 2.1.3 The Gauss formula

Let $\nabla$ be the Chern connection of $(M,L)$, and $\widehat{\nabla}$ that of $(\widehat{M},\hat{L})$. Let $V \in \mathfrak{X}(M)$ with values in $\mathbb{A}$ and $X, Y, Z \in \mathfrak{X}(M)$ tangent to $\widehat{M}$ and $v, x, y, z$ their image at some point of $\widehat{M}$. By the Koszul formula (1.39), we have

$$\begin{aligned}2g_v(\nabla^v_x Y, z) = {} & x g_V(Z,Y) + y g_V(X,Z) - z g_V(X,Y) \\ & + g_v([Z,X],y) + g_v(x,[Z,Y]) + g_v([X,Y],z) \\ & - 2C_v(z,y,\nabla^v_x V) - 2C_v(x,z,\nabla^v_y V) + 2C_v(x,y,\nabla^v_z V)\end{aligned}$$

and

$$\begin{aligned}2g_v(\widehat{\nabla}^v_x Y, z) = {} & x g_V(Z,Y) + y g_V(X,Z) - z g_V(X,Y) \\ & + g_v([Z,X],y) + g_v(x,[Z,Y]) + g_v([X,Y],z) \\ & - 2C_v(z,y,\widehat{\nabla}^v_x V) - 2C_v(x,z,\widehat{\nabla}^v_y V) + 2C_v(x,y,\widehat{\nabla}^v_z V).\end{aligned}$$

Their difference yields

$$\begin{aligned}g_v(\nabla^v_x Y - \widehat{\nabla}^v_x Y, z) = {} & -C_v(z,y,\nabla^v_x V - \widehat{\nabla}^v_x V) - C_v(x,z,\nabla^v_y V - \widehat{\nabla}^v_y V) \\ & + C_v(x,y,\nabla^v_z V - \widehat{\nabla}^v_z V).\end{aligned} \tag{2.6}$$

By Proposition 1.2.7, we obtain

$$g_v(\nabla^v_v V - \widehat{\nabla}^v_v V, z) = 0.$$

Therefore, by non-degeneracy, $(\nabla^v_v V)^\top_v - \widehat{\nabla}^v_v V = 0$ or in other words $\nabla^v_v V - \widehat{\nabla}^v_v V = \mathrm{II}_v(v,v)$. Thus, setting $Y = V$ in (2.6) gives us

$$g_v(\nabla^v_x V - \widehat{\nabla}^v_x V, z) = -C_v(x,z,\mathrm{II}_v(v,v)),$$

allowing us to pin down $\nabla - \widehat{\nabla}$ as the following tensor.

**Definition 2.1.9.** *Let $\widehat{Q}$ be the symmetric anisotropic tensor of $\widehat{M}$ defined for $v \in \mathbb{A} \cap \pi^{-1}(\widehat{M})$ and $x, y, z \in T_{\pi(v)}\widehat{M}$ by*

$$
\begin{aligned}
g_v(\widehat{Q}_v(x,v),z) &= -C_v(x,z,\mathrm{II}_v(v,v)), \\
g_v(\widehat{Q}_v(x,y),z) &= -C_v(z,y,\widehat{Q}_v(x,v)) - C_v(x,z,\widehat{Q}_v(y,v)) + C_v(x,y,\widehat{Q}_v(z,v))
\end{aligned} \tag{2.7}
$$

*uniquely determined by non-degeneracy of $g_v$.*

**Theorem 2.1.10** (Gauss Formula). *$\widehat{Q}$ verifies $\nabla^\top = \widehat{\nabla} + \widehat{Q}$ where $\nabla$ is the Chern connection of $(M,L)$ and $\widehat{\nabla}$ that of $(\widehat{M},\widehat{L})$. More precisely, let $v \in \mathbb{A}$, $x, y \in T_{\pi(v)}\widehat{M}$. Then, for all extension $Y \in \mathfrak{X}(M)$ of $y$,*

$$\nabla^v_x Y = \widehat{\nabla}^v_x Y + \widehat{Q}_v(x,y) + \mathrm{II}_v(x,y). \tag{2.8}$$

Taking the vertical derivative of (2.8), we can also read the Gauss formula under the form

$$P_v(x,y,\cdot) = \widehat{P}_v(x,y,\cdot) + (\dot{\partial}\widehat{Q})_v(x,y,\cdot) + (\dot{\partial}\mathrm{II})_v(x,y,\cdot), \tag{2.9}$$

recalling that $P$ and $\widehat{P}$ are as defined in (1.50). Note that in the anisotropic case we are far from being able to identify the induced connection $\nabla^\top$ with $\widehat{\nabla}$, and must instead deal with several additional terms compared with the classical Riemannian or pseudo-Riemannian case.

**Lemma 2.1.11.** *$\widehat{Q}$ satisfies the following identities for each admissible vector $v$ and vector $u$ and $w$ tangent to the submanifold:*

*(i)* $g_v(\widehat{Q}_v(u,v),v) = g_v(\widehat{Q}_v(v,u),v) = 0$,

*(ii)* $g_v(\widehat{Q}_v(u,w),v) = -g_v(\widehat{Q}_v(u,v),w) = -g_v(\widehat{Q}_v(v,u),w)$,

*(iii)* $g_v((\dot{\partial}\widehat{Q})_v(u,v,w),v) = g_v((\dot{\partial}\widehat{Q})_v(v,u,w),v) = 0$,

*(iv)* $g_v((\widehat{\nabla}_w\widehat{Q})_v(u,v),v) = g_v((\widehat{\nabla}_w\widehat{Q})_v(v,u),v) = 0$.

*Proof.* The first two identities are a direct consequence of the cancellation of the Cartan tensor by property of the Cartan tensor. Note that $\dot{\partial}\widehat{Q}$ is symmetric in its first two components. To prove that $g_v((\dot{\partial}\widehat{Q})_v(u,v,w),v)$ is zero, let us compute

$$
\begin{aligned}
\tfrac{\partial}{\partial t} g_{v+tw}(\widehat{Q}_{v+tw}(u,v+tw),v+tw)|_{t=0} = 2C_v(\widehat{Q}_v(u,v),v,w) + g_v((\dot{\partial}\widehat{Q})_v(u,v,w),v) \\
+ g_v(\widehat{Q}_v(u,w),v) + g_v(\widehat{Q}_v(u,v),w).
\end{aligned}
$$

The left hand side and last two terms of the right hand side are zero as mentioned above, the first term of the right hand side is zero by property of the Cartan tensor and we have

$$g_v((\dot{\partial}\widehat{Q})_v(u,v,w),v) = 0.$$

Note that $(\widehat{\nabla}_w\widehat{Q})_v$ is symmetric. To prove that $g_v((\widehat{\nabla}_w\widehat{Q})_v(u,v),v)$ is zero, consider a locally admissible extension $V$ of $v$ tangent to the submanifold and satisfying $\widehat{\nabla}^v V = 0$ and an extension $U$ of $u$ also tangent to the submanifold. Discarding the terms in $\widehat{\nabla}^v V$, we have

$$w(g_V(\widehat{Q}_V(U,V),V)) = g_v((\widehat{\nabla}_w\widehat{Q})_v(u,v),v) + g_v(\widehat{Q}_v(\widehat{\nabla}^v_w U,v),v)$$

whose left hand side and last term of the right hand side are zero as mentioned above. □

## 2.2 The Gauss and Codazzi equations

In this part, the anisotropic analogue of the Gauss and Codazzi equations are derived, relating the parts tangent and $g$-orthogonal to $\widehat{M}$ of the anisotropic curvature tensor $R$ associated to the Chern connection $\nabla$ of $(M,L)$ to terms that only depend on the submanifolds $\widehat{M}$.

### 2.2.1 Submanifold curvature tensors

In order to face the question of the Gauss and Codazzi equations in the next subsection, let us introduce the following tensors.

**Definition 2.2.1.** *Let $\nabla\mathrm{II}$ be the tensor of $\widehat{M}$ defined for each locally admissible $V$ extending $v\in\mathbb{A}\cap\pi^{-1}(\widehat{M})$ and $x,y,z\in T_{\pi(v)}\widehat{M}$ with any extensions $X,Y,Z\in\mathfrak{X}(M)$ by the relation*

$$(\nabla^v_x(\mathrm{II}(Y,Z)))^\perp_v=(\nabla_x\mathrm{II})_v(y,z)+\mathrm{II}_v(\widehat{\nabla}^v_xY,z)+\mathrm{II}_v(y,\widehat{\nabla}^v_xZ)\,. \tag{2.10}$$

**Definition 2.2.2.** *Let $\nabla\widehat{Q}$ be the tensor of $\widehat{M}$ defined for $V\in\mathfrak{X}(M)$ extending $v\in\mathbb{A}\cap\pi^{-1}(\widehat{M})$ with values in $\mathbb{A}$ and $x,y,z\in T_{\pi(v)}\widehat{M}$ with any extensions $X,Y,Z\in\mathfrak{X}(M)$*

$$\widehat{\nabla}^v_x\widehat{Q}_V(Y,Z)=(\nabla_x\widehat{Q})_v(y,z)+\widehat{Q}_v(\widehat{\nabla}^v_xY,z)+\widehat{Q}_v(y,\widehat{\nabla}^v_xZ)+(\dot{\partial}\widehat{Q})_v(y,z,\widehat{\nabla}^v_xV)\,. \tag{2.11}$$

The following lemma provides an elementary proof in our context of the particular case of Corollary 2.17 appearing in [50] applied to the torsion-free Chern connection and induced connection.

**Lemma 2.2.3.** *Let $R^\top$ the anisotropic curvature tensor associated to the connection $\nabla^\top$ induced on $\widehat{M}$ by the Chern connection $\nabla$ of $(M,L)$, and $\widehat{R}$ that of the Chern connection $\widehat{\nabla}$ of $(\widehat{M},\hat{L})$. Then for $v\in\hat{\mathbb{A}}$ and $x,y,z\in T_{\pi(v)}\widehat{M}$*

$$\begin{aligned}R^\top_v(x,y)z&=\widehat{R}_v(x,y)z+(\widehat{\nabla}_x\widehat{Q})_v(y,z)-(\widehat{\nabla}_y\widehat{Q})_v(x,z)+\widehat{Q}_v(x,\widehat{Q}_v(y,z))-\widehat{Q}_v(y,\widehat{Q}_v(x,z))\\&+(\dot{\partial}\widehat{Q})_v(y,z,\widehat{Q}_v(x,v))-(\dot{\partial}\widehat{Q})_v(x,z,\widehat{Q}_v(y,v))-\widehat{P}_v(y,z,\widehat{Q}_v(x,v))+\widehat{P}_v(x,z,\widehat{Q}_v(x,v))\,,\end{aligned} \tag{2.12}$$

*where $\widehat{P}$ is the vertical derivatives of $\widehat{\nabla}$ in the sense of (1.50).*

*Proof.* Let $V,X,Y,Z\in\mathfrak{X}(M)$ tangent to $\widehat{M}$ extend $v,x,y,z$ respectively, with $V$ having values in $\mathbb{A}$. By the decomposition (2.8), we have

$$(\nabla^v_{[X,Y]}Z)^\top_v=\widehat{\nabla}^v_{[X,Y]}Z+\widehat{Q}_v([X,Y],z)\,, \tag{2.13}$$

while by symmetry of $\widehat{Q}$ and our definition (2.11)

$$\begin{aligned}(\nabla^v_x(\nabla^V_YZ)^\top_V)^\top_v&=\left(\widehat{\nabla}^v_x+\widehat{Q}_v(x,\cdot)\right)\left(\widehat{\nabla}^V_YZ+\widehat{Q}_V(Y,Z)\right)\\&=\widehat{\nabla}^v_x\widehat{\nabla}^V_YZ+\widehat{Q}_v(x,\widehat{\nabla}^v_yZ)+\widehat{\nabla}^v_x\widehat{Q}_V(Y,Z)+\widehat{Q}_v(x,\widehat{Q}_v(y,z))\\&=\widehat{\nabla}^v_x\widehat{\nabla}^V_YZ+\widehat{Q}_v(\widehat{\nabla}^v_xY,z)+\widehat{Q}_v(y,\widehat{\nabla}^v_xZ)+\widehat{Q}_v(x,\widehat{\nabla}^v_yZ)\\&\quad+(\widehat{\nabla}_x\widehat{Q})_v(y,z)+\widehat{Q}_v(x,\widehat{Q}_v(y,z))+(\dot{\partial}\widehat{Q})_v(y,z,\widehat{\nabla}^v_xV)\,.\end{aligned}$$

The same is true when exchanging $x,X$ and $y,Y$. Therefore,

$$\begin{aligned}(R^\top)^V_{\pi(v)}(x,y)z&=\widehat{R}^V_{\pi(v)}(x,y)z+(\widehat{\nabla}_x\widehat{Q})_v(y,z)-(\widehat{\nabla}_y\widehat{Q})_v(x,z)\\&+\widehat{Q}_v(x,\widehat{Q}_v(y,z))-\widehat{Q}_v(y,\widehat{Q}_v(x,z))+(\dot{\partial}\widehat{Q})_v(y,z,\widehat{\nabla}^v_xV)-(\dot{\partial}\widehat{Q})_v(x,z,\widehat{\nabla}^v_yV)\,.\end{aligned}$$

By (1.54) and (2.8), we have

$$(R^\top)^V_{\pi(v)}(x,y)z = R^\top_v(x,y)z + P^\top_v(y,z,(\nabla^v_x V)^\top_v) - P^\top_v(x,z,(\nabla^v_y V)^\top_v)$$
$$= R^\top_v(x,y)z + P^\top_v(y,z,\widehat{\nabla}^v_x V + \widehat{Q}_v(x,v)) - P^\top_v(x,z,\widehat{\nabla}^v_y V + \widehat{Q}_v(y,v)),$$

while, again by (1.54),

$$\widehat{R}^V_{\pi(v)}(x,y)z = \widehat{R}_v(x,y)z + \widehat{P}_v(y,z,\widehat{\nabla}^v_x V) - \widehat{P}_v(x,z,\widehat{\nabla}^v_y V).$$

From there, (2.9) implies that $\widehat{P}_v + (\dot{\partial}\widehat{Q})_v$ is the vertical derivative of $\nabla^\top$ on $T_{\pi(v)}\widehat{M}$, concluding the proof. □

### 2.2.2 Deriving the equations

**Theorem 2.2.4** (Gauss and Codazzi Equations). *Let $v \in \hat{\mathbb{A}}$, $w \in T_{\pi(v)}M$ $g_v$-orthogonal to $T_{\pi(v)}\widehat{M}$ and $u,x,y,z \in T_{\pi(v)}\widehat{M}$. Let $R$ the anisotropic curvature tensor of the Chern connection $\nabla$ of $(M,L)$ and $\widehat{R}$ that of the Chern connection $\widehat{\nabla}$ of the submanifold $(\widehat{M},\hat{L})$. Then*

$$\begin{aligned} g_v(R_v(x,y)z,u) &= g_v(\widehat{R}_v(x,y)z,u) + g_v(\mathrm{II}_v(x,z),\mathrm{II}_v(y,u)) - g_v(\mathrm{II}_v(y,z),\mathrm{II}_v(x,u)) \\ &+ g_v((\nabla_x\widehat{Q})_v(y,z) - (\nabla_y\widehat{Q})_v(x,z),u) + 2C_v(\mathrm{II}_v(x,z),\mathrm{II}_v(y,v),u) - 2C_v(\mathrm{II}_v(y,z),\mathrm{II}_v(x,v),u) \\ &+ g_v(P_v(x,z,\mathrm{II}_v(y,v)) - P_v(y,z,\mathrm{II}_v(x,v)),u) + g_v(\widehat{P}_v(x,z,\widehat{Q}_v(x,v)) - \widehat{P}_v(y,z,\widehat{Q}_v(x,v)),u) \\ &+ \widehat{Q}_v(x,\widehat{Q}_v(y,z)) - \widehat{Q}_v(y,\widehat{Q}_v(x,z)) + (\dot{\partial}\widehat{Q})_v(y,z,\widehat{Q}_v(x,v)) - (\dot{\partial}\widehat{Q})_v(x,z,\widehat{Q}_v(y,v)), \end{aligned} \tag{2.14}$$

*where $\nabla\widehat{Q}$ is defined in (2.11), and*

$$\begin{aligned} g_v(R_v(x,y)z,w) &= g_v(\mathrm{II}_v(x,\widehat{Q}_v(y,z)) - \mathrm{II}_v(y,\widehat{Q}_v(x,z)) + (\nabla_x\mathrm{II})_v(y,z) - (\nabla_y\mathrm{II})_v(x,z),w) \\ &+ g_v((\dot{\partial}\mathrm{II})_v(y,z,\mathrm{II}_v(x,v)) - (\dot{\partial}\mathrm{II})_v(x,z,\mathrm{II}_v(y,v)),w) \\ &+ g_v(P_v(x,z,\mathrm{II}_v(y,v)) - P_v(y,z,\mathrm{II}_v(x,v)),u) \end{aligned} \tag{2.15}$$

*where $\nabla\mathrm{II}$ is defined in (2.10).*

*Proof.* Let $V$ a locally admissible extension of $v \in \mathbb{A}$ tangent to $\widehat{M}$ satisfying $\widehat{\nabla}^v V = 0$ and $U,X,Y,Z \in \mathfrak{X}(M)$ respective extensions of $u,x,y,z$ tangent to $\widehat{M}$. By (2.8), we have on the one hand

$$\nabla^v_{[X,Y]}Z = \widehat{\nabla}^v_{[X,Y]}Z + \widehat{Q}_v([X,Y],Z) + \mathrm{II}_v([X,Y],z),$$

while on the other

$$\begin{aligned} \nabla^v_x\nabla^V_Y Z &= \nabla^v_x\widehat{\nabla}^V_Y Z + \nabla^v_x\widehat{Q}_V(Y,Z) + \nabla^v_x(\mathrm{II}_V(Y,Z)) \\ &= \widehat{\nabla}^v_x\widehat{\nabla}^V_Y Z + \widehat{Q}_v(x,\widehat{\nabla}^v_y Z) + \mathrm{II}_v(x,\widehat{\nabla}^v_y Z) + (\widehat{\nabla}_x\widehat{Q})_v(y,z) + \widehat{Q}_v(\widehat{\nabla}^v_x Y,z) \\ &+ \widehat{Q}_v(y,\widehat{\nabla}^v_x Z) + (\dot{\partial}\widehat{Q})_v(y,z,\widehat{\nabla}^v_x V) + \widehat{Q}_v(x,\widehat{Q}_v(y,z)) + \mathrm{II}_v(x,\widehat{Q}_v(y,z)) \\ &(\nabla^v_x(\mathrm{II}_V(Y,Z)))^\top_v + (\nabla_x\mathrm{II})_v(y,z) + \mathrm{II}_v(\widehat{\nabla}^v_x Y,z) + \mathrm{II}_v(y,\widehat{\nabla}^v_x Z), \end{aligned}$$

and similarly when swapping $x, X$ with $y, Y$. These combine into

$$\begin{aligned} R^V_{\pi(v)}(x,y)z &= \widehat{R}^V_{\pi(v)}(x,y)z + (\widehat{\nabla}_x\widehat{Q})_v(y,z) - (\widehat{\nabla}_y\widehat{Q})_v(x,z) \\ &\quad + (\dot{\partial}\widehat{Q})_v(y,z,\widehat{Q}_v(x,v)) - (\dot{\partial}\widehat{Q})_v(x,z,\widehat{Q}_v(y,v)) + \widehat{Q}_v(x,\widehat{Q}_v(y,z)) - \widehat{Q}_v(y,\widehat{Q}_v(x,z)) \\ &\quad + \mathrm{II}_v(x,\widehat{Q}_v(y,z)) - \mathrm{II}_v(y,\widehat{Q}_v(x,z)) + (\nabla^v_x \mathrm{II}_V(Y,Z))^\top_v - (\nabla^v_y \mathrm{II}_V(X,Z))^\top_v \\ &\quad + (\nabla_x \mathrm{II})_v(y,z) - (\nabla_y \mathrm{II})_v(x,z) + ((\dot{\partial}\mathrm{II})_v(y,z,\nabla^v_x V))^\perp_v - ((\dot{\partial}\mathrm{II})_v(x,z,\nabla^v_y V))^\perp_v \,. \end{aligned}$$

By (1.54) and (2.9), this means

$$\begin{aligned} R_v(x,y)z &= \widehat{R}_v(x,y)z + (\widehat{\nabla}_x\widehat{Q})_v(y,z) - (\widehat{\nabla}_y\widehat{Q})_v(x,z) \\ &\quad + (\dot{\partial}\widehat{Q})_v(y,z,\widehat{Q}_v(x,v)) - (\dot{\partial}\widehat{Q})_v(x,z,\widehat{Q}_v(y,v)) + \widehat{Q}_v(x,\widehat{Q}_v(y,z)) - \widehat{Q}_v(y,\widehat{Q}_v(x,z)) \\ &\quad + \mathrm{II}_v(x,\widehat{Q}_v(y,z)) - \mathrm{II}_v(y,\widehat{Q}_v(x,z)) + (\nabla^v_x \mathrm{II}_V(Y,Z))^\top_v - (\nabla^v_y \mathrm{II}_V(X,Z))^\top_v \\ &\quad + (\nabla_x \mathrm{II})_v(y,z) - (\nabla_y \mathrm{II})_v(x,z) + \widehat{P}_v(y,z,\widehat{\nabla}^v_x V) \\ &\quad - \widehat{P}_v(x,z,\widehat{\nabla}^v_y V) - P_v(y,z,\nabla^v_x V) + P_v(x,z,\nabla^v_y V) \\ &= \widehat{R}_v(x,y)z + (\widehat{\nabla}_x\widehat{Q})_v(y,z) - (\widehat{\nabla}_y\widehat{Q})_v(x,z) + \widehat{Q}_v(x,\widehat{Q}_v(y,z)) - \widehat{Q}_v(y,\widehat{Q}_v(x,z)) \\ &\quad + \mathrm{II}_v(x,\widehat{Q}_v(y,z)) - \mathrm{II}_v(y,\widehat{Q}_v(x,z)) + (\nabla^v_x \mathrm{II}_V(Y,Z))^\top_v - (\nabla^v_y \mathrm{II}_V(X,Z))^\top_v \\ &\quad + (\nabla_x \mathrm{II})_v(y,z) - (\nabla_y \mathrm{II})_v(x,z) + (\dot{\partial}\mathrm{II})_v(y,z,\widehat{Q}_v(x,v) + \mathrm{II}_v(x,v))^\perp_v \\ &\quad - (\dot{\partial}\mathrm{II})_v(x,z,\widehat{Q}_v(y,v) + \mathrm{II}_v(y,v))^\perp_v - P_v(y,z,\widehat{Q}_v(x,v) + \mathrm{II}_v(x,v)) \\ &\quad + P_v(x,z,\widehat{Q}_v(x,v) + \mathrm{II}_v(x,v)) \,. \end{aligned}$$

Alternatively, in order to alleviate the computation sparing half the terms, one may instead use (2.8) under the form $\nabla = \nabla^\top + \mathrm{II}$ on $\mathfrak{X}(\widehat{M})$, resorting to (2.12) to arrive at the same expression.

Applying $g_v(\cdot,u)$ or $g_v(\cdot,w)$ yields (2.14) and (2.15) respectively, up to (1.38) under the form

$$\begin{aligned} g_v(\nabla^v_x \mathrm{II}_V(Y,Z),u) &= x g_V(\mathrm{II}_V(Y,Z),U) - g_v(\mathrm{II}_v(y,z),\nabla^v_x U) - 2C_v(\mathrm{II}_v(y,z),u,\nabla^v_x V) \\ &= -g_v(\mathrm{II}_v(y,z),\mathrm{II}_v(x,u)) - 2C_v(\mathrm{II}_v(y,z),\mathrm{II}_v(x,v),u) - 2C_v(\mathrm{II}_v(y,z),(\nabla^v_x V)^\top_v,u) \,. \end{aligned}$$

By taking for $\alpha = (\nabla^v_x V)^\top_v$ the derivative of $g_{v+t\alpha}(\nabla^{v+t\alpha}_y Z,u) = g_{v+t\alpha}((\nabla^{v+t\alpha}_y Z)^\top_{v+t\alpha},u)$ at $t=0$, one finds by (2.8) that

$$\begin{aligned} 2C_v(\mathrm{II}_v(y,z),\alpha,u) &= g_v(\widehat{P}_v(y,z,\alpha) + (\dot{\partial}\widehat{Q})_v(y,z,\alpha) - P_v(y,z,\alpha),u) \\ &= -g_v((\dot{\partial}\mathrm{II})_v(y,z,\alpha),u) \,, \end{aligned}$$

which simplifies the $((\dot{\partial}\mathrm{II})_v(y,z,\nabla^v_x V))^\top_v$ term to $(\dot{\partial}\mathrm{II})_v(y,z,\mathrm{II}_v(x,v))$ in (2.14), same goes when swapping $x$ and $y$.

Also note that by (2.9) $P_v - (\dot{\partial}\mathrm{II})_v = \widehat{P} + (\dot{\partial}\widehat{Q})_v$ on $\mathfrak{X}(\widehat{M})$, and is therefore vertical when evaluated at $\widehat{Q}$, simplifying $P_v(y,z,\widehat{Q}_v(x,v) + \mathrm{II}_v(x,v))$ and $((\dot{\partial}\mathrm{II})_v(y,z,\widehat{Q}_v(x,v) + \mathrm{II}_v(x,v)))^\perp_v$ terms in (2.15) by removing $\widehat{Q}_v(x,v)$, same goes when swapping $x$ and $y$. □

Note that this theorem can be extended to a generalised form {0'} and {4'} of Chapter 3 which holds for an admissible $v$ not necessarily tangent to the submanifold, as will be discussed later.

**Corollary 2.2.5.** *Let $(M,L)$ be a pseudo-Finsler manifold and $\widehat{M}$ a non-degenerate submanifold. The flag curvature of $(\widehat{M},L|_{\widehat{M}})$ with flagpole $v \in \mathbb{A} \cap T\widehat{M}$ and flag $u \in T_{\pi(v)}\widehat{M}$ with*

*$L(v)g_v(u,u) \neq g_v(v,u)^2$ is given by*

$$K_v(u) = \widehat{K}_v(u) + \frac{g_v(\mathrm{I\!I}_v(u,u),\mathrm{I\!I}_v(v,v)) - g_v(\mathrm{I\!I}_v(v,u),\mathrm{I\!I}_v(v,u))}{L(v)g_v(u,u) - g_v(v,u)^2} + \frac{g_v(P_v(u,u,\mathrm{I\!I}_v(v,v)) - (\widehat{\nabla}_v\widehat{Q})_v(u,u),v) + C_v(u,\widehat{Q}_v(v,u),\mathrm{I\!I}_v(v,v))}{L(v)g_v(u,u) - g_v(v,u)^2}.$$

*Proof.* By the Gauss equation (2.14), we have

$$\begin{aligned} g_v(R_v(v,u)u,v) &= g_v(\widehat{R}_v(v,u)u,v) - g_v(\mathrm{I\!I}_v(u,u),\mathrm{I\!I}_v(v,v)) \\ &\quad + g_v(\mathrm{I\!I}_v(v,u),\mathrm{I\!I}_v(v,u)) - 2C_v(\mathrm{I\!I}_v(v,v),\mathrm{I\!I}_v(u,u),v) + 2C_v(\mathrm{I\!I}_v(v,u),\mathrm{I\!I}_v(v,u),v) \\ &\quad + g_v(P_v(v,u,\mathrm{I\!I}_v(v,u)) - P_v(u,u,\mathrm{I\!I}_v(v,v)),v) + g_v((\widehat{\nabla}_v\widehat{Q})_v(u,u) - (\widehat{\nabla}_u\widehat{Q})_v(v,u),v) \\ &\quad + g_v((\dot{\partial}\widehat{Q})_v(v,u,\widehat{Q}_v(u,v)) - (\dot{\partial}\widehat{Q})_v(u,u,\widehat{Q})_v(v,v)),v) \\ &\quad + g_v(\widehat{Q}_v(v,\widehat{Q}_v(u,u)) - \widehat{Q}_v(u,\widehat{Q}_v(v,u)),v) \end{aligned}$$

with many vanishing terms: the Cartan tensor terms vanish by property of the Cartan tensor, the first $P$ and $\widehat{P}$ terms vanish by (1.44), and by Lemma 2.1.11 all the terms in $\widehat{Q}$ are zero except $g_v((\widehat{\nabla}_u\widehat{Q})_v(u,u),v)$ and $-g_v(\widehat{Q}_v(u,\widehat{Q}_v(v,u)),v)$. Note that by definition of $\widehat{Q}$ this term is $-C_v(u,\widehat{Q}_v(v,u),\mathrm{I\!I}_v(v,v))$. The previous equation thus reduces to

$$\begin{aligned} g_v(R_v(v,u)u,v) &= g_v(\widehat{R}_v(v,u)u,v) - g_v(\mathrm{I\!I}_v(u,u),\mathrm{I\!I}_v(v,v)) + g_v(\mathrm{I\!I}_v(v,u),\mathrm{I\!I}_v(v,u)) \\ &\quad - g_v(P_v(u,u,\mathrm{I\!I}_v(v,v)),v) + g_v((\widehat{\nabla}_u\widehat{Q})_v(u,u),v) - C_v(u,\widehat{Q}_v(v,u),v). \end{aligned}$$

Conclude by definition of the flag curvature. □

## 2.3 Randers-Minkowski submanifolds

This section concerns itself with the particular case of submanifolds $S \subset \mathscr{V}$ of a vector space $\mathscr{V}$ equipped with a Randers norm $F$ with Zermelo data $(\langle\cdot,\cdot\rangle, W)$. By expressing various invariants of $S$ in terms of those of $\mathscr{V}$, we can deduce from (2.14) an expression of the flag curvature of $S$.

Letting $v \in \mathscr{V}$, the tangent space $T_v\mathscr{V}$ can be canonically identified with $\mathscr{V}$ itself. In that sense, $(\mathscr{V}, F)$ defines a Finsler manifold, with $\mathbb{A} = \mathscr{V} \setminus \{0\}$ up to that identification, and whose Chern connection coincides with the Levi-Civita connection of $(\mathscr{V}, \langle\cdot,\cdot\rangle)$. For clarity, however, we shall still occasionally distinguish $T\mathscr{V} \ni v$ from $\mathscr{V} \ni \pi(v)$.

Let for simplicity $\nabla$ denote the Chern connection of $(S, F|_{TS})$, instead of the more cumbersome $\widehat{\nabla}$, since it will be the only anisotropic connection we will concern ourselves with. The Levi-Civita connection $\bar{\nabla}$ of each of the fibres of $(T\mathscr{V}, \langle\cdot,\cdot\rangle)$ can incidentally be identified with that of $(\mathscr{V}, \langle\cdot,\cdot\rangle)$ itself by the identification mentioned above, which should hopefully make sure our notation remains clear and unambiguous, as we have so far only denoted isotropic connections by $\bar{\nabla}$ and anisotropic connections by $\nabla$.

### 2.3.1 Second fundamental form

Recall that, in this context, equations (1.23), (1.24) and (1.25) hold. Tangency to the indicatrix $\Sigma = F^{-1}(\{1\})$ at $\frac{v}{F(v)} \in \Sigma$ is by translation equivalent to tangency to the indicatrix of $\langle\cdot,\cdot\rangle$ at $\frac{v}{F(v)} - W$. In other words, (1.24) applies to vectors $\langle\cdot,\cdot\rangle$-orthogonal to $\frac{v}{F(v)} - W$. Note the relation

$$g_v(\tfrac{v}{F(v)},W)=\frac{\langle\frac{v}{F(v)}-W,W\rangle}{\langle\frac{v}{F(v)}-W,\frac{v}{F(v)}\rangle}=1-\frac{1}{\langle\frac{v}{F(v)}-W,\frac{v}{F(v)}\rangle},\tag{2.16}$$

where the first identity is applying (1.23), and the second (1.22) under the form

$$\langle\frac{v}{F(v)}-W,W\rangle=\langle\frac{v}{F(v)}-W,\frac{v}{F(v)}\rangle-1\,.$$

**Proposition 2.3.1.** *For each admissible $v$ and $x$ tangent to $(F^{-1}(\{1\})\cap T_{\pi(v)}M)$,*

$$\langle\tfrac{v}{F(v)},x\rangle=-\langle\tfrac{v}{F(v)}-W,\tfrac{v}{F(v)}\rangle^2 g_v(W,x)\tag{2.17}$$

*Proof.* Recall that, by property of the indicatrix, the vector $x$ tangent to the indicatrix is both $g_v$-orthogonal to $v$ and $\langle\cdot,\cdot\rangle$-orthogonal to $\frac{v}{F(v)}-W$, such that by Propositions 2.1.8 and 2.1.6

$$\begin{aligned}\langle v,x\rangle&=\langle v-\langle\tfrac{v}{F(v)}-W,v\rangle(\tfrac{v}{F(v)}-W),x\rangle\\&=-g_v(v-\langle\tfrac{v}{F(v)}-W,v\rangle(\tfrac{v}{F(v)}-W),x)\langle\tfrac{v}{F(v)}-W,v\rangle=-g_v(W,x)\langle\tfrac{v}{F(v)}-W,v\rangle^2\end{aligned}$$

□

**Lemma 2.3.2.** *Let $\mathrm{II}'$ denote the fundamental form of $S$ along $N\in\mathfrak{X}(S)$ $\langle\cdot,\cdot\rangle$-orthogonal to $S$, while $\mathrm{II}$ is the anisotropic fundamental form of $S$ as defined in Definition 2.1.2. Then, for $v\in\mathbb{A}\cap TS$*

$$\mathrm{II}_v=\mathrm{II}'+\frac{\langle\mathrm{II}',W\rangle}{\langle\frac{v}{F(v)}-W,\frac{v}{F(v)}\rangle}(\tfrac{v}{F(v)}-W_v)\,,\tag{2.18}$$

*where $W_v=W^\top_{T_vS\cap T_{v/F(v)}\Sigma}$ represents the $\langle\cdot,\cdot\rangle$-orthogonal projection of $W$ onto $T_vS\cap T_{v/F(v)}\Sigma$.*

*Proof.* Since the Chern connection of $g$ coincides with the Levi-Civita connection of $\langle\cdot,\cdot\rangle$, we have for any $x,y\in T_{\pi(v)}S$

$$\mathrm{II}_v(x,y)=(\mathrm{II}'(x,y))^\perp_v=\mathrm{II}'(x,y)-(\mathrm{II}'(x,y))^\top_v\,,$$

whose second term we can express by projecting onto a base of $T_vS$. By (1.23), we have

$$g_v(\mathrm{II}'(x,y),\tfrac{v}{F(v)})\tfrac{v}{F(v)}=\frac{\langle\mathrm{II}'(x,y),\frac{v}{F(v)}-W\rangle}{\langle\frac{v}{F(v)}-W,\frac{v}{F(v)}\rangle}\frac{v}{F(v)}=-\frac{\langle\mathrm{II}'(x,y),W\rangle}{\langle\frac{v}{F(v)}-W,\frac{v}{F(v)}\rangle}\frac{v}{F(v)}\,,\tag{2.19}$$

while for any $e\in T_vS$ $g_v$-orthogonal to $v$, and therefore $\langle\cdot,\cdot\rangle$-orthogonal to $\frac{v}{F(v)}-W$, we have by (1.24)

$$\begin{aligned}g_v(\mathrm{II}'(x,y),e)&=g_v(\chi_v(x,y),e)+\langle\mathrm{II}'(x,y),\tfrac{v}{F(v)}-W\rangle g_v(-W,e)\\&=\frac{\langle\chi_v(x,y),e\rangle}{\langle\frac{v}{F(v)}-W,\frac{v}{F(v)}\rangle}+\langle\mathrm{II}'(x,y),W\rangle g_v(W,e)\end{aligned}$$

where $\chi_v(x,y)=\mathrm{II}'(x,y)-\langle\mathrm{II}'(x,y),\frac{v}{F(v)}-W\rangle(\frac{v}{F(v)}-W)$ is $\langle\cdot,\cdot\rangle$-orthogonal to $\frac{v}{F(v)}-W$. Since both $\langle\mathrm{II}'(x,y),e\rangle$ and $\langle\frac{v}{F(v)}-W,e\rangle$ are zero, so is $\langle\chi_v(x,y),e\rangle$. Then, by (2.17),

$$g_v(\mathrm{II}'(x,y),e)e=\langle\mathrm{II}'(x,y),W\rangle g_v(W,e)e=\frac{\langle\mathrm{II}'(x,y),W\rangle\langle W,e\rangle}{\langle\frac{v}{F(v)}-W,\frac{v}{F(v)}\rangle^2}e\,.\tag{2.20}$$

If $e$ spans a $g_v$-orthonormal set $\mathbb{E}$ of elements $e_i \in T_vS$ that complete $\frac{v}{F(v)}$ into a basis of $T_vS$, then by (1.24) $\mathbb{E}$ is a set of mutually $\langle\cdot,\cdot\rangle$-orthogonal elements with norm $\langle e_i, e_i\rangle = \langle\frac{v}{F(v)} - W, \frac{v}{F(v)}\rangle^{-1}$. Summing (2.20) over $e \in \mathbb{E}$ together with (2.19) then produces

$$(\mathrm{II}'(x,y))_v^\top = -\frac{\langle \mathrm{II}'(x,y), W\rangle}{\langle\frac{v}{F(v)} - W, \frac{v}{F(v)}\rangle}\left(\frac{v}{F(v)} - W_v\right).$$

□

### 2.3.2 Intermediate identities

Detailed out of elementarity but left apart from the main proof of the formula for the flag curvature of a Randers-Minkowski submanifold for clarity's sake, the following tedious yet straightforward computations prove indispensable.

**Lemma 2.3.3.** *Let $v \in T\mathscr{V} \setminus \{0\}$ and $u \in T_{\pi(v)}S$ such that $g_v(u,v) = 0$, and $Q$ defined as in (2.7) and therefore given by (2.8). Then*

$$C_v(u,u,\mathrm{II}_v(v,v)) = -\frac{g_v(u,u)\langle \mathrm{II}_v(v,v), v\rangle}{2F^2(v)\langle\frac{v}{F(v)} - W, \frac{v}{F(v)}\rangle}, \tag{2.21}$$

$$C_v(u,\widehat{Q}_v(u,v),\mathrm{II}_v(v,v)) = -\frac{g_v(u,u)\langle \mathrm{II}_v(v,v), v\rangle^2}{4F^4(v)\langle\frac{v}{F(v)} - W, \frac{v}{F(v)}\rangle^2}. \tag{2.22}$$

*Proof.* Since $g_v(v,\mathrm{II}_v(v,v)) = 0$, we can apply (1.24) to find

$$\langle u, \mathrm{II}_v(v,v)\rangle = \langle\frac{v}{F(v)} - W, \frac{v}{F(v)}\rangle g_v(u,\mathrm{II}_v(v,v)) = 0.$$

Consequently, (1.25) reduces to

$$C_v(u,u,\mathrm{II}_v(v,v)) = -\frac{\langle u,u\rangle\langle \mathrm{II}_v(v,v), v\rangle}{2F^2(v)\langle\frac{v}{F(v)} - W, \frac{v}{F(v)}\rangle},$$

proving the first equality up to (1.24) under the form $\langle u,u\rangle = \langle\frac{v}{F(v)} - W, \frac{v}{F(v)}\rangle g_v(u,u)$. To prove the second equality, note that $g_v(\widehat{Q}_v(u,v),v) = 0$ by (2.7) and Proposition (1.2.7), such that we can again apply (1.24) to find

$$\langle\widehat{Q}_v(u,v), \mathrm{II}_v(v,v)\rangle = \langle\frac{v}{F(v)} - W, \frac{v}{F(v)}\rangle g_v(\widehat{Q}_v(u,v),\mathrm{II}_v(v,v)) = 0.$$

From (1.25) we obtain

$$C_v(u,\widehat{Q}_v(u,v),\mathrm{II}_v(v,v)) = -\frac{\langle u,\widehat{Q}_v(u,v)\rangle\langle \mathrm{II}_v(v,v), v\rangle}{2F^2(v)\langle\frac{v}{F(v)} - W, \frac{v}{F(v)}\rangle^2},$$

where we can identify by (1.24) followed by (2.7)

$$\langle u,\widehat{Q}_v(u,v)\rangle = \langle\frac{v}{F(v)} - W, \frac{v}{F(v)}\rangle g_v(u,\widehat{Q}_v(u,v)) = -\langle\frac{v}{F(v)} - W, \frac{v}{F(v)}\rangle C_v(u,u,\mathrm{II}_v(v,v)).$$

Conclude by (2.21). □

**Lemma 2.3.4.** *Let $v \in TS \setminus \{0\}$ and $u \in T_{\pi(v)}S$ such that $g_v(u,v) = 0$. Then*

$$g_v((\nabla_v \widehat{Q})_v(u,u),v) = -\frac{g_v(u,u)\|\frac{v}{F(v)}-W_v\|^2}{2F(v)\langle\frac{v}{F(v)}-W,\frac{v}{F(v)}\rangle}\Big[\langle(\bar\nabla_v \mathrm{II}')(v,v),W\rangle - \langle \mathrm{II}'(v,v),\mathrm{II}'(v,W^\top)\rangle - \frac{4\langle \mathrm{II}'(v,v),W\rangle\langle \mathrm{II}'(v,W_v),W\rangle}{\langle\frac{v}{F(v)}-W,\frac{v}{F(v)}\rangle} + \frac{2\langle \mathrm{II}'(v,v),W\rangle^2}{F(v)\langle\frac{v}{F(v)}-W,\frac{v}{F(v)}\rangle^2}\Big(2\langle\frac{v}{F(v)}-W,\frac{v}{F(v)}\rangle - \|\frac{v}{F(v)}-W_v\|^2\Big)\Big] \tag{2.23}$$

*where $W^\top$ and $W_v$ are the respective $\langle\cdot,\cdot\rangle$ and $g_v$-orthogonal projections of $W$ onto $T_{\pi(v)}S$ and $T_{\pi(v)}S \cap F^{-1}(\{1\})$ and $\bar\nabla \mathrm{II}$ is given for $X,Y,Z \in \mathfrak{X}(S)$ by*

$$\bar\nabla_X \mathrm{II}'(Y,Z) = (\bar\nabla_X \mathrm{II})(Y,Z) + \mathrm{II}'(\bar\nabla_X Y,Z) + \mathrm{II}'(Y,\bar\nabla_X Z).$$

*Proof.* For respective extensions $U,V \in \mathfrak{X}(\mathscr{V})$ tangent to $S$ of $u,v$ verifying $\nabla^v V = 0$ and $[U,V] = 0$, thus in particular $\nabla^v_v U = \nabla^v_u V = 0$, by (2.7),

$$g_v((\nabla_v \widehat{Q})_v(u,u),v) = v g_V(\widehat{Q}_V(U,U),V) = v C_V(U,U,\mathrm{II}_V(V,V)).$$

Let $W_V$ the $g_V$-orthogonal projection of $W$ onto $TS \cap F^{-1}(\{1\})$ in each fibre of $T\mathscr{V}$. By (2.21) and (2.18),

$$C_V(U,U,\mathrm{II}_V(V,V)) = \frac{g_V(U,U)\langle \mathrm{II}'(V,V),W\rangle\langle\frac{V}{F(V)}-W_V,V\rangle}{2F^2(V)\langle\frac{V}{F(V)}-W,\frac{V}{F(V)}\rangle^2}.$$

Since $vF^2(V) = 2g_v(\nabla^v_v V,v) = 0$ and $v g_V(U,U) = 2g_v(\nabla^v_v U,v) = 0$ for our choice of extensions $U$ and $V$, we can differentiate the previous equation to

$$g_v((\nabla_v \widehat{Q})_v(u,u),v) = -\frac{g_v(u,u)}{2F^2(v)\langle\frac{v}{F(v)}-W,\frac{v}{F(v)}\rangle^2}\Big(-2\frac{v\langle\frac{V}{F(V)}-W,\frac{V}{F(V)}\rangle}{\langle\frac{v}{F(v)}-W,\frac{v}{F(v)}\rangle}\langle \mathrm{II}'(v,v),W\rangle\langle\frac{v}{F(v)}-W_v,v\rangle + v\langle \mathrm{II}'(V,V),W\rangle\langle\frac{v}{F(v)}-W_v,v\rangle + \langle \mathrm{II}'(v,v),W\rangle v\langle\frac{V}{F(V)}-W_V,V\rangle\Big). \tag{2.24}$$

To compute $v\langle\frac{V}{F(V)}-W,\frac{V}{F(V)}\rangle$, recall that by (2.7) and (2.8)

$$\bar\nabla_v V = \nabla^v_v V + \widehat{Q}_v(v,v) + \mathrm{II}_v(v,v) = \mathrm{II}_v(v,v), \tag{2.25}$$

where $\bar\nabla$ is the Levi-Civita connection of $(\mathscr{V},\langle\cdot,\cdot\rangle)$ and $\nabla$ the Chern connection of $(S,F|_{TS})$. $\mathrm{II}_v(v,v)$ is by definition $g_v$-orthogonal to $T_{\pi(v)}S \ni v$, hence $\langle\frac{v}{F(v)}-W,\bar\nabla_v V\rangle = 0$. Therefore, by (2.18),

$$v\langle\frac{V}{F(V)}-W,\frac{V}{F(V)}\rangle = \frac{1}{F^2(v)}\langle \mathrm{II}_v(v,v),v\rangle = \frac{1}{F^2(v)}\langle \mathrm{II}'(v,v),W\rangle\langle\frac{v}{F(v)}-W_v,v\rangle.$$

To compute $v\langle \mathrm{II}'(V,V),W\rangle$, denote by $W^\top$ and $W^\perp$ the $\langle\cdot,\cdot\rangle$-orthogonal projection of $W$ onto the fibre of $TS$ and its $\langle\cdot,\cdot\rangle$-orthogonal, respectively. For $W$ constant, we have $\bar\nabla_v W^\perp = -\bar\nabla_v W^\top$, which implies by (2.18) and $\bar\nabla_v V = \mathrm{II}_v(v,v)$ that

$$\begin{aligned} v\langle \mathrm{II}'(V,V),W\rangle &= \langle\bar\nabla_v \mathrm{II}'(V,V),W^\perp\rangle + \langle \mathrm{II}'(v,v),\bar\nabla_v W^\perp\rangle \\ &= \langle(\bar\nabla_v \mathrm{II}')(v,v),W\rangle + 2\langle \mathrm{II}'(\bar\nabla_v V,v),W\rangle + \langle \mathrm{II}'(v,v),\mathrm{II}'(v,W^\top)\rangle \\ &= \langle(\bar\nabla_v \mathrm{II}')(v,v),W\rangle + \langle \mathrm{II}'(v,v),\mathrm{II}'(v,W^\top)\rangle + \frac{2\langle \mathrm{II}'(v,v),W\rangle}{\langle\frac{v}{F(v)}-W,\frac{v}{F(v)}\rangle}\langle \mathrm{II}'(\frac{v}{F(v)}-W_v,v),W\rangle. \end{aligned}$$

To compute $v\langle\frac{V}{F(V)}-W_V,V\rangle$, let $\mathbb{E}|_{\pi(v)}S$ the completion of $\frac{v}{F(v)}$ into a $g_v$-orthonormal basis of $T_{\pi(v)}S$. Its $\nabla^V$-parallel transport along the integral curves of $V$, which are geodesics of

$(S, L_{TS})$ at any point of $S$ since we chose $V$ to be an extension of $v \in TS$ tangent to $S$, defines a $g_V$-orthogonal frame of $TS$ such that each $E \in \mathbb{E}$ verifies $\nabla^v_v E = 0$.

Recall that $\frac{V}{F(V)} - W$ is $\langle\cdot,\cdot\rangle$-orthogonal to $\mathbb{E}$, in particular $\langle W, E\rangle = \langle \frac{V}{F(v)}, E\rangle$ which is to say

$$\langle W,E\rangle\langle E,V\rangle = F(V)\langle W,E\rangle^2 . \tag{2.26}$$

By (1.24), $\langle E,E\rangle = \langle \frac{V}{F(V)} - W, \frac{V}{F(V)}\rangle$, from which we obtain

$$W_v = \frac{1}{\langle \frac{V}{F(V)} - W, \frac{V}{F(V)}\rangle} \sum_{E\in\mathbb{E}} \langle W,E\rangle E .$$

Note that by (2.7) and (2.8)

$$\bar{\nabla}_v E = \mathrm{II}_v(v,E) + \widehat{Q}_v(v,E) .$$

Consequently, using (2.26) and (2.25)

$$\begin{aligned} v\langle \tfrac{V}{F(V)} - W_V, V\rangle = \frac{2\langle \mathrm{II}_v(v,v),v\rangle}{F(v)} + \frac{v\langle \frac{V}{F(V)} - W, V\rangle}{\langle \frac{v}{F(v)} - W, v\rangle^2} \sum_{E\in\mathbb{E}} \langle W,E\rangle\langle E,v\rangle \\ - \frac{2F(v)}{\langle \frac{v}{F(v)} - W, v\rangle} \sum_{E\in\mathbb{E}} \langle W, \mathrm{II}_v(v,E) + Q_v(v,E)\rangle\langle E,W\rangle . \end{aligned} \tag{2.27}$$

The second term can be computed with $v\langle \frac{V}{F(V)} - W, \frac{V}{F(V)}\rangle = \frac{1}{F^2(v)}\langle \mathrm{II}_v(v,v),v\rangle$ as

$$\frac{v\langle \frac{V}{F(V)} - W, V\rangle}{\langle \frac{v}{F(v)} - W, v\rangle^2} \sum_{E\in\mathbb{E}} \langle W,E\rangle\langle E,v\rangle = \frac{\langle \mathrm{II}_v(v,v),v\rangle}{F^2(v)\langle \frac{v}{F(v)} - W, \frac{v}{F(v)}\rangle} \sum_{E\in\mathbb{E}} \langle\langle W,E\rangle E, v\rangle = \frac{\langle \mathrm{II}_v(v,v),v\rangle\langle W_v,v\rangle}{F^2(v)\langle \frac{v}{F(v)} - W, \frac{v}{F(v)}\rangle} .$$

The last term can be computed as

$$\begin{aligned} &\frac{2F(v)}{\langle \frac{v}{F(v)} - W, \frac{v}{F(v)}\rangle} \sum_{E\in\mathbb{E}} \langle W, \mathrm{II}_v(v,E) + \widehat{Q}_v(v,E)\rangle\langle E,W\rangle \\ &\qquad = 2F(v)\langle W, \mathrm{II}_v(v,W_v) + \widehat{Q}_v(v,W_v)\rangle = 2\langle v, \mathrm{II}_v(v,W_v) + \widehat{Q}_v(v,W_v)\rangle , \end{aligned}$$

noting that, by (2.7), $\mathrm{II}_v(v,W_v) + \widehat{Q}_v(v,W_v)$ is $g_v$-orthogonal to $v$. Equivalently, it is also $\langle\cdot,\cdot\rangle$-orthogonal to $\frac{v}{F(v)} - W$. Reinserting into (2.27), we have so far found

$$v\langle \tfrac{V}{F(V)} - W_V, V\rangle = \frac{2\langle \mathrm{II}_v(v,v),v\rangle}{F(v)} + \frac{\langle \mathrm{II}_v(v,v),v\rangle\langle W_v,v\rangle}{F^2(v)\langle \frac{v}{F(v)} - W, \frac{v}{F(v)}\rangle} - 2\langle v, \mathrm{II}_v(v,W_v)\rangle - \frac{\langle W_v,W_v\rangle\langle \mathrm{II}_v(v,v),v\rangle}{F(v)\langle \frac{v}{F(v)} - W, \frac{v}{F(v)}\rangle} ,$$

up to the following simplification, applying (2.17) to $\widehat{Q}_v(v,W_v)$ which is $g_v$-orthogonal to $v$ by (2.7) and concluding by (1.25) that

$$\begin{aligned} &\langle \widehat{Q}_v(v,W_v), v\rangle = F(v)\langle \tfrac{v}{F(v)} - W, \tfrac{v}{F(v)}\rangle^2 g_v(\widehat{Q}_v(v,W_v), W) \\ &\quad = F(v)\langle \tfrac{v}{F(v)} - W, \tfrac{v}{F(v)}\rangle^2 g_v(\widehat{Q}_v(v,W_v), W_v^\top) = F(v)\tfrac{v}{F(v)} - W, \tfrac{v}{F(v)}\rangle g_v(\widehat{Q}_v(v,W_v), W_v) \\ &\qquad = -F(v)\langle \tfrac{v}{F(v)} - W, \tfrac{v}{F(v)}\rangle C_v(W_v, W_v, \mathrm{II}_v(v,v)) = \frac{\langle W_v,W_v\rangle\langle \mathrm{II}_v(v,v),v\rangle}{2F(v)\langle \frac{v}{F(v)} - W, \frac{v}{F(v)}\rangle} , \end{aligned}$$

where $W_v^\top$ denotes the $g_v$-orthogonal projection of $W$ onto $T_{\pi(v)}S$, and where we have used the intermediate relation

$$\begin{aligned} W_v^\top - g_v(W, \tfrac{v}{F(v)})\tfrac{v}{F(v)} = \sum_{E\in\mathbb{E}} g_v(W,E)E = \sum_{E\in\mathbb{E}} \frac{\langle v,E\rangle E}{F(v)\langle \frac{v}{F(v)} - W, \frac{v}{F(v)}\rangle^2} = \sum_{E\in\mathbb{E}} \frac{\langle W,E\rangle E}{\langle \frac{v}{F(v)} - W, \frac{v}{F(v)}\rangle^2} \\ = \frac{1}{\langle \frac{v}{F(v)} - W, \frac{v}{F(v)}\rangle} W_v . \end{aligned}$$

To conclude, reinsert into (2.24) applying (2.18) and observing that

$$\langle \tfrac{v}{F(v)}, W_v\rangle = \langle W, W_v\rangle = \langle W_v, W_v\rangle\,.$$

□

### 2.3.3 Flag curvature

Recall from the previous chapter that the flag curvature is given by (1.59). In the case $M = \mathscr{V}$, the ambient curvature tensor $R$ is zero. Using (2.14) will allow us to calculate the flag curvature $\widehat{K}$ intrinsic to the submanifold $S \subset \mathscr{V}$ given for each admissible $v$ tangent to $S$ and $u \in T_{\pi(v)}S$ such that $g_v(u,v) = 0$ by:

$$\widehat{K}_v(u) = \frac{g_v(\widehat{R}_v(v,u)u,v)}{F^2(v)g_v(u,u)} \tag{2.28}$$

**Theorem 2.3.5.** *The flag curvature $\widehat{K}$ of the submanifold $S$ of a vector space $\mathscr{V}$ equipped with a Randers metric $F$ with Zermelo data $(W,\langle\cdot,\cdot\rangle)$ is given for $v \in TS$ non-zero and $u = w - \frac{\langle \frac{v}{F(v)} - W, w\rangle}{\langle \frac{v}{F(v)} - W, \frac{v}{F(v)}\rangle}\frac{v}{F(v)}$ with $w \in T_{\pi(v)}\mathscr{V}$ transverse to $v$ by*

$$\begin{aligned} K_v(w) = {}& \left(\langle \tfrac{v}{F(v)}, \tfrac{v}{F(v)}\rangle - \langle \tfrac{w}{\|w\|}, \tfrac{v}{F(v)}\rangle^2\right)\bar{K}(v,w) \\ &+ \frac{\|\frac{v}{F(v)} - W_v\|^2}{F^2(v)\langle \frac{v}{F(v)} - W, \frac{v}{F(v)}\rangle\langle u,u\rangle}\left(\langle \mathrm{I\!I}'(w,w), W\rangle\langle \mathrm{I\!I}'(v,v), W\rangle - \langle \mathrm{I\!I}'(v,w), W\rangle^2\right) \\ &+ \frac{\|\frac{v}{F(v)} - W_v\|^2}{2F^3(v)\langle \frac{v}{F(v)} - W, \frac{v}{F(v)}\rangle^2}\Big[\langle(\bar{\nabla}_v \mathrm{I\!I}')(v,v), W\rangle - \langle \mathrm{I\!I}'(v,v), \mathrm{I\!I}'(v,W_v)\rangle \\ &+ \frac{4\langle \frac{v}{F(v)} - W, \frac{v}{F(v)}\rangle - \frac{5}{2}\|\frac{v}{F(v)} - W_v\|^2}{F(v)\langle \frac{v}{F(v)} - W, \frac{v}{F(v)}\rangle^2}\langle \mathrm{I\!I}'(v,v), W\rangle^2 - \frac{4\langle \mathrm{I\!I}'(v,W_v), W\rangle}{\langle \frac{v}{F(v)} - W, \frac{v}{F(v)}\rangle}\langle \mathrm{I\!I}'(v,v), W\rangle\Big] \end{aligned} \tag{2.29}$$

*where $\bar{K}(v,w) = \frac{\langle \bar{R}(v,w)w,v\rangle}{\langle v,v\rangle\langle w,w\rangle - \langle v,w\rangle^2}$ is the sectional curvature with respect to $\langle\cdot,\cdot\rangle$, being $\bar{R}$ the curvature tensor of its Levi-Civita connection.*

*Proof.* Let us assume $g_v(v,u) = 0$ without loss of generality. From Corollary 2.2.5,

$$\begin{aligned} K_v(u) = {}& \frac{\langle \mathrm{I\!I}_v(u,u), \mathrm{I\!I}_v(v,v)\rangle - \langle \mathrm{I\!I}_v(v,u), \mathrm{I\!I}_v(v,u)\rangle}{F^2(v)g_v(u,u)\langle \frac{v}{F(v)} - W, \frac{v}{F(v)}\rangle} \\ &+ \frac{C_v(u, \widehat{Q}_v(u,v), \mathrm{I\!I}_v(v,v)) - g_v((\widehat{\nabla}_v\widehat{Q})_v(u,u),v)}{F^2(v)g_v(u,u)}. \end{aligned} \tag{2.30}$$

By (2.18),

$$\begin{aligned} \langle \mathrm{I\!I}_v(u,u), \mathrm{I\!I}_v(v,v)\rangle - \langle \mathrm{I\!I}_v(v,u), \mathrm{I\!I}_v(v,u)\rangle = {}& \langle \mathrm{I\!I}'(u,u), \mathrm{I\!I}'(v,v)\rangle - \langle \mathrm{I\!I}'(v,u), \mathrm{I\!I}'(v,u)\rangle \\ &+ \|\tfrac{v}{F(v)} - W_v\|^2 \frac{\langle \mathrm{I\!I}'(u,u), W\rangle\langle \mathrm{I\!I}'(v,u), W\rangle - \langle \mathrm{I\!I}'(v,u), W\rangle^2}{\langle \frac{v}{F(v)} - W, \frac{v}{F(v)}\rangle}. \end{aligned}$$

Recalling Proposition 1.2.4, we may compute

$$F^2(v)g_v(u,u)\langle \tfrac{v}{F(v)} - W, \tfrac{v}{F(v)}\rangle = F^2(v)\langle u,u\rangle$$

from which

$$\frac{\langle u,u\rangle\langle v,v\rangle-\langle v,u\rangle^2}{F^2(v)g_v(u,u)\langle\frac{v}{F(v)}-W,\frac{v}{F(v)}\rangle}=\langle\tfrac{v}{F(v)},\tfrac{v}{F(v)}\rangle-\langle\tfrac{u}{\|u\|},\tfrac{v}{F(v)}\rangle^2. \tag{2.31}$$

Given an arbitrary $u\in T_{\pi(v)}S$, observe that $\tilde u=u-g_v(\frac{v}{F(v)},u)\frac{v}{F(v)}$ spans with $v$ the same plane span$(\{u,v\})$ while satisfying $g_v(\tilde u,v)=0$. We have

$$\tilde u=u-g_v(\tfrac{v}{F(v)},u)\tfrac{v}{F(v)}=u-\frac{\langle\frac{v}{F(v)}-W,u\rangle}{\langle\frac{v}{F(v)}-W,\frac{v}{F(v)}\rangle}\frac{v}{F(v)}.$$

So far, we have obtained that the first term of the right hand side of (2.30) transform into the first two terms of the right hand side of (2.29). Furthermore, by (2.22),

$$\begin{aligned}C_v(u,\widehat{Q}_v(v,u),\mathrm{I\!I}_v(v,v))&=-\frac{g_v(u,u)\langle\mathrm{I\!I}'(v,v),W\rangle^2\langle\frac{v}{F(v)}-W_v,v\rangle^2}{4F^4(v)\langle\frac{v}{F(v)}-W,\frac{v}{F(v)}\rangle^4}\\&=-\frac{g_v(u,u)}{4F^2(v)\langle\frac{v}{F(v)}-W,\frac{v}{F(v)}\rangle^4}\langle\mathrm{I\!I}'(v,v),W\rangle^2\|\tfrac{v}{F(v)}-W_v\|^4.\end{aligned}$$

By Lemma 2.3.4, we conclude that the last term of the right hand side of (2.30) transforms into the last three terms of (2.29). □

### 2.3.4 Hypersurfaces in Randers-Minkowski spaces

In the case that $S$ is a hypersurface of $\mathcal{V}$, given $v\in TS\setminus\mathbf{0}$ there exists a vector $\xi_v\in\mathcal{V}$ unique up to a sign change such that $\xi_v$ is $g_v$-orthogonal to $T_{\pi(v)}S$ and $g_v(\xi_v,\xi_v)=1$. Moreover, there exists $\sigma_v\colon T_{\pi(v)}S\times T_{\pi(v)}S\to\mathbb{R}$ satisfying

$$\mathrm{I\!I}_v(u,w)=\sigma_v(u,w)\xi_v$$

for all $u,w\in T_{\pi(v)}S$. Analogously, at every $p\in S$, there exists $N_p\in\mathcal{V}$ $\langle\cdot,\cdot\rangle$-orthogonal to $T_pS$ with $\langle N_p,N_p\rangle=1$ and $\sigma'_p\colon T_pS\times T_pS\to\mathbb{R}$ satisfying

$$\mathrm{I\!I}'(u,w)=\sigma'_p(u,w)N_p$$

for all $u,w\in T_pS$.

**Lemma 2.3.6.** *Let $S$ be a hypersurface of a Randers-Minkowski space $(\mathcal{V},F)$. With the above notation,*

$$\xi_v=\left(\frac{\langle\frac{v}{F(v)}-W,\frac{v}{F(v)}\rangle}{1-\langle N_{\pi(v)},W\rangle^2}\right)^{\frac12}\Big(N_{\pi(v)}+\langle N_{\pi(v)},W\rangle(\tfrac{v}{F(v)}-W)\Big) \tag{2.32}$$

*is the vector $g_v$-orthogonal to $T_{\pi(v)}S$ with $g_v(\xi_v,\xi_v)=1$, and*

$$\sigma_v(u,w)=\left(\frac{1}{\langle\frac{v}{F(v)}-W,\frac{v}{F(v)}\rangle(1-\langle N_{\pi(v)},W\rangle^2)}\right)^{\frac12}\sigma'(u,w) \tag{2.33}$$

*for all $v\in TS\setminus\mathbf{0}$ and $u,w\in T_{\pi(v)}S$.*

*Proof.* Let us show that $\xi_v$ is $g_v$-orthogonal to $T_{\pi(v)}S$. First observe that $\langle N_{\pi(v)},v\rangle$ is zero by definition, therefore

$$\langle N_{\pi(v)},W\rangle=-\langle N_{\pi(v)},\tfrac{v}{F(v)}-W\rangle.$$

This implies that $\xi_v$ is $\langle\cdot,\cdot\rangle$-orthogonal to $\frac{v}{F(v)}-W$, recalling that $\langle \frac{v}{F(v)}-W, \frac{v}{F(v)}-W\rangle = 1$, and therefore by property of the indicatrix $\Sigma$ of $F$ $\xi_v$ is $g_v$-orthogonal to $v$. If $u \in T_{\frac{v}{F(v)}}\Sigma \cap T_{\pi(v)}S$, then

$$g_v(\xi_v,u) = \frac{1}{\langle \frac{v}{F(v)}-W, \frac{v}{F(v)}\rangle}\langle \xi_v, u\rangle = 0$$

because $u$ is $\langle\cdot,\cdot\rangle$-orthogonal to $N_{\pi(v)}$ and $\frac{v}{F(v)}-W$, since it lies respectively in $T_{\pi(v)}S$ and $T_{\frac{v}{F(v)}}\Sigma$. This concludes the proof that $\xi_v$ is $g_v$-orthogonal to $T_{\pi(v)}S$, because $T_{\frac{v}{F(v)}}\Sigma\cap T_{\pi(v)}S$ has dimension $\dim S-1$ and $v$ is not in the space tangent to $\Sigma$ at $\frac{v}{F(v)}$. Observing that

$$g_v(\xi,\xi) = \frac{1}{\langle \frac{v}{F(v)}-W, \frac{v}{F(v)}\rangle}\langle \xi_v, \xi_v\rangle\,,$$

it is straightforward to check that $g_v(\xi_v,\xi_v)=1$. Finally, as $\mathrm{II}'(u,w)-\mathrm{II}_v(u,w)$ is tangent to $T_{\pi(v)}S$ by Lemma 2.3.2, it follows that

$$\sigma'(u,w)N_{\pi(v)} = \mathrm{II}'(u,w) = \mathrm{II}_v(u,w)^{\perp_{\pi(v)}} = \sigma_v(u,w)\xi_v^{\perp_{\pi(v)}} = \sigma_v(u,w)\langle N_{\pi(v)},\xi_v\rangle N_{\pi(v)}\,,$$

and then one has

$$\sigma'(u,w) = \sigma_v(u,w)\langle N_{\pi(v)},\xi_v\rangle\,.$$

As

$$\langle N_{\pi(v)},\xi_v\rangle = \Big(\langle \tfrac{v}{F(v)}-W, \tfrac{v}{F(v)}\rangle(1-\langle N_{\pi(v)},W\rangle^2)\Big)^{\frac{1}{2}}\,,$$

the previous identity is equivalent to (2.33), which concludes the proof. □

**Corollary 2.3.7.** *Let $(\mathscr{V},F)$ be a Randers-Minkowski space with Zermelo data $(\langle\cdot,\cdot\rangle,W)$ and $S$ a hypersurface of $\mathscr{V}$. For $v\in TS\setminus\mathbf{0}$ and $u\in T_{\pi(v)}S$,*

$$\begin{aligned}(1-\langle N_{\pi(v)},W\rangle^2)K_v(u) = \Big(\langle \tfrac{v}{F(v)},\tfrac{v}{F(v)}\rangle - \langle \tfrac{\tilde u}{\|\tilde u\|},\tfrac{v}{F(v)}\rangle^2\Big)\bar K(v,u) \\ +\frac{1}{2F^3(v)}\ \bar\nabla_v\sigma'(v,v)\langle N_{\pi(v)},W\rangle - \sigma'(v,v)\sigma'(v,W^{\top_{\pi(v)}}) \\ +\frac{\langle N_{\pi(v)},W\rangle^2}{\langle\frac{v}{F(v)}-W,v\rangle}\big(4-\tfrac{5}{2(1-\langle N_{\pi(v)},W\rangle^2)}\big)\sigma'(v,v) \\ -\frac{4\langle N_{\pi(v)},W\rangle^2}{\langle\frac{v}{F(v)}-W,\frac{v}{F(v)}\rangle}\sigma'(v,v)\sigma'(v,W^{\top_v}_{S\cap\sigma})\Big)\,,\end{aligned} \tag{2.34}$$

*where $\bar K(v,u)$ is the Riemannian curvature in the plane spanned by $\{v,u\}$ computed with the metric induced by $\langle\cdot,\cdot\rangle$ on $S$, and $\tilde u = u-\frac{\langle\frac{v}{F(v)}-W,u\rangle}{\langle\frac{v}{F(v)}-W\frac{v}{F(v)}\rangle}\frac{v}{F(v)}$.*

*Proof.* Let us compute the flag curvature using Corollary 2.2.5. The second term of the right hand side can be computed using Lemma 2.3.6, resulting in

$$\frac{\sigma'(u,u)\sigma'(v,v)-\sigma'(v,u)^2}{\langle\frac{v}{F(v)}-W,\frac{v}{F(v)}\rangle(1-\langle N_{\pi(v)},W\rangle^2L(v)g_v(u,u))}$$

with a choice of $u$ satisfying $g_v(v,u)=0$. Proceeding as in (2.31), we obtain the first term of the right hand side of (2.34). Now observe that

$$W^{\top_v}_{S\cap\Sigma} = W - \frac{\langle\frac{v}{F(v)}-W,\frac{v}{F(v)}\rangle\langle N_{\pi(v)},W\rangle}{1-\langle N_{\pi(v)},W\rangle^2}N_{\pi(v)} - \left(-1+\frac{\langle\frac{v}{F(v)}-W,\frac{v}{F(v)}\rangle}{1-\langle N_{\pi(v)},W\rangle^2}\right)\left(\frac{v}{F(v)}-W\right),$$

and then

$$\|\tfrac{v}{F(v)}-W_{S\cap\Sigma}^{\top_v}\|^2=\frac{\langle\frac{v}{F(v)}-W,\frac{v}{F(v)}\rangle^2}{1-\langle N_{\pi(v)},W\rangle^2}.$$

Taking into account the above identity and the fact that $\bar{\nabla}_X \mathrm{II}'(Y,Z)=(\bar{\nabla}_X\sigma')(Y,Z)N$ for any $X,Y,Z\in\mathfrak{X}(S)$, we obtain the remaining terms of (2.34) from the last terms in (2.29). □

**Corollary 2.3.8.** *A hypersurface $S$ of a Randers-Minkowski space $(\mathscr{V},F)$ with Zermelo data $(\langle\cdot,\cdot\rangle,W)$ is of scalar flag curvature if and only if*

$$\left(\langle\tfrac{v}{F(v)},\tfrac{v}{F(v)}\rangle-\langle\tfrac{u}{\|u\|},\tfrac{v}{F(v)}\rangle\right)\bar{K}(v,u)$$

*is independent of $u\in T_{\pi(v)}S$ satisfying $g_v(v,u)=0$.*

# Chapter 3

# Pseudo-Finsler submersions

This final chapter studies the dual notion to pseudo-Finsler submanifolds, successfully generalising the fundamental equations of a pseudo-Riemannian submersion (first derived in [14]) to the pseudo-Finsler setting in Corollary 3.2.11. Contrary to the study of submanifolds, which has its roots in 19th century mathematics and was generalised to Finsler geometry in the early 20th century, the study of submersions is a recent innovation with its origin in the 1966 research paper by Barrett O'Neill [14]. Theorem 3.2.11 derives the fundamental equations of a pseudo-Finsler submersion analogously to the fundamental equations obtained by O'Neill. Various applications are provided for submersions whose fibres are totally geodesic, that is to say that the geodesics intrinsic to the fibres are also geodesics of the ambient manifold of the submersion.

**Definition 3.0.1.** *A smooth map $\sigma\colon M \to \widetilde{M}$ from an ambient manifold $M$ onto a lower dimensional base manifold $\widetilde{M}$ is a* submersion *when the restriction of its differential to the tangent space at each point $p$ is surjective onto $T_{\sigma(p)}\widetilde{M}$.*

Recall from elementary differential geometry that submersions can be locally trivialised much the same way as with immersions.

**Theorem 3.0.2.** *There exist* adapted *local coordinates $(x_1,\cdots,x_n)$ around any $p \in M$ and $(\bar{x}_1,\cdots,\bar{x}_k)$ around $\sigma(p)$ for which $\sigma$ takes the form*

$$(x_1,\cdots,x_k,\cdots,x_n) \mapsto (\bar{x}_1,\cdots,\bar{x}_k)\,. \tag{3.1}$$

This ensures that the submersion fibres $\sigma^{-1}(\{b\}) = \{p \in M,\ \sigma(p) = b\}$ indeed constitute a submanifold of $M$ above any base point $b \in \widetilde{M}$. Namely, choosing adapted coordinates $(x_1,\cdots,x_k,\cdots,x_n)$ on the neighbourhood $U \subset M$ of any $p \in \sigma^{-1}(\{b\}) \cap U$ incidentally defines the local coordinates $(x_1,\cdots,x_k)$ within the fibre which is, locally, the set of points with vanishing $(x_{k+1},\cdots,x_n)$.

We shall think of $\sigma$ as if projecting an ambient atmosphere down to a levelled landscape, and therefore refer to the set $\mathbb{T} = \ker \mathrm{d}\sigma$ of vectors tangent to the submersion fibres $\sigma^{-1}(\{b\})$ whose image by $\ker \mathrm{d}\sigma$ are 0 as *vertical*.

If $(M,L)$ is a pseudo-Finsler manifold, at each point $p \in M$ the level sets $L^{-1}(\{t\}) \cap T_pM$ for $t \in \mathbb{R}$ will be tangent to $\mathbb{T}_p = \ker \mathrm{d}\sigma \cap T_pM$ at $v \in T_pM$ such that $L(v) = t$ precisely when $g_v(v,\cdot)|_{\ker \mathrm{d}\sigma \cap T_pM}$ is identically zero. Those are the directions we will call *horizontal*. With the identity $(\dot{\partial} L)_v = 2g_v(v,\cdot)$ from Proposition 1.2.3 in mind, where $g$ is the fundamental metric tensor associated to the pseudo-Finsler metric $L$, we can rephrase this definition as follows.

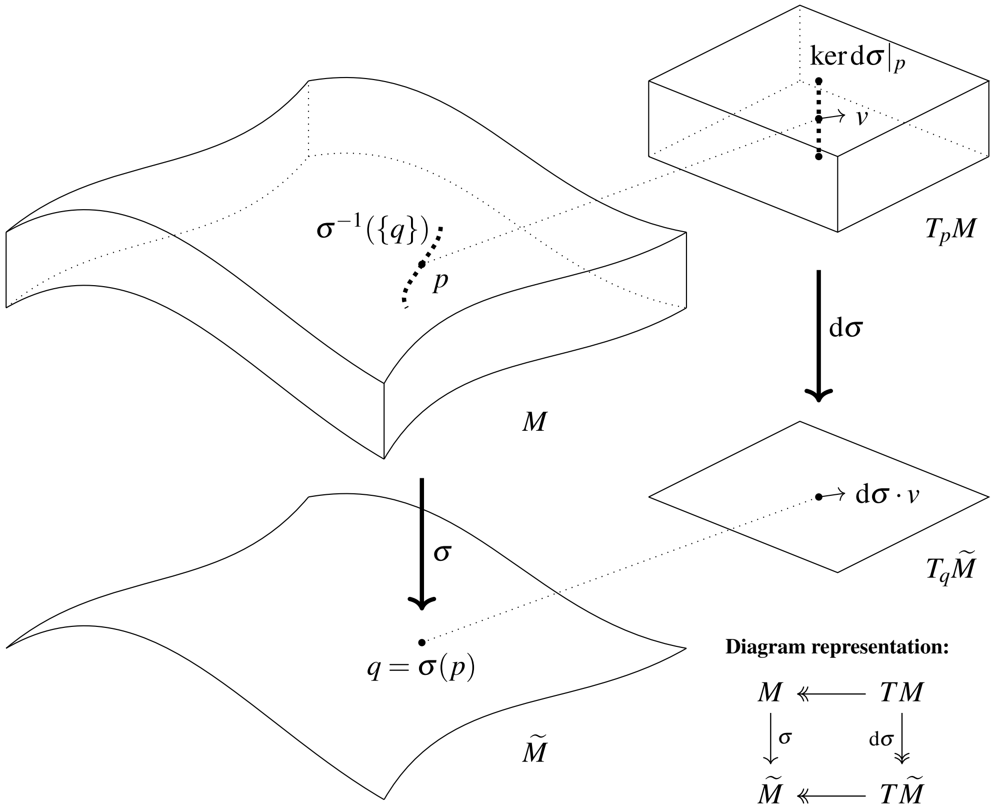


FIGURE 3.1: Visual representation of a submersion.
(The sequence $\ker d\sigma|_p \hookrightarrow T_pM \twoheadrightarrow T_q\widetilde{M}$ is exact.)

**Definition 3.0.3.** $v \in \mathbb{A}$ *is called* horizontal *when*

$$g_v(v,\cdot)|_{\mathbb{T}_{\pi(v)}} = 0. \tag{3.2}$$

Let $\mathbb{\Lambda} = \{v \in \mathbb{A},\ g_v(v,\cdot)|_{\mathbb{T}_{\pi(v)}} = 0\}$ be the set of admissible horizontal vectors of $TM$, and let us write $\mathbb{\Lambda}_p = \mathbb{\Lambda} \cap T_pM$ for convenience. By Proposition 1.2.5, they are cones that do not contain the origin.

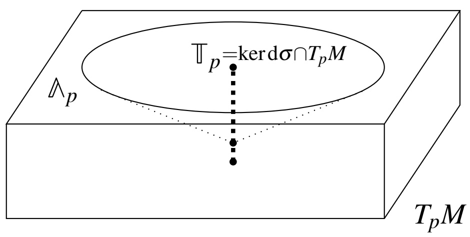


FIGURE 3.2: Illustration of the horizontal cone $\mathbb{\Lambda}$ and vertical space $\mathbb{T}$.

**Lemma 3.0.4.** *$\mathbb{\Lambda}_p$ is a submanifold of $T_pM$ at each $p \in M$, and $\mathbb{\Lambda}$ is a submanifold of $TM$.*

*Proof.* Consider a basis $(e_1, \cdots, e_r)$ of $\mathbb{T}_p$ and the map $\mu \colon v \mapsto (g_v(v,e_1), \cdots, g_v(v,e_r))$ with differential $d\mu \colon w \mapsto (g_v(w,e_1), \cdots, g_v(w,e_r))$ whose kernel is the $(n-r)$-dimensional space of vectors $g_v$-orthogonal to $\mathbb{T}_p$ (where $n$ is the dimension of the ambient manifold $M$). Thus

$\mu$ is a submersion and the submersion fibre $\mathbb{A}_p = \mu^{-1}(\{0\})$ is a submanifold of $T_pM$. The same map applied to a frame $(E_1, \cdots, E_r)$ of vertical vector fields (which can be obtained from local coordinates adapted to the submersion $\sigma$) similarly prove that $\mathbb{A}$ is a submanifold of $TM$. □

**Definition 3.0.5.** *A submersion $\sigma\colon M \to \widetilde{M}$ between differentiable manifolds equipped with respective pseudo-Finsler metrics $L$ and $\tilde{L}$ is called a* pseudo-Finsler submersion *provided that it satisfies that:*

*(i) the submersion fibres are non-degenerate pseudo-Finsler submanifolds of the ambient manifold $(M, L)$, namely, the restriction of $L$ to the space tangent to the fibres provides a pseudo-Finsler metric on the fibres, and*

*(ii) for each $v \in \mathbb{A}$*

$$L(v) = \tilde{L}(\mathrm{d}\sigma \cdot v). \tag{3.3}$$

Note that this places restrictions on both $(M, L)$ and $(\widetilde{M}, \tilde{L})$, since on the one hand $\tilde{L}$ is entirely determined by $L$ and on the other hand merely disposing of an arbitrary submersion $\sigma\colon M \to \widetilde{M}$ together with some pseudo-Finsler metric $L$ on $M$ does not guarantee that the condition (3.3) consistently defines a pseudo-Finsler metric on $\widetilde{M}$.

Our implicit assumption is that the domains of definition $\mathbb{A}$ of $L$ and $\tilde{\mathbb{A}}$ of $\tilde{L}$ are compatible with the submersion $\sigma$. By hypothesis, we want to additionally require that there be some unique horizontal vector over each admissible tangent vector of the base manifold. It is sufficient to assume that the Legendre map $v \mapsto g_v(v, \cdot)$ is bijective on the subset of admissible vectors at each point (see Lemma 3.1.7).

## 3.1 Elementary identities

Contrary to the case of pseudo-Finsler submanifolds, where tangency to the submanifold is an intrinsic property, pseudo-Finsler submersions do not offer an immediate identification of the anisotropic metric tensor between the ambient manifold $M$ and the base manifold $\widetilde{M}$; we must proceed with caution when attempting to deduce (3.7) and (3.12) from (3.3).

**Definition 3.1.1.** *$e \in T_{\pi(v)}M$ is called $g_v$-horizontal when*

$$g_v(e, \cdot)|_{\mathbb{T}_{\pi(v)}} = 0. \tag{3.4}$$

As a warning to the reader, a decision has been made to use the Latin letter $e$ to designate an arbitrary tangent vector, following closely the choice of notation of [14]. O'Neill intended to use $x, y, z$ for horizontal vectors, $X, Y, Z$ for horizontal vector fields, $u, v, w$ for vertical vectors, and- $U, V, W$ for vector fields. In our case, $v$ already designates the anisotropic variable, and going one letter further will create a collision with the O'Neill tensor $T$.

The decision has been to consistently use $s, u, w$ and $S, U, W$ for vertical vector fields. Along similar lines, an additional decision has been made to use $b, e, h$ and $B, E, H$ for arbitrary vectors and vector fields, avoiding the collision with the O'Neill tensor $A$, and otherwise the letters $C$ that designates the Cartan tensor, d that designates the differential operation of pushing forward, $F$ that designates a generic Finsler metric, and $g$ the fundamental metric tensor associated to a pseudo-Finsler metric. We hope these choices of notation will avoid any ambiguity while remaining fairly easy to keep track of.

Let $\mathbb{H}_v = \{e \in T_{\pi(v)}M,\ g_v(e,\cdot)|_{\mathbb{T}_{\pi(v)}} = 0\}$ denote the set of $g_v$-horizontal vectors, for any admissible $v \in \mathbb{A}$, and let us write $\mathbb{H}_V = \bigcup_{p\in M} \mathbb{H}_{V(p)}$ whenever $V$ has image in $\mathbb{A}$, for convenience.

**Proposition 3.1.2.** *For $\gamma$ a smooth path of admissible vectors, $g_\gamma(\gamma, w)$ is stationary at $t \in \mathbb{R}$ for all $w \in \mathbb{T}_{\pi(v)}$ if and only if $\dot\gamma(t)$ is $g_{\gamma(t)}$-horizontal.*

*Proof.* By Proposition 1.2.7, the Cartan tensor term vanishes from

$$(g_\gamma(\gamma,w))' = g_\gamma(\dot\gamma,w) + 2C_\gamma(\gamma,w,\dot\gamma) = g_\gamma(\dot\gamma,w)\,, \tag{3.5}$$

thus $g_{\gamma(t)}(\gamma(t),w)$ is stationary at $t$ if and only if $g_{\gamma(t)}(\dot\gamma(t),w)$ is zero. By definition, the condition is verified for all vertical $w$ if and only if $\dot\gamma$ is $g_\gamma$-horizontal. □

**Lemma 3.1.3.** *For $v \in \mathbb{\Lambda}$, $g_v$-horizontal vectors are precisely tangent to $\mathbb{\Lambda}$. More succinctly,*

$$\mathbb{H}_v = T_v \mathbb{\Lambda}_{\pi(v)}\,. \tag{3.6}$$

*Proof.* Recalling by the previous lemma that $\mathbb{\Lambda}$ and $\mathbb{\Lambda}_{\pi(v)}$ are submanifolds of the tangent bundle, set $\gamma$ to a smooth path of the horizontal cone $\mathbb{\Lambda}_{\pi(v)}$ in the previous proposition, by which $\dot\gamma$ is $g_\gamma$-horizontal, then by definition $g_\gamma(\gamma,w)$ is identically zero for all vertical $w$. □

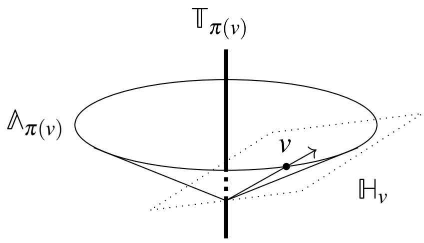


FIGURE 3.3: Illustration of the $g_v$-horizontal space $\mathbb{H}_v = T_v\mathbb{\Lambda}_{\pi(v)}$ for $v \in \mathbb{\Lambda}$.

Note that the vertical space must be non degenerate with respect to $g_v$ for all $v \in \mathbb{A}$ in order for the $g_v$-orthogonal decomposition into vertical and $g_v$-horizontal parts to be uniquely defined. By definition of a pseudo-Finsler submersion, this will be true for vertical and horizontal vectors, and we can reduce the domain of definition of $L$ to ensure this condition which always holds when $g_v$ is positive definite. As a consequence of this reduction, it may be that $\mathbb{A}$ is non-connected.

**Definition 3.1.4.** *For $v \in \mathbb{A}$ and any $e \in T_{\pi(v)}M$, let $(e_v^\top, e_v^\perp)$ denote this unique pair $(w,x)$ of the vertical part $e_v^\top = w \in \mathbb{T}_{\pi(v)}$ and the $g_v$-horizontal part $e_v^\perp = x \in \mathbb{H}_v$ of $e$, such that*

$$e = e_v^\top + e_v^\perp\,.$$

*By extension, for $V$ with values in $\mathbb{A}$ on some region $\Omega \subset M$ and any $E \in \mathfrak{X}(M)$, let $(E_V^\top, E_v^\perp)$ be given at $p \in \Omega$ by*

$$(E_V^\top)(p) = \big(E(p)\big)_{V(p)}^\top\,,$$
$$(E_V^\perp)(p) = \big(E(p)\big)_{V(p)}^\perp\,.$$

*Furthermore, $E^\top$ and $E^\perp$ shall denote the anisotropic vector fields given at $v \in \mathbb{A}$ by*

$$(E^\top)(v) = \big(E(\pi(v))\big)_v^\top\,,$$
$$(E^\perp)(v) = \big(E(\pi(v))\big)_v^\perp\,.$$

**Proposition 3.1.5.** *The maps $E \mapsto E^\top$ and $E \mapsto E^\perp$ are smooth on the space of vector fields $E \in \mathfrak{X}(M)$.*

*Proof.* Consider a non-degenerate basis $(e_1, \cdots, e_k, \cdots, e_n)$ such that the first vectors form a basis $(e_1, \cdots, e_k)$ of the space $\mathbb{T}$ of vertical vectors and $(e_1, \cdots, e_m)$ is the basis of a non-degenerate subspace with respect to $g_v$ for each $m \in [\![1,n]\!]$. Such a basis can be locally obtain explicitly by some constant combination of the vector fields $\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}$ induced by local coordinates $(x^1, \cdots, x^n)$ adapted to the submersion. By applying the Gram-Schmidt process, we may generate a $g_{v'}$-orthonormal basis $(e_1(v'), \cdots, e_k(v'), \cdots, e_n(v'))$ for each admissible $v'$ in the neighbourhood of $v$ such that the first vectors again form a basis of the space of vertical vectors. Locally,

$$E_{v'}^\top = \sum_{i \in [\![1,k]\!]} \tfrac{g_{v'}(e_i(v'),E)}{g_{v'}(e_i(v'),e_i(v'))} e_i(v')$$

depends smoothly on $v'$. □

Note that $v \in \mathbb{A}$ belongs to $\mathbb{A}$ if and only if it satisfies the equation

$$v = v_v^\perp\,.$$

Our assumption will be that the set $\mathbb{A}$ of horizontal vectors covers the base tangent bundle $T\widetilde{M}$, such that this equation has a solution in each vertical equivalence class $v + \mathbb{T}_{\pi(v)}$, unique up to restricting the domain $\mathbb{A}$. Under this assumption, we may always lift a base vector $\tilde{v} \in T\widetilde{M}$ to some $v \in \mathbb{A}$ with $\mathrm{d}\sigma \cdot v = \tilde{v}$, and further assume that this lift is always unique (see Lemma 3.1.7).

### 3.1.1 Fundamental tensors

By some involved conceptual gymnastics, we may relate very succinctly the anisotropic metric tensors of ambient and base manifolds of pseudo-Finsler submersions.

**Proposition 3.1.6.** *Let $v \in \mathbb{A}$ horizontal, a $g_v$-horizontal vector $x \in \mathbb{H}_v$, an arbitrary vector $e \in T_{\pi(v)}M$, and denote by $\tilde{v}$, $\tilde{x}$ and $\tilde{e}$ their projections by $\mathrm{d}\sigma$. We have*

$$g_v(x,e) = \tilde{g}_{\tilde{v}}(\tilde{x},\tilde{e})\,. \tag{3.7}$$

*Proof.* As lightlike vectors are in the closure of $\{L \neq 0\}$, we may assume without loss of generality that $L(v) \neq 0$ and extend by continuity to the light cone. By homogeneity of $g$, we may furthermore assume that $L(v) = 1$, or $L(v) = -1$. Let us assume $L(v) = 1$. Repeating the steps for the pseudo-Finsler metrics $-L$ and $-\tilde{L}$ would complete the proof.

Let $u = e_v^\top \in \mathbb{T}_{\pi(v)}$ and $y = e_v^\perp \in \mathbb{H}_v = T_v\mathbb{A}_{\pi(v)}$, recalling (3.5). Note how $\mathrm{d}\sigma \cdot y = \tilde{e}$. We aim to prove the identity

$$g_v(x,e) = g_{\mathrm{d}\sigma\cdot v}(\mathrm{d}\sigma \cdot x, \mathrm{d}\sigma \cdot e) = \tilde{g}_{\mathrm{d}\sigma\cdot v}(\mathrm{d}\sigma \cdot x, \mathrm{d}\sigma \cdot y)\,.$$

Since $\mathbb{\Lambda} \cap L^{-1}(\{-1,1\})$ projects by $\mathrm{d}\sigma$ onto $\tilde{L}^{-1}(\{-1,1\})$ by our assumption (3.3), let us proceed by resorting to equation (2.5) and work on relating the second fundamental forms of the indicatrices of $L$ and $\tilde{L}$.

Note that, by our hypothesis that $x$ is $g_v$-horizontal and by linearity of $g_v$, we have

$$g_v(x, u+y) = g_v(x,u) + g_v(x,y) = g_v(x,y)\,.$$

Let

$$\bar{\mathbb{\Lambda}}_{\pi(v)} = \mathbb{\Lambda}_{\pi(v)} \cap L^{-1}(\{1\})$$

denote the set of unit horizontal vectors. There exists a non-trivial space within $T_v\mathbb{\Lambda}_{\pi(v)}$ of vectors that are $g_v$-orthogonal to $v \in T_v\mathbb{\Lambda}_{\pi(v)}$.

That non-trivial space $\{\bar{v} \in T_v\mathbb{\Lambda}_{\pi(v)},\ g_v(v,\bar{v}) = 0\}$ does not contain $v$ under our assumption that $L(v) \neq 0$. Let us decompose $g_v$-orthogonally along $v$ the vectors $x = \lambda v + \bar{x}$ and $y = \mu v + \bar{y}$ for $\lambda, \mu \in \mathbb{R}$ and $\bar{x}, \bar{y} \in T_v\bar{\mathbb{\Lambda}}_{\pi(v)}$ to obtain

$$g_v(x,y) = \lambda\mu g_v(v,v) + g_v(\bar{x},\bar{y})\,. \tag{3.8}$$

By Proposition 1.2.3 and by definition of $\tilde{L}$ we have

$$g_v(v,v) = L(v) = \tilde{L}(\mathrm{d}\sigma\cdot v) = \tilde{g}_{\mathrm{d}\sigma\cdot v}(\mathrm{d}\sigma\cdot v, \mathrm{d}\sigma\cdot v)\,, \tag{3.9}$$

which reduces our study to $T_v\bar{\mathbb{\Lambda}}_{\pi(v)}$. Indeed, all we have left to prove is that

$$g_v(\bar{x},\bar{y}) = \tilde{g}_{\mathrm{d}\sigma\cdot v}(\mathrm{d}\sigma\cdot\bar{x}, \mathrm{d}\sigma\cdot\bar{y})\,, \tag{3.10}$$

as reinserting (3.9) and (3.10) into (3.8) gives by linearity

$$g_v(x,y) = \lambda\mu\tilde{g}_{\mathrm{d}\sigma\cdot v}(\mathrm{d}\sigma\cdot v, \mathrm{d}\sigma\cdot v) + \tilde{g}_{\mathrm{d}\sigma\cdot v}(\mathrm{d}\sigma\cdot\bar{x}, \mathrm{d}\sigma\cdot\bar{y}) = \tilde{g}_{\mathrm{d}\sigma\cdot v}(\mathrm{d}\sigma\cdot x, \mathrm{d}\sigma\cdot y)\,. \tag{3.11}$$

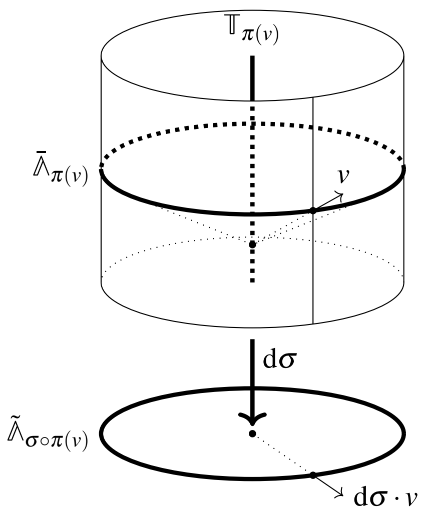


FIGURE 3.4: Illustration of the $\sigma$-saturated cylinder of the set $\bar{\mathbb{\Lambda}}_{\pi(v)}$ of unit horizontal vectors of $T_{\pi(v)}M$, and their projection onto $\tilde{\mathbb{\Lambda}}_{\sigma\circ\pi(v)}$.

In order to prove (3.10), let $\mathbf{C} = \bar{\mathbb{\Lambda}}_{\pi(v)} + \mathbb{T}_{\pi(v)}$ be the cylinder obtained when saturating $\bar{\mathbb{\Lambda}}_{\pi(v)}$ with respect to $\mathrm{d}\sigma$, and let $\tilde{\mathbb{\Lambda}}_{\sigma\circ\pi(v)} = \{\mathrm{d}\sigma\cdot w,\ w \in \bar{\mathbb{\Lambda}}_{\pi(v)}\}$. By equation (3.3) and by our assumption on $\tilde{\mathbb{A}}$,

$$\tilde{\mathbb{A}}_{\sigma\circ\pi(v)} = \tilde{L}^{-1}(\{1\}) \cap T_{\sigma\circ\pi(v)}\widetilde{M}\,.$$

Let $\mathrm{II}^{\mathbf{C}}_v$ be the $g_v$-orthogonal second fundamental form of $\mathbf{C}$ in the following sense. Being $\langle\cdot,\cdot\rangle = g_v$ a well-defined pseudo-Euclidean inner product on $T_{\pi(v)}M$ with pseudo-Riemannian metric $e \mapsto \langle e,e\rangle$, and $\bar{\nabla}$ the trivial Levi-Civita connection of the flat vector space $T_{\pi(v)}M$ equipped with any pseudo-Euclidean metric, let $\mathrm{II}^{\mathbf{C}}_v$ satisfy on the 1-dimensional linear space $\mathrm{span}(\{v\})$ spanned by vectors $\langle\cdot,\cdot\rangle$-orthogonal to $T_v\mathbf{C}$

$$\langle \mathrm{II}^{C}_v(\bar{x},\bar{y}),\cdot\rangle|_{\mathrm{span}(\{v\})} = \langle \bar{\nabla}_{\bar{x}}\bar{Y},\cdot\rangle|_{\mathrm{span}(\{v\})}\,,$$

where $\bar{Y} \in \mathfrak{X}(T_{\pi(v)}M)$ is any extension of $\bar{y}$ locally tangent to $\mathbf{C}$. Recall from equation (2.4) that we also have

$$g_v(\bar{x},\bar{y}) = -\langle \bar{\nabla}_{\bar{x}}\bar{Y},v\rangle\,,$$

such that

$$g_v(\bar{x},\bar{y}) = -\langle \bar{\nabla}_{\bar{x}}\bar{Y},v\rangle = -\langle \mathrm{II}^{\mathbf{C}}_v(\bar{x},\bar{y}),v\rangle\,.$$

Note that $\mathrm{II}^{\mathbf{C}}_v(\bar{x},\bar{y})$ depends only on the horizontal parts of $\bar{x}$ and $\bar{y}$, and is hence only a function of their projection by $\mathrm{d}\sigma$. Indeed, considering the constant vector field extension $W \in \mathfrak{X}(T_{\pi(v)}M)$ of $w \in \mathbb{T}_{\pi(v)}$, incidentally tangent to $\mathbf{C}$, we would have

$$\langle \mathrm{II}^{\mathbf{C}}_v(\cdot,w),v\rangle = \langle \bar{\nabla}W,v\rangle = 0$$

on $T_v\mathbf{C}$, such that, by non-degeneracy of $\langle\cdot,\cdot\rangle = g_v$ and linearity and symmetry of $\mathrm{II}^{\mathbf{C}}_v$,

$$\mathrm{II}^{\mathbf{C}}_v(\bar{x},\bar{y}) = \mathrm{II}^{\mathbf{C}}_v(\bar{x},\bar{y}^{\top}_v) + \mathrm{II}^{\mathbf{C}}_v(\bar{y}^{\perp}_v,\bar{x}^{\top}_v) + \mathrm{II}^{\mathbf{C}}_v(\bar{x}^{\perp}_v,\bar{y}^{\perp}_v) = \mathrm{II}^{\mathbf{C}}_v(\bar{x}^{\perp}_v,\bar{y}^{\perp}_v)\,.$$

Given a coordinate system adapted to the submersion, we can canonically identify with $T_{\mathrm{d}\sigma\cdot v}\tilde{\mathbb{A}}_{\sigma\circ\pi(v)}$ the subset of $T_v\mathbf{C}$ with the $g_v$-horizontal space. Let us adjust the vectors $\bar{x},\bar{y} \in T_v\mathbf{C}$ by vertical corrections $w_{\bar{x}},w_{\bar{y}} \in \mathbb{T}_{\pi(v)}$, such that the vectors $\bar{x}+w_{\bar{x}},\bar{y}+w_{\bar{y}} \in T_v\mathbf{C}$ have zero vertical coordinate and are, up to the canonical identification and by our assumption (3.3), tangent to $\tilde{\mathbb{A}}_{\sigma\circ\pi(v)}$. We have

$$\mathrm{II}^{\mathbf{C}}_v(\bar{x},\bar{y}) = \mathrm{II}^{\mathbf{C}}_v(\bar{x}^{\perp}_v,\bar{y}^{\perp}_v) = \mathrm{II}^{\mathbf{C}}_v(\bar{x}+w_{\bar{x}},\bar{y}+w_{\bar{y}})\,.$$

Then, again by equation (2.4) and for an extension $\bar{Y}+W_{\bar{Y}} \in \mathfrak{X}(T_{\pi(v)}M)$ locally tangent to $\mathbf{C}$ and therefore incidentally to $\tilde{\mathbb{A}}_{\sigma\circ\pi(v)}$ up to the canonical identification, we have

$$\begin{aligned}\langle \mathrm{II}^{\mathbf{C}}_v(\bar{x}+w_{\bar{x}},\bar{y}+w_{\bar{y}}),v\rangle = \langle \bar{\nabla}_{\bar{x}+w_{\bar{x}}}(\bar{Y}+W_{\bar{Y}}),v\rangle &= -\tilde{g}_{\mathrm{d}\sigma\cdot v}(\mathrm{d}\sigma\cdot(\bar{x}+w_{\bar{x}}),\mathrm{d}\sigma\cdot(\bar{y}+w_{\bar{y}}))\\ &= -\tilde{g}_{\mathrm{d}\sigma\cdot v}(\mathrm{d}\sigma\cdot\bar{x},\mathrm{d}\sigma\cdot\bar{y})\,.\end{aligned}$$

Thus we have proved (3.10), from which we obtain (3.11), concluding the proof. □

**Lemma 3.1.7.** *If the Legendre map $v \mapsto g_v(v,\cdot)$ is injective over the admissible vectors at $p \in M$, then each admissible $\tilde{v}$ at $\sigma(p)$ admits at most one horizontal vector $v$ at $p$ satisfying $\mathrm{d}\sigma\cdot v = \tilde{v}$. If bijective, then $\tilde{v}$ admits exactly one such $v$.*

*Proof.* Injectivity implies uniqueness: for any two horizontal vectors $v_1$ and $v_2$ satisfying $\mathrm{d}\sigma\cdot v_1 = \mathrm{d}\sigma\cdot v_2 = \tilde{v}$, their images $g_{v_1}(v_1,\cdot)$ and $g_{v_2}(v_2,\cdot)$ by the Legendre map are both equal to $\tilde{g}_{\tilde{v}}(\tilde{v},\tilde{\cdot})$ due to (3.7). Indeed, for each $e \in T_pM$ with $\mathrm{d}\sigma\cdot e$

$$g_{v_1}(v_1,e) = \tilde{g}_{\tilde{v}}(\tilde{v},\tilde{e}) = g_{v_2}(v_2,e)\,,$$

which implies $v_1 = v_2$ by injectivity of $v \mapsto g_v(v, \cdot)$.

Invertibility implies existence: there always exists a 1-form $\omega$ on $T_pM$ with prescribed kernel $\mathbb{T}_p$ and acting on each $x$ in some fixed transversal subspace as $\tilde{g}_{\tilde{v}}(\tilde{v}, \cdot)$ acts on $\mathrm{d}\sigma \cdot x$. Its preimage $v_0 = (v \mapsto g_v(v, \cdot))^{-1}(\omega)$ by the Legendre map must satisfy by construction that $g_{v_0}(v_0, \cdot)|_{\mathbb{T}_p}$ is identically zero, hence $v_0$ is horizontal, and by the hypothesis that $g_{v_0}(v_0, \cdot)$ acts on some fixed transversal subspace (and therefore on the whole $T_pM$) as $\tilde{g}_{\tilde{v}}(\tilde{v}, \cdot)$ on its projection by $\mathrm{d}\sigma$ we necessarily have $\mathrm{d}\sigma \cdot v_0 = \tilde{v}$. □

Note that if the pseudo-Finsler metric is defined in the whole tangent space, then the Legendre map $v \mapsto g_v(v, \cdot)$ is bijective whenever $M$ has dimension greater or equal to 3 (see [44] and [45]).

**Proposition 3.1.8.** *Given $v \in \mathbb{A}$ and any $\tilde{x} \in T_{\sigma\circ\pi(v)}\widetilde{M}$, there exists a unique $g_v$-horizontal vector $x \in T_v\mathbb{A}_{\pi(v)}$ satisfying $\mathrm{d}\sigma \cdot x = \tilde{x}$.*

*Proof.* The restriction of $\mathrm{d}\sigma$ to $T_{\pi(v)}M$ is a pseudo-Riemannian submersion onto $T_{\sigma\circ\pi(v)}\widetilde{M}$ if we respectively equip the two vector spaces with the metric tensors $g_v$ and $g_{\mathrm{d}\sigma\cdot v}$, since they are related by (3.7). This reduces the proof to the pseudo-Riemannian case. Basically, it suffices to observe that the restriction of $\mathrm{d}\sigma$ to horizontal vectors is an isomorphism.

Specifically, since $\sigma$ is a submersion, $T_{\sigma\circ\pi(v)}\widetilde{M} = \{\mathrm{d}\sigma \cdot e,\ e \in T_{\pi(v)}M\}$ such that there exists some $e \in (\mathrm{d}\sigma)^{-1}(\{\tilde{x}\})$, from which we obtain by linearity that

$$(\mathrm{d}\sigma)^{-1}(\{\tilde{x}\}) \supset \mathrm{span}(\{e\}) + \mathbb{T}_{\pi(v)}\,,$$

which necessarily intersects the hyperplane $\mathbb{H}_v$ of vectors $g_v$-orthogonal to the vector subspace $\mathbb{T}_{\pi(v)}$ of the Euclidean space $T_{\pi(v)}M$. Thus there exists some $x$ in this intersection, satisfying both $x \in \mathbb{H}_v$ and $\mathrm{d}\sigma \cdot x = \tilde{x}$.

Consider another $y \in \mathbb{H}_v$ such that $\mathrm{d}\sigma \cdot x = \mathrm{d}\sigma \cdot y = \tilde{x}$. By definition (3.4) of $g_v$-horizontality, for any $w \in \mathbb{T}_p$

$$g_v(x - y, w) = g_v(x, w) - g_v(y, w) = 0\,,$$

while, by (3.7) and using $\mathrm{d}\sigma \cdot x = \mathrm{d}\sigma \cdot y = \tilde{x}$, for any $z \in \mathbb{H}_v$

$$g_v(x - y, z) = g_v(x, z) - g_v(y, z) = g_{\mathrm{d}\sigma\cdot v}(\tilde{x}, \mathrm{d}\sigma \cdot z) - g_{\mathrm{d}\sigma\cdot v}(\tilde{x}, \mathrm{d}\sigma \cdot z) = 0\,.$$

By non-degeneracy of $g_v$, we have that both $(x - y)_v^{\top}$ and $(x - y)_v^{\perp}$ are zero, thus $x = y$. □

**Definition 3.1.9.** *For $v \in \mathbb{A}$ and $\tilde{x} \in T_{\sigma\circ\pi(v)}\widetilde{M}$, let $\tilde{x}_v^* \in \mathbb{H}_v$ denote this unique $g_v$-horizontal lift $x$ of $\tilde{x}$. By extension, for a locally horizontal extension $V$ and $\tilde{X}$ a vector field of the base manifold, let $\tilde{X}_V^*$ be given at $p \in M$ by*

$$(\tilde{X}_V^*)(p) = \left(\tilde{X}(\sigma(p))\right)_{V(p)}^*\,.$$

*Furthermore, let $\tilde{X}^*$ denote the anisotropic vector field given at $v \in \mathbb{A}$ by*

$$(\tilde{X}^*)(v) = \left(\tilde{X}(\sigma \circ \pi(v))\right)_v^*\,.$$

Note that $\tilde{X}$ and $\tilde{X}^*$ are related by $\mathrm{d}\sigma \cdot \tilde{X}_V^* = \tilde{X} \circ \sigma$ for any locally admissible choice of $V$.

**Definition 3.1.10.** *An ambient vector field $V$ is called* projectable *when there exists a base vector field $\tilde{V}$ satisfying*

$$\mathrm{d}\sigma \cdot V = \tilde{V} \circ \sigma\,.$$

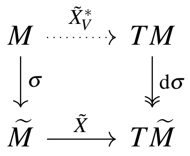


FIGURE 3.5: Commutative diagram of the $g_V$-horizontal lift $\tilde{X}_V^*$ of $\tilde{X}$.

*In that case, projectable $g_V$-horizontal vector fields are called $g_V$-basic.*

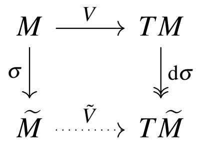


FIGURE 3.6: Commutative diagram of the projection $\tilde{V}$ of $V$.

**Proposition 3.1.11.** *For a vector field $V$ locally admissible on some neighbourhood, every $\tilde{X} \in \mathfrak{X}(\widetilde{M})$ admits a unique local $g_V$-horizontal lift $X \in \mathfrak{X}(M)$, in the sense that $X$ is projectable onto $\tilde{X}$ and $g_V$-horizontal on that neighbourhood, and any other such vector field must be equal to $X$ on that neighbourhood.*

Recall that we are assuming for our purposes that there exists at a fixed $p \in M$ a unique horizontal lift $v$ of each admissible $\tilde{v}$ at $\sigma(p)$, which we can equivalently phrase as the bijectivity of the Legendre map $v \mapsto g_v(v, \cdot)$ (see Lemma 3.1.7). We shall furthermore assume that lifting a base vector field pointwise in this way is smooth, and will yield a locally horizontal vector field. (There might not generally exist vector fields with values in $\mathbb{A}$ globally, one simple reason to rule out this possibility in general is that such a vector field would be nowhere-zero.)

**Definition 3.1.12.** *For each admissible vector $\tilde{v}$ of the base manifold, let $\tilde{v}_{\tilde{v}}^*|_p$ denote by abuse of notation this unique horizontal lift $v \in \mathbb{A}_p$ at each $p \in \sigma^{-1}(\{\pi(\tilde{v})\})$ satisfying*

$$\mathrm{d}\sigma \cdot \tilde{v}_{\tilde{v}}^*|_p = \tilde{v}\,.$$

*Furthermore, for $\tilde{V} \in \mathfrak{X}(\widetilde{M})$, let $\tilde{V}_{\tilde{V}}^*$ denote the locally unique horizontal vector field $V$ satisfying*

$$\mathrm{d}\sigma \cdot \tilde{V}_{\tilde{V}}^* = V \circ \sigma\,.$$

**Proposition 3.1.13.** *Let $v \in \mathbb{A}$ and $e \in T_{\pi(v)}M$, and denote $\tilde{v} = \mathrm{d}\sigma \cdot v$ and $\tilde{x} = \mathrm{d}\sigma \cdot e$. Furthermore, consider extensions $V \in \mathfrak{X}(M)$ of $v$ with values in $\mathbb{A}$ in some neighbourhood $\Omega \subset M$ of $\pi(v)$ and $E \in \mathfrak{X}(M)$ of $e$ projectable onto $\tilde{V}, \tilde{X} \in \mathfrak{X}(\widetilde{M})$. Then, assuming existence and uniqueness of horizontal lifts,*

$$\begin{aligned} v &= v_v^{\perp} = \tilde{v}_v^* = \tilde{v}_{\tilde{v}}^*|_{\pi(v)}\,,\\ V &= \tilde{V}_V^{\perp} = \tilde{V}_V^* = \tilde{V}_{\tilde{V}}^*\,, \end{aligned}$$

*and*

$$\begin{aligned} e_v^{\perp} &= \tilde{x}_v^*\,,\\ E_V^{\perp} &= \tilde{X}_V^*\,, \end{aligned}$$

*on $\Omega$, such that $^{\perp}$ informally stands for the pullback of $^{*}$ along $\sigma$ in the sense that*

$$e^{\perp} = \tilde{x}^{*}\,,$$
$$E^{\perp} = \tilde{X}^{*}\,,$$

*as anisotropic tensors. Furthermore, $x \in T_{\pi(v)}M$ is $g_v$-horizontal if and only if*

$$x = \tilde{x}^{*}_{v}\,,$$

*and similarly $X \in \mathfrak{X}(M)$ projectable onto $\tilde{X}$ is, on $\Omega$, $g_V$-horizontal if and only if*

$$X = \tilde{X}^{*}_{V}\,.$$

**Proposition 3.1.14.** *The Cartan tensors of $L$ and $\tilde{L}$ can be identified for horizontal vectors in the following sense: let $v \in \mathbb{A}$ and $x, y, z \in T_{\pi(v)}M$ $g_v$-horizontal, then*

$$C_v(x,y,z) = \tilde{C}_{\tilde{v}}(\tilde{x},\tilde{y},\tilde{z})\,, \tag{3.12}$$

*where $\tilde{}$ denotes the image by $\mathrm{d}\sigma$. Furthermore, let $u \in \mathbb{T}_{\pi(v)}$. Then*

$$C_v(x,u,z) = -\tfrac{1}{2} g_v(\mathrm{II}^{\mathbb{A}}_v(x,z),u)\,, \tag{3.13}$$

*being $\mathrm{II}^{\mathbb{A}}_v$ the second fundamental form of $\mathbb{A}_{\pi(v)}$ defined as the part $g_v$-orthogonal to $\mathbb{A}_{\pi(v)}$ at $v$ in the $g_v$-orthogonal decomposition of the affine space connection of $T_{\pi(v)}M$.*

*Proof.* Let $\gamma\colon \mathbb{R} \to \mathbb{A}_{\pi(v)}$ a path of $\mathbb{A}_{\pi(v)}$ satisfying $\gamma(0) = v$ and $\dot{\gamma}(0) = z$. Let $x_t, y_t \in \mathbb{H}_v = T_{\gamma(t)}\mathbb{A}_{\pi(v)}$. Let $\tilde{x}_t = \mathrm{d}\sigma \cdot x_t$ and $\tilde{y}_t = \mathrm{d}\sigma \cdot y_t$. By (3.7),

$$g_{\gamma(t)}(x_t,y_t) = \tilde{g}_{\tilde{\gamma}(t)}(\tilde{x}_t,\tilde{y}_t)\,. \tag{3.14}$$

Assimilate the left hand side of the equation above to $f(t,t,t)$ for the smooth $\mathbb{R}^3 \to \mathbb{R}$ function $f\colon (r,s,t) \mapsto g_{\gamma(t)}(x_r,y_s)$. By the chain rule $\frac{\partial f(r,s,t)}{\partial t} = 2C_{\gamma(t)}(x_r,y_s,\cdot) \circ \frac{\partial \gamma(t)}{\partial t}$, differentiating the left hand side of (3.14) yields

$$\begin{aligned}\frac{\partial g_{\gamma(t)}(x_t,y_t)}{\partial t} &= \left.\frac{\partial g_{\gamma(t)}(x_s,y_t)}{\partial s}\right|_{s=t} + \left.\frac{\partial g_{\gamma(t)}(x_t,y_s)}{\partial s}\right|_{s=t} + \left.\frac{\partial g_{\gamma(s)}(x_t,y_t)}{\partial s}\right|_{s=t} \\ &= g_{\gamma(t)}(\dot{x}_t,y_t) + g_{\gamma(t)}(x_t,\dot{y}_t) + 2C_{\gamma(t)}(x_t,y_t,\dot{\gamma}(t))\,.\end{aligned}$$

Similarly, differentiating the right hand side of (3.14) yields

$$\frac{\partial \tilde{g}_{\tilde{\gamma}(t)}(\tilde{x}_t,\tilde{y}_t)}{\partial t} = \tilde{g}_{\tilde{\gamma}(t)}(\dot{\tilde{x}}_t,\tilde{y}_t) + \tilde{g}_{\tilde{\gamma}(t)}(\tilde{x}_t,\dot{\tilde{y}}_t) + 2\tilde{C}_{\tilde{\gamma}(t)}(\tilde{x}_t,\tilde{y}_t,\dot{\tilde{\gamma}}(t))\,.$$

Note how, by definition of differentials, the restriction $\mathrm{d}\sigma|_{T_{\pi(v)}M}\colon T_{\pi(v)}M \to T_{\sigma\circ\pi(v)}\widetilde{M}$ of $\mathrm{d}\sigma$ to $T_{\pi(v)}M$ is its own differential as a smooth map from $T_{\pi(v)}M$ to $T_{\sigma\circ\pi(v)}\widetilde{M}$. Therefore, it takes the derivative of $t \mapsto x_t$ as a path of $T_{\pi(v)}M$ to the derivative of $t \mapsto \mathrm{d}\sigma \cdot x_t$ as a path of $T_{\sigma\circ\pi(v)}\widetilde{M}$. We have

$$\dot{\tilde{x}}_t = \frac{\partial(\mathrm{d}\sigma\cdot x_t)}{\partial t} = \mathrm{d}\sigma \cdot \frac{\partial x_t}{\partial t} = \tilde{\dot{x}}_t\,, \tag{3.15}$$

and similarly $\dot{\tilde{y}}_t = \tilde{\dot{y}}_t$. By (3.7), we obtain

$$g_{\gamma(t)}(\dot{x}_t,y_t) + g_{\gamma(t)}(x_t,\dot{y}_t) = \tilde{g}_{\tilde{\gamma}(t)}(\tilde{\dot{x}}_t,\tilde{y}_t) + \tilde{g}_{\tilde{\gamma}(t)}(\tilde{x}_t,\tilde{\dot{y}}_t) = \tilde{g}_{\tilde{\gamma}(t)}(\dot{\tilde{x}}_t,\tilde{y}_t) + \tilde{g}_{\tilde{\gamma}(t)}(\tilde{x}_t,\dot{\tilde{y}}_t)\,,$$

and thus by (3.14) from (3.15)

$$C_v(x_0,y_0,z) - \tilde{C}_{\tilde{v}}(\tilde{x}_0,\tilde{y}_0,\tilde{z}) = \frac{1}{2}\left.\frac{\partial(g_{\gamma(t)}(x_t,y_t) - \tilde{g}_{\tilde{\gamma}(t)}(\tilde{x}_t,\tilde{y}_t))}{\partial t}\right|_{t=0} = 0$$

proving (3.12). For the second identity (3.13), note that $u$ is $g_{\gamma(t)}$-orthogonal to $\mathbb{A}_{\pi(v)}$ at each $\gamma(t)$ by definition of $\mathbb{A}_{\pi(v)} \subset \mathbb{A}$. Recalling that by Lemma 3.1.3 $T_{\gamma(t)}\mathbb{A}_{\pi(v)}$ is the space of $g_{\gamma(t)}$-horizontal vectors. A similar computation can then be performed on

$$g_{\gamma(t)}(x_t,u) = 0$$

to obtain, since $g_{\gamma(t)}(\cdot,u)$ is zero on $T_{\gamma(t)}\mathbb{A}_{\pi(v)}$, that differentiating the left hand side to

$$\begin{aligned}\frac{\partial g_{\gamma(t)}(x_t,u)}{\partial t} &= \left.\frac{\partial g_{\gamma(t)}(x_s,u)}{\partial s}\right|_{s=t} + \left.\frac{\partial g_{\gamma(s)}(x_t,u)}{\partial s}\right|_{s=t} \\ &= g_{\gamma(t)}(\dot{x}_t,u) + 2C_{\gamma(t)}(x_t,u,\dot{\gamma}(t)) = g_{\gamma(t)}(\mathrm{II}^{\mathbb{A}}_{\gamma(t)}(\dot{\gamma}(t),x_t),u) + 2C_{\gamma(t)}(x_t,u,\dot{\gamma}(t))\end{aligned}$$

gives zero, which is to say

$$g_v(\mathrm{II}^{\mathbb{A}}_v(z,x),u) + 2C_v(x,u,z) = g_{\gamma(0)}(\mathrm{II}^{\mathbb{A}}_{\gamma(0)}(\dot{\gamma}(0),x_0),u) + 2C_{\gamma(0)}(x_0,u,\dot{\gamma}(0)) = 0\,,$$

concluding the proof of equation (3.13). □

With these simple identities (3.7) and (3.12), we can import much of what was true in pseudo-Riemannian submersions, mainly the correspondence between ambient $g_V$-horizontal frames of $M$ and basic $g_{\mathrm{d}\sigma\cdot V}$-orthogonal frames of $\widetilde{M}$.

**Lemma 3.1.15.** *Given $V \in \mathfrak{X}(M)$ with image in $\mathbb{A}$ in the neighbourhood of some $p \in M$, there exists a $g_V$-orthogonal local frame $(X_1,\cdots,X_k,W_{k+1},\cdots,W_n)$ such that for $q \in M$ close enough to $p$ $(W_{k+1}(q),\cdots,W_n(q))$ forms a basis of $\mathbb{T}_q$, while $(X_1(q),\cdots,X_k(q))$ forms a basis of $\mathbb{H}_{V(q)}$.*

*Proof.* Let $(x_1,\cdots,x_k,\cdots,x_n)$ and $(\bar{x}_1,\cdots,\bar{x}_k)$ be adapted coordinates around $p$ and $\sigma(p)$. Since in this system of coordinates the submersion $\sigma$ takes the form

$$(x_1(q),\cdots,x_k(q),\cdots,x_n(q)) \mapsto (\bar{x}_1(\sigma(q)),\cdots,\bar{x}_k(\sigma(q)))\,,$$

necessarily

$$\mathrm{d}\sigma\cdot\frac{\partial}{\partial x^{k+1}} = \cdots = \mathrm{d}\sigma\cdot\frac{\partial}{\partial x^n} = 0$$

such that the $n-k$ vector fields $\frac{\partial}{\partial x^{k+1}},\cdots,\frac{\partial}{\partial x^n}$ form at each point a basis of $\mathbb{T}$.

To conclude, we may proceed as follows. Where $v = V(p)$, choose a $g_v$-orthonormal basis $(e_1,\cdots,e_k,u_{k+1},\cdots,u_n)$ of $T_pM$ such that the last $n-k$ vectors are vertical, decompose them in the basis

$$\left(\frac{\partial}{\partial x^1}\Big|_p,\cdots,\frac{\partial}{\partial x^k}\Big|_p,\cdots,\frac{\partial}{\partial x^n}\Big|_p\right)$$

whose last $n-k$ vectors are vertical, such that $u_{k+1},\cdots,u_n$ are linear combinations of the vertical vectors $\frac{\partial}{\partial x^k}|_p,\cdots,\frac{\partial}{\partial x^n}|_p$, and consider the constant coefficient extension of the original vectors $e_1,\cdots,e_k,u_{k+1},\cdots,u_n$ in that basis, which yields a local frame of vector fields $(E_1,\cdots,E_k,U_{k+1},\cdots,U_n)$. Then apply the Gram-Schmidt process starting with the vertical vectors, in order to obtain first the locally mutually $g_V$-orthogonal vertical vector fields $W_{k+1},\cdots,W_n$ and then the complete $g_V$ local frame $(X_1,\cdots,X_k,W_{k+1},\cdots,W_n)$.

By (3.7), we can check that $X_1(q),\cdots,X_k(q)$ forms a $g_{V(q)}$-orthogonal frame of the $g_{V(q)}$-horizontal space $T_{V(q)}\mathbb{A}_q$ for $q \in M$ close to $p$. □

**Proposition 3.1.16.** *For $E,H \in \mathfrak{X}(M)$ and $\tilde{X},\tilde{Y} \in \mathfrak{X}(\widetilde{M})$ such that $\mathrm{d}\sigma \cdot E = \tilde{X}\circ\sigma$ and $\mathrm{d}\sigma\cdot H = \tilde{Y}\circ\sigma$ are constant on submersion fibres, we also have that*

$$\mathrm{d}\sigma \cdot [E,H] = [\tilde{X},\tilde{Y}]\circ\sigma \tag{3.16}$$

*are constant on submersion fibres. Consequently, if $W \in \mathfrak{X}(M)$ is a vertical vector field, then*

$$[E,W]_v^{\perp} = 0\,. \tag{3.17}$$

*Proof.* By the chain rule for differentials, we have for any $e \in TM$ and $\tilde{f} \in C^\infty(\widetilde{M})$

$$(\mathrm{d}\sigma\cdot e)\tilde{f} = \mathrm{d}\tilde{f}\cdot(\mathrm{d}\sigma\cdot e) = (\mathrm{d}\tilde{f}\cdot\mathrm{d}\sigma)\cdot e = \mathrm{d}(\tilde{f}\circ\sigma)\cdot e = e(\tilde{f}\circ\sigma)\,, \tag{3.18}$$

such that *pushing forward* $E$ to $\tilde{X}$ and letting it act on $\tilde{f}$ is the same as *pulling back* $\tilde{f}$ to $f = \tilde{f}\circ\sigma$ and letting it be acted on by $E$. In particular, by definition of the Lie brackets,

$$\begin{aligned}(\mathrm{d}\sigma\cdot[E,H])\tilde{f} = [E,H]f = E(H(f)) - H(E(f))&\\ = (\tilde{X}(\tilde{Y}\tilde{f}))\circ\sigma - (\tilde{Y}(\tilde{X}\tilde{f}))\circ\sigma = ([\tilde{X},\tilde{Y}]\circ\sigma)\tilde{f}&\end{aligned}$$

where, applying the same reasoning to $\tilde{Y}\tilde{f}$ we applied to $\tilde{f}$ in (3.18), we have implicitly computed

$$E(H(f)) = E((\mathrm{d}\sigma\cdot H)\tilde{f}) = E((\tilde{Y}\tilde{f})\circ\sigma) = (\mathrm{d}\sigma\cdot E)(\tilde{Y}\tilde{f}) = (\tilde{X}(\tilde{Y}\tilde{f}))\circ\sigma$$

and similarly $H(E(f)) = (\tilde{Y}(\tilde{X}\tilde{f}))\circ\sigma$. □

**Lemma 3.1.17.** *Let $v \in \mathbb{A}$ with a locally horizontal extension $V \in \mathfrak{X}(M)$ projectable onto some $\tilde{V}\in\mathfrak{X}(\widetilde{M})$, and $x \in \mathbb{H}_v$ with a locally $g_V$-horizontal extension $X \in \mathfrak{X}(M)$. Let $\tilde{v} = \mathrm{d}\sigma\cdot v$ and $\tilde{x} = \mathrm{d}\sigma\cdot x$. Then*

$$\nabla^v_v V = (\widetilde{\nabla}^{\tilde{v}}_{\tilde{v}}\tilde{V})^*_v\,, \tag{3.19}$$

*and*

$$\nabla^v_x V = (\widetilde{\nabla}^{\tilde{v}}_{\tilde{x}}\tilde{V})^*_v + \tfrac{1}{2}[X,V]^{\top}_v - C^{\sharp}_v(x,\nabla^v_v V)^{\top}_v\,. \tag{3.20}$$

*Furthermore, let $y,z \in \mathbb{H}_v$, a locally $g_V$-horizontal extensions $X,Y \in \mathfrak{X}(M)$ of $x,y$ projectable onto $\tilde{Y}\in\mathfrak{X}(\widetilde{M})$, and $w \in \mathbb{T}_{\pi(v)}$. Then*

$$g_v(\nabla^v_x Y,w) = g_v(\tfrac{1}{2}[X,Y] - C^{\sharp}_v(\nabla^v_x V,y) - C^{\sharp}_v(x,\nabla^v_y V),w) + C_v(x,y,\nabla^v_w V)\,, \tag{3.21}$$

*and*

$$g_v(\nabla^v_x Y,z) = g_v((\widetilde{\nabla}^{\tilde{v}}_{\tilde{x}}\tilde{Y})^*_v - C^{\sharp}_v((\nabla^v_x V)^{\top}_v,y) - C^{\sharp}_v(x,(\nabla^v_y V)^{\top}_v),z) + C_v(x,y,(\nabla^v_z V)^{\top}_v)\,. \tag{3.22}$$

*Lastly, we have for each projectable $g_V$-horizontal extension $Z$ of $z$*

$$g_v(\nabla^v_w Y,z) = g_v(w,\tfrac{1}{2}[Z,Y]^{\top}_v - C^{\sharp}_v(\nabla^v_y V,z) + C^{\sharp}_v(y,\nabla^v_z V)) - C_v(z,y,\nabla^v_w V)\,. \tag{3.23}$$

*When $V$ is locally the horizontal lift of an extension $\tilde{V}\in\mathfrak{X}(M)$ of $\tilde{v}$ satisfying $\widetilde{\nabla}^{\tilde{v}}\tilde{V} = 0$ as per Proposition 1.3.18, we have in particular*

$$\nabla^v_x V = \tfrac{1}{2}[X,V]^{\top}_v \tag{3.24}$$

*for any $x \in \mathbb{H}_v$ with any extension $X \in \mathfrak{X}(M)$, and*

$$g_v(\nabla^v_x Y,w) = \tfrac{1}{2}g_v([X,Y]^{\top}_v - C^{\sharp}_v([X,V]^{\top}_v,y) - C^{\sharp}_v(x,[Y,V]^{\top}_v),w) + C_v(x,y,\nabla^v_w V) \tag{3.25}$$

*for any $y \in \mathbb{H}_v$ with a locally $g_V$-horizontal projectable extension $Y \in \mathfrak{X}(M)$.*

*Proof.* Consider the Koszul formula for the Chern connection $\nabla$ of $(M,L)$

$$\begin{aligned}2g_v(\nabla^v_x Y, z) = x g_V(Z,Y) + y g_V(X,Z) - z g_V(X,Y) \\ + g_v([Z,X],y) + g_v(x,[Z,Y]) + g_v([X,Y],z) \\ - 2C_v(z,y,\nabla^v_x V) - 2C_v(x,z,\nabla^v_y V) + 2C_v(x,y,\nabla^v_z V)\,,\end{aligned}$$

and that of the Chern connection $\widetilde{\nabla}$ of $(\widetilde{M},\tilde{L})$ which lifts by (3.7), (3.12) and (3.16) to

$$\begin{aligned}2g_v((\widetilde{\nabla}^{\tilde{v}}_{\tilde{x}}\tilde{Y})^*_v, z) = x g_V(Z,Y) + y g_V(X,Z) - z g_V(X,Y) \\ + g_v([Z,X],y) + g_v(x,[Z,Y]) + g_v([X,Y],z) \\ - 2C_v(z,y,(\widetilde{\nabla}^{\tilde{v}}_{\tilde{x}}\tilde{V})^*_v) - 2C_v(x,z,(\widetilde{\nabla}^{\tilde{v}}_{\tilde{y}}\tilde{V})^*_v) + 2C_v(x,y,(\widetilde{\nabla}^{\tilde{v}}_{\tilde{z}}\tilde{V})^*_v)\,.\end{aligned} \tag{3.26}$$

We can combine these two results into

$$\begin{aligned}g_v((\nabla^v_x Y)^\perp_v, z) = g_v((\widetilde{\nabla}^{\tilde{v}}_{\tilde{x}}\tilde{Y})^*_v - C^\sharp_v(y, \nabla^v_x V - (\widetilde{\nabla}^{\tilde{v}}_{\tilde{x}}\tilde{V})^*_v) - C^\sharp_v(x, \nabla^v_y V - (\widetilde{\nabla}^{\tilde{v}}_{\tilde{y}}\tilde{V})^*_v), z) \\ + C_v(x,y,\nabla^v_z V - (\widetilde{\nabla}^{\tilde{v}}_{\tilde{z}}\tilde{V})^*_v)\,.\end{aligned} \tag{3.27}$$

By setting $Y$ to $V$, we obtain by Proposition 1.2.7

$$(\nabla^v_x V)^\perp_v = (\widetilde{\nabla}^{\tilde{v}}_{\tilde{x}}\tilde{V})^*_v - C^\sharp_v(x, \nabla^v_v V - (\widetilde{\nabla}^{\tilde{v}}_{\tilde{v}}\tilde{V})^*_v)^\perp_v\,. \tag{3.28}$$

By setting $x$ to $v$, we have

$$(\nabla^v_v V)^\perp_v = (\widetilde{\nabla}^{\tilde{v}}_{\tilde{v}}\tilde{V})^*_v\,.$$

Similarly,

$$2g_v(\nabla^v_x Y, w) = g_v([X,Y]^\top_v, w) - 2C_v(w,y,\nabla^v_x V) - 2C_v(x,w,\nabla^v_y V) + 2C_v(x,y,\nabla^v_w V)\,,$$

where $g_V(W,Y) = g_V(X,W) = 0$ since vertical and $g_V$-horizontal vector fields are by definition $g_V$-orthogonal, $w g_V(X,Y)$ since by (3.7) $g_V(X,Y) = \tilde{g}_{\tilde{V}}(\tilde{X},\tilde{Y}) \circ \sigma$ is constant on the submersion fibre through $\pi(v)$, which is a submanifold of $M$ to which $w$ is tangent, and $g_v([W,X],y) = g_v(x,[W,Y]) = 0$ by (3.7) and (3.16), proving (3.21). Then,

$$2g_v(\nabla^v_x V, w) = g_v([X,V]^\top_v, w) - 2C_v(x,w,\nabla^v_v V)\,,$$

or by non-degeneracy

$$(\nabla^v_x V)^\top_v = \tfrac{1}{2}[X,V]^\top_v - C^\sharp_v(x, \nabla^v_v V)^\top_v\,, \tag{3.29}$$

and finally

$$2g_v(\nabla^v_v V, w) = 0\,,$$

such that

$$(\nabla^v_v V)^\top_v = 0\,,$$

proving (3.19). We can reinsert this result into equation (3.29) as

$$(\nabla^v_x V)^\top_v = \tfrac{1}{2}[X,V]^\top_v - C^\sharp_v(x, (\widetilde{\nabla}^{\tilde{v}}_{\tilde{v}}\tilde{V})^*_v)^\top_v\,,$$

and into equation (3.28) as

$$(\nabla^v_x V)^\perp_v = (\widetilde{\nabla}^{\tilde{v}}_{\tilde{x}}\tilde{V})^*_v\,,$$

proving (3.20). We can in turn reinsert that result into equation (3.27) as

$$g_v((\nabla^v_x Y)^\perp_v, z) = g_v((\widetilde{\nabla}^{\tilde{v}}_{\tilde{x}} \tilde{Y})^*_v, z) - C_v(z, y, (\nabla^v_x V)^\top_v) - C_v(x, z, (\nabla^v_y V)^\top_v) + C_v(x, y, (\nabla^v_z V)^\top_v),$$

proving (3.22). The Koszul formula also yields

$$2g_v(\nabla^v_w Y, z) = g_v(w, [Z, Y]^\top_v) - 2C_v(z, y, \nabla^v_w V) - 2C_v(w, z, \nabla^v_y V) + 2C_v(w, y, \nabla^v_z V),$$

with the same simplifications as mentioned earlier, proving (3.23). □

### 3.1.2 O'Neill tensors

Let $\nabla$ be the Chern connection of $(M, L)$. When generalising from submanifolds to submersions, in which the ambient manifold $M$ is fibred by submanifolds which in some sense are the individual points of the base manifold, the evaluation $\nabla^v_e H$ of the Chern connection in $v \in TM$, $e \in T_{\pi(v)}M$ and $H \in \mathfrak{X}(M)$ splits as

$$\begin{aligned}\nabla^v_e H &= \left(\nabla^v_e (H^\top + H^\perp)\right)^\top_v + \left(\nabla^v_e (H^\top + H^\perp)\right)^\perp_v \\ &= (\nabla^v_e H^\top)^\top_v + (\nabla^v_e H^\top)^\perp_v + (\nabla^v_e H^\perp)^\top_v + (\nabla^v_e H^\perp)^\perp_v. \end{aligned} \tag{3.30}$$

Note how precisely the terms $(\nabla^v_e H^\top)^\perp_v$ and $(\nabla^v_e H^\perp)^\top_v$ are tensorial, by an argument similar to the one presented for equation (1.31), for $f \in C^\infty(M)$,

$$\begin{aligned}(\nabla^v_e (fH)^\top)^\perp_v &= \left(e(f) H^\top_v + f \nabla^v_e H^\top\right)^\perp_v = f(\nabla^v_e H^\top)^\perp_v, \\ (\nabla^v_e (fH)^\perp)^\top_v &= \left(e(f) H^\perp_v + f \nabla^v_e H^\perp\right)^\top_v = f(\nabla^v_e H^\perp)^\top_v.\end{aligned}$$

Consequently,

$$\chi^v_e h = (\nabla^v_e H^\top)^\perp_v + (\nabla^v_e H^\perp)^\top_v$$

is tensorial in $H \in \mathfrak{X}(M)$, depending only on $h = H(\pi(v))$.

Notational consistency would demand that we write the evaluation $\chi^v_e h$ of the anisotropic tensor $\chi$ as $\chi_v(e, h)$, and this is the notation we have used in Chapter 2. Nevertheless, as will become evident in the computational steps leading up to the main theorem of this chapter and the unified fundamental equation (3.83) of pseudo-Finsler submersions, the tensor $\chi$ will be persistently chained with itself and in combination with $\nabla$.

Following O'Neill's choice of a notation, and despite the rupture with the notation from the previous chapter, let us prefer to write $\chi^v_e h$ instead of $\chi_v(e, h)$ so as to be able to write $\chi^v_e \chi^v_h$, $(\nabla_e \chi)^v_h$ and $\chi^v_e \nabla^v_h$ rather than the more cumbersome $\chi_v(e, \chi_v(h, \cdot))$, $(\nabla_e \chi)_v(h, \cdot)$ and $\chi_v(e, \nabla^v_h \cdot)$. We hope that this decision will make it easier to compare the results with O'Neill's original article [14].

While we may prefer to use $\chi$ in order to write simpler intermediate calculations, let us nevertheless define the O'Neill tensors $A^v_e = \chi^v_{e^\perp_v}$ and $T^v_e = \chi^v_{e^\top_v}$ by analogy to [14]. In order to define rigorously the tensors $T$ and $A$ as anisotropic tensors, recall the necessary condition that the vertical space must be non degenerate with respect to $g_v$ for all $v \in \mathbb{A}$. As discussed earlier, by definition of a pseudo-Finsler submersion, this will be true for vertical and horizontal vectors, and we can reduce the domain of definition of $L$ to ensure this condition which always holds when $g_v$ is positive definite and as a consequence of this reduction it may be that $\mathbb{A}$ is non-connected.

**Definition 3.1.18.** *Let $A, T$ be the anisotropic tensors given for $v \in \mathbb{A}$, $e \in T_{\pi(v)}M$ with $w = e_v^\top$ and $x = e_v^\perp$, $H \in \mathfrak{X}(M)$ and $h = H(\pi(v))$ by*

$$A_e^v h = (\nabla_x^v H^\perp)_v^\top + (\nabla_x^v H^\top)_v^\perp, \tag{3.31}$$

*and*

$$T_e^v h = (\nabla_w^v H^\perp)_v^\top + (\nabla_w^v H^\top)_v^\perp. \tag{3.32}$$

We may obtain (3.40) and (3.41) by expressing (3.30) in terms of the vector fields $U = H_V^\top$ and $Y = H_V^\perp$ for any locally admissible choice of extension $V \in \mathfrak{X}(M)$ of $v$ up to a $C^\sharp$ term given by the following proposition.

**Proposition 3.1.19.** *For $v \in \mathbb{A}$ and any $e \in T_{\pi(v)}M$,*

$$(\dot{\partial} e^\top)_v = 2(C_v^\sharp(e_v^\perp, \cdot))_v^\top, \quad (\dot{\partial} e^\perp)_v = -2(C_v^\sharp(e_v^\perp, \cdot))_v^\top. \tag{3.33}$$

*In particular, if $v \in \mathbb{A}$, then*

$$(\dot{\partial} v^\top)_v = 0. \tag{3.34}$$

*Proof.* Let $w = e_v^\top$ and $x = e_v^\perp$, and observe how $(\dot{\partial} w^\top)_v = (\dot{\partial} w)_v = 0$ implies

$$(\dot{\partial} e^\top)_v = (\dot{\partial} w^\top + \dot{\partial} x^\top)_v = (\dot{\partial} x^\top)_v.$$

Let $h \in T_{\pi(v)}M$, and consider a smooth one parameter family of $g_{v+th}$-orthonormal bases $(u_1^t, \cdots, u_m^t)$ of $\mathbb{T}_{\pi(v)}$ over $t \in \mathbb{R}$ close to 0 and let $\dot{u}_i^0$ denote the derivative at $t = 0$ of each $u_i \colon \mathbb{R} \to T_{\pi(v)}M$. Note that $\dot{u}_i^0 \in \mathbb{T}_{\pi(v)}$. Denote $\varepsilon_i = g_{v+te}(u_i(t), u_i(t))$. By definition

$$\begin{aligned}(\dot{\partial} x^\top)_v(h) &= \frac{\partial}{\partial t} \sum_i \varepsilon_i g_{v+th}(e, u_i^t) u_i^t \Big|_{t=0} \\ &= \sum_i \varepsilon_i \left( g_v(x, \dot{u}_i^0) u_i^0 + 2C_v(x, u_i^0, h) u_i^0 + g_v(x, u_i^0) \dot{u}_i^0 \right) \\ &= 2\sum_i \varepsilon_i g_v(C_v^\sharp(x, h), u_i^0) u_i^0 = 2(C_v^\sharp(x, h))_v^\top.\end{aligned}$$

Conclude (3.33) up to noting that $\dot{\partial} x^\perp = \dot{\partial}(x^\top + x^\perp) - \dot{\partial} x^\top = -\dot{\partial} x^\top$.

In particular,

$$(\dot{\partial} v^\top)_v = 2(C_v^\sharp(v_v^\perp, \cdot))_v^\top = -2(C_v^\sharp(v_v^\top, \cdot))_v^\top$$

is identically zero for $v \in \mathbb{A}$ which necessarily satisfies $v_v^\top = 0$. □

Observe how, in the anisotropic case, the O'Neill tensors $A_e^v x$ and $T_e^v x$ are no longer the vertical part of the $g_v$-orthogonal decomposition of $\nabla_e^v X$ for any extension $X \in \mathfrak{X}(M)$ of $x \in \mathbb{H}_v$, due to this $C^\sharp$ term.

**Lemma 3.1.20.** *Let $v \in \mathbb{A}$, $C^\sharp$ defined as in (1.19), vertical vectors $w, u \in \mathbb{T}_{\pi(v)}$ and $g_v$-horizontal vectors $x, y \in \mathbb{H}_v$. Then for any vertical extension $U \in \mathfrak{X}(M)$ of $u$*

$$\nabla_w^v U = (\nabla_w^v U)_v^\top + T_w^v u, \tag{3.35}$$

$$\nabla_x^v U = (\nabla_x^v U)_v^\top + A_x^v u. \tag{3.36}$$

*Furthermore, let $V \in \mathfrak{X}(M)$ extend $v$ with image in $\mathbb{A}$ in the neighbourhood of $\pi(v)$. Then for any locally $g_V$-horizontal extension $Y \in \mathfrak{X}(M)$ of $y$*

$$\nabla^v_w Y = (\nabla^v_w Y)^\perp_v + T^v_w y - 2(C^\sharp_v(y, \nabla^v_w V))^\top_v \,, \tag{3.37}$$

$$\nabla^v_x Y = (\nabla^v_x Y)^\perp_v + A^v_x y - 2(C^\sharp_v(y, \nabla^v_x V))^\top_v \,. \tag{3.38}$$

*In particular, for arbitrary vectors $e = w + x$ and $h = u + y$ with an arbitrary extension $H = U + Y$,*

$$(\nabla^v_e H)^\perp_v - (\nabla^v_x Y)^\perp_v = A^v_y w + A^v_x u + T^v_w u \,. \tag{3.39}$$

*Proof.* By (1.36) and (3.33), we may compute for any vertical extension $U \in \mathfrak{X}(M)$ of $u$

$$\nabla^v U = (\nabla^v U^\top)^\top_v + A^v u + T^v u + (\nabla^v U^\perp)^\perp_v = (\nabla^v U)^\top_v + A^v u + T^v u \,, \tag{3.40}$$

proving (3.35) and (3.36).

Meanwhile, again by (3.33) for $y \in \mathbb{H}_v$ with a locally $g_V$-horizontal extension $Y \in \mathfrak{X}(M)$

$$\begin{aligned}
\nabla^v Y &= (\nabla^v Y^\top)^\top_v + A^v y + T^v y + (\nabla^v Y^\perp)^\perp_v \\
&= (\nabla^v Y^\top_V)^\top_v + A^v y + T^v y + (\nabla^v Y^\perp_V)^\perp_v - \Big[(\dot\partial y^\top)_v(\nabla^v V)\Big]^\top_v - \Big[(\dot\partial y^\perp)_v(\nabla^v V)\Big]^\perp_v \\
&= A^v y + T^v y + (\nabla^v Y)^\perp_v - 2C^\sharp_v(y, \nabla^v V)^\top_v \,,
\end{aligned} \tag{3.41}$$

proving (3.37) and (3.38) for this extension.

In order to prove (3.39), recall that by (3.17) the mixed Lie brackets of vertical and $g_V$-horizontal vector fields vanish under $^\perp_v$, such that for any vertical extension $W$ of $w$

$$(\nabla^v_e H)^\perp_v = (\nabla^v_x Y)^\perp_v + (\nabla^v_y W)^\perp_v + (\nabla^v_x U)^\perp_v + (\nabla^v_w U)^\perp_v = (\nabla^v_x Y)^\perp_v + A^v_y w + A^v_x u + T^v_w u \,.$$

□

Nevertheless, the simple properties of the classical O'Neill tensors that $A^v$ is a $g_v$-horizontal operator and $T^v$ a vertical one, and that they flip $g_v$-horizontality and verticality of their input, remain true in the anisotropic case.

**Proposition 3.1.21.** *Let $v \in \mathbb{A}$, and $e \in T_{\pi(v)}M$ with $w = e^\top_v$ and $x = e^\perp_v$. Then $A^v$ is a $g_v$-horizontal in the sense that*

$$A^v_e = A^v_x \,,$$

*$T^v$ is a vertical operator in the sense that*

$$T^v_e = T^v_w \,,$$

*and they interchange verticality and $g_v$-horizontality in the sense that*

$$\begin{aligned}
A^v w &= (A^v e)^\perp_v \,, & A^v x &= (A^v e)^\top_v \,, \\
T^v w &= (T^v e)^\perp_v \,, & T^v x &= (T^v e)^\top_v \,.
\end{aligned}$$

*Proof.* The first two equations are immediate consequences of the definitions (3.31) of $A$ and (3.32) of $T$. For the other four, note how $A^v w$ and $T^v w$ are $g_v$-horizontal due to, respectively, equations (3.36) and (3.35), while $A^v x$ and $T^v x$ are vertical by equations (3.38) and (3.37). That is to say $A^v w = (A^v w)^\perp_v$ and $T^v w = (T^v w)^\perp_v$ are $g_v$-horizontal, while $A^v x = (A^v x)^\top_v$ and

$T^v x = (T^v x)_v^\top$ are vertical. By linearity, $A^v w + A^v x = A^v e$ and $T^v w + T^v x = T^v e$, concluding the proof. □

**Proposition 3.1.22.** *Let $v \in \mathbb{A}$. The tensors $A$ and $T$ are skew symmetric in the following sense: for any $e, h \in T_{\pi(v)}M$,*

$$g_v(A^v e, h) = -g_v(e, A^v h)\,, \tag{3.42}$$
$$g_v(T^v e, h) = -g_v(e, T^v h)\,. \tag{3.43}$$

Furthermore, if $v$ is vertical then $T^v$ is symmetric.

*Proof.* Symmetry of $T^v$ on $\mathbb{T}_{\pi(v)}$ for $v$ vertical is merely Proposition 2.1.5, as indeed its restriction to $\mathbb{T}_{\pi(v)}$ is the second fundamental form of the submersion fiber $\sigma^{-1}(\{\sigma \circ \pi(v)\})$ at $\pi(v)$ and with respect to $g_v$. We can check by Lemma 3.1.20 and for vertical extensions $U, W \in \mathfrak{X}(M)$ of $u, w \in \mathbb{T}_{\pi(v)}$ that

$$[W,U]|_{\pi(v)} = [W,U]_v^\top + T_w^v u - T_u^v w$$

is a vertical vector, since by (3.7) and (3.16) $g_v(x, [W,U]) = 0$ for every $x \in \mathbb{H}_v$.

Skew symmetry (3.42) and (3.43) for $v$ an arbitrary admissible vector can be obtained in rigorously the same way as each other, for example using Lemma 3.1.20 under the form

$$A_x^v y = (\nabla_x^v Y)_v^\top + 2C_v^\sharp(y, \nabla_x^v V)_v^\top\,,$$

for $x, y \in \mathbb{H}_v$, we can compute

$$\begin{aligned} g_v(A_x^v y, u) &= g_v((\nabla_x^v Y)_v^\top + 2C_v^\sharp(y, \nabla_x^v V)_v^\top, u) = g_v(\nabla_x^v Y, u) + 2C_v(y, u, \nabla_x^v V) \\ &= x g_V(Y, U) - g_v(y, \nabla_x^v U) = -g_v(y, (\nabla_x^v U)_v^\perp) = -g_v(y, A_x^v u)\,, \end{aligned}$$

and similarly

$$g_v(T_w^v y, u) = -g_v(y, T_w^v u)\,.$$

By decomposing $e = w + x$ and $h = u + y$ for $w = e_v^\top$, $x = e_v^\perp$, $u = h_v^\top$ and $y = h_v^\perp$ and using Proposition 3.1.21, we have

$$\begin{aligned} g_v(A^v e, h) &= g_v(A^v e, u) + g_v(A^v e, y) = g_v(A^v x, u) + g_v(A^v w, y) \\ &= -g_v(x, A^v u) - g_v(w, A^v y) = -g_v(x, A^v h) - g_v(w, A^v h) = -g_v(e, A^v h)\,, \end{aligned}$$

and similarly

$$\begin{aligned} g_v(T^v e, h) &= g_v(T^v w, h) + g_v(T^v x, h) = g_v(T^v w, y) + g_v(T^v x, u) \\ &= -g_v(w, T^v y) - g_v(x, T^v u) = -g_v(e, T^v y) - g_v(e, T^v u) = -g_v(e, T^v h)\,. \end{aligned}$$

□

**Lemma 3.1.23.** *Let $v \in \mathbb{A}$ and $x, y \in \mathbb{H}_v$. Then*

$$A_v^v v = 0\,, \tag{3.44}$$
$$A_x^v y + A_y^v x = 2(A+T)^v_{C_v^\sharp(x,y)} v\,,\,. \tag{3.45}$$

*Let $V \in \mathfrak{X}(M)$ be a locally horizontal extension of $v$ and $X, Y \in \mathfrak{X}(M)$ locally $g_V$-horizontal extensions of $x$ and $y$. Then*

$$A^v_v x = -A^v_x v = \tfrac{1}{2}[X,V]^\top_v\,, \tag{3.46}$$
$$A^v_x y = \tfrac{1}{2}[X,Y]^\top_v + (A+T)^v_{C^\sharp_v(x,y)} v + C^\sharp_v(A^v_x v, y)^\top_v - C^\sharp_v(x, A^v_y v)^\top_v - C^\sharp_v(C^\sharp_v(x,y)^\perp_v, \nabla^v_v V)\,. \tag{3.47}$$

*Proof.* By (3.38), (3.24), Lemma 3.1.20 and by homogeneity of the Cartan tensor,

$$A^v_x v = \nabla^v_x V = \tfrac{1}{2}[X,V]^\top_v = \tfrac{1}{2}(A^v_x v - A^v_v x)$$

proving (3.44). Similarly,

$$\tfrac{1}{2}[Y,V]^\top_v = \nabla^v_y V = (\nabla^v_y V)^\top_v = A^v_y v\,.$$

Equation (3.45) is an immediate consequence of (3.47), which can be obtained from Lemma 3.1.20 and (3.21) as follows. Recalling that for any vertical extension $W$ of $w$ the Lie brackets $[V,W] = \nabla^v_v W - \nabla^v_w V$ are vertical, by Lemma 3.1.20

$$\nabla^v_w V = (\nabla^v_w V)^\top_v + (\nabla^v_v W)^\perp_v = T^v_w v + A^v_v w\,. \tag{3.48}$$

Then, we can compute by (3.38) and (3.21)

$$\begin{aligned} g_v(A^v_x y, w) &= g_v(A^v_x y - 2C^\sharp_v(y, \nabla^v_x V) + 2C^\sharp_v(y, \nabla^v_x V), w) = g_v(\nabla^v_x Y + 2C^\sharp_v(\nabla^v_x V, y), w)\\ &= g_v(\tfrac{1}{2}[X,Y] + C^\sharp_v(\nabla^v_x V, y) - C^\sharp_v(x, \nabla^v_y V), w) + C_v(x,y,\nabla^v_w V)\,, \end{aligned}$$

noting that by Propositions 3.1.21 and 3.1.22 together with (3.46)

$$\begin{aligned} C_v(x,y,\nabla^v_w V) &= C_v(x,y,A^v_v w + T^v_w v - C^\sharp_v(w, \nabla^v_v V)^\perp_v)\\ &= g_v(C^\sharp_v(x,y)^\perp_v, A^v_v w) + g_v(C^\sharp_v(x,y)^\top_v, T^v_w v) - g_v(C^\sharp_v(x,y), C^\sharp_v(w, \nabla^v_v V)^\perp_v)\\ &= -g_v(A^v_v C^\sharp_v(x,y)^\perp_v, w) - g_v(T^v_{C^\sharp_v(x,y)^\top_v} w, v) - C_v(C^\sharp_v(x,y)^\perp_v, w, \nabla^v_v V)\\ &= g_v(A^v_{C^\sharp_v(x,y)} v, w) + g_v(T^v_{C^\sharp_v(x,y)} v, w) - g_v(C^\sharp_v(C^\sharp_v(x,y)^\perp_v, \nabla^v_v V), w)\\ &= g_v((A+T)^v_{C^\sharp_v(x,y)} v - C^\sharp_v(C^\sharp_v(x,y)^\perp_v, \nabla^v_v V), w)\,. \end{aligned}$$

□

### 3.1.3 Gauss formula and dual Gauss formula

**Definition 3.1.24.** *Let $\widehat{Q}$ be the anisotropic tensor defined for each $v \in \mathbb{A}$ and $g_v$-horizontal $x$ by $\widehat{Q}^v_x = \widehat{Q}^v x = 0$ and for each vertical $u$ and $w$ by*

$$\widehat{Q}^v_u w = -\Big(T^v_v C^\sharp_v(u,w) + C^\sharp_v(T^v_v u, w) + C^\sharp_v(u, T^v_v w) + C^\sharp_v(C^\sharp_v(u,w)^\top_v, T^v_v v) - C^\sharp_v(C^\sharp_v(u, T^v_v v)^\top_v, w) - C^\sharp_v(u, C^\sharp_v(w, T^v_v v)^\top_v)\Big)^\top_v$$

**Lemma 3.1.25** (Gauss formula). *For $v \in \mathbb{A}$ and $u, w \in \mathbb{T}_p$ and any vertical extension $W$ of $w$,*

$$(\nabla^v_u W)^\top_v = \widehat{\nabla}^v_u W + \widehat{Q}^v_u w\,.$$

*Proof.* Completely analogous to the proof of Theorem 2.1.10. □

**Definition 3.1.26.** *Let $\widetilde{Q}$ be the anisotropic tensor defined for each $v \in \mathbb{A}$ and vertical $w$ by $\widetilde{Q}^v_w = \widetilde{Q}^v w = 0$ and for $x, y \in \mathbb{H}_v$ by*

$$\widetilde{Q}^v_x y = \left(A^v_v C^\sharp_v(x,y) + C^\sharp_v(A^v_v x, y) + C^\sharp_v(x, A^v_v y)\right)^\perp_v .$$

Note how, by skew-symmetry (3.42), we can equivalently characterise $\widetilde{Q}$ as follows: for each $v \in \mathbb{A}$ and $x, y, z \in \mathbb{H}_v$,

$$g_v(\widetilde{Q}^v_x y, z) = C_v(A^v_v x, y, z) + C_v(x, A^v_v y, z) - C_v(x, y, A^v_v z) .$$

**Lemma 3.1.27** (Dual Gauss Formula)**.** *For $v \in \mathbb{A}$ and $x, y \in \mathbb{H}_v$ with projections $\tilde{v} = \mathrm{d}\sigma \cdot$, $\tilde{x} = \mathrm{d}\sigma \cdot x$ and $\tilde{y} = \mathrm{d}\sigma \cdot y$, and any extension $E \in \mathfrak{X}(M)$ of $y$ projectable onto some $\tilde{Y} \in \mathfrak{X}(\widetilde{M})$,*

$$(\nabla^v_x E)^\perp_v = (\widetilde{\nabla}^{\tilde{v}}_{\tilde{x}} \tilde{Y})^*_v + \widetilde{Q}^v_x y . \tag{3.49}$$

*Proof.* Let $V$ be a locally admissible extension of $v$ projectable onto some $\tilde{V}$ satisfying $\widetilde{\nabla}^{\tilde{v}}\tilde{V}$, and $Y \in \mathfrak{X}(M)$ a locally $g_V$-horizontal extension of $y$ projectable onto $\tilde{Y}$. By Lemma 3.1.20, we have on $\mathbb{H}_v$

$$(\nabla^v V)^\top_v = A^v v ,$$

allowing us to simplify (3.22) and obtain

$$g_v(\nabla^v_x Y, z) = g_v((\widetilde{\nabla}^{\tilde{v}}_{\tilde{x}} \tilde{Y})^*_v - C^\sharp_v(A^v_x v, y) - C^\sharp_v(x, A^v_y v), z) + C_v(x, y, A^v_z v)$$

where, by (3.42) and (3.46), the last term can be manipulated into

$$g_v(C^\sharp_v(x,y), A^v_z v) = -g_v(C^\sharp_v(x,y), A^v_v z) = g_v(A^v_v C^\sharp_v(x,y), z) .$$

Conclude by Lemma 3.1.20, from which we obtain

$$(\nabla^v_x E)^\perp_v = (\nabla^v_x Y)^\perp_v + (\nabla^v_x (E^\top_V))^\perp_v = (\nabla^v_x Y)^\perp_v + A^v_x E^\top_v = (\nabla^v_x Y)^\perp_v .$$

□

Observe that, by (3.39), we can rewrite the dual Gauss formula under the slightly more general form (3.79).

### 3.1.4 Alternation properties

Recall from the definition of anisotropic tensor derivation that for each $v \in \mathbb{A}$ and $E, H \in \mathfrak{X}(M)$ extending $e, h \in T_{\pi(v)}M$, the tensor $\nabla A$ is unambiguously determined by

$$\nabla^v (A_E H) = (\nabla A)^v_e h + A^v_{\nabla^v E} h + A^v_e \nabla^v H , \tag{3.50}$$

and let $\nabla T$ be the anisotropic tensor unambiguously determined by

$$\nabla^v (T_E H) = (\nabla T)^v_e h + T^v_{\nabla^v E} h + T^v_e \nabla^v H . \tag{3.51}$$

Similarly, let $\nabla \widetilde{Q}$ be the anisotropic tensor unambiguously determined by

$$\nabla^v (\widetilde{Q}_E H) = (\nabla \widetilde{Q})^v_e h + \widetilde{Q}^v_{\nabla^v E} h + \widetilde{Q}^v_e \nabla^v H . \tag{3.52}$$

To simplify later computation, recall our short hand $\chi = A + T$ and $\nabla\chi = \nabla A + \nabla T$. Furthermore,

$$(\nabla\chi)^v_x h = \nabla^v(A^V_X H) - \chi^v_{\nabla^v X} h - A^v_x \nabla^v H - (\dot{\partial} A_x h)_v(\nabla^v V) = (\nabla A)^v_x h - T^v_{\nabla^v X} h\,,$$

thus, since $(T^v h)^\top_v = T^v(h^\perp_v)$,

$$((\nabla\chi)^v_x w)^\top_v = ((\nabla A)^v_x w)^\top_v\,. \tag{3.53}$$

Similarly,

$$(\nabla\chi)^v_w h = \nabla^v(T^V_W H) - \chi^v_{\nabla^v W} h - T^v_w \nabla^v H - (\dot{\partial} T_w h)_v(\nabla^v V) = (\nabla T)^v_w h - T^v_{\nabla^v W} h\,,$$

thus, since $(A^v h)^\perp_v = A^v(h^\top_v)$,

$$((\nabla\chi)^v_w x)^\perp_v = ((\nabla T)^v_w x)^\perp_v\,. \tag{3.54}$$

These two identities (3.53) and (3.54) motivate the following Lemma.

**Lemma 3.1.28.** *Let $v \in \mathbb{A}$, $x, y \in \mathbb{H}_v$, $w, u \in \mathbb{T}_{\pi(v)}$ and $e \in T_{\pi(v)}M$. Then*

$$\big((\nabla_w A)^v_e u\big)^\top_v = T^v_w A^v_e u - A^v_e T^v_w u\,, \tag{3.55}$$

$$\big((\nabla_x A)^v_e u\big)^\top_v = A^v_x A^v_e u - A^v_e A^v_x u\,, \tag{3.56}$$

$$\big((\nabla_w T)^v_e u\big)^\top_v = T^v_w T^v_e u - T^v_e T^v_w u\,, \tag{3.57}$$

*and*

$$\big((\nabla_x T)^v_e u\big)^\top_v = A^v_x T^v_e u - T^v_e A^v_x u\,. \tag{3.58}$$

*Furthermore,*

$$\big((\nabla_w A)^v_e y\big)^\perp_v = T^v_w A^v_e y - A^v_e T^v_w y\,, \tag{3.59}$$

$$\big((\nabla_x A)^v_e y\big)^\perp_v = A^v_x A^v_e y - A^v_e A^v_x y\,, \tag{3.60}$$

$$\big((\nabla_w T)^v_e y\big)^\perp_v = T^v_w T^v_e y - T^v_e T^v_w y\,, \tag{3.61}$$

*and*

$$\big((\nabla_x T)^v_e y\big)^\perp_v = A^v_x T^v_e y - T^v_e A^v_x y\,. \tag{3.62}$$

*Proof.* Using the shorthand $\chi = A + T$, and for arbitrary tangent vectors $e, h \in T_{\pi(v)}M$,

$$(\nabla_{e^\top_v} A)^v_h + (\nabla_{e^\perp_v} A)^v_h + (\nabla_{e^\top_v} T)^v_h + (\nabla_{e^\perp_v} T)^v_h = (\nabla_e \chi)^v_h\,,$$

furthermore

$$(T^v_{e^\top_v} A^v_h - A^v_h T^v_{e^\top_v}) + (A^v_{e^\perp_v} A^v_h - A^v_h A^v_{e^\perp_v}) + (T^v_{e^\top_v} T^v_h - T^v_h T^v_{e^\top_v}) + (A^v_{e^\perp_v} T^v_h - T^v_h A^v_{e^\perp_v}) = (\chi^v_e \chi^v_h - \chi^v_h \chi^v_e)\,.$$

In light of (3.53) and (3.54), the eight identities can thus be summed up for every $w \in \mathbb{T}_{\pi(v)}$ as

$$\big((\nabla_e \chi)^v_h w\big)^\top_v = \chi^v_e \chi^v_h w - \chi^v_h \chi^v_e w\,, \tag{3.63}$$

and for every $x \in \mathbb{H}_v$

$$\big((\nabla_e \chi)^v_h x\big)^\perp_v = \chi^v_e \chi^v_h x - \chi^v_h \chi^v_e x\,, \tag{3.64}$$

which we shall prove for any $e,h \in T_{\pi(v)}M$ by considering a locally admissible extension $V \in \mathfrak{X}(M)$ of $v$ satisfying $\nabla^v_e V = 0$, some extension $H \in \mathfrak{X}(M)$ of $h$, a vertical extension $W \in \mathfrak{X}(M)$ of $w$ and a locally $g_V$-horizontal extension $X$ of $x \in \mathbb{H}_v$. Then,

$$(\nabla_e \chi)^v_h w = \nabla^v_e(\chi^V_H W) - \chi^v_{\nabla^v_e H} w - \chi^v_h \nabla^v_e W\,, \tag{3.65}$$

where the middle term of the right hand side is $g_v$-horizontal by Proposition 3.1.21 and will vanish under ${}^\top_v$. Similarly, by Lemma 3.1.20, the last term leaves only

$$(\chi^v_h \nabla^v_e W)^\top_v = (A^v_h \nabla^v_e W)^\top_v + (T^v_h \nabla^v_e W)^\top_v = (A^v_h(A^v_e w + T^v_e w))^\top_v + (T^v_h(A^v_e w + T^v_e w))^\top_v = \chi^v_h \chi^v_e w\,.$$

That is to say, being $u \in \mathbb{T}_{\pi(v)}$ with a vertical extension $U \in \mathfrak{X}(M)$, by skew-symmetry (3.42) and (3.43) and Lemma 3.1.20,

$$\begin{aligned} g_v((\nabla_e \chi)^v_h w, u) &= g_v(\nabla^v_e(\chi^V_H W), u) - g_v(\chi^v_h \chi^v_e w, u) \\ &= e g_V(\chi^V_H W, U) - g_v(\chi^v_h w, \nabla^v_e U) - 2C_v(\chi^v_h w, u, \nabla^v_e V) - g_v(\chi^v_h \chi^v_e w, u) \\ &= -g_v(\chi^v_h w, \chi^v_e u) - g_v(\chi^v_h \chi^v_e w, u) = g_v(\chi^v_e \chi^v_h w - \chi^v_h \chi^v_e w, u)\,, \end{aligned}$$

recalling that $\chi^V_H W$ is locally $g_V$-horizontal, such that $g_V(\chi^V_H W, U) = 0$ in the neighbourhood of $\pi(v)$. Conclude (3.63) by Proposition 3.1.21 and by non-degeneracy.

By an identical computation, we find that

$$(\nabla_e \chi)^v_h x = \nabla^v_e(\chi^V_H X) - \chi^v_{\nabla^v_e H} x - \chi^v_h \nabla^v_e X\,, \tag{3.66}$$

where the middle term of the right hand side is vertical by Proposition 3.1.21 and will vanish under ${}^\perp_v$. Similarly, by Lemma 3.1.20, the last term leaves only

$$(\chi^v_h \nabla^v_e X)^\top_v = (A^v_h \nabla^v_e X)^\top_v + (T^v_h \nabla^v_e X)^\top_v = (A^v_h(A^v_e x + T^v_e x))^\top_v + (T^v_h(A^v_e x + T^v_e x))^\top_v = \chi^v_h \chi^v_e w\,,$$

noting that the $-2C^\sharp_v(x, \nabla^v_e V)^\top_v$ term in the decomposition is zero for our choice of extensions satisfying $\nabla^v_e V = 0$. That is to say, being $y \in \mathbb{H}_v$ with a locally $g_V$-horizontal extension $Y \in \mathfrak{X}(M)$, by skew-symmetry (3.42) and (3.43) and Lemma 3.1.20,

$$\begin{aligned} g_v((\nabla_e \chi)^v_h x, y) &= g_v(\nabla^v_e(\chi^V_H X), y) - g_v(\chi^v_h \chi^v_e x, y) \\ &= e g_V(\chi^V_H X, Y) - g_v(\chi^v_h x, \nabla^v_e Y) - 2C_v(\chi^v_h x, y, \nabla^v_e V) - g_v(\chi^v_h \chi^v_e x, y) \\ &= -g_v(\chi^v_h x, \chi^v_e y) - g_v(\chi^v_h \chi^v_e x, y) = g_v(\chi^v_e \chi^v_h x - \chi^v_h \chi^v_e x, y)\,, \end{aligned}$$

recalling that $\chi^V_H X$ is locally vertical, such that $g_V(\chi^V_H X, Y) = 0$ in the neighbourhood of $\pi(v)$. Conclude (3.64) by Proposition 3.1.21 and by non-degeneracy.

To conclude the proof, note how for any choice of extension $E$ of $e$

$$(\nabla A)^v_{e^\top_v} u = \nabla^v(A^V_{E^\top_V} U) - A^v_{(\nabla^v E^\top_V)^\perp_v} u - A^v_{e^\top_v} \nabla^v U = -A^v_{(\nabla^v E^\top_V)^\perp_v} u$$

since $A^v_{e^\top_v}$ and $A^V_{E^\top_V}$ are both zero by definition of $A$, and therefore is $g_v$-horizontal by property of $A$ (since $u$ is vertical) and vanishes under $\top_v$. By the same computation, $(\nabla T)^v_{e^\perp_v} u$ also vanishes under ${}^\top_v$, similarly $(\nabla A)^v_{e^\top_v} y$ and $(\nabla T)^v_{e^\perp_v} y$ both vanish under ${}^\perp_v$. Thus, indeed, we may deduce the first four equations from (3.63) and the last four from (3.64). □

Equations (3.65) and (3.66) can be split into an equation on $A$ (3.73) and an equation on $T$ (3.74), providing us with the following simplifications (see Lemma 4 from [14]).

**Lemma 3.1.29.** *For $v \in \mathbb{A}$, $x, y \in \mathbb{H}_v$ and $w, u \in \mathbb{T}_{\pi(v)}$,*

$$(\nabla^v_w A)^v_u = -A^v_{T^v_w u}\,, \tag{3.67}$$
$$(\nabla^v_x T)^v_y = -T^v_{A^v_x y}\,, \tag{3.68}$$
$$(\nabla^v_x A)^v_u = -A^v_{A^v_x u}\,, \tag{3.69}$$
$$(\nabla^v_w T)^v_y = -T^v_{T^v_w y}\,. \tag{3.70}$$

*Proof.* Note how we may combine these equations by pairs into the equivalent forms, for $e \in T_{\pi(v)}M$ arbitrary,

$$(\nabla^v_e A)^v_u = -A^v_{\chi^v_e u} \tag{3.71}$$

and

$$(\nabla^v_e T)^v_y = -T^v_{\chi^v_e y}\,, \tag{3.72}$$

using as before our shorthand $\chi = A + T$. Consider any extension $E \in \mathfrak{X}(M)$, and the usual locally admissible extension $V$, locally $g_V$-horizontal extensions $X, Y, Z$ and vertical extensions $W, U, S$. By Lemma 3.1.20, from the derivation

$$\begin{aligned}(\nabla^v A)^v_u e = \nabla^v(A^V_U E) - A^v_{\nabla^v U} e - A^v_u \nabla^v E - (\dot{\partial} A)_v(u, e, \nabla^v V)\\ = -A^v_{\chi^v u} e - (\dot{\partial} A)_v(u, e, \nabla^v V)\end{aligned} \tag{3.73}$$

we obtain the first equation, and from the derivation

$$\begin{aligned}(\nabla^v T)^v_y e = \nabla^v(T^V_Y E) - T^v_{\nabla^v Y} e - T^v_y \nabla^v E - (\dot{\partial} T)_v(y, e, \nabla^v V)\\ = -T^v_{\chi^v y} e - (\dot{\partial} T)_v(y, e, \nabla^v V)\end{aligned} \tag{3.74}$$

the other equation, for a locally admissible choice of extension satisfying $\nabla^v V = 0$. □

Using Lemmas 3.1.28 and 3.1.29, we can eliminate covariant derivation from any terms of the form

$$(\nabla_e A)^v_h,\ (\nabla_e T)^v_h$$

when splitting along vertical and $g_v$-horizontal parts, except for the following cases. Being $x, y \in \mathbb{H}_v$ arbitrary $g_v$-horizontal vectors and $w, u \in \mathbb{T}_{\pi(v)}$ arbitrary vertical vectors, neither terms

$$((\nabla A)^v_x y)^\top_v\,,\ ((\nabla T)^v_w u)^\perp_v$$

can generally be further simplified in the way of Lemma 3.1.28 or Lemma 3.1.29, and they are the only two instances of $\nabla A$ and $\nabla T$ that ever need to be computed by hand in specific contexts. As O'Neill mentioned in Lemma 6 of [14], the best that can be done about these two remaining terms in the isotropic case is to note that they exhibit symmetry and anti-symmetry; in our anisotropic case, symmetry and almost-antisymmetry.

**Proposition 3.1.30.** *Let $v \in \mathbb{A}$, $w \in \mathbb{T}_{\pi(v)}$ and $x, y, z \in \mathbb{H}_v$. Then*

$$g_v(T^v_{A^v_x y} z, w) = g_v(T^v_w z, A^v_x y)\,. \tag{3.75}$$

*Proof.* By Proposition 3.1.22, $g_v(T^v_{A^v_x y} z, w) = -g_v(z, T^v_{A^v_x y} w) = -g_v(z, T^v_w A^v_x y) = g_v(T^v_w z, A^v_x y)$. □

### 3.1.5 Geodesic properties

The geodesics of $(M,L)$ and $(\widetilde{M},\tilde{L})$, defined as smooth paths verifying the differential equation (1.46) or its local equivalent (1.47) substituting $\nabla$ and $\Gamma$ for $\widetilde{\nabla}$ and $\widetilde{\Gamma}$ as well as summing over the fewer indices of the corresponding coordinate functions in the latter case of the base manifold, are remarkably well behaved in the context of pseudo-Finsler submersions.

This subsection contains the succinct proof of two facts: that the geodesics of $M$ come in two kinds, horizontal and nowhere-horizontal, with no in-between; and that those horizontal geodesics are precisely the lifts of the geodesics of the base manifold.

This means we can strengthen our previous Lemma 3.1.15 to guarantee that geodesic frames around any $p \in M$ that split between horizontal and vertical geodesics induce geodesic frames around $\sigma(p)$, and conversely that geodesic frames around $b \in \widetilde{M}$ lift to horizontal geodesics that can be completed into a full geodesic frame around any $p \in \sigma^{-1}(\{b\})$.

**Lemma 3.1.31.** *Horizontal geodesics of $(M,L)$ project by $\sigma$ onto geodesics of $(\widetilde{M},\tilde{L})$, and conversely horizontal lifts of geodesics of $(\widetilde{M},\tilde{L})$ are geodesics of $(M,L)$.*

*Proof.* Consider a geodesic $\gamma$ of $(M,L)$ such that $\dot\gamma$ has image in $\mathbb{A}$. By the dual Gauss formula (3.49), we can write

$$\widetilde{\nabla}^{\tilde{\dot\gamma}}_{\tilde{\dot\gamma}}\tilde{\dot\gamma} = \mathrm{d}\sigma \cdot \nabla^{\dot\gamma}_{\dot\gamma}\dot\gamma = 0\,,$$

which is to say $\sigma \circ \gamma$ is then a geodesic of $\widetilde{M}$ with respect to $(\widetilde{M},\tilde{L})$. Assuming conversely that $\sigma \circ \gamma$ is a geodesic, we have, again by the formula (3.49),

$$(\nabla^{\dot\gamma}_{\dot\gamma}\dot\gamma)^{\perp}_{\dot\gamma} = (\widetilde{\nabla}^{\tilde{\dot\gamma}}_{\tilde{\dot\gamma}}\tilde{\dot\gamma})^{*}_{\dot\gamma} = 0\,,$$

while by Lemma 3.1.20, (3.44) and Proposition 1.2.7 recalling that $\dot\gamma = (\dot\gamma)^{\perp}_{v}$ is horizontal

$$(\nabla^{\dot\gamma}_{\dot\gamma}\dot\gamma)^{\top}_{\dot\gamma} = A^{\dot\gamma}_{\dot\gamma}\dot\gamma - 2C^{\sharp}_{v}(\dot\gamma, \nabla^{\dot\gamma}_{\dot\gamma}\dot\gamma)^{\top}_{v} = 0\,.$$

□

**Theorem 3.1.32.** *Anywhere horizontal geodesics are horizontal everywhere.*

*Proof.* If $\dot\gamma(t)$ is horizontal for any $t \in \mathbb{R}$, then projecting to $(\sigma\circ\gamma)'(t)$ there exists by Proposition 1.3.14 a unique geodesic of $\widetilde{M}$ through $\sigma\circ\gamma(t)$ with velocity $(\sigma\circ\gamma)'(t)$ at that point whose horizontal lift is by the previous lemma an everywhere horizontal geodesic of $M$ with velocity $\dot\gamma(t)$ at $\gamma(t)$. By uniqueness, $\gamma$ must be that geodesic. □

## 3.2 O'Neill equations

The aim of this section is to generalise the fundamental equations $\{0\}$ to $\{4\}$ in [14]. To this end, let us introduce the following curvature tensors.

**Proposition 3.2.1.** *Let $v \in \mathbb{A}$ with a locally admissible extension $V \in \mathfrak{X}(M)$ and $B,E,H \in \mathfrak{X}(M)$. Then*

$$\left(\nabla^{v}_{B}(\nabla^{V}_{E}H^{\top})^{\top}_{V} - \nabla^{v}_{E}(\nabla^{V}_{B}H^{\top})^{\top}_{V} - \nabla^{v}_{[B,E]}H^{\top}\right)^{\top}_{v}$$

$$\left(\nabla^{v}_{B}(\nabla^{V}_{E}H^{\perp})^{\perp}_{V} - \nabla^{v}_{E}(\nabla^{V}_{B}H^{\perp})^{\perp}_{V} - \nabla^{v}_{[B,E]}H^{\perp}\right)^{\perp}_{v}$$

*are $C^\infty(M)$-multilinear in $B,E,H$ and*

$$\left(\nabla^v_B(\nabla_E H^\top)^\top - \nabla^v_E(\nabla_B H^\top)^\top - \nabla^v_{[B,E]}H^\top\right)^\top_v$$

$$\left(\nabla^v_B(\nabla_E H^\perp)^\perp - \nabla^v_E(\nabla_B H^\perp)^\perp - \nabla^v_{[B,E]}H^\perp\right)^\perp_v$$

*define anisotropic tensors in $B(\pi(v)),E(\pi(v)),H(\pi(v))$.*

*Proof.* Proceed as with the ordinary anisotropic curvature tensor. □

**Definition 3.2.2.** *Let us denote these tensors by $R^\top$ and $R^\perp$ respectively, such that for $v\in\mathbb{A}$ and $b,e,h\in T_{\pi(v)}M$ with extensions $B,E,H\in\mathfrak{X}(M)$*

$$R^\top_v(b,e)h = \left(\nabla^v_b(\nabla_E H^\top)^\top - \nabla^v_e(\nabla_B H^\top)^\top - \nabla^v_{[B,E]}H^\top\right)^\top_v,$$

*and*

$$R^\perp_v(b,e)h = \left(\nabla^v_b(\nabla_E H^\perp)^\perp - \nabla^v_e(\nabla_B H^\perp)^\perp - \nabla^v_{[B,E]}H^\perp\right)^\perp_v.$$

### 3.2.1 Preliminary simplifications for a unified fundamental equation

**Proposition 3.2.3.** *Let $v\in\mathbb{A}$ with $\tilde v = \mathrm{d}\sigma\cdot v$ and a locally admissible extension $V\in\mathfrak{X}(M)$ projectable onto some $\tilde V\in\mathfrak{X}(\widetilde M)$, $w\in\mathbb{T}_{\pi(v)}$ and $x\in\mathbb{H}_v$ with $\tilde x = \mathrm{d}\sigma\cdot x$. Then*

$$\nabla^v_w V = T^v_w v + A^v_v w \tag{3.76}$$

*and*

$$\nabla^v_x V = A^v_x v + (\widetilde{\nabla}^{\tilde v}_{\tilde x}\tilde V)^*_v. \tag{3.77}$$

*Proof.* Let $e = w+x$. By Lemma 3.1.20 together with Proposition 1.2.7, we have

$$\nabla^v_w V = (\nabla^v_w V)^\perp_v + T^v_w v$$

while

$$\nabla^v_x V = (\nabla^v_x V)^\perp_v + A^v_x v.$$

Note that we can simplify $\widetilde{Q}^v_v = 0$ by Proposition 1.2.7, such that by (3.49) and recalling that $[V,W]|_{\pi(v)} = \nabla^v_v W - \nabla^v_w V$ is vertical for every vertical extension $W$ of $w$

$$\nabla^v_e V = T^v_w v + (\nabla^v_e V)^\perp_v + A^v_x v = T^v_w v + A^v_v w + (\widetilde{\nabla}^{\tilde v}_{\tilde x}\tilde V)^\perp_v + A^v_x v.$$

□

**Proposition 3.2.4.** *Let $v\in\mathbb{A}$ and $e,h\in T_{\pi(v)}M$ with any extension $H\in\mathfrak{X}(M)$. Then*

$$\nabla^v_e H = (\nabla^v_e H^\top)^\top_v + A^v_e h + T^v_e h + (\nabla^v_e H^\perp)^\perp_v. \tag{3.78}$$

*Consequently,*

$$\nabla^v_e H^\top = (\nabla^v_e H^\top)^\top_v + \chi^v_e h^\top_v$$

*and*

$$\nabla^v_e H^\perp = \chi^v_e h^\perp_v + (\nabla^v_e H^\perp)^\perp_v.$$

*Proof.* We obtain (3.78) by definition of the O'Neill tensors,

$$\nabla^v_e H = (\nabla^v_e H^\top)^\top_v + \underbrace{(\nabla^v_e H^\top)^\perp_v + (\nabla^v_e H^\perp)^\top_v}_{A^v_e h + T^v_e h} + (\nabla^v_e H^\perp)^\perp_v\,.$$

□

**Proposition 3.2.5.** *Let $v \in \mathbb{A}$ and $e, h \in T_{\pi(v)}M$ with any extension $H \in \mathfrak{X}(M)$ of $h$ projectable onto some $\tilde{H} \in \mathfrak{X}(\widetilde{M})$. Then, using the shorthand $\chi = T + A$,*

$$\nabla^v_e H = (\nabla^v_e H^\top)^\top_v + \chi^v_e h + (\widetilde{\nabla}^{\tilde{v}}_{\tilde{e}} \tilde{H})^*_v + \widetilde{Q}^v_e h + (A^v_h e)^\perp_v\,. \tag{3.79}$$

*Proof.* For a locally projectable horizontal extension $V$ of $v$, by definition

$$\nabla^v_e H = (\nabla^v_e H^\top)^\top_v + \chi^v_e h + (\nabla^v_e H^\perp)^\perp_v\,.$$

Consider a locally $g_V$-horizontal projectable extension $X$ of $e^\perp_v$ and a vertical extension $U$ of $e^\top_v$. Using (3.33) and Lemma 3.1.20, from the dual Gauss formula and noting that the Lie bracket of projectable vector fields is vertical when either vector field is vertical,

$$\begin{aligned}(\nabla^v_e H^\perp)^\perp_v = (\nabla^v_e (H^\perp_V))^\perp_v = (\nabla^v_X (H^\perp_V))^\perp_v + (\nabla^v_U (H^\perp_V))^\perp_v \\ = (\widetilde{\nabla}^{\tilde{v}}_{\tilde{e}} \tilde{H})^*_v + \widetilde{Q}^v_e h + (\nabla^v_{H^\perp_V} U)^\perp_v = (\widetilde{\nabla}^{\tilde{v}}_{\tilde{e}} \tilde{H})^*_v + \widetilde{Q}^v_e h + A^v_h(e^\top_v)\,.\end{aligned}$$

This concludes the proof up to $A^v_h(e^\top_v) = (A^v_h e)^\perp_v$ by property of $A$. □

**Proposition 3.2.6.** *For $v \in \mathbb{A}$ with a locally admissible vector field extension $V$, $b, e, h \in T_pM$ with vector field extensions $B, E, H$ and $w \in \mathbb{T}_p$ with vertical extension $W$,*

$$\begin{aligned}R^\top_v(b,e)w = \Big(\nabla^v_b(\nabla^V_E W)^\top_V - \nabla^v_E(\nabla^V_B W)^\top_V - \nabla^v_{[B,E]} W\Big)^\top_v - 2C^\sharp_v(T^v_e w, \nabla^v_b V)^\top_v \\ + 2C^\sharp_v(T^v_b w, \nabla^v_e V)^\top_v - P_v(e, w, \nabla^v_b V)^\top_v + P_v(b, w, \nabla^v_e V)^\top_v\,,\end{aligned} \tag{3.80}$$

*and*

$$\begin{aligned}R^\perp_v(b,e)h = \Big(\nabla^v_b(\nabla^V_E H^\perp_V)^\perp_V - \nabla^v_e(\nabla^V_B H^\perp_V)^\perp_V - \nabla^v_{[B,E]} H^\perp_V\Big)^\perp_v + 2(T+A)^v_e C^\sharp_v(h^\perp_v, \nabla^v_b V)^\top_v \\ - 2(T+A)^v_b C^\sharp_v(h^\perp_v, \nabla^v_e V)^\perp_v - P_v(e, h^\perp_v, \nabla^v_b V)^\perp_v + P_v(b, h^\perp_v, \nabla^v_e V)^\perp_v\,.\end{aligned} \tag{3.81}$$

*Proof.* In the first equation, as $W^\top$ is identically equal to $W$, we may compute by (3.33)

$$\begin{aligned}\nabla^v_b(\nabla_E W)^\top = \nabla^v_b(\nabla^V_E W)^\top_V - (\dot{\partial}(\nabla_e W)^\top)_v(\nabla^v_b V) \\ = \nabla^v_b(\nabla^V_E W)^\top_V - 2C^\sharp_v(T^v_e w, \nabla^v_b V)^\top_v - P_v(e, w, \nabla^v_b V)^\top_v\,,\end{aligned}$$

and a similar identity when commuting $e$ and $b$. In the second case, we have

$$\nabla^v_{[B,E]} H^\perp = \nabla^v_{[B,E]} H^\perp_V - (\dot{\partial}(H^\perp))_v(\nabla^v_{[B,E]} V) = \nabla^v_{[B,E]} H^\perp_V + 2C^\sharp_v(h^\perp_v, \nabla^v_{[B,E]} V)^\top_v\,,$$

whose last term vanishes under ${}^\perp_v$, while

$$(\nabla^v_b(\nabla_E H^\perp)^\perp)^\perp_v = (\nabla^v_b(\nabla^V_E H^\perp)^\perp_V)^\perp_v - ((\dot{\partial}(\nabla_e H^\perp)^\perp)_v(\nabla^v_b V))^\perp_v\,.$$

By (1.45),

$$((\dot{\partial}(\nabla_e H^\perp)^\perp)_v(\nabla_b^v V))_v^\perp = ((\dot{\partial}(\nabla_e^v H^\perp)^\perp)_v(\nabla_b^v V))_v^\perp + (((\dot{\partial}(\nabla_e H^\perp))_v(\nabla_b^v V))_v^\perp)_v^\perp$$
$$= ((\dot{\partial}(\nabla_e H^\perp))_v(\nabla_b^v V))_v^\perp = P_v(e, h_v^\perp, \nabla_b^v V)_v^\perp + ((\nabla_e(\dot{\partial} H^\perp))_v(\nabla_b^v V))_v^\perp ,$$

having ignored vertical terms that vanish under $_v^\perp$. Lastly,

$$(\nabla_E^V H^\perp)_V^\perp = (\nabla_E^V H_V^\perp)_V^\perp - ((\dot{\partial} H^\perp)_V(\nabla_E^V V))_V^\perp = (\nabla_E^V H_V^\perp)_V^\perp ,$$

again ignoring the vertical term from (3.33). All that remains is

$$(\nabla_e(\dot{\partial} H^\perp))_v(\nabla_b^v V) = -2(\nabla_e C^\sharp(H^\perp, \cdot)^\top)_v(\nabla_b^v V)$$

and from there we obtain (up to extending formally the definition of the tensors $T$ and $A$ to also be defined for anisotropic tensor fields as inputs)

$$(\nabla_e(\dot{\partial} H^\perp))_v(\nabla_b^v V) = -2(\nabla_e^v C^\sharp(H^\perp, \nabla_B V)^\top)_v^\perp = -2(T+A)_e^v C_v^\sharp(h_v^\perp, \nabla_b^v V)_v^\top .$$

Thus we have computed

$$(\nabla_b^v(\nabla_E H^\perp)^\perp)_v^\perp = (\nabla_b^v(\nabla_E^V H_V^\perp)_V^\perp)_v^\perp - P_v(e, h_v^\perp, \nabla_b^v V)_v^\perp + 2(T+A)_e^v C_v^\sharp(h_v^\perp, \nabla_b^v V)_v^\top ,$$

and similarly we would compute a similar identity commuting $b$ and $e$, concluding the proof. □

**Proposition 3.2.7.** *For $v \in \mathbb{A}$, $e, h \in T_{\pi(v)}M$, $w \in \mathbb{T}_{\pi(v)}$ and $x \in \mathbb{H}_v$,*

$$R_v^\perp(e,h)w = 0 , \quad R_v^\top(e,h)x = -2C_v^\sharp(R_v(e,h)v, x)_v^\top .$$

*Proof.* The first identity is merely a consequence of the fact that any vertical extension $W$ of $w$ satisfies that $W^\top$ is identically equal to $W$ and $W^\perp$ identically equal to 0. As for the second identity, consider a locally admissible extension $V$ of $v$ satisfying $\nabla^v V = 0$, extensions $E$ and $H$ of $e$ and $h$ with vanishing Lie brackets at $\pi(v)$, and a locally $g_V$-horizontal extension $X$ of $x$. By (3.33),

$$(\nabla_H^V X^\top)_V^\top = (\nabla_H^V X_V^\top)_V^\top - ((\dot{\partial} X^\top)_V(\nabla_H^V V))_V^\top = -2C_V^\sharp(X, \nabla_H^V V)_V^\top .$$

Differentiating the tensor $-2C_V^\sharp(\cdot,\cdot)_V^\top$ with $\nabla_h^v V = \nabla_e^v V = 0$ for our choice of $V$ yields

$$\nabla_e^v((\nabla_H^V X^\top)_V^\top) = -2C_v^\sharp(x, \nabla_e^v \nabla_H^V V)_v^\top .$$

Reinserting into

$$(\nabla_e^v(\nabla_H X^\top)^\top)_v^\top = (\nabla_e^v(\nabla_H^V X^\top)_V^\top)_v^\top = -2C_v^\sharp(x, \nabla_e^v \nabla_H^V V)_V^\top = -2C_v^\sharp(x, \nabla_e^v \nabla_H V)_V^\top$$

concludes the proof, up to a similar identity commuting $e$ and $h$. □

**Proposition 3.2.8.** *For each vertical admissible $v$ and vertical $s$, $u$ and $w$ at $p$,*

$$R_v^\top(w,u)s = \widehat{R}_v(w,u)s + (\widehat{\nabla}_w \widehat{Q})_u^v s - (\widehat{\nabla}_u \widehat{Q})_w^v s + \widehat{Q}_w^v \widehat{Q}_u^v s - \widehat{Q}_u^v \widehat{Q}_w^v s$$
$$- P_v(u, s, \widehat{Q}_w^v v + T_w^v v)_v^\top + P_v(w, s, \widehat{Q}_u^v v + T_u^v v)_v^\top$$
$$- 2C_v^\sharp(T_u^v s, \widehat{Q}_w^v v + T_w^v s)_v^\top + 2C_v^\sharp(T_w^v s, \widehat{Q}_u^v v + T_u^v s)_v^\top .$$

*Proof.* Let us proceed as in Proposition 2.16 of [50]. Consider a locally admissible vertical extension $V$ of $v$ satisfying $\widehat{\nabla}^v V = 0$, and therefore by the Gauss formula $\nabla^v_w V = \widehat{\nabla}^v_w V + \widehat{Q}^v_w v + T^v_w v = \widehat{Q}^v_w v + T^v_w v$. For vertical extensions $S$ and $U$ and $W$ of $s$ and $u$ and $w$ such that $[W,U]$ vanishes at $p$,

$$R^\top_v(w,u) = (\nabla^v_w(\nabla_U S)^\top)^\top_v - (\nabla^v_u(\nabla_W S)^\top)^\top_v \,.$$

By the Gauss formula,

$$(\nabla^v_w(\nabla_U S)^\top)^\top_v = (\nabla^v_w \widehat{\nabla}^V_U S)^\top_v + (\nabla^v_w(\widehat{Q}^V_U S))^\top_v - (\dot{\partial}(\nabla_u S)^\top)_v(\widehat{Q}^v_w v + T^v_w v)\,,$$

whose last term can be manipulated into the following expression: using the chain rule and (3.33),

$$(\dot{\partial}(\nabla_u S)^\top)_v = (\dot{\partial}(\nabla^v_u S)^\top)_v + P_v(u,s,\cdot)^\top_v = 2C^\sharp_v(T^v_u s,\cdot)^\top_v + P_v(u,s,\cdot)^\top_v \,.$$

The term $(\nabla^v_w \widehat{\nabla}^V_U S)^\top_v$ can be expanded using the Gauss formula

$$(\nabla^v_w \widehat{\nabla}^V_U S)^\top_v = \widehat{\nabla}^v_w \widehat{\nabla}_U S + \widehat{Q}^v_w \widehat{\nabla}^v_u S + \widehat{P}_v(u,s,\widehat{\nabla}^v_w V) = \widehat{\nabla}^v_w \widehat{\nabla}_U S + \widehat{Q}^v_w \widehat{\nabla}^v_u S\,,$$

while the term

$$(\nabla^v_w(\widehat{Q}^V_U S))^\top_v = \widehat{\nabla}^v_w(\widehat{Q}^V_U S) + \widehat{Q}^v_w \widehat{Q}^v_u s$$

can be further expanded by definition of $\widehat{\nabla}\widehat{Q}$

$$\widehat{\nabla}^v_w(\widehat{Q}^V_U S) = (\widehat{\nabla}_w \widehat{Q})^v_u s + \widehat{Q}^v_{\widehat{\nabla}^v_w U} s + \widehat{Q}^v_u \widehat{\nabla}^v_w S + ((\dot{\partial}\widehat{Q})^v_u s)(\widehat{\nabla}^v_w V) = (\widehat{\nabla}_w \widehat{Q})^v_u s + \widehat{Q}^v_{\widehat{\nabla}^v_w U} s + \widehat{Q}^v_u \widehat{\nabla}^v_w S\,.$$

These combine into

$$\begin{aligned}(\nabla^v_w(\nabla_U S)^\top)^\top_v = {} & \widehat{\nabla}^v_w \widehat{\nabla}_U S + \widehat{Q}^v_w \widehat{\nabla}^v_u S + (\widehat{\nabla}_w \widehat{Q})^v_u s + \widehat{Q}^v_{\widehat{\nabla}^v_w U} s + \widehat{Q}^v_u \widehat{\nabla}^v_w S + \widehat{Q}^v_w \widehat{Q}^v_u s \\ & - 2C^\sharp_v(T^v_u s, \widehat{Q}^v_w v + T^v_w s)^\top_v - P_v(u,s,\widehat{Q}^v_w v + T^v_w s)^\top_v \,.\end{aligned}$$

When subtracting a similar formula for $(\nabla^v_u(\nabla_W S)^\top)^\top_v$ commuted in $w$ and $u$, the terms $\widehat{Q}^v_{\widehat{\nabla}^v_w U} s$ and $\widehat{Q}^v_{\widehat{\nabla}^v_u W} s$ cancel each other out by our assumption that $[U,W]$ vanishes at $p$, and the pair of terms $\widehat{Q}^v_w \widehat{\nabla}^v_u S$ and $\widehat{Q}^v_u \widehat{\nabla}^v_w S$ appears twice with opposite signs. □

**Proposition 3.2.9.** *For $v \in \mathbb{A}$ and $x, y, z \in \mathbb{H}_v$,*

$$\begin{aligned}R^\perp_v(x,y)z = {} & \widetilde{R}^*_v(\tilde{x},\tilde{y})\tilde{z} + A^v_z A^v_y x - A^v_z A^v_x y + ((\nabla_x \widetilde{Q})^v_y z - (\nabla_y \widetilde{Q})^v_x z)^\perp_v + \widetilde{Q}^v_y \widetilde{Q}^v_x z - \widetilde{Q}^v_x \widetilde{Q}^v_y z \\ & + 2A^v_z C^\sharp_v(y, A^v_x v)^\top_v + 2A^v_y C^\sharp_v(z, A^v_x v)^\top_v - 2A^v_z C^\sharp_v(x, A^v_y v)^\top_v - 2A^v_x C^\sharp_v(z, A^v_y v)^\top_v \\ & - P_v(y,z,A^v_x v)^\perp_v + P_V(x,z,A^v_y v)^\perp_v + ((\dot{\partial}\widetilde{Q})^v_y z(A^v_x v))^\perp_v - ((\dot{\partial}\widetilde{Q})^v_x z(A^v_y v))^\perp_v \,. \end{aligned} \quad (3.82)$$

*Proof.* Consider an extension $\tilde{V}$ of $\tilde{v}$ satisfying $\widetilde{\nabla}^{\tilde{v}}\tilde{V} = 0$ and an extension $V$ of $v$ locally the horizontal lift of $\tilde{V}$. Recall from the Gauss formula that $(\nabla^v V)^\perp_v$ vanishes on $\mathbb{H}_v$. Furthermore, by Lemma 3.1.20 $\nabla^v V$ is equal to $A^v v$ on $\mathbb{H}_v$. Let $\tilde{X}$, $\tilde{Y}$ and $\tilde{Z}$ be extensions of $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$ with mutual Lie brackets vanishing at $\pi(v)$. Let $X$, $Y$ and $Z$ be vector field extensions of $x$, $y$ and $z$ that are, locally, the $g_V$-horizontal lifts of $\tilde{X}$, $\tilde{Y}$ and $\tilde{Z}$. In particular, their mutual Lie brackets are vertical. By Proposition 3.2.6, it suffices to compute $(\nabla^v_{[X,Y]}Z)^\perp_v$. By the dual Gauss formula,

$$(\nabla^v_x(\nabla^V_Y Z)^\perp_V)^\perp_v = (\nabla^v_x(\widetilde{\nabla}^{\tilde{V}}_{\tilde{Y}}\tilde{Z})^*_V)^\perp_v + (\nabla^v_x \widetilde{Q}^V_Y Z)^\perp_v \,.$$

By (3.45) and (3.47),

$$\tfrac{1}{2}A^v_x y = \tfrac{1}{2}[X,Y]^\top_v + \tfrac{1}{2}A^v_y x + C^\sharp_v(A^v_x v, y)^\top_v - C^\sharp_v(x, A^v_y v)^\top_v .$$

Using (3.39) and Proposition 3.1.21 yields

$$(\nabla^v_{[X,Y]} Z)^\perp_v = A^v_z [X,Y] = A^v_z A^v_x y - A^v_z A^v_y x + 2A^v_z C^\sharp_v (A^v_v x, y)^\top_v - 2A^v_z C^\sharp_v (x, A^v_v y)^\top_v .$$

Again by the dual Gauss formula and since for our choice of extension $\widetilde{\nabla}^{\tilde v}\tilde V = 0$,

$$\begin{aligned}(\nabla^v_x(\widetilde{\nabla}^{\tilde V}_{\tilde Y}\tilde Z)^*_V)^\perp_v + (\nabla^v_x \widetilde{Q}^V_Y Z)^\perp_v &= (\widetilde{\nabla}^{\tilde v}_{\tilde x}\widetilde{\nabla}_{\tilde Y}\tilde Z)^*_v + \widetilde{Q}^v_x(\widetilde{\nabla}^{\tilde v}_{\tilde y}\tilde Z)^*_v \\ &+ ((\nabla_x \widetilde{Q})^v_y z)^\perp_v + \widetilde{Q}^v_{\nabla^v_x Y} z + \widetilde{Q}^v_y(\widetilde{\nabla}^{\tilde v}_{\tilde x}\tilde Z)^*_v + \widetilde{Q}^v_y \widetilde{Q}^v_x z + ((\dot\partial \widetilde{Q})^v_y z(A^v_x v))^\perp_v .\end{aligned}$$

When subtracting to the formula we have just obtained a similar formula commuted in $x$ and $y$, the pairs of $\widetilde{Q}^v(\widetilde{\nabla}^{\tilde v}\tilde Z)^*_v$ terms cancel each other out, as well as the pair of $\widetilde{Q}^v_{\nabla^v_x Y} z$ and $\widetilde{Q}^v_{\nabla^v_y X} z$ under our assumption that $[X,Y]$ is vertical. ☐

In terms of these tensors $R^\top$ and $R^\perp$, and using our shorthand $\chi = A + T$, we may generalise O'Neill's fundamental equations of a Riemannian submersion (equations $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$ and $\{4\}$ of [14]) into the singled unified closed form (3.83).

### 3.2.2 Fundamental equations

**Lemma 3.2.10** (Unified fundamental equation)**.** *For $v \in \mathbb{A}$ and $e, h \in T_{\pi(v)}M$, using the shorthand $\chi = A + T$,*

$$R_v(e,h) = R^\top_v(e,h) + R^\perp_v(e,h) + (\nabla_e \chi)^v_h - (\nabla_h \chi)^v_e + \chi^v_h \chi^v_e - \chi^v_e \chi^v_h . \tag{3.83}$$

*Proof.* Let $b \in T_{\pi(v)}M$ and recall that by definition and for a choice of extensions $E, H, B \in \mathfrak{X}(B)$ of $b$, $e$ and $h$ such that the Lie brackets $[E,H]$ are zero at $\pi(v)$ we have

$$R_v(e,h) = \nabla^v_e \nabla_H B - \nabla^v_h \nabla_E B .$$

By Proposition 3.2.4,

$$\nabla^v_e \nabla_H B = \nabla^v_e (\nabla_H B^\top)^\top + \nabla^v_e(\chi_H B) + \nabla^v_e(\nabla_H B^\perp)^\perp , \tag{3.84}$$

where we shall further expand, again by Proposition 3.2.4,

$$\nabla^v_e(\nabla_H B^\top)^\top = (\nabla^v_e(\nabla_H B^\top)^\top)^\top_v + \chi^v_e(\nabla^v_h B^\top)^\top_v \tag{3.85}$$

and similarly

$$\nabla^v_e(\nabla_H B^\perp)^\perp = \chi^v_e(\nabla^v_h B^\perp)^\perp_v + (\nabla^v_e(\nabla_H B^\perp)^\perp)^\perp_v \tag{3.86}$$

such that, together with the definition (3.50) and (3.51) of $\nabla\chi = \nabla A + \nabla T$, we obtain

$$\begin{aligned}\nabla^v_e \nabla_H B &= (\nabla^v_e(\nabla_H B^\top)^\top)^\top_v + \chi^v_e(\nabla^v_h B^\top)^\top_v \\ &\quad + (\nabla_e \chi)^v_h b + \chi^v_{\nabla^v_e H} b + \chi^v_h \nabla^v_e B + \chi^v_e(\nabla^v_h B^\perp)^\perp_v + (\nabla^v_e(\nabla_H B^\perp)^\perp)^\perp_v \\ &= (\nabla^v_e(\nabla_H B^\top)^\top)^\top_v + \chi^v_e(\nabla^v_h B^\top)^\top_v + (\nabla_e \chi)^v_h b + \chi^v_{\nabla^v_e H} b + \chi^v_h \left( (\nabla^v_e B^\top)^\top_v + \chi^v_e b + (\nabla^v_e B^\perp)^\perp_v \right) \\ &\quad + \chi^v_e(\nabla^v_h B^\perp)^\perp_v + (\nabla^v_e(\nabla_H B^\perp)^\perp)^\perp_v\end{aligned}$$

and compute similarly $\nabla^v_h\nabla^v_E B$. Cancelling out $\chi^v_e(\nabla^v_h B^\top)^\top_v + \chi^v_h(\nabla^v_e B^\top)^\top_v$ and $\chi^v_e(\nabla^v_h B^\perp)^\perp_v + \chi^v_h(\nabla^v_e B^\perp)^\perp_v$ with similar terms,

$$\begin{aligned} R_v(e,h)b &= (\nabla^v_e(\nabla_H B^\top)^\top)^\top_v - (\nabla^v_h(\nabla_E B^\top)^\top)^\top_v + (\nabla_e\chi)^v_h b - (\nabla_h\chi)^v_e b \\ &\quad + \chi^v_{[E,H]}b + \chi^v_h\chi^v_e b - \chi^v_e\chi^v_h b + (\nabla^v_e(\nabla_H B^\perp)^\perp)^\perp_v - (\nabla^v_h(\nabla_E B^\perp)^\perp)^\perp_v \\ &= R^\top_v(e,h)b + (\nabla_e\chi)^v_h b - (\nabla_h\chi)^v_e b + \chi^v_h\chi^v_e b - \chi^v_e\chi^v_h b + R^\perp_v(e,h)b\,. \end{aligned}$$

□

**Corollary 3.2.11.** *[Fundamental equations of a pseudo-Finsler submersion] For $v \in \mathbb{A}$, vertical vectors $s', s, u, w \in \mathbb{T}_{\pi(v)}$ and $g_v$-horizontal vectors $x, y, z, z' \in \mathbb{H}_v$,*

$$\begin{aligned}
&g_v(R_v(w,u)s,s') = g_v(R^\top_v(w,u)s,s') + g_v(T^v_w s, T^v_u s') - g_v(T^v_u s, T^v_w s') && \{0\}\\
&g_v(R_v(w,u)s,z) = g_v((\nabla_w T)^v_u s,z) - g_v((\nabla_u T)^v_w s,z) && \{1\}\\
&g_v(R_v(x,u)s,w) = g_v(R^\top_v(x,u)s,w) + g_v(T^v_u s, A^v_x w) - g_v(A^v_x s, T^v_u w) && \{1'\}\\
&g_v(R_v(x,u)s,z) = g_v((\nabla_x T)^v_u s,z) - g_v((\nabla_u A)^v_x s,z) - g_v(A^v_{A^v_x u}s,z) - g_v(T^v_u x, T^v_s z) && \{2\}\\
&g_v(R_v(x,y)s,w) = g_v(R^\top_v(x,y)s,w) + g_v(A^v_y s, A^v_x w) - g_v(A^v_x s, A^v_y w) && \{2'\}\\
&g_v(R_v(x,y)s,z) = g_v((\nabla_x A)^v_y s,z) - g_v((\nabla_y A)^v_x sw,z) + g_v(A^v_y x, T^v_s z) - g_v(A^v_x y, T^v_s z) && \{3\}\\
&g_v(R_v(x,y)z,z') = g_v(R^\perp_v(x,y)z,z') + g_v(A^v_x z, A^v_y z') - g_v(A^v_y z, A^v_x z')\,. && \{4\}
\end{aligned}$$

*Proof.* Recall that by Proposition 3.2.7 $R^\perp_v(w,u)s = R^\perp_v(x,u)s = R^\perp_v(x,y)s = 0$. By using Lemma 3.1.28, for arbitrary vectors $e,h \in T_{\pi(v)}M$ and with the shorthand $\chi = T + A$

$$(\nabla_e\chi)^v_h s - (\nabla_h\chi)^v_e s + \chi^v_h\chi^v_e s - \chi^v_e\chi^v_h s = ((\nabla_e\chi)^v_h s - (\nabla_h\chi)^v_e s)^\perp_v + \chi^v_e\chi^v_h s - \chi^v_h\chi^v_e s\,. \quad (3.87)$$

All identities but the last can be obtained from (3.83) using the previous simplification, up to Lemma 3.1.29 and the properties of $A$ and $T$. In particular,

$$((\nabla_w A)^v_u s - (\nabla_u A)^v_w s)^\perp_v = -A^v_{T^v_w u}s + A^v_{T^v_u w}s = 0$$

from which we get {0} and {1},

$$((\nabla_x A)^v_u s - (\nabla_u T)^v_x s)^\perp_v = -A^v_{A^v_x u}s + T^v_s T^v_u x$$

from which {1'} and {2},

$$\left((\nabla_x T)^v_y s - (\nabla_y T)^v_x s\right)^\perp_v = -T^v_s A^v_x y + T^v_s A^v_y x$$

from which {2'} and {3}. In order to prove {4}, we may proceed analogously: indeed by Lemma 3.1.28

$$(\nabla_e\chi)^v_h z - (\nabla_h\chi)^v_e z + \chi^v_h\chi^v_e z - \chi^v_e\chi^v_h z = ((\nabla_e\chi)^v_h z - (\nabla_h\chi)^v_e z)^\top_v + \chi^v_e\chi^v_h z - \chi^v_h\chi^v_e z$$

and similarly using Lemma 3.1.29

$$\left((\nabla_x T)^v_y z - (\nabla_y T)^v_x z\right)^\top_v = -T^v_{A^v_x y}z + T^v_{A^v_y x}z\,.$$

□

**Theorem 3.2.12** (Generalised Gauss equation and dual Gauss equation)**.** *For $v \in \mathbb{A} \cap \ker d\sigma$ and $s,u,w \in \mathbb{T}_{\pi(v)}$,*

$$\begin{aligned}(R_v(w,u)s)_v^\top = \widehat{R}_v(w,u)s + T_w^v T_u^v s - T_u^v T_w^v s \\ + (\widehat{\nabla}_u \widehat{Q})_w^v s - (\widehat{\nabla}_w \widehat{Q})_u^v s + \widehat{Q}_u^v \widehat{Q}_w^v s - \widehat{Q}_w^v \widehat{Q}_u^v s \\ - P_v(w,s,\widehat{Q}_u^v v + T_u^v v)_v^\top + P_v(u,s,\widehat{Q}_w^v v + T_w^v v)_v^\top \\ - 2C_v^\sharp(T_w^v s, \widehat{Q}_u^v v + T_u^v v)_v^\top + 2C_v^\sharp(T_u^v s, \widehat{Q}_w^v v + T_w^v v)_v^\top . \quad \{0'\}\end{aligned}$$

*For $v \in \mathbb{A}$ and $x,y,z \in \mathbb{H}_v$,*

$$\begin{aligned}(R_v(x,y)z)_v^\perp = \widetilde{R}_v^*(\tilde{x},\tilde{y})\tilde{z} + A_x^v A_y^v z - A_y^v A_x^v z + A_z^v A_y^v x - A_z^v A_x^v y \\ + 2A_z^v C_v^\sharp(y, A_x^v v)_v^\top + 2A_y^v C_v^\sharp(z, A_x^v v)_v^\top - 2A_z^v C_v^\sharp(x, A_y^v v)_v^\top - 2A_x^v C_v^\sharp(z, A_y^v v)_v^\top \\ + ((\nabla_x \widetilde{Q})_y^v z - (\nabla_y \widetilde{Q})_x^v z)_v^\perp + \widetilde{Q}_y^v \widetilde{Q}_x^v z - \widetilde{Q}_x^v \widetilde{Q}_y^v z + ((\dot{\partial}\widetilde{Q})_y^v z(A_x^v v))_v^\perp - ((\dot{\partial}\widetilde{Q})_x^v z(A_y^v v))_v^\perp \\ - P_v(y,z,A_x^v v)_v^\perp + P_v(x,z,A_y^v v)_v^\perp . \quad \{4'\}\end{aligned}$$

*Proof.* From {0} and {4} together with Proposition 3.2.8 and equation (3.82). □

### 3.2.3 Flag curvature

Recall from Chapter 1 the definition (1.59) of flag curvature. Let us now introduce the vertical and horizontal sectional curvature analogously to Definition 3.2.2 of the vertical and horizontal curvature tensors $R^\top$ and $R^\perp$.

**Definition 3.2.13.** *Denote by $K_v^\top(e)$ the vertical flag curvature of $(M,L)$ defined for a flagpole $v \in \mathbb{A}$ and a flag held by $e \in T_{\pi(v)}M$ such that $L(v)g_v(e,e) \neq g_v(v,e)^2$ by*

$$K_v^\top(e) = \frac{g_v(R_v^\top(v,e)e,v)}{L(v)g_v(e,e) - g_v(v,e)^2},$$

*and by $K_v^\perp(e)$ the horizontal flag curvature defined by*

$$K_v^\perp(e) = \frac{g_v(R_v^\perp(v,e)e,v)}{L(v)g_v(e,e) - g_v(v,e)^2}.$$

**Corollary 3.2.14.** *For $v \in \mathbb{A}$, $w \in \mathbb{T}_{\pi(v)}$ and $x \in \mathbb{H}_v$,*

$$\begin{aligned}K_v(w) = K_v^\top(w) + \frac{g_v((\nabla_v T)_w^v w - (\nabla_w (T+A))_v^v w, v)}{L(v)g_v(w,w) - g_v(v,w)^2} \\ + \frac{-g_v(A_{(T+A)_v^v w}^v w, v) + g_v(T_w^v w, (T+A)_v^v v) - g_v((T+A)_v^v w, T_w^v v)}{L(v)g_v(w,w) - g_v(v,w)^2}\end{aligned}$$

*and*

$$\begin{aligned}K_v(x) = K_v^\perp(x) + \frac{g_v((\nabla_v A)_x^v x - (\nabla_x (T+A))_v^v x, v)}{L(v)g_v(x,x) - g_v(v,x)^2} \\ + \frac{-g_v(T_{(T+A)_v^v x}^v x, v) + g_v(A_x^v x, (T+A)_v^v v) - g_v((T+A)_v^v x, A_x^v v)}{L(v)g_v(x,x) - g_v(v,x)^2}.\end{aligned}$$

*Proof.* Recalling that $R^{\perp}_v(\cdot,\cdot)$ is zero when applied to vertical vectors, by Lemma 3.1.28

$$g_v(R_v(v,w)w,v) = g_v(R^{\top}_v(v,w)w + (\nabla_v(T+A))^v_w w - (\nabla_w(T+A))^v_v w, v) \\ + g_v(T^v_w w,(T+A)^v_v v) - g_v((T+A)^v_v w, T^v_w v)\,,$$

and similarly

$$g_v(R_v(v,x)x,v) = g_v(R^{\top}_v + R^{\perp}_v)(v,x)x + (\nabla_v(T+A))^v_x x - (\nabla_x(T+A))^v_v x, v) \\ + g_v(A^v_x x,(T+A)^v_v v) - g_v((T+A)^v_v x, A^v_x v)\,.$$

Conclude using by Lemma 3.1.29 $(\nabla_v A)^v_w = -A^v_{(T+A)^v_v w}$ and $(\nabla_v T)v_x = -T^v_{(T+A)^v_v x}$. □

**Corollary 3.2.15.** *For $v \in \mathbb{A} \cap \ker \mathrm{d}\sigma$ and $w \in \mathbb{T}_{\pi(v)}$ such that $L(v)g_v(w,w) \neq g_v(v,w)^2$*

$$K_v(w) = \widehat{K}_v(w) - \frac{g_v(T^v_w w, T^v_v v) - g_v(T^v_v w, T^v_v w)}{L(v)g_v(w,w) - g_v(v,w)^2} \\ - \frac{g_v(P_v(w,w,T^v_v v) - (\widehat{\nabla}_v \widehat{Q})^v_w w, v) + C_v(w, \widehat{Q}^v_v u, T^v_v v)}{L(v)g_v(w,w) - g_v(v,w)^2}\,,$$

*where $\widehat{K}$ denotes the flag curvature intrinsic to the submersion fibres of $\sigma$.*

*Proof.* The proof is completely analogous to Corollary 2.2.5. □

**Corollary 3.2.16.** *For $v \in \mathbb{A}$, $w \in \mathbb{T}_{\pi(v)}$ and $x \in \mathbb{H}_v$ such that $L(v)g_v(w,w) \neq g_v(v,w)$ and $L(v)g_v(x,x) \neq g_v(v,x)$,*

$$K_v(w) = \frac{g_v((\nabla_v T)^v_w w, v) + g_v(A^v_v w, A^v_v w) - g_v(T^v_w v, T^v_w v)}{L(v)g_v(w,w) - g_v(v,w)^2}$$

*and*

$$K_v(x) = \widetilde{K}^*_v(\tilde{x}) - \frac{3g_v(A^v_x v, A^v_x v)}{L(v)g_v(x,x) - g_v(v,x)^2}\,,$$

*where $\widetilde{K}$ denotes the flag curvature of the base manifold of $\sigma$.*

*Proof.* The first identity follows from {2} using

$$g_v(A^v_{A^v_v w} w, v) = -g_v(w, A^v_{A^v_v w} v) = g_v(w, A^v_v A^v_v w) = -g_v(A^v_v w, A^v_v w)$$

by properties of $A$, as well as

$$g_v((\nabla_w A)^v_v w, w) = 0$$

which we can obtain as follows: for any extension $V$ of $v$ satisfying $\nabla^v_v V = 0$ and vertical extension $W$ of $w$,

$$g_v((\nabla_w A)^v_v w, w) = g_v(\nabla^v_w(A^V_V W), v) - g_v(A^v_{\nabla^v_w V} w, v) - g_v(A^v_v(\nabla^v_w W), v)\,.$$

By properties of $A$, the last term is

$$g_v(A^v_v(\nabla^v_w W), v) = -g_v(A^v_v v, \nabla^v_w W) = 0$$

while the first term is

$$g_v(\nabla^v_w(A^V_V W), v) = w(g_V(A^V_V W, V)) - g_v(A^v_v w, \nabla^v_w V) = -g_v(A^v_v w, A^v_v w)$$

because $g_V(A^V_V W, V) = -g_V(W, A^V_V V) = 0$ and $(\nabla^v_w V)^\perp_v = A^v_v w$ for our choice of extension. The middle term can be manipulated into

$$g_v(A^v_{\nabla^v_w V} w, v) = -g_v(A^v_{\nabla^v_w V} v, w) = g_v(A^v_v(\nabla^v_w V), w) = -g_v(A^v_v w, \nabla^v_w V) = g_v(A^v_v w, A^v_v w)\,.$$

These three terms indeed cancel to $g_v((\nabla_w A)^v_v w, w) = 0$.

From {4'}, by property of the Cartan tensor, by the identity $\widetilde{Q}^v_v = \widetilde{Q}^v v = 0$ for $v \in \mathbb{A}$ and since $g_v(A^v_v \cdot, v) = -g_v(\cdot, A^v_v v) = 0$,

$$\begin{aligned} g_v(R_v(v,x)x, v) = g_v(\widetilde{R}^*_{\tilde{v}}(\tilde{v}, \tilde{x})\tilde{x} + 3A^v_x A^v_x v + (\nabla_v \widetilde{Q})^v_x x - (\nabla_x \widetilde{Q})^v_v x, v) \\ - g_v((\dot{\partial}\widetilde{Q})^v_v x(A^v_x v) + P_v(v, x, A^v_x v), v)\,. \end{aligned} \quad (3.88)$$

By definition of $\nabla\widetilde{Q}$, for a locally horizontal extension $V$ of $v$ and a locally $g_V$-horizontal extension $X$ of $x$ satisfying $\nabla^v_x V = A^v_x v$ and $\nabla^v_v V = 0$

$$(\nabla_v \widetilde{Q})^v_x x = \nabla^v_v \widetilde{Q}^V_X X - \widetilde{Q}^v_{\nabla^v_v X} x - \widetilde{Q}^v_x \nabla^v_v X$$

such that, using $g_v(\widetilde{Q}^v, v) = 0$ for $v \in \mathbb{A}$ by definition of $\widetilde{Q}$ and property of the Cartan tensor, and in particular $g_V(\widetilde{Q}^V_X X, V) = 0$,

$$g_v((\nabla_v \widetilde{Q}^v_x x, v)) = v(g_V(\widetilde{Q}^V_X X, V)) - g_v(\widetilde{Q}^v_x x, \nabla^v_v V) - 2C_v(\widetilde{Q}^v_x x, v, \nabla^v_v V) = 0$$

and furthermore since $\widetilde{Q}^V_V = 0$

$$(\nabla_x \widetilde{Q})^v_v x = \nabla^v_x \widetilde{Q}^V_V X - \widetilde{Q}_{A^v_x v} x - \widetilde{Q}^v_v \nabla^v_x X - (\dot{\partial}\widetilde{Q})^v_v x(A^v_x v) = -(\dot{\partial}\widetilde{Q})^v_v x(A^v_x v)\,.$$

Finally, by (1.44) (see equation (56) of [50])

$$g_v(P_v(v, x, A^v_x v), v) = 0\,.$$

This means that only the first two terms are non-zero in (3.88), concluding the proof up to skew-symmetry of $A$. □

The expressions for $K_v(w)$ and $K_v(x)$ in the last Corollary were previously obtained as Theorem 5.1 of [32] and Theorem 5.12 of [47]. For the first identity, beware that in [32] the definition of $\nabla T$ is different from the one provided here.

In the positive definite case, $g_v(A^v_x v, A^v_x)$ is non-negative, such that Finsler submersions never increase the flag curvature along horizontal flags.

**Theorem 3.2.17.** *Let $\sigma$ be a a Finsler submersion, and denote by $K$ and $\tilde{K}$ the respective flag curvatures of the ambient and base Finsler manifolds of $\sigma$. Then for each horizontal $v$ and $g_v$-horizontal $x$,*

$$K_v(x) \leq \tilde{K}_{\tilde{v}}(\tilde{x}) \quad (3.89)$$

*where $\tilde{v}$ and $\tilde{x}$ denote the images of $v$ and $x$ by* $\mathrm{d}\sigma$.

This fact has been previously derived by a different method as Theorem 6.1 of [23].

## 3.3 Application to submersions whose fibres are totally geodesic

This final section presents results my supervisor and I have encountered and expect to publish promptly (see [52] for a preprint version).

A pseudo-Finsler submanifold is said to be *totally geodesic* when geodesics intrinsic to the submanifold are also geodesic of the ambient manifold. A pseudo-Finsler submersion $\sigma\colon M \to \widetilde{M}$ from $(M,L)$ to $(\widetilde{M},\tilde{L})$ is then said to be totally geodesic when its fibres are totally geodesics as submanifolds of $(M,L)$. Classically, a Riemannian submersion has totally geodesic fibres if and only if the O'Neill tensor $T$ vanishes, and equivalently the inclusion map into $(M,L)$ is isometric on each fibre. In the anisotropic case, a pseudo-Finsler submersion has totally geodesic fibres if and only if $T^v_v v = 0$ for each vertical admissible $v$, or equivalently $T^v_v w$ for each vertical admissible $v$ and vertical $w$ at $\pi(v)$ (see Proposition 4 of [51]).

**Proposition 3.3.1.** *The composition of pseudo-Finsler submersions is a pseudo-Finsler submersion.*

*Proof.* Consider three pseudo-Finsler manifolds $(M_1,L_1)$, $(M_2,L_2)$ and $(M_3,L_3)$ and pseudo-Finsler submersions $\sigma_1\colon M_1 \to M_2$ and $\sigma_2\colon M_2 \to M_3$ between them. Denote by $g^1$ and $g^2$ the anisotropic metric tensors associated with the pseudo-Finsler metrics $L_1$ and $L_2$. By hypothesis, every $\sigma_1$-horizontal $v_2$ satisfies

$$L_1(v_1) = L_2(\mathrm{d}\sigma_1 \cdot v_1)\,, \tag{3.90}$$

and every $\sigma_2$-horizontal $v_2$ satisfies

$$L_2(v_2) = L_3(\mathrm{d}\sigma_2 \cdot v_2)\,.$$

In particular,

$$L_1(v_1) = L_3(\mathrm{d}(\sigma_2 \circ \sigma_1) \cdot v_1) \tag{3.91}$$

provided $\mathrm{d}\sigma_1 \cdot v_1$ is $\sigma_2$-horizontal in addition to $v_1$ being $\sigma_1$-horizontal.

By definition, a vector $v$ is $\sigma_1$-horizontal (respectively $\sigma_2$-horizontal) if and only if $g^1_v(v,\cdot)$ (respectively $g^2_v(v,\cdot)$) vanishes on the $\sigma_1$-vertical subspace (respectively the $\sigma_2$-vertical subspace). However, since $\ker \mathrm{d}\sigma_1 \subset \ker \mathrm{d}(\sigma_2 \circ \sigma_1)$, the $\sigma_2 \circ \sigma_1$-vertical subspace is contained in the $\sigma_1$-vertical subspace. Therefore every $\sigma_2 \circ \sigma_1$-horizontal vector $v$ is a $\sigma_1$-horizontal vector and satisfies (3.90). By (3.7), such a $v$ also satisfies for every $e \in T_{\pi(v)}M_1$

$$g^1_v(v,e) = g^2_{\mathrm{d}\sigma_1 \cdot v}(\mathrm{d}\sigma_1 \cdot v, \mathrm{d}\sigma_1 \cdot e)\,.$$

By surjectivity, every $\sigma_2$-vertical vector can be written as $\mathrm{d}\sigma_1 \cdot e$ for some $e \in T_{\pi(v)}M_1$, and then necessarily $\mathrm{d}(\sigma_2 \circ \sigma_1) \cdot e = 0$ such that $e$ would be $\sigma_2 \circ \sigma_1$-vertical and therefore also $\sigma_1$-vertical. This would mean that $g^1_v(v,e)$, and therefore $g^2_{\mathrm{d}\sigma_1 \cdot v}(\mathrm{d}\sigma_1 \cdot v, \mathrm{d}\sigma_1 \cdot e)$, are zero; in other words $\mathrm{d}\sigma_1 \cdot v$ is $\sigma_2$-horizontal. Consequently, $v$ also satisfies (3.91). □

**Lemma 3.3.2.** *If the composition of pseudo-Finsler submersions has totally geodesic fibres, then so does the last submersion of the composition.*

*Proof.* Consider as before three pseudo-Finsler manifolds $(M_1,L_1)$, $(M_2,L_2)$ and $(M_3,L_3)$ and pseudo-Finsler submersions $\sigma_1\colon M_1 \to M_2$ and $\sigma_2\colon M_2 \to M_3$ between them. Let $v_2$ be an arbitrary $\sigma_2$-vertical vector with some $\sigma_2$-vertical extension $V_2$ and denote by $v_1$ and $V_1$ their $\sigma_1$-horizontal lift. In particular, $v_1$ and $V_1$ are $\sigma_2 \circ \sigma_1$-vertical.

Let $\nabla_{v_1}$ denote the covariant derivative along $v_1$ using the Chern connection of $(M_1,L_1)$ and $\nabla_{v_2}$ the covariant derivative along $v_1$ using that of $(M_2,L_2)$. By the dual Gauss formula for

$\sigma_1$, (recalling that $\widetilde{Q}^v_v v = 0$ for $v$ horizontal)

$$\mathrm{d}\sigma_1 \cdot \nabla^{v_1}_{v_1} V_1 = \nabla^{v_2}_{v_2} V_2\,,$$

while by the Gauss formula for $\sigma_2 \circ \sigma_1$ (noting that, by our hypothesis that $\sigma_2 \circ \sigma_1$ has totally geodesic fibres, the O'Neill tensor $T$ of $\sigma_2 \circ \sigma_1$ vanishes when applied to $v_1$ since it is $\sigma_2 \circ \sigma_1$-vertical)

$$\mathrm{d}(\sigma_2 \circ \sigma_1) \cdot \nabla^{v_1}_{v_1} V_1 = 0\,.$$

Combining those two identities yields by the chain rule for differentials

$$\mathrm{d}\sigma_2 \cdot \nabla^{v_2}_{v_2} V_2 = \mathrm{d}\sigma_2 \cdot \mathrm{d}\sigma_1 \cdot \nabla^{v_1}_{v_1} V_1 = \mathrm{d}(\sigma_2 \circ \sigma_1) \cdot \nabla^{v_1}_{v_1} V_1 = 0\,,$$

such that $\nabla^{v_2}_{v_2} V_2$ is $\sigma_2$-vertical, such that the $T$ tensor of $\sigma_2$ satisfies

$$T^{v_2}_{v_2} v_2 = (\nabla^{v_2}_{v_2} V_2)^{\perp}_{v_2} = 0\,,$$

for our arbitrary choice of a $\sigma_2$-vertical vector $v_2$. That is to say $\sigma_2$ has totally geodesic fibres. □

Let us return to our earlier notation of a pseudo-Finsler submersion $\sigma\colon M \to \widetilde{M}$ between pseudo-Finsler manifolds $(M, L)$ and $(\widetilde{M}, \tilde{L})$. Assume both pseudo-Finsler metrics $L$ and $\tilde{L}$ are defined on the slit tangent bundle and that the ambient manifold $M$ has dimension $n \geq 3$. Under these conditions, the Legendre map $v \mapsto g_v(v, \cdot)$ is bijective (see [44] and [45]).

For each piecewise smooth path $\tilde{\gamma}$ of the base manifold joining end points $\tilde{p}$ and $\tilde{q}$, let $F_{\tilde{\gamma}}$ be the map between the fibres $\widehat{M} = \sigma^{-1}(\{\tilde{p}\})$ and $\sigma^{-1}(\{\tilde{q}\})$ defined as follows: for each $p \in \widehat{M}$, let $\gamma$ be the horizontal lift of $\tilde{\gamma}$ through $p$ guaranteed to exist uniquely due to Lemma 3.1.7; $F_{\tilde{\gamma}}(p)$ is the endpoint of $\gamma$ (opposite to the initial endpoint $p$).

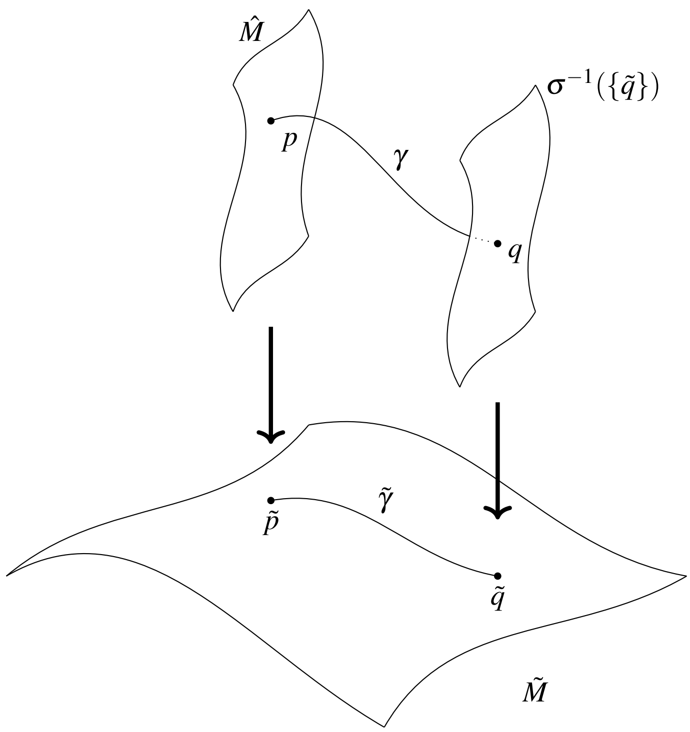


FIGURE 3.7: Visual representation of $F_{\tilde{\gamma}}\colon p \mapsto q$.

Since horizontal lifts are determined by ordinary differential equations (whose solutions necessarily depend smoothly on the initial values), the map $F_{\tilde{\gamma}}$ is smooth. Furthermore, if $\tilde{\gamma}^{-1}$ is the reverse parametrisation of $\tilde{\gamma}$ from $\tilde{q}$ to $\tilde{p}$, then we may recognise $F_{\tilde{\gamma}^{-1}}$ and the functional inverse of $F_{\tilde{\gamma}}$, and it is also smooth by the previous argument. Thus $F_{\tilde{\gamma}}$ is a diffeomorphism.

Due to the special relationship of Lemma 3.1.31 between horizontal geodesics of the ambient manifold and geodesics of the base manifold, we may prove under the right conditions that $F_{\tilde{\gamma}}$ is an isometry.

**Definition 3.3.3.** *A pseudo-Finsler submersion is said to be* horizontally regular *when for each vertical admissible vector $v$ and projectable horizontal vector field $X$*

$$(\nabla_v^v X^\top)_v^\top = 0\,.$$

**Proposition 3.3.4.** *For horizontally regular Finsler submersions between complete Finsler manifolds, $F_{\tilde{\gamma}}$ is an isometry for each $\tilde{\gamma}$ if and only if the submersion fibres are totally geodesic.*

*Proof.* Assume first that the fibres are totally geodesic. To show that $F_{\tilde{\gamma}}$ is an isometry, parametrise $\tilde{\gamma}$ on $[0,1]$ and consider a smooth admissible curve $\alpha_0$ of the initial fibre $\widehat{M}_0 = \sigma^{-1}(\{\tilde{\gamma}(0)\})$ (whose tangent vector $\alpha_0(t)$ is admissible for each $t \in [0,1]$) and define

$$\lambda: \begin{array}{ccc} [0,1]\times[0,1] & \to & M \\ (s,t) & \mapsto & \lambda(s,t) \end{array}$$

such that $s \mapsto \lambda(s,t)$ is the horizontal lift of $\tilde{\gamma}$ through $\alpha_0(t)$. Denote by $\alpha_s$ the vertical curve $t \mapsto \lambda(s,t)$ in the fibre $\widehat{M}_s = \sigma^{-1}(\{\tilde{\gamma}(s)\})$. Denote by $\beta_t$ the horizontal curve $s \mapsto \lambda(s,t)$ through $\alpha_0(t)$. Let $X_s = \dot{\beta}_s$ to emphasise the fact that it consists, at each $t \in [0,1]$, of a horizontal vector. Recalling the commutativity of covariant derivation of two-parameter maps (see Proposition 3.2 of [37]), we have

$$(\nabla_{\beta_t}^{\dot{\alpha}_s}\dot{\alpha}_s)_{\dot{\alpha}_s}^\top = (\nabla_{\beta_t}^{\dot{\alpha}_s}X_s)_{\dot{\alpha}_s}^\top = (\nabla_{\beta_t}^{\dot{\alpha}_s}(X_s)^\top)_{\dot{\alpha}_s}^\top + T_{\dot{\alpha}_s}^{\dot{\alpha}_s}((X_s)_{\dot{\alpha}_s}^\perp) \tag{3.92}$$

whose first term is zero by horizontal regularity of the submersion and whose second term is zero by total geodesicity of its fibres. (Note that we have denoted by $D_{\beta_s}$ the formal covariant derivation along the curve $\beta_s$, which can be defined explicitly as explained in (1.47).) Therefore

$$g_{\dot{\alpha}_s}(D_{\beta_t}^{\dot{\alpha}_s}\dot{\alpha}_s, \dot{\alpha}_s) = 0\,.$$

We may now check that $\alpha_1$ has the same length as $\alpha_0$ and prove that $F_{\tilde{\gamma}}$ is indeed an isometry: assuming that $g_{\dot{\alpha}_s}(\dot{\alpha}_s, \dot{\alpha}_s)$ is non-negative, and by the Leibniz integral rule, the variation

$$\frac{\partial}{\partial s}\int_0^1 \sqrt{g_{\dot{\alpha}_s}(\dot{\alpha}_s,\dot{\alpha}_s)}\mathrm{d}t = \int_0^1 \frac{\partial}{\partial s}\sqrt{g_{\dot{\alpha}_s}(\dot{\alpha}_s,\dot{\alpha}_s)}\mathrm{d}t = \int_0^1 \frac{\frac{\partial}{\partial s}g_{\dot{\alpha}_s}(\dot{\alpha}_s,\dot{\alpha}_s)}{\sqrt{g_{\dot{\alpha}_s}(\dot{\alpha}_s,\dot{\alpha}_s)}}\mathrm{d}t$$

of the length of $\alpha_s$ is zero because of the vanishing integrand's numerator

$$\frac{\partial}{\partial s}g_{\dot{\alpha}_s}(\dot{\alpha}_s,\dot{\alpha}_s) = 2g_{\dot{\alpha}_s}(D_{\beta_t}^{\dot{\alpha}_s}\dot{\alpha}_s,\dot{\alpha}_s) + 2C_{\dot{\alpha}_s}(\dot{\alpha}_s,\dot{\alpha}_s,D_{\beta_t}^{\dot{\alpha}_s}\dot{\alpha}_s) = 0$$

since we have shown that $D_{\beta_t}^{\dot{\alpha}_s}\dot{\alpha}_s$ is $g_{\dot{\alpha}_s}$-horizontal and by property of the Cartan tensor. Consequently, the image by $F_{\tilde{\gamma}}$ of any vertical geodesic between two fixed points has the same length. Some geodesic attains the minimal length in $\widehat{M}_0$ by completeness, its image must have the same length in $\widehat{M}_1$, such that the distance between the end points of $\alpha_1$ must be at most the distance between those of $\alpha_0$. The same argument applied to $F_{\tilde{\gamma}^{-1}}$ shows that the distance between the end points of $\alpha_0$ is at most the distance between those of $\alpha_1$, and must therefore be equal.

Conversely, assume that the maps $F_{\tilde{\gamma}}$ are always isometries. By the identity (3.92) computed above and skew symmetry of the O'Neill tensor $T$

$$g_{\dot{\alpha}_s}(D^{\dot{\alpha}_s}_{\beta_t}\dot{\alpha}_s, \dot{\alpha}_s) = g_{\dot{\alpha}_s}(T^{\dot{\alpha}_s}_{\dot{\alpha}_s} X_s, \dot{\alpha}_s) = -g_{\dot{\alpha}_s}(X_s, T^{\dot{\alpha}_s}_{\dot{\alpha}_s}\dot{\alpha}_s)\,,$$

we have that if

$$\int_0^1 \frac{\frac{\partial}{\partial s} g_{\dot{\alpha}_s}(\dot{\alpha}_s, \dot{\alpha}_s)}{\sqrt{g_{\dot{\alpha}_s}(\dot{\alpha}_s, \dot{\alpha}_s)}} \mathrm{d}t = \frac{\partial}{\partial s}\int_0^1 \sqrt{g_{\dot{\alpha}_s}(\dot{\alpha}_s, \dot{\alpha}_s)}\mathrm{d}t = 0$$

then by the mean value theorem for some $t_0 \in [0,1]$

$$g_{\dot{\alpha}_s}(X_s, T^{\dot{\alpha}_s}_{\dot{\alpha}_s}\dot{\alpha}_s)|_{t_0} = \frac{\partial}{\partial s} g_{\dot{\alpha}_s}(\dot{\alpha}_s, \dot{\alpha}_s)|_{t_0} = 0\,.$$

By the same argument, we obtain a sequence of values $t_n \in [0, \frac{1}{2^n}]$ converging to 0 for which

$$g_{\dot{\alpha}_s}(X_s, T^{\dot{\alpha}_s}_{\dot{\alpha}_s}\dot{\alpha}_s)|_{t_n} = 0$$

and therefore by continuity

$$g_v(x, T^v_v v) = 0$$

for the arbitrary (vertical) initial vector $v = \alpha_s(0)$ and the arbitrary $g_v$-horizontal vector $x = X_s(0)$. Conclude by non-degeneracy. □

**Corollary 3.3.5.** *For a horizontally regular Finser submersion with totally geodesic fibres, the flow of a projectable horizontal geodesic vector field gives rise to an isometry between the fibres.*

*Proof.* Consider on some neighbourhood a projectable vector field satisfying the geodesic equation. Provided it is horizontal, each of the geodesic arcs described by its flow is the horizontal lift of some geodesic arc $\tilde{\gamma}$ described by the flow of its projection, which also satisfies the geodesic equation by the dual Gauss formula; by construction $F_{\tilde{\gamma}}$ is that horizontal flow, which we have just whosn to be an isometry between the fibres in the previous proposition. □

**Theorem 3.3.6.** *A Finsler submersion whose ambient manifold is connected and geodesically complete has a geodesically complete base. If it is horizontally regular and has totally geodesic fibres, then it is the projection of a bundle associated with a principal fibre bundle whose structure group is the Lie group of isometries of the fibre.*

*Proof.* First recall that the isometries of a Finsler manifold form a Lie group (see [25] for further details).

For completeness, observe that each geodesic arc is extended by the projection of the extension of its horizontal lift.

For the second part of the theorem, fix a point $\tilde{p}$ in the base manifold $\widetilde{M}$ denote by $\mathbf{F}$ the submersion fibre $\sigma^{-1}(\{\tilde{p}\})$ and by $\mathbf{G}$ the Lie group of isometries of $\mathbf{F}$. Let $\mathbf{E}_{\tilde{q}}$ be the set of isometries from $\mathbf{F}$ to the submersion fibre $\sigma^{-1}(\{\tilde{q}\})$ over $\tilde{q}$, and denote by $\mathbf{E}$ their union.

Note how $\mathbf{G}$ acts diffeomorphically and freely on $\mathbf{E}$ as

$$\begin{array}{rcl} \mathbf{G}\times\mathbf{E} & \to & \mathbf{E} \\ (\mathbf{g},\mathbf{e}) & \mapsto & \mathbf{e}\circ\mathbf{g}. \end{array}$$

Let us show that $\mathbf{E}$ is a funbre bundle over the base $\mathbf{B}=\widetilde{M}$ for the map $\mu\colon \mathbf{E}\to\mathbf{B}$ sending each $\mathbf{E}_{\tilde{q}}$ onto $\tilde{q}$, with structure group $\mathbf{G}$, equipped with the differentiable structure obtained as follows: for the choice of a point $\tilde{q}_i$ in each set of some covering of $\mathbf{B}$ by geodesically convex sets $\mathbf{U}_i$, and the choice of a (piecewise) smooth path $\tilde{\gamma}_i$ joining $\tilde{p}$ to $\tilde{q}_i$, there exists, at each point $\tilde{q}$ and for each $\mathbf{U}_i$ that contains it, a unique geodesic arc $\tilde{\gamma}_{i,\tilde{q}}$ from $\tilde{q}_i$ to $\tilde{q}$. By the previous proposition, both $\tilde{\gamma}_i$ and $\tilde{\gamma}_{i,\tilde{q}}$ induce an isometry between the corresponding submersion fibres, respectively $F_{\tilde{\gamma}_i}$ and $F_{\tilde{\gamma}_{i,\tilde{q}}}$. Let us define a local section $\mathbf{e}_i$ on each $\mathbf{U}_i\ni\tilde{q}$ as the unique isometry

$$\mathbf{e}_i(\tilde{q}) = F_{\tilde{\gamma}i,\tilde{q}}\circ F_{\tilde{\gamma}_i}\in\mathbf{E}_{\tilde{q}}\,.$$

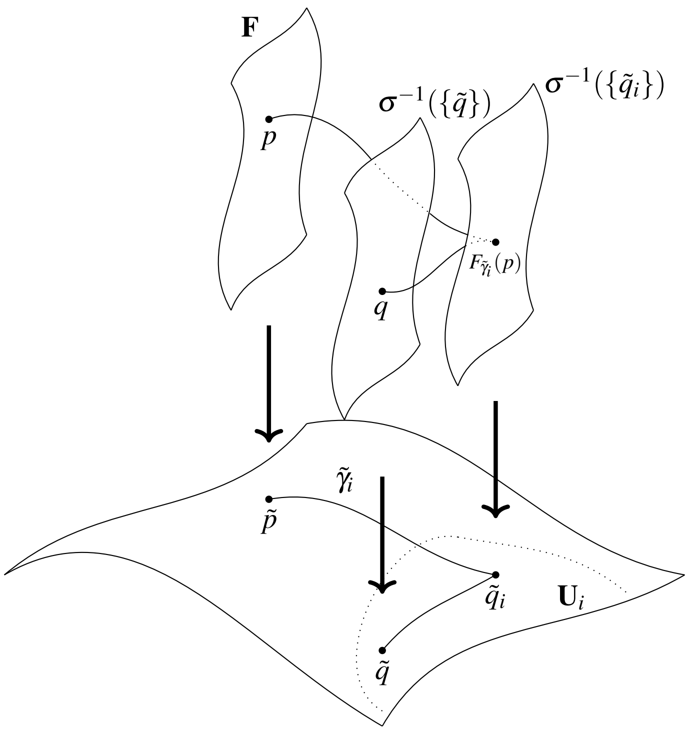


FIGURE 3.8: Visual representation of $\mathbf{e}_i(\tilde{q})\colon p\mapsto q$.

When $\mathbf{U}_i\cap\mathbf{U}_j\ni\tilde{q}$, we can define the transition functions

$$\mathbf{g}_{ij}(\tilde{q}) = \mathbf{e}_j^{-1}(\tilde{q})\circ\mathbf{e}_i(\tilde{q})\in\mathbf{G}$$

satisfying

$$\mathbf{e}_i(\tilde{q}) = \mathbf{e}_j(\tilde{q})\circ\mathbf{g}_{ij}(\tilde{q})$$

and the cocycle condition

$$\mathbf{g}_{ik}(\tilde{q}) = \mathbf{g}_{jk}(\tilde{q})\circ\mathbf{g}_{ij}(\tilde{q})$$

with respect to the group action. By the fibre bundle construction theorem, $\mathbf{E}$ is a fibre bundle over $\mathbf{B}$ for the map $\mu$. Denote by $\mathbf{E}'$ the quotient of $\mathbf{E}\times\mathbf{F}$ by the diffeomorphic action

$$\begin{array}{rcl} \mathbf{G}\times\mathbf{E}\times\mathbf{F}\times\mathbf{E} & \to & \mathbf{E}\times\mathbf{F} \\ (\mathbf{g},(\mathbf{e},p)) & \mapsto & \mathbf{e}\circ(\mathbf{e}\circ\mathbf{g},\mathbf{g}^{-1}(p))\,. \end{array}$$

By definition of the action, the map

$$\begin{array}{ccc} \mathbf{E}\times\mathbf{F}\times\mathbf{E} & \to & \mathbf{B} \\ (\mathbf{e},p) & \mapsto & \mu(\mathbf{e}) \,. \end{array}$$

is identically equal to $\tilde{q}$ on the whole equivalence class of $(\mathbf{e},p)$ in $\mathbf{E}'$, and induces a well-defined projection $\mu'\colon \mathbf{E}'\to\mathbf{B}$, defining a fibre bundle associated with the principal fibre bundle $\mu\colon \mathbf{E}\to\mathbf{B}$.

Let us now show that the submersion coincides with this associated fibre bundle: indeed, the action on $\mathbf{E}\times\mathbf{F}$ is free, by freedom of the action of $\mathbf{G}$ on $\mathbf{E}$, and proper, because the action of $\mathbf{G}$ on $\mathbf{F}$ is proper (recall that the isometries of a Finsler manifold is a closed subgroup of the isometries of the Riemannian metric obtained by aferaging, see [28]). To conclude, the evaluation map

$$\phi\colon \begin{array}{ccc} \mathbf{E}\times\mathbf{F} & \to & M \\ (\mathbf{e},p) & \mapsto & \mathbf{e}(p) \end{array}$$

induces a map $\phi'$ on the quotient $\mathbf{E}'$ which is a diffeomorphism and maps fibres of $\mu'$ to fibres of the submersion $\sigma$. To check that $\phi'$ is surjective, observe that for $\mathbf{e}\in\mathbf{E}$ fixed $\phi'(\mathbf{e},\cdot)$ maps $\mathbf{F}$ to a fibre of $\sigma$ diffeomorphically. Moreover, if $\phi(\mathbf{e},p)=\phi(\mathbf{f},q)$, then $\mathbf{e}^{-1}\circ\mathbf{f}\in\mathbf{G}$ such that $(\mathbf{e},p)$ and $(\mathbf{f},q)$ belong to the same equivalence class in $\mathbf{E}'$. □

# Bibliography

[1] Bernhard Riemann, *Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse*, E. A. Huth, Göttingen, (1851) p. 26 "These conditions which form a continuous *manifold* [...]" ("Diese Bedingungen, deren Gesammtheit eine stetige *Mannigfaltigkeit* bildet [...]")

DOI: 10.3931/e-rara-49487

[2] Bernhard Riemann, *Über die Hypothesen, welche der Geometrie zu Grunde liegen*, H. Weyl, Springer-Verlag Berlin Heidelberg (1921).

DOI: 10.1007/978-3-662-24861-4

[3] Albert Einstein, *Zur Elektrodynamik bewegter Körper*, Annalen der Physik 17 (1905) p. 891-921.

DOI: 10.1002/andp.19053221004

[4] Albert Einstein and Marcel Grossmann, *Entwurf einer verallgemeinerten Relativitätstheorie und einer Theorie der Gravitation*, Monatshefte für Mathematik und Physik, Band 26, Wien, (1915) S. A38.

DOI: 10.1007/BF01999515

[5] Paul Finsler, *Über Kurven und Flächen in Allgemeinen Räumen*, unaltered reprint of the 1918 PhD Thesis at the University of Göttingen (1952).

DOI: 10.1002/zamm.19520321012

[6] Elie Cartan, *La méthode du repère mobile, la théorie des groupes continus, et les espaces généralisés*, Actualités scientifiques et industrielles 194 (1935).

ARK: 12148/bpt6k38182z

[7] Manuel Haimovici, *Les formules fondamentales dans la theorie des hypersurfaces d'un espace général*, Annales scientifiques de l'Université de Iaçi 20 (1935) p. 39–58.

[8] Johannes Wegener, *Hyperflächen in Finslerschen Räumen als Transversalflächen einer Schar von Extremalen*, Monatshefte für Mathematik und Physik 44, (1936) p. 115–130.

[9] Gunnar Randers, *On an Asymmetrical Metric in the Four-Space of General Relativity*, Physics Review 59 (1941).

DOI: 10.1103/PhysRev.59.195

[10] Shiing-Shen Chern, *On the Euclidean connections in a Finsler space*, Proceedings of the National Academy of Sciences of the United States of America 29 (1943), p. 33–37.

[11] Shiing-Shen Chern, *Local Equivalence and Euclidean Connections in Finsler Spaces*, Science Reports of the National Tsing Hua University Series A 5 (1948), p. 95–121.

ISBN: 978-1-4612-8151-1

[12] Hanno Rund, *The Differential Geometry of Finsler Spaces*, Grundlehren der mathematischen Wissenschaften, Band 101 (1959) p. 201-203.

ISBN: 978-3-642-51610-8

[13] Banesh Hoffman and Helen Dukas, *Albert Einstein: Creator and Rebel*, Hart-Davis, MacGibbon, 1st edition, (1973) p. 119 *"Because of the metrical tensor's newly acquired gravitational role, Einstein and Grossmann denoted it by the letter g [. . . ]"*.

ISBN: 978-0-246-10606-3

[14] Barrett O'Neill, *The Fundamental Equations of a Submersion*, Michigan Mathematical Journal 13, (1966) p. 459-469.

DOI: 10.1307/mmj/1028999604

[15] Manula Verma, *Generalised Gauss-Codazzi equations for Berwald's curvature tensors in subspaces of a Finsler space*, Journal of Mathematics Physics and Astronomy Volume 35, Istanbul University (1972).

[16] Barrett O'Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, (1983).

ISBN: 978-0-12-526740-3

[17] Peter Antonelli, Roman Ingarden and Makoto Matsumoto, *The Theory of Sprays and Finsler Spaces with Applications in Physics and Biology*, Fundamental Theories of Physics 58, Springer (1993).

DOI: 10.1007/978-94-015-8194-3

[18] Shiing-Shen Chern, *On Finsler Geometry*, Comptes Rendus de l'Académie des Sciences Série I Mathématiques 314, Gauthier-Villars (1992), p. 757–761.

ARK: 12148/bpt6k58688425

[19] Shiing-Shen Chern, Weihuan Chen and Kai Shue Lam, *Lectures on Differential Geometry*, Series on University Mathematics 1, World Scientific, (1999).

ISBN: 978-981-02-4182-7

[20] Zhongmin Shen, *Differential Geometry of Spray and Finsler Spaces*, Springer, (1999).

ISBN: 978-0-7923-6868-7

[21] David Bao, Shiing-Shen Chern and Zhongmin Shen, *An Introduction to Riemann-Finsler Geometry*, Graduate Texts in Mathematics 200, Springer, (2000).

ISBN: 978-1-4612-1268-3

[22] John Marshall Lee, *Introduction to smooth manifolds*, Springer (2000), p. 153-157.

ISBN: 978-0-3879-5448-6

[23] Juan Carlos Alvarez and Carlos Duran, *Isometric submersions of Finsler manifolds*, Proceedings of the American Mathematical Society 129 (2001), p. 2409-2417.

DOI: 10.1090/S0002-9939-01-05910-X

[24] Zhongmin Shen, *Lectures on Finsler Geometry*, Series on Multivariate Analysis, World Scientific, (2001).

ISBN: 978-981-02-4531-3

[25] Shaoqiang Deng and Zixin Hou, *The group of isometries of a Finsler space*, Pacific Journal of Mathematics Vol. 207 No. 1 (2002), p. 149-155.

DOI:10.2140/pjm.2002.207.149

[26] Xinyue Cheng, Xiaohuan Mo and Zhongmin Shen, *On the flag curvature of Finsler metrics of scalar curvature* Journal of the London Mathematical Society 68 3 (2003).

DOI: 10/1112S0024610703004599

[27] David Bao, Colleen Robles and Zhongmin Shen, *Zermelo navigation on Riemannian manifolds*, Journal on Differential Geometry 66 (2004), p. 377-435.

DOI: 10.4310/jdg/1098137838

[28] Ricardo Gallego Torromé, *Average structures assoiated with a Finsler space* (2005).

arXiv: math/0501058 [math.DG]

[29] Gary William Gibbons, Joaquim Gomis and Christopher Neville Pope, *General very special relativity is Finsler geometry*, Physical Review D 76, 081701 (2007).

DOI: 10.1103/PhysRevD.76.081701

[30] Xinyue Cheng and Zhongmin Shen, *Classification of Randers metrics of scalar flag curvature*, Acta Mathematica Academiae Paedagogicae Nyiregyhaensis 1 (2008).

[31] Jintang Li, *The Fundamental Formulas of Finsler Submanifolds*, Bulletin of the Korean Mathematical Society 47, (2010) Issue 4, p. 767-775.

DOI: 10.4134/BKMS.2010.47.4.767

[32] Henrique Vitorio, *The geometry of fanning curves and of their simplectic reductions*, PhD. thesis, Universidade Estadual de Campinas (2010).

[33] Xinyue Cheng and Zhongmin Shen, *Finsler Geometry, an approach via Randers Spaces*, Springer, (2012).

ISBN: 978-3-642-24887-0

[34] Miguel Ángel Javaloyes, *Conformally standard and stationary spacetimes and Fermat metrics*, (2012).

DOI: 10.48550/arXiv.1201.1841

[35] Zhongmin Shen and Qiaoling Xia, *A class of Randers metrics of scalar flag curvature*, International Journal of Mathematics 24 (2013).

DOI: 10.1142/S0129167X13500559

[36] Jędrzej Śniatycki, *Differential Geometry of Singular Spaces and Reduction of Symmetry*, Cambridge University Press (2013), p. 56.

DOI: 10.1017/CBO9781139136990

[37] Miguel Ángel Javaloyes and Miguel Sánchez, *On the definition and examples of Finsler metrics*, Annali della Scuola normale superiore di Pisa, Classe di scienze 13 (2014), p. 813-858.

DOI: 10.2422/2036-2145.201203-002

[38] Miguel Ángel Javaloyes and Bruno Learth Soares, *Geodesics and Jacobi Fields of Pseudo-Finsler Manifolds*, Publicationes Mathematicae 87, (2014).

DOI: 10.5486/PMD.2015.7028

[39] Miguel Ángel Javaloyes and Henrique Vitório, *Zermelo Navigation in pseudo-Finsler metrics*, (2014).

DOI: 10.48550/arXiv.1412.0465

[40] Michael Ruzhansky and Mitsuru Sugimoto, *On global inversion of homogeneous maps*, Bulletin of Mathematical Sciences 5 (2015), p. 13-18.

DOI: 10.1007/s13373-014-0059-1

[41] Amir Babak Aazami and Miguel Ángel Javaloyes, *Penrose's singularity theorem in a Finsler spacetime*, Classical and Quantum Gravity 33 (2015), 025003.

DOI: 10.1088/0264-9381/33/2/025003

[42] Marcos Alexandrino and Renato Bettiol, *Lie groups and geometric aspects of isometric actions*, Springer (2015), p. 69-70.

[43] Dragos Hrimiuc, *On pseudo-Finsler manifolds of scalar flag curvature*, Journal of Geometry and Physics 98 (2015).

DOI: 10.1016/j.geomphys.2015.09.002

[44] Ettore Minguzzi, *Light cones in Finsler spacetime*, Communications in Mathematical Physics 334, p. 1529-1551.

DOI: 10.1007/s00220-014-2215-6

[45] Michael Ruzhansky and Mitsuru Sugimoto, *On global inversion of homogeneous maps*, Bulletin of Mathematical Sciences 5 (2015), p. 13 -18.

[46] Miguel Ángel Javaloyes, *Anisotropic Tensor Calculus*, International Journal of Geometric Methods in Modern Physics 16, (2016) No. supp02.

DOI: 10.1142/S0219887819410019

[47] Carlos Duran and Henrique Vitorio, *Moving planes, Jacobi curves and the dynamical approach to Finsler geometry*, European Journal of Mathematics 3 (2017).

DOI: 10.1007/s40879-017-0190-0

[48] Ricardo Gallego Torromé, *Average structures associated to a Finsler space* (2017).

DOI:10.48550/arXiv.math/0501058

[49] Claus Lämmerzahl and Volker Perlick, *Finsler geometry as a model for relativistic gravity*, International Journal of Geometric Methods in Modern Physics 15, 1850166 (2018).

DOI: 10.1142/S0219887818501669

[50] Miguel Ángel Javaloyes, *Curvature Computations in Finsler Geometry Using a Distinguished Class of Anisotropic Connections*, Mediterranean Journal of Mathematics, (2020).

DOI: 10.1007/s00009-020-01560-0

[51] Matthieu Huber and Miguel Ángel Javaloyes, *The Flag Curvature of a Submanifold of a Randers-Minkowski Space in Terms of Zermelo Data*, Results in Mathematics 77, (2022) 124.

DOI: 10.1007/s00025-022-01661-0

[52] Matthieu Huber and Miguel Ángel Javaloyes, *The fundamental equations of a pseudo-Finsler submersion* (preprint 2022).

DOI: 10.48550/arXiv.2208.04839