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Super weak compactness
and its applications to Banach space theory

Super compacidad débil
y sus aplicaciones a la teoría de espacios de Banach

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Resumen

La mayor parte de este trabajo se centra en el estudio de la super compacidad débil, que es una versión localizada de la superreflexividad. Los ultraproductos son también una noción central en este trabajo. A continuación se resume el contenido de esta tesis.

Capítulo 1: Contenido preliminar

Este primer capítulo contiene los conceptos básicos utilizados a lo largo de este documento. La mayoría de los resultados que se presentan en esta sección se dan sin pruebas, mientras que los nuevos resultados se presentan con su demostración.

Comenzamos brevemente con la definición de filtros y algunas de sus propiedades elementales. Éstas nos permiten definir la noción de ultraproductos de los espacios de Banach. En particular, recordamos que es difícil acceder al dual de un ultraproducto (excepto en el caso superreflexivo) y que el ultraproducto de un ultraproducto es también un ultraproducto. A continuación definimos los ultraproductos de subconjuntos acotados de un espacio de Banach y enunciamos propiedades fundamentales que se utilizarán repetidamente en este documento.

Recordamos la definición de representabilidad finita de los espacios de Banach y su caracterización en términos de ultraproducto. Enunciamos resultados fundamentales como el Teorema de Dvoretsky o el Principio de reflexividad local. A continuación, extendemos la definición de representabilidad finita a subconjuntos de espacios de Banach y obtenemos resultados similares a los relativos a los espacios de Banach.

En la siguiente parte, introducimos los *spreading models* de un espacio de Banach X, que tienen la particularidad de ser finitamente representables en X. Se recuerdan algunas propiedades elementales y se enuncian los resultados establecidos por Beauzamy que vinculan los modelos de extensión con las propiedades de Banach-Saks.

Terminamos esta sección definiendo la noción fundamental en torno a la cual se articula este trabajo: *la super compacidad débil* (SWC).

Definición 0.0.1. Sea X un espacio de Banach y sea $A \subset X$ un conjunto acotado. Decimos que A es *relativamente super débilmente compacto* si todos sus ultraproductos son relativamente débilmente compactos. Si A es además débilmente cerrado, decimos que A es super débilmente compacto (SWC).

Establecemos caracterizaciones conocidas de los conjuntos super débilmente compactos que son similares a las de la superreflexividad. También podemos definir los *operadores super débilmente compactos* (SWC) y estudiamos algunas de sus propiedades. También nos interesan los espacios generados por un conjunto SWC ya que tienen notables propiedades de renormamiento.

Capítulo 2: Funciones uniformemente convexas

En este capítulo, introducimos la noción de función ε-uniformemente convexa. Comenzamos con la definición de la noción principal que se estudia en esta sección:

Definición 0.0.2. Sea $(X, \|.\|)$ un espacio de Banach y sea $\varepsilon > 0$. Una función $f : X \to \overline{\mathbb{R}}$ se dice *ε-uniformemente convexa* si existe $\delta > 0$ tal que si $\|x - y\| \geq \varepsilon$, entonces

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} - \delta.$$

Una función se dice *uniformemente convexa* si es ε-uniformemente convexa para todo $\varepsilon > 0$.

Resulta que admitir una función convexa continua ε-uniformemente convexa es en realidad una propiedad geométrica-topológica del dominio. Se sabe que un espacio de Banach admite una función uniformemente convexa acotada en conjuntos acotados si y sólo si es superreflexivo. Raja demostró en [147] que un conjunto cerrado convexo acotado admite una función continua uniformemente convexa acotada si y sólo si es SWC. Nuestras técnicas nos permiten dar una versión cuantitativa muy precisa de la relación entre la existencia de árboles separados y la existencia de una función uniformemente convexa para un conjunto.

Teorema 0.0.1. Sea C un subconjunto convexo cerrado acotado de espacio de Banach X. Entonces los siguientes números son iguales:

(a) el mínimo de los $\varepsilon > 0$ tal que existe un límite común para las alturas de todos los árboles diádicos ε-separados;

(b) el ínfimo de los $\varepsilon > 0$ tal que existe una función acotada ε-uniformemente convexa (y convexa, Lipschitz…) definida en C.

Es sabido que los conjuntos SWC convexos no admiten árboles diádicos ε-separados de altura arbitraria para todo $\varepsilon > 0$ (ver el Teorema 1.5.12). Entonces, el teorema anterior puede verse como una cuantificación de la super compacidad débil. Esta idea se desarrollará en el próximo capítulo. En este capítulo, se obtendrán otras caracterizaciones de la SWC demostrada por Raja como consecuencias de estos nuevos resultados (ver los Corolarios 2.4.9 y 2.4.12).

Nuestro afán por estudiar la super compacidad débil nos ha llevado a establecer resultados de interés independiente sobre las funciones uniformemente convexas. A lo largo de este documento, $\breve{f}$ denotará la envoltura convexa semicontinua inferior de una función f. El siguiente resultado muestra el comportamiento global de las funciones ε-uniformemente convexas y la estabilidad relativa de los minimizadores por perturbaciones lineales.

Teorema 0.0.2. Sea X un espacio de Banach y sea $f : X \to \overline{\mathbb{R}}$ una función ε-uniformemente convexa tal que $\breve{f}$ es propia. Entonces f es acotada inferiormente y coerciva, más precisamente se tiene que

$$\liminf_{\|x\|\to+\infty} \frac{f(x)}{\|x\|^2} > 0.$$

Además, para todo $\varepsilon' > \varepsilon$ existen $\delta, \eta > 0$ tal que si $x_0^* \in X^*$ y $x_0 \in X$ son tales que

$$f(x_0) + x_0^*(x_0) < \inf(f + x_0^*) + \delta,$$

si $x^* \in X^*$ es tal que $\|x^* - x_0^*\| < \eta$ y si $x \in X$ minimiza $f + x^*$, entonces $\|x - x_0\| \leq \varepsilon'$. La existencia de tal par minimizante (x, x^*) está garantizada si $f = \breve{f}$.

La prueba del primer resultado se basa en la posibilidad de volver convexa una función ε-uniformemente convexa sin perder la ε-convexidad uniforme. Diremos que una función f es ε^+*-uniformemente convexa* si es ε'-uniformemente convexa para todo $\varepsilon' > \varepsilon$. Tenemos el siguiente resultado.

Teorema 0.0.3. Sea X un espacio de Banach y sea $f : X \to \overline{\mathbb{R}}$ una función ε-uniformemente convexa tal que $\breve{f}$ es propia. Entonces $\breve{f}$ es ε^+-uniformemente convexa.

Ejemplos sencillos, como el Ejemplo 2.1.7, muestran que la convexidad ε-uniforme de f no garantiza que $\breve{f}$ sea propia. Para cumplir ese requisito en términos de f, dirigimos al lector al Corolario 2.4.2. Supongamos ahora que ya tenemos una función convexa propia semicontinua inferior y ε-uniformemente convexa f. Nos preguntamos si podríamos "mejorar" f para obtener una nueva función que comparta esas propiedades y que, además, sea localmente Lipschitz (no se permite un comportamiento Lipschitz global para funciones uniformemente convexas). En ese sentido, tenemos el siguiente resultado.

Teorema 0.0.4. Sea $(X, \|\cdot\|)$ un espacio de Banach y sea $f \in \Gamma(X)$ una función ε-uniformemente convexa. Entonces existe una norma equivalente $\vert\kern-0.25ex\vert\kern-0.25ex\vert \cdot \vert\kern-0.25ex\vert\kern-0.25ex\vert$ en X tal que la función $x \mapsto \vert\kern-0.25ex\vert\kern-0.25ex\vert x \vert\kern-0.25ex\vert\kern-0.25ex\vert^2$ es ε^+-uniformemente convexa sobre los subconjuntos de $\mathrm{dom}(f)$ donde f es acotada superiormente. Además, esta norma $\vert\kern-0.25ex\vert\kern-0.25ex\vert \cdot \vert\kern-0.25ex\vert\kern-0.25ex\vert$ se puede elegir tan cercana de $\|\cdot\|$ como uno quiera.

La estructura de este capítulo es la siguiente. La Sección 2.1 trata de las propiedades básicas de las funciones ε-uniformemente convexas y ε-uniformemente cuasi-convexas, principalmente bajo la hipótesis de convexidad. Se dan algunos ejemplos para mostrar que las definiciones no garantizan algunas buenas propiedades adicionales. La Sección 2.2 está dedicada a la demostración del Teorema 0.0.3 que permitirá la reducción al caso convexo de otros resultados. La construcción de funciones uniformemente convexas a partir de cero (árboles y conjuntos) se realiza en la Sección 2.3. La Sección 2.4 trata de las propiedades generales de las funciones ε-uniformemente convexas y la posibilidad de añadir más propiedades como ser Lipschitz o la homogeneidad (renormación). También probamos una estimación de la aproximación por diferencias de funciones convexas. En la sección siguiente esbozaremos una demostración comprensible del teorema del renormamiento uniforme convexo de los espacios superreflexivos de Enflo basada en las ideas expuestas a lo largo del trabajo. La última sección está dedicada a la suavidad uniforme, que es la propiedad dual de la convexidad uniforme.

Los resultados presentados en este capítulo son consecuencia de una colaboración con M. Raja y pueden encontrarse en [88].

Capítulo 3: Cuantificación de la super compacidad débil

La super compacidad débil está más presente de lo que parece. Por ejemplo, cualquier operador débilmente compacto con rango $L_1(\mu)$ (μ cualquier medida) o dominio $C(K)$ (K cualquier compacto Hausdorff) es SWC, ver [122, Proposición 6.1]. En realidad, algunos resultados de la teoría de los espacios de Banach podrían entenderse en términos de super compacidad débil. Por ejemplo, el clásico resultado de Szlenk, que establece que una sucesión débilmente convergente en $L_1(\mu)$ tiene una subsucesión cuyos promedios de Cesàro convergen (al mismo límite), es una consecuencia de dos hechos: los subconjuntos débilmente compactos de $L_1(\mu)$ son SWC; y los conjuntos SWC tienen la propiedad Banach-Saks [122, Corolario 6.3].

El objetivo de este capítulo es mostrar que, en realidad, la super compacidad débil y, en particular, su cuantificación, pueden arrojar luz sobre la estructura de los subespacios de los espacios de Banach generados por espacios de Hilbert. En efecto, nos hemos dado cuenta de que varias “hipótesis técnicas” en los trabajos de Troyanski [162], Argyros y Farmaki [9], y la serie de trabajos de Fabian, Godefroy, Hájek, Montesinos y Zizler [71, 66, 65, 70] sobre la estructura de los espacios generados por un espacio de Hilbert y el renormamiento uniformemente Gâteaux, pueden entenderse en términos de una versión cuantificada de la super compacidad débil.

Introducimos una medida de no super compacidad débil Γ. Sea A un subconjunto acotado de un espacio de Banach X, entonces definimos

$$\gamma(A) = \inf\{\varepsilon > 0 : \overline{A}^{w^*} \subset X + \varepsilon B_{X^{**}}\}.$$

Tenemos que un conjunto A es relativamente débilmente compacto si y sólo si $\gamma(A) = 0$. Esta medida ha sido estudiada en [69, 84, 42], véase también [91, Sección 3.6], y hay varias medidas de no compacidad débil que resultan ser equivalentes [7]. Si $\mathcal{U}$ es un ultrafiltro libre, definimos Γ por

$$\Gamma(A) := \gamma(A_{\mathcal{U}}).$$

Veremos que la elección de $\mathcal{U}$ no es importante. Obtenemos el siguiente resultado de cuantificación:

Teorema 0.0.5. Sea C un subconjunto convexo cerrado acotado de un espacio de Banach X. Definimos los siguientes números:

(μ_1) el supremo de los $\varepsilon > 0$ tal que para todo $n \in \mathbb{N}$ existen $x_1, \dots, x_n \in C$ tal que

$$\mathrm{d}(\mathrm{conv}\{x_1, \dots, x_k\}, \mathrm{conv}\{x_{k+1}, \dots, x_n\}) \geq \varepsilon \text{ para todo } k = 1, \dots, n-1;$$

(μ_2) el supremo de los $\varepsilon > 0$ tal que existen árboles diádicos ε-separados de tamaño arbitrariamente grande;

(μ_3) $= \Delta(C_{\mathcal{U}})$, para un ultrafiltro libre $\mathcal{U}$ en $\mathbb{N}$;

(μ_4) $= \Gamma(C)$, (calculado para un ultrafiltro libre $\mathcal{U}$ en $\mathbb{N}$);

(μ_5) el ínfimo de los $\varepsilon > 0$ tal que $\mathrm{Dz}(C, \varepsilon) < \omega$;

(μ_6) el ínfimo de los $\varepsilon > 0$ tal que C soporta una función convexa acotada ε-uniformemente convexa.

Entonces $\mu_1 \leq \mu_2 \leq 2\mu_3 \leq 2\mu_4 \leq 2\mu_1$ and $\mu_4 \leq 2\mu_5 \leq 2\mu_6 \leq 2\mu_2$.

También podemos cuantificar la super compacidad débil para operadores. De hecho, si $T : X \to Y$ es un operador acotado, podemos definir $\Gamma(T) = \Gamma(T(B_X))$. En particular, demostramos una versión cuantificada de la simetría del bi-ideal $\mathfrak{W}^{super}$ (el conjunto de los operadores SWC), así como una versión cuantificada del renormamiento de Beauzamy para volver uniformemente convexo un operador super débilmente compacto.

En la última parte, establecemos la superversión del siguiente resultado:

Teorema 0.0.6 ([22, 68])**.** Para un espacio de Banach X, los siguientes resultados son equivalentes:

(i) X es un subespacio de espacio WCG;

(ii) (B_{X^*}, w^*) es un compacto Eberlein;

(iii) Para todo $\varepsilon > 0$ existen conjuntos $(A_n^\varepsilon)_n$ tal que $B_X = \bigcup_{n=1}^\infty A_n^\varepsilon$ y $\gamma(A_n^\varepsilon) < \varepsilon$.

La equivalencia (i) $\Leftrightarrow$(ii) se debe a Benyamini, Rudin y Wage [22]. La caracterización interna (iii) fue obtenida por Fabian, Montesinos y Zizler [68]. Recordemos que WCG significa *weakly compactly generated*, es decir, un espacio de Banach que contiene un subconjunto débilmente compacto cuyo espacio generado es denso. Gracias al famoso resultado de interpolación de Davis, Figiel, Johnson y Pełczyński [56] (véase también [67, Teorema 13.22]), un espacio de Banach X es WCG si y sólo si existe un espacio reflexivo Z y un operador $T : Z \to X$ con rango denso. Además, si el espacio Z puede tomarse como un espacio de Hilbert, decimos que X es *Hilbert generado*. El nombre *Eberlein* se aplica a los espacios compactos que son homeomorfos a un subconjunto débilmente compacto de un espacio de Banach. Es conocido, después de Amir y Lindenstrauss (ver [67, Corolario 13.17], por ejemplo), que un compacto Eberlein se inyecta linealmente como un subconjunto débilmente compacto de $c_0(I)$ para I suficientemente grande. Si tal inyección puede hacerse en un espacio de Hilbert $\ell_2(I)$, entonces se dice que el compacto es *uniformemente Eberlein*. Nótese que el tercer enunciado del Teorema 0.0.6 es en realidad una caracterización interna ya que está escrito en términos del espacio X, no de un sobreespacio o de su dual. También veremos que el cálculo de γ se puede hacer sin apelar al espacio bidual (ver la Proposición 3.1.2).

El principal resultado de este capítulo es el siguiente. Nótese el paralelismo con el teorema anterior.

Teorema 0.0.7. Sea X un espacio de Banach. Las siguientes proposiciones son equivalentes:

(i) X es un subsespacio de un espacio Hilbert generado;

(ii) (B_{X^*}, w^*) es un compacto uniformemente Eberlein;

(iii) Para todo $\varepsilon > 0$ existen conjuntos $(B_n^\varepsilon)_n$ tal que $B_X = \bigcup_{n=1}^\infty B_n^\varepsilon$ y $\Gamma(B_n^\varepsilon) < \varepsilon$.

Este capítulo está estructurado como sigue. La Sección 3.1 está dedicada a demostrar el Teorema 0.0.5 y el hecho de que Γ define una medida de no-compacidad (según la Definición 3.1.1). En la Sección 3.2, cuantificamos la convexidad uniforme de un operador. En la última parte, demostramos el Teorema 0.0.7. Concluimos este capítulo con una caracterización de los espacios *super débilmente compactamente generados* (SWCG), en particular en el caso de los espacios $\mathcal{C}(K)$, y una aplicación de los resultados anteriores a las álgebras de Jordan.

La mayoría de los resultados presentados en este capítulo se encuentran en [87] (véase también la Sección 6 en [88]) y se han establecido con M. Raja.

Capítulo 4: Nuevos resultados sobre la super compacidad débil

Los trabajos anteriores sobre el renormamiento uniformemente Gâteaux de Fabian, Godefroy, Hájek y Zizler [65], así como los primeros resultados de Troyanski [162], contienen estimaciones de Γ (ver el capítulo 3). La explicación vendrá a través del siguiente resultado.

Proposición 0.0.3. Sea A un subconjunto acotado de un espacio de Banach X. Definimos los siguientes números:

(ε_1) es el ínfimo de los $\varepsilon > 0$ tal que existe $n_1 \in \mathbb{N}$ tal que para todo $x^* \in B_{X^*}$ se tiene que

$$|\{x \in A : |x^*(x)| > \varepsilon\}| \leq n_1;$$

(ε_2) es el ínfimo de los $\varepsilon > 0$ tal que existe $n_2 \in \mathbb{N}$ tal que para todo conjunto finito $B \subset A$ tal que $|B| \geq n_2$ se tiene que

$$\left\| \frac{1}{|B|} \sum_{x \in B} x \right\| < \varepsilon.$$

Entonces $\varepsilon_1 = \varepsilon_2$ y en este caso $\Gamma(A) \leq \varepsilon_1$.

Llamaremos conjuntos *uniformemente debilmente nulos* a los conjuntos que satisfacen el enunciado de la Proposición 0.0.3 con $\varepsilon_1 = \varepsilon_2 = 0$. Obsérvese que un conjunto uniformemente débilmente nulo se convierte en un conjunto SWC añadiendo $\{0\}$. Junto con las bolas unitarias de los espacios sobrerreflexivos, los conjuntos uniformemente nulos son los ejemplos más prototípicos de conjuntos SWC. La primera parte de este capítulo trata de las propiedades de los conjuntos uniformemente débiles nulos. Como veremos, los conjuntos SWC con alguna hipótesis de discretización razonable son uniformemente débilmente nulos. Nótese que la segunda proposición (ε_2) es una especie de propiedad de Banach-Saks uniforme (con límite único 0). Esto nos permitirá aplicar resultados de la combinatoria infinita, como los resultados de selección de Erdös-Magidor [63] y de Mercourakis [134]. Investigamos cuándo una base de Schauder es uniformemente débilmente nula, y la relación de los conjuntos uniformemente débilmente nulos con la representación de los compactos uniformemente Eberlein.

La segunda parte de este capítulo está dedicada a las propiedades de ergodicidad y puntos fijos de los conjuntos SWC. En [50], los autores demostraron que un subconjunto convexo cerrado y acotado K de un espacio de Banach X es SWC si y sólo si tiene la super propiedad del punto fijo para isometrías afines $T : X \to X$ preservando K. En esta parte, mejoramos esta caracterización de dos maneras diferentes. Por un lado, demostramos el siguiente resultado:

Teorema 0.0.8. Sea C un subconjunto convexo cerrado de un espacio de Banach X. Las siguientes proposiciones son equivalentes:

(i) C es SWC;

(ii) C es super-ergódico.

Este resultado se demuestra estudiando los puntos fijos del *shift* de un *spreading model* (ver Proposición 4.2.5). Entonces las propiedades de punto fijo de K se deducen fácilmente de una adaptación del Teorema ergódico medio (ver Teorema 4.2.3). También observamos que un conjunto SWC K puede tener propiedades de puntos fijos más fuertes, ya que el espacio subyacente puede ser renormado de tal manera que K tiene estructura normal (ver el Teorema 1.5.33). Por otro lado, el principal inconveniente de la caracterización dada en [50] es que pueden existir isometrías afines definidas sobre K que no pueden extenderse a todo el espacio. Además, la super compacidad débil es una versión localizada de la superreflexividad y es natural esperar que dicha caracterización sólo dependa de K, lo cual se demuestra en el Teorema 4.2.11. Aplicamos algunos de estos resultados a los espacios de Banach *fuertemente super débilmente compactamente generados* (ver Definición 1.5.30). Para terminar esta segunda parte del capítulo, demostramos que si un espacio de Banach X tiene la *M-(FPP para isometrías afines)* (es decir, todos los *spreading models* de X tienen la propiedad del punto fijo para isometrías afines) entonces X es reflexivo. Más precisamente, obtenemos:

Teorema 0.0.9. La M-(FPP para isometrías afines) implica la propiedad de Banach-Saks.

De ello se desprende que la reflexividad se encuentra estrictamente entre la propiedad del punto fijo y la M-(propiedad del punto fijo) (ver la definición 4.2.18).

La primera parte de este capítulo proviene de [87]. Los resultados de la segunda parte son parte de un *preprint* (ver [85]).

Capítulo 5: Propiedades de Banach-Saks uniformes

La superreflexividad y su versión localizada, la super compacidad débil, están fuertemente relacionadas con la propiedad de Banach-Saks. De hecho, la super compacidad débil es equivalente a la super propiedad de Banach-Saks por el Teorema 1.5.15. En esta sección, estamos interesados en las propiedades de Banach-Saks uniformes, lo que significa que tratamos de controlar la velocidad de convergencia de una subsucesión Cesaro-convergente de una sucesión acotada. Muchos de los resultados de este capítulo se inspiran en el trabajo de Beauzamy y su uso de *spreading models* para caracterizar las propiedades de Banach-Saks (ver el capítulo 1). El concepto principal de este capítulo se presenta en la siguiente definición. Las demás propiedades de Banach-Saks que se estudiarán son variaciones de esta definición.

Definición 0.0.4. Sea X un espacio de Banach. Decimos que X tiene la *la propiedad uniforme de Banach-Saks débil* (WBS uniforme) si existe una sucesión $(a_n)_n$ de reales positivos tal que $a_n \to 0$ y para todo $(x_n)_n \subset B_X$ tal que $x_n \xrightarrow{w} 0$, existe una subsucesión $(x'_n)_n$ de $(x_n)_n$ tal que

$$\frac{1}{m}\left\|\sum_{k=1}^{m} x'_k\right\| \leq a_m$$

para todo $m \geq 1$.

La Sección 5.1 está dedicada al estudio de la WBS uniforme. En la literatura (ver [151, 148]), aparece un caso especial de la WBS uniforme cuando a_n es de la forma $a_n = Cn^{\frac{1}{p}-1}$ para algún $C > 0$ y $p > 1$. En ese caso, decimos que X tiene la p-WBS fuerte. En [148], Rakov demostró que si X tiene la WBS uniforme entonces existe algún $p > 1$ tal que X tiene la p-WBS fuerte. Este resultado se establecerá aquí con otra prueba en la Subsección 5.1.2 (ver Teorema 5.1.9). La idea principal es demostrar que la (A_∞)-débil, introducida por Partington en [139], y la WBS uniforme son equivalentes. Esto se hace en la Subsección 5.1.1.

Nótese que la constante C presente en la definición de la p-WBS fuerte no depende de la elección de la subsucesión débilmente nula. Introducimos la propiedad de p-Banach-Saks débil (p-WBS) eliminando esta independencia. Aparte de algunos casos especiales, parece que no se sabía si ambas nociones coinciden. En la Subsección 5.1.3, demostramos con toda generalidad que la p-WBS coincide con la p-WBS fuerte cuando $p \in (1, \infty)$ (ver Teorema 5.1.14). En resumen, obtenemos el siguiente resultado:

Teorema 0.0.10. Sea X un espacio de Banach. Las siguientes proposiciones son equivalentes:

(i) X tiene la WBS uniforme;

(ii) X tiene la (A_∞)-débil;

(iii) X tiene la p-WBS fuerte para algún $p \in (1, +\infty)$;

(iv) X tiene la p-WBS para algún $p \in (1, +\infty)$.

La equivalencia entre la ∞-WBS fuerte y la ∞-WBS también es cierta, pero no se puede demostrar con el mismo argumento. Por ello, se establecerá utilizando la teoría de Ramsey y una poderosa caracterización de la propiedad hereditaria de Dunford-Pettis (ver el Teorema 3.1 en [117]).

Si X tiene la WBS uniforme, es interesante encontrar el mayor p tal que X tenga la p-WBS. Por eso introducimos la siguiente definición:

Definición 0.0.5. Sea X un espacio de Banach. Definimos *el indice WBS uniforme* UWBS(X) de X como el supremo de los $p > 1$ tal que X tiene la p-WBS (fuerte). Si X no tiene la p-WBS (fuerte) para ningún $p > 1$, fijamos UWBS(X) $= 1$.

También introducimos el siguiente índice, que caracteriza las propiedades (A_k) introducidas por Partington:

Definición 0.0.6. Sea X un espacio de Banach. Para todo $m \geq 1$, definimos $\phi_X(m)$ como el ínfimo de los b tal que para toda sucesión $(x_n)_n \subset B_X$ tal que $x_n \xrightarrow{w} 0$, existen $p_1 < p_2 < ... < p_m$ tal que

$$\frac{1}{m}\left\|\sum_{i=1}^{m} x_{p_i}\right\| \leq b.$$

En la Subsección 5.1.4, establecemos uno de nuestros principales resultados:

Teorema 0.0.11. Sea X un espacio de Banach.

(a) Si $\phi_X(n) > \frac{1}{n}$ para todo $n \geq 2$, entonces

$$\mathrm{UWBS}(X) = \sup_{n\geq 2} \frac{\ln(n)}{\ln(n\phi_X(n))}.$$

(b) Si $\phi_X(n) \leq \frac{1}{n}$ para algún $n \geq 2$, entonces

$$\mathrm{UWBS}(X) = +\infty.$$

La fórmula anterior ha sido establecida por Rakov en [148] por un método diferente, pero sin ninguna hipótesis sobre ϕ_X. Sin embargo, es fácil ver que esta fórmula no es cierta si no se supone nada sobre ϕ_X (ver Proposición 5.1.27).

En la Sección 5.2, introducimos la *propiedad uniforme de Banach-Saks* (BS uniforme), que se obtiene básicamente añadiendo la reflexividad a la WBS uniforme. Se dan muchos ejemplos de espacios que tienen la BS uniforme como los espacios (β), los espacios k-NUCε o los espacios NUS. La *propiedad uniforme de Banach-Saks alternante débil* (WABS uniforme) se introduce en la Sección 5.3. Resulta que la WABS uniforme y la WBS uniforme son equivalentes (ver Teorema 5.3.6). Como aplicación, obtenemos que cualquier espacio de Banach con un tipo no trivial p tiene la p-WBS (fuerte). En la última sección, introducimos la *propiedad uniforme de Banach-Saks alternante* (ABS uniforme) y estudiamos sus vínculos con las demás propiedades uniformes de Banach-Saks. Concluimos este capítulo con una aplicación a las constantes simétricas de Kottman.

Los resultados de este capítulo forman parte de un *preprint* (ver [86]).

Capítulo 6: Estructura extremal en ultraproductos

El ultraproducto de los espacios de Banach ha demostrado ser una herramienta muy útil en el estudio de las propiedades locales de los espacios de Banach. Por ejemplo, en [1, Teorema 11.1.4] se utilizan los ultraproductos para demostrar que un espacio de Banach X no tiene tipo $p > 1$ si y sólo si ℓ_1 es finitamente representable en X. Este vínculo entre la estructura local de un espacio de Banach X y la estructura global de sus ultraproductos $X_{\mathcal{U}}$ nos ha permitido obtener resultados estructurales en espacios de Banach.

Estudios más recientes sobre la geometría de los ultraproductos de espacios de Banach pueden encontrarse en [94] para los espacios de Banach octaédricos y casi cuadrados o en [25, 110] para la propiedad de Daugavet. En realidad, el ejemplo de la propiedad de Daugavet es paradigmático de dos hechos básicos que, la mayoría de las veces, aparecen cuando se trata de una propiedad geométrica en espacios de Banach. El primero es que, cuando se requiere que un ultraproducto $X_{\mathcal{U}}$ goce de una propiedad geométrica, se debe buscar una "versión uniforme" de esta propiedad geométrica en X (esto ocurre, por ejemplo, con la propiedad de Daugavet y la propiedad de

Daugavet uniforme [25, Teorema 6.4], para la convexidad estricta y la convexidad uniforme o para la reflexividad y superreflexividad [97, Proposición 6.4]). La segunda es que hay que evitar en lo posible tratar con el dual de un espacio de ultraproductos (esto se hace en [25] utilizando su Teorema 6.2). La razón es que, en la mayoría de los casos (es decir, fuera de los espacios de Banach superreflexivos [97, Corolario 7.2]), no hay un buen acceso al dual de $X_{\mathcal{U}}$.

Teniendo en cuenta los dos hechos anteriores, el objetivo de este capítulo es estudiar la estructura extrema de los subconjuntos de un ultraproducto. Esta estructura codifica mucha información de los conjuntos convexos acotados (podemos pensar por ejemplo en los teoremas de Krein-Milman) y es extremadamente útil en otras áreas del Análisis Funcional tal que el alcanzamiento de la norma (ver [41, 126]). El caso particular de la estructura extrema de los ultraproductos ha sido considerado previamente por J. Talponen en [159], donde el autor estudió las propiedades que unen un punto x de la esfera unitaria de un espacio de Banach y su imagen $\mathcal{J}(x)$ en el ultraproducto a través de la isometría canónica. Algunos de sus resultados serán generalizados en este documento ya que tratamos con conjuntos más generales (no sólo con la esfera unitaria) y ultrafiltros más generales (no sólo en $\mathbb{N}$). También establecemos cómo las propiedades sobre los x_i se trasladan a $(x_i)_{\mathcal{U}}$ y recíprocamente. Se obtienen los resultados que relacionan x y $\mathcal{J}(x)$ como caso particular.

Las tres primeras secciones contienen nuestros resultados principales con total generalidad. Después de proporcionar una serie de ejemplos que sugieren cuales son las propiedades que debemos buscar en X, establecemos varios resultados de estabilidad relativos a la estructura extremal. Por ejemplo, extendemos el resultado de Talponen en el Teorema 6.1.5 mostrando que x es un punto fuertemente extremo de un conjunto convexo acotado C si y sólo si $\mathcal{J}(x)$ es un punto (fuertemente) extremo de su ultraproducto $C_{\mathcal{U}}$. Además, mostramos que los puntos extremos y fuertemente extremos de $C_{\mathcal{U}}$ coinciden bajo suposiciones débiles sobre $\mathcal{U}$, es decir:

Teorema 0.0.12. Sea C un subconjunto convexo acotado de un espacio de Banach X y sea $\mathcal{U}$ un ultrafiltro CI sobre un conjunto infinito I. Entonces $\operatorname{ext}(C_{\mathcal{U}}) = \text{str-ext}(C_{\mathcal{U}})$.

También caracterizamos en el Teorema 6.1.7 los elementos $(x_i)_{i\in I} \in C^I$ tal que $(x_i)_{\mathcal{U}}$ es un punto extremo de $C_{\mathcal{U}}$ para todo ultrafiltro libre $\mathcal{U}$ en I. En el contexto de los puntos dientes (respectivamente puntos fuertemente expuestos) demostramos que $(x_i)_{\mathcal{U}} \in C_{\mathcal{U}}$ es un punto diente (respectivamente punto fuertemente expuesto) si $\{x_i\}$ satisface una “condición *denting* uniforme” (respectivamente una condición uniforme de exposición fuerte), ver los Teoremas 6.2.2 y 6.3.2. Finalmente, demostramos que todo elemento de $C_{\mathcal{U}}$ que está expuesto por un funcional en $(X^*)_{\mathcal{U}}$ está fuertemente expuesto bajo suposiciones débiles sobre $\mathcal{U}$:

Teorema 0.0.13. Sea C un subconjunto convexo acotado de un espacio de Banach X y sea $\mathcal{U}$ un ultrafiltro CI sobre un conjunto infinito I. Entonces $\exp_{(X^*)_{\mathcal{U}}}(C_{\mathcal{U}}) = \text{str-exp}_{(X^*)_{\mathcal{U}}}(C_{\mathcal{U}})$.

En la última parte, examinamos con más detalle los conjuntos convexos SWC donde esperamos un buen comportamiento de la estructura extremal. También investigamos los *conjuntos uniformemente convexos*, que es un caso particular de conjuntos SWC. La principal herramienta en este estudio es que si C es uniformemente convexo entonces $C_{\mathcal{U}}$ también lo es (ver Proposición 6.4.9). El objetivo de esta sección es recuperar dentro de lo posible las propiedades extremales de la bola unitaria de un espacio de Banach uniformemente convexo. La mayor dificultad es que un conjunto uniformemente convexo puede tener interior vacío. Sin embargo, demostramos que cualquier punto extremo de un conjunto de este tipo es diente y que cualquier punto expuesto es fuertemente expuesto (Proposición 6.4.14). También caracterizamos los puntos extremos de su ultraproducto (Teorema 6.4.13).

Los resultados aquí presentados han sido establecidos con L. García-Lirola y A. Rueda Zoca y se pueden encontrar en [77].

Capítulo 7: Ultraproductos en espacios Lipschitz libres

En este capítulo, consideramos la noción de ultraproducto de espacios métricos, lo cual es una generalización de la correspondiente para espacios de Banach. Lo aplicamos para obtener una caracterización de ultraproducto de espacios métricos que son Lipschitz finitamente representables (en el sentido introducido por Lee, Naor y Peres [123]) en un espacio de Banach. Además, analizamos la relación entre la Lipschitz representabilidad finita de los espacios métricos y la representabilidad finita de los espacios Lipschitz libres correspondientes. Estos espacios (también llamados espacios de Arens-Eells y espacios de coste de transporte) se han convertido en un tema de investigación muy activo debido a sus aplicaciones en el Análisis No Lineal [81], así como en la Informática y en el Transporte Óptimo.

En la primera sección, introducimos las propiedades fundamentales de los ultraproductos de espacios métricos y de los espacios Lipschitz libres utilizados en este documento. En la siguiente sección, demostramos que, dado un espacio métrico M y un ultrafiltro $\mathcal{U}$, el espacio Lipschitz libre del ultraproducto de M, $\mathcal{F}(M_{\mathcal{U}})$, es linealmente isométrico a un subespacio del ultraproducto del espacio Lipschitz libre, $\mathcal{F}(M)_{\mathcal{U}}$. De hecho, tenemos que:

Teorema 0.0.14. Sea $\mathcal{U}$ un ultrafiltro sobre un conjunto I y sea $(M_i)_{i\in I}$ una familia de espacios métricos. Entonces $\mathcal{F}((M_i)_{\mathcal{U}})$ es linealmente isométrico a $\overline{\operatorname{span}(\delta(M_i)_{\mathcal{U}})} \subset \mathcal{F}(M_i)_{\mathcal{U}}$.

En particular, $\mathcal{F}(M_{\mathcal{U}})$ es finitamente representable en $\mathcal{F}(M)$. Además, demostramos que un espacio métrico M es Lipschitz finitamente representable en un espacio de Banach X si y sólo si M se inyecta biLipschitz en un ultraproducto de X (Teorema 7.3.6). Esto se hace en la Sección 7.3 donde vinculamos los ultraproductos de los espacios métricos con la noción de representabilidad finita. Como consecuencia obtenemos el siguiente resultado:

Teorema 0.0.15. Sea M un espacio métrico y sea X un espacio de Banach. Supongamos que M es λ-Lipschitz finitamente representable en X. Entonces $\mathcal{F}(M)$ es λ-finitamente representable en $\mathcal{F}(X)$.

Este resultado tiene algunas consecuencias sobre el cótipo de los espacios Lipschitz libres que pueden encontrarse en la Sección 7.3. Por ejemplo, se tiene la siguiente dicotomía: o bien $\mathcal{F}(\ell_2)$ tiene un cótipo no trivial, o bien $\mathcal{F}(X)$ no tiene cótipo para ningún espacio de Banach de dimensión infinita X. Finalmente, aunque se sabe que varias clases de espacios de Banach (como los retículos de Banach, las álgebras C* y los espacios $\mathcal{C}(K)$) son estables por ultraproductos, mostramos que $(\mathcal{F}(M))_{\mathcal{U}}$ no es isomorfo a ningún espacio Lipschitz libre siempre que M sea un espacio métrico infinito y $\mathcal{U}$ sea contablemente incompleto. En la Sección 7.5, comparamos la estabilidad de $\mathrm{Lip}_0(K)$ y $\mathcal{C}(K)$ bajo ultraproductos y observamos algunas similitudes y diferencias entre ellos.

Este capítulo está basado en [76] y es consecuencia de un trabajo con L. García-Lirola.

Capítulo 8: Una nota sobre espacios Lipschitz libres no separables

En la primera sección, demostramos que varias propiedades clásicas de los espacios de Banach son equivalentes a la separabilidad para la clase de espacios Lipschitz libres, incluyendo la propiedad de Corson ($\mathcal{C}$), ser un espacio de diferenciabilidad de Gâteaux, o la propiedad de separación contable. Por otro lado, señalamos propiedades, más generales que las anteriores, en las que falla la equivalencia con la separabilidad en la clase de espacios libres de Lipschitz. En particular, la cuestión de si los espacios Lipschitz libres no separables pueden tener una bola dual débil* secuencialmente compacta es indecidible. De hecho, si denotamos por $\mathfrak{s}$ el *splitting cardinal* tenemos que:

Teorema 0.0.16. Sea M un espacio métrico completo. Las siguientes proposiciones son equivalentes:

(i) $\mathrm{dens}(M) \geq \mathfrak{s}$,

(ii) M contiene un subconjunto uniformemente discreto de cardinal $\mathfrak{s}$,

(iii) $B_{\mathrm{Lip}_0(M)}$ no es w^*-secuencialmente compacto.

En particular, $B_{\ell_\infty(\Gamma)}$ es w^*-secuencialmente compacto si y solo si $|\Gamma| < \mathfrak{s}$. Por lo tanto es indecidible si $B_{\ell_\infty(\omega_1)}$ es w^*-secuencialmente compacto.

Finalmente, en la segunda sección proporcionamos un ejemplo de un espacio Lipschitz libre dual no separable que no cumple la propiedad de Radon-Nikodým. Este ejemplo se basa en la siguiente proposición que afirma que el espacio $\mathcal{M}(S)$ de medidas de Radon sobre un espacio polaco S es un espacio Lipschitz libre:

Proposición 0.0.7. Sea S un espacio polaco. Entonces $\mathcal{M}(S)$ es linealmente isométrico a $\mathcal{F}(M)$ para cierto espacio métrico M. Más precisamente,

$$\mathcal{M}(S) \cong \ell_1(|S|) \oplus_1 \left(\bigoplus_{\kappa} L_1 \right)_1 \cong \mathcal{F}(M)$$

donde κ es un cardinal tal que $\kappa \leq \mathfrak{c}$ y M es la suma métrica de $|S|$ espacios con dos puntos y de κ copias de $[0, 1]$.

Este pequeño capítulo forma parte de un trabajo en curso con R. Aliaga y A. Procházka.

Capítulo 9: Proximinalidad y conjuntos uniformemente aproximables en L^p

Este capítulo es totalmente independiente de los anteriores. Sin embargo, este trabajo proviene del estudio de los espacios superreflexivos. En efecto, es sabido que si X es superreflexivo entonces también lo es $L^2(X)$. Esto se deduce directamente del Teorema de Enflo, ya que $L^2(X)$ es uniformemente convexo si X lo es. Esta transición a $L^2(X)$ es fundamental en la demostración del Teorema de Pisier y, hasta la fecha, no existe ninguna demostración de este teorema que no requiera el Teorema de Enflo. Los resultados de este capítulo provienen del cuestionamiento del autor de por qué $L^2(X)$ es superreflexivo si X es superreflexivo sin pasar por el Teorema de Enflo.

En este capítulo se estudia la aproximación de funciones medibles por funciones simples que toman como máximo k valores, para $k \in \mathbb{N}$. Este problema tiene importantes consecuencias en múltiples aplicaciones, donde por ejemplo se busca la reducción de la dimensionalidad, entre otras muchas. Por ejemplo, la inyección de espacios métricos en espacios normados de dimensión finita es uno de los principales problemas del análisis no lineal (ver [32, 108, 131]). Estos resultados tienen consecuencias profundas para diseñar algoritmos de aproximación, como por ejemplo el *Sparsest Cut problem* (ver [10]). Cuando aproximamos una función dada $f \in L^p(\Omega, \mathcal{F}, \mu)$ mediante funciones simples, el número de términos en esas aproximaciones crece hasta el infinito en general. Aquí, una de las principales preocupaciones es lo que podemos decir si restringimos el número de términos en las aproximaciones. En particular, lo que podemos decir sobre los subconjuntos de $L^p(\Omega, \mathcal{F}, \mu)$ que pueden ser uniformemente aproximados por funciones simples que toman k valores a medida que k crece a ∞. Como veremos, este nuevo concepto es más general que la integrabilidad uniforme o la compacidad, y lo caracterizamos completamente en términos de una nueva medida de variación definida para funciones en $L^p(\Omega, \mathcal{F}, \mu)$ para $p \in [1, \infty)$, y en términos de números de recubrimiento en el caso $p = \infty$.

Fijemos algunas notaciones necesarias para explicar los principales resultados de este trabajo. Consideremos $(\Omega, \mathcal{F}, \mu)$ un espacio de medida. Para cualquier $k \geq 1$, denotamos por $\mathscr{G}_{p,k}(\Omega, \mathcal{F}, \mu)$, o simplemente $\mathscr{G}_{p,k}$ cuando el espacio de medidas $(\Omega, \mathcal{F}, \mu)$ es claro por el contexto, el conjunto de funciones simples dadas por

$$\mathscr{G}_{p,k} = \left\{ \sum_{i=1}^{l} a_i \mathbb{1}_{A_i} \in L^p(\Omega, \mathcal{F}, \mu) : \ \{A_i\}_{1 \leq i \leq l} \text{ partición medible de } \Omega,\ a_i \in \mathbb{R} \text{ para todo } i,\, l \leq k \right\}.$$

Observación 0.0.8. Supongamos que μ es una medida finita. Entonces $\mathscr{G}_{p,k} = \mathscr{G}_{1,k}$ para todo $p \in [1, \infty]$ es simplemente el conjunto de funciones medibles simples que toma como máximo k valores. Si μ es una medida infinita, entonces $h = \sum_{i=1}^{k} a_i \mathbb{1}_{A_i}$, donde $\{A_i\}_{1 \leq i \leq k}$ es una partición medible, pertenece a $\mathscr{G}_{p,k}$, para $p \in [1, \infty)$, si y sólo si $\mu(A_i) = \infty$ implica $a_i = 0$. Así, de nuevo para todo $p \in [1, \infty)$ se cumple que $\mathscr{G}_{p,k} = \mathscr{G}_{1,k} \subset \mathscr{G}_{\infty,k}$, y este último es el conjunto de todas las funciones simples medibles que toman como máximo k valores.

Recordamos algunas nociones de la teoría de la aproximación. Sea X un espacio de Banach y sea K un subconjunto cerrado de X. La *proyección métrica* sobre K es la aplicación multivaluada $P_K : X \rightrightarrows K$ definida por $P_K(x) = \{y \in K \ : \|x - y\| = d(x, K)\}$ (donde $d(A, B)$ es la distancia entre dos subconjuntos A y B de X). Si $P_K(x)$ no está vacío para todo $x \in X$, decimos que K es *proximinal*. Si $P_K(x)$ es un conjunto unitario para todas las $x \in X$, decimos que K es *Chebyshev*. La Sección 9.1 está dedicada a mostrar el siguiente resultado.

Teorema 0.0.17. Sea $(\Omega, \mathcal{F}, \mu)$ un espacio de medida y sea $p \in [1, +\infty]$. Entonces $\mathscr{G}_{p,k}$ es proximinal en $L^p(\Omega, \mathcal{F}, \mu)$ para todo $k \geq 1$.

En otras palabras, la distancia de una función f a $\mathscr{G}_{p,k}$ se alcanza en algún $g \in \mathscr{G}_{p,k}$. Obsérvese que la mayoría de los resultados clásicos sobre la existencia de una solución no pueden utilizarse en este caso, ya que obviamente $\mathscr{G}_{p,k}$ no es compacto en la topología fuerte, ni convexo, y como veremos, en general no es cerrado en la topología débil. La demostración de este resultado se divide en varios pasos. Primero tratamos el caso $p \in [1, \infty)$ y también demostramos que un mínimo puede ser elegido para tener una forma particular (ver Teorema 9.1.5 cuando μ es finito y el Teorema 9.1.12 si no lo es). La demostración es bastante técnica ya que tratamos con cualquier tipo de medida (no sólo finita o σ-finita). En el caso de que μ sea finita también damos condiciones para asegurar que hay un minimizador único (ver Teorema 9.1.10). En general, el conjunto de minimizadores no es un conjunto unitario, incluso puede existir un *continuum* de minimizadores. Entonces, tiene sentido estudiar si la proyección métrica $P_{\mathscr{G}}$ tiene una selección continua. En general, no hay selección continua, a menos que $L^p(\Omega, \mathcal{F}, \mu)$ sea de dimensión finita (ver Observación 9.1.21).

Cuando $p = \infty$, en la Proposición 9.1.14 demostramos que $\mathscr{G}_{\infty,k}$ es proximinal. Las pruebas que proporcionamos son de alguna manera constructivas, pero todavía hay un largo camino por recorrer para obtener algoritmos útiles, que en sí mismo, pensamos, será importante en muchas aplicaciones.

Un papel importante en esta sección lo desempeña $\mathcal{M}_p(f, A)$, la media p-ésima de f en un conjunto A (ver Definición 9.1.2). En particular, para $p = 2$, tenemos $\mathcal{M}_2(f, A) = \frac{1}{\mu(A)} \int_A f(x)\, d\mu(x)$. Una aproximación conocida asociada a una partición medible finita $\mathcal{P} = \{A_i\}_{1 \leq i \leq k}$ viene dada por:

$$\mathbb{E}^{\mathcal{P}}(f) = \sum_{i=1}^{k} \mathcal{M}_2(f, A_i)\mathbb{1}_{A_i},$$

y corresponde a la esperanza condicional de f con respecto a la σ-algebra generada por $\mathcal{P}$.

En la Sección 9.2, introducimos la p-variación $\mathrm{Var}_{p,k}(f)$ de una función $f \in L^p(\Omega, \mathcal{F}, \mu)$, para $p \in [1, \infty)$, y estudiamos algunas de sus propiedades. La p-variación de una función nos permite controlar la distancia de f a los conjuntos $\mathscr{G}_{p,k}$, bajo un factor 2 (ver Proposición 9.2.6). Esta noción será una herramienta útil para caracterizar la aproximabilidad uniforme de los conjuntos en la siguiente sección y cuya definición es la siguiente:

Definición 0.0.9. Sea $(\Omega, \mathcal{F}, \mu)$ un espacio de medida y sea $p \in [1, +\infty]$. Sea $\mathscr{A} \subset L^p(\Omega, \mathcal{F}, \mu)$. Para $\varepsilon > 0$, definimos

$$N_{p,\varepsilon}(\mathscr{A}) = \inf\{k \geq 1 \ : \ \forall f \in \mathscr{A}, \ \exists h \in \mathscr{G}_{p,k} \ \ \|f - h\|_p \leq \varepsilon\}.$$

Como siempre, si el conjunto donde se toma el ínfimo es vacío, definimos $N_{p,\varepsilon}(\mathscr{A}) = \infty$. Decimos que $\mathscr{A}$ es *uniformemente aproximable* (UA) en $L^p(\Omega, \mathcal{F}, \mu)$ si $N_{p,\varepsilon}(\mathscr{A}) < \infty$ para todo $\varepsilon > 0$.

Concretamente, un conjunto $\mathscr{A}$ es UA en $L^p(\Omega, \mathcal{F}, \mu)$ si para cualquier $\varepsilon > 0$ existe $k \geq 1$ tal que cualquier función en $\mathscr{A}$ puede ser ε-aproximada en $L^p(\Omega, \mathcal{F}, \mu)$ por funciones simples que toman menos de k valores diferentes. Obsérvese que $\mathscr{A}$ es UA si y sólo si

$$\lim_{k\to\infty} \sup_{f\in\mathscr{A}} \inf\{\|f - g\|_p : \ g \in \mathscr{G}_{p,k}\} = 0.$$

Señalamos que una cantidad similar conduce a la compacidad relativa de $\mathscr{A}$. En efecto, si $1 \leq p < \infty$, un resultado inspirado por M. Riesz (ver Teorema 4.7.28 en [26]) afirma que $K \subset L^p(\Omega, \mathcal{F}, \mu)$ es relativamente compacto si y sólo si K está acotado en $L^p(\Omega, \mathcal{F}, \mu)$ y

$$\inf_{\mathcal{P}} \sup_{f\in K} \|f - \mathbb{E}^{\mathcal{P}}(f)\|_p = 0.$$

Notar que para toda partición medible finita $\mathcal{P}$ con un máximo de k átomos, se cumple

$$\inf\{\|f - g\|_p : \ g \in \mathscr{G}_{p,k}\} \leq \|f - \mathbb{E}^{\mathcal{P}}(f)\|_p,$$

por lo que la compacidad relativa implica UA, un hecho que puede ser fácilmente demostrado directamente.

La última parte del trabajo, la Sección 9.3, está dedicada al estudio de los conjuntos uniformemente aproximables. Daremos algunos ejemplos de conjuntos UA y demostraremos que es una clase más grande que la de los conjuntos uniformemente integrables. También caracterizamos esta propiedad en términos de números de recubrimiento si $p = \infty$ y en términos de la p-variación si $p < \infty$. Los números de recubrimiento $\mathcal{N}(f, \varepsilon)$ de una función f se definen simplemente como el número de recubrimiento de su imagen, salvo un conjunto de medida 0. Demostraremos los dos resultados siguientes:

Teorema 0.0.18. Sea $(\Omega, \mathcal{F}, \mu)$ un espacio de medida y sea $\mathscr{A} \subset L^\infty(\Omega, \mathcal{F}, \mu)$. Las siguientes proposiciones son equivalentes:

(i) $\mathscr{A}$ es UA;

(ii) $\sup_{f\in\mathscr{A}} \mathcal{N}(f, \varepsilon) < \infty$ para todo $\varepsilon > 0$.

Teorema 0.0.19. Sean $(\Omega, \mathcal{F}, \mu)$ un espacio de medida, $p \in [1, \infty)$ y $\mathscr{A} \subset L^p(\Omega, \mathcal{F}, \mu)$. Las siguientes proposiciones son equivalentes:

(i) $\mathscr{A}$ es UA en $L^p(\Omega, \mathcal{F}, \mu)$;

(ii) $\lim\limits_{k\to\infty} \sup_{f\in\mathscr{A}} \mathrm{Var}_{p,k}(f) = 0$.

Luego investigamos cuando la bola unitaria de $L^p(\Omega, \mathcal{F}, \mu)$ es UA. Si $1 \leq p < \infty$, esto ocurre, como es de esperar, si y sólo si $L^p(\Omega, \mathcal{F}, \mu)$ es de dimensión finita (ver Teorema 9.3.10). Concluimos esta sección estableciendo algunas propiedades de estabilidad de la clase de conjuntos UA. En particular, un buen uso del tipo de Rademacher nos permite demostrar que si $\mathscr{A}$ es un conjunto UA acotado en $L^p(\Omega, \mathcal{F}, \mu)$ para $p \in (1, \infty)$ entonces su envoltura convexa cerrada también es UA (ver Teorema 9.3.15). Para más información sobre el tipo y cotipo de Rademacher, referimos el lector a [1] (capítulo 6).

Este capítulo es un preprint ([89]) que es consecuencia de una colaboración con J. San Martín.

Abstract

Most of this work focuses on the study of super weak compactness, which is a localized version of superreflexivity. Ultraproducts are also a central notion in this paper. We summarize the contents of this thesis.

Chapter 1: Some preliminary content

This first chapter contains the basic concepts used throughout this document. Most of the results presented in this section are given without proof. The few new results are proved.

We start briefly with the definition of filters and some of their elementary properties. These allow us to define the notion of ultraproducts of Banach spaces. We recall in particular that it is difficult to access the dual of an ultraproduct (except in the superreflexive case) and that the ultraproduct of an ultraproduct is also an ultraproduct. We then define the ultraproducts of bounded subsets of a Banach space and state fundamental properties that will be used repeatedly in this document.

We recall the definition of finite representability of Banach spaces and its characterization in terms of ultraproduct. We state fundamental results such as Dvoretsky's Theorem or the Principle of local reflexivity. We then extend the definition of finite representability to subsets of Banach spaces and obtain results similar to those concerning Banach spaces.

In the next part, we introduce the spreading models of a Banach space X, which have the particularity of being finitely representable in X. Some elementary properties are recalled and we state the results established by Beauzamy which link spreading models to Banach-Saks properties.

We end this section by defining the fundamental notion around which this work is articulated: *the super weak compactness* (SWC).

Definition 0.0.10. Let X be a Banach space and let $A \subset X$ be a bounded set. We say that A is *relatively super weakly compact* all of its ultraproducts are relatively weakly compact. If furthemore A is weakly closed, we say that A is *super weakly compact* (in short, SWC).

We establish well-known characterizations of super weakly compact sets which are similar to those of superreflexivity. We can define the *super weakly compact operators* (SWC) and we study some of their properties. We are also interested in the spaces generated by a SWC set since they have remarkable renorming properties.

Chapter 2: Uniformly convex functions

In this chapter, we introduce the notion of ε-uniformly convex function. We start with the definition of the main notion studied here:

Definition 0.0.11. Let $(X, \|.\|)$ be a Banach space and let $\varepsilon > 0$. A function $f : X \to \overline{\mathbb{R}}$ is said to be *ε-uniformly convex* if there is $\delta > 0$ such that

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} - \delta,$$

whenever $\|x - y\| \geq \varepsilon$. The function is said to be *uniformly convex* if it is ε-uniformly convex for all $\varepsilon > 0$.

It turns out that supporting a convex continuous ε-uniformly convex function is actually a geometrical-topological property of the domain. It is known that a Banach space admits a uniformly convex function bounded on bounded sets if and only if it is superreflexive. Raja proved in [147] that a closed convex bounded set admits a bounded continuous uniformly convex function if and only if it is SWC. Our techniques allow us to give a very precise quantitative version of the relation between containment of separated trees and supporting a uniformly convex function for a set.

Theorem 0.0.12. *Let C be a closed bounded convex subset of a Banach space X. Then these two numbers coincide:*

(ν_1) *the infimum of the $\varepsilon > 0$ such that there is a common bound for the heights of all the ε-separated dyadic trees;*

(ν_2) *the infimum of the $\varepsilon > 0$ such that there is a bounded ε-uniformly convex (and convex, Lipschitz. . .) function defined on C.*

It is well-known that convex SWC sets do not admit ε-separated dyadic trees of arbitrary height for all $\varepsilon > 0$ (see Theorem 1.5.12). Then the previous theorem can be seen as a quantification of the super weak compactness. This idea will be developed in the next chapter. In this chapter, further characterizations of the SWC proved by Raja will be obtained as consequences of these new results (see Corollaries 2.4.9 and 2.4.12).

In our desire to study super weak compactness, we have been led to establish results of independent interest on uniformly convex functions. Along this document, $\breve{f}$ will denote the lower semicontinuous convex envelope of a function f. The next result shows the global behaviour of ε-uniformly convex functions and the relative stability of minimizers by linear perturbations.

Theorem 0.0.13. *Let X Banach space and let $f : X \to \overline{\mathbb{R}}$ be an ε-uniformly convex function such that $\breve{f}$ is proper. Then f is bounded below and coercive, more precisely we have*

$$\liminf_{\|x\|\to+\infty} \frac{f(x)}{\|x\|^2} > 0.$$

Moreover, for any $\varepsilon' > \varepsilon$ there exist $\delta, \eta > 0$ such that if given $x_0^* \in X^*$ and $x_0 \in X$ with

$$f(x_0) + x_0^*(x_0) < \inf(f + x_0^*) + \delta,$$

and $x^* \in X^*$ such that $\|x^* - x_0^*\| < \eta$ and $x \in X$ that minimizes $f + x^*$, then $\|x - x_0\| \leq \varepsilon'$. The existence of such minimizer pair (x, x^*) is guaranteed if $f = \breve{f}$.

The proof of the former result relies in the possibility of "making convex" an ε-uniformly convex function without loosing the ε-uniformly convexity. We will say that a function f is *ε^+-uniformly convex* if it is ε'-uniformly convex for every $\varepsilon' > \varepsilon$. We have the following result.

Theorem 0.0.14. *Let X Banach space and let $f : X \to \overline{\mathbb{R}}$ be an ε-uniformly convex function such that $\breve{f}$ is proper. Then $\breve{f}$ is ε^+-uniformly convex.*

Simple examples, such as Example 2.1.7, show that the ε-uniformly convexity of f does not guarantee that $\bar{f}$ would be proper. In order to fulfil that requirement in terms of f, we direct the reader to Corollary 2.4.2. Suppose now that we already have a proper lower semicontinuous convex and ε-uniformly convex function f. We wonder if we could "upgrade" f to a new function sharing those properties and, besides, being locally Lipschitz (global Lipschitzness is not allowed for uniformly convex functions). In that sense, we have the following result.

Theorem 0.0.15. *Let $(X, \|\cdot\|)$ be a Banach space and let $f \in \Gamma(X)$ be ε-uniformly convex. Then there exists an equivalent norm $|\!|\!|\cdot|\!|\!|$ on X such that the function $x \mapsto |\!|\!|x|\!|\!|^2$ is ε^+-uniformly convex on the subsets of* $\operatorname{dom}(f)$ *where f is bounded above. Moreover, the norm $|\!|\!|\cdot|\!|\!|$ can be taken as close to $\|\cdot\|$ as we wish.*

The structure of this chapter is the following. Section 2.1 deals with basic properties of ε-uniformly convex and ε-uniformly quasi-convex functions, mostly under the hypothesis of convexity. A few examples are given to show that the definitions do not guarantee some additional nice properties. Section 2.2 is devoted to the proof of Theorem 0.0.14 that will allow the reduction to the convex case of other results. The construction of uniformly convex functions from scratch (trees and sets) is done in Section 2.3. Section 2.4 treats general properties of ε-uniformly convex functions and the possibility of adding more properties like Lipschitzness or homogeneity (renorming). We also prove an estimation of the approximation by differences of convex functions. In the following section we will sketch an understandable proof of Enflo's uniformly convex renorming of superreflexive spaces theorem based on the ideas exposed along the paper. The last section is dedicated to the uniform smoothness, which is the dual property of the uniform convexity.

The results presented in this chapter are the consequence of a collaboration with M. Raja and can be found in [88].

Chapter 3: Quantification of the super weak compactness

Super weak compactness is more widespread than it may appear. For instance, any weakly compact operator with range $L_1(\mu)$ (μ any measure) or domain $C(K)$ (K any Hausdorff compact) is SWC, see [122, Proposition 6.1]. Actually, some results in Banach space theory could be understood in terms of super weak compactness. As for instance, the classic Szlenk result establishing that a weakly convergent sequence in $L_1(\mu)$ has a subsequence whose Cesàro means converge (to the same limit) is a consequence of two facts: the weakly compact subsets of $L_1(\mu)$ are SWC; and the SWC sets have the Banach-Saks property [122, Corollary 6.3].

The aim of this chapter is to show that, actually, super weak compactness and, particularly, its quantification, may cast light on the structure of the subspaces of Hilbert generated Banach spaces. Indeed, we have realized that several "technical hypotheses" in papers of Troyanski [162], Argyros and Farmaki [9], and the series by Fabian, Godefroy, Hájek, Montesinos and Zizler [71, 66, 65, 70] on the structure of Hilbert generated spaces and uniformly Gâteaux renorming, can be understood in terms of a quantified version of super weak compactness.

We introduce a measure of super weak noncompactness Γ. Let A be a bounded subset of a Banach space X, then take

$$\gamma(A) = \inf\{\varepsilon > 0 : \overline{A}^{w^*} \subset X + \varepsilon B_{X^{**}}\}.$$

We have that a set A is relatively weakly compact if and only if $\gamma(A) = 0$. This measure has been studied in [69, 84, 42], see also [91, Section 3.6], and there are several measures of weak noncompactness that turn out to be equivalent [7]. If $\mathcal{U}$ is a free ultrafilter, we define Γ by

$$\Gamma(A) := \gamma(A_{\mathcal{U}}).$$

We will see that the choice of $\mathcal{U}$ does not matter. We obtained the following quantification result:

Theorem 0.0.16. *Let C be a bounded closed convex subset of a Banach space X. Consider the following numbers:*

(μ_1) *the supremum of the numbers $\varepsilon > 0$ such that for any $n \in \mathbb{N}$ there are $x_1, \dots, x_n \in C$ such that $\mathrm{d}(\mathrm{conv}\{x_1, \dots, x_k\}, \mathrm{conv}\{x_{k+1}, \dots, x_n\}) \geq \varepsilon$ for all $k = 1, \dots, n-1$;*

(μ_2) *the supremum of the $\varepsilon > 0$ such that there are ε-separated dyadic trees of arbitrary height;*

(μ_3) $= \Delta(C_{\mathcal{U}})$, *for $\mathcal{U}$ a free ultrafilter on $\mathbb{N}$;*

(μ_4) $= \Gamma(C)$, *(computed for an arbitrary free ultrafilter $\mathcal{U}$ on $\mathbb{N}$);*

(μ_5) *the infimum of the $\varepsilon > 0$ such that $\mathrm{Dz}(C, \varepsilon) < \omega$;*

(μ_6) *the infimum of the $\varepsilon > 0$ such that C supports a convex bounded ε-uniformly convex function.*

Then $\mu_1 \leq \mu_2 \leq 2\mu_3 \leq 2\mu_4 \leq 2\mu_1$ and $\mu_4 \leq 2\mu_5 \leq 2\mu_6 \leq 2\mu_2$.

We can also quantify the super weak compactness for operators. In fact, if $T : X \to Y$ is a bounded operator, we can define $\Gamma(T) = \Gamma(T(B_X))$. In particular, we prove a quantified version of the symmetry of the bideal $\mathfrak{W}^{super}$ (the set of SWC operators), as well as a quantified version of Beauzamy's renorming to make uniformly convex a super weakly compact operator.

In the last part, we establish the super-version of the following result:

Theorem 0.0.17 ([22, 68])**.** *For a Banach space X the following statements are equivalent:*

(i) X is a subspace of a WCG space;

(ii) (B_{X^*}, w^*) is an Eberlein compact;

(iii) For every $\varepsilon > 0$ there are sets $(A_n^\varepsilon)_n$ such that $B_X = \bigcup_{n=1}^\infty A_n^\varepsilon$ and $\gamma(A_n^\varepsilon) < \varepsilon$.

The equivalence (i)⇔(ii) is due to Benyamini, Rudin and Wage [22]. The inner characterization (iii) was obtained by Fabian, Montesinos and Zizler [68]. Recall that WCG stands for *weakly compactly generated*, that is, a Banach space that contains a weakly compact subset whose linear span is dense. Thanks to the celebrated interpolation result of Davis, Figiel, Johnson and Pełczyński [56] (see also [67, Theorem 13.22]), a Banach space X is WCG if and only if there exists a reflexive space Z and an operator $T : Z \to X$ with dense range. Moreover, if the space Z can be taken a Hilbert space, we say that X is *Hilbert generated*. The name *Eberlein* applies to the compact spaces which are homeomorphic to a weakly compact set of a Banach space. It is well known after Amir and Lindenstrauss (see [67, Corollary 13.17], for instance) that an Eberlein compact embeds as a weakly (equivalent, bounded and pointwise) compact subset of $c_0(I)$ for I large enough. If such an embedding can be done into a Hilbert space $\ell_2(I)$, then the compact is said to be *uniformly Eberlein*. Note that the third statement in Theorem 0.0.17 is actually an internal characterization as it is written in terms of the space X, not an over-space or its dual. We will see that the computation of γ can be done without appealing to the bidual space (see Proposition 3.1.2).

The main result of this chapter is the following. Please note the parallelism with the previous theorem.

Theorem 0.0.18. *Let X be a Banach space. The following statements are equivalent:*

(i) X is a subspace of a Hilbert generated space;

(ii) (B_{X^*}, w^*) is a uniform Eberlein compact;

(iii) For every $\varepsilon > 0$ there are sets $(B_n^\varepsilon)_n$ such that $B_X = \bigcup_{n=1}^\infty B_n^\varepsilon$ and $\Gamma(B_n^\varepsilon) < \varepsilon$.

This chapter is structured as follows. Section 3.1 is devoted to prove Theorem 0.0.16 and the fact that Γ defines a measure of noncompactness (according to Definition 3.1.1). In Section 3.2, we quantify the uniform convexity of an operator. In the last part, we prove Theorem 0.0.18. We conclude this chapter with a characterization of super weakly compactly generated spaces (SWCG), in particular in the case of $\mathcal{C}(K)$-spaces, and an application of the previous results to Jordan algebras.

Most of the results presented in this chapter can be found in [87] (see also Section 6 in [88]) and have been established with M. Raja.

Chapter 4: New results on super weak compactness

Previous works on uniformly Gâteaux renorming by Fabian, Godefroy, Hájek and Zizler [65], as well as early results by Troyanski [162], unawarely contain estimations of Γ (see Chapter 3). The explanation will come through the following result.

Proposition 0.0.19. *Let A a bounded subset of a Banach space X and consider the two following numbers:*

(ε_1) *is the infimum of the $\varepsilon > 0$ such that there is $n_1 \in \mathbb{N}$ such that for every $x^* \in B_{X^*}$ then*

$$|\{x \in A : |x^*(x)| > \varepsilon\}| \leq n_1;$$

(ε_2) *is the infimum of the $\varepsilon > 0$ such that there is $n_2 \in \mathbb{N}$ such that for any finite set $B \subset A$ with $|B| \geq n_2$ then*

$$\left\| \frac{1}{|B|} \sum_{x \in B} x \right\| < \varepsilon.$$

Then $\varepsilon_1 = \varepsilon_2$ and in such a case $\Gamma(A) \leq \varepsilon_1$.

The sets satisfying the statements of Proposition 4.1.3 with $\varepsilon_1 = \varepsilon_2 = 0$ will be called *uniformly weakly null* sets. Note that a uniformly weakly null set becomes SWC by adding $\{0\}$. Together with unit balls of superreflexive spaces, uniformly weakly null sets are the most prototypical examples of SWC sets. The first part deals with properties of uniformly weakly null set. As we will see, SWC sets with some reasonable discreteness assumption are uniformly weakly null. Note that the second statement (ε_2) is a sort of uniform Banach-Saks property (with unique limit 0). That will allow us to apply results of infinite combinatorics, such as the Erdös-Magidor [63] and Mercourakis [134] selections. We investigate when a Schauder basis is uniformly weakly null, and the relation of uniformly weakly null sets with the representation of uniformly Eberlein compacts.

The second part of this chapter is dedicated to ergodicity and fixed points properties of SWC sets. In [50], the authors proved that a closed bounded convex subset K of a Banach space X is SWC if and only if it has the super fixed point property for affine isometries $T : X \to X$ preserving K. In this part, we improve this characterization in two different ways. On the one hand, we prove the following result:

Theorem 0.0.20. *Let C be a closed convex subset of a Banach space X. The following assertions are equivalent:*

(i) C is SWC;

(ii) C is super-ergodic.

This result is proved studying the fixed points of the shift of a spreading model (see Proposition 4.2.5). Then the fixed point properties of K are easily deduced from an adaptation of the classical mean ergodic Theorem (see Theorem 4.2.3). We also note that a SWC set K may enjoy stronger fixed points properties since the underlying space can be renormed such K has normal structure (see Theorem 1.5.33). On the other hand, the main drawback of the characterization given in [50] is that it can exist affine isometries defined on K which can not be extended to the all space. Moreover, the super weak compactness is a localized version of superreflexivity and it is natural to expect that such a characterization only depends on K. This is done in Theorem 4.2.11. We apply some of these results to *strongly super weakly compactly generated* Banach spaces (see Definition 1.5.30). To end this second part of the chapter, we prove that if a Banach space X has the *M-(FPP for affine isometries)* (i.e. all the spreading models of X have the fixed point property for affine isometries) then X is reflexive. More precisely, we obtain:

Theorem 0.0.21. *The M-(FPP for affine isometries) implies the Banach-Saks property.*

It follows that the reflexivity strictly lies between the fixed point property and the M-(fixed point property) (see Definition 4.2.18).

The first part of this chapter comes from [87]. The results of the second part are part of a preprint (see [85]).

Chapter 5: Uniform Banach-Saks properties

Superreflexivity and its localized version, the super weak compactness, are strongly related to the Banach-Saks property. In fact, the super weak compactness is equivalent to the super-Banach-Saks property by Theorem 1.5.15. In this section, we are interested in the uniform Banach-Saks properties, meaning that we try to control the speed of convergence of a Cesaro convergent subsequence of a bounded sequence. Many results in this chapter are inspired by the work of Beauzamy and his use of spreading models for the characterization of Banach-Saks properties (see Chapter 1). The main concept of this chapter is presented in the following definition. The other Banach-Saks properties that will be studied are variations of this definition.

Definition 0.0.22. Let X be a Banach space. We say that X has the *uniform weak Banach-Saks property* (in short uniform WBS) if there exists a sequence $(a_n)_n$ of real positive numbers such that $a_n \to 0$ and for all $(x_n)_n \subset B_X$ such that $x_n \xrightarrow{w} 0$, there is a subsequence $(x'_n)_n$ of $(x_n)_n$ such that

$$\frac{1}{m}\left\|\sum_{k=1}^{m} x'_k\right\| \leq a_m$$

for all $m \geq 1$.

Section 5.1 is dedicated to the study of the uniform WBS. In the literature (see [151, 148]), a special case of the uniform WBS appears when a_n is of the form $a_n = Cn^{\frac{1}{p}-1}$ for some $C > 0$ and $p > 1$. In that case, we say that X has the strong p-weak Banach-Saks property (in short, strong p-WBS). In [148], Rakov proved that if X has the uniform WBS then there exists some $p > 1$ such that X has the strong p-WBS. This result will be established here with another proof in

Subsection 5.1.2 (see Theorem 5.1.9). The main idea is to prove that the weak-(A_∞), introduced by Partington in [139], and the uniform WBS are equivalent. This is done in Subsection 5.1.1.

Note that the constant C present is the definition of the strong p-WBS does not depend on the choice of the weakly null subsequence. We introduce the p-weak Banach-Saks property (in short, p-WBS) by releasing this independance. Apart from a few special cases, it seemed unknown if both notions coincide. In Subsection 5.1.3, we prove in full generality that the p-WBS coincide with the strong p-WBS when $p \in (1, \infty)$ (see Theorem 5.1.14). To sum up, we obtain the following result:

Theorem 0.0.23. *Let X be a Banach space. The following assertions are equivalent:*

(i) X has the uniform WBS;

(ii) X has the weak-(A_∞);

(iii) X has the strong p-WBS for some $p \in (1, +\infty)$;

(iv) X has the p-WBS for some $p \in (1, +\infty)$.

The equivalence between the strong ∞-Banach-Saks property and the ∞-Banach-Saks property is also true but can not be proved with the same argument. It will be established using Ramsey's theory and a powerful characterization of the hereditary Dunford-Pettis property (see Theorem 3.1 in [117]).

If X has the uniform WBS, it is interesting to find the greatest p such that X has the p-WBS. That is why we introduce the following definition:

Definition 0.0.24. Let X be a Banach space. We define the *uniform WBS-index* $\mathrm{UWBS}(X)$ of X as the supremum of the $p > 1$ such that X has the (strong) p-WBS. If X does not have the (strong) p-WBS for any $p > 1$, we set $\mathrm{UWBS}(X) = 1$.

We also introduce the following index, which caracterizes the properties (A_k) introduced by Partington:

Definition 0.0.25. Let X be a Banach space. For all $m \geq 1$, we define $\phi_X(m)$ as the infimum of the b such that for all $(x_n)_n \subset B_X$ with $x_n \xrightarrow{w} 0$, there exist $p_1 < p_2 < ... < p_m$ such that

$$\frac{1}{m}\left\|\sum_{i=1}^{m} x_{p_i}\right\| \leq b.$$

In Subsection 5.1.4, we establish one of our main result:

Theorem 0.0.26. *Let X be a Banach space.*

(a) If $\phi_X(n) > \frac{1}{n}$ for all $n \geq 2$, then

$$UWBS(X) = \sup_{n \geq 2} \frac{\ln(n)}{\ln(n\phi_X(n))}.$$

(b) If $\phi_X(n) \leq \frac{1}{n}$ for some $n \geq 2$, then

$$UWBS(X) = +\infty.$$

The previous formula has been established by Rakov in [148] by a different method, but without any hypothesis on ϕ_X. However, it is easy to see that this formula is not true if we do not make the assumption on ϕ_X (see Proposition 5.1.27).

In Section 5.2, we introduce the *uniform Banach-Saks property* (in short, uniform BS), which is basically obtained by adding the reflexivity to the uniform WBS. Many examples of spaces enjoying the uniform BS are given such that (β)-spaces, k-NUCε spaces or NUS spaces. The *uniform alternating weak Banach-Saks property* (in short, uniform WABS) is introduced in Section 5.3. It turns out that the uniform WABS and the uniform WBS are equivalent (see Theorem 5.3.6). As an application, we obtain that any Banach space with non-trivial type p has the (strong) p-WBS. In the last section, we introduce the *uniform alternating Banach-Saks property* (in short, uniform ABS) and we study its links with the other uniform Banach-Saks properties. We conclude this chapter with an application to symmetric Kottman's constants.

The results of this chapter are part of a preprint (see [86]).

Chapter 6: Extremal structure in ultraproducts

The ultraproduct of Banach spaces has shown to be a very useful tool in the study of local properties of Banach spaces. For instance, in [1, Theorem 11.1.4] ultraproducts are used in order to prove that a Banach space X fails to have type $p > 1$ if and only if ℓ_1 is finitely representable in X. This link between the local structure of a Banach space X and the global one of its ultraproducts $X_{\mathcal{U}}$ has allowed us to obtain structural results in Banach spaces.

More recent studies about the geometry of ultraproduct Banach spaces can be found in [94] for octahedral and almost square Banach spaces or in [25, 110] for the Daugavet property. Actually, the example of the Daugavet property is paradigmatic of two basic facts that, more often than not, appear when dealing with a geometric property in Banach spaces. The first one is that, when requiring an ultraproduct $X_{\mathcal{U}}$ to enjoy a geometric property, one has to look for a "uniform version" of this geometric property in X (this happens for instance with the Daugavet property and the uniform Daugavet property [25, Theorem 6.4], for the strict convexity and uniform convexity or for the reflexivity and superreflexivity [97, Proposition 6.4]). The second one is that one should avoid as much as possible to deal with the dual of an ultraproduct space (this is done in [25] by using their Theorem 6.2). The reason is that, in most of the cases (i.e. out of superreflexive Banach spaces [97, Corollary 7.2]), there is not good access to the dual of $X_{\mathcal{U}}$.

Taking the above two facts in mind, the aim of this chapter is to study the extremal structure of subsets of an ultraproduct. This structure codifies much information of bounded convex sets (we can think for instance in Krein-Milman theorems) and it is extremely useful in other areas of the Functional Analysis like the norm-attainment (see [41, 126]). In the particular case of the extremal structure of ultraproducts, it has been previously considered by J. Talponen in [159], where the author studied the properties that link a point x of the unit sphere of a Banach space and its image $\mathcal{J}(x)$ in the ultraproduct through the canonical isometry. Some of his results will be generalised in this document since we deal with more general sets (not only with the unit ball) and more general ultrafilters (not only on $\mathbb{N}$). We also establish how the properties on the x_i's are transferred to $(x_i)_{\mathcal{U}}$ and reciprocally. Then the results linking x and $\mathcal{J}(x)$ are obtained as a particular case.

The first three sections contain our main results in complete generality. After providing a number of examples that suggests which properties we need to look for on X, we establish several stability results concerning the extremal structure. For instance, we extend Talponen's result in Theorem 6.1.5 showing that x is a strongly extreme point of a bounded convex set C if and only if $\mathcal{J}(x)$ is a (strongly) extreme point of its ultraproduct $C_{\mathcal{U}}$. Moreover, we show that extreme and strongly extreme points of $C_{\mathcal{U}}$ coincide under mild assumptions on $\mathcal{U}$, that is:

Theorem 0.0.27. *Let C be a bounded convex subset of a Banach space X, $\mathcal{U}$ be a CI ultrafilter on an infinite set I. Then $\operatorname{ext}(C_{\mathcal{U}}) = \text{str-ext}(C_{\mathcal{U}})$.*

We also characterise in Theorem 6.1.7 elements $(x_i)_{i\in I} \in C^I$ giving that $(x_i)_{\mathcal{U}}$ is an extreme point of $C_{\mathcal{U}}$ for every free ultrafilter $\mathcal{U}$ on I. In the context of denting points (respectively strongly exposed points) we prove that $(x_i)_{\mathcal{U}} \in C_{\mathcal{U}}$ is a denting point (respectively strongly exposed point) if $\{x_i\}$ satisfy a "uniform denting condition" (respectively a uniform condition of strong exposition), see Theorems 6.2.2 and 6.3.2. Finally, we prove that every element of $C_{\mathcal{U}}$ which is exposed by a functional in $(X^*)_{\mathcal{U}}$ is in fact strongly exposed under mild assumptions on $\mathcal{U}$:

Theorem 0.0.28. *Let C be a bounded convex subset of a Banach space X, and $\mathcal{U}$ be a CI ultrafilter on an infinite set I. Then $\exp_{(X^*)_{\mathcal{U}}}(C_{\mathcal{U}}) = \text{str-exp}_{(X^*)_{\mathcal{U}}}(C_{\mathcal{U}})$.*

In the last part, we take a closer look at SWC convex sets where we expect a nice behaviour of the extremal structure. We also investigate the *uniformly convex sets*, which is a particular case of SWC sets. The main tool in this study is that if C is uniformly convex then $C_{\mathcal{U}}$ too (see Proposition 6.4.9). The aim of this section is to recover as much as possible the extremal properties of the unit ball of a uniformly convex Banach space. The biggest difficulty is that a uniformly convex set can have empty interior. However, we prove that any extreme point of such a set is denting and any exposed point is strongly exposed (Proposition 6.4.14). We also characterise the extreme points of its ultraproduct set (Theorem 6.4.13).

The results presented here have been established with L. García-Lirola and A. Rueda Zoca and can be found in [77].

Chapter 7: Ultraproducts in Lipschitz-free spaces

In this chapter, we consider the notion of ultraproduct of metric spaces (which is a generalization of the corresponding one for Banach spaces). We apply it to obtain an ultraproduct characterization of the metric spaces that are finitely Lipschitz representable (in the sense introduced by Lee, Naor and Peres [123]) in a Banach space. Also, we analyze the relation between finite Lipschitz representability of metric spaces and finite representability of the corresponding Lipschitz-free spaces. These spaces (also called Arens-Eells spaces and transportation cost spaces) have become a very active research topic due to their applications in Non-Linear Analysis [81], as well as Computer Science and Optimal Transport.

In the first section, we introduce the fundamental properties of ultraproducts of metric spaces and Lipschitz-free spaces used in this document. In the following section, we prove that, given a metric space M and an ultrafilter $\mathcal{U}$, the Lipschitz-free space on the ultraproduct of M, $\mathcal{F}(M_{\mathcal{U}})$, is linearly isometric to a subspace of the ultraproduct of the Lipschitz-free space, $\mathcal{F}(M)_{\mathcal{U}}$. In fact, we have:

Theorem 0.0.29. *Let $\mathcal{U}$ be an ultrafilter on a set I and let $(M_i)_{i\in I}$ be a family of metric spaces. Then $\mathcal{F}((M_i)_{\mathcal{U}})$ is linearly isometric to $\overline{\operatorname{span}(\delta(M_i)_{\mathcal{U}})} \subset \mathcal{F}(M_i)_{\mathcal{U}}$.*

In particular, $\mathcal{F}(M_{\mathcal{U}})$ is finitely representable in $\mathcal{F}(M)$. Also, we prove that a metric space M is finitely Lipschitz representable into a Banach space X if and only if M biLipschitz embeds in an ultraproduct of X (Theorem 7.3.6). This is done in Section 7.3 where we link the ultraproducts of metric spaces with the notion of finite representability. As a consequence we obtain the following result:

Theorem 0.0.30. *Let M be a metric space and X be a Banach space. Assume that M is finitely λ-Lipschitz representable in X. Then $\mathcal{F}(M)$ is λ-finitely representable in $\mathcal{F}(X)$.*

This result has some consequences on the cotype of Lipschitz-free spaces that can be found in Section 7.3. For instance, the following dichotomy holds: either $\mathcal{F}(\ell_2)$ has non-trivial cotype or $\mathcal{F}(X)$ does not have cotype for any infinite-dimensional Banach space X. Finally, although several classes of Banach spaces (as Banach lattices, C*-algebras, and $\mathcal{C}(K)$-spaces) are known to be stable by ultraproducts, we show that $(\mathcal{F}(M))_{\mathcal{U}}$ is not isomorphic to any Lipschitz-free space whenever M is an infinite metric space and $\mathcal{U}$ is countably incomplete. In Section 7.5, we then compare the stability of $\mathrm{Lip}_0(K)$ and $\mathcal{C}(K)$ under ultraproducts and remark some similarities and differences between them.

This chapter is based on [76] and is the consequence of a work with L. García-Lirola.

Chapter 8: A note on non-separable Lipschitz-free spaces

In the first section, we prove that several classical Banach space properties are equivalent to separability for the class of Lipschitz-free spaces, including Corson's property $(\mathcal{C})$, being a Gâteaux differentiability space, or the Countable Separation Property. On the other hand, we single out properties, more general than the previously mentioned where the equivalence with separability in class of Lipschitz-free spaces fails. In particular, the question whether non-separable Lipschitz-free spaces can have a weak* sequentially compact dual ball is undecidable. In fact, if we denote by $\mathfrak{s}$ the *splitting cardinal* we have that:

Theorem 0.0.31. *Let M be a complete metric space. Then the following assertions are equivalent:*

(i) $\mathrm{dens}(M) \geq \mathfrak{s}$,

(ii) M contains a uniformly discrete subset of cardinality $\mathfrak{s}$,

(iii) $B_{\mathrm{Lip}_0(M)}$ is not w^*-sequentially compact.

In particular, $B_{\ell_\infty(\Gamma)}$ is w^*-sequentially compact if and only if $|\Gamma| < \mathfrak{s}$. Therefore it is undecidable whether $B_{\ell_\infty(\omega_1)}$ is w^*-sequentially compact.

Finally, in the second section, we provide an example of a non-separable dual Lipschitz-free space that fails the Radon-Nikodým property. This is example is based on the following proposition asserting that the space $\mathcal{M}(S)$ of Radon measures on a Polish space S is a Lipschitz-free space:

Proposition 0.0.32. *Let S be a Polish space. Then $\mathcal{M}(S)$ is linearly isometric to $\mathcal{F}(M)$ for some metric space M. Specifically*

$$\mathcal{M}(S) \cong \ell_1(|S|) \oplus_1 \left(\bigoplus_{\kappa} L_1 \right)_1 \cong \mathcal{F}(M)$$

for some cardinal $\kappa \leq \mathfrak{c}$, and M is the metric sum of $|S|$ two-point spaces and κ copies of $[0,1]$.

This little chapter is part of a ongoing work with R. Aliaga and A. Procházka.

Chapter 9: Proximinality and uniformly approximable sets in L^p

This chapter is totally independant from the previous ones. However, this work comes from the study of superreflexive spaces. Indeed, it is well known that if X is superreflexive then so is $L^2(X)$. This follows directly from Enflo's Theorem since $L^2(X)$ is uniformly convex if X is. This transition to $L^2(X)$ is fundamental in the proof of Pisier's Theorem and, to date, there is no proof of this theorem that does not require Enflo's Theorem. The results in this chapter come from the author's questioning of why $L^2(X)$ is superreflexive if X is superreflexive without going through Enflo's theorem.

In this chapter we study the approximation of measurable functions by simple functions taking at most k values, for $k \in \mathbb{N}$. This problem has important consequences in multiple applications, where for example, one seeks for reduction of dimensionality, among many others. For example, the embedding of metric spaces into finite-dimensional normed spaces with small dimension is one of the main issue in non-linear analysis (see [32, 108, 131]). These results have deep consequences in order to design approximation algorithms, for instance for the Sparsest Cut problem (see [10]). When we aproximate a given function $f \in L^p(\Omega, \mathcal{F}, \mu)$ by simple functions, the number of terms in those approximations growths to infinity in general. Here, a main concern is what we can say if we restrict the number of terms in the approximations. In particular, what we can say about subsets of $L^p(\Omega, \mathcal{F}, \mu)$ that can be uniformly approximated by simple functions taking k values, as k growth to ∞. As we shall see, this new concept is more general than uniform integrability or compactness, and we fully characterize it in terms of a new measure of variation defined for functions in $L^p(\Omega, \mathcal{F}, \mu)$ for $p \in [1, \infty)$, and in terms of covering numbers in the case of $p = \infty$.

Let us fix some notations we need to explain the main results of this paper. Consider $(\Omega, \mathcal{F}, \mu)$ a measure space. For any $k \geq 1$, we denote by $\mathscr{G}_{p,k}(\Omega, \mathcal{F}, \mu)$, or simply $\mathscr{G}_{p,k}$ when the measure space $(\Omega, \mathcal{F}, \mu)$ is clear from the context, the set of simple functions given by

$$\mathscr{G}_{p,k} = \left\{ \sum_{i=1}^{l} a_i \mathbb{1}_{A_i} \in L^p(\Omega, \mathcal{F}, \mu) : \ \{A_i\}_{1 \leq i \leq l} \text{ measurable partition of } \Omega, \ a_i \in \mathbb{R} \text{ for all } i,\, l \leq k \right\}.$$

Remark 0.0.33. Assume μ is a finite measure. Then $\mathscr{G}_{p,k} = \mathscr{G}_{1,k}$ for all $p \in [1, \infty]$ is just the set of simple measurable functions that takes at most k values. If μ is an infinite measure, then $h = \sum_{i=1}^{k} a_i \mathbb{1}_{A_i}$, where $\{A_i\}_{1 \leq i \leq k}$ is a measurable partition, belongs to $\mathscr{G}_{p,k}$, for $p \in [1, \infty)$, if and only if $\mu(A_i) = \infty$ implies $a_i = 0$. So, again for all $p \in [1, \infty)$ it holds that $\mathscr{G}_{p,k} = \mathscr{G}_{1,k} \subset \mathscr{G}_{\infty,k}$, and the latter is the set of all simple measurable functions that takes at most k values.

We recall some notions from approximation theory. Let X be a Banach space and let K be a closed subset of X. The *metric projection* on K is the multi-valued mapping $P_K : X \rightrightarrows K$ defined by $P_K(x) = \{y \in K \ : \ \|x - y\| = d(x, K)\}$ (where $d(A, B)$ is the distance between two subsets A and B of X). If $P_K(x)$ is not empty for all $x \in X$, we say that K is *proximinal*. If $P_K(x)$ is a singleton for all $x \in X$, we say that K is *Chebyshev*. Section 9.1 is devoted to show the following result.

Theorem 0.0.34. *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $p \in [1, +\infty]$. Then $\mathscr{G}_{p,k}$ is proximinal in $L^p(\Omega, \mathcal{F}, \mu)$ for all $k \geq 1$.*

In other words, the distance of a function f to $\mathscr{G}_{p,k}$ is attained at some $g \in \mathscr{G}_{p,k}$. Note that most of the classical results on the existence of a solution cannot be used in this case since $\mathscr{G}_{p,k}$ is obviously not compact in the strong topology, nor convex, and as we will see, it is not closed in the weak topology, in general. The proof of this result is divided into several steps. We first deal with the case $p \in [1,\infty)$ and we also prove that a minimum can be chosen to have a particular form (see Theorem 9.1.5 when μ is finite and Theorem 9.1.12 if not). The proof is rather technical since we deal with any kind of measure (not only finite or σ-finite). In case μ is finite we also give conditions to ensure that there is a unique minimizer (see Theorem 9.1.10). In general, the set of minimizers is not a singleton, it can even exists a continuum of minimizers. Then, it makes sense to study if the metric projection $P_{\mathscr{G}_{p,k}}$ has a continuous selection. In general, there is no continuous selection, unless $L^p(\Omega,\mathcal{F},\mu)$ is finite dimensional (see Remark 9.1.21).

When $p=\infty$, in Proposition 9.1.14 we prove that $\mathscr{G}_{\infty,k}$ is proximinal. The proofs we provide are somehow constructive in nature, but still there is a long way to go for obtaining useful algorithms, which in itself, we think, will be important in many applications.

An important role in this section is played by $\mathcal{M}_p(f,A)$, the p-th mean of f on a set A (see Definition 9.1.2). In particular, for $p=2$ we have $\mathcal{M}_2(f,A)=\frac{1}{\mu(A)}\int_A f(x)\,d\mu(x)$. A well-known approximation associated to a finite measurable partition $\mathcal{P}=\{A_i\}_{1\le i\le k}$ is given by

$$\mathbb{E}^{\mathcal{P}}(f)=\sum_{i=1}^{k}\ \mathcal{M}_2(f,A_i)\mathbb{1}_{A_i},$$

which corresponds to the conditional expectation of f over the σ-field generated by $\mathcal{P}$.

In Section 9.2, we introduce the p-variation $\mathrm{Var}_{p,k}(f)$ of a function $f\in L^p(\Omega,\mathcal{F},\mu)$, for $p\in[1,\infty)$, and we studied some of its properties. The p-variation of a function allow us to control the distance of f to the sets $\mathscr{G}_{p,k}$, up to a factor of 2 (see Proposition 9.2.6). This notion will be a useful tool to characterize the uniform approximability of sets in the following section and whose definition is the following:

Definition 0.0.35. Let $(\Omega,\mathcal{F},\mu)$ be a measure space and $p\in[1,+\infty]$. Let $\mathscr{A}\subset L^p(\Omega,\mathcal{F},\mu)$. For $\varepsilon>0$, we define

$$N_{p,\varepsilon}(\mathscr{A})=\inf\{k\ge 1\ :\ \forall f\in\mathscr{A},\ \exists h\in\mathscr{G}_{p,k}\ \ \|f-h\|_p\le\varepsilon\}.$$

As usual if the set where the infimum is taken is empty we set $N_{p,\varepsilon}(\mathscr{A})=\infty$. We say that $\mathscr{A}$ is *uniformly approximable* (in short UA) in $L^p(\Omega,\mathcal{F},\mu)$ if $N_{p,\varepsilon}(\mathscr{A})<\infty$ for any $\varepsilon>0$.

Concretely a set $\mathscr{A}$ is UA in $L^p(\Omega,\mathcal{F},\mu)$ if for any $\varepsilon>0$ there exists $k\ge 1$ such that any function in $\mathscr{A}$ can be ε-approximated in $L^p(\Omega,\mathcal{F},\mu)$ by simple functions taking less than k different values. Notice that $\mathscr{A}$ is UA if and only if

$$\lim_{k\to\infty}\sup_{f\in\mathscr{A}}\inf\{\|f-g\|_p:\ g\in\mathscr{G}_{p,k}\}=0.$$

We point out that a similar quantity leads to relatively compactness of $\mathscr{A}$. Indeed, if $1\le p<\infty$, a result inspired by M. Riesz (see Theorem 4.7.28 in [26]) says that $K\subset L^p(\Omega,\mathcal{F},\mu)$ is relatively compact if and only if K is bounded in $L^p(\Omega,\mathcal{F},\mu)$ and

$$\inf_{\mathcal{P}}\sup_{f\in K}\|f-\mathbb{E}^{\mathcal{P}}(f)\|_p=0.$$

We point out that for every finite measurable partition $\mathcal{P}$, with at most k atoms it holds

$$\inf\{\|f-g\|_p:\ g\in\mathscr{G}_{p,k}\}\le\|f-\mathbb{E}^{\mathcal{P}}(f)\|_p,$$

so relatively compactness implies UA, a fact that can be easily proved directly.

The last part of the paper, Section 9.3, is dedicated to the study of uniformly approximable sets. We will give some examples of UA sets and prove that it is a larger class than the class of uniformly integrable sets. We also characterize this property in terms of covering numbers if $p = \infty$ and in terms of the p-variation if $p < \infty$. The covering numbers $\mathcal{N}(f, \varepsilon)$ of a function f is simply defined as the covering number of its range, up to measure 0. We will prove the following two results:

Theorem 0.0.36. *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $\mathscr{A} \subset L^\infty(\Omega, \mathcal{F}, \mu)$. The following assertions are equivalent:*

(i) *$\mathscr{A}$ is UA;*

(ii) $\sup_{f \in \mathscr{A}} \mathcal{N}(f, \varepsilon) < \infty$ *for all* $\varepsilon > 0$.

Theorem 0.0.37. *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, $p \in [1, \infty)$ and let $\mathscr{A} \subset L^p(\Omega, \mathcal{F}, \mu)$. The following assertions are equivalent:*

(i) *$\mathscr{A}$ is UA in $L^p(\Omega, \mathcal{F}, \mu)$;*

(ii) $\lim\limits_{k \to \infty} \sup_{f \in \mathscr{A}} Var_{p,k}(f) = 0$.

Then we investigate when the unit ball of $L^p(\Omega, \mathcal{F}, \mu)$ is UA. If $1 \leq p < \infty$, this happens, as one can expect, if and only if $L^p(\Omega, \mathcal{F}, \mu)$ is finite dimensional (see Theorem 9.3.10). We conclude this section by establishing some stability properties of the class of UA sets. In particular, a nice use of the Rademacher type allows us to prove that if $\mathscr{A}$ is a bounded UA set in $L^p(\Omega, \mathcal{F}, \mu)$ for $p \in (1, \infty)$ then its closed convex hull also is UA (see Theorem 9.3.15). For more information about Rademacher type and cotype, we refer the reader to [1] (chapter 6).

This chapter is a preprint (see [89]) which is the consequence of a collaboration with J. San Martín.

Notation

Our notation is standard and follows books as [67] and [1].

All the Banach spaces are supposed to be real. In this document, $(X, \|.\|)$ will always be a Banach space and we denote

B_X	the closed unit ball of X;
X^*	the dual space of X;
w	the weak topology;
w^*	the weak* topology.

If $A \subset X$, conv(A) denotes the convex hull of A. If $T : X \to Y$ is a linear operator between two Banach spaces X and Y, T^* is the adjoint operator.

If (T, τ) is the topological space and if $A \subset T$, then $\overline{A}^{\tau}$ (or simply $\overline{A}$) is the closure of A.

A more detailed list of notations is given at the end of this work.

Contents

Chapter 1

Some preliminary content

1.1 Filters

We begin by stating the basic definitions and properties of filters that will be very useful later. If I is a set, we denote its power set by $\mathcal{P}(I)$.

Definition 1.1.1. Given a set I and $\mathcal{U} \subseteq \mathcal{P}(I)$, we say that $\mathcal{U}$ is a *filter* if

(a) $\emptyset \notin \mathcal{U}$.

(b) $A, B \in \mathcal{U}$ implies $A \cap B \in \mathcal{U}$.

(c) $A \in \mathcal{U}$ and $A \subseteq B$ implies $B \in \mathcal{U}$.

In that case, we say that $\mathcal{U}$ is *free* if $\bigcap_{A \in \mathcal{U}} A = \emptyset$ and that $\mathcal{U}$ is *proper* if $\mathcal{U} \neq \mathcal{P}(I)$. An *ultrafilter* is a filter which is maximal in the set of proper ultrafilters ordered by inclusion.

Using Zorn's lemma it is easy to see that any filter can be extended to an ultrafilter, that is:

Proposition 1.1.2. *Let $\mathcal{U}$ be a proper filter on a set I. Then there exists an ultrafilter $\mathcal{G}$ on I such that $\mathcal{U} \subset \mathcal{G}$*

We have the following well-known characterization of ultrafilter:

Proposition 1.1.3. *Let $\mathcal{U}$ be a filter on a set I. The following assertions are equivalent:*

(i) $\mathcal{U}$ is an ultrafilter;

(ii) for all $A \subset I$, $A \in \mathcal{U}$ or $A^c \in \mathcal{U}$;

(iii) for all $A, B \subset I$, if $A \cup B \in \mathcal{U}$, then $A \in \mathcal{U}$ or $B \in \mathcal{U}$.

The notion of filter plays an important role in topology. For more details about filters and topology, we refer the reader to [30]. It is possible to define a notion of convergence according to a filter and which will allow us to define ultraproducts in the next part.

Definition 1.1.4. Let (T, τ) be a topological space. Consider I a set and $\mathcal{U} \subset \mathcal{P}(I)$ an ultrafilter. Let $(x_i)_{i \in I} \in \mathrm{T}^I$ and $x \in X$. We say that $(x_i)_{i \in I}$ *converges to x with respect to $\mathcal{U}$* if $\{i \in I \ : \ x_i \in U\} \in \mathcal{U}$ for all open neighborhood U of x. In this case, we say that x is a *$\mathcal{U}$-ultralimit* (or simply an *ultralimit* is there is no confusion) of $(x_i)_{i \in I}$ and we write $\lim_{\mathcal{U}, i} x_i = x$ or simply $\lim_{\mathcal{U}} x_i = x$ if there is no confusion.

The following proposition is elementary:

Proposition 1.1.5. *Let* (T, τ) *be a topological space and* $\mathcal{U}$ *an ultrafilter on a set* I. *Let* $(x_i)_{i\in I} \in \mathrm{T}^I$. *Then*

(a) if (T, τ) *is Hausdorff and if* $(x_i)_{i\in I}$ *admits a* $\mathcal{U}$*-ultralimit, then this latter is unique;*

(b) if (T, τ) *is compact, then* $(x_i)_{i\in I}$ *admits a* $\mathcal{U}$*-ultralimit.*

We need to recall that if $\mathcal{U}$ is an ultrafilter on I and $\mathcal{V}$ is an ultrafilter on J, *the product* $\mathcal{U} \times \mathcal{V}$ of $\mathcal{U}$ and $\mathcal{V}$ is the ultrafilter defined on $I \times J$ by

$$L \in \mathcal{U} \times \mathcal{V} \iff \{j \in J \mid \{i \in I \mid (i,j) \in L\} \in \mathcal{U}\} \in \mathcal{V}.$$

Note that this product is not commutative in general.

The product limit of an ultrafilter is simply obtained by iterating the limits.

Lemma 1.1.6. *Let* (M,d) *be a metric space,* $\mathcal{U}$ *an ultrafilter on a set* I *and* $\mathcal{V}$ *an ultrafilter on a set* J. *Let* $(x_{i,j})_{i,j} \in M^{I\times J}$. *Then* $\lim_{\mathcal{U}\times\mathcal{V}} x_{i,j} = \lim_{\mathcal{V},j} \lim_{\mathcal{U},i} x_{i,j}$ *whenever all of these limits exist.*

Proof. Let $x = \lim_{\mathcal{U}\times\mathcal{V}} x_{i,j}$ and $y = \lim_{\mathcal{V},j} \lim_{\mathcal{U},i} x_{i,j}$. For $j \in J$, define also $y_j = \lim_{\mathcal{U},i} x_{i,j}$. Fix $\varepsilon > 0$ and note that by definition of the limit we have

$$\{(i,j) \in I \times J \mid d(x_{i,j}, x) < \varepsilon\} \in \mathcal{U} \times \mathcal{V},$$

that is

$$J_\varepsilon := \{j \in J \mid \{i \in I \mid d(x_{i,j}, x) < \varepsilon\} \in \mathcal{U}\} \in \mathcal{V}.$$

Then, for all $j \in J_\varepsilon$, we have that $\{i \in I \mid d(x_{i,j}, x) < \varepsilon\} \in \mathcal{U}$ which implies that $d(y_j, x) \leq \varepsilon$. Since $J_\varepsilon \in \mathcal{V}$, we obtain that $d(x,y) \leq \varepsilon$. The arbitrariness of ε allows us to conclude that $x = y$. $\square$

The next definition introduces a type of ultrafilters which will be very usefull in this document since it allows to imitate what happens for free ultrafilters on $\mathbb{N}$:

Definition 1.1.7. An ultrafilter $\mathcal{U}$ on a set I is said to be *countably incomplete* (in short, CI) if it contains a sequence of sets $(I_n)_{n\in\mathbb{N}}$ such that $\cap_{n\in\mathbb{N}} I_n = \emptyset$.

If $\mathcal{U}$ is CI, the sequence $(I_n)_{n\in\mathbb{N}}$ given by the previous definition can always be chosen to be decreasing. Note that every CI ultrafilter is free and that any free ultrafilter on $\mathbb{N}$ is CI. We have the following characterization of CI ultrafilter:

Proposition 1.1.8. *Let* $\mathcal{U}$ *be ultrafilter on a set* I. *The following assertions are equivalent:*

(i) $\mathcal{U}$ *is CI;*

(ii) there exists $(a_i)_{i\in I} \subset \mathbb{R}$ *such that* $\lim_{\mathcal{U}} a_i = 0$ *and* $a_i > 0$ *for all* $i \in I$.

Proof. Suppose first that $\mathcal{U}$ is CI and let $(I_n)_{n\geq 1} \subset \mathcal{U}$ with $\bigcap_{n\geq 1} I_n = \emptyset$. We can suppose that $I_1 = I$ and $I_{n+1} \subsetneq I_n$ for all $n \in \mathbb{N}$. Define $(a_i)_{i\in I}$ by $a_i = \frac{1}{n}$ if $i \in I_n \setminus I_{n+1}$. Given $\varepsilon > 0$, take n_0 such that $\frac{1}{n_0} < \varepsilon$. It is easily seen that

$$I_{n_0} \subset \{i \in I : a_i < \varepsilon\}.$$

It follows that the last set belongs to $\mathcal{U}$. Thus, $\lim_{\mathcal{U}} a_i = 0$.

Now suppose that (ii) holds. For $n > 0$, define

$$I_n := \left\{ i \in I : a_i < \frac{1}{n} \right\} \in \mathcal{U}.$$

Since $a_i \neq 0$ for all $i \in I$, it is clear that $\bigcap_n I_n = \emptyset$, i.e. $\mathcal{U}$ is CI. $\square$

1.2 Ultraproduct of Banach spaces

We start with the construction of the ultraproduct of a Banach space.

Given a Banach space X and a set I, we denote $\ell_\infty(I,X) := \{(x_i)_{i\in I} \ : \ \sup_{i\in I}\|x_i\| < \infty\}$. Given a ultrafilter $\mathcal{U}$ over I, consider $N_{\mathcal{U}} := \{f \in \ell_\infty(I,X) : \lim_{\mathcal{U}}\|x_i\| = 0\}$. The *ultraproduct of X with respect to $\mathcal{U}$* is the Banach space

$$X_{\mathcal{U}} := \ell_\infty(I,X)/N_{\mathcal{U}}.$$

We denote by $(x_i)_{\mathcal{U},i}$ or simply by $(x_i)_{\mathcal{U}}$, if no confusion is possible, the coset in $X_{\mathcal{U}}$ given by $(x_i)_{i\in I} + N_{\mathcal{U}}$.

The norm on $X_{\mathcal{U}}$ is the associated quotient norm. Fortunately, there is a simpler way to compute it:

Proposition 1.2.1. *Let X be a Banach space and $\mathcal{U}$ be an ultrafilter on a set I. If $(x_i)_{\mathcal{U}} \in X_{\mathcal{U}}$, we have that $\|(x_i)_{\mathcal{U}}\| = \lim_{\mathcal{U}}\|x_i\|$*

The previous proposition implies that the *canonical inclusion* $\mathcal{J} : X \longrightarrow X_{\mathcal{U}}$ given by the formula $\mathcal{J}(x) := (x)_{\mathcal{U}}$ is an into linear isometry. Note that, if X is finite dimensional, $X_{\mathcal{U}} = X$ isometrically.

The properties that are possessed by the ultraproducts are called super-properties:

Definition 1.2.2. Let $(\mathcal{P})$ be a property of Banach spaces. We say that a Banach has *super-*$(\mathcal{P})$ if $X_{\mathcal{U}}$ has $(\mathcal{P})$ for any ultrafilter $\mathcal{U}$.

For example, we can talk about *superreflexivity*, which will be one of the main notion used in this document.

One big issue dealing with ultraproducts is the lack of representation for $(X_{\mathcal{U}})^*$, except when X is superreflexive. In fact, we have that $(X_{\mathcal{U}})^* = (X^*)_{\mathcal{U}}$ if and only if X is superreflexive (see Proposition 6.4 in [97]). Otherwise, $(X^*)_{\mathcal{U}}$ is a proper subspace of $(X_{\mathcal{U}})^*$. The identification is done by the assignment

$$\langle (x_i^*), (x_i)\rangle := \lim_{\mathcal{U}} x_i^*(x_i).$$

Recall that a subset $B \subset B_{X^*}$ is called a *boundary* (of X) if for every $x \in X$ there exists $x^* \in B$ such that $\|x\| = x^*(x)$. A norming subspace $Z \in X^*$ is called a *boundary* if $B_Z = Z \cap B_{X^*}$ is a boundary in the previous sense. Even if the dual can be difficult to determine, we have the following positive result:

Theorem 1.2.3. *Let X be Banach space and let $\mathcal{U}$ be any free ultrafilter on a set I. Then $(X^*)_{\mathcal{U}}$ is a boundary for $X_{\mathcal{U}}$. Therefore, the relatively weakly compact subsets of $X_{\mathcal{U}}$ are exactly those which are relatively compact for the topology of pointwise convergence on the elements from $(X^*)_{\mathcal{U}}$.*

Proof. Indeed, for every $i \in I$ take $x_i^* \in B_{X^*}$ such that $x_i^*(x_i) = \|x_i\|$. The second statement comes from Pfitzner's solution [141] to Godefroy's boundary problem. $\square$

We can also define the ultraproduct of bounded sets in the natural way:

Definition 1.2.4. Let X be a Banach space and let $A \subset X$ be a bounded set. Let $\mathcal{U}$ be a ultrafilter on a infinite set I. We define the *ultraproduct of A with respect to $\mathcal{U}$* by

$$A_{\mathcal{U}} := \{(x_i)_{\mathcal{U}} : x_i \in A\ \forall i \in I\}.$$

Concerning the question of when $A_{\mathcal{U}}$ is closed we have the following result. The first part can be found in [47, Proposition 3.2].

Proposition 1.2.5. *Let X be a Banach space and let $A \subset X$ be a bounded set. Let $\mathcal{U}$ be a CI ultrafilter on a infinite set I. Then $A_{\mathcal{U}}$ is closed and $(\overline{A})_{\mathcal{U}} = A_{\mathcal{U}} = \overline{A_{\mathcal{U}}}$.*

Proof. Let $(I_n)_{n\in\mathbb{N}}$ be a sequence of sets as in the definition of CI ultrafilter, we may assume that $I_{n+1} \subsetneq I_n$ for every n. Let $x \in \overline{A_{\mathcal{U}}}$ and let $(x^n)_{n\in\mathbb{N}}$ be a sequence of $A_{\mathcal{U}}$ such that $\|x - x^n\| < \frac{1}{n}$. Consider $U_n = \{i \in I \;:\; \|x_i - x_i^n\| < \frac{1}{n}\}$ for all n and note that $U_n \in \mathcal{U}$. Then define $I'_n = I_n \cap U_n \in \mathcal{U}$. Define $y \in A_{\mathcal{U}}$ by $y_i = x_i^m$ if $i \in I'_m \setminus I'_{m+1}$ for some m, and $y_i = x_0$ in the other case, where x_0 is a arbitrary element of A. One can check that $x = y \in A_{\mathcal{U}}$.

Now we need to show that $(\overline{A})_{\mathcal{U}} = A_{\mathcal{U}}$. Let $(x_i)_{\mathcal{U}} \in (\overline{A})_{\mathcal{U}}$ with $x_i \in \overline{A}$ for all $i \in I$. By Proposition 1.1.8, there exists $(a_i)_{i\in I} \subset \mathbb{R}$ such that $\lim_{\mathcal{U}} a_i = 0$ and $a_i > 0$ for all $i \in I$. For $i \in I$, choose $y_i \in A$ such that $\|x_i - y_i\| < a_i$. We deduce that $\lim_{\mathcal{U}} \|x_i - y_i\| = 0$, that is, $(x_i)_{\mathcal{U}} = (y_i)_{\mathcal{U}} \in A_{\mathcal{U}}$. The other inclusion is obvious. □

The following technical result which will be useful in order to deal with slices in an ultraproduct.

Lemma 1.2.6. *Let A be a bounded subset of a Banach space X and $\mathcal{U}$ be a free ultrafilter on an infinite set I. If $(x_i^*)_{\mathcal{U}} \in (X^*)_{\mathcal{U}}$, then $\sup_{A_{\mathcal{U}}} (x_i^*)_{\mathcal{U}} = \lim_{\mathcal{U}} \sup_A x_i^*$.*

Proof. Let $a = \sup_{A_{\mathcal{U}}} (x_i^*)_{\mathcal{U}}$ and $b = \lim_{\mathcal{U}} \sup_A x_i^*$. Let $\varepsilon > 0$. By definition of a, there exists $(x_i)_{\mathcal{U}} \in A_{\mathcal{U}}$ such that $\langle (x_i^*)_{\mathcal{U}}, (x_i)_{\mathcal{U}} \rangle > a - \varepsilon$. Then there exists $J \in \mathcal{U}$ such that $x_i^*(x_i) > a - \varepsilon$ for all $i \in J$. It follows that

$$\sup_A x_i^* \geq x_i^*(x_i) > a - \varepsilon,$$

for all $i \in J$ and taking limit on $\mathcal{U}$, we conclude that $b \geq a - \varepsilon$.

Now, by definition of b, the set $J := \{i \in I : \sup_A x_i^* > b - \varepsilon\}$ belongs to $\mathcal{U}$. For all $i \in J$ there exists $x_i \in A$ such that $x_i^*(x_i) > b - \varepsilon$. Define $y_i = x_i$ if $i \in J$ and $y_i = x_0$ if not, where x_0 is an arbitrary element of A. It is clear that

$$\langle (x_i^*)_{\mathcal{U}}, (y_i)_{\mathcal{U}} \rangle \geq b - \varepsilon.$$

It follows that $a \geq b - \varepsilon$. □

The next proposition (see [156, Proposition 2.1]) shows that the ultraproduct of an ultraproduct is still an ultraproduct:

Proposition 1.2.7. *Let $\mathcal{U}$ and $\mathcal{V}$ be two ultrafilters on sets I and $\mathcal{V}$ respectively and let X be a Banach space. Then $(X_{\mathcal{U}})_{\mathcal{V}} = X_{\mathcal{U}\times\mathcal{V}}$ isometrically, via $T : (X_{\mathcal{U}})_{\mathcal{V}} \to X_{\mathcal{U}\times\mathcal{V}}$ defined by $T(((x_{i,j})_{\mathcal{U},i})_{\mathcal{V},j}) = (x_{i,j})_{\mathcal{U}\times\mathcal{V}}$.*

Note that the isometry given by the previous proposition preserved the ultraproducts of sets. In fact, we have that $T((A_{\mathcal{U}})_{\mathcal{V}}) = A_{\mathcal{U}\times\mathcal{V}}$ for all bounded sets $A \subset X$.

1.3 Finite representability

1.3.1 Finite representability of Banach spaces

Definition 1.3.1. We say that a Banach space X is *finitely representable* in a Banach space Y (in short, X is f.r. in Y) if for all finite-dimensional subspaces E of X and all $\varepsilon > 0$, there exist a finite-dimensional subspace F of Y and an isomorphism $T : E \to F$ such that $\|T\|\|T^{-1}\| < 1+\varepsilon$.

The following fundamental result characterizes the finite representability in terms of ultraproduct (see Theorem 6.3 in [97]):

Theorem 1.3.2. *Let X and Y be two Banach spaces. The following assertions are equivalent:*

(i) X is f.r. in Y;

(ii) there exists an ultrafilter $\mathcal{U}$ such that X is isometric to a subspace of $Y_{\mathcal{U}}$.

In that case, if X is moreover separable then X is isometric to a subspace of $Y_{\mathcal{U}}$ for any CI ultrafilter $\mathcal{U}$.

It follows that a Banach space X is superreflexive if and only if any ultraproduct of X is reflexive. In fact, we will see that it is enough to consider only one free ultrafilter on $\mathbb{N}$ (see Theorem 1.5.6).

When X is separable, we can just check the finite representability on a particular increasing sequence of finite-dimensional subspaces:

Proposition 1.3.3. *Let X, Y be two Banach spaces and suppose that there exists an increasing sequence of finite-dimensional subspaces $(E_n)_n$ of X such that $X = \overline{\bigcup_{n=1}^{\infty} E_n}$. The following assertions are equivalent:*

(i) X is f.r. in Y;

(ii) for all $n \in \mathbb{N}$ and all $\varepsilon > 0$, there exist a finite-dimensional subspace F_n of Y and an isomorphism $T : E_n \to F_n$ such that $\|T\|\|T^{-1}\| < 1 + \varepsilon$.

The previous proposition is particularly interesting when X admits a Schauder basis $(e_n)_n$. In this case, it is enough to check the finite representability on the subspaces $\operatorname{span}\{e_k\}_{1 \leq k \leq n}$ for all n.

We now recall some particularly well-known theorems on finite representability. We start with a consequence of the famous Dvoretzky's Theorem:

Theorem 1.3.4. *ℓ_2 is f.r. in any infinite-dimensional Banach space.*

The next result is a consequence of the Principle of local reflexivity:

Theorem 1.3.5. *X^{**} is f.r. in X for any Banach space X.*

We end this subsection recalling that the type and the cotype can be characterized in terms of finite representability. We refer the reader to [1] for more informations about type and cotype and to Theorem 12.1.15 in [1] for the proof of the next result:

Theorem 1.3.6. *Let X be a Banach space. Then*

(a) X has non-trivial type if and only if ℓ_1 is not f.r. in X;

(b) X has non-trivial cotype if and only if ℓ_∞ is not f.r. in X.

1.3.2 Finite representability of sets

One definition of finite representability for sets has been introduced in [47], but we believe that the next definition is more natural:

Definition 1.3.7. Let X and Y be Banach spaces. We say that a set $B \subset Y$ is *finitely represented in a set* $A \subset X$ (in short, B is f.r. in A) if for all $\varepsilon > 0$, for all finite linearly independent sets $B_0 \subset B$, there exist $A_0 \subset A$ and an isomorphism $T : \mathrm{span}(B_0) \to \mathrm{span}(A_0)$ such that $T(B_0) \subset A$ and $\|T\|\|T^{-1}\| < 1 + \varepsilon$.

Note that the previous definition generalizes the usual one for Banach spaces since a Banach space X is f.r. in a Banach space Y if and only if B_X is f.r. in B_Y.

The two following results are similar to Propositions 6.1 and 6.2 in [97] and the proofs require minor adjustments.

Proposition 1.3.8. *Let X be a Banach space and $A \subset X$. Then $A_{\mathcal{U}}$ is f.r. in A for any free ultrafilter $\mathcal{U}$.*

Proof. Let $x^1, ..., x^n$ be a finite linearly independant subset of $A_{\mathcal{U}}$ and let $\varepsilon \in (0, 1)$. Let $M = \mathrm{span}\{x^1, ..., x^n\}$. Each x^k can be write $x^k = (x_i^k)_{\mathcal{U}}$ with $x_i^k \in A$. For all i, let $M_i = \mathrm{span}\{x_i^1, ..., x_i^n\}$ and define an bounded operator $T_i : M \to M_i$ by $T_i(x^k) = x_i^k$ for all k. Obviously, T_i is surjective and $T_i(x^k) \in A$ for all k.
Note that it exists $C \in \mathbb{R}$ such that $\|T_i\| \leq C$ for all i. In fact, define on M an equivalent norm by $\|x\|_\infty = \max_{1\leq k\leq n} |\lambda_k|$ where $x = \sum_{k=1}^n \lambda_k x^k$. If α is such that $\|.\|_\infty \leq \alpha\|.\|$ then

$$\|T_i(x)\| = \left\|\sum_{k=1}^n \lambda_k x_i^k\right\| \leq \|x\|_\infty \max_{1\leq k\leq n} \|x_i^k\| \leq \alpha \max_{1\leq k\leq n} \|(x_i^k)_i\|_{l_\infty(I)}\|x\| \leq C\|x\|$$

with $C = \alpha \max_{1\leq k\leq n} \|(x_i^k)_i\|_{l_\infty(I)}$. It implies that if $x = \sum_{k=1}^n \lambda_k x^k \in M$ then $(T_i(x))_{i\in I}$ and we have

$$\lim_{\mathcal{U},i} \|(T_i x)_i\| = \lim_{\mathcal{U},i} \left\|\left(\sum_{k=1}^n \lambda_k x_i^k\right)_i\right\| = \left\|\sum_{k=1}^n \lambda_k (x_i^k)_{\mathcal{U}}\right\| = \|x\|.$$

Then, for all $x \in M$, it exists $I_x \in \mathcal{U}$ such that $(1 + \frac{\varepsilon}{2})^{-1}\|x\| \leq \|T_i x\| \leq (1 + \frac{\varepsilon}{2})\|x\|$.
Let $\delta > 0$ and take a δ-net $(y_n)_{n\in J}$ in $\mathcal{S}_M$ where J is a finite set. Define $I_0 = \cap_{n\in J} I_{y_n} \in \mathcal{U}$. Let $x \in \mathcal{S}_M$ and choose $n_0 \in J$ such that $\|x - y_{n_0}\| \leq \delta$. For $i \in I_0$, we have that:

$$\|T_i(x)\| \leq \|T_i(x - y_{n_0})\| + \|T_i(y_{n_0})\| \leq C\delta + 1 + \frac{\varepsilon}{2}$$

and

$$\|T_i(x)\| \geq \|T_i(y_{n_0})\| - \|T_i(x - y_{n_0})\| \geq \left(1 + \frac{\varepsilon}{2}\right)^{-1} - C\delta.$$

If we choose δ such that $C\delta + 1 + \frac{\varepsilon}{2} < 1 + \varepsilon$ and $(1 + \frac{\varepsilon}{2})^{-1} - C\delta > (1 + \varepsilon)^{-1}$, we have that $1 - \varepsilon \leq (1 + \varepsilon)^{-1} \leq \|T_i(x)\| \leq 1 + \varepsilon$ and the proof is complete. □

Proposition 1.3.9. *Let X and Y be Banach spaces. Suppose that $B \subset Y$ is f.r. in $A \subset X$. If $B_0 \subset B$ is a linearly independant set, then there exist a free ultrafilter $\mathcal{U}$ and a linear isometry $T : span(B_0) \to (span(A))_{\mathcal{U}}$ such that $T(B_0) \subset A_{\mathcal{U}}$.*

Proof. Define $I = \{(C, \varepsilon) : C \subset B_0$ finite and linearly independent set, $\varepsilon > 0\}$ with the order $(C_1, \varepsilon_1) \leq (C_2, \varepsilon_1)$ if and only if $C_1 \subset C_2$ and $\varepsilon_1 \geq \varepsilon_2$. Let $\mathcal{U}$ be an ultrafilter that contains the filter basis $\{\{i \in I : i_0 < i\} : i_0 \in I\}$. For $i = (C_i, \varepsilon_i) \in I$, there exist $A_i \subset A$ and an operator $T_i : \mathrm{span}(C_i) \to \mathrm{span}(A_i)$ such that $\|T_i\|, \|T_i^{-1}\| \leq 1 + \varepsilon_i$ and $T_i(C_i) \subset A$. Define $T : \mathrm{span}(B_0) \to (\mathrm{span}(A))_{\mathcal{U}}$ by $T(x) = (y_i)_{\mathcal{U}}$ where $y_i = T_i x$ if $x \in C_i$ and $y_i = 0$ in the other case. Let $\varepsilon_0 > 0$ and let $x \in \mathrm{span}(B_0)$. By definition of $\mathcal{U}$, we have that $I_x := \{(C, \varepsilon) : x \in C, \ \varepsilon \leq \varepsilon_0\} \in \mathcal{U}$. Then, for $i \in I_x$, one has that $(1 + \varepsilon_0)^{-1}\|x\| \leq \|y_i\| \leq (1 + \varepsilon_0)\|x\|$ and then $(1 + \varepsilon_0)^{-1}\|x\| \leq \|T(x)\| = \lim_{\mathcal{U},i} \|y_i\| \leq (1 + \varepsilon_0)\|x\|$. Since ε_0 is arbitrary, we deduce that T is an isometry. Furthemore, for $i \in I_x$, we have that $y_i = T_i x \in A$ and, since $I_x \in \mathcal{U}$, we deduce that $T(x) \in A_{\mathcal{U}}$. □

We will need the following lemma:

Lemma 1.3.10. *Let $A \subset X$ be a subset of a Banach space X and let $\varepsilon > 0$. Then for any finite linearly independent set $x_1, ..., x_N$ in $\overline{A}$, there exists a finite linearly independent set $y_1, ..., y_N$ in A such that $\|x_k - y_k\| < \varepsilon$ for all $1 \leq k \leq N$.*

Proof. There exist sequences $(y_n^k)_n \subset A$ such that $(y_n^k)_n \underset{n}{\rightarrow} x_k$ for all $1 \leq k \leq N$. Without loss of generality, we can suppose that $\|y_n^k - x_k\| < \varepsilon$ for all $n \in \mathbb{N}$ and $1 \leq k \leq N$. Since $\operatorname{span}\{x_k\}_{1\leq k\leq N}$ is finite dimensional, it is complemented in X and then there exists a bounded onto projection $p : X \to \operatorname{span}\{x_k\}_{1\leq k\leq N}$. By continuity of p, we have that $p(y_n^k)_n \underset{n}{\rightarrow} p(x_k) = x_k$ for all $1 \leq k \leq N$. Now using the continuity of the determinant in $\operatorname{span}\{x_k\}_{1\leq k\leq N}$, we deduce that there exists $n_0 \in \mathbb{N}$ such that the family $p(y_{n_0}^1), ..., p(y_{n_0}^N)$ is linearly independent for all $n \geq n_0$. The family $y_{n_0}^1, ..., y_{n_0}^N$ is linearly independent and fulfills that $\|x_k - y_{n_0}^k\| < \varepsilon$ for all $1 \leq k \leq N$. □

The following result is an adaptation of Proposition 1.3.3 to the finite representability of sets:

Proposition 1.3.11. *Let $A \subset X$ and $B \subset Y$ be subsets of two Banach spaces X and Y. Suppose that A can be written $A = \overline{\bigcup_{n=1}^{\infty} A_n}$ where $(A_n)_n$ is an increasing sequence of sets such that A_n is f.r. in B. Then A is f.r. in B. In particular, A is f.r. in B if and only if $\overline{A}$ is f.r. in B.*

Proof. Let $e_1, ..., e_N$ be a finite linearly independent set in $\overline{A}$ and let $\varepsilon > 0$. Define $E = \operatorname{span}\{x_k\}_{1\leq k\leq N}$. Since E is finite dimensional, there exists $C > 0$ such that for all $a_1, ..., a_N \in \mathbb{R}$

$$\frac{1}{C} \max_{1\leq k\leq N} |a_k| \leq \left\| \sum_{k=1}^{N} a_k e_k \right\| \leq C \max_{1\leq k\leq N} |a_k|.$$

Choose $\nu > 0$ such that $(1+\varepsilon)\frac{1+CN\nu}{1-CN\nu} < 1 + 2\varepsilon$. By the previous lemma, there exist a finite linearly independant set $x_1, ..., x_N \in \bigcup_{n=1}^{\infty} A_n$ such that $\|e_k - x_k\| < \varepsilon$ for all $1 \leq k \leq N$. Let $n \in \mathbb{N}$ such that $x_1, ..., x_n \in A_n$. Since A_n is f.r. in B, there exist $B_0 \subset B$ and an isomorphism $T : \operatorname{span}(A_0) \to \operatorname{span}(B_0)$ such that $T(A_0) \subset B$ and $\|T\|\|T^{-1}\| < 1+\varepsilon$. Define a linear operator $S : E \to \operatorname{span}(B_0)$ by $S(e_k) = T(x_k)$ for all $1 \leq k \leq N$. Take $e = \sum_{k=1}^{N} a_k e_k \in E$. Note that

$$\left\| \sum_{k=1}^{N} a_k e_k - \sum_{k=1}^{N} a_k x_k \right\| \leq N\nu \max_{1\leq k\leq N} |a_j| \leq CN\nu \left\| \sum_{k=1}^{N} a_k e_k \right\|.$$

On one hand, we have that

$$\begin{aligned}
\|S(e)\| = \left\| T\left(\sum_{k=1}^{N} a_k x_k \right) \right\| &\leq \|T\| \left\| \sum_{k=1}^{N} a_k e_k \right\| \\
&\leq \|T\| \left(\left\| \sum_{k=1}^{N} a_k e_k - \sum_{k=1}^{N} a_k x_k \right\| + \left\| \sum_{k=1}^{N} a_k e_k \right\| \right) \\
&\leq \|T\|\|e\|(1 + CN\nu)
\end{aligned}$$

and on the other hand

$$\begin{aligned}\|S(e)\| = \left\|T\left(\sum_{k=1}^N a_k x_k\right)\right\| &\geq \frac{1}{\|T^{-1}\|}\left\|\sum_{k=1}^N a_k x_k\right\| \\ &\geq \frac{1}{\|T^{-1}\|}\left(\left\|\sum_{k=1}^N a_k e_k\right\| - \left\|\sum_{k=1}^N a_k e_k - \sum_{k=1}^N a_k x_k\right\|\right) \\ &\geq \frac{1}{\|T^{-1}\|}\|e\|(1-CN\nu).\end{aligned}$$

We conclude that $\|S\|\|S^{-1}\| \leq \|T\|\|T^{-1}\|\frac{1+CN\nu}{1-CN\nu} \leq (1+\varepsilon)\frac{1+CN\nu}{1-CN\nu} < 1+2\varepsilon$. $\square$

1.4 Spreading models

1.4.1 Definition and first properties

In this part, we recall the definition of spreading models initially introduced by Brunel and Sucheston in [40]. We refer the reader to [19] for a great presentation of spreading models. Almost all the results presented in this section can be found in [19].

If A is a set and $n \in \mathbb{N}$, we define $A^{(n)} = \{B \subset A \ : \ \mathrm{Card}(B) = n\}$. The construction of spreading models is based on Ramsey's Theorem (see Proposition 6.4 in [67]).

Theorem 1.4.1. *Let $n \in \mathbb{N}^*$ and let $(S_i)_{1\leq i\leq k}$ be a finite partition of $\mathbb{N}^{(n)}$. Then there exist $i \in \{1, ..., k\}$ and an infinite set $M \subset \mathbb{N}$ such that $M^{(n)} \subset S_i$.*

Definition 1.4.2. Let X be a Banach space and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. We said that $(x_n)_{n\in\mathbb{N}}$ is a *good sequence* if the limit

$$\lim_{n_1\to\infty} \|a_1 x_{n_1} + ... + a_1 x_{n_k}\|$$

exists whenever $n_1 < ... < n_k$, $k \in \mathbb{N}$ and $a_1, ..., a_k \in \mathbb{R}$. Using Theorem 1.4.1, one can prove that every bounded sequence has a good subsequence. If $(x_n)_{n\in\mathbb{N}}$ is a good sequence, the formula $\|(a_1, ..., a_k)\| = \lim_{n_1\to\infty} \|a_1 x_{n_1} + ... + a_k x_{n_k}\|$, with $n_1 < ... < n_k$, defines a semi-norm on c_{00}. It is easily seen that it defines a norm if and only if $(x_n)_{n\in\mathbb{N}}$ is not convergent. In this case, the completion Z of c_{00} with this new norm is called *spreading model of X built on* $(x_n)_{n\in\mathbb{N}}$. We say that $(e_n)_{n\in\mathbb{N}}$ (where $(e_n)_{n\in\mathbb{N}}$ is the canonical basis of c_{00}) is the *fundamental sequence* of the spreading model.

Any spreading model of X is finitely representable in X. More precisely, one has that:

Proposition 1.4.3. *Let Z be a spreading model built on a sequence $(x_n)_{n\in\mathbb{N}}$ of X. Then*

$$\forall \varepsilon > 0 \ \ \forall N \geq 1 \ \ \exists p \in \mathbb{N} \ \ \forall n_1 < ... < n_N \ \ \textit{with} \ \ n_1 \geq p \ \ \forall a_1, ..., a_N \in \mathbb{R}$$

$$(1-\varepsilon)\left\|\sum_{i=1}^N a_i x_{n_i}\right\| \leq \left\|\sum_{i=1}^N a_i e_i\right\| \leq (1+\varepsilon)\left\|\sum_{i=1}^N a_i x_{n_i}\right\|.$$

Note that the finite representability of Z into X follows then from Proposition 1.3.3.

Definition 1.4.4. A non-constant sequence $(e_n)_{n\in\mathbb{N}}$ of a Banach space X is said to be *spreading* if

$$\left\|\sum_{i=1}^{n} a_i e_i\right\| = \left\|\sum_{i=1}^{n} a_i e_{n_i}\right\|$$

for all $k \in \mathbb{N}$, all $a_1, ..., a_k \in \mathbb{R}$ and all $n_1 < ... < n_k$.

By construction of the norm of a spreading model, its fundamental sequence is spreading.

The following proposition characterized the spreading sequences which are basic (see Proposition 2 p.17 in [19]):

Proposition 1.4.5. *Let $(e_n)_{n\in\mathbb{N}}$ be a spreading sequence in a Banach space X. The following assertions are equivalent:*

(i) $(e_n)_{n\in\mathbb{N}}$ *is basic;*

(ii) $(e_n)_{n\in\mathbb{N}}$ *does not weakly converge to a non-zero vector.*

The behaviour of the fundamental sequence is strongly linked with the behaviour of the sequence on which is built the spreading model. For example, we have (see Theorem 3 p.25 in [19]):

Proposition 1.4.6. *Let X be a Banach space and let Z be a spreading model built on $(x_n)_{n\in\mathbb{N}} \subset X$ with fundamental sequence $(e_n)_n$. Suppose that $(e_n)_{n\in\mathbb{N}}$ is not equivalent to the canonical basis of ℓ_1. The following assertions are equivalent:*

(i) $(x_n)_{n\in\mathbb{N}}$ *is weakly convergent;*

(ii) $(e_n)_{n\in\mathbb{N}}$ *is weakly convergent.*

We end this section with the following result:

Proposition 1.4.7. *Let Z be a spreading model of a Banach space X built on $(x_n)_n$ with spreading sequence $(e_n)_n$. Then $\overline{\mathrm{conv}}\{e_n\}_n$ is f.r. in* $\mathrm{conv}\{x_n\}_n$.

Proof. By the Proposition 1.3.11, it is enough to prove that $\mathrm{conv}\{e_1, ..., e_p\}$ is f.r. in $\{x_n\}_n$ for all $p \geq 1$. But that follows directly from Proposition 1.4.3. □

1.4.2 Relations with Banach-Saks properties

Definition 1.4.8. A sequence $(x_n)_n$ is said to be *Cesàro convergent* if the sequence of its arithmetic means

$$n^{-1}\sum_{k=1}^{n} x_k$$

converges to some $x \in X$.

Definition 1.4.9. Let A be a subset of a Banach space X. We say that A has:

(a) the *Banach-Saks property* (in short, BS) if every sequence $(x_n)_n \subset A$ has a Cesàro convergent subsequence;

(b) the *weak Banach-Saks property* (in short, WBS) if every weakly convergent sequence $(x_n)_n \subset A$ has a Cesàro convergent subsequence;

(c) the *alternate Banach-Saks property* (in short, ABS) if every sequence $(x_n)_n \subset A$ has a subsequence $(x'_n)_n$ such that $n^{-1}\sum_{k=1}^n x'_k$ converges.

If B_X has the BS (resp. WBS, ABS), we say that X has the BS (resp. WBS, ABS).

Spreading sequences have a particular behavior (see Proposition 4 p.21 in [19]):

Proposition 1.4.10. *Let $(e_n)_{n\in\mathbb{N}}$ be a spreading sequence in a Banach space X. The following assertions are equivalent:*

(i) $0 \in \overline{\text{conv}}\{e_n\}_n$*;*

(ii) $(e_n)_n$ *is Cesàro convergent;*

(iii) $(e_n)_n$ *weakly converges to* 0*;*

(iv) $(e_n)_n$ *is basic unconditional and is not equivalent to the canonical basis of* ℓ_1*.*

The following lemma characterizes the spreading models which are isomorphic to ℓ_1 (see Lemma 1 p.39 in [19]):

Lemma 1.4.11. *Let Z be a spreading model with fundamental sequence $(e_n)_n$. The following assertions are equivalent:*

(i) Z *is isomorphic to* ℓ_1*;*

(ii) $(e_n)_n$ *is equivalent to the canonical basis of* ℓ_1*.*

The following result characterizes the ABS and is due to Beauzamy (see Theoreme 5 p.47 in [19]):

Theorem 1.4.12. *Let A be a bounded subset of a Banach space X. The following assertions are equivalent:*

(i) A *has the ABS property;*

(ii) A *does not contain any spreading sequence whose spreading model is isomorphic to* l_1*.*

The WBS is characterized in the same spirit (see Theorem 5 p.49 in [19]):

Theorem 1.4.13. *Let A be a bounded subset of a Banach space X. The following assertions are equivalent:*

(i) A *has the WBS property;*

(ii) A *does not contain any weakly null spreading sequence whose spreading model is isomorphic to* l_1*.*

It follows directly from the previous theorems that the ABS imply the WBS. Moreover, if A does not contain any sequence equivalent to the canonical basis of ℓ_1, then the converse is true (see Proposition 2 p.50 in [19]). The following result gives the link between the BS and the ABS (see Proposition 1 p.50 in [19]):

Proposition 1.4.14. *Let A be a bounded subset of a Banach space X. If A has the BS, then A has the ABS. Moreover, if A is relatively weakly compact, then the converse is true.*

The previous results give the following characterization of the BS:

Theorem 1.4.15. *Let A be a bounded subset of a Banach space X. The following assertions are equivalent:*

(i) A has the BS;

(ii) A is relatively weakly compact and does not contain any spreading sequence whose spreading model is isomorphic to l_1.

Proof. Using the previous results, the only thing to show is that a bounded set with the BS is relatively weakly compact. So, let A such a set. Let $(x_n)_{n\in\mathbb{N}} \subset A$. Since the canonical basis of l_1 does not have any Cesaro convergent subsequence, the l_1-Rosenthal's Theorem (see Theorem 5.37 in [67]) implies that $(x_n)_{n\in\mathbb{N}}$ has a weakly Cauchy subsequence $(x'_n)_{n\in\mathbb{N}}$. Furthemore, $(x'_n)_{n\in\mathbb{N}}$ admits a subsequence $(x''_n)_{n\in\mathbb{N}}$ such that $\left(\frac{1}{n}\sum_{i=1}^n x''_i\right)_n$ converges to some $x \in X$. It is easily seen that $x''_n \xrightarrow{w} x$ and the proof is complete. □

1.5 Super weak compactness

1.5.1 Definitions and first properties

In this section, a localized notion of superreflexivity is introduced. We will see that most of the characterizations of superreflexivity transpose without major problems. The notion of *super weakly compact* set has been introduced in [146] under the name *finitely dentable* set. We will see that both notions are equivalent (see Corollary 2.4.12). The name super weakly compact is used for the first time in [48].

Definition 1.5.1. Let X be a Banach space and let $A \subset X$ be a bounded set. We say that A is *relatively super weakly compact* if $A_{\mathcal{U}}$ is relatively weakly compact for all free ultrafilters $\mathcal{U}$. If furthemore A is weakly closed, we say that A is *super weakly compact* (in short, SWC).

It is clear that any bounded subset of a superreflexive Banach space is relatively SWC. Through this document, we will see many other examples. We start this subsection with the following stability result proved by Tu in [163], which is a super-version of the Krein-Smulian Theorem:

Theorem 1.5.2. *Let K be a relatively SWC subset of a Banach space X and let $\mathcal{U}$ be an ultrafilter. Then*

$$\overline{(\text{conv}(K))_{\mathcal{U}}} = \overline{\text{conv}}(K_{\mathcal{U}}).$$

In particular, $\overline{\text{conv}}(K)$ is SWC.

As the superreflexivity, super weak compactness can be characterized in terms of finite representability of sets:

Theorem 1.5.3. *Let X be a Banach space and let $K \subset X$ be a bounded weakly closed set. The following assertions are equivalent:*

(i) K is SWC

(ii) Every set f.r. in K is relatively weakly compact.

Proof. By Proposition 1.3.8, $(ii) \implies (i)$ is direct. Now, suppose by contradiction that there exists a set A f.r. in K which is not relatively weakly compact. Then A contains a sequence $(x_n)_{n\in\mathbb{N}}$ without any weakly convergent subsequence. By Zorn's lemma, consider a maximal linearly independant subset $\mathcal{C}$ of $\{x_n\}_{n\in\mathbb{N}}$. Suppose that $\mathcal{C}$ is finite. By maximality, we have that $x_n \in \text{span}(\mathcal{C})$ for all $n \in \mathbb{N}$, which is a subspace of finite dimension. Then $(x_n)_n$ has a convergent subsequence since it is a bounded sequence, which is a contradiction. So $\mathcal{C}$ is infinite and can be seen as a subsequence $(x'_n)_n$ of $(x_n)_n$. By Proposition 1.3.9, there exist a free ultrafilter $\mathcal{U}$ and an isometry $T : \mathrm{span}(B_0) \to (\mathrm{span}(A))_{\mathcal{U}}$ such that $T(B_0) \subset A_{\mathcal{U}}$. That contradicts the fact that $A_{\mathcal{U}}$ is relatively weakly compact since $(T(x'_n))_{n\in\mathbb{N}}$ does not admit any weakly convergent subsequence. $\square$

We recall the following two theorems of James (see [105]) which caracterize the weak compactness:

Theorem 1.5.4. *Let C be a subset of a Banach space X. The following assertions are equivalent:*

(i) C is not relatively weakly compact;

(ii) there exist $\theta > 0$ and a sequence $(x_n)_{n\in\mathbb{N}} \subset K$ such that $d(\mathrm{conv}\{x_j\}_{j\leq k}, \mathrm{conv}\{x_j\}_{j>k}) \geq \theta$ for all $k \in \mathbb{N}$.

Theorem 1.5.5. *Let C be a closed convex subset of a Banach space X. The following assertions are equivalent:*

(i) C is not weakly compact;

(ii) there exist $\theta > 0$, a sequence $(x_n)_{n\in\mathbb{N}} \subset K$ and a sequence $(x^*_n)_{n\in\mathbb{N}} \subset B_{X^*}$ such that $x^*_n(x_k) = 0$ if $n > k$ and $x^*_n(x_k) = \theta$ if $n \leq k$.

Many characterizations of superreflexive Banach spaces are also true for SWC sets. For example, the "finite" version of James' theorems characterizes the super weak compactness:

Theorem 1.5.6. *Let X be a Banach space and let $K \subset X$ be a bounded set. The following assertions are equivalent:*

(i) K is not relatively SWC;

(ii) there exists an free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ such that $K_{\mathcal{U}}$ is not relatively weakly compact;

(iii) for all free ultrafilters $\mathcal{U}$ on $\mathbb{N}$, $K_{\mathcal{U}}$ is not relatively weakly compact;

(iv) $\exists\theta > 0\ \forall n \geq 1\ \exists x_1, ..., x_n \in K$ such that $\forall k \in \{1, ..., n-1\}$ $d(\mathrm{conv}\{x_j\}_{j\leq k}, \mathrm{conv}\{x_j\}_{j>k}) \geq \theta$.

Proof. $(iii) \implies (ii) \implies (i)$ is trivial. Suppose that (iv) holds. For all $n \geq 1$, let $x_{n,1}, ..., x_{n,n}$ such that (iv) holds. Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. For $n \geq 1$, define $x^n \in K_{\mathcal{U}}$ by $x^n = (0, ..., 0, x_{n,n}, x_{n+1,n}, ...)_{\mathcal{U}}$ (with n zeros at the beginning). It is easy to see that $d(\mathrm{conv}\{x_j\}_{j\leq k}, \mathrm{conv}\{x_j\}_{j>k}) \geq \theta$ for all $k \geq 1$. By Theorem 1.5.4, $K_{\mathcal{U}}$ can not be relatively weakly compact, i.e. (iii) holds.
Now suppose that K is not relatively SWC. By Theorem 1.5.3, there exists a set A f.r. in K such that A is not relatively weakly compact. By James' theorem, there exist $\theta > 0$ and $(x_n)_{n\geq 1} \subset A$ such that $d(\mathrm{conv}\{x_j\}_{j\leq k}, \mathrm{conv}\{x_j\}_{j>k}) \geq \theta$ for all $k \geq 1$. Since K is bounded and by taking a subsequence if necessary, we can and do suppose that $\{x_n\}_{n\in\mathbb{N}}$ is linearly independant. Let $n \geq 1$. If $M_n = \mathrm{span}\{x_1, ...x_n\}$, there exists $K_n \subset K$ and $T : M_n \to \mathrm{span}(K_n)$ such that $T(x_i) \in K$ for all $i \in \{1, ..., n\}$ and $\frac{1}{2}\|x\| \leq \|T(x)\| \leq \|x\|$ for all $x \in M_n$. The finite sequence $(y_i = T(x_i))_{1\leq i\leq n}$ fulfils (iv) with $\frac{\theta}{2}$. $\square$

Note that the previous theorem together with Proposition 1.2.5 imply that a bounded set K is relatively SWC if and only if $\overline{K}^w$ is SWC.

Lemma 1.5.7. *Let C be a convex subset of a Banach space X. If $T : C \to C$ is an affine continuous mapping then T is weakly continuous.*

Proof. Let $(x_a)_{a\in A}$ be a net in C that weakly converges to some $x \in C$. Suppose that $(T(x_a))_{a\in A}$ does not weakly converge to $T(x)$. Then there exists a weak open neighborhood V of $T(x)$ and a subnet $(x_b)_{b\in B}$ of $(x_a)_{a\in A}$ such that $T(x_b) \notin V$ for all $b \in B$. We can write $V = \bigcap_{i=1}^{p} U_i$ with $U_i = \{y \in X \mid x_i^*(y - T(x)) < \varepsilon\}$. So, by taking another subnet if necessary, we can suppose that there exists $i_0 \in \{1, ..., p\}$ such that $x_b \notin U_{i_0}$ for all $b \in B$. Since $(x_b)_{b\in B}$ weakly converges to x, we have that $x \in \overline{\mathrm{conv}}\{x_b\}_{b\in B}$ and then there exists a sequence $(y_n)_{n\in\mathbb{N}} \subset \mathrm{conv}\{x_b\}_{b\in B}$ such that $y_n \to x$. By continuity of T, we have that $T(y_n) \to T(x)$. However using that T is affine, the convexity of U_i^c and the fact that $T(x_b) \notin U_{i_0}$ for all $b \in B$, it is easy to see that $T(y_n) \notin U_{i_0}$ for all $n \in \mathbb{N}$. This is a contradiction and the proof is complete. $\square$

Corollary 1.5.8. *Suppose that X and Y are Banach spaces and let $K \subset X$ be a (relatively) convex SWC set. Let $T : K \to Y$ be a uniformly continuous affine function. Then $T(K)$ is (relatively) SWC.*

Proof. Fix $\theta > 0$ and take $\varepsilon > 0$ such that for all $x, y \in K$, if $\|x-y\| < \varepsilon$ then $\|T(x)-T(y)\| < \theta$. Let $n \in \mathbb{N}$ and let $y_1, ..., y_n \in T(K)$ and let $x_i \in K$ such that $T(x_i) = y_i$ for all $i \in \{1, ..., n\}$. By Theorem 1.5.6, there exists $k \in \{1, ..., n-1\}$ such that $d(\mathrm{conv}\{x_j\}_{j\le k}, \mathrm{conv}\{x_j\}_{j>k}) < \varepsilon$. So, there exists $a_1, ..., a_k, b_{k+1}, ..., b_n \ge 0$ such that $\sum_{j=1}^{k} a_j = 1$, $\sum_{j=k+1}^{n} b_j = 1$ and

$$\left\| \sum_{j=1}^{k} a_j x_j - \sum_{j=k+1}^{n} b_j x_j \right\| < \varepsilon.$$

It follows that

$$\left\| \sum_{j=1}^{k} a_j y_j - \sum_{j=k+1}^{n} b_j y_j \right\| = \left\| \sum_{j=1}^{k} a_j T(x_j) - \sum_{j=k+1}^{n} b_j T(x_j) \right\| = \left\| T\left(\sum_{j=1}^{k} a_j x_j\right) - T\left(\sum_{j=k+1}^{n} b_j x_j\right) \right\| < \theta.$$

By the previous lemma, $T(K)$ is (relatively) weakly compact and therefore bounded. The conclusion follows by Theorem 1.5.6. $\square$

Theorem 1.5.9. *Let X be a Banach space and let $K \subset X$ be a bounded closed convex set. The following assertions are equivalent:*

(i) *K is not SWC*

(ii) $\exists \theta > 0\ \forall n \ge 1\ \exists x_1, ..., x_n \in K\ \exists x_1^*, ..., x_n^* \in B_{X^*}$ *such that* $x_i^*(x_j) = \theta$ *if* $i \le j$ *and* $x_i^*(x_j) = 0$ *if* $i > j$.

Proof. It is easy to see that (ii) implies (iv) of Theorem 1.5.6, so $(ii) \implies (i)$ holds. Suppose that (i) is true and let $\mathcal{U}$ be an ultrafilter such that $K_{\mathcal{U}}$ is not weakly compact. By Theorem 1.5.5, there exist $\theta > 0$, $(f_n)_{n\ge1} \subset B_{(X^{\mathcal{U}})^*}$ and $(x^n)_{n\ge1} \subset K_{\mathcal{U}}$ such that $f_i(x^j) = \theta$ if $i \le j$ and $f_i(x^j) = 0$ if $i > j$. Let $n \ge 1$. Since $\{x^1, ..., x^n\}$ is linearly independent and $K_{\mathcal{U}}$ is f.r. in K, there exist $K_n \subset K$ and an isomorphism $T : \mathrm{span}\{x^1, ..., x^n\} \to \mathrm{span}(K_n)$ such that $y_i := T(x^i) \in K$ for all $i \in \{1, ..., n\}$ and $\frac{1}{2}\|x\| \le \|T(x)\| \le \|x\|$ for all $x \in \mathrm{span}\{x^1, ..., x^n\}$. Let y_j^* be a Hahn-Banach extension of $f_j \circ T^{-1}$. Then $\|y_j\| \le \|f_j\|\|T^{-1}\| \le 2$ and we have that $y_i^*(x_j) = \theta$ if $i \le j$ and $y_i^*(x_j) = 0$ if $i > j$. Then (ii) holds with $x_i^* = \frac{1}{2} y_i^*$ and $\theta' = \frac{\theta}{2}$. $\square$

Definition 1.5.10. A *dyadic tree* of height $n \in \mathbb{N} \cup \{\infty\}$ is a set of the form $\{x_s : |s| \le n\}$, indexed by finite sequences $s \in \bigcup_{k=0}^{n} \{0,1\}^k$ of length $|s| \le n$, such that $x_s = 2^{-1}(x_{s\frown 0} + x_{s\frown 1})$ for every $|s| < n$, where $\{0,1\}^0 := \{\emptyset\}$ indexes the root $x_\emptyset$ and the symbol "$\frown$" stands for concatenation. We say that a dyadic tree $\{x_s : |s| \le n\}$ is *ε-separated* if $\|x_{s\frown 0} - x_{s\frown 1}\| \ge \varepsilon$ for every $|s| < n$. A ε-separated tree is *strongly separated* if the points of the tree are at a distance greater than $\varepsilon/2$ from each other. Bushes are defined in a very similar way, however the index set is $\bigcup_{k=0}^n \mathbb{N}^k$ and $x_s = \sum_k \lambda_{s\frown k} x_{s\frown k}$ where $\lambda_{s\frown k} \geq 0$, $\lambda_{s\frown k} = 0$ except for finitely many k's and $\sum_k \lambda_{s\frown k} = 1$. We say that a bush $\{x_s : |s| \leq n\}$ is ε-separated if $\|x_{s\frown k} - x_s\| \geq \varepsilon$ for all k such that $\lambda_{s\frown k} > 0$.

Definition 1.5.11. A subset of Banach space has the *finite tree property* (resp. *strong finite tree property*) if there exists $\varepsilon > 0$ such that it contains a ε-separated tree (resp. a ε-strongly separated tree) of height n for all $n \in \mathbb{N}$.

Theorem 1.5.12. *Let K be a bounded closed convex subset of a Banach space X. The following assertions are equivalent:*

(i) K is SWC;

(ii) K does not have the finite tree property;

(iii) K does not have the strong finite tree property.

Proof. $(ii) \implies (iii)$ is obvious.

$(iii) \implies (i)$ Suppose that K is not SWC. Then there exists an ultrafilter such that $K_{\mathcal{U}}$ is not weakly compact. By Theorem 1.5.5, there exist $\theta > 0$, a sequence $(y_n)_{n\in\mathbb{N}} \subset K$ and a sequence $(y_n^*)_{n\in\mathbb{N}}$ such that $y_n^*(y_k) = 0$ if $n > k$ and $y_n^*(y_k) = \theta$ if $n \leq k$. Note that for all $k < p$ and all real numbers α_i, one has that

$$\left\|\sum_{i=1}^{k} \alpha_i y_i - \sum_{i=k+1}^{p} y_i\right\| \geq y_{k+1}^*\left(\sum_{i=1}^{k} \alpha_i y_i - \sum_{i=k+1}^{p} y_i\right) \geq \theta(p-k). \tag{1.1}$$

Let $n \in \mathbb{N}$. Define a finite sequence $(z_i)_{0\leq i\leq 2^{n+1}-2} \subset K$ by

$$z_0 = \frac{1}{2^n}\sum_{i=1}^{2^n} y_i$$

$$z_1 = \frac{1}{2^{n-1}}\sum_{i=1}^{2^{n-1}} y_i, \quad z_2 = \frac{1}{2^{n-1}}\sum_{i=2^{n-1}+1}^{2^n} y_i$$

$$z_3 = \frac{1}{2^{n-2}}\sum_{i=1}^{2^{n-2}} y_i, \quad z_4 = \frac{1}{2^{n-2}}\sum_{i=2^{n-2}+1}^{2^{n-1}} y_i, \quad z_5 = \frac{1}{2^{n-2}}\sum_{i=2^{n-1}+1}^{3.2^{n-1}} y_i, \quad z_6 = \frac{1}{2^{n-2}}\sum_{i=3.2^{n-1}+1}^{2^n} x_i$$

and so on until $z_{2^{n+1}-2}$. Using (1.1), it is easily seen that $(z_i)_{0\leq i\leq 2^{n+1}-2} \subset K$ is a strong (n,θ)-tree. Using the finite-representability of $K_{\mathcal{U}}$ in K (note that $(z_i)_{0\leq i\leq 2^{n+1}-2}$ is linearly independant since $(y_n)_{n\in\mathbb{N}}$ is), we deduce that K contains a strong $(n, \frac{\theta}{2})$-tree for all $n \in \mathbb{N}$.

$(i) \implies (ii)$ Now suppose that K has the finite tree property. Let $\{x_s^n : |s| \leq n\}$ be a ε-separated tree of height n in K for a suitable ε. For all n, define a semi-norm in c_{00} by $\|(a_i)_i\|_n = \left\|\sum_{|s|\leq n} a_i x_s^n\right\|$. A diagonal procedure based on a separability argument permits to find a strictly increasing sequence $(n_k)_k$ such that $\|(a_i)\| := \lim_k \|(a_i)_i\|_{n_k}$ is well-defined for all $(a_i)_i \in c_{00}$. Suppose that c_{00} is endowed this semi-norm. Let Y the completion of c_{00}/N where $N = \{a \in c_{00} \ : \ \|a\| = 0\}$. Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. Define $T : Y \to X_{\mathcal{U}}$ by

$$T((a_i)_i + N) = \left(\sum_{|s|\leq n_k} a_i x_s^{n_k}\right)_{\mathcal{U},k}.$$

Let $(e_i)_i$ is the canonical basis of c_{00}. It is easy to see that T is an isometry and that $(T(e_i + N))_{i\in\mathbb{N}}$ is a infinite ε-separated tree, called T, in $K_{\mathcal{U}}$. Suppose that K is SWC, then $K_{\mathcal{U}}$ is weakly compact and $C := \overline{\text{co}}(T)$ also is. By Theorem 8.13 in [67], C has a strongly exposed point x. Let f be a functional that strongly exposes x. Let $\delta > 0$ such that $\|z - x\| < \frac{\varepsilon}{8}$ if $f(z) > f(x) - \delta$ with $z \in T$. Since $f(x) = \sup_T f$, there exist $t \in T$ such that $f(t) > f(x) - \delta$. Let $t_1, t_2 \in T$ such that $t = \frac{1}{2}(t_1 + t_2)$. Suppose without loss of generality that $f(t_1) > f(x) - \delta$. It follows that $\|t - x\| < \frac{\varepsilon}{8}$ and $\|t_1 - x\| < \frac{\varepsilon}{8}$. So $\|t_1 - t\| < \frac{\varepsilon}{4}$, which contradicts the fact that $\|t - t_1\| = \frac{1}{2}\|t_1 - t_2\| \geq \frac{\varepsilon}{2}$. Then K is not SWC. □

If $(x_n)_n$ is a basic sequence (eventually a finite sequence), $\text{bc}(x_n)_n$ denotes its basic constant:

Theorem 1.5.13. *Let K be a weakly closed bounded subset of a Banach space X such that $0 \notin K$. The following propositions are equivalent:*

(i) K is not SWC;

(ii) for all $n \in \mathbb{N}$, there exists $x_1^n, ..., x_n^n \in K$ linearly independant such that

$$\sup_{n\in\mathbb{N}} bc(x_i^n)_{1\leq i\leq n} < +\infty.$$

Proof. $(i) \implies (ii)$ Suppose that K is not SWC. There exist a Banach space Y and a closed subset $A \subset Y$ which is finitely representable in K. We can suppose that $0 \notin A$. In fact, if A is an isolated point of A, just consider $A' = A \setminus \{0\}$. Otherwise, there exists $(y_n)_n \subset A$ such that $y_n \to 0$ and $y_n \neq 0$. For all $n \in \mathbb{N}$, by the definition of finite representability, there exist $x_n \in K$ and $T_n : \text{span}\{y_n\} \to \text{span}\{x_n\}$ such that $T(y_n) = x_n$ and $\|T_n\| \leq 1$. It follows that $\|x_n\| \leq \|y_n\| \to 0$ and then $0 \in K$, which is a contradiction. By Theorem 1.5.6 in [1], A contains a basic sequence $(y_n)_n$. Let $b = \text{bc}(y_n)_n$. Let $n \geq 1$. Since $\{y_1, ..., y_n\}$ is a finite linearly independent set, the definition of finite representability implies that there exist $x_1^n, ..., x_n^n \in K$ and an isomorphism $T : \text{span}\{y_1, ..., y_n\} \to \text{span}\{x_1^n, ..., x_n^n\}$ such that $T(y_i) = x_i^n$ and for all $y \in \text{span}\{y_1, ..., y_n\}$

$$\frac{1}{2}\|y\| \leq \|T(y)\| \leq \frac{3}{2}\|y\|.$$

If $a_1, ...a_n \in \mathbb{R}$ and $p \leq n$, it follows that

$$\left\|\sum_{i=1}^{p} a_i x_i^n\right\| = \left\|\sum_{i=1}^{p} a_i T(y_i)\right\| \leq \frac{3}{2}\left\|\sum_{i=1}^{p} a_i y_i\right\| \leq \frac{3b}{2}\left\|\sum_{i=1}^{n} a_i y_i\right\| \leq \frac{3b}{2}\left\|\sum_{i=1}^{n} a_i T^{-1}(x_i^n)\right\| \leq 3b\left\|\sum_{i=1}^{n} a_i x_i^n\right\|$$

and this part is complete.

$(ii) \implies (i)$ Suppose that (ii) holds. If K is not weakly compact then (i) holds so we can suppose obviously that K is weakly compact. For all n, define a semi-norm in c_{00} by $\|(a_i)_i\|_n = \|\sum_{i=1}^n a_i x_i^n\|$. As in the previous proof, let $(n_k)_k$ be a strictly increasing sequence such that $\|(a_i)\| := \lim_k \|(a_i)_i\|_{n_k}$ is well-defined for all $(a_i)_i \in c_{00}$. Let Y the completion of c_{00}/N (c_{00} with this new semi-norm) where $N = \{a \in c_{00} \ : \ \|a\| = 0\}$. Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. Define as before the isometry $T : Y \to X_{\mathcal{U}}$ by

$$T((a_i)_i + N) = \left(\sum_{i=1}^{n_k} a_i x_i^{n_k}\right)_{\mathcal{U},k}.$$

Let show that $(T(e_i + N))_{i\in\mathbb{N}}$ is a basic sequence. For simplicity, we write $\tilde{e_i} = T(e_i + N) = (x_i^{n_k})_{\mathcal{U}}$. Let $A = \sup_{n\in\mathbb{N}} \text{bc}(x_i^n)_{1\leq i\leq n} < +\infty$. Take $1 \leq p \leq q$ and $a_1, ..., a_q \in \mathbb{R}$. We have that

$$\left\|\sum_{i=1}^{p} a_i\tilde{e_i}\right\| = \left\|\sum_{i=1}^{p} a_i(x_i^{n_k})_{\mathcal{U}}\right\| \leq \lim_{\mathcal{U}}\left\|\sum_{i=1}^{p} a_i x_i^{n_k}\right\| \leq A\left\|\sum_{i=1}^{q} a_i x_i^{n_k}\right\| = A\left\|\sum_{i=1}^{q} a_i(x_i^{n_k})_{\mathcal{U}}\right\| = A\left\|\sum_{i=1}^{q} a_i\tilde{e_i}\right\|,$$

i.e. $(T(e_i+N))_{i\in\mathbb{N}}$ is a basic sequence in $K_{\mathcal{U}}$. By Theorem 1.5.6 in [1] again, it implies that $K^{\mathcal{U}}$ is not relatively weakly compact or $0 \in \overline{K_{\mathcal{U}}}^w$. To conclude that K is not SWC, let show that 0 can not be in $\overline{K_{\mathcal{U}}}^w$. In fact, suppose that there exist a net $((x_i^\lambda)_{\mathcal{U}})_\lambda$ in $K_{\mathcal{U}}$ which converges weakly to 0. By weak compactness, $x_\lambda = \lim_{\mathcal{U}}(x_i^\lambda)$ exists and belongs to K. It follows that for every $x^* \in X^*$, one has that

$$\lim_{\mathcal{U},i} x^*(x_i^\lambda) = x^*\left(\lim_{\mathcal{U},i} x_i^\lambda\right) = x^*(x_\lambda) \to 0.$$

That means that $(x_\lambda)_\lambda$ weakly converges to 0, which is a contradiction since $0 \notin K$. □

Definition 1.5.14. Let A be a bounded subset of a Banach space X. We say A has the super-Banach-Saks (in short, super-BS) if $A_{\mathcal{U}}$ has the BS for all free ultrafilters $\mathcal{U}$. We also say that X has the super-BS if B_X has the super-BS.

Note that the same arguments used in the proof of theorem 1.5.3 show that A has the super-BS property if and only if every set f.r. in A has the BS property.

The BS and the SWC are closely related notions. In fact, we have the following result (see [122, Corollary 2.4]):

Theorem 1.5.15. *Let K be a bounded subset of a Banach space X. The following assertions are equivalent:*

(i) K is relatively SWC;

(ii) K has the super-BS.

Proof. $(ii) \implies (i)$ Let A be a set f.r. in K. The set A is bounded with the BS. Then A is relatively weakly compact by Theorem 1.4.15. By Theorem 1.5.3, K is relatively SWC.

$(i) \implies (ii)$ Suppose that K is relatively SWC and let A be a set f.r. in K. Suppose that A does not have the BS. Since A is relatively weakly compact, Theorem 1.4.15 implies that A contain a spreading sequence $(x_n)_{n\in\mathbb{N}}$ whose spreading model Z is isomorphic to l_1. Its fundamental sequence $(e_n)_{n\in\mathbb{N}}$ is equivalent to the canonical basis of l_1 (by Lemma 1.4.11) and then is not relatively weakly compact. By Proposition 1.4.3, $\{e_n\}_{n\in\mathbb{N}}$ is f.r. in $\{x_n\}_{n\in\mathbb{N}}$ and then in K, which is a contradiction since K is relatively SWC. □

1.5.2 Super weakly compact operators

Beauzamy [16] introduced an operator version of superreflexivity under the name of *uniformly convexifying* property (of an operator), but it was later renamed. The equivalence between these two notions will be quantified in Theorem 3.2.9.

Definition 1.5.16. A bounded operator $T : X \to Y$ between two Banach spaces is said to be super weakly compact (in short, SWC) if for all free ultrafilters $\mathcal{U}$ on a set I, the operator $T_{\mathcal{U}} : X_{\mathcal{U}} \to Y_{\mathcal{U}}$ defined by $T_{\mathcal{U}}((x_i)_{\mathcal{U}}) = (T(x_i))_{\mathcal{U}}$ is weakly compact.

The following easy proposition is left to the reader:

Proposition 1.5.17. *Let $T : X \to Y$ be a bounded operator between two Banach spaces. The following assertions are equivalent:*

(i) T is SWC;

(ii) $T(B_X)$ is relatively SWC;

(iii) there exists a free ultrafilter on $\mathbb{N}$ *such that* $T_{\mathcal{U}}$ *is weakly compact.*

Now let us prove that the set of super weakly compact operators is an operator ideal denoted by $\mathfrak{W}^{super}$. We start recalling the following lemma due to Grothendiek (see [67], Lemma 13.32):

Lemma 1.5.18. *Let* X *be a Banach space and let* $A \subset X$*. If for all* $\varepsilon > 0$*, there exists a weakly compact set* $A_\varepsilon \subset X$ *such that* $A \subset A_\varepsilon + \varepsilon B_X$ *then* A *is relatively weakly compact.*

The previous lemma leads easily to the following characterization of super weak compactness:

Proposition 1.5.19. *Let* K *be a subset of a Banach space* X*. The following assertions are equivalent:*

(i) K *is relatively SWC;*

(ii) for all $\varepsilon > 0$*, there exists a relatively SWC set* $K_\varepsilon \subset X$ *such that* $K \subset K_\varepsilon + \varepsilon B_X$*.*

Proposition 1.5.20. *The space of SWC operators between two Banach spaces* X *and* Y *is a closed subspace of* $\mathcal{B}(X,Y)$*, the space of bounded operators between* X *and* Y*.*

Proof. Suppose that $(T_n)_{n\in\mathbb{N}}$ is a sequence of SWC operators such that $T_n \to T$ in $\mathcal{B}(X,Y)$. Let $\varepsilon > 0$. There exists $n_0 \in \mathbb{N}$ such that $\|T_{n_0} - T\| < \varepsilon$. In particular, we have that $T(B_X) \subset T_{n_0}(B_X) + \varepsilon B_Y$ where $T_{n_0}(B_X)$ is relatively SWC. By Proposition 1.5.19, we deduce that $T(B_X)$ is relatively SWC, i.e. T is SWC. $\square$

Corollary 1.5.8 leads to the following result:

Proposition 1.5.21. *Let* X*,* Y *and* Z *three Banach spaces. Let* $T : X \to Y$ *and* $R : Y \to Z$ *two bounded operators. Then* RT *is SWC whenever* R *or* T *is SWC.*

Notably, $\mathfrak{W}^{super}$ is a symmetric ideal, that is:

Proposition 1.5.22. *Let* $T : X \to Y$ *be a bounded operator between two Banach spaces. The following assertions are equivalent:*

(i) T *is SWC;*

(ii) T^* *is SWC.*

We do not provide the proof of the previous result since it will be quantified and thus generalized in Theorem 3.2.2.

We refer the reader to [16, 17, 96] and Chapter 11 in [109] for more properties of $\mathfrak{W}^{super}$ and its relation with other operator ideals. See also [172] for characterizations in terms of martingale type and cotype, and [44] for a nonlinear characterization.

1.5.3 Super weakly compactly generated spaces

The definitions and results presented in this section can be found in [147].

Definition 1.5.23. A Banach space X is said to be *super weakly compactly generated* (in short, SWCG) if there exists a SWC set $K \subset X$ such that $X = \overline{\operatorname{span}(K)}$.

In this case, note that K can be taken to be convex and symmetric by Theorem 1.5.2.

Proposition 1.5.24. *Let X be a SWCG Banach space and let K be a SWC set such that $X = \overline{span(K)}$. Then there exist a reflexive Banach space Y and an one-to-one SWC operator $T : Y \to X$ such that $K \subset T(B_Y)$.*

Proof. Without loss of generality, we can suppose that K is symetric and convex. For $n \in \mathbb{N}$, define $U_n = 2^n K + \frac{1}{2^n} B_X$ and let $\|.\|_n$ its Minkovski funcional and note that it is an equivalent norm on X. For $x \in X$, define $|x| = \left(\sum_{n\in\mathbb{N}} \|x\|_n^2\right)^{\frac{1}{2}}$. Let $Y = \{x \in X \ : \ |x| < \infty\}$. With the norm $|.|$, it is easily seen that Y is a Banach space. Define $T : Y \to X$ by $T(y) = y$. The operator T is one-to-one and bounded and verifies that $K \subset T(B_Y)$. Let $C = B_{(Y,|.|)}$. Since $C \subset 2^n K + \frac{1}{2^n} B_X$ for all $n \in \mathbb{N}$, it follows that C is SWC by Proposition 1.5.19. The fact that Y is reflexive is classic and has been proved by Davis, Figiel, Johnson and Pełczyinski (see Theorem 13.22 of [67]). □

As a consequence, we obtain the two following corollaries:

Corollary 1.5.25. *Let X be a Banach space. The following assertions are equivalent:*

(i) X is SWCG;

(ii) there exist a reflexive Banach space Y and a (one-to-one) SWC operator $T : Y \to X$ such that $\overline{T(Y)} = X$.

Corollary 1.5.26. *Let K be a closed bounded subset of a Banach space X. The following assertions are equivalent:*

(i) K is SWC;

(ii) there exist a reflexive Banach space Y and a (one-to-one) SWC operator $T : Y \to X$ such that $K \subset T(B_Y)$.

Remark 1.5.27. In [147], Raja gave an example of a SWCG space X such that the space Y in the previous corollaries can not be chosen to be superreflexive.

Definition 1.5.28. The norm of the Banach space $(X, \|\cdot\|)$ is said to be *uniformly Gâteaux (UG) smooth* if for every $h \in X$

$$\sup\{\|x + th\| + \|x - th\| - 2 : x \in S_X\} = o(t) \text{ when } t \to 0.$$

Given a bounded set $H \subset X$, the norm is said to be *H-UG smooth* if

$$\sup\{\|x + th\| + \|x - th\| - 2 : x \in S_X, h \in H\} = o(t) \text{ when } t \to 0.$$

The norm is *uniformly Fréchet (UF) smooth* if it is B_X-UG smooth. Finally, the norm is said to be *strongly UG smooth* if it is H-UG smooth for some bounded and linearly dense subset $H \subset X$.

In [147], the author characterized the SWCG spaces in terms of renorming:

Theorem 1.5.29. *Let X be a Banach space. The following assertions are equivalent:*

(i) X is SWCG;

(ii) X admits an equivalent strongly UG smooth norm.

Definition 1.5.30. A Banach space X is said to be *strongly super weakly compactly generated* (in short, S^2WCG) if there is a SWC set $K \subset X$ that strongly generates X, i.e. for any weakly compact set $H \subset X$ and $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that $H \subset nK + \varepsilon B_X$.

In this case, we can suppose again that K is also convex and symmetric.

Note that if X is S^{2}WCG then any weakly compact subset of X is SWC by Proposition 1.5.19. A fundamental example of a S^{2}WCG space is $L^1(\Omega, \mathcal{A}, \mu, X)$ where $(\Omega, \mathcal{A}, \mu)$ is a finite measure space and X is a superreflexive Banach space (see [147] for the proof).

We just recall the following well-known definition:

Definition 1.5.31. Let $(X, \|\cdot\|)$ be a Banach space. The *modulus of convexity* of the norm is the function $\delta_X : [0,2] \to \mathbb{R}$ defined by

$$\delta_X(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| \ : \ x, y \in B_X \ \|x-y\| \geq \varepsilon\right\}.$$

We say that the norm is *uniformly convex* if $\delta_X(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$.

The following result is Theorem 3.2 in [147]. It will be used repeatedly in section 4.2 since it has a strong connexion with normal structure and fixed point properties.

Theorem 1.5.32. *Let K be a SWC absolutely convex subset of a Banach space X. Then X admits an equivalent norm $|.|$ such that the restriction of $|.|^2$ to any convex set strongly generated by K is uniformly convex.*

As a consequence, we obtain the following result (see Theorem 1.9 in [147]):

Corollary 1.5.33. *Let X be a S^2WCG Banach space. Then there is an equivalent norm on X such that its restriction to any reflexive subspace is uniformly convex. In particular, any reflexive subspace of X is superreflexive.*

Using a procedure called Asplund averaging (see [57] p.52), the previous norm can be chosen such that its restriction to any reflexive subspace is also UF smooth.

Chapter 2

Uniformly convex functions

In this chapter, we are dealing with real functions defined on X. If there is not specific hypothesis on the domain, we will follow the convention typical from Convex Analysis [28, 175] that a function f is defined everywhere and takes values in $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$. A function f is said to be *proper* if $f > -\infty$ and $\mathrm{dom}(f) := \{x \in X : f(x) < +\infty\} \neq \emptyset$. In the following, all the functions are supposed to be proper. However some operations performed on proper functions could lead to non-proper functions. The class of lower semicontinuous convex proper functions on X will be denoted $\Gamma(X)$.

2.1 Definitions and first properties

We start with the definition of ε-uniformly convex. For technical purposes, we need to introduce a slightly more general definition than Definition 0.0.11:

Definition 2.1.1. Let $(X, \|.\|)$ be a Banach space. Let $f : X \to \overline{\mathbb{R}}$ be a function and let $d : \mathrm{dom}(f) \to \mathbb{R}$ be a pseudo-metric which is uniformly continuous with respect to $\|.\|$. The function f is said to be *ε-uniformly convex with respect to d* if for a given $\varepsilon > 0$, there is $\delta > 0$ such that if $d(x, y) \geq \varepsilon$ then

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x) + f(y)}{2} - \delta.$$

The function is said *uniformly convex with respect to d* if it is ε-uniformly convex for all $\varepsilon > 0$. If f is ε-uniformly convex (resp. uniformly convex) with respect to $\|.\|$, we simply say that f is ε-uniformly convex (resp. uniformly convex). To finish, we say that f is ε^+-uniformly convex (resp. with respect to d) if it is ε'-uniformly convex (resp. with respect to d) for every $\varepsilon' > \varepsilon$.

The suggestive name *discrete uniformly convex functions* applied to functions which are ε-uniformly convex for some $\varepsilon > 0$ could be misleading here. Clearly, a uniformly convex function is midpoint-convex, that is, the inequality $f(\frac{x+y}{2}) \leq \frac{f(x)+f(y)}{2}$ holds whenever $x, y \in X$. Therefore, a uniformly convex function is convex provided some regularity holds (e.g., if f is lower semicontinuous). The notion of uniform convexity for functions was introduced by Levitin and Polyak [124], and based on Clarkson's uniform convexity for normed spaces [52]. Since then, the properties of uniformly convex functions have been studied in several papers, notably [166, 167, 174, 14, 27, 29], the Section 3.5 in Zalinescu's book [175], and part of chapter 5 in Borwein-Vaderwerff's book [28] devoted to them. In relation to the standard theory, let us point out the notion of modulus of uniform convexity

$$\delta_f(\varepsilon) = \inf\left\{\frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right) : x, y \in \operatorname{dom}(f), \|x-y\| \geq \varepsilon\right\}.$$

Note that δ_f could take negative values unless f is supposed to be (midpoint-)convex. Analogously, it is possible to define ε-uniformly concave functions, however it will not be necessary to treat them here because all the theory extends trivially.

We will discuss in this section results of almost arithmetical nature. The first proposition contains some easy facts whose proof is left to the reader.

Proposition 2.1.2. *Let X be a Banach space. Let $\varepsilon > 0$ and let $f : X \to \overline{\mathbb{R}}$ be an ε-uniformly convex function. Then:*

(a) If g is convex, then $f+g$ is ε-uniformly convex with $\delta_{f+g} \geq \delta_f$.

(b) The supremum of finitely many ε-convex functions is ε-convex too.

(c) If $f \geq 0$, then f^2 is ε-uniformly convex.

(d) The lower semicontinuous envelope of f is ε-uniformly convex.

Recall that the infimal convolution of two functions f, g is defined as

$$(f \,\square\, g)(x) = \inf\{f(x-y) + g(y) : y \in X\}, \text{ for } x \in X.$$

Proposition 2.1.3. *Let X be a Banach space. Let $f_1, f_2 : X \to \overline{\mathbb{R}}$ be two convex functions such that f_1 is ε_1-uniformly convex and f_2 is ε_2-uniformly convex for $\varepsilon_1, \varepsilon_2 > 0$. Then $f_1 \,\square\, f_2$ is $(\varepsilon_1+\varepsilon_2)$-uniformly convex with modulus $\min\{\delta_{f_1}(\varepsilon_1), \delta_{f_2}(\varepsilon_2)\}$.*

Proof. Given $x_1, x_2 \in \operatorname{dom}(f_1 \,\square\, f_2) = \operatorname{dom}(f_1) + \operatorname{dom}(f_2)$ with $\|x_1 - x_2\| \geq \varepsilon_1 + \varepsilon_2$ and $\eta > 0$ we may find $y_1, y_2 \in \operatorname{dom}(f_2)$ such that

$$f_1(x_1 - y_1) + f_2(y_1) < (f_1 \,\square\, f_2)(x_1) + \eta,$$

$$f_1(x_2 - y_2) + f_2(y_2) < (f_1 \,\square\, f_2)(x_2) + \eta.$$

We have

$$\|(x_1 - y_1) - (x_2 - y_2)\| + \|y_1 - y_2\| \geq \|x_1 - x_2\| \geq \varepsilon_1 + \varepsilon_2.$$

Therefore, one of the inequalities either

$$\|(x_1 - y_1) - (x_2 - y_2)\| \geq \varepsilon_1 \quad \text{or} \quad \|y_1 - y_2\| \geq \varepsilon_2$$

holds. Assume the first one does (the other case is similar)

$$(f_1 \,\square\, f_2)\left(\frac{x_1+x_2}{2}\right) \leq f_1\left(\frac{x_1+x_2}{2} - \frac{y_1+y_2}{2}\right) + f_2\left(\frac{y_1+y_2}{2}\right)$$

$$\leq \frac{f_1(x_1-y_1) + f_1(x_2-y_2)}{2} - \delta_{f_1}(\varepsilon_1) + \frac{f_2(y_1)+f_2(y_2)}{2}$$

$$\leq \frac{(f_1 \,\square\, f_2)(x_1) + (f_1 \,\square\, f_2)(x_2)}{2} - \delta_{f_1}(\varepsilon_1) + \eta$$

which implies the statement as $\eta > 0$ was arbitrary. □

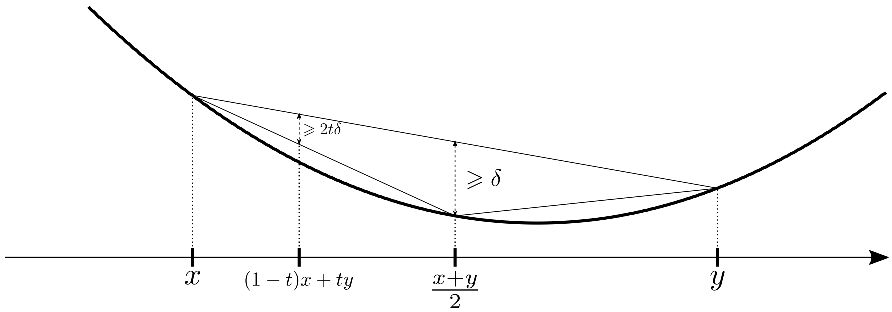

Figure 2.1: Geometric interpretation of Proposition 2.1.4.

Now we will discuss some properties of the modulus of uniform convexity in the classic case, that is, when the function is assumed to be also convex. The following property can be deduced easily with the help of a picture.

Proposition 2.1.4. *Let X be a Banach space. Let $f : X \to \overline{\mathbb{R}}$ be convex and $\varepsilon > 0$. Then*

$$(1-t)f(x) + tf(y) - f((1-t)x + ty) \geq 2\delta_f(\varepsilon) \min\{t, 1-t\}$$

whenever $x, y \in \mathrm{dom}(f)$, $\|x - y\| \geq \varepsilon$ and $t \in [0, 1]$ (cf Fig. 2.1).

Proof. Without loss of generality we may assume $t \in [0, 1/2]$ so $t = \min\{t, 1-t\}$. Note now that

$$(1-t)x + ty = (1-2t)x + 2t\frac{x+y}{2}.$$

By convexity of f we have

$$f((1-t)x + ty) \leq (1-2t)f(x) + 2tf\left(\frac{x+y}{2}\right)$$

$$\leq (1-2t)f(x) + 2t\left(\frac{f(x)+f(y)}{2} - \delta_f(\varepsilon)\right) = (1-t)f(x) + tf(y) - 2t\delta_f(\varepsilon)$$

as wished. □

The *gage of uniform convexity* is introduced in [166] (see also [175, p. 203]) for convex function as

$$p_f(\varepsilon) = \inf\left\{\frac{(1-t)f(x) + tf(y) - f((1-t)x + ty)}{t(1-t)} : 0 < t < 1, \|x - y\| \geq \varepsilon\right\}.$$

Corollary 2.1.5. *Let X be a Banach space. For any convex function $f : X \to \overline{\mathbb{R}}$ and $\varepsilon > 0$, we have*

$$2\delta_f(\varepsilon) \leq p_f(\varepsilon) \leq 4\delta_f(\varepsilon).$$

Proof. The first inequality is a consequence of Proposition 2.1.4 together with the fact that $\min\{t, 1-t\} \geq t(1-t)$, for $t \in \mathbb{R}$. The second inequality follows just taking $t = 1/2$. □

Therefore, for convex functions, ε-uniformly convexity can be expressed as $p_f(\varepsilon) > 0$. The gage of uniform convexity has the following remarkable property

$$p_f(\lambda\varepsilon) \geq \lambda^2 p_f(\varepsilon)$$

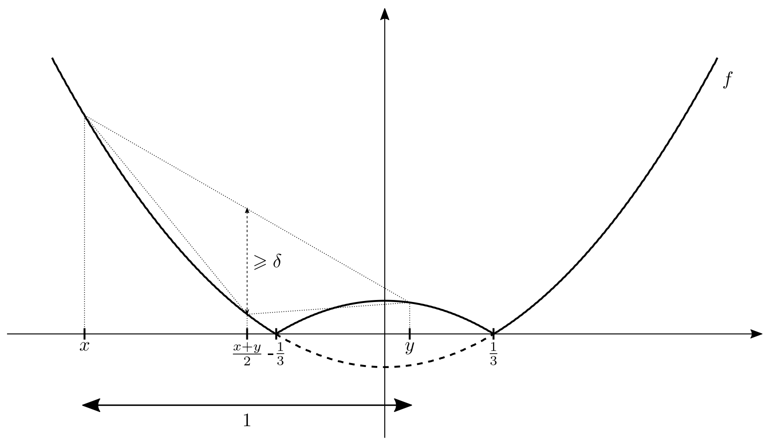

Figure 2.2: Example 2.1.6.

whenever $\varepsilon \geq 0$ and $\lambda \geq 1$, see [175, Proposition 3.5.1] and note that the proof does not requiere the uniform convexity of f. In particular $\varepsilon \to \varepsilon^{-2} p_f(\varepsilon)$ is a non decreasing function.

Now we will discuss some examples showing the limitations of the notions we are dealing with.

Example 2.1.6. $f(x) = |x^2 - 1/9|$ is a continuous nonconvex 1-uniformly convex function on $\mathbb{R}$.

Proof. This can be deduced by inspection of the drawing (see Fig. 2.2). A more detailed computation shows that $\delta = 1/36$. □

Example 2.1.7. A (proper) ε-uniformly convex function may have a non-proper lower semicontinuous convex envelope.

Proof. Take a function f which is finite and unbounded below on $B(0, \varepsilon/3)$ and takes the value $+\infty$ outside. By the very definition, f is ε-uniformly convex and necessarily $\breve{f} = -\infty$ on $B(0, \varepsilon/3)$. □

Example 2.1.8. A uniformly convex continuous function taking finite values which is unbounded on a bounded convex closed set.

Proof. The function will be defined on ℓ^2. Firstly note that $\|x\|^2$ is uniformly convex. Consider the convex function $h : \mathbb{R} \to \mathbb{R}$ defined by

$$h(t) = \max\{0, t - 1/2, -t - 1/2\}.$$

The series $g(x) = \sum_{n=1}^{\infty} n h(x_n)$, for $x = (x_n) \in \ell^2$, defines a convex continuous function. Indeed, at each point, only a finite number of summands can be positive at once. The continuity comes from the fact that the same is true on any ball of radius less than 1. Now, the function $f(x) = \|x\|^2 + g(x)$ is continuous, unbounded on B_{ℓ^2} and, by Proposition 2.1.2, it is also uniformly convex. □

The following notions will be useful in relation with ε-uniform convexity.

Definition 2.1.9. Let X be a Banach space. Let $f : X \to \overline{\mathbb{R}}$ be a function. Then f is said to be:

(a) quasi-convex if

$$f(\lambda x + (1-\lambda)y) \le \max\{f(x), f(y)\}$$

for every $x, y \in X$ and $\lambda \in [0,1]$.

(b) ε-uniformly quasi-convex if, for a given $\varepsilon > 0$, there is some $\delta > 0$ such that

$$f\left(\frac{x+y}{2}\right) \le \max\{f(x), f(y)\} - \delta$$

whenever $x, y \in X$ with $\|x-y\| \ge \varepsilon$ (or $d(x,y) \ge \varepsilon$ for a pseudometric d).

(c) uniformly quasi-convex if it is ε-uniformly quasi-convex for every $\varepsilon > 0$.

Whereas the notion of quasi-convexity is well known, our definition of uniform quasi-convexity is weaker than the one given in [167]. As with convexity, the midpoint version does not implies the "λ-version" unless some regularity (e.g. lower semicontinuity) is assumed. The following result shows one relation between the quantified versions of uniform convexity and uniform quasi-convexity for functions.

Proposition 2.1.10. *Let X be a Banach space. Let $\varepsilon > 0$ and let $f : X \to \overline{\mathbb{R}}$ be a positive, convex and ε-uniformly quasi-convex function. Then f^2 is ε-uniformly convex.*

Proof. The following inequality can be checked easily: if for some real numbers a, b, c we have $a + b \ge 2c \ge 0$ then

$$\left(\frac{a+b}{2} - c\right)^2 + \left(\frac{a-b}{2}\right)^2 \le \frac{a^2+b^2}{2} - c^2. \tag{2.1}$$

Assume $\|x-y\| \ge \varepsilon$ and let $\delta > 0$ be given by the definition of ε-uniform quasi-convexity. If $|f(x) - f(y)| > \delta$ the previous inequality implies

$$\frac{f(x)^2 + f(y)^2}{2} - f\left(\frac{x+y}{2}\right)^2 \ge \frac{\delta^2}{4}.$$

On the other hand, if $|f(x) - f(y)| \le \delta$ then

$$f\left(\frac{x+y}{2}\right) \le \max\{f(x), f(y)\} - \delta \le \frac{f(x)+f(y)}{2} - \frac{\delta}{2}$$

and thus

$$\frac{f(x)^2 + f(y)^2}{2} - f\left(\frac{x+y}{2}\right)^2 \ge \frac{\delta^2}{4}$$

using again the inequality (2.1). □

Example 2.1.11. A uniformly quasi-convex non-convex (concave) function.

Proof. Take $f(x) = x$ for $x < 0$, and $f(x) = x/2$ for $x \ge 0$. □

2.2 Convexifying the ε-uniform convexity

In order to cover previous developments around finite dentability [146], we will consider uniformly convex functions with respect to a pseudometric d defined on the domain of f. The norm of the Banach space will still play an important role and we require that d be uniformly continuous with respect to the norm. Therefore, along this section we will assume that ε-uniform convexity refers to d. We will refer as d-diameter of a subset in $X \times \mathbb{R}$ the diameter with respect to d of the projection of the set onto X. Let ϖ be the modulus of uniform continuity (the standard symbol is "ω" but we are using it as the first countable ordinal later), that is, the following inequality holds

$$d(x, y) \leq \varpi(\|x - y\|)$$

and $\lim_{t \to 0^+} \varpi(t) = 0$.

Proposition 2.2.1. *Let X be a Banach space. Let $f : X \to \overline{\mathbb{R}}$ be a function and let $\varepsilon > 0$. Then*

(a) If f is ε-uniformly convex then every slice of $\operatorname{epi}(f)$ disjoint from $\operatorname{epi}(f + \delta_f(\varepsilon))$ has d-diameter less than ε.

(b) If $f \in \Gamma(X)$ and there is $\delta > 0$ such that every slice of $\operatorname{epi}(f)$ disjoint from $\operatorname{epi}(f + \delta)$ has d-diameter less than ε then f is ε-uniformly convex with modulus $\delta_f(\varepsilon) \geq \delta/2$.

Proof. For the first statement, assume that $(x, f(x)), (y, f(y))$ belong to such a slice. The separation from $\operatorname{epi}(f + \delta_f(\varepsilon))$ implies

$$\frac{f(x) + f(y)}{2} < f\left(\frac{x + y}{2}\right) + \delta_f(\varepsilon)$$

and so $d(x, y) < \varepsilon$. On the other hand, let $\delta > 0$ as in statement (b) and take $x, y \in X$ such that the following inequality holds

$$\frac{f(x) + f(y)}{2} - f\left(\frac{x + y}{2}\right) < \frac{\delta}{2}.$$

It implies that $\left(\frac{x+y}{2}, \frac{f(x)+f(y)}{2}\right)$ does not belong to $\operatorname{epi}(f + \delta/2)$. We may take an affine function h such that $h < f + \delta/2$ and $h(\frac{x+y}{2}) > \frac{f(x)+f(y)}{2}$. It is evident that either $f(x) < h(x)$ or $f(y) < h(y)$. We may assume without loss of generality that the first inequality holds as the scenario is symmetric for x and y. Now we have

$$f(y) < 2h\left(\frac{x + y}{2}\right) - f(x) = h(x) + h(y) - f(x) < h(y) + \frac{\delta}{2}.$$

That implies both $(x, f(x))$ and $(y, f(y))$ belong to the slice defined by $h + \delta/2$

$$S = \{(x, t) \in \operatorname{epi}(f) : t < h(x) + \delta/2\}.$$

By our choices, we have $S \cap \operatorname{epi}(f + \delta) = \emptyset$ and thus $d(x, y) < \varepsilon$ by the hypothesis. We deduce in this way that $\delta/2 \leq \delta_f(\varepsilon)$. ☐

Corollary 2.2.2. *Let X be a Banach space. Let $\varepsilon > 0$ and let $f : X \to \overline{\mathbb{R}}$ be a convex and ε-uniformly convex function. Then*

$$f(x) \leq \sum_{k=1}^{n} \lambda_k f(x_k) - \delta_f(\varepsilon)$$

whenever $x, x_1, \ldots, x_n \in \operatorname{dom}(f)$ satisfy that $d(x, x_k) \geq \varepsilon$ and $x = \sum_{k=1}^{n} \lambda_k x_k$ with $\lambda_k \geq 0$ and $\sum_{k=1}^{n} \lambda_k = 1$.

Proof. If the inequality does not hold, then $(x, \sum_{k=1}^{n} \lambda_k f(x_k))$ does not belong to $\text{epi}(f + \delta_f(\varepsilon))$ so it can be separated from that set with a slice. Necessarily, one of the points $(x_k, f(x_k))$ belongs to the slice. That implies that the d-diameter of the slice is at least ε which contradicts the previous proposition. □

The following result is based on the techniques of the geometrical study of the Radon–Nikodym property (RNP), see [35]. Note that the technique works only on bounded domains.

Lemma 2.2.3. *Let X be a Banach space. Let $\varepsilon > 0$ and let $f : X \to \overline{\mathbb{R}}$ be a bounded below function with bounded domain. Let $m > 0$ be an upper bound for the norm diameter of dom(f) and let $\tau > 0$ be such that $\tau/m < 1$. Assume that the set*

$$\{x \in X : f(x) < \inf f + \delta\}$$

has d-diameter less than ε. Then the set

$$\{x \in X : \breve{f}(x) < \inf f + \delta\tau/m\}$$

has d-diameter less than $\varepsilon + 2\varpi(\tau)$.

Proof. Consider the sets

$$A = \{(x, r) \in X \times \mathbb{R} : f(x) \le r < \inf f + \delta\};$$

$$B = \{(x, r) \in X \times \mathbb{R} : r \ge \inf f + \delta, f(x) \le r\}.$$

Note that the epigraph of f is $A \cup B$. Consider their closed convex hulls $\breve{A} = \overline{\text{conv}}(A)$ and $\breve{B} = \overline{\text{conv}}(B)$ and note that $\text{conv}(\breve{A} \cup \breve{B})$ is dense in the epigraph of $\breve{f}$. Assume that $(x, r) \in \text{conv}(\breve{A} \cup \breve{B})$ and $r < \inf f + \delta\tau/m$. There is $\lambda \in [0, 1]$ such that $(x, r) = \lambda(y, t) + (1 - \lambda)(z, s)$ where $(y, t) \in \breve{A}$ and $(z, s) \in \breve{B}$. The condition $\lambda t + (1 - \lambda)s < \inf f + \delta\tau/m$ implies $1 - \lambda < \tau/m$. Indeed, suppose $1 - \lambda \ge \tau/m$. As $s \ge \inf f + \delta$, then

$$(1 - \lambda)s \ge (1 - \lambda) \inf f + \delta\tau/m$$

On the other hand, $\lambda t \ge \lambda \inf f$. Adding these inequalities we get $\lambda t + (1 - \lambda)s \ge \inf f + \delta\tau/m$, which contradicts the assumption. Therefore

$$\|x - y\| = \|(\lambda - 1)y + (1 - \lambda)z\| = (1 - \lambda)\|y - z\| < \tau.$$

In order to estimate the d-diameter of

$$S = \{(x, t) : x \in X, \breve{f}(x) \le t < \inf f + \delta\tau/m\},$$

we may consider only points on the dense set $S \cap \text{conv}(\breve{A} \cup \breve{B})$. Therefore, consider $(x_1, r_1), (x_2, r_2) \in \text{conv}(\breve{A} \cup \breve{B})$ with $r_1, r_2 < \inf f + \delta\tau/m$. The convex decomposition above shows that for some $\lambda_1, \lambda_2 \in [0, 1]$ and points $(y_1, t_1), (y_2, t_2) \in \breve{A}$ and $(z_1, s_1), (z_2, s_2) \in \breve{B}$ we have

$$(x_1, r_1) = \lambda_1(y_1, t_1) + (1 - \lambda_1)(z_1, s_1),$$

$$(x_2, r_2) = \lambda_2(y_2, t_2) + (1 - \lambda_2)(z_2, s_2).$$

By the previous estimations, we have $\|x_1 - y_1\|, \|x_2 - y_2\| \le \tau$, which implies that $d(x_1, y_1), d(x_2, y_2) \le \varpi(\tau)$, and thus, by the assumption on A,

$$d(x_1, x_2) \le d(x_1, y_1) + d(y_1, y_2) + d(x_2, y_2) \le \varepsilon + 2\varpi(\tau)$$

as desired. □

We will deal now with the transfer of the ε-uniform convexity property to the lower semicontinuous convex envelope. Note that ε-uniform convexity is referred to a uniformly continuous pseudometric, however we require the hypothesis that the domain be norm bounded.

Theorem 2.2.4. *Let X be a Banach space. Let $\varepsilon > 0$ and let $f : X \to \overline{\mathbb{R}}$ be a bounded below ε-uniformly convex function with bounded domain . Then $\breve{f}$ is ε^+-uniformly convex and given $\varepsilon' > \varepsilon$, the modulus of convexity $\delta_{\breve{f}}(\varepsilon')$ depends only on ε', $\delta_f(\varepsilon)$, ϖ and the norm diameter of* $\mathrm{dom}(f)$.

Proof. Let m an upper bound for the diameter of $\mathrm{dom}(f)$ and $\delta > 0$ the parameter given by the definition of ε-uniform convexity. Take $\tau > 0$ such that $\tau/m < 1$. We will estimate the d-diameter of any slice of $\mathrm{epi}(\breve{f})$ not meeting $\mathrm{epi}(\breve{f} + \delta\tau/m)$. Suppose that the slice is given by $x^* \in X^*$. Note that the estimation of the d-diameter of the slice we need is equivalent to the same for an horizontal slice of $\mathrm{epi}(\breve{f} - x^*)$ not meeting $\mathrm{epi}(\breve{f} - x^* + \delta\tau/m)$, which is the same as taking the points of $\mathrm{epi}(\breve{f} - x^*)$ whose scalar coordinate is less than $\inf(\breve{f} - x^*) + \delta\tau/m$. Since $\breve{f} - x^*$ equals the convex envelope of the function $f - x^*$, which is ε-uniformly convex with parameter δ, the set

$$\{x \in X : f(x) - x^*(x) < \inf(f - x^*) + \delta\}$$

has diameter less than ε by Proposition 2.2.1. The previous lemma applies to get that

$$\{x \in X : \breve{f}(x) - x^*(x) < \inf(f - x^*) + \delta\tau/m\}$$

has diameter less than $\varepsilon + 2\varpi(\tau)$. Thanks to Proposition 2.2.1, it follows that $\breve{f}$ is $\varepsilon + 2\varpi(\tau)$-uniformly convex. Given $\varepsilon' > \varepsilon$, we only have to set $\tau > 0$ such that $2\varpi(\tau) < \varepsilon' - \varepsilon$. $\square$

The following result is the key to deal with unbounded domains.

Proposition 2.2.5. *Let X be a Banach space. Let $\varepsilon > 0$ and let $f : X \to \overline{\mathbb{R}}$ be an ε-uniformly convex function such that $\breve{f}$ is proper. Then the value of $\breve{f}(x)$ for $x \in \mathrm{dom}(f)$ depends only on the set of values $\{f(y) : \|y - x\| < \varepsilon\}$. Namely, if g is the function defined by $g(y) = f(y)$ if $\|y - x\| < \varepsilon$ and $g(x) = +\infty$ otherwise, then $\breve{f}(x) = \breve{g}(x)$.*

Proof. Let us roughly explain the idea of the proof before going into details. A priori, the computation of $\breve{f}(x)$ may involve values of f at points arbitrarily far away from x. Namely, $(x, \breve{f}(x))$ can be approximated by a convex combination of points of the form $(x_k, f(x_k))$. As we want the points x_k to be close to x, we will describe an algorithm that will modify the set $\{x_k\}$ by the substitution of one (or several points) at each step until the resulting set is contained in $B(x, \varepsilon)$. The algorithm consists in switching a farthest point x_i by the middle point between it and an "opposite point" x_j which is not farther from x as x_i is. If $d(x_i, x_j) \geq \varepsilon$, the ε-uniform convexity of f will imply that we do not loose information about $\breve{f}(x)$ when switching x_i by $(x_i + x_j)/2$. Once, x_i has disappeared from the the set, we choose a new farthest point and start over. Actually, the method brings the points closer to x with respect to a prefixed direction $x^* \in X^*$. The repetition of the algorithm with several directions will eventually finish with the modified set of points contained into $B(x, \varepsilon)$. Now we will resume the proof.
The definition of $\breve{f}$ implies that the following set

$$\left\{(x, t) : t \geq \sum_{k=1}^{n} \lambda_k f(x_k) \ \text{ with } \ x = \sum_{k=1}^{n} \lambda_k x_k \ \text{ a convex combination}\right\}$$

is dense in $\mathrm{epi}(\breve{f})$. Fix $x \in \mathrm{dom}(f)$ and suppose $x = \sum_{k=1}^{n} \lambda_k x_k$ is a convex combination. Now, we are going to describe the announced algorithm that will transform the set of points

$S = \{x_1, \dots, x_n\}$ into a set $S' = \{x'_1, \dots, x'_{n'}\} \subset B(x, \varepsilon)$ such that still we have $\sum_{k=1}^{n'} \lambda'_k x'_k = x$, where $\sum_{k=1}^{n'} \lambda'_k x'_k = x$ with $\lambda'_k \geq 0$, $\sum_{k=1}^{n'} \lambda'_k = 1$, and

$$\sum_{k=1}^{n'} \lambda'_k f(x'_k) \leq \sum_{k=1}^{n} \lambda_k f(x_k).$$

In order to do that, without loss of generality, we may assume $x = 0$. Fix $x^* \in S_{X^*}$. Let $a = \sup_S x^* \geq 0$ and $b = -\inf_S x^*$. As a and b can be exchanged just taking $-x^*$ instead, without loss of generality we may assume $a \geq \max\{b, \varepsilon\}$. Also, without loss of generality, we may assume $x^*(x_1) = a$. Since x_1 is the farther point (with respect to x^*), its "mass" λ_1 compensates with masses on the side $x^* \leq 0$. Suppose firstly that $x^*(x_2) \leq 0$ and $\lambda_2 \geq \lambda_1$. We have $\|x_1 - x_2\| \geq \varepsilon$. We claim that it is possible to switch x_1 by $x'_1 = (x_1 + x_2)/2$. Indeed,

$$2\lambda_1 x'_1 + (\lambda_2 - \lambda_1)x_2 + \lambda_3 x_3 + \cdots + \lambda_n x_n = 0$$

which is still a convex combination. Note that

$$\begin{aligned} &2\lambda_1 f(x'_1) + (\lambda_2 - \lambda_1) f(x_2) + \lambda_3 f(x_3) + \cdots + \lambda_n f(x_n) \\ \leq\ &\lambda_1(f(x_1) + f(x_2)) + (\lambda_2 - \lambda_1) f(x_2) + \lambda_3 f(x_3) + \cdots + \lambda_n f(x_n) \\ =\ &\lambda_1 f(x_1) + \lambda_2 f(x_2) + \lambda_3 f(x_3) + \cdots + \lambda_n f(x_n) \end{aligned}$$

where we have used $f(x'_1) \leq (f(x_1) + f(x_2))/2$ (see the definition of ε-uniform convexity). The inequality means that $S_1 = \{x'_1, x_2, \dots, x_n\}$ is an improvement of S in the sense of the approximation to $\breve{f}$. Note also that $x^*(x'_1) \leq a/2$.
In case, $\lambda_1 > \lambda_2$, we will use several vectors x_k with $x^*(x_k) \leq 0$ to compensate x_1. This is possible because $a \geq b$ implies that the "mass" lying on the halfspace $x^* < 0$ is not lesser than λ_1. In this case, λ_1 could be cancelled with several λ_k's. In any case, we will get a new set S_1 whose cardinal is not larger than that of S and $\mathrm{conv}(S_1) \subset \mathrm{conv}(S)$. After that, suppose that, unfortunately, we still have $\sup_{S_1} x^* = a$. In such a case, the maximizing vector cannot be x_1, so it is a new vector, say x_3. We will apply the argument with x_3 in order to replace it by another vector x'_3 and S_1 by a new set S_2. Eventually, we will get $\sup_{S_n} x^* \leq a/2$ after a finite number of steps. Then, with the same x^*, we have to change the constants $a, b > 0$ by new ones. This can be done with the same x^* until we get $\max\{a, b\} < \varepsilon$, so it is not possible to go further.
If the set of points it is not yet inside $B(0, \varepsilon)$ then find a new $x^* \in S_{X^*}$ such that $\sup_{S_n} x^* \geq \varepsilon$ and then run again the algorithm. Since $\mathrm{conv}(S)$ is finite dimensional, it is enough to do this procedure over finitely many $x^* \in S_{X^*}$ in order to get $S_n \subset B(0, \varepsilon)$ eventually. □

We are now ready to prove the main result of this section.

Theorem 2.2.6. *Let X Banach space and let $f : X \to \overline{\mathbb{R}}$ be an ε-uniformly convex function such that $\breve{f}$ is proper. Then $\breve{f}$ is ε^+-uniformly convex.*

Proof. For the proof it would be convenient to represent a convex combination in X by means of a vector integral instead of the usual symbol "$\sum$". Namely, given a convex combination $\sum_{k=1}^{n} \lambda_k x_k$, the weights λ_k are changed by n disjoint intervals I_k of lengths λ_k and whose union is $[0, 1]$. In this way, the convex combination is represented as the integral of the simple function $\overline{x}$ defined on $[0, 1]$ by $\overline{x}(t) = x_k$ whenever $t \in I_k$. As we will deal only with simple functions, no further knowledge of vector integration theory is required.
We resume the proof. If $\breve{f}$ is proper then it is bounded below by an affine function, so by adding an affine function (that does not alter the ε-uniform convexity), we may suppose that

f is bounded below (actually that is true without modifications, see Corollary 2.4.2). Given $x, y \in \operatorname{dom}(f)$, if $\|x-y\| \geq 3\varepsilon$ then

$$\breve{f}\left(\frac{x+y}{2}\right) \leq \frac{\breve{f}(x)+\breve{f}(y)}{2} - \delta_f(\varepsilon).$$

Indeed, fix $\eta > 0$. By Proposition 2.2.5, we may take $(\overline{x}_n)_n$ a sequence of simple functions defined on $[0,1]$ such that $\|\overline{x}_n(t) - x\| < \varepsilon$ for all $t \in [0,1]$, $n \in \mathbb{N}$, with

$$\lim_n \int_0^1 \overline{x}_n(t)\,dt = x, \quad \text{and} \quad \lim_n \int_0^1 f(\overline{x}_n(t))\,dt = \breve{f}(x).$$

Let $(\overline{y}_n)_n$ an analogous sequence of simple functions playing the same role for y and $\breve{f}(y)$. Clearly we have $\|\overline{x}_n(t) - \overline{y}_n(t)\| \geq \varepsilon$ for all $t \in [0,1]$ and $n \in \mathbb{N}$. Therefore

$$\breve{f}\left(\frac{x+y}{2}\right) \leq \liminf_n \int_0^1 f\left(\frac{\overline{x}_n(t)+\overline{y}_n(t)}{2}\right) dt$$

$$\leq \lim_n \int_0^1 \left(\frac{f(\overline{x}_n(t))+f(\overline{y}_n(t))}{2} - \delta_f(\varepsilon)\right) dt \leq \frac{\breve{f}(x)+\breve{f}(y)}{2} - \delta_f(\varepsilon).$$

Since $\eta > 0$ was arbitrary we get the claimed inequality provided $\|x-y\| \geq 3\varepsilon$.
Now we will suppose $\varepsilon \leq \|x-y\| < 3\varepsilon$. Proposition 2.2.5 implies that reducing the domain of f to $[x,y] + B(0,\varepsilon)$ does not affect to the values of $\breve{f}(x)$, $\breve{f}(y)$ and $\breve{f}(\frac{x+y}{2})$. Fix $\varepsilon' > \varepsilon$. Theorem 2.2.4 says that $\delta_{\breve{f}}(\varepsilon')$ depends only on ε', $\delta_f(\varepsilon)$, ϖ, which are fixed, and the diameter of the domain, which is bounded by 5ε. □

2.3 Building uniformly convex functions

Most of the constructions of uniformly convex functions on a Banach spaces that one can find in the literature are based on modifications of a uniformly convex norm, see [29]. Nevertheless, the existence of a finite uniformly convex function whose domain has nonempty interior implies that X has an equivalent uniformly convex norm. In any case, the constructions dealing with the composition of a uniformly convex norm and a suitably chosen function can be quite tricky, except for the Hilbert space. Here we will exploit a method based on "discretization" and uniformly quasi-convex functions.

Lemma 2.3.1. *Let X Banach space. Let $\varepsilon > 0$ and let $f : X \to \mathbb{R}$ be a bounded below ε-uniformly quasi-convex with modulus $\delta > 0$. Then the function $h \circ f$ is ε-uniformly convex where $h(t) = 3^{t/\delta}$.*

Proof. The function h is increasing and satisfies the property $3h(t) = h(t-\delta)$. Take

$$\eta = 4^{-1} \inf\{h(t+\delta) - h(t) : t \geq \inf f\} = 2^{-1} \cdot 3^{\inf f/\delta}$$

and note that it depends only on f. If $x, y \in C$ are such that $d(x,y) \geq \varepsilon$ take $a = f(x)$, $b = f(y)$ and $c = f(\frac{x+y}{2})$. The hypothesis says that $c \leq \max\{a,b\} - \delta$. With no loss of generality, we may assume $b \leq a$. We have

$$h(c) \leq h(a) - 4\eta.$$

Since $3h(c) \leq h(a)$ and $h(b) > 0$, we also have

$$3h(c) \leq h(a) + 2h(b)$$

and adding the previous inequality, we get

$$4h(c) \leq 2h(a) + 2h(b) - 4\eta$$

and thus

$$h(c) \leq \frac{h(a) + h(b)}{2} - \eta$$

which is the ε-uniform convexity of $h \circ f$. $\square$

If X is uniformly convex, it is well known that $x \mapsto \|x\|^2$ is a uniformly convex function on bounded convex subsets. The usual construction of a global uniformly convex functions involves additional properties of the norm, such as having a power type modulus of uniform convexity. Here there is a simple alternative construction based in our methods.

Proposition 2.3.2. *If a Banach space X has a uniformly convex norm then there exists a real function ϕ such that $x \mapsto \phi(\|x\|)$ is a uniformly convex function defined on X.*

Proof. Fix $\varepsilon > 0$. Take $a_1 = \varepsilon/2$ and define inductively a sequence $(a_n)_n$ by the implicit equation

$$a_{n-1} = \left(1 - \delta_X\left(\frac{\varepsilon}{a_n}\right)\right) a_n$$

which has a unique solution thanks to the continuity of δ_X on $[0, 2)$, [83, Lemma 5.1]. The sequence $(a_n)_n$ is increasing with $\lim_n a_n = +\infty$ and has the following property: if $\|x\|, \|y\| \leq a_n$ and $\|x - y\| > \varepsilon$ then $\|(x + y)/2\| \leq a_{n-1}$.
Define a function as $f_\varepsilon(x) = n$ if $a_{n-1} < \|x\| \leq a_n$ Note that f_ε satisfies the hypothesis of Lemma 2.3.1 with $\delta = 1$, and so $h \circ f_\varepsilon$ is ε^+-uniformly convex. Now, for $\varepsilon = 1/n$ take f_n the convex envelope of $h \circ f_\varepsilon$ and $c_n = 2^{-n} \sup_{nB_X} f_n$. The series $\sum_{n=1}^{\infty} c_n f_n$ converges uniformly on bounded sets to a uniformly convex function f. By construction, $f(x)$ depends only on $\|x\|$. Therefore, we may define a real function by $\phi(t) = f(x)$ if $t = \|x\|$, for $t \geq 0$. Clearly, $f(x) = \phi(\|x\|)$. $\square$

Now we will explain constructions using trees. Separated trees and bushes are obstructions to the existence of bounded uniformly convex functions.

Proposition 2.3.3. *Let $\varepsilon > 0$ and let C be a convex subset of a Banach space X that supports an ε-uniformly convex function f with values in $[a, b]$. Then $(b-a)/\delta_f(\varepsilon)$ is the maximum height of*

(a) any ε-separated tree contained in C;

(b) any ε^+-separated bush contained in C.

Proof. If $\{x_s\}$ is an ε-separated tree then we have

$$f(x_s) \leq \max\{f(x_{s\frown 0}), f(x_{s\frown 1})\} - \delta_f(\varepsilon)$$

that gives the estimation. In the case of bushes, the argument is the same after passing to $\breve{f}$, which is ε^+-uniformly convex by Theorem 2.2.6, and applying Corollary 2.2.2. $\square$

Our following result is quite a converse.

Theorem 2.3.4. *Let $\varepsilon > 0$ and let C be a convex subset of a Banach space X that does not contain arbitrarily high ε-separated trees (with respect to some uniformly continous pseudometric). Then C supports a bounded ε-uniformly convex function, and a bounded convex ε^+-uniformly convex function (with respect the same pseudometric).*

Proof. Define a function for $x \in C$ by

$$f(x) = -\max\{\mathrm{height}(x_s) : (x_s) \subset C \ \varepsilon\text{-sep. tree}, x_\emptyset = x\}$$

and $f(x) = +\infty$ otherwise. We claim that f is ε-uniformly quasi-convex. Indeed, consider points $x, y \in C$ with $d(x,y) \geq \varepsilon$. Take ε-separated trees contained into C $\{x_{s'} : |s'| \leq n'\}$ and $\{y_{s''} : |s''| \leq n''\}$ of maximal length with the property that $x_\emptyset = x$ and $y_\emptyset = y$. The trees can be "glued" as follows. Take $n = \min\{n', n''\}$. Define a new tree (z_s), for $|s| \leq n+1$, as $z_\emptyset = \frac{x+y}{2}$, $z_{0\frown s} = x_s$ and $z_{1\frown s} = y_s$ for $|s| \leq n$. Now (z_s) is a ε-separated tree rooted at $\frac{x+y}{2}$ of height $\min\{n', n''\} + 1$. That means in terms of the function f the uniform quasi-convex inequality

$$f\left(\frac{x+y}{2}\right) \leq \max\{f(x), f(y)\} - 1$$

for $d(x,y) \geq \varepsilon$. Now, Lemma 2.3.1 says that $h \circ f$ is ε-uniformly convex and its convex hull is ε^+-uniformly convex after Theorem 2.2.6. □

Theorem 2.3.5. *Let C be a closed bounded convex subset of a Banach space X. Then these two numbers coincide:*

(ν_1) *the infimum of the $\varepsilon > 0$ such that there is a common bound for the heights of all the ε-separated dyadic trees;*

(ν_2) *the infimum of the $\varepsilon > 0$ such that there is a bounded ε-uniformly convex (and convex, Lipschitz. . .) function defined on C.*

Proof. It just follows from Theorem 2.3.4 and Proposition 2.3.3. □

We obtain the following well-known characterization of convex SWC sets.

Corollary 2.3.6. *Let C be a closed bounded convex subset of a Banach space X. The following assertions are equivalent:*

(i) C is SWC;

(ii) there exists a bounded uniformly convex function $f : C \to \mathbb{R}$.

Proof. $(ii) \implies (i)$ The infimums of the previous theorem are zero. So C does not have the finite tree property. By Theorem 1.5.12, C is SWC.

$(i) \implies (ii)$ By Theorem 1.5.12, the infimum of the previous theorem are again zero. Then, for all $\varepsilon > 0$, there exists a bounded ε-uniformly convex function on C. Taking a serie, we deduce easily that there exists a bounded uniformly convex function on C. □

Finally we will explain constructions based on the dentability index.

Definition 2.3.7. Let C be a bounded closed convex subset of a Banach space X, (M, d) be a pseudometric space and $F : C \to M$ be a map. We say that F is *dentable* if for any nonempty closed convex subset $D \subset C$ and $\varepsilon > 0$, it is possible to find an open halfspace H intersecting D such that $\mathrm{diam}(F(D \cap H)) < \varepsilon$, where the diameter is computed with respect to d. If F is dentable, we may consider the following "derivation"

$$[D]'_{F,\varepsilon} = \{x \in D : \mathrm{diam}(F(D \cap H)) > \varepsilon, \ \forall H \in \mathbb{H}, \, x \in H\},$$

where $\mathbb{H}$ denotes the set of all the open halfspaces of X. Clearly, $[D]'_{F,\varepsilon}$ is what remains of D after removing all the slices of diameter less or equal than ε. Consider the sequence of sets defined by $[C]^0_{F,\varepsilon} = C$ and, for every $n \in \mathbb{N}$, inductively by

$$[C]^n_{F,\varepsilon} = [[C]^{n-1}_{F,\varepsilon}]'_{F,\varepsilon}.$$

If there is $n \in \mathbb{N}$ such that $[C]^{n-1}_{F,\varepsilon} \neq \emptyset$ and $[C]^{n}_{F,\varepsilon} = \emptyset$ we say that $\mathrm{Dz}(F,\varepsilon) = n$. We say that F is *finitely dentable* if $\mathrm{Dz}(F,\varepsilon) < \omega$ for every $\varepsilon > 0$ (ω stands for the first infinite ordinal number).

A useful trick is the so called (nonlinear) Lancien's midpoint argument: if a segment satisfies $[x,y] \subset D$ and $[x,y] \cap [D]'_{F,\varepsilon} = \emptyset$ then $d(F(x),F(y)) \leq 2\varepsilon$, see the beginning of [146, Theorem 2.2].

All these notions can be applied to the identity map of a convex set where there is a pseudometric defined. The following result is the quantified version of [146, Theorem 2.2]. For convenience we will write

$$\Delta_\Phi(x,y) = \frac{\Phi(x)+\Phi(y)}{2} - \Phi\left(\frac{x+y}{2}\right).$$

Theorem 2.3.8. *Let C be a bounded closed convex subset of a Banach space X, let M be a pseudometric space, let $F : C \to M$ be a uniformly continuous map, and let $\varepsilon > 0$.*

(a) Suppose that there exists a bounded lower semi-continuous convex function Φ defined on C and $\delta > 0$ such that $d(F(x),F(y)) \leq \varepsilon$ whenever $x,y \in C$ satisfy $\Delta_\Phi(x,y) \leq \delta$. Then $\mathrm{Dz}(F,\varepsilon) < \omega$.

(b) On the other hand, if $\mathrm{Dz}(F,\varepsilon) < \omega$ then for every $\varepsilon' > 2\varepsilon$ there exits a bounded lower semi-continuous convex function Φ defined on C and $\delta > 0$ such that $d(F(x),F(y)) \leq \varepsilon'$ whenever $x,y \in C$ satisfy $\Delta_\Phi(x,y) \leq \delta$.

Proof. Let $s = \sup_C F$. The hypothesis implies $[C]'_{F,\varepsilon} \subset \{F \leq s-\delta\}$. Iterating this we will eventually get to the empty set. For the second part, we need to introduce some notation. Firstly put $d'(x,y) = d(F(x),F(y))$ which is a pseudometric uniformly continuous with respect to $\|\cdot\|$. Derivations and diameters will be referred to d'. The slice of a set A with parameters $x^* \in X^*$ and $\alpha > 0$ is

$$S(A,x^*,\alpha) = \{x \in A : x^*(x) > \sup_A x^* - \alpha\}.$$

The "half-derivation" of a convex set is defined as

$$\langle D\rangle'_\varepsilon = \{x \in D : x^*(x) \leq \alpha,\ \forall x^*, \alpha > 0 \ s.t.\ \mathrm{diam}(S(D,x^*,2\alpha)) > \varepsilon\}.$$

The geometric interpretation is that we remove half of the slice, in sense of the width, for every slice of d'-diameter less than ε. This derivation can be iterated by taking $\langle C\rangle^n_\varepsilon = \langle\langle C\rangle^{n-1}_\varepsilon\rangle'_\varepsilon$. It is not difficult, but rather tedious, to show that if $\mathrm{Dz}(F,\varepsilon) < \omega$ then for every $\varepsilon' > 2\varepsilon$ there is some $n \in \mathbb{N}$ such that $\langle C\rangle^n_{\varepsilon'} = \emptyset$. The idea is the following. Firstly note that every slice of C not meeting $[C]'_{F,\varepsilon}$ has diameter 2ε at most by Lancien's argument. Taking "half a slice" of the slice given by some $x^* \in X^*$, we deduce that

$$\sup_{\langle C\rangle'_{2\varepsilon}} x^* - \sup_{[C]'_{F,\varepsilon}} x^* \leq 2^{-1}\left(\sup_C x^* - \sup_{[C]'_{F,\varepsilon}} x^*\right).$$

Iterating, we would get

$$\sup_{\langle C\rangle^n_{2\varepsilon}} x^* - \sup_{[C]'_{F,\varepsilon}} x^* \leq 2^{-n}\left(\sup_C x^* - \sup_{[C]'_{F,\varepsilon}} x^*,\right)$$

for every $x^* \in X^*$. If $\eta > 0$, we will get for some n large enough that

$$\langle C\rangle^n_{2\varepsilon} \subset [C]'_{F,\varepsilon} + B_X(0,\eta).$$

We can do that for every set $[C]^k_{F,\varepsilon}$. A perturbation argument, using the room between ε and ε', will allow us to fill the gap between the sequences of sets. In this way we will get that $\langle C\rangle^n_{\varepsilon'} = \emptyset$ for some $n \in \mathbb{N}$ large enough.
Now we define a function g on C by $g(x) = -n$ if $x \in \langle C\rangle^n_{\varepsilon'} \setminus \langle C\rangle^{n+1}_{\varepsilon'}$ following the notation above. We claim that g satisfies Lemma 2.3.1 with separation ε'. Indeed, if $d'(x,y) > \varepsilon'$ and $n = -\max\{g(x), g(y)\}$ then $x, y \in \langle C\rangle^n_{\varepsilon'}$. If $\frac{x+y}{2} \notin \langle C\rangle^{n+1}_{\varepsilon'}$ then the segment $[x,y]$ would be fully contained into a slice of diameter less than ε' and so $d'(x,y) \leq \varepsilon'$ which is a contradiction. Therefore $\frac{x+y}{2} \in \langle C\rangle^{n+1}_{\varepsilon'}$ and so $g(\frac{x+y}{2}) \leq -n-1$. Now $f(x) = 3^{g(x)}$ is ε'-uniformly convex with respect to d'. Take $\Phi = \breve{f}$ to get the desired function. □

If F in Theorem 2.3.8 (2) were finitely dentable, a standard argument using a convergent series would lead to this results, which is essentially [79, Proposition 3.2] with a uniformly convex function instead of a norm.

Corollary 2.3.9. *Let C be a bounded closed convex subset of a Banach space X, let M be a pseudometric space, and let $F : C \to M$ be a uniformly continuous finitely dentable map. Then there exists a bounded convex function Φ defined on C such that for every $\varepsilon > 0$ there is $\delta > 0$ such that $d(F(x), F(y)) \leq \varepsilon$ whenever $x, y \in C$ are such that $\Delta_\Phi(x,y) \leq \delta$.*

2.4 Improving functions and domains

So far the best improvement we have done on an existing ε-uniformly convex function is taking its lower semicontinuous convex envelope provided this last one is proper. The aim in this section is to manipulate the functions in order to improve their qualities. We will begin by proving the results about global behaviour.

The next result shows global behaviour of ε-uniformly convex functions and the relative stability of minimizers by linear perturbations.

Theorem 2.4.1. *Let X Banach space and let $f : X \to \overline{\mathbb{R}}$ be an ε-uniformly convex function such that $\breve{f}$ is proper. Then f is bounded below and coercive, more precisely we have*

$$\liminf_{\|x\|\to+\infty} \frac{f(x)}{\|x\|^2} > 0.$$

Moreover, for any $\varepsilon' > \varepsilon$ there exist $\delta, \eta > 0$ such that if given $x_0^* \in X^*$ and $x_0 \in X$ with

$$f(x_0) + x_0^*(x_0) < \inf(f + x_0^*) + \delta,$$

and $x^* \in X^*$ such that $\|x^* - x_0^*\| < \eta$ and $x \in X$ that minimizes $f + x^*$, then $\|x - x_0\| \leq \varepsilon'$. The existence of such minimizer pair (x, x^*) is guaranteed if $f = \breve{f}$.

Proof. Since $\breve{f} \leq f$, it is enough to prove that the property holds for an ε-convex and convex proper function. Actually the same proof for a uniformly convex function done in Zalinescu's book [175, Proposition 3.5.8] works in this case because $\liminf_{t\to+\infty} t^{-2} p_f(t) \geq \varepsilon^{-2} p_f(\varepsilon) > 0$.
For the second part, without loss of generality we may assume that $x_0^* = 0$ (just change f by $f + x_0^*$). Let $\delta = \delta_{\breve{f}}(\varepsilon')$ and take $\eta = \inf f + \delta - \breve{f}(x_0) > 0$. Note that $\inf f = \inf \breve{f}$. By the property established in the first part applied to $\breve{f} - x^*$, there is $R > 0$ such that $\breve{f}(x) \geq \breve{f}(x_0) - x^*(x - x_0)$ for any $x^* \in B_{X^*}$ and $\|x - x_0\| \geq R$. Now, fix x^* such that $\|x^*\| \leq \eta/R$. Then we have

$$\breve{f}(x) + x^*(x) \geq \breve{f}(x_0) + x^*(x_0) - \delta$$

for all $x \in X$ such that $\|x - x_0\| \leq R$, and therefore the inequality holds for all $x \in X$. That implies $\mathrm{epi}(\breve{f} + x^* + \delta)$ does not meet the horizontal slice

$$S = \{(x,t) \in \mathrm{epi}(\breve{f} + x^*) : t \leq \breve{f}(x_0) + x^*(x_0)\}$$

By Proposition 2.2.1, the projection of S on X has diameter less than ε'. Moreover, if $f + x^*$ attains a minimum at x, then the same holds for $\breve{f} + x^*$ and so $x \in S$. Since $x_0 \in S$ we have $\|x - x_0\| \leq \varepsilon'$. The existence of a dense set of x^*'s such that $\breve{f} + x^*$ attains a minimum is guaranteed by Brøndsted–Rockafellar [28, Theorem 4.3.2] (or Bishop–Phelps [67, Theorem 7.4.1] applied to the epigraph). $\square$

As a consequence, we characterize when an ε-uniformly convex function has a proper convex envelope.

Corollary 2.4.2. *Let X Banach space. Let $\varepsilon > 0$ and let $f : X \to \overline{\mathbb{R}}$ be an ε-uniformly convex function. Then the following statements are equivalent:*

(i) $\breve{f}$ is proper;

(ii) f is bounded below;

(iii) f is bounded below by an affine continuous function.

For a ε-uniformly quasi-convex function we can say the following

Proposition 2.4.3. *Let X Banach space. Let $\varepsilon > 0$ and let $f : X \to \overline{\mathbb{R}}$ be an ε-uniformly quasi-convex function that is bounded below. Then f is coercive and moreover*

$$\liminf_{\|x\| \to +\infty} \frac{f(x)}{\|x\|} > 0.$$

Proof. By adding a constant, we may suppose that $\inf f = 0$. Take $x_0 \in X$ such that $f(x_0) < \delta/2$. For any $x \in X$ such that $\|x - x_0\| \geq \varepsilon$ we have $f(x) \geq \delta$. Indeed, otherwise it would be $f(x) < \delta$ and by the ε-uniformly quasi-convexity, $f(\frac{x+x_0}{2}) < \inf f$, an obvious contradiction. Now, if $\|x - x_0\| \geq 2\varepsilon$, then $\|\frac{x+x_0}{2} - x_0\| \geq \varepsilon$. That implies $f(\frac{x+x_0}{2}) \geq \delta$ and therefore $f(x) \geq 2\delta$. Inductively, we will get that if $\|x - x_0\| \geq 2^n\varepsilon$ then $f(x) \geq 2^n\delta$. Now, the statement follows easily. $\square$

The following results will show that, given a ε-uniformly convex function, we can make modification in both the function and its domain in order to get a new function with additional properties.

Proposition 2.4.4. *Let X Banach space. Let $\varepsilon > 0$, let $f : X \to \overline{\mathbb{R}}$ be an ε-uniformly convex function that is locally bounded below and let $\eta > 0$. Then there exists a lower semicontinuous $(\varepsilon + 2\eta)$-uniformly convex function defined on $\mathrm{dom}(f) + B(0, \eta)$. In particular, $\overline{\mathrm{dom}(f)}$ admits a lower semicontinuous ε^+-uniformly convex function.*

Proof. Define $g(x) = \inf\{f(y) : \|y - x\| < \eta\}$ on $\mathrm{dom}(f) + B(0, \eta)$. This function g is $(\varepsilon + 2\eta)$-uniformly convex (the simple verification of this fact is left to the reader). Now take its lower semicontinuous envelope. $\square$

The following result will be done for ε-uniformly convexity with respect to a metric because such a degree of generality will be needed later.

Proposition 2.4.5. *Let X Banach space. Let $\varepsilon > 0$, let $f : X \to \overline{\mathbb{R}}$ be an ε-uniformly convex function (with respect to a metric d with modulus of uniform continuity ϖ) and let $C \subset \operatorname{dom}(f)$ be convex such that f is bounded on it. Then for any $\varepsilon' > \varepsilon$, there exists $g \in \Gamma(X)$ Lipschitz (with respect to the norm of X) such that $g|_C$ is ε'-uniformly convex.*

Proof. Without loss of generality, we may assume that f is convex. Take $\eta > 0$ such that $\varpi(\eta) < (\varepsilon' - \varepsilon)/2$, $m = \sup\{f(x) - f(y) : x, y \in C\}$ and $c = m/\eta$. Define

$$g(x) = \inf\{f(y) + c\|x - y\| : y \in C\}$$

which is convex and c-Lipschitz. Let $x \in C$ and $\xi > 0$. Then either $g(x) = f(x)$ and the infimum is attained with $y = x$, or $g(x) < f(x)$. In the last case, the infimum can be computed over the $y \in C$ such that $f(y) + c\|x - y\| < f(x)$. Therefore, we can find $y \in C$ such that $f(y) + c\|x - y\| < g(x) + \xi$ and $\|x - y\| < m/c = \eta$, which implies $d(x, y) < (\varepsilon' - \varepsilon)/2$. Now, for $x_1, x_2 \in C$ with $d(x_1, x_2) \geq \varepsilon'$ find $y_1, y_2 \in C$ as above. Clearly we have $d(y_1, y_2) \geq \varepsilon$, and so

$$g\left(\frac{x_1 + x_2}{2}\right) \leq f\left(\frac{y_1 + y_2}{2}\right) + c\left\|\frac{x_1 + x_2}{2} - \frac{y_1 + y_2}{2}\right\|$$

$$\leq \frac{f(y_1) + f(y_2)}{2} - \delta + \frac{c}{2}\|x_1 - y_1\| + \frac{c}{2}\|x_2 - y_2\| \leq \frac{g(x_1) + g(x_2)}{2} - \delta + 2\xi.$$

Since $\xi > 0$ was arbitrary, we get the ε'-uniform convexity of g as wished. □

Remark 2.4.6. A Baire category argument shows that an ε-uniformy convex function f is bounded in an open ball if $\operatorname{dom}(f)$ has nonempty interior. However we do not know how to ensure that f will be bounded on a larger set.

Now we will show how to change an ε-uniformly convex function by a norm with the same property.

Theorem 2.4.7. *Let $(X, \|\cdot\|)$ be a Banach space, let $f \in \Gamma(X)$ be a non negative function and let $C \subset \operatorname{dom}(f)$ be a bounded convex set. Assume f is Lipschitz on C. Then given $\delta > 0$ there exists an equivalent norm $|||\cdot|||$ on X and $\zeta > 0$ such that $\Delta_f(x, y) < \delta$ whenever $x, y \in C$ satisfy $\Delta_{|||\cdot|||^2}(x, y) < \zeta$. Therefore, if f was moreover ε-uniformly convex for some $\varepsilon > 0$ (with respect to a pseudometric) on C, then $|||\cdot|||^2$ would be ε-uniformly convex on C (with respect to the same pseudometric).*

Proof. Taking $f(x)+f(-x)+\|x\|$ instead, we may indeed assume that f is symmetric and attains a strong minimum at 0. Let $M = \sup_C f$ and $m = \min_C f + \delta/2$. The Lipschitz condition easily implies that there is $\eta > 0$ such that if $r \leq M$ then

$$\{f \leq r - \delta\} + B(0, \eta) \subset \{f \leq r\}.$$

For $r \in [m, M]$ let $\|\cdot\|_r$ the Minkowski functional of the set $\{f \leq r\}$, which is an equivalent norm on X. Let $N = \sup\{\|x\| : x \in C\}$ and note that $\lambda = (1 + \eta/N)^{-1}$ has the property

$$\{f \leq r - \delta\} \subset \lambda\{f \leq r\}.$$

We deduce the following property: if $x, y \in C$, $\|x\|_r, \|y\|_r \leq 1$ and $\Delta_f(x, y) \geq \delta$ then

$$\left\|\frac{x + y}{2}\right\|_r \leq \lambda.$$

Consider a partition $m = a_1 < a_2 < \cdots < a_k = M$ such that $a_j/a_{j+1} < \lambda^{1/2}$ and put $|||\cdot|||_j = |||\cdot|||_{a_j}$. Let $x, y \in C$ such that $\Delta_f(x,y) \geq \delta$. Assume $f(x) \geq f(y)$ for instance. There is some $1 \leq j < k$ such that $1 \geq |||x|||_j \geq \lambda^{1/2}$. Since $|||\frac{x+y}{2}|||_j \leq \lambda$, we have

$$\left|\left|\left|\frac{x+y}{2}\right|\right|\right|_j \leq \max\{|||x|||_j, |||y|||_j\} - (\lambda^{1/2} - \lambda).$$

Following the same computations that in the proof of Proposition 2.1.10, we have

$$\left|\left|\left|\frac{x+y}{2}\right|\right|\right|_j^2 \leq \frac{|||x|||_j^2 + |||y|||_j^2}{2} - \frac{(\lambda^{1/2} - \lambda)^2}{4}.$$

Therefore, if we define an equivalent norm by $|||\cdot|||^2 = \sum_{j=1}^k |||\cdot|||_j^2$ we will have

$$\left|\left|\left|\frac{x+y}{2}\right|\right|\right|^2 \leq \frac{|||x|||^2 + |||y|||^2}{2} - \frac{(\lambda^{1/2} - \lambda)^2}{4}.$$

whenever $x, y \in C$ satisfies $\Delta_f(x,y) \geq \delta$, meaning that the statement is true with $\zeta = 4^{-1}(\lambda^{1/2} - \lambda)^2$. □

Theorem 2.4.8. *Let $(X, \|\cdot\|)$ be a Banach space and let $f \in \Gamma(X)$ be ε-uniformly convex. Then there exists an equivalent norm $|||\cdot|||$ on X such that the function $x \mapsto |||x|||^2$ is ε^+-uniformly convex on the subsets of* $\operatorname{dom}(f)$ *where f is bounded above. Moreover, the norm $|||\cdot|||$ can be taken as close to $\|\cdot\|$ as we wish.*

Proof. Consider the sets $C_n = \{f \leq n\}$ that eventually will "capture" any set where f is bounded. Fixed $n \in \mathbb{N}$, by Proposition 2.4.5, we may assume that f is already norm-Lipschitz and finite on X provided we change ε by ε^+. Let $|||\cdot|||_n$ the norm given by Theorem 2.4.7, which is ε^+-uniformly convex on C_n. Let $\alpha > 0$ and let (α_n) be a sequence of positive numbers such that

$$|||\cdot|||^2 = \alpha\|\cdot\|^2 + \sum_{n=1}^{\infty} \alpha_n |||\cdot|||_n^2$$

converges uniformly on bounded sets. Clearly, $|||\cdot|||$ will be ε^+-uniformly convex on bounded sets too. The last affirmation follows just taking $\alpha > 0$ small enough. □

Corollary 2.4.9. *Let C be a closed bounded convex subset of a Banach space X. The following assertions are equivalent:*

(i) C is SWC;

(ii) X admits an equivalent norm $|||\cdot|||$ such that $|||\cdot|||^2$ is uniformly convex on C.

Proof. $(ii) \implies (i)$ is direct by Corollary 2.3.6.

$(i) \implies (ii)$ By Corollary 2.3.6, there exists a bounded uniformly convex function $f : C \to \mathbb{R}$. We can also assume that f is Lipschitz. By the previous theorem, for all $\varepsilon > 0$ there exists an equivalent norm $|||\cdot|||_\varepsilon$ on X such that $|||\cdot|||_\varepsilon^2$ is uniformly convex on C. Taking a serie, we deduce that (ii) holds. □

Finally we will discuss the approximation by differences of convex functions in terms of the index of dentability improving [146, Theorem 1.4] and [79, Theorem 4.1], which in turn are based in the seminal work [46]. A real function defined on a convex set is called $\mathcal{DC}$-Lipschitz if it is the difference of two convex Lipschitz functions. The symbol $\|\cdot\|_C$ stands for the supremum norm on the set C.

Theorem 2.4.10. *Let C be a bounded closed convex subset of a Banach space X and let $f : C \to \mathbb{R}$ be a uniformly continuous function. Consider the following numbers:*

(ε_1) *the infimum of the $\varepsilon > 0$ such that $\mathrm{Dz}(f, \varepsilon) < \omega$;*

(ε_2) *the infimum of the $\varepsilon > 0$ such that there exists a $\mathcal{DC}$-Lipschitz function g such that $\|f - g\|_C < \varepsilon$.*

Then $\varepsilon_1/2 \le \varepsilon_2 \le 2\varepsilon_1$.

Proof. Let $\varepsilon > \varepsilon_2$ and find a $\mathcal{DC}$-Lipschitz function g such that $\|f-g\|_C < \varepsilon$. We know by [146, Proposition 5.1] that g is finitely dentable, which easily implies that f is 2ε-finitely dentable. For the reverse inequality, take $\varepsilon > 2\varepsilon_1$ and $M = \sup\{f(x) - f(y) : x, y \in C\} < +\infty$. Apply Theorem 2.3.8 to get a function Φ such that $|f(x) - f(y)| \le \varepsilon$ if $\Delta_\Phi(x, y) < \delta$. By Proposition 2.4.5 we may suppose that Φ is Lipschitz too, and by Theorem 2.4.7, there is an equivalent norm $\|\!|\cdot|\!\|$ defined on X such that $\Delta_{\|\!|\cdot|\!\|^2}(x, y) < \zeta$ implies $\Delta_\Phi(x, y) < \delta$. Take $c > M/\zeta$. Consider the function

$$g(x) = \inf_{y\in C}\left\{f(y) + c\left(\frac{\|\!|x|\!\|^2 + \|\!|y|\!\|^2}{2} - \left\|\!\left|\frac{x+y}{2}\right|\!\right\|^2\right)\right\} = \inf_{y\in C}\{f(y) + c\,\Delta_{\|\!|\cdot|\!\|^2}(x, y)\}$$

which is actually an inf-convolution with the *Cepedello's kernel*, see [46] or [21, Theorem 4.21]. For every $x \in C$, the infimum can be computed just on the set

$$A(x) = \{y \in C : f(y) + c\,\Delta_{\|\!|\cdot|\!\|^2}(x, y)\} \le f(x)\}.$$

If $x \in C$ and $y \in A(x)$, we have

$$0 \le c\,\Delta_{\|\!|\cdot|\!\|^2}(x, y) \le f(x) - f(y) \le M.$$

Then $\Delta_{\|\!|\cdot|\!\|^2}(x, y) \le \zeta$ by the choice of c and thus $0 \le f(x) - f(y) \le \varepsilon$. Fix $\eta > 0$ and take $y \in A(x)$ such that

$$f(y) + c\,\Delta_{\|\!|\cdot|\!\|^2}(x, y) \le g(x) + \eta.$$

Then

$$f(x) - g(x) \le f(x) - f(y) - c\,\Delta_{\|\!|\cdot|\!\|^2}(x, y) + \eta \le \varepsilon + \eta.$$

We deduce that $\|f(x) - g(x)\|_C \le \varepsilon$ and

$$g(x) = \frac{c}{2}\,\|\!|x|\!\|^2 - \sup_{y\in C}\left\{c\left\|\!\left|\frac{x+y}{2}\right|\!\right\|^2 - \frac{c}{2}\,\|\!|y|\!\|^2 - f(y)\right\}$$

which is an explicit decomposition of g as a difference of two convex Lipschitz functions on C, as wanted. $\square$

In definition 2.3.7, we introduced the dentability of a map. The dentability of a set is defined in the obvious way:

Definition 2.4.11. Let C be a closed convex subset of a Banach space X. We say that C is dentable (resp. finitely dentable) if the identity $\mathrm{id} : C \to C$ is dentable (resp. finitely dentable). For all $n \in \mathbb{N}$ and all $\varepsilon > 0$, we simply write $[C]^n_\varepsilon$ instead of $[C]^n_{id,\varepsilon}$. Similarly, we defined $\mathrm{Dz}(C, \varepsilon)$ by $\mathrm{Dz}(C, \varepsilon) := \mathrm{Dz}(\mathrm{id}, \varepsilon)$.

Corollary 2.4.12. *Let C be a closed bounded convex subset of a Banach space X. The following assertions are equivalent:*

(i) C *is SWC;*

(ii) C *is finitely dentable;*

(iii) every Lipschitz function defined on C *is finitely dentable;*

(iv) every Lipschitz function defined on C *can be uniformly approximated by a* $\mathcal{DC}$*-Lipschitz function.*

Proof. The equivalence between (iii) and (iv) follows from the previous theorem.

$(ii) \implies (i)$ We can assume that C is symmetric by Theorem 1.5.2. Suppose that C is not SWC. By Theorem 1.5.12, C has the finite tree property. So, there exists $\varepsilon > 0$ such that C contains a 2ε-separated tree T_n of height n for all $n \in \mathbb{N}$. Let $n \geq 1$. It is easily seen that the 2ε-separated tree of height $n-1$ obtained by cutting the last level of T_n is a subset of $[C]'_\varepsilon$. A trivial induction shows that $[C]^n_\varepsilon \neq \emptyset$. Since $[C]^n_\varepsilon \neq \emptyset$ is convex and symmetric, we have that $0 \in [C]^n_\varepsilon \neq \emptyset$ for all $n \in \mathbb{N}$. It follows that C is not finitely dentable.

$(i) \implies (ii)$ By Corollary 2.3.6, there exists a function $f : C \to \mathbb{R}$ which is bounded and uniformly convex. Let $s = \sup_C f$ and fix $\varepsilon > 0$. It is easy to see that $[C]'_\varepsilon \subset \{f \leq s - \delta_f(\varepsilon)\}$. Iterating this we will eventually get to the empty set.

$(iv) \implies (i)$ Without loss of generality, we can suppose that $0 \in C$. Suppose that C is not SWC. In particular, C is not compact. So there exist $\varepsilon > 0$ and a sequence $(x_n)_{n\in\mathbb{N}} \subset C$ such that $\|x_p - x_q\| > 3\varepsilon$ for all $p \neq q$. Since C is bounded, there exists $\alpha \in (0, \frac{1}{2})$ such that $\alpha C \subset \frac{\varepsilon}{2} B_X$. Define $y_n = \frac{1}{2} x_n \in C$ (since $0 \in C$) and $C_n = y_n + \alpha C$ for all $n \in \mathbb{N}$. Note that $C_n = \frac{1}{2} x_n + \alpha C \subset C$ for all $n \in \mathbb{N}$. Furthemore, it is easily seen by triangle inequality that $d(C_p, C_q) > \varepsilon$ for all $p \neq q$. Note that C_n has the same properties of C, i.e. it is a closed bounded convex non SWC set. By Theorem 1.5.12, there exists $\theta > 0$ such that C_n contains a θ-strongly separated tree of height n for all $n \in \mathbb{N}$. So, for a suitable η, we have constructed a family $(T_n)_{n\in\mathbb{N}}$ of trees in C such that T_n is a strong η-strongly separated tree of height n and $d(T_p, T_q) > \eta$ for all $p \neq q$. For $n \in \mathbb{N}$, denote by S_n the set of nodes at odd levels of T_n and consider the subset $S = \bigcup_{n=1}^\infty S_n$. Define a 1-Lipschitz function $f : C \to \mathbb{R}$ by $f(x) = d(x, S)$. Let $\delta \in (0, \frac{\eta}{4})$. By (iv), there exists two Lipschitz convex functions $h, g : C \to \mathbb{R}$ such that $\|f - (h+g)\|_C < \delta$. We write $T_n = \{x^n_s \ : \ |s| \leq n\}$. Let $n \in \mathbb{N}$. Since $f(x^n_\emptyset) = 0$, one has that $h(x^n_\emptyset) \geq g(x^n_\emptyset) - \delta$. By convexity of h and since $x^n_\emptyset = \frac{x^n_0 + x^n_1}{2}$, there exists $\alpha_1 \in \{0, 1\}$ such that

$$h(x^n_{\alpha_1}) \geq h(x^n_\emptyset) \geq g(x^n_\emptyset) - \delta$$

Then, we have that

$$g(x^n_{\alpha_1}) \geq f(x^n_{\alpha_1}) + h(x^n_{\alpha_1}) - \delta \geq \frac{\eta}{2} + g(x^n_\emptyset) - 2\delta. \tag{2.2}$$

Using the convexity of g, there exists $\alpha_2 \in \{0, 1\}$ such that $g(x^n_{\alpha_1 \frown \alpha_2}) \geq g(x^n_{\alpha_1})$. By (2.2), we deduce that

$$g(x^n_{\alpha_1 \frown \alpha_2}) - g(x^n_\emptyset) \geq \frac{\eta}{2} - 2\delta > 0.$$

Iterating this process until the last level of T_n, we find a point $x^n_{\alpha_1 \frown \alpha_2 \frown \ldots \frown \alpha_n} \in T_n$ such that

$$g(x^n_{\alpha_1 \frown \alpha_2 \frown \ldots \frown \alpha_n}) - g(x^n_\emptyset) \geq \left(\frac{\eta}{2} - 2\delta\right) \frac{n}{2},$$

contradicting the fact that g is Lipschitz.

$(ii) \implies (iii)$ Let $f : C \to \mathbb{R}$ be a L-Lipschitz function. One has that $[C]^n_{f,\varepsilon} \subset [C]^n_{\varepsilon L^{-1}}$ for all $\varepsilon > 0$ and for all $n \in \mathbb{N}$. So (iii) holds. □

2.5 A new glance at Enflo's theorem

Let us show how Enflo's theorem follows from our results.

Theorem 2.5.1 (Enflo [62]). *Let X be a superreflexive Banach space. Then X has an equivalent uniformly convex norm.*

Proof. The unit ball B_X endowed with the weak topology is SWC. Therefore, there is a bounded convex ε-uniformly convex function defined on B_X for every $\varepsilon > 0$ by Theorem 2.3.5. Now, by Theorem 2.4.8, there is an equivalent norm $\|\cdot\|_\varepsilon$ on X whose square is an ε-uniformly convex function on B_X. Without loss of generality, we may assume that $\|\cdot\| \leq \|\cdot\|_\varepsilon \leq 2\|\cdot\|$. The series $|\!|\!|\cdot|\!|\!|^2 = \sum_{n=1}^{\infty} 2^{-n}\|\cdot\|_{1/n}^2$ defines an equivalent uniformly convex norm. □

At this point, we want to stress that we barely get Enflo's but not Pisier's, see [57] for instance, because we are mainly focused on "ε" (the separation of dyadic trees) instead of "δ" (the quality of the modulus of convexity).

Enflo's original proof of the uniformly convex renorming of superreflexive Banach spaces has remain practically unchanged in books, see [67, Pages 438-442] for instance. We believe that the reason is that the proof is difficult to follow from a geometrical point of view. One of the original aims of this paper was to cast some light on the renorming of superreflexive spaces. Since the geometrical ideas are now diluted along this paper, we would like to offer to the interested reader a more direct pathway to Enflo's theorem as a successive improvement of functions.

- By Theorem 1.5.12, B_X does not have the finite tree property. The maximal height of an ε-separated tree with root $x \in B_X$ is an ε-uniformly concave function $h(x)$. This is the main idea in the proof of Theorem 2.3.4. Note that this function is also symmetric.

- $g(x) = 3^{-h(x)}$ is a symmetric ε-uniformly convex function taking values in $[0,1]$. This comes from Lemma 2.3.1 and is just an arithmetical fact.

- $f = \breve{g}$ is convex, symmetric and 3ε-uniformly convex. The key idea is that $f(x)$ is computed with the values of $g(y)$ with $\|y - x\| < \varepsilon$. The technical details can be carried out as in the proof of Theorem 2.2.6, which relies on Proposition 2.2.5.

- Let f_n the function given in the previous steep for $\varepsilon = 1/n$. The function

$$F(x) = \|x\| + \sum_{n=1}^{\infty} 2^{-n} f_n(x)$$

  is uniformly convex, symmetric, Lipschitz on the balls rB_X for $0 < r < 1$ and it attains a strong minimum at 0. Moreover, elementary computations can show that $F(0) \leq 1/17$ and $F(x) \geq 1$ for $x \in S_X$.

- The set $B = \{x : F(x) \leq \inf F + 1/2\}$ is the unit ball of an equivalent uniformly convex norm $|\!|\!|\cdot|\!|\!|$. The idea is to use the Lipschitz property of F to show that for any $\delta > 0$, there is $\lambda(\delta) \in (0,1)$ such that

$$\{x : F(x) \leq \inf F + 1/2 - \delta\} \subset \lambda(\delta)B.$$

  Therefore, if $|\!|\!|x|\!|\!| = |\!|\!|y|\!|\!| = 1$ and $\|x - y\| \geq \varepsilon$ then $F(x) = F(y) = \inf F + 1/2$ and $F(\frac{x+y}{2}) \leq \inf F + 1/2 - \delta$ for some $\delta = \delta(\varepsilon) > 0$ and thus $|\!|\!|\frac{x+y}{2}|\!|\!| \leq \lambda(\delta)$.

2.6 A remark on ε-uniformly smooth functions

In this small subsection, we show that there exists a duality between ε-uniformly convex functions and ε-uniformly smooth functions. This point will probably be exploited in a future work.

If $f : X \to \overline{\mathbb{R}}$ is a function, then f^* denotes its Fenchel conjugate.

Proposition 2.6.1. *Let X be a Banach space and let $f : X \to \overline{\mathbb{R}}$ be an ε-uniformly convex function which is not identically $+\infty$. The following are equivalent:*

(i) $\breve{f}$ is proper

(ii) f is bounded below

(iii) f^* is bounded on bounded sets

Proof. $(i) \iff (ii)$ follows from Corollary 2.4.2. Suppose that (ii) holds. Fix $r > 0$. Since $\liminf \frac{f(x)}{\|x\|^2} > 0$, there exists $R > 0$ such that $f(x) \geq r\|x\|$ whenever $\|x\| > R$. By hypotesis, there exists $\alpha > 0$ such that $f \geq -\alpha$ on X. It follows that $f \geq r\|.\| - rR - \alpha$. By taking the conjugate, we conclude that $f^* \leq rR + \alpha$ on $B(0, \alpha)$. Furthermore f^* is a lower semicontinous convex function which is also proper (f is bounded below by a affine function), it follows that f^* is bounded below on bounded sets.
Now, suppose that (iii) holds. Recall that

$$f^*(x^*) = \sup_{x \in X} \{x^*(x) - \breve{f}(x)\}$$

for all $x^* \in X^*$. If there exists $x_0 \in X$ such that $\breve{f}(x_0) = -\infty$ then $f^* \equiv +\infty$, which contradicts (ii). Now if $\breve{f} \equiv +\infty$, then $f^* \equiv -\infty$, which is again a contradiction. □

Corollary 2.6.2. *Let X be a Banach space and let $f : X \to \overline{\mathbb{R}}$ be an ε-uniformly convex bounded below function. Then f^* is continuous.*

Proof. By the previous proposition, f^* is locally bounded above. Since f^* is convex, it implies that f^* is continuous. □

Nor we point out that we can define a dual notion to ε-uniform convexity.

Definition 2.6.3. Let X be a Banach space and let $f : X \to \overline{\mathbb{R}}$ be a function. We define its *modulus of smoothness* as the function $\rho_f : \mathbb{R}^+ \to \overline{\mathbb{R}}$ defined by

$$\rho_f(\varepsilon) = \sup \left\{ \frac{f(x) + f(y)}{2} - f\left(\frac{x+y}{2}\right) \ : \ \|x - y\| \geq \varepsilon \right\}.$$

We say that f is *ε-uniformly smooth* if there exists $\eta_\varepsilon > 0$ such that $\frac{\rho_f(t)}{t} \leq \varepsilon$ for all $t \in (0, \eta_\varepsilon)$. Finally, we say that f is *uniformly smooth* if f is ε-uniformly smooth for all $\varepsilon > 0$.

Note that if f is mid-convex, then ρ_f is a positive function and f is uniformly smooth if and only if

$$\lim_{t \to 0^+} \frac{\rho_f(t)}{t} = 0.$$

The two following results shows that the notions of ε-uniform convexity and ε-uniformly smooth are dual notions.

Proposition 2.6.4. *Let X be a Banach space and let $f : X \to \overline{\mathbb{R}}$ be a ε-uniformly convex function such that f^* is proper. Then f^* is ε^+- uniformly smooth.*

Proof. Since $f^* = (\breve{f})^*$ and $\breve{f}$ is ε-uniformly convex, we can suppose that f is convex. Pick $\varepsilon' > \varepsilon$. Define $\eta_{\varepsilon'} = \frac{\delta_f(\varepsilon')}{\varepsilon'} > 0$ and take $s \in (0, \eta_{\varepsilon'})$. By the previous proposition, f^* is $(\delta_f)^*$-smooth. We have that

$$(\delta_f)^*(s) = \sup\{ts - \delta_f(t) \ : \ t \geq 0\}$$

$$\leq \max\{\sup\{ts - \delta_f(t) \ : \ t \in [0, \varepsilon']\}, \sup\{ts - \delta_f(t) \ : \ t > \varepsilon'\}\} \leq \varepsilon' s.$$

It follows that

$$\frac{\rho_{f^*}(s)}{s} \leq \frac{(\delta_f)^*(s)}{s} \leq \varepsilon',$$

i.e. f^* is ε'- uniformly smooth. □

Proposition 2.6.5. *Let X be a Banach space and let $f : X \to \overline{\mathbb{R}}$ be a ε-uniformly smooth function such that f^* is proper. Then f^* is ε^+- uniformly convex.*

Proof. There exists $\eta_\varepsilon > 0$ such that

$$\frac{\rho_f(t)}{t} \leq \varepsilon$$

for all $t \in (0, \eta_\varepsilon)$. Define a function $g : \mathbb{R} \to \overline{\mathbb{R}}$ by $g(t) = \varepsilon t$ if $t \in [0, \eta_\varepsilon]$ and $g(t) = +\infty$ if not. We have that $\rho_f \leq g$ and it follows that $(\rho_f)^* \geq g^*$. In particular, for $\varepsilon' > \varepsilon$, we have that

$$(\rho_f)^*(\varepsilon') \geq \sup\{\varepsilon' t - \varepsilon t \ : \ t \in [0, \eta_\varepsilon]\} \geq (\varepsilon' - \varepsilon)\eta_\varepsilon > 0.$$

Since f^* is $(\rho_f)^*$-convex, it follows that

$$\delta_{f^*}(\varepsilon') \geq (\rho_f)^*(\varepsilon') > 0,$$

i.e. f^* is ε'-uniformly convex. □

We can deduce the dual version of Theorem 2.2.6:

Corollary 2.6.6. *Let X be a Banach space. Let $f : X \to \overline{\mathbb{R}}$ be a ε-uniformly smooth function such that $\breve{f}$ is proper. Then $\breve{f}$ is ε^+-uniformly smooth.*

Proof. We have that $f^{**} = \breve{f}$. In particular, it implies that f^* is proper. It follows that f^* is ε^+-uniformly convex and then f^{**} is ε^+-uniformly smooth. □

Chapter 3

Quantification of the super weak compactness

3.1 Different ways to quantify SWC

Measures of noncompactness can be defined in very general settings. Here we will restrict ourselves to the frame of topological vector spaces. Let X be a topological vector space and let $\mathfrak{K}$ be a vector bornology of compact subsets (that just means the class is stable under some elementary operations).

Definition 3.1.1. A *measure of noncompactness* associated to $\mathfrak{K}$ is a nonnegative function μ defined on the bounded subsets of X that satisfies the following properties:

(a) $\mu(\overline{A}) = \mu(A)$

(b) $\mu(A) = 0$ if and only if $\overline{A} \in \mathfrak{K}$

(c) $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$

(d) $\mu(\lambda A) = |\lambda|\mu(A)$ for all $\lambda \in \mathbb{R}$

(e) $\mu(A + B) \leq \mu(A) + \mu(B)$

(f) there exists $k > 0$ such that $\mu(\operatorname{conv}(A)) \leq k\, \mu(A)$

This list of conditions comes from the usual requirements in literature [13] and some properties enjoyed by several measures that are interesting for Banach space geometry, such as γ or the family of measures introduced in [121] in relation with the Szlenk index. Condition (6) is usually the most tricky and necessarily requieres that the class $\mathfrak{K}$ be stable by closed convex hulls (Krein-type theorem).

The quantification of the super weak non-compactness is linked to the quantification of the weak non-compactness. De Blasi (see [7], for instance) introduced a measure of weak noncompactness ω as follows

$$\omega(A) = \inf\{\varepsilon > 0 : \exists K \subset X \text{ weakly compact}, A \subset K + \varepsilon B_X\}.$$

It is not hard to check that ω enjoys all the properties above. In particular, we have

$$\omega(\operatorname{conv}(A)) = \omega(A),$$

that is, its "convexifiability constant" is 1. Another quite natural way to measure weak noncompactness, is the function γ defined by

$$\gamma(A) = \inf\{\varepsilon > 0 : \overline{A}^{w^*} \subset X + \varepsilon B_{X^{**}}\} = \sup\{d(X, x^{**}) : x^{**} \in \overline{A}^{w^*}\}.$$

It is easy to check that $\gamma(A) \leq \omega(A)$ for any bounded set $A \subset X$. However, there is no constant $c > 0$ such that $\omega(A) \leq c\,\gamma(A)$ in general, see [7, Corollary 3.4]. That fact says that ω and γ are not *equivalent*. The measure γ was introduced in [68] where the authors also proved ([84] independently, see also [91, Theorem 3.64]) that

$$\gamma(\mathrm{conv}(A)) \leq 2\,\gamma(A)$$

for any bounded $A \subset X$. Notably, there are many different equivalent ways to deal with γ which are interesting to us because they have a "super" version. We will requiere the following *measure of super weak noncompactness*: for a bounded set $A \subset X$ take

$$\Gamma(A) := \gamma(A_{\mathcal{U}})$$

where $\mathcal{U}$ is a free ultrafilter and γ is computed in $X_{\mathcal{U}}$. Later we will see that Γ does not depend, essentially, on the choice of the ultrafilter $\mathcal{U}$ and that Γ is a *measure of noncompactness* according to definition 3.1.1 (see Theorem 3.1.11). Obviously, we have that A is relatively SWC if and only if $\Gamma(A) = 0$, and an operator $T : X \to Y$ is SWC if and only if $\Gamma(T(B_X)) = 0$

The following contains the quantified version of two classic James' characterizations of relative weak compactness together with the quantified version of Grothendieck's commutation of limits criterion.

Proposition 3.1.2. *Let A be a bounded subset of a Banach space X. Consider the following numbers:*

(γ_1) $= \gamma(A)$*;*

(γ_2) *the supremum of the numbers $\varepsilon \geq 0$ such that there are sequences $(x_n)_n \subset A$ and $(x_n^*)_n \subset B_{X^*}$ such that $x_n^*(x_m) = 0$ if $m < n$ and $x_n^*(x_m) \geq \varepsilon$ if $m \geq n$;*

(γ_3) *the supremum of the numbers $\varepsilon > 0$ such that for any $n \in \mathbb{N}$ there are $x_1, \ldots, x_n \in C$ such that $\mathrm{d}(\mathrm{conv}\{x_1, \ldots, x_k\}, \mathrm{conv}\{x_{k+1}, \ldots, x_n\}) \geq \varepsilon$ for all $k = 1, \ldots, n-1$;*

(γ_4) *the infimum of the numbers $\varepsilon \geq 0$ such that*

$$|\lim_n \lim_m x_n^*(x_m) - \lim_m \lim_n x_n^*(x_m)| \leq \varepsilon$$

whenever $(x_n)_n \subset A$, $(x_n^*)_n \subset B_{X^*}$ and the iterated limits exist.

Then $\gamma_1 \leq \gamma_2 \leq \gamma_3 \leq \gamma_4 \leq 2\gamma_1$.

Proof. Take $\varepsilon < \gamma(A)$ and let $x^{**} \in \overline{A}^{w^*}$ with $d(X, x^{**}) > \varepsilon$. We will build sequences satisfying the second statement for such an ε. Indeed, there exists $x_1^* \in B_{X^*}$ with $|x^{**}(x_1^*)| > \varepsilon$. Now take $x_1 \in A$ such that $x_1^*(x_1) \geq \varepsilon$. Assume we have x_k and x_k^* already built for $1 \leq k < n$ and it is satisfied $x^{**}(x_k^*) > \varepsilon$. An elementary application of Helly's theorem [67, p. 159] to X^{**} allows us to choose $x_n^* \in B_{X^*}$ such that $x_n^*(x_k) = 0$ for $1 \leq k < n$ and $x^{**}(x_n^*) > \varepsilon$. Now we take

$$x_n \in A \cap \{x \in X : x_k^*(x) > \varepsilon, 1 \leq k \leq n\}$$

since the set is nonempty. That finishes the construction of the sequence and proves $\gamma_1 \leq \gamma_2$. The inequality $\gamma_2 \leq \gamma_3$ follows straight. In order to prove $\gamma_3 \leq \gamma_4$, take $\varepsilon < \gamma_3$, and sequence

(x_n) as in the statement (γ_3). For every $n \in \mathbb{N}$, take $x_n^* \in B_{X^*}$ such that $x_n^*(y) \leq \varepsilon + x_n^*(z)$ for every $y \in \operatorname{conv}\{x_1, \dots, x_n\}$ and $z \in \operatorname{conv}\{x_{n+1}, x_{n+2}, \dots\}$. The sequences satisfies the following property

$$x_n^*(x_p) \leq \varepsilon + x_n^*(x_q)$$

whenever $p \leq n < q$. Passing to a subsequence, we may assume the existence of the limits $\lim_n x_n^*(x_m)$ and $\lim_m x_n^*(x_m)$, as well as the existence of the iterated limits. In such a case we will get

$$\lim_m \lim_n x_n^*(x_m) \leq \varepsilon + \lim_n \lim_m x_n^*(x_m)$$

which implies $\varepsilon \leq \gamma_4$, and therefore $\gamma_3 \leq \gamma_4$. Finally, $\gamma_4 \leq 2\gamma_1$ is proved in [7]. □

Now we will state the "super" version of Proposition 3.1.2, for which we prefer to avoid a uniform version of *Grothendieck's commutation of limits* (fourth statement).

Proposition 3.1.3. *Let A be a subset of a Banach space X. Consider the following numbers:*

(Γ_1) $= \gamma(A_{\mathcal{U}})$ measured in $X_{\mathcal{U}}$ for $\mathcal{U}$ a free ultrafilter;

(Γ_2) the infimum of the numbers $\varepsilon > 0$ such that there are no arbitrarily long sequences $(x_k)_1^n \subset A$, $(x_k^*)_1^n \subset B_{X^*}$ with $x_k^*(x_j) = 0$ if $j < k$ and $x_k^*(x_j) > \varepsilon$ if $j \geq k$;

(Γ_3) the supremum of the numbers $\varepsilon > 0$ such that for any $n \in \mathbb{N}$ there are $x_1, \dots, x_n \in C$ such that $\mathrm{d}(\operatorname{conv}\{x_1, \dots, x_k\}, \operatorname{conv}\{x_{k+1}, \dots, x_n\}) \geq \varepsilon$ for all $k = 1, \dots, n-1$;

Then $\Gamma_1 \leq \Gamma_2 \leq \Gamma_3 \leq 2\Gamma_1$.

Proof. The fact $\Gamma_1 \leq \Gamma_2$ follows straight by applying finite representatibity to inequality $\varepsilon_1 \leq \varepsilon_2$ in Proposition 3.1.2. It is quite easy to get $\Gamma_2 \leq \Gamma_3$, and $\Gamma_3 \leq 2\Gamma_1$ follows using the standard ultraproduct technique, (see also Theorem 3.1.7 below where the convex case is considered). □

Recall that Γ_1 is the measure introduced at the beginning of this section

$$\Gamma(A) := \gamma(A_{\mathcal{U}})$$

that depends on the choice of $\mathcal{U}$. From now on, we will assume the free ultrafilter $\mathcal{U}$ is fixed when speaking of Γ or dealing with the ultraproducts. Note that the equivalent measures Γ_2 and Γ_3 does not depend on any ultrafilter. Moreover, Γ_3 does not involves explicitly the dual space. In the next section we will use $\Gamma_s(A) = \Gamma_2$ as an alternative to $\Gamma(A)$.

Proposition 3.1.4. *Let $T : X \to Y$ be an operator between two Banach spaces and let $A \subset X$ be a bounded set. Then $\Gamma(T(A)) \leq \|T\| \, \Gamma(A)$.*

Proof. Firstly, we will prove a similar statement for γ. Consider $T^{**} : X^{**} \to Y^{**}$ which is weak* to weak* continuous. For any bounded set $A \subset X$ we have

$$\overline{T(A)}^{w^*} = T^{**}(\overline{A}^{w^*}) \subset T(X) + \varepsilon T^{**}(B_{X^{**}}) \subset Y + \varepsilon \|T\| B_{Y^{**}}$$

where $\varepsilon > \gamma(A)$. Therefore $\gamma(T(A)) \leq \|T\| \, \gamma(A)$. In order to prove the statement for Γ, consider the induced operator $T_{\mathcal{U}} : X_{\mathcal{U}} \to Y_{\mathcal{U}}$. Then we have

$$\Gamma(T(A)) = \gamma(T_{\mathcal{U}}(A_{\mathcal{U}})) \leq \|T_{\mathcal{U}}\| \gamma(A_{\mathcal{U}}) = \|T\| \, \Gamma(A),$$

as we wished. □

Definition 3.1.5. Given a convex subset C of a Banach space X, let us define $\mathrm{Dent}(C)$ as the infimum of the numbers $\varepsilon > 0$ such that C has nonempty slices contained in balls of radius less than ε, and take $\Delta(C) = \sup\{\mathrm{Dent}(B) : B \subset C\}$.

The measure Δ was introduced in [43] as a way to quantify the lack of *Radon-Nikodym property* (RNP).

Lemma 3.1.6. *Let A be a closed convex bounded subset of a Banach space X, $\mathcal{U}$ a free ultrafilter on $\mathbb{N}$ and $\varepsilon > 0$. Then*

$$[A_{\mathcal{U}}]'_{2\varepsilon} \subset ([A]'_{\varepsilon})_{\mathcal{U}}.$$

Proof. Given $(x_n)_{\mathcal{U}} \in A_{\mathcal{U}} \setminus ([A]'_{\varepsilon})_{\mathcal{U}}$, we have to find a slice of $A_{\mathcal{U}}$ containing $(x_n)_{\mathcal{U}}$ of diameter not greater than 2ε. As $(x_n)_{\mathcal{U}} \notin ([A]'_{\varepsilon})^{\mathcal{U}}$, then for some $\alpha > 0$

$$\{n : d(x_n, [A]'_{\varepsilon}) \geq \alpha\} \in \mathcal{U}.$$

Indeed, otherwise the sequence $(x_n)_n$ would be equivalent to a sequence in $[A]'_{\varepsilon}$. It is possible to find $x_n^* \in B_{X^*}$ such that $x_n^*(x_n) \geq \alpha + \sup_{[A]'_{\varepsilon}} x_n^*$ for those indices n from the previous set, for the other n's the choice of $x_n \in B_{X^*}$ does not make a difference. Consider the functional $(x_n^*)_{\mathcal{U}} \in (X^*)_{\mathcal{U}} \subset (X_{\mathcal{U}})^*$. By construction,

$$\langle (x_n^*)_{\mathcal{U}}, (x_n)_{\mathcal{U}} \rangle \geq \alpha + \sup_{([A]'_{\varepsilon})_{\mathcal{U}}} (x_n^*)_{\mathcal{U}}.$$

Now, we will estimate the diameter of the slice defined by $(x_n^*)_{\mathcal{U}}$. Suppose that

$$(y_n)_{\mathcal{U}}, (z_n)_{\mathcal{U}} \in A_{\mathcal{U}} \cap \{(x_n^*)_{\mathcal{U}} \geq \alpha + \sup_{([A]'_{\varepsilon})_{\mathcal{U}}} (x_n^*)_{\mathcal{U}}\}.$$

Then for a subset in $\mathcal{U}$ of indices n, we have $y_n, z_n \in A \cap \{x_n^* \geq \alpha + \sup\{x_n^*, A_n\}\}$ and thus $\|y_n - z_n\| \leq 2\varepsilon$ by Lancien's midpoint argument. That implies $\|(y_n)_{\mathcal{U}} - (z_n)_{\mathcal{U}}\| \leq 2\varepsilon$, so the diameter of the slice is not greater than 2ε as wished. $\square$

The following result is a quantification of super weak compactness (see Theorems 1.5.6, 1.5.9, 1.5.12 and Corollaries 2.3.6, 2.4.12).

Theorem 3.1.7. *Let C be a bounded closed convex subset of a Banach space X. Consider the following numbers:*

(μ_1) *the supremum of the numbers $\varepsilon > 0$ such that for any $n \in \mathbb{N}$ there are $x_1, \dots, x_n \in C$ such that $\mathrm{d}(\mathrm{conv}\{x_1, \dots, x_k\}, \mathrm{conv}\{x_{k+1}, \dots, x_n\}) \geq \varepsilon$ for all $k = 1, \dots, n-1$;*

(μ_2) *the supremum of the $\varepsilon > 0$ such that there are ε-separated dyadic trees of arbitrary height;*

(μ_3) $= \Delta(C_{\mathcal{U}})$, *for $\mathcal{U}$ a free ultrafilter on $\mathbb{N}$;*

(μ_4) $= \Gamma(C)$, *(computed for an arbitrary free ultrafilter $\mathcal{U}$ on $\mathbb{N}$);*

(μ_5) *the infimum of the $\varepsilon > 0$ such that $\mathrm{Dz}(C, \varepsilon) < \omega$;*

(μ_6) *the infimum of the $\varepsilon > 0$ such that C supports a convex bounded ε-uniformly convex function.*

Then $\mu_1 \leq \mu_2 \leq 2\mu_3 \leq 2\mu_4 \leq 2\mu_1$ and $\mu_4 \leq 2\mu_5 \leq 2\mu_6 \leq 2\mu_2$.

Proof. We will label the steps of the proof by the couple of numbers associated to the inequality.
(1-2) If $\varepsilon < \mu_1$, the separation between convex hulls applied to 2^n elements allows the construction of a ε-separated dyadic trees of height n. Therefore $\mu_2 \geq \mu_1$.
(2-3) If $\varepsilon < M_2$ then $\Delta(C_{\mathcal{U}}) \geq \varepsilon/2$. Indeed, $C_{\mathcal{U}}$ contains an infinite ε-separated dyadic tree T, therefore any nonempty slice of T cannot be covered by finitely many balls of radius less than $\varepsilon/2$.
(3-4) By [43, Proposition 6.1], $\mathrm{Dent}(A) \leq \gamma(A)$, therefore $\Delta(C_{\mathcal{U}}) \leq \gamma(C_{\mathcal{U}}) = \Gamma(C)$.
(4-1) Let $\varepsilon < \mu_4$. Then there is $x \in \overline{C}^{w^*}$ which is at distance greater than ε from X. Following Oja's proof of James theorem [67, Theorem 3.132], it is possible to find an infinite sequence $(x_n)_n$ with convex separation greater than ε. Finite representativity gives arbitrarily large sequences in X with the same separation, thus $\mu_1 \geq \mu_4$.
(4-5) If $\varepsilon > M_5$ then there is a finite sequence of sets $C = C_1 \supset C_2 \supset \cdots \supset C_n$ given by the ε-dentability process. Taking weak* closures in the bidual, we have

$$\overline{C}^{w^*} = (\overline{C_1}^{w^*} \setminus \overline{C_2}^{w^*}) \cup \cdots \cup (\overline{C_{n-1}}^{w^*} \setminus \overline{C_n}^{w^*}) \cup \overline{C_n}^{w^*}.$$

Now, take any $x \in \overline{C}^{w^*}$ that belongs to one of those sets. The w^*-open slice separating x from the smaller set, say $\overline{C_{k+1}}^{w^*}$ ($\emptyset$ for the last set) in the difference is contained in the w^*-closure of a slice of C_k not meeting C_{k+1} which has diameter less than 2ε (Lancien's midpoint argument). Since w^*-closures does not increase the diameter, we have $d(x, X) \leq 2\varepsilon$. The argument actually implies $\gamma(C) \leq 2\varepsilon$. However, we can apply it to the sequence of sets in $X_{\mathcal{U}}$

$$C_{\mathcal{U}} = (C_1)_{\mathcal{U}} \supset (C_2)_{\mathcal{U}} \supset \cdots \supset (C_n)_{\mathcal{U}}$$

which has the same slice-separation property by Lemma 3.1.6.
(5-6) If $\varepsilon > \mu_6$, there is a bounded convex and ε-uniformly convex function f that, without loss of generality, we may suppose lower semicontinuous. By Proposition 2.2.1, any slice of the set $\{x \in C : f(x) \leq a\}$ not meeting the set $\{x \in C : f(x) \leq a + \delta\}$ has diameter less than ε. A judicious arranging of these sets shows that C is ε-finitely dentable. Thus $\mu_5 \leq \mu_6$.
(6-2) Take $\varepsilon > \mu_2$. Then the ε-separated dyadic trees are uniformly bounded in height. By Theorem 2.3.4, that implies the existence of ε'-uniformly convex function for every $\varepsilon' > \varepsilon$. Thus $\mu_6 \leq \mu_2$. □

Remark 3.1.8. The equivalence between μ_3 and μ_4 is both a local and a quantitative version of the well know statement saying that super-RNP is the same that superreflexivity. Note that Theorem 2.4.10 gives another expression of μ_5 in terms of approximation $\mathcal{DC}$-Lipschitz function. Also notice that some estimations can be improved. For example, we have that $\mu_2 = \mu_6$ by Theorem 2.3.5. Let us point out that some other relations between the quantities μ_i for $i = 1, \ldots, 6$ can be established and so improving the equivalence constants. For instance $\mu_2 \leq 2\mu_5$ which is somehow straightforward or $\mu_6 \leq \mu_5$ as a consequence of Proposition 2.3.3.

We will need the following estimation of how much thicken the closure with respect to the topology induced by a norming subspace of the dual.

Lemma 3.1.9. *Let X a Banach space and $F \subset X^*$ an 1-norming subspace. Then for any bounded convex $A \subset X$ and any $\varepsilon > \gamma(A)$ we have*

$$\overline{A}^{\sigma(X,F)} \subset A + 2\varepsilon B_X.$$

Proof. By [91, Proposition 3.59], $\overline{A}^{w^*} \subset A + 2\varepsilon B_{X^{**}}$. The linear map $p : X^{**} \to F^*$ defined by $p(x^{**}) = x^{**}|_F$ has norm 1 and satisfies $p(\overline{A}^{w^*}) = \overline{A}^{\sigma(F^*,F)}$. We may identify $p(X) = X$ isometrically into F^* and so we have $\overline{A}^{\sigma(F^*,F)} \subset A + 2\varepsilon B_{F^*}$. Therefore $\overline{A}^{\sigma(X,F)} \subset A + 2\varepsilon B_X$ as wished. □

We will need the following result that appears as a fact inside the proof of [163, Theorem 3.1]. The 1-norming subspace $(X^*)_{\mathcal{U}} \subset (X_{\mathcal{U}})^*$ will play an important role.

Lemma 3.1.10. *Let X be a Banach space and let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. For any $(x_n^*)_{\mathcal{U}} \in (X^*)_{\mathcal{U}}$ and $(a_n)_{\mathcal{U}} \in \operatorname{conv}(A)_{\mathcal{U}}$, there is $(b_n)_{\mathcal{U}} \in \operatorname{conv}(A_{\mathcal{U}})$ such that $\langle (x_n^*)_{\mathcal{U}}, (a_n)_{\mathcal{U}} \rangle \leq \langle (x_n^*)_{\mathcal{U}}, (b_n)_{\mathcal{U}} \rangle$.*

Let us finish this section by showing that Γ fulfils the all requirements for a genuine measure of noncompactness listed at the beginning.

Theorem 3.1.11. *The function Γ defined on bounded subsets of a Banach space X has the following properties:*

(a) $\Gamma(\overline{A}) = \Gamma(A)$;

(b) $\Gamma(A) = 0$ *if and only if* $\overline{A}$ *is SWC;*

(c) $\Gamma(A \cup B) = \max\{\Gamma(A), \Gamma(B)\}$;

(d) $\Gamma(\lambda A) = |\lambda|\Gamma(A)$ *for all* $\lambda \in \mathbb{R}$;

(e) $\Gamma(A + B) \leq \Gamma(A) + \Gamma(B)$;

(f) $\Gamma(\operatorname{conv}(A)) \leq 4\,\Gamma(A)$.

Proof. (a) and (b) follow straightly from the definition of Γ. (c), (d) and (e) follow from the following set identities: $(A\cup B)_{\mathcal{U}} = A_{\mathcal{U}} \cup B_{\mathcal{U}}$, $(\lambda A)_{\mathcal{U}} = \lambda A_{\mathcal{U}}$ and $(A+B)_{\mathcal{U}} = A_{\mathcal{U}} + B_{\mathcal{U}}$. Consider $F = (X^*)_{\mathcal{U}}$ which is an 1-norming subspace of $(X_{\mathcal{U}})^*$. Take $\varepsilon > \gamma(A_{\mathcal{U}})$. By Lemma 3.1.9,

$$\overline{\operatorname{conv}}^{\sigma(X,F)}(A_{\mathcal{U}}) \subset \operatorname{conv}(A_{\mathcal{U}}) + 2\varepsilon B_{X_{\mathcal{U}}}.$$

We claim $(\operatorname{conv}(A))_{\mathcal{U}} \subset \operatorname{conv}(A_{\mathcal{U}}) + 2\varepsilon B_{X_{\mathcal{U}}}$. If it not the case, then we could separate a point $(\operatorname{conv}(A))_{\mathcal{U}}$ from $\overline{\operatorname{conv}}^{\sigma(X,F)}(A_{\mathcal{U}})$ by a functional from F. That leads to a contradiction with Lemma 3.1.10. Now, we have

$$\gamma((\operatorname{conv}(A))_{\mathcal{U}}) \leq 2\gamma(A_{\mathcal{U}}) + 2\varepsilon$$

which implies (f). □

Note that (f) is a quantitative version (in terms of μ_6) of [163, Theorem 3.1] establishing that the super weak compactness is stable by closed convex hulls. The measure of non super weak compactness introduced by K.Tu in [163] is different from ours and so our result is not equivalent to [163, Theorem 4.2].

3.2 Quantifying uniform convexity for operators

In this section we will discuss the application of the measure of weak noncompactness. For an operator $T : X \to Y$, we will write $\Gamma(T) := \Gamma(T(B_X))$. We have the following.

Proposition 3.2.1. *Let A be convex symmetric bounded subset of a Banach space X with $\Gamma(A) < \varepsilon$. Then there exists a Banach space Z and an operator $T : Z \to X$ such that $\|T\| = 1$, $A \subset T(B_Z)$ and $\Gamma(T) < \varepsilon$.*

Proof. Without loss of generality we may assume that A is closed. Then, just take $Z = \operatorname{span}(A)$, endow it with the norm given by the Minkowski functional of A and take T the identity operator. □

If we consider the alternative measure of weak noncompactness Γ_s introduced after Proposition 3.1.3, we have the following quantified version of the symmetry of the operator ideal $\mathfrak{W}^{super}$.

Theorem 3.2.2. *Let $T : X \to Y$ be an operator between two Banach spaces. Then $\Gamma_s(T^*) = \Gamma_s(T)$.*

Proof. We will assume firstly that $\Gamma_s(T) > 0$. Take $0 < \varepsilon < \Gamma_s(T)$. Then, for every $N \in \mathbb{N}$ there are elements $(x_n)_{n=1}^N \subset B_X$ and $(x_n^*)_{n=1}^N \subset B_{X^*}$ such that

$$\langle x_n^*, T(x_m)\rangle = 0 \text{ for } m < n,$$

$$\langle x_n^*, T(x_m)\rangle \geq \varepsilon \text{ for } m \geq n.$$

But this is exactly the same that

$$\langle T^*(x_n^*), x_m\rangle = 0 \text{ for } m < n,$$

$$\langle T^*(x_n^*), x_m\rangle \geq \varepsilon \text{ for } m \geq n.$$

By reversing the order of $1, \ldots, N$, we get $\Gamma_s(T^*) \geq \varepsilon$. That gives $\Gamma_s(T^*) \geq \Gamma_s(T)$. Suppose now that $\Gamma_s(T^*) > 0$ and take $0 < \varepsilon < \Gamma_s(T^*)$. Then, for every $N \in \mathbb{N}$ there are elements $(x_n^{**})_{n=1}^N \subset B_{X^{**}}$ and $(x_n^*)_{n=1}^N \subset B_{X^*}$ such that

$$\langle x_n^{**}, T^*(x_m^*)\rangle = 0 \text{ for } m < n,$$

$$\langle x_n^{**}, T^*(x_m)\rangle \geq \varepsilon \text{ for } m \geq n.$$

Fix $\lambda > 1$. Helly's theorem [67, p. 159] allows us to find $(x_n)_{n=1}^N \subset \lambda B_X$ such that

$$\langle x_n^{**}, T^*(x_m^*)\rangle = \langle x_n, T^*(x_m^*)\rangle$$

for every $1 \leq n, m \leq N$. That implies $\Gamma_s(T) \geq \lambda^{-1}\varepsilon$, after reversing the order of $1, \ldots, N$. By the arbitrarily choice of constants, we get $\Gamma_s(T) \geq \Gamma_s(T^*)$.
So far we have proved that $\Gamma_s(T) > 0$ if and only if $\Gamma_s(T^*) > 0$ and, in such a case, $\Gamma_s(T) = \Gamma_s(T^*)$. That also implies $\Gamma_s(T) = 0$ if and only if $\Gamma_s(T^*) = 0$, therefore the proof is complete. □

Corollary 3.2.3. *Let $T : X \to Y$ be an operator between two Banach spaces. Then $2^{-1}\Gamma(T) \leq \Gamma(T^*) \leq 2\Gamma(T)$.*

Remark 3.2.4. Using γ_2 as a measure of weak noncompactness for sets and operators, the quantified version of Gantmacher theorem [7] would become an equality.

De Blasi's measure applied to operators does not satisfy a similar quantified Gantmacher result, as observed in [7] after an example from [11], neither does the measure on super weak noncompactness introduced by Tu [164], inspired by De Blasi's definition, as

$$\sigma(T) = \inf\{\varepsilon > 0 : \exists K \subset Y,\ K \text{ is SWC},\ T(B_X) \subset K + \varepsilon B_Y\}$$

Indeed, Tu shows provides a sequence of operators T_n such that and $\sigma(T_n^*) = 1$ for all $n \in \mathbb{N}$ and $\lim_n \sigma(T_n) = 0$.

Remember that if f function is a function then Δ_f is the function defined by

$$\Delta_f(x, y) = \frac{f(x) + f(y)}{2} - f\left(\frac{x+y}{2}\right).$$

Definition 3.2.5. An operator $T : X \to Y$ between two Banach spaces is called *uniformly convex* (UC) if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\|T(x) - T(y)\| \leq \varepsilon$ whenever $x, y \in B_X$ are such that $\Delta_{\|\cdot\|^2}(x, y) < \delta$. An operator $T : X \to Y$ is called *uniformly convexifying* if it becomes uniformly convex after a suitable renorming of X.

Lemma 3.2.6. *Let $T : X \to Y$ be a bounded operator between two Banach spaces. The following assertions are equivalent:*

(i) T is UC;

(ii) $\lim_n \|T(x_n) - T(y_n)\| = 0$ whenever $x_n, y_n \in B_X$ are such

$$\lim_n \Delta_{\|\cdot\|^2}(x_n, y_n) = 0;$$

(iii) $\lim_n \|T(x_n) - T(y_n)\| = 0$ whenever $x_n, y_n \in B_X$ are such $\|x_n + y_n\| \to 2$.

Proof. $(i) \implies (iii)$ Suppose that (iii) does not hold. Then, there exists $\varepsilon > 0$ such that for all $n \in \mathbb{N}$ there exist $x_n, y_n \in B_X$ such that $\|x_n + y_n\| > 2(1 - \frac{1}{n})$ and $\|T(x_n) - T(y_n)\| > \varepsilon$. It is easily seen that $\|x_n\| \to 1$ and $\|y_n\| \to 1$. Then $\Delta_{\|\cdot\|^2}(x_n, y_n) = 2\|x_n\|^2 + 2\|y_n\|^2 - \|x_n + y_n\|^2 \to 0$ and $\|T(x_n) - T(y_n)\| > \varepsilon$, i.e. (i) T is not UC.

$(iii) \implies (ii)$ Let $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \subset B_X$ such that $\Delta_{\|\cdot\|^2}(x_n, y_n) \to 0$. Define $\alpha = \limsup_n \|T(x_n) - T(y_n)\|$ and let us prove that $\alpha = 0$. Taking subsequence if necessary, we assume that $\alpha = \lim_n \|T(x_n) - T(y_n)\|$. Since

$$2\|x_n\|^2 + 2\|y_n\|^2 - \|x_n + y_n\|^2 \geq 2\|x_n\|^2 + 2\|y_n\| - (\|x_n\|^2 - \|y_n\|)^2 = (\|x_n\| - \|y_n\|)^2,$$

we have that $\|x_n\| - \|y_n\| \to 0$. Without loss of generality, we can suppose that $\lim_n \|x_n\| = \lim_n \|y_n\| =: a$. If $a = 0$, we have that $\|T(x_n) - T(y_n)\| \leq \|T\|(\|x_n\| + \|y_n\|) \to 0$ and then $\alpha = 0$. Suppose now that $a > 0$. Since $2\|x_n\|^2 + 2\|y_n\|^2 - \|x_n + y_n\|^2 \to 0$, we have that $\|x_n + y_n\| \to 2a$ and then $\left\| \frac{x_n}{\|x_n\|} + \frac{y_n}{\|y_n\|} \right\| \to 2$. By (iii), we deduce that $\left\| \frac{T(x_n)}{\|x_n\|} - \frac{T(y_n)}{\|y_n\|} \right\| \to 0$ and then $\alpha = \lim_n \|T(x_n) - T(y_n) = 0$.

$(ii) \implies (i)$ follows directly from the definition. ☐

Now we define the ε-version of UC operator:

Definition 3.2.7. We will say that an operator $T : X \to Y$ between two Banach spaces is *ε-uniformly convex* (ε-UC) if there is $\delta > 0$ such that $\|T(x) - T(y)\| \leq \varepsilon$ whenever $x, y \in B_X$ are such that $\Delta_{\|\cdot\|^2}(x, y) < \delta$.

The following result contains two alternative forms of the ε-UC property that we will need later.

Lemma 3.2.8. *For an operator $T : X \to Y$ and $\varepsilon > 0$, the following statements are equivalent:*

(i) T is ε-UC;

(ii) $\limsup_n \|T(x_n) - T(y_n)\| \leq \varepsilon$ whenever $x_n, y_n \in B_X$ are such

$$\lim_n \Delta_{\|\cdot\|^2}(x_n, y_n) = 0;$$

(iii) there is $\delta > 0$ such that $\|T(x) - T(y)\| \leq \varepsilon$ whenever $x, y \in X$ are such that $\|x\| = \|y\| = 1$ and $\|x + y\| > 2(1 - \delta)$.

Proof. The proof is left to the reader. ☐

We are ready to prove the quantified Beauzamy's renorming result.

Theorem 3.2.9. *Let $(X, \|\cdot\|), Y$ be two Banach spaces, and let $T : X \to Y$ be an operator such that $\Gamma(T) < \varepsilon$. Then there exists an equivalent norm $\|\!|\cdot|\!\|$ on X such that $\|\!|\cdot|\!\| \leq \|\cdot\|$ and such that T is ε-UC on $(X, \|\!|\cdot|\!\|)$.*
Moreover, in case X and Y are dual Banach spaces and T is an adjoint operator, then the norm $\|\!|\cdot|\!\|$ making T is ε-UC can be taken to be a dual one.

Proof. Take $\Gamma(T) = \varepsilon' < \varepsilon$ and $1 < \lambda < \varepsilon/\varepsilon'$. By Theorem 3.1.7, the set $B = \lambda\overline{T(B_X)}$ supports a convex bounded ε-uniformly convex function f that we may assume it is also Lipschitz (see Proposition 2.4.5). The function $f \circ T$ is ε-uniformly convex with respect to the pseudo-metric $d(x, y) = \|T(x) - T(y)\|$ on λB_X. By Theorem 2.4.7, there is an equivalent norm $\|\cdot\|_u$ on X whose square is ε-uniformly convex with respect to d on the set λB_X. All the norms defined by the formula

$$\|\!|\cdot|\!\|^2 = \lambda^{-2}\|\cdot\|^2 + \xi\|\cdot\|_u^2$$

fulfills that $\|\!|\cdot|\!\|^2$ is ε-uniformly convex with respect to d on the set λB_X. By taking $\xi > 0$ small enough we may assume that

$$\lambda^{-1}\|\cdot\| \leq \|\!|\cdot|\!\| \leq \|\cdot\|.$$

Since the unit ball of $\|\!|\cdot|\!\|$ contains λB_X, we get that T becomes ε-UC when X is endowed with $\|\!|\cdot|\!\|$.
Assume now that X and Y are dual spaces and T is an adjoint operator, and therefore it is weak* to weak* continuous. By the first part, we may assume that X is already endowed with a (non dual) norm such that T is ε-UC. We claim that the norm $\|\!|.|\!\|$ on X having $\overline{B_X}^{w^*}$ as the unit ball makes T ε-UC too. By Lemma 3.2.8 there is $\delta > 0$ such that $x, y \in B_X$ and $\|x + y\| > 2(1 - \delta)$ implies $\|T(x) - T(y)\| \leq \varepsilon$. Therefore, $\operatorname{diam}(T(H \cap B_X)) \leq \varepsilon$ whenever H is a halfspace such that $H \cap (1-\delta)B_X = \emptyset$. Take $x, y \in X$ with $\|\!|x|\!\| = \|\!|y|\!\| = 1$ and $\|\!|x+y|\!\| > 2(1-\delta/2)$. Note that the condition implies that the segment $[x, y]$ does not meet $(1 - \delta)\overline{B_X}^{w^*}$ Take H a weak*-open halfspace such that $[x, y] \cap (1 - \delta)\overline{B_X}^{w^*} = \emptyset$. We have $\|x - y\| \leq \operatorname{diam}(H \cap \overline{B_X}^{w^*})$. Now, by the weak* to weak*-continuity of T we have

$$T(H \cap \overline{B_X}^{w^*}) \subset \overline{T(H \cap B_X)}^{w^*}.$$

As $\operatorname{diam}(\overline{T(H \cap B_X)}^{w^*}) = \operatorname{diam}(T(H \cap B_X)) \leq \varepsilon$ by the weak* semicontinuity of the norm of Y and the previous observation, we get that $\|T(x) - T(y)\| \leq \varepsilon$ as wished. □

It turns out that the class of uniformly convexifying operators agrees with $\mathfrak{W}^{super}$.

Proposition 3.2.10. *Let $T : X \to Y$ be an operator between two Banach spaces. The following assertions are equivalent:*

(i) T is SWC;

(ii) T is uniformly convexifying.

Proof. $(i) \implies (ii)$ By Corollary 2.3.6, $\overline{T(B_X)}$ supports a uniformly convex function f. To conclude, just follow the previous proof without taking care of ε.

$(ii) \implies (i)$ Without loss of generality, we can suppose that T is uniformly convex and that $\|T\| \leq 1$. Let $\varepsilon > 0$. By Lemma 3.2.6, there exists $\delta > 0$ such that for all $x, y \in B_X$ with $\left\|\frac{x+y}{2}\right\| > 1 - \delta$, one has that $\|T(x) - T(y)\| \leq \varepsilon$. If S is an open slice of B_X such that $(1 - \delta)B_X \cap S = \emptyset$ then $\operatorname{diam}(T(S)) < \varepsilon$. It is easy to deduce that if S is a open slice of $\overline{T(B_X)}$

such that $(1-\delta)\overline{T(B_X)} \cap S = \emptyset$ then $\operatorname{diam}(S) < \varepsilon$. It shows that $[\overline{T(B_X)}]'_\varepsilon \subset (1-\delta)\overline{T(B_X)}$. By an easy induction, we deduce that $\overline{T(B_X)}$ is finitely dentable. By Corollary 2.4.12, we conclude that $\overline{T(B_X)}$ is SWC. $\square$

As a consequence, we obtain the following well-known result (see Theorem 1.3 in [146]). We juste recall that the norm of a Banach space is *weak uniformly rotund* (in short, WUR) if $w - \lim_n(x_n - y_n) = 0$ (limit in the weak topology) provided that $\|x_n\| = \|y_n\| = 1$ and $\lim_n \|x_n + y_n\| = 2$. *Weak* uniformly rotund norms* ($\mathrm{W^*UR}$) are defined analogously for dual Banach spaces. We include that the original proof since this Corollary will be used afterwards.

Corollary 3.2.11. *Let $K \subset X$ be a SWC subset of a Banach space X. Then K is uniformly Eberlein in its weak topology.*

Proof. By Corollary 1.5.26, there exists a reflexive Banach space Y and a one-to-one SWC operator $T : Y \to X$ such that $K \subset T(B_Y)$. Note that $T_{/B_Y} : (B_Y, w) \to (T(B_Y), w)$ is an homeomorphism. Since $K \subset T(B_Y)$, it is enough to show that $(T(B_Y), w)$ and then (B_Y, w) is uniformly Eberlein. By the previous proposition, we can suppose that T is uniformly convex. If $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \subset S_Y$ are such that $\|x_n + y_n\| \to 2$, then $\|T(x_n) - T(y_n)\| \to 0$ and, in particular, $T(x_n) - T(y_n) \xrightarrow{w} 0$. Since $T_{/B_Y}$ is a homeomorphism, we deduce that $x_n - y_n \xrightarrow{w} 0$. It shows that the norm of Y is WUR. Since Y is reflexive, its norm is $\mathrm{W^*}$-UR. The duality between UG smooth norms and $\mathrm{W^*UR}$ norms (see Theorem II-6-7 in [57]) and the use of Theorem 2 in [66] imply that $(B_Y, w^*) = (B_Y, w)$ is uniformly Eberlein. $\square$

3.3 Uniformly Gâteaux renorming and other applications

The objective of this section is to establish the super-version of the following result.

Theorem 3.3.1 ([22, 68])**.** *For a Banach space X, the following statements are equivalent:*

(i) X is a subspace of a WCG space;

(ii) (B_{X^*}, w^*) is an Eberlein compact;

(iii) For every $\varepsilon > 0$ there are sets $(A_n^\varepsilon)_n$ such that $B_X = \bigcup_{n=1}^\infty A_n^\varepsilon$ and $\gamma(A_n^\varepsilon) < \varepsilon$.

It is well known [57, Theorem 6.7] that the norm on X is UG smooth if and only if the dual norm on X^* is *weak* uniformly rotund* ($\mathrm{W^*UR}$), that is, weak*-$\lim_n(x_n^* - y_n^*) = 0$ whenever $x_n^*, y_n^* \in B_{X^*}$ are such that $\lim_n \Delta_{\|\cdot\|^2}(x_n^*, y_n^*) = 0$.

Lemma 3.3.2. *Let A be a subset of a Banach space X and let $\varepsilon > 0$. Assume that $A = \bigcup_{k=1}^\infty A_k$ with A_k bounded and $\Gamma(A_k) < \varepsilon$ for every $k \in \mathbb{N}$. Then, there exists an equivalent norm $|\!|\!| \cdot |\!|\!|$ on X such that the dual norm on X has the following property: whenever $(x_n^*), (y_n^*) \subset B_{X^*}$ are such that $\lim_n \Delta_{|\!|\!|\cdot|\!|\!|^2}(x_n^*, y_n^*) = 0$, then*

$$\limsup_n |x_n^*(x) - y_n^*(x)| \leq 8\varepsilon$$

for every $x \in A$.

Proof. Let B_k be the symmetric convex hull of A_k. By Proposition 3.1.11, we have $\Gamma(B_k) < 4\varepsilon$. Let $T_k : Z_k \to X$ the operator given by Proposition 3.2.1 such that $\Gamma(T_k) < 4\varepsilon$ and $A_k \subset B_k \subset T_k(B_{Z_k})$. Now, by Corollary 3.2.3 $\Gamma(T_k^*) < 8\varepsilon$, and, by Theorem 3.2.9, T_k^* became 8ε-UC with an equivalent dual norm $\|\cdot\|_k \leq \|\cdot\|$. Consider the equivalent dual norm on X^* defined by the formula

$$|\!|\!| \cdot |\!|\!|^2 = \sum_{k=1}^\infty 2^{-k} \|\cdot\|_k^2.$$

Suppose given $(x_n^*), (y_n^*) \subset B_{X^*}$ with $\lim_n \Delta_{\|\cdot\|^2}(x_n^*, y_n^*) = 0$. Then, for every $k \in \mathbb{N}$, we have $\lim_n \Delta_{\|\cdot\|_k^2}(x_n^*, y_n^*) = 0$ and therefore $\limsup_n \|T_k^*(x_n^*) - T_k^*(y_n^*)\| \leq 8\varepsilon$ on Z_k^*. In particular, for every $z \in Z_k$, we get

$$\limsup_n |\langle T_k(z), x_n^*\rangle - \langle T_k(z), y_n^*\rangle| = \limsup_n |\langle z, T_k^*(x_n^*)\rangle - \langle z, T_k^*(y_n^*)\rangle| \leq 8\varepsilon.$$

Having in mind that $A_k \subset T(B_{Z_k})$, we obtain $\limsup_n |x_n^*(x) - y_n^*(x)| \leq 8\varepsilon$ for every $x \in A_k$. Since this is true for every $k \in \mathbb{N}$, the lemma is proved. □

Now we are ready to state the main result of this section.

Theorem 3.3.3. *Let X be a Banach space. The following statements are equivalent:*

(i) X is a subspace of a Hilbert generated space;

(ii) For every $\varepsilon > 0$ there are sets $(B_n^\varepsilon)_n$ such that $B_X = \bigcup_{n=1}^\infty B_n^\varepsilon$ and $\Gamma(B_n^\varepsilon) < \varepsilon$;

(iii) There exists a linearly dense set $A \subset X$ such that for every $\varepsilon > 0$ it can be decomposed as $A = \bigcup_{n=1}^\infty A_n^\varepsilon$ where each A_n^ε is bounded and $\Gamma(A_n^\varepsilon) < \varepsilon$;

(iv) X admits an equivalent UG smooth norm.

Proof. $(i) \implies (ii)$ It is enough to prove statement (ii) for a Hilbert generated space since that property is clearly inherited by subspaces. Let H be a Hilbert space and $T : H \to X$ an operator with dense range. For every $0 < \varepsilon' < \varepsilon$ we have

$$B_X \subset \bigcup_{n=1}^\infty (nT(B_H) + \varepsilon' B_X).$$

We have $\Gamma(nT(B_H) + \varepsilon' B_X) \leq \varepsilon'$ and we can take $B_n^\varepsilon = B_X \cap (nT(B_H) + \varepsilon' B_X)$.
$(ii) \implies (iii)$ It is obvious.
$(iii) \implies (iv)$ By Lemma 3.3.2, for every $k \in \mathbb{N}$ there exists an equivalent dual norm $\|\cdot\|_k$ on X^* such that: whenever $(x_n^*), (y_n^*) \subset B_{X^*}$ are such that $\lim_n \Delta_{\|\cdot\|_k^2}(x_n^*, y_n^*) = 0$, then

$$\limsup_n |x_n^*(x) - y_n^*(x)| \leq 1/k$$

for every $x \in A$. The dual norm defined by

$$\|\!|\cdot|\!\|^2 = \sum_{k=1}^\infty 2^{-k} \|\cdot\|_k^2$$

satisfies then $\limsup_n |x_n^*(x) - y_n^*(x)| = 0$ whenever $x \in \operatorname{span}(A)$ and $(x_n^*), (y_n^*) \subset B_{X^*}$ are such that $\lim_n \Delta_{\|\!|\cdot|\!\|^2}(x_n^*, y_n^*) = 0$. As the sequences $(x_n^*), (y_n^*)$ are bounded and $\operatorname{span}(A)$ is dense, we have $\limsup_n |x_n^*(x) - y_n^*(x)| = 0$ for every $x \in X$. Therefore, the norm $\|\!|\cdot|\!\|$ is W*UR and its predual norm on X is uniformly Gâteaux.
$(iv) \iff (i)$ It was proved in [66] (see also [91, Theorem 6.30]). □

The result of Fabian, Godefroy and Zizler [66] (see also [91, Theorem 6.30]) gives actually more information: the linearly dense set can be decomposed, for every $\varepsilon > 0$ in countably many pieces which are uniformly weakly null up to ε in the sense of Proposition 4.1.3. That cannot be done on every set generally, however it applies to Markushevich bases as a consequence of the following dual interpretation of [9].

Proposition 3.3.4. *Let X be a subspace of a Hilbert generated Banach space and let $A \subset X$ be a bounded set such that 0 is its only cluster point and $A \cup \{0\}$ is weakly compact. Then, for every $\varepsilon > 0$ there is a decomposition $A = \bigcup_{n=1}^{\infty} A_n^{\varepsilon}$ such that for every $n \in \mathbb{N}$ and for every $x^* \in B_{X^*}$ then*

$$|\{x \in A_n^{\varepsilon} : |x^*(x)| > \varepsilon\}| \leq n.$$

Proof. Without loss of generality we may assume $A \subset C(K)$ where K is uniform Eberlein. Indeed, take $K = (B_{X^*}, w^*)$ that is uniform Eberlein after a result from [22]. Consider the embedding of K into $\ell_\infty(A)$ given by $K \ni t \to (f(t))_{f \in A}$ and note that, actually, it take values into $c_0(A)$. By [9] (see also [91, Theorem 6.33]), there is a decomposition of the index set $A = \bigcup_{n=1}^{\infty} A_n^{\varepsilon}$ such that for every $t \in K$

$$|\{f \in A_n^{\varepsilon} : |f(t)| > \varepsilon\}| \leq n.$$

Since K is a norming set on $C(K)$, we get the conclusion for every norm one functional, see Remark 4.1.4. □

It is interesting to investigate the case where statement (iii) Theorem 3.3.3 happens without countable decomposition of the linearly dense set, that is, when that set is relatively SWC.

Theorem 3.3.5. *Let X be a Banach space. The following are equivalent:*

(i) X is SWCG;

(ii) X has a Markushevich basis $\{x_i, x_i^*\}_{i \in I}$ such that $\{x_i : i \in I\} \cup \{0\}$ is SWC;

(iii) There exists an one-to-one bounded linear operator $T : X^* \to c_0(I)$, for some set I, which is weak to pointwise continuous and SWC.*

Proof. $(i) \implies (ii)$ Without loss of generality we may assume that X is generated by a balanced convex SWC set K. The proof of existence of Markushevich basis on WCG spaces allows the choice $\{x_i : i \in I\} \subset K$, see [67, Theorem 13.16]. Clearly, the only cluster point of $\{x_i : i \in I\}$ is 0, and thus $\{x_i : i \in I\} \cup \{0\}$ is SWC.
$(ii) \implies (iii)$ We may assume that $\{x_i^* : i \in I\}$ is uniformly bounded. Define $T(x^*) = (x^*(x_i))_{i \in I}$ which, initially, takes values into $\ell_\infty(I)$. It can be proved that $T(X^*) \subset c_0(I)$, see [67, Theorem 12.20] for the details. In order to see that T is SWC, we will see that T^* is SWC. Indeed, T^* takes the basis $(e_i)_{i \in I}$ of $\ell_1(I)$ to the set $\{x_i : i \in I\}$. Since $B_{\ell_1(I)}$ is the closed convex hull of $(e_i)_{i \in I}$, we deduce that $T(B_{\ell_1(I)})$ is contained in the balanced convex hull of $\{x_i : i \in I\}$, and therefore it is relatively SWC.
$(iii) \implies (i)$ Consider the adjoint operator $T^* : \ell_1(I) \to X^{**}$, which is SWC, and note that every element from $(e_i)_{i \in I}$, the basis of $\ell_1(I)$, goes through T^{**} to a weak* continuous element of X^{**}. Therefore $T^*(\{e_i : i \in I\}) \subset X$, and thus $T^*(\ell_1(I)) \subset X$. Now, as T is one-to-one, T^* has a dense range and therefore X is super WCG. □

Remark 3.3.6. Note that if X is SWCG then X is a subspace of a Hilbert generated space. In fact, let T given by the previous theorem. Then $T(B_{X^*})$ is SWC and then is uniformly Eberlein with the weak topology (see Theorem 1.3 in [146]) and so is B_{X^*}. In particular, X is a subspace of SWCG Banach space if and only if X is a subspace of a Hilbert generated space.

For $\mathcal{C}(K)$-spaces, we have the following result:

Proposition 3.3.7. *Let K be a Haussdorf compact set. The following assertions are equivalent:*

(i) K is uniformly Eberlein;

(ii) $\mathcal{C}(K)$ is Hilbert generated;

(iii) $\mathcal{C}(K)$ *is SWCG;*

(iv) there exists a SWC subset $L \subset C(K)$ *that separates points of* K.

Proof. $(i) \iff (ii)$ is Theorem 2 in [66]. $(iii) \implies (iv)$ is obvious. Let us suppose that (iv) holds. By Corollary 1.5.26, there exist a reflexive Banach space Z and a SWC operator $T : Z \to C(K)$ such that $L \subset T(B_Z)$. In particular, $T(B_Z)$ separates points of K. By Theorem 1.5.22, $T^* : C(K)^* \to Z^*$ is SWC. We identificate K with a subset of $B_{C(K)^*}$ (via the homeomorphism $x \mapsto \delta_x$). Since K is compact, $T^*(K)$ is w^*-compact and so w-compact since Z is reflexive. Furthermore, $T^*(B_{C(K)^*})$ is SWC. We deduce that $T^*(K)$ is SWC since $T^*(K)$ is a weakly closed subset of $T^*(B_{C(K)^*})$. By Proposition 3.2.11, $T^*(K)$ is uniformly Eberlein. By compactness and since $T(B_Z)$ separates points of K, $T^*_{/K} : K \to T^*(K)$ is an homeomorphism and the proof is complete. $\square$

Remark 3.3.8. c_0 has a weakly compact set which is not SWC. Consider for all $n \in \mathbb{N}$,

$$W_n = \left\{ \sum_{i=0}^{2^{n+1}-1} \varepsilon_i e_{2^n+i} \ : \ \varepsilon_i \in \{-1, 1\} \right\}$$

where $(e_n)_{n\in\mathbb{N}}$ is the canonical basic of c_0. Then $W = \cap_{n\in\mathbb{N}} W_n$ is weakly compact but not SWC (see Example 5.3 in [47] and Theorem 12 in [72]). If K is infinite compact Haussdorf space, $\mathcal{C}(K)$ contains an isometric copy of c_0 (see Proposition 4.3.11 in [1]). In this case, $\mathcal{C}(K)$ contains a weakly compact set which is not SWC. It follows that $\mathcal{C}(K)$ is S^{2}WCG if and only if K is finite.

Now we will give an application to *Jordan algebras*. We refer the reader to [93] for the necessary definitions. In [93] the authors have proved that the measures of weak noncompactness γ and ω (De Blasi's measure) agree on a *JBW*-triple predual*. The next result shows that we can add Γ to them.

Proposition 3.3.9. *Let* X *be JBW*-triple predual. Then* ω, γ *and* Γ *agree on* X.

Proof. Let $A \subset X$ be bounded and take $\varepsilon > \gamma(A)$. Since $\gamma = \omega$ by [93], there is $K \subset X$ weakly compact such that $A \subset K + \varepsilon B_X$. By [122, Theorem 6.3], K is SWC. Therefore, the inclusion $A \subset K + \varepsilon B_X$ implies $\Gamma(A) \leq \varepsilon$. We deduce $\Gamma(A) \leq \gamma(A)$. Since the other inequality always holds $\Gamma(A) = \gamma(A) = \omega(A)$. $\square$

This result implies for a JBW*-triple predual that the notions of WCG and super WCG are equivalent. Moreover, in [93, Theorem 9.3] the authors provide characterizations for JBW*-triple predual to be WCG or strongly WCG. It turns out that in such cases the spaces become super WCG or strongly super WCG (S^{2}WCG), respectively, which implies nice geometrical properties under renorming, see Theorem 1.6 and Theorem 1.9 in [147].

Chapter 4

New results on super weak compactness

4.1 Uniformly weakly null set

The most typical example of SWC set is the unit ball of a superreflexive Banach space. Now we will introduce another family of (relatively) SWC sets.

Definition 4.1.1. We say that a subset A of a Banach space X is *uniformly weakly null* if for every $\varepsilon > 0$ there is $n(\varepsilon) \in \mathbb{N}$ such that, for every $x^* \in B_{X^*}$,

$$|\{x \in A : |x^*(x)| > \varepsilon\}| \leq n(\varepsilon).$$

Note that any sequence made of different points of a uniformly weakly null set is a weakly null sequence. Therefore, uniformly weakly null sets are relatively weakly compact (and become weakly compact just by adding 0). We have something better.

Theorem 4.1.2. *Let $A \subset X$ be a uniformly weakly null set and let $\mathcal{U}$ be any free ultrafilter. Then $A_{\mathcal{U}}$ is uniformly weakly null in $X_{\mathcal{U}}$ and, therefore, A is relatively SWC in X.*

Proof. Let $\overline{x}_1, \ldots, \overline{x}_n \in A_{\mathcal{U}}$ be different vectors, $\overline{x}^* \in B_{(X_{\mathcal{U}})^*}$ and $\varepsilon > 0$ such that $|\overline{x}^*(x_k)| > \varepsilon$ for every $1 \leq k \leq n$. We claim that for $\varepsilon' < \varepsilon$, there are different elements $x_1, \ldots, x_n \in A$ and $x^* \in B_{X^*}$ with $|x^*(x_k)| > \varepsilon'$. Indeed, the proof that X is finitely representable in $X_{\mathcal{U}}$ (see [18, p. 222] for instance), provides those $x_1, \ldots, x_n \in X$ in such a way that $\overline{Y} = \operatorname{span}\{\overline{x}_1, \ldots, \overline{x}_n\}$ and $Y = \operatorname{span}\{x_1, \ldots, x_n\}$ are ε/ε'-isomorphic. Moreover, the vector x_k is found on the "coordinates" of $\overline{x}_k$, so we may assume $x_k \in A$ for all k. Then $T : Y \to \overline{Y}$ be the isomorphism. Let x^* be the Hahn-Banach extension of $(\varepsilon'/\varepsilon)\overline{x}^* \circ T$. Then, $x^* \in B_{X^*}$ and $|x^*(x_k)| > \varepsilon'$ for all $1 \leq k \leq n$ as desired. That claim shows that $A_{\mathcal{U}}$ have to be uniformly weakly null. Now we have $A_{\mathcal{U}}$ is weakly compact in $X_{\mathcal{U}}$ and thus A is relatively SWC. $\square$

A sequence $(x_n)_n$ that is a uniformly weakly null set is called *uniformly weakly null sequence.* A sequence $(x_n)_n$ is *uniformly weakly convergent* to x if $(x_n - x)_n$ is a uniformly weakly null sequence. The fact that a uniformly weakly convergent sequence together its limit is SWC set was noted in [50]. Uniformly weakly convergent sequences are closely related to the Banach-Saks property.

The following result quantifies the uniform weak nullity in terms of a kind of uniform Banach-Saks property (with unique limit 0):

Proposition 4.1.3. *Let A a bounded subset of a Banach space X and consider the two following numbers:*

(ε_1) *is the infimum of the $\varepsilon > 0$ such that there is $n_1 \in \mathbb{N}$ such that for every $x^* \in B_{X^*}$ then*
$$|\{x \in A : |x^*(x)| > \varepsilon\}| \leq n_1;$$

(ε_2) *is the infimum of the $\varepsilon > 0$ such that there is $n_2 \in \mathbb{N}$ such that for any finite set $B \subset A$ with $|B| \geq n_2$ then*
$$\left\| \frac{1}{|B|} \sum_{x \in B} x \right\| < \varepsilon.$$

Then $\varepsilon_1 = \varepsilon_2$ and in such a case $\Gamma(A) \leq \varepsilon_1$.

Proof. Let $r > 0$ such that $A \subset rB_X$. Take $\varepsilon > \varepsilon_1$ and fix the corresponding number n_1. For $n > n_1$ and any $B \subset A$ with $|B| = n$ we have
$$|x^*(\sum_{x \in B} x)| < n_1 r + (n - n_1)\varepsilon$$
for every $x^* \in B_{X^*}$. Therefore
$$n^{-1} \left\| \sum_{x \in B} x \right\| < \frac{n_1 r}{n} + \left(1 - \frac{n_1}{n}\right) \varepsilon$$
Since the bound can be taken arbitrarily closed to ε independently from B if n is large enough, we have that $\varepsilon_2 \leq \varepsilon_1$. That proves the equality $\varepsilon_1 = \varepsilon_2$ in case $\varepsilon_1 = 0$. Assume now that $\varepsilon_1 > 0$ and take $0 < \varepsilon < \varepsilon_1$. Then, for every $n \in \mathbb{N}$ we can find $C \subset A$ with $|C| = 2n$ and $x^* \in B_{X^*}$ such that $x^*(x) > \varepsilon$ or $x^*(x) < -\varepsilon$ for all $x \in C$. Since at least one half of the elements satisfies the same inequality, we may find $B \subset C$ such that $|B| = n$ and
$$|x^*(\sum_{x \in B} x)| > n\varepsilon.$$
Therefore, we have
$$n^{-1} \left\| \sum_{x \in B} x \right\| > \varepsilon,$$
that implies $\varepsilon_2 \geq \varepsilon_1$. Now, note that the first statement implies
$$\overline{A}^{w^*} \subset A \cup \varepsilon_1 B_{X^{**}} \subset X + \varepsilon_1 B_{X^{**}}$$
and so $\gamma(A) \leq \varepsilon_1$. In order to pass to Γ, just follow the ideas in the proof of Theorem 4.1.2 or check that the property of the second statement is stable by ultraproducts. In any case, we get that $\Gamma(A) \leq \varepsilon_1$. □

Remark 4.1.4. The proof of the equivalence shows that it is enough to check condition (ε_2) for x^* from a norming subset of B_{X^*}.

Definition 4.1.5. A subset A of a Banach space X is said to be *uniformly Banach-Saks null* if for every $\varepsilon > 0$ there is $n(\varepsilon)$ such that whenever $B \subset A$ is finite with $|B| \geq n(\varepsilon)$ then
$$|B|^{-1} \left\| \sum_{x \in B} x \right\| < \varepsilon.$$

Proposition 4.1.3 has the following consequence.

Corollary 4.1.6. *Let A be a bounded subset of a Banach space X. Then A is uniformly Banach-Saks null if and only if it is uniformly weakly null.*

Mercourakis [134] improvement of the Erdös-Magidor [63] dichotomy for bounded sequences can be stated in this way (see also [129] for related results and references).

Theorem 4.1.7 (Mercourakis)**.** *Let $(x_n)_n$ a bounded sequence in a Banach space X. Then there exists a subsequence $(x_{n_k})_k$ of $(x_n)_n$ for which one of the following statements holds:*

(a) either, $(x_{n_k})_k$ is uniformly weakly convergent;

(b) or, no subsequence of $(x_{n_k})_k$ is Cesàro convergent.

The celebrated Eberlein-Šmulian theorem, see [67] for instance, says that weak compactness is determined by sequences. As an application, we get that there is no Eberlein-Šmulian for super weak compactness. That is, the fact that every sequence has a relatively SWC subsequence does not imply that the set is relatively SWC.

Corollary 4.1.8. *Let A be a relatively SWC subset of a Banach space X. Then every sequence $(x_n)_n \subset A$ contains a uniformly weakly convergent subsequence. However, this property does not characterize the super weak compactness. Actually, it characterizes the Banach-Saks property.*

Proof. For a Banach-Saks set the dichotomy 4.1.7 always produces a uniformly weakly convergent subsequence. On the other hand, every uniformly convergent sequence is Cesàro convergent. Therefore, the Banach-Saks property is characterized by sequences. The other statements follow from the fact that relatively SWC sets are Banach-Saks and there exist Banach-Saks sets which are not relatively SWC [122, Corollary 2.5]. □

A set that contains a sequence equivalent to the basis of ℓ_1 cannot be uniformly weakly null. Whether a Schauder basis is a uniformly weakly null set or not will be characterized among the symmetric basis. Recall that an unconditional Schauder basis is said *symmetric* if it is uniformly equivalent to all its permutations. The following is a result due to Troyanski [161] reformulated in our terms.

Theorem 4.1.9. *Let X be a Banach space with a symmetric basis $(e_i)_{i\in I}$. Then the following statements are equivalent:*

(i) $\{e_i : i \in I\}$ is uniformly weakly null;

(ii) $\{e_i : i \in I\} \cup \{0\}$ is SWC;

(iii) 0 is a weak cluster point of $\{e_i : i \in I\}$;

(iv) X is not isomorphic to $\ell_1(I)$.

In case I is not countable (equivalently, X is not separable), these conditions characterize the existence of an equivalent uniformly Gâteaux norm on X.

Proof. Note that $(i) \iff (iv)$ and characterization of uniform Gâteaux renorming for Banach spaces with symmetric bases is the original result of Troyanski [161], see also [91, Lemma 7.52] and [91, Theorem 7.54]. Clearly $(i) \implies (ii)$ and $(i) \implies (iii)$. On the other hand, assume (iii) and let $c_s \geq 1$ the symmetric unconditionality constant of the basis. For every $\varepsilon > 0$ there are indices $(i_k)_{k=1}^n \subset I$ and positive numbers λ_k, $1 \leq k \leq n$, with $\sum_{k=1}^n \lambda_k = 1$ such that

$$\|\sum_{k=1}^n \lambda_k e_{i_k}\| \leq \varepsilon.$$

Consider a cyclic permutation $\lambda_k^1 = \lambda_{k+1}$ if $k < n$ and $\lambda_n^1 = \lambda_1$. Then we have

$$\| \sum_{k=1}^{n} \lambda_k^1 e_{i_k} \| \leq c_s \varepsilon.$$

If we take the other $n-2$ cyclic permutations obtained by iterating the first one, the sum gives

$$\| \sum_{k=1}^{n} e_{i_k} \| \leq \| \sum_{k=1}^{n} \lambda_k e_{i_k} \| + \| \sum_{k=1}^{n} \lambda_k^1 e_{i_k} \| + \cdots + \| \sum_{k=1}^{n} \lambda_k^{(n-1)} e_{i_k} \| \leq n\, c_s\, \varepsilon.$$

Again, by the symmetry of the basis, for any $J \subset I$ with n elements we have

$$n^{-1} \| \sum_{i \in J} e_i \| \leq c_s^2\, \varepsilon$$

that implies $\{e_i : i \in I\}$ is uniformly weakly null. Finally, assume (ii). By Corollary 4.1.8, there exists an infinite sequence in the set $\{e_i : i \in I\}$ which is uniformly weakly convergent. Since the unique allowed cluster point is 0, the sequence is uniformly weakly null. That behavior can easily be extended to all the basis $(e_i)_{i \in I}$ by the symmetry. □

Without the hypothesis of symmetry for the basis, we have the following result.

Proposition 4.1.10. *Let X be a Banach space with nontrivial type. Then every unconditional seminormalized basic sequence (or set) is uniformly weakly null.*

Proof. Let (e_n) be an unconditional basic sequence with unconditionality constant $c_u \geq 1$, let $p \in (1, 2]$ be the type of X and c_τ the type constant. Without loss of generality we may assume (e_n) is normalized. We have

$$\| \sum_{n \in F} e_n \| \leq c_u \, \| \sum_{n \in F} \epsilon_n e_n \|$$

whenever $\epsilon_n \in \{-1, 1\}$ and $F \subset \mathbb{N}$ finite. Let $(r_n(t))$ denote the sequence of Rademacher functions. Applying the definition of type we get

$$\| \sum_{n \in F} e_n \| \leq c_u \int_0^1 \| \sum_{n \in F} r_n(t)\, e_n \| \, dt \leq c_u c_\tau (\sum_{n \in F} \|e_n\|^p)^{1/p} = c_u c_\tau \, n^{1/p}$$

that implies (e_n) is a uniformly weakly null set. □

In [122] it is proved a result about the coordinate combinatoric behavior of the SWC compact subsets of $c_0(\mathbb{N})$ that are made up of characteristic functions. The following result shows that uniformly weakly null subsets in $c_0(I)$ made up of characteristic functions are more boring.

Proposition 4.1.11. *Let $\mathcal{F}$ be a family of finite subsets of a set I. Then $A = \{\mathbb{1}_F : F \in \mathcal{F}\}$ is uniformly weakly null as a subset of $c_0(I)$ if and only if there is a Hilbert space H and an operator $T : H \to c_0(I)$ such that A is covered by the image of an orthonormal basis of H. Moreover, an analogous result fails if $c_0(I)$ is replaced by another space with a long unconditional basis.*

Proof. Consider the Hilbert space $H = \ell_2(A)$ with the basis $\{e_x : x \in A\}$. As A is uniformly weakly null, there exists N such that

$$|\{i \in I : |x_i| > 0\}| \leq N$$

for every $x = (x_i)_{i\in I} \in A$. That implies that the assignation $e_x \to x$ can be extended to a linear operator. Indeed, for $(a_x)_{x\in A} \subset \mathbb{R}$ finitely supported, the sum $\sum_{x\in A} a_x x$ takes values in $c_0(I)$ and the bound

$$|(\sum_{x\in A} a_x x)_i| \leq \sum_{x_i\neq 0} |a_x| \leq N \sup_{x\in A}\{|a_x|\} \leq N\|\sum_{x\in A} a_x e_x\|$$

implies that the operator

$$T((a_x)_{x\in A}) = \sum_{x\in A} a_x x$$

can be extended to all $(a_x)_{x\in A} \in H$ with $\|T\| \leq N$. On the other hand, if the set A is covered by the image of an orthonormal basis of a Hilbert space H through an operator T, fix for every $x \in A$ an element $e_x \in H$ such that $(e_x)_{x\in A}$ is orthonormal. The fact that A is uniformly weakly null follows easily from the fact that

$$\|x_1 + \cdots + x_n\| \leq \|T\| \, \|e_{x_1} + \cdots + e_{x_n}\| = \|T\| \, n^{1/2}$$

for different points $x_1, \dots, x_n \in A$. For the last statement, we claim that $c_0(I)$ cannot be replaced by $\ell_{3/2}(I)$. Indeed, the canonical basis of $\ell_{3/2}(I)$ is a uniformly weakly null set, however it cannot be covered by the image of an operator from a Hilbert space (for I uncountable). Otherwise $\ell_{3/2}(I)$ would be Hilbert generated, which is not the case by [65, p. 316]. □

Remark 4.1.12. According to a classic result of Davis, Johnson, Lindenstrauss and Pełczyński [56], every relatively weakly compact set whose unique accumulation point is 0 (like as in the hypothesis of Proposition 3.3.4) is the image through an operator of an unconditional basis in a reflexive space.

The second named author proved that convex SWC sets considered with the weak topology are uniformly Eberlein [146]. These last results will deal with a more restrictive property. Following [66], we say that a compact subset $K \subset E$ in a locally convex space is *linearly uniformly Eberlein* if there exists a linear injection $T : E \to c_0(I)$ which is continuous to the pointwise topology of $c_0(I)$ and for every $\varepsilon > 0$ there is $n(\varepsilon)$ such that

$$|\{i \in I : |T(x)_i| > \varepsilon\}| \leq n(\varepsilon)$$

for every $x \in K$. In case K is moreover convex, we say that K is affinely uniformly Eberlein if an affine map can be defined on K with values on $c_0(I)$ with similar properties.

Proposition 4.1.13. *Let X be a Banach space. Then X contains a linearly dense uniformly weakly null set if and only if (B_{X^*}, w^*) is linearly uniformly Eberlein.*

Proof. Let $A \subset X$ be uniformly weakly null. Observe that $T : X^* \to c_0(A)$ given by $T(x^*) = (x^*(x))_{x\in A}$ is well defined, one-to-one, and linearly represents B_{X^*} as uniform Eberlein. On the other hand, if $T : X^* \to c_0(I)$ witnesses that (B_{X^*}, w^*) is linearly uniformly Eberlein, then the coordinate maps define elements $\{x_i : i \in I\} \subset X$, as they are weak* continuous. It is not difficult to check that $\{x_i : i \in I\}$ is uniformly weakly null and linearly dense. □

Remark 4.1.14. Note that (B_{X^*}, w^*) can be uniformly Eberlein but not linearly uniformly Eberlein. For that, just take a uniformly Gâteaux Banach space which is not WCG, for instance, Rosenthal's non WCG subspace of some $L_1(\mu)$ space, see [65] for more details.

This is the main question we cannot answer with the techniques of this paper.

Problem 4.1.15. Is every SWCG Banach space generated by a uniformly weakly null set?

Next result is motivated by [65, Theorem 4].

Theorem 4.1.16. *Let K be a SWC convex subset of a Banach space X such that K has density (equivalently, weight) ω_1. Then K is affinely uniformly Eberlein.*

Proof. Let Z be a reflexive Banach space and $T : Z \to X$ a one-to-one super weakly compact operator such that $K \subset T(B_Z)$, [146, Theorem 1.3]. Then K is linearly homeomorphic to a weakly compact subset of Z of density ω_1. Without loss of generality we may assume that Z has density ω_1 too. Now, $T^* : X^* \to Z^*$ is a super weakly compact operator with dense range. We deduce that Z^* is super WCG and thus it is a uniformly Gâteaux renormable Banach space of density ω_1. By [65, Theorem 4], B_Z is linearly uniformly Eberlein, which implies that K is affinely uniformly Eberlein. □

There are stronger results for weakly compact convex sets of weight strictly less than ω_1, that is, the compact is metrizable. For instance, Keller's theorem, see [24], showing an affine homeomorphism to the Hilbert cube.

4.2 Ergodicity and fixed point properties

4.2.1 Ergodicity and (super) weak compactness

Definition 4.2.1. Let C be a convex subset of a Banach space X. We say that an affine function $T : C \to C$ is

(a) ergodic if the Cesaro mean sequence $\left(\frac{1}{n}\sum_{k=0}^{n-1} T^k(x)\right)_n$ converges for all $x \in C$;

(b) Cesaro equicontinuous if $\left\{\frac{1}{n}\sum_{k=0}^{n-1} T^k\right\}_{n\geq 1}$ is an equicontinuous set.

We say that C is *ergodic* if any Cesaro equicontinuous affine function $T : C \to C$ is ergodic. We say that C is *super-ergodic* if any convex set which is f.r. in C is ergodic.

Note the that the previous definition of ergodicity extends the usual one in a natural way. In fact, remember that a Banach space X is *ergodic* if any Cesaro bounded operator $T : X \to X$ (i.e. $\sup_{n\geq 1}\left\|\frac{1}{n}\sum_{k=0}^{n-1} T^n\right\| < \infty$) is ergodic (see [75]).

Proposition 4.2.2. *A Banach space X is ergodic if and only if B_X is ergodic.*

Proof. Suppose that X is ergodic. Let $T : B_X \to B_X$ be a Cesaro equicontinuous affine function. Without loss of generality, we can suppose that $T(0) = 0$. Note that T can be extended to X by $T' : X \to X$ by $T'(x) = T\left(\frac{x}{\|x\|}\right)\|x\|$. It is easy to prove that T' is linear. Moreover, from the Cesaro equicontinuity of T, it is clear that T' is Cesaro bounded. We deduce that T' is ergodic and then T also is.

Now let us suppose that B_X is ergodic. Let $T : X \to X$ such that T is Cesaro bounded. Without loss of generality, we can suppose that $\|T\| \leq 1$. So $T_{/B_X} : B_X \to B_X$ is well-defined, Cesaro equicontinuous and then is ergodic by hypothesis. It follows that T is ergodic. □

We start with an adaptation of the mean ergodic Theorem (see Theorem 1.1 p.72 in [118]):

Theorem 4.2.3. *Let C be a bounded convex subset of a Banach space X and let $x, y \in C$. Let $T : C \to C$ be a Cesaro equicontinuous affine function. The following assertions are equivalent:*

(i) $Ty = y$ *and* $y \in \overline{\mathrm{conv}}\{T^n x\}_{n\geq 0}$;

(ii) $\frac{1}{n}\sum_{k=0}^{n-1} T^k(x) \to y$;

(iii) $\frac{1}{n}\sum_{k=0}^{n-1} T^k(x) \xrightarrow{w} y$;

(iv) $\left(\frac{1}{n}\sum_{k=0}^{n-1} T^k(x)\right)_{n\geq 1}$ *has a subsequence that converges weakly to* y.

Proof. For simplicity, we write $S_n = \frac{1}{n}\sum_{k=0}^{n-1} T^k$ for all $n \geq 1$. Obviously we have that $(ii) \implies (iii) \implies (iv)$. Suppose that (iv) holds, i.e. $(S_{\phi(n)}(x))_{n\geq 1}$ weakly converges to y for some stricly increasing function $\phi : \mathbb{N} \to \mathbb{N}$. It is clear that $y \in \overline{\mathrm{conv}}\{T^n x\}_{n\geq 0}$. Note that for all $n \in \mathbb{N}$

$$T\left(S_{\phi(n)}(x)\right) = S_{\phi(n)}(x) + \frac{1}{\phi(n)}T^{\phi(n)}(x) - \frac{1}{\phi(n)}x$$

and since C is bounded, we deduce that $T\left(S_{\phi(n)}(x)\right) \xrightarrow{w} y$. Moreover T is weakly continuous, so $T\left(S_{\phi(n)}(x)\right) \xrightarrow{w} T(y)$. It follows that $Ty = y$.

Now suppose that (i) is true and fix $\varepsilon > 0$. Since T is Cesaro equicontinuous, there exists $\eta > 0$ such that whenever $z \in C$ fulfills $\|z - y\| < \eta$ then $\|S_n(y) - S_n(z)\| < \varepsilon$ for all $n \in \mathbb{N}$. There exists a convex combination $\sum_{k=0}^{p} a_k T^k(x)$ such that $\left\|y - \sum_{k=0}^{p} a_k T^k(x)\right\| < \eta$. Define an affine function on C by $S = \sum_{k=0}^{p} a_k T^k$. For all $n > p$, one has that

$$\begin{aligned}
\|S_n Sx - S_n x\| &= \frac{1}{n}\left\|\sum_{k=0}^{n-1} T^k\left(\sum_{j=0}^{p} a_j T^j(x)\right) - \sum_{k=0}^{n-1} T^k(x)\right\| \\
&= \frac{1}{n}\left\|\sum_{k=0}^{n-1}\sum_{j=0}^{p} a_j T^{k+j}(x) - \sum_{k=0}^{n-1} T^k(x)\right\| \\
&= \frac{1}{n}\left\|\sum_{j=0}^{p} a_j \sum_{k=0}^{n-1}\left(T^{k+j}(x) - \sum_{k=0}^{n-1} T^k(x)\right)\right\| \\
&= \frac{1}{n}\left\|\sum_{j=1}^{p} a_j \sum_{k=n}^{n-1+j} T^k(x)\right\|.
\end{aligned}$$

Since $\lim_n \frac{1}{n}T^n(x) = 0$ (C is bounded), we deduce that there exists $N > p$ such that $\|S_n Sx - S_n x\| < \varepsilon$ for all $n \geq N$. It follows that

$$\|y - S_n x\| = \|S_n y - S_n x\| \leq \|S_n y - S_n Sx\| + \|S_n Sx - S_n x\| < \varepsilon + \varepsilon = 2\varepsilon$$

for all $n \geq N$ since $\|y - Sx\| < \eta$. We conclude that $y = \lim_n S_n x$. □

We obtain the following caracterization of weak compactness:

Theorem 4.2.4. *Let C be a closed convex subset of a Banach space X. The following assertions are equivalent:*

(i) C is weakly compact;

(ii) any closed convex subset of C is ergodic.

Proof. $(i) \implies (ii)$ follows directly from the previous theorem. Now suppose that C is not weakly compact. By Proposition 1 in [20], there exists a basic sequence $(y_n)_n \subset C$ and an affine homeomorphism $\Phi : A \to B$ such that $\Phi(y_n) = e_n$ for all $n \in \mathbb{N}$ where

$$A := \left\{\sum_{n=1}^{\infty} a_n y_n \mid a_n \geq 0 \text{ and } \sum_{n=1}^{\infty} a_n = 1\right\}$$

and

$$B := \left\{ \sum_{n=1}^{\infty} a_n e_n \mid a_n \geq 0 \text{ and } \sum_{n=1}^{\infty} a_n = 1 \right\}$$

with $(e_n)_n$ the canonical basis of l_1. Define the bilateral shift on l_1 by

$$S\left(\sum_{n=1}^{\infty} a_n e_n\right) = a_2 e_1 + \sum_{n=1}^{\infty} a_{2n-1} e_{2n+1} + \sum_{n=2}^{\infty} a_{2n} e_{2n+2}.$$

Finally we define an affine continous mapping by $T = \Phi^{-1} S \Phi : A \to A$. It is clear that T is Cesaro equicontinuous. In fact, we have that $\frac{1}{n}\sum_{k=0}^{n-1} T^k = \Phi^{-1}\left(\frac{1}{n}\sum_{k=0}^{n-1} S^k\right)\Phi$ with $\left\|\frac{1}{n}\sum_{k=0}^{n-1} S^k\right\| \leq 1$ for all $n \geq 1$. Moreover, it is proved in theorem 3.2 of [20] that T does not have any fixed point. By Theorem 4.2.3, we deduce that T is not ergodic. So A is a non-ergodic subset of C and the proof is complete. □

Before proving the super-version of the previous theorem, we need to introduce the shift of a spreading model. Let Z be a spreading model of a Banach space X built on a bounded good sequence $(x_n)_n$ with fundamental sequence $(e_n)_n$. Following ideas of Brunel and Sucheston (see [39]), we can define a linear isometry $T : c_{00} \to Z$ by

$$T\left(\sum_{i \geq 1} a_i e_i\right) = \sum_{i \geq 2} a_{i-1} e_i.$$

Then T extends to a linear isometry from Z to Z. It is clear that $T(\overline{\text{conv}}\{e_n\}_n) \subset \overline{\text{conv}}\{e_n\}_n$. In this document, we refer to T as *the shift of the spreading model Z*.

Proposition 4.2.5. *Let X be a Banach space and let Z be a spreading model of X with fundamental sequence $(e_n)_n$. Let T be the shift of Z. The following assertions are equivalent:*

(i) $T_{/\overline{\text{conv}}\{e_n\}_n}$ has a fixed point;

(ii) $(e_n)_n$ is weakly convergent.

Proof. $(ii) \implies (i)$ Suppose that $(e_n)_n$ weakly converges to $e \in Z$. By weak continuity of T, we have that $e_{n+1} = T(e_n) \xrightarrow[n]{w} T(e)$ and since $e_{n+1} \xrightarrow[n]{w} e$ we deduce that $T(e) = e$. Obviously we also have that $e \in \overline{\text{conv}}\{e_n\}_n$.

$(i) \implies (ii)$ Let $e \in \overline{\text{conv}}\{e_n\}_n$ be a fixed point of T. For $n \geq 1$, define $F_n = \overline{\text{span}(e_i)_{i \geq n}}$. It is clear that $e \in F_\infty := \bigcap_{n \geq 1} F_n$. Note that we can suppose that $(e_n)_n$ is a basic sequence. In fact, if $(e_n)_n$ is not a basic sequence then $(e_n)_n$ weakly converges (by Proposition 1.5.13) and we are done. So let us suppose that $(e_n)_n$ is a basic sequence. Then $F_\infty = \{0\}$ and it follows that $e = 0 \in \overline{\text{conv}}\{e_n\}_n$. Since $(e_n)_n$ is a spreading sequence, this is equivalent to the fact that $e_n \xrightarrow[n]{w} 0$ by Proposition 1.4.10. □

We are ready to prove the super-version of Theorem 4.2.4:

Theorem 4.2.6. *Let C be a closed convex subset of a Banach space X. The following assertions are equivalent:*

(i) C is SWC;

(ii) C is super-ergodic;

(iii) any affine isometry from a set f.r. in C into itself is ergodic.

Proof. $(i) \implies (ii)$ follows directly from the previous theorem. $(ii) \implies (iii)$ is obvious.

Suppose that C is not SWC and let us show that (iii) does not hold. There exists a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ such that $C_{\mathcal{U}}$ is not weakly compact. So there exists a sequence $(x_n)_n \in C_{\mathcal{U}}$ without any convergent subsequence. Taking subsequence if necessary, we can suppose that $(x_n)_n$ is a good sequence. Let Z be the spreading model built on $(x_n)_n$ with fundamental sequence $(e_n)_n$ and consider let T be the shift of Z.

If $(e_n)_n$ is equivalent to the canonical basis of l_1 then the mean sequence $(\frac{1}{n}\sum_{k=0}^{n-1} T^k(e_1))_n$ does not converge since $\frac{1}{n}\sum_{k=0}^{n-1} T^k(e_1) = \frac{1}{n}\sum_{k=0}^{n-1} e_k$. Remember that $\overline{\text{conv}}\{e_n\}_n$ is f.r. in $\overline{\text{conv}}\{x_n\}_n \subset C_{\mathcal{U}}$ by Proposition 1.4.7 and that $C_{\mathcal{U}}$ is f.r. in C by Proposition 1.3.8. Since $\overline{\text{conv}}\{e_n\}_n$ is not ergodic, it follows that (ii) does not hold.

Now we suppose that $(e_n)_n$ is not equivalent to the canonical basis of l_1. To conclude, we are going to show again that the Cesaro mean sequence $(\frac{1}{n}\sum_{k=0}^{n-1} T^k(e_1))_n$ can not converge. Suppose on the contrary that this sequence converges. This implies that $(e_n)_n$ weakly converges by Proposition 1.4.10. Since $(e_n)_n$ is not equivalent to the canonical basis of l_1, it follows that $(x_n)_n$ weakly converges by Proposition 1.4.6, which is a contradiction. □

Theorem 4.2.7. *Superreflexivity is equivalent to super-ergodicity.*

Proof. Suppose that X is not super-ergodic and let Y be a Banach space f.r. in X which is not ergodic. So B_Y is not ergodic. By Theorem 4.2.4, it follows that B_Y is not weakly compact. So Y is not reflexive and X is not superreflexive. Now suppose that X is super-ergodic and let Y be a Banach space which is f.r. in X. Then B_Y is f.r. in B_X and then B_Y is ergodic by the previous theorem. So Y is ergodic by Proposition 4.2.2. □

4.2.2 Fixed point property and (super) weak compactness

The objective of this section is to generalize Theorem 3.6 in [50]. The authors proved that a closed bounded convex subset C of a Banach space X is SWC if and only if any affine isometry $T : C \to C$ which can be extended to an affine isometry on X has a fixed point. However, it could exist affine isometries on C without any affine isometric extension to X. That is why we propose a intrinsic characterization.

Definition 4.2.8. Let $\mathcal{C}$ be a class of convex mappings. We say that a closed convex bounded subset C of a Banach space X has *the fixed point property* (FPP in short) for $\mathcal{C}$, if every mapping from C into itself belonging to $\mathcal{C}$ has a fixed point. We say that C has the *super-FPP* for $\mathcal{C}$ if any convex set which is f.r. in C has the FPP for $\mathcal{C}$. Finally, if any closed convex bounded subset of X has the FPP for $\mathcal{C}$, we say that X has the FPP for $\mathcal{C}$.

Lemma 4.2.9. *Let C be a convex subset of a Banach space $(X, \|.\|)$ such that $\|.\|^2$ is uniformly convex on C. If D is a convex subset of a Banach space $(Y, |.|)$ which is f.r. in C, then $|.|^2$ is uniformly convex on D.*

Proof. Define $\delta(t) = \min\{\delta_{\|.\|^2}\left(\frac{t}{2}\right), \delta_g(t)\} > 0$ where $g(s) = s^2$ for all $s \in \mathbb{R}$. Let $x, y \in D$. Suppose first that x and y are linearly independant. For all $n \in \mathbb{N}$, there exist $C_n \subset C$ and an isomorphism $T_n : \text{span}\{x, y\} \to \text{span}(C_n)$ such that $T_n(x), T_n(y) \in C$ and

$$\left(1 - \frac{1}{n}\right)|z| \le \|T_n(z)\| \le \left(1 + \frac{1}{n}\right)|z|$$

for all $z \in \operatorname{span}\{x, y\}$. For all $n \geq 2$, it follows that

$$\begin{aligned}\left(1-\frac{1}{n}\right)^2\left|\frac{x+y}{2}\right|^2 &\leq \left\|\frac{T_n(x)+T_n(y)}{2}\right\|^2 \\ &\leq \frac{\|T_n(x)\|^2+\|T_n(y)\|^2}{2} - \delta_{\|.\|^2}(\|T_n(x)-T_n(y)\|) \\ &\leq \left(1+\frac{1}{n}\right)^2\frac{|x|^2+|y|^2}{2} - \delta_{\|.\|^2}\left(\frac{|x-y|}{2}\right)\end{aligned}$$

and letting $n \to \infty$ we obtain that

$$\left|\frac{x+y}{2}\right|^2 \leq \frac{|x|^2+|y|^2}{2} - \delta_{\|.\|^2}\left(\frac{|x-y|}{2}\right) \leq \frac{|x|^2+|y|^2}{2} - \delta(|x-y|).$$

Now if x and y are linearly dependant, one can easily prove that

$$\left|\frac{x+y}{2}\right|^2 \leq \frac{|x|^2+|y|^2}{2} - \delta_g(|x-y|) \leq \frac{|x|^2+|y|^2}{2} - \delta(|x-y|)$$

and the proof is complete. □

We recall that a closed convex subset C of a Banach space X has *normal structure* (see [116]) if any bounded closed convex subset D of C containing more than one point has a *diametral point* $x \in D$, i.e. such that $\sup\{\|x-y\| : y \in D\} < \operatorname{diam}(D)$.

Lemma 4.2.10. *Suppose that C is a convex subset of a Banach space X such that $\|.\|^2$ is uniformly convex on C. Then C has normal structure.*

Proof. Let D be a bounded closed convex subset of C and let $d = \operatorname{diam}(D)$. Fix $x, y \in D$ two distinct point and let us show that $\frac{x+y}{2}$ is a diametral point of D. Suppose that it is not diametral. Then for all $n \in \mathbb{N}$ there exists $x_n \in D$ such that $\left\|\frac{x+y}{2} - x_n\right\|^2 > d^2 - \frac{1}{n}$. It follows that:

$$\begin{aligned}d^2 - \frac{1}{n} &< \left\|\frac{x+y}{2} - x_n\right\|^2 \\ &= 2\left\|\frac{1}{2}\left(\frac{x-x_n}{2} + \frac{y-x_n}{2}\right)\right\|^2 \\ &\leq \left\|\frac{x-x_n}{2}\right\|^2 + \left\|\frac{y-x_n}{2}\right\|^2 - \delta_{\|.\|^2}(\|x-y\|) \\ &\leq d^2 - \delta_{\|.\|^2}(\|x-y\|)\end{aligned}$$

and we obtain a contradiction by letting $n \to \infty$. □

Theorem 4.2.11. *Let C be a bounded closed convex subset of a Banach space X. The following assertions are equivalent:*

(i) C is SWC;

(ii) C has the super-(FPP for affine isometries);

(iii) C has the super-(FPP for continuous affine mappings);

(iv) there exists an equivalent norm on X such that C the super-(FPP for non-expansive mappings).

Proof. Any continuous affine self-mapping of a closed convex set is weakly continuous by Lemma 1.5.7. It follows that $(i) \implies (iii)$ by Schauder-Tychonoff theorem. The implication $(iii) \implies (ii)$ is obvious.

$(ii) \implies (i)$ That follows directly from Theorem 4.2.6 and Theorem 4.2.3.

$(i) \implies (iv)$ By Corollary 2.4.9, there exists an equivalent $|.|$ norm on X such that $|.|^2$ is uniformly convex on C. Let us show that $(C, |.|)$ has the super-(FPP for non-expansive mappings). Let D be any convex set f.r. in $(C, |.|)$. By the two previous lemmas, D has normal structure. By Kirk's theorem (see [116]), it follows that D has the FPP for non-expansive mapping.

$(iv) \implies (i)$ Under this new norm, C has the super-(FPP for affine isometries) and we deduce that C is SWC thanks to the implication $(ii) \implies (i)$. □

Remark 4.2.12. In general, it is not true that a convex SWC set has the super-(FPP for isometries). In fact, Alspach constructed a weakly compact set K (and then SWC, see the next part) of $L_1[0,1]$ and an isometry $T : K \to K$ without any fixed point (see [6]).

Remark 4.2.13. The implication $(i) \implies (ii)$ can also be proved directly using the existence of lower semi-continuous uniformly convex functions on C. Let $T : C \to C$ be any isometric affine mapping. For $\varepsilon > 0$, we define $f_\varepsilon(x)$ as the height of the tallest ε-separated dyadic tree with root x. As in the proof of Theorem 2.3.4, we have that f_ε is ε-quasi concave. Since a ε-separated dyadic tree with root x gives a ε-separated dyadic tree with root $T(x)$ through T, we have that $f_\varepsilon \circ T \geq f_\varepsilon$. By Lemma 2.3.1, we deduce that the function $h_\varepsilon := 3^{-f_\varepsilon}$ is ε-uniformly convex and verifies that $h_\varepsilon \circ T \leq h_\varepsilon$. Theorem 2.2.6 implies that $g_\varepsilon = \overline{\text{conv}}(h_\varepsilon)$ is ε^+-uniformly convex convex and lower semi-continuous. Moreover, note that $g_\varepsilon \circ T \leq g_\varepsilon$. In fact, we have that $g_\varepsilon \circ T = \overline{\text{conv}}(h_\varepsilon) \circ T \leq h_\varepsilon \circ T \leq h_\varepsilon$ where $g_\varepsilon \circ T$ is convex lower semi-continuous and then $g_\varepsilon \circ T \leq \overline{\text{conv}}(h_\varepsilon) = g_\varepsilon$. The closed envelope g of the function $\sum_{n\geq 1} \frac{1}{2^n \|g_{1/n}\|_\infty} g_{1/n}$ is uniformly convex, lower semi-continuous and verifies that $g \circ T \leq g$. The function g is convex lower semi-continuous (thus convex w-lower semi-continuous) on a weakly compact set and then reaches his minimum on C at some point $x \in C$. Since g is uniformly convex, this miminum is unique. Moreover, we have that $g(T(x)) \leq g(x)$. It follows that $T(x) = x$.

4.2.3 Application to S^{2}WCG Banach spaces

Theorem 4.6 of [50] is obtained as an easy consequence of the previous results:

Proposition 4.2.14. *Let X be a S^2WCG Banach space. Then X admits an equivalent norm such that any weakly compact convex subset has the FPP for non-expansive mappings.*

Proof. Let K be a SWC absolutely convex set that strongly generates X. Consider that X is endowed with the norm given by Theorem 1.5.33. It follows that the square of the norm is uniformly convex on any weakly compact subset of X and then any weakly compact subset has normal structure by Lemma 4.2.10. The conclusion is obtained thanks to Kirk's theorem. □

Combining the results of the previous parts, we obtain:

Proposition 4.2.15. *Let C be a closed convex subset of a S^2WCG Banach space X. The following assertions are equivalent:*

(i) C is weakly compact;

(ii) C is SWC;

(iii) *C is superergodic;*

(iv) *any closed convex subset of C is ergodic;*

(v) *any closed convex subset of C has the FPP for continuous affine mappings.*

Proof. $(i) \implies (ii)$ follows from the fact that X is S^2WCG. $(ii) \implies (iii)$ follows from Theorem 4.2.6. $(iii) \implies (iv)$ is obvious. $(iv) \implies (v)$ follows theorem 4.2.3. $(v) \implies (i)$ is Theorem 3.2 in [20]. □

Proposition 4.2.16. *Let Y be a subspace of a S^2WCG Banach space X. The following assertions are equivalent:*

(i) *Y is reflexive;*

(ii) *Y is superreflexive;*

(iii) *Y is super-ergodic;*

(iv) *Y is ergodic;*

(iv) *Y has the FPP for continuous affine mappings.*

Proof. It follows directly from the previous results. □

Remark 4.2.17. In the case of $L^1(\Omega, \mathcal{A}, \mu, \mathbb{R})$, the previous corollary can be considerably improved. In that case, Y is reflexive if and only if Y has the FPP for non-expansive mappings. One implication is due to Maurey (Theorem 1 in [132]) and the other one is due to Dowling and Lennard (theorem 1.4 in [61]).

4.2.4 A remark on the M-FPP

Definition 4.2.18. Let $(\mathcal{P})$ be a property of Banach spaces. We say that a Banach space X has the property M-$(\mathcal{P})$ if any spreading model of X has $(\mathcal{P})$.

It is worth noting that it does not imply that X has $(\mathcal{P})$ in general. The notion of M-property has been introduced by Beauzamy in [19] (see Chapter 5). In this book, the author claims that there does not exist any characterization of the M-reflexivity. As far as we know, this question is still opened. Before stating the result, we just need to recall the following result due to James (see Theorem 2.2 in [103]). However we give a simplest proof:

Proposition 4.2.19. *Let X be a Banach lattice. The following assertions are equivalent:*

(i) *X is superreflexive;*

(ii) *X has non-trivial type.*

Proof. Theorem 1.3.6 implies that any superreflexive Banach spaces has non-trivial type. So let us suppose that (ii) holds. Let $\mathcal{U}$ be any ultrafilter. Then $X_{\mathcal{U}}$ is a lattice (see Proposition 3.2 in [97]) with non-trivial type. In particular, $X_{\mathcal{U}}$ can not contain an isomorphic copy of c_0 or ℓ_1. By Theorem in 1.c.5 in [128], it follows that $X_{\mathcal{U}}$ is reflexive, i.e. X is superreflexive. □

Note that the previous result is true for Banach spaces with an unconditional Schauder basis.

If X has a non-trivial type, it is possible to characterize the M-reflexivity:

Proposition 4.2.20. *Let X be a Banach space with non-trivial type. The following assertions are equivalent:*

(i) X is reflexive;

(ii) X has M-reflexivity;

(iii) X has M-superreflexivity.

Proof. $(iii) \implies (ii) \implies (i)$ is obvious. Let suppose that (i) holds. Suppose that Z is a spreading model built on $(x_n)_{n\in\mathbb{N}}$ whose basic sequence $(e_n)_{n\in\mathbb{N}}$. Since X is reflexive (and using the fact that the spreading model built on $(x_n)_n$ is isomorfic to the spreading model built on $(x_n - x)_n)$, we can suppose $x_n \xrightarrow{w} 0$. In this case, $(e_n)_{n\in\mathbb{N}}$ is a unconditional basic sequence by Proposition 1.4.10. Then Z is a Banach space with an unconditionnal Schauder basis and non-trivial type. Thus Z is superreflexive by Proposition 4.2.19. □

For the study of the M-FPP, we start with the following lemma:

Lemma 4.2.21. *The M-(FPP for affine isometries) implies the ABS.*

Proof. Let X be a Banach space and suppose that X does not have the ABS. By Theorem 1.4.12, it follows that X has a spreading model Z isomorphic to l_1. Then the fundamental sequence $(e_n)_n$ of Z is equivalent to the canonical basis of l_1 (see Lemma 1 p.39 in [19]). In particular, $(e_n)_n$ is not weakly convergent. By Proposition 4.2.5, we deduce that $\overline{\text{conv}}\{e_n\}_n$ and thus Z do not have the FPP for affine isometries. □

It is well-known that the fixed point property does not imply reflexivity. In fact, l_1 can be renormed to have the FPP for non-expansive mappings (see [125]). However the M-FPP implies reflexivity. More precisely we have that:

Theorem 4.2.22. *The M-(FPP for affine isometries) implies the BS.*

Proof. Let X be a Banach space with the M-(FPP for affine isometries). By the previous lemma and Proposition 1.4.14, we need to prove that X is reflexive. By contradiction, suppose that X is not reflexive. There exists a bounded sequence $(x_n)_n$ without any weakly convergent subsequence. By taking subsequence if necessary, we can suppose that $(x_n)_n$ is a good sequence generating a spreding model Z with fundamental sequence $(e_n)_n$. By Theorem 1.4.12 and Lemma 1.4.11, $(e_n)_n$ is not equivalent to the canonical basis of l_1. Since $(x_n)_n$ is not weakly convergent, it follows that $(e_n)_n$ is not weakly convergent by Proposition 1.4.6. By Proposition 4.2.5, we conclude that Z can not have the FPP for affine isometries, which is a contradiction. □

Maurey proved that any isometry $T : C \to C$ on a closed convex subset C of a superreflexive Banach space has a fixed point (see for example Theorem F p.112 in [154]). Any continuous affine mapping $T : C \to C$ also enjoys this property:

Theorem 4.2.23. *Let X be a Banach space. The following assertions are equivalent:*

(i) X is superreflexive;

(ii) X has the super-(FPP for affine isometries);

(iii) X has the super-(FPP for continuous affine mappings);

(iv) X has the super-(FPP for isometries).

Proof. By the previous theorem, we have that the super-(FPP for affine isometries) implies the BS. By Theorem 1.5.15, superreflexivity is equivalent to the super-BS and it follows that $(ii) \implies (i)$. We have that $(i) \implies (iii)$ by Schauder-Tychonoff theorem and $(iii) \implies (ii)$ is obvious. $(i) \implies (iv)$ is Maurey's Theorem. Since $(iv) \implies (ii)$ is obvious, the proof is complete. □

Chapter 5

Uniform Banach-Saks properties

The main objective of this section is to establish the following graph of implications:

$$
\begin{array}{ccccccc}
\text{(strong) } p\text{-BS} & \Longleftrightarrow & \text{uniform BS} & \Longleftrightarrow & (A_\infty) & \Longrightarrow & \text{BS} \\
 & & \Downarrow & & \Downarrow & & \Downarrow \\
 & & \text{uniform ABS} & \Longleftrightarrow & \text{alternate-}(A_\infty) & \Longrightarrow & \text{ABS} \\
 & & \Downarrow & & \Downarrow & & \Downarrow \\
 & & \text{uniform WABS} & \Longleftrightarrow & \text{weak alternate-}(A_\infty) & \Longrightarrow & \text{WABS} \\
 & & \Updownarrow & & \Updownarrow & & \Updownarrow \\
\text{(strong) } p\text{-WBS} & \Longleftrightarrow & \text{uniform WBS} & \Longleftrightarrow & \text{weak-}(A_\infty) & \Longrightarrow & \text{WBS}
\end{array}
$$

where BS holds for Banach-Saks, A for alternate and W for weak.

5.1 The uniform WBS

5.1.1 The weak-(A_∞) property

The property (A_k) (see definition 5.2.1) has been introduced by Partington in [139]. Removing the reflexivity of the space, we define the weak-(A_k):

Definition 5.1.1. Let X be a Banach space and let $k \geq 2$. We say that X has the *weak-(A_k)* if there exists $\theta \in (0,1)$ such that for all $(x_n)_n \subset B_X$ such that $x_n \xrightarrow{w} 0$, there exist $p_1 < p_2 < ... < p_k$ such that

$$\frac{1}{k}\left\|\sum_{i=1}^{k} x_{p_i}\right\| \leq \theta.$$

We say that X has the weak-(A_∞) if X has the weak-(A_k) for some $k \geq 2$.

Núnez introduced in [137] a definition of uniform weak Banach-Saks. We find that the following definition is more natural. It turns out that both definitions are in fact equivalent (see Proposition 5.1.4).

Definition 5.1.2. Let X be a Banach space. We say that X has the *uniform weak Banach-Saks property* (in short uniform WBS) if there exists a sequence $(a_n)_n$ of real positive numbers such that $a_n \to 0$ and for all $(x_n)_n \subset B_X$ such that $x_n \xrightarrow{w} 0$, there is a subsequence $(x'_n)_n$ of $(x_n)_n$ such that

$$\frac{1}{m}\left\|\sum_{k=1}^{m} x'_k\right\| \leq a_m$$

for all $m \geq 1$.

We start with the following easy arithmetical result:

Lemma 5.1.3. *Let $(b_n)_n$ a sequence of strictly positive numbers. The following assertions are equivalent:*

(i) there exists a sequence of strictly positive numbers $(a_n)_n$ such that for all $n \geq 1$

$$\frac{1}{n}\sum_{k=1}^{n} a_k \leq b_n$$

(ii) there exists a sequence of strictly positive numbers $(a_n)_n$ such that for all subsequence $(a'_n)_n$ of $(a_n)_n$ and all $n \geq 1$

$$\frac{1}{n}\sum_{k=1}^{n} a'_k \leq b_n$$

(iii) $\inf_n nb_n > 0$.

Proof. $(iii) \implies (ii)$ Let $l \in (0, \inf_n nb_n)$. Define $a_n = \frac{l}{2^n}$ for all $n \in \mathbb{N}$. Let $(a_{\phi(n)})_n$ be a subsequence of $(a_n)_n$. Then

$$\sum_{k=1}^{n} a_{\phi(k)} = l\sum_{k=1}^{n}\frac{1}{2^{\phi(k)}} \leq l \leq nb_n,$$

that is

$$\frac{1}{n}\sum_{k=1}^{n} a_{\phi(k)} \leq b_n$$

for all $n \geq 1$.

$(ii) \implies (i)$ is obvious.

$(i) \implies (iii)$ Suppose that (iii) does not hold. Then $\inf_n nb_n = 0$. There exists a subsequence such that $\phi(n)b_{\phi(n)} \to 0$. If there exists a sequence $(a_n)_n$ as in (i) then

$$a_1 \leq \sum_{k=1}^{\phi(n)} a_k \leq \phi(n)b_{\phi(n)} \to 0,$$

which is a contradiction since $a_1 > 0$. □

Proposition 5.1.4. *Let X be a Banach space. The following assertions are equivalent:*

(i) X has the uniform WBS;

(ii) there exists a sequence $(b_n)_n$ of real positive numbers such that $b_n \to 0$ and for all $(x_n)_n \subset B_X$ such that $x_n \xrightarrow{w} 0$, there is a subsequence $(x'_n)_n$ of $(x_n)_n$ such that for every subsequence $(x''_n)_n$ of $(x'_n)_n$;

$$\frac{1}{m}\left\|\sum_{k=1}^{m} x''_k\right\| \leq b_m$$

for all $m \geq 1$.

(iii) there exists a sequence $(c_n)_n$ of real positive numbers such that $c_n \to 0$ and for all $(x_n)_n \subset B_X$ such that $x_n \xrightarrow{w} 0$ and for all $m \geq 1$, there exist $p_1 < ... < p_m$ such that

$$\frac{1}{m}\left\|\sum_{i=1}^{m} x_{p_i}\right\| \leq c_m.$$

In such a case, if one of the sequences $(a_n)_n$ (where $(a_n)_n$ is given by the definition of uniform WBS), $(b_n)_n$ or $(c_n)_n$ is $O(n^{\frac{1}{p}-1})$ for some $p \in (1, \infty)$, then the others can be chosen such that they also are $O(n^{\frac{1}{p}-1})$.

Proof. $(ii) \implies (i) \implies (iii)$ is obvious. Suppose that (iii) holds. Let $(c_n)_n$ as in (iii). Define

$$b_n = \frac{3q_n}{n} + 3c_n$$

where $q_n = E\left(\frac{\ln(n)}{\ln(2)}\right)$ for all $n \in \mathbb{N}$. Note that $b_n \to 0$ and $nb_n \to \infty$, so in particular $\inf_n nb_n > 0$. Let $(x_n)_n \subset B_X$ such that $x_n \xrightarrow{w} 0$. First, suppose that $(x_n)_n$ admits a convergent subsequence to 0. Take $(a_n)_n$ as in (ii) of the previous lemma. Choose a subsequence $(x'_n)_n$ of $(x_n)_n$ such that $\|x'_n\| \leq a_n$ for all $n \in \mathbb{N}$. Then if $(x'_{\phi(n)})_n$ is a subsequence of $(x'_n)_n$, we have that

$$\frac{1}{n}\left\|\sum_{k=1}^{n} x'_{\phi(k)}\right\| \leq \frac{1}{n}\sum_{k=1}^{n} a_{\phi(k)} \leq b_n$$

for all $n \in \mathbb{N}$ and we are done. Now, let suppose that $(x_n)_n$ does not admit any convergent subsequence. By taking a subsequence if necessary, we can suppose that $(x_n)_n$ is a good sequence generating a spreading model Z with fundamental basis $(e_n)_n$. For all $k \in \mathbb{N}$, define $N_k = 2^k$. By Proposition 1.4.3, there exists a strictly increasing sequence $(p_k)_k$ such that for all $n_1 < ... < n_{N_k}$ with $n_1 \geq p_k$ and all $a_1, ..., a_{N_k}$

$$\frac{1}{2}\left\|\sum_{i=1}^{N_k} a_i x_{n_i}\right\| \leq \left\|\sum_{i=1}^{N_k} a_i e_i\right\| \leq \frac{3}{2}\left\|\sum_{i=1}^{N_k} a_i x_{n_i}\right\|.$$

For all $k \in \mathbb{N}$, we define $x'_k = x_{p_k}$. Let $m \in \mathbb{N}$ and take k such that $m \leq N_k$. By considering the sequence $(x_n)_{n \geq p_k}$ and by (iii), there exists $p_k \leq n_1 < ... < n_m$ such that

$$\frac{1}{m}\left\|\sum_{i=1}^{m} x_{n_i}\right\| \leq c_m.$$

It follows that

$$\frac{1}{m}\left\|\sum_{i=1}^{m} e_i\right\| \leq \frac{3}{2m}\left\|\sum_{i=1}^{m} x_{n_i}\right\| \leq \frac{3}{2}c_m$$

for all $m \geq 1$. Now, let $(x''_n)_n$ be a subsequence of $(x'_n)_n$. Let $n \geq 1$. We have that

$$\begin{aligned}
\frac{1}{n}\left\|\sum_{i=1}^{n} x''_i\right\| &\leq \frac{1}{n}\left\|\sum_{i=1}^{q_n} x''_i\right\| + \frac{1}{n}\left\|\sum_{i=q_n+1}^{n} x''_i\right\| \\
&\leq \frac{q_n}{n} + \frac{2}{n}\left\|\sum_{i=q_n+1}^{n} e_i\right\| \\
&\leq \frac{q_n}{n} + \frac{2}{n}\left(\left\|\sum_{i=1}^{n} e_i\right\| + \left\|\sum_{i=1}^{q_n} e_i\right\|\right) \\
&\leq \frac{3q_n}{n} + 3c_n = b_n
\end{aligned}$$

and the proof of the equivalence is complete. The last assertion is easily deduced from the proof. □

Theorem 5.1.5. *The uniform WBS and the weak-(A_∞) are equivalent.*

Proof. Suppose that X has the uniform WBS. Take $(c_n)_n$ as in (iii) of the previous proposition. Choose $n_0 \geq 2$ such that $c_{n_0} < 1$. Let $(x_n)_n \subset B_X$ such that $x_n \xrightarrow{w} 0$. There exists $p_1 < ... < p_{n_0}$ such that

$$\frac{1}{n_0}\left\|\sum_{i=1}^{n_0} x_{p_i}\right\| \leq c_{n_0}.$$

Then X has property (A_{n_0}).

Now let suppose that X has property (A_{j_0}) for some $j_0 \geq 2$ and let θ given by the definition. For $m \in \mathbb{N}$, define

$$c_m = 2\frac{j_0^{p_m+1}}{m}\theta^{p_m+1}$$

where $p_m = \left[\frac{\ln(m)}{\ln(j_0)}\right]$. Note that $c_m \to 0$. Let $(x_n)_n \subset B_X$ such that $x_n \xrightarrow{w} 0$. If $(x_n)_n$ has a subsequence that converges to 0, then for all $m \geq 1$ there exists $p_1 < ... < p_m$ such that

$$\frac{1}{m}\left\|\sum_{i=1}^{m} x_{p_i}\right\| \leq c_m.$$

So we can suppose that $\inf_n \|x_n\| > 0$. So, by taking subsequence if necessary, we can suppose that $(x_n)_n$ is a basic sequence with basic constant less that 2. By induction, we construct a subsequence $(x_{m_k})_k$ of $(x_n)_n$ such that

$$\frac{1}{j_0}\|x_{m_{kj_0+1}} + ... + x_{m_{(k+1)j_0}}\| \leq \theta$$

for all $k \in \mathbb{N}$. For all n, define

$$x^1_n = \frac{1}{j_0}(x_{m_{nj_0+1}} + ... + x_{m_{(n+1)j_0}}).$$

Since $\|x^1_n\| \leq \theta$ and $x^1_n \xrightarrow{w} 0$, we can construct a strictly increasing sequence $(m_n(1))_n$ such that

$$\frac{1}{j_0}\|x^1_{m_{kj_0+1}(1)} + ... + x^1_{m_{(k+1)j_0}(1)}\| \leq \theta^2$$

for all $k \in \mathbb{N}$. For all n, define

$$x_n^2 = \frac{1}{j_0}(x^1_{m_{nj_0+1}(1)} + ... + x^1_{m_{(n+1)j_0}(1)}).$$

Following by induction, we construct a sequence $(x_n^p)_p$ for all $p \in \mathbb{N}$ such that $\|x_n^p\| \leq \theta^p$ and $x_n^p \xrightarrow{w} 0$. Now we can choose a strictly increasing sequence $(m_n(p))_n$ such that

$$\frac{1}{j_0}\|x^p_{m_{kj_0+1}(p)} + ... + x^p_{m_{(k+1)j_0}(p)}\| \leq \theta^{p+1}.$$

For all n, define

$$x_n^{p+1} = \frac{1}{j_0}(x^p_{m_{nj_0+1}(p)} + ... + x^p_{m_{(n+1)j_0}(p)}).$$

Let $m \in \mathbb{N}$. By definition of p_m, one has that $j_0^{p_m} \leq m < j_0^{p_m+1}$. By construction, $x_1^{p_m+1}$ is the arithmetic mean of $j_0^{p_m+1}$ terms of $(x_n)_n$, say $x'_1, ..., x'_{j_0^{p_m+1}}$. Using the fact that $(x_n)_n$ is a basic sequence, we obtain that

$$\frac{1}{m}\left\|\sum_{i=1}^{m} x'_i\right\| \leq \frac{2}{m}\left\|\sum_{i=1}^{j_0^{p_m+1}} x'_i\right\| \leq \frac{2}{m} j_0^{p_m+1}\theta^{p_m+1} = c_m,$$

i.e. X has the uniform WBS by the previous proposition. □

Remark 5.1.6. The previous theorem could have been proved using Kakutani's method (see proof of Theorem 5.4.7). However, Proposition 5.1.4 has its own interest and will be usefull in the following section.

Remark 5.1.7. There exist Banach spaces with the BS but without the uniform WBS (see Theorem 7 in [137]).

5.1.2 The strong p-WBS

Definition 5.1.8. Let X be a Banach space and $p \in (1, +\infty]$. We say that X has the *strong p-weak Banach-Saks property* (in short, strong p-WBS) if there exists $C > 0$ such that for all $(x_n)_n \subset B_X$ such that $x_n \xrightarrow{w} 0$, there exists a subsequence $(x'_n)_n$ of $(x_n)_n$ such that

$$\left\|\sum_{k=1}^{n} x'_k\right\| \leq C n^{\frac{1}{p}}$$

for all $n \geq 1$ (with the convention $n^{\frac{1}{\infty}} = 1$).

In the previous definition, C does not depend on the sequence $(x_n)_n$. In the litterature, one can find two definitions of the p-WBS: the previous one and a definition where C can depend on $(x_n)_n$ and called p-WBS in this document (see Definition 5.1.10). The term "strong" lays the emphasis upon the fact that C does not depend on the weakly null sequence. It is obvious that the strong p-WBS implies the p-WBS. The reciproque is not clear and has been proved only in some cases (for example in some rearrangement invariant function spaces, see Lemma 4.2 in [151]). The reciproque will be established later.

Theorem 5.1.9. *A Banach space has the uniform WBS if and only if it has the strong p-WBS for some $p \in (1, +\infty)$.*

Proof. We only prove the non-trivial implication. Suppose that X has the uniform WBS. Then there exists $k_0 \geq 2$ such that X has the weak-(A_{k_0}). Let $\theta \in \left(\frac{1}{k_0}, 1\right)$ given by the definition of weak-(A_{k_0}). By the proof of Theorem 5.1.5, we deduce that (iii) of Proposition 5.1.4 holds with

$$c_m = 2\frac{k_0^{p_m+1}}{m}\theta^{p_m+1}$$

where $p_m = \left[\frac{\ln(m)}{\ln(k_0)}\right]$. Note that

$$mc_m \leq 2k_0 m\theta^{\frac{\ln(m)}{\ln(k_0)}}.$$

If $p \in \left(1, \frac{1}{1+\frac{\ln(\theta)}{\ln(k_0)}}\right)$ (note that $\frac{1}{1+\frac{\ln(\theta)}{\ln(k_0)}} > 1$ since $\theta > \frac{1}{k_0}$), it follows that

$$\frac{mc_m}{m^{\frac{1}{p}}} \to 0.$$

In particular, we deduce that $c_m = O(m^{\frac{1}{p}-1})$. The sequence $(a_n)_n$ given by the definition of uniform WBS can be chosen such that $a_m = O(m^{\frac{1}{p}-1})$ (by Proposition 5.1.4), i.e. X has the strong p-WBS. □

5.1.3 Equivalence between the strong p-WBS and p-WBS

Definition 5.1.10. Let X be a Banach space and $p \in (1, +\infty]$. We say that X has the *p-weak Banach-Saks property* (in short, p-WBS) if for all $(x_n)_n \subset B_X$ such that $x_n \xrightarrow{w} 0$, there exist $C > 0$ and a subsequence $(x'_n)_n$ of $(x_n)_n$ such that

$$\left\|\sum_{k=1}^{n} x'_k\right\| \leq Cn^{\frac{1}{p}}$$

for all $n \geq 1$ (with the convention $n^{\frac{1}{\infty}} = 1$).

Definition 5.1.11. Let X be a Banach space and let $(x_k)_k$ be a bounded good sequence. For all $n \geq 1$, we define

$$s((x_k)_k, n) = \lim_m \left\|\sum_{k=1}^{n} x_{m+k}\right\| = \lim_{p_1\to\infty, p_1<\ldots<p_n} \left\|\sum_{k=1}^{n} x_{p_k}\right\|.$$

Remark 5.1.12. Note that $s((x_k)_k, n)$ is just the norm of $\sum_{k=1}^{n} e_k$ where $(e_k)_k$ is the fundamental basis of the spreading model generated by $(x_k)_k$.

The following lemma is contained in [148] (without using spreading models in the second part of the proof):

Lemma 5.1.13. *Let X be a Banach space with the p-WBS for some $p \in (1, +\infty]$. Then there exists $C > 0$ such that for all weakly null good sequences $(y_k)_k \subset B_X$*

$$s((y_k)_k, n) \leq Cn^{\frac{1}{p}}$$

for all $n \geq 1$.

Proof. Suppose that the conclusion of the lemma is false. We claim that there exists a weakly null good sequence $(y_k)_k \subset B_X$ such that

$$\sup_n \frac{s((y_k)_k, n)}{n^{\frac{1}{p}}} = \infty.$$

Let $(y_k^1)_k \subset B_X$ be any weakly null good sequence. Define $q_1 = 1$. By induction, we construct sequences $(y_k^n)_k$, $(\mu'_n)_n$, $(x_k^n)_k$, $(q_n)_n$, (μ_n) and $(z_k^n)_k$ such that for all $n \geq 1$

$$\mu'_{n+1} = \min\left\{\frac{1}{2^n}, \min_{1\leq l\leq n}\left\{\frac{lq_l^{\frac{1}{p}-1}}{2^{n-l+1}}\right\}\right\}, \tag{5.1}$$

$$s((x_k^{n+1})_k, q_{n+1}) \geq \frac{4(n+1)}{\mu'_n} q_{n+1}^{\frac{1}{p}}, \tag{5.2}$$

$$\mu_{n+1} = \begin{cases} \mu'_{n+1} & \text{if } \ s((y_k^n)_k, q_{n+1}) \leq 2(n+1)q_{n+1}^{\frac{1}{p}}, \\ 0 & \text{if not} \end{cases} \tag{5.3}$$

$$z_k^{n+1} = \begin{cases} y_k^n & \text{if } \ k \leq n+1 \\ y_k^n + \mu_{n+1}x_k^{n+1} & \text{if } \ k > n+1 \end{cases} \tag{5.4}$$

where $(x_k^n)_k \subset B_X$ is a weakly null good sequence and $(y_k^{n+1})_k$ is a good subsequence of $(z_k^{n+1})_k$ such that $y_k^{n+1} = z_k^{n+1}$ for all $k \in \{1, ..., n+1\}$. Note that the existence of $(x_k^n)_k$ and $(q_n)_n$ in (5.2) is guaranteed by our initial hypothesis. By construction, we have that

$$s((y_k^{n+1})_k, q_{n+1}) \geq 2(n+1)q_{n+1}^{\frac{1}{p}} \tag{5.5}$$

for all $n \geq 1$. In fact, write $y_k^{n+1} = z_{\phi(k)}^{n+1}$ and suppose first that $\mu_{n+1} = 0$. By (5.3), one has that $s((y_k^n)_k, q_{n+1}) > 2(n+1)q_{n+1}^{\frac{1}{p}}$ and it follows that

$$\begin{aligned} s((y_k^{n+1})_k, q_{n+1}) = \lim_m \left\|\sum_{k=1}^{q_{n+1}} z_{\phi(m+k)}^{n+1}\right\| &= \lim_m \left\|\sum_{k=1}^{q_{n+1}} y_{\phi(m+k)}^{n}\right\| \\ &= s((y_k^n)_k, q_{n+1}) > 2(n+1)q_{n+1}^{\frac{1}{p}}. \end{aligned}$$

Now if $\mu_{n+1} = \mu'_{n+1}$, then $s((y_k^n)_k, q_{n+1}) \leq 2(n+1)q_{n+1}^{\frac{1}{p}}$ and it follows that

$$\begin{aligned} s((y_k^{n+1})_k, q_{n+1}) &= \lim_m \left\|\sum_{k=1}^{q_{n+1}} z_{\phi(m+k)}^{n+1}\right\| = \lim_m \left\|\sum_{k=1}^{q_{n+1}} y_{\phi(m+k)}^{n} + \mu'_{n+1}x_{\phi(m+k)}^{n+1}\right\| \\ &\geq \mu'_{n+1}s((x_k^{n+1})_k, q_{n+1}) - s((y_k^n)_k, q_{n+1}) \\ &\geq 2(n+1)q_{n+1}^{\frac{1}{p}}. \end{aligned}$$

We are now ready to define the sequence $(y_k)_k$. For $k \geq 1$, define $y'_k = y_k^k = z_k^k$. By construction, it is not hard to see that there exist a increasing function $\phi : \mathbb{N} \to \mathbb{N}$ and a function $\psi : \mathbb{N}^2 \to \mathbb{N}$ such that $\psi(k, n) \geq k$ and

$$y'_k = y_{\phi(k)}^1 + \sum_{n=2}^{k-1} \mu_n x_{\psi(k,n)}^n \tag{5.6}$$

for all $k \geq 3$. From the facts that $\mu_n \leq \frac{1}{2^n}$, $y_k^1 \xrightarrow{w} 0$ and $x_k^n \xrightarrow[k]{w} 0$ for all n, it is now clear that $y'_k \xrightarrow{w} 0$. Now, let $(y_k)_k$ be any good subsequence of $(y'_k)_k$. Using the fact that y'_k can be expressed in terms of y_k^n $(n < k)$ and $(x_k^j)_k$ (with $n < j < k$) in a similar way as in (5.6), it follows by triangle inequality that

$$\begin{aligned} s((y_k)_k, q_n) &\geq s((y_k^n)_k, q_n) - \sum_{j=n+1}^{\infty} \mu_j s((x_k^j)_k, q_n) \\ &\geq 2nq_n^{\frac{1}{p}} - \sum_{j=n+1}^{\infty} \mu_j q_n \\ &\geq 2nq_n^{\frac{1}{p}} - q_n \sum_{j=n+1}^{\infty} \frac{nq_n^{\frac{1}{p}-1}}{2^{j-n}} = nq_n^{\frac{1}{p}} \end{aligned}$$

and the claim is proved.
Now let Z be the spreading model generated by $(y_k)_k$ with fundamental basis $(e_k)_k$. Let $(N_k)_k$ be a increasing sequence of natural numbers such that $N_k^{\frac{1}{p}} \geq 6$ and

$$\sup_n \frac{s((y_k)_k, N_k)}{N_k^{\frac{1}{p}}} \geq k$$

for all $k \geq 1$. By Proposition 1.4.3, there exists a increasing sequence $(p_k)_k$ such that

$$\frac{1}{2} \left\| \sum_{i=1}^{N_k} a_i y_{n_i} \right\| \leq \left\| \sum_{i=1}^{N_k} a_i e_i \right\| \leq \frac{3}{2} \left\| \sum_{i=1}^{N_k} a_i y_{n_i} \right\|$$

for all $k \geq 1$, $n_1 < ... < n_{N_k}$ with $n_1 \geq p_k$ and $a_1, ..., a_{N_k} \in \mathbb{R}$. Define $y'_k = y_{p_k}$ for all $k \geq 1$ and let $(y''_k)_k$ be an arbitrary subsequence of $(y'_k)_k$. For all $k \geq 1$, one has that

$$\begin{aligned} \frac{1}{N_k^{\frac{1}{p}}} \left\| \sum_{i=1}^{N_k} y''_i \right\| &= \frac{1}{N_k^{\frac{1}{p}}} \left\| \sum_{i=k+1}^{N_k+k} y''_i + \sum_{i=1}^{k} y''_i - \sum_{i=N_k+1}^{N_k+k} y''_i \right\| \\ &\geq \frac{2}{3N_k^{\frac{1}{p}}} \left\| \sum_{i=1}^{N_k} a_i e_i \right\| - \frac{2k}{N_k^{\frac{1}{p}}} \geq \frac{2k}{3} - \frac{k}{3} = \frac{k}{3}. \end{aligned}$$

This implies that X can not have the p-WBS, which is a contradiction. $\square$

Theorem 5.1.14. *For all $p \in (1, +\infty)$, the p-WBS and the strong p-WBS are equivalent.*

Proof. Suppose that X has the p-WBS. By the previous lemma, let $C > 0$ such that for all weakly null good sequence $(y_k)_k \subset B_X$

$$s((y_k)_k, n) \leq Cn^{\frac{1}{p}}$$

for all $n \geq 1$. Let $(x_n)_n \subset B_X$ be a weakly null sequence. Let $(x'_n)_n$ be a good subsequence of $(x_n)_n$. For all $n \geq 1$, we have that $s((x'_k)_k, n) \leq Cn^{\frac{1}{p}}$. In particular, for all $n \geq 1$, there exists $p_1 < ... < p_n$ such that

$$\left\| \sum_{k=1}^{n} x'_{p_k} \right\| \leq 2Cn^{\frac{1}{p}}.$$

Then (iii) of Proposition 5.1.4 holds with $c_n = 2Cn^{\frac{1}{p}-1}$. It follows that (i) of Proposition 5.1.4 holds with $a_n = O(n^{\frac{1}{p}-1})$, which means exactly that X has the strong p-WBS. $\square$

The case $p = \infty$ can not be deduced with the same arguments. This can be seen directly in the proof of Proposition 5.1.4. In fact, we do not know if $(a_n)_n$ can be chosen such that $na_n = O(1)$, even if $nc_n = O(1)$ is. That is why we will used another strategy.

We will need some tools from Ramsey's theory which can be found in [58] (chapter 10). If $M \subset \mathbb{N}$, we denote by $\mathcal{P}_\infty(M)$ the set of all infinite subsets of M. We consider $\mathcal{P}_\infty(\mathbb{N})$ endowed with the Ellentuck topology. We recall that this topology is stronger than the product topology. A subset $\mathcal{S} \subset \mathcal{P}_\infty(\mathbb{N})$ is called a Ramsey set if there exists $M \in \mathcal{P}_\infty(\mathbb{N})$ such that $\mathcal{P}_\infty(M) \subset \mathcal{S}$ or $\mathcal{P}_\infty(M) \subset \mathcal{P}_\infty(\mathbb{N}) \backslash \mathcal{S}$. It is well known that any Borel set (in the Ellentuck topology) is Ramsey.

In [82], the authors proved the following result:

Proposition 5.1.15. *Let X be a Banach space. The following assertions are equivalent:*

(i) X has ∞-WBS;

(ii) for all $(x_n)_n \subset B_X$ such that $x_n \xrightarrow{w} 0$, there exist $C > 0$ and a subsequence $(x'_n)_n$ of $(x_n)_n$ such that for every subsequence $(x''_n)_n$ of $(x'_n)_n$

$$\left\| \sum_{k=1}^{n} x''_k \right\| \leq C$$

for all $n \geq 1$;

(iii) any normalized weakly null sequence contains a subsequence equivalent to the canonical basis of c_0.

Remark 5.1.16. Note that the previous assertions are equivalent to the fact that X has the hereditary Dunford-Pettis property (Proposition 2 in [45]).

Using the same tools, we prove the following similar result:

Proposition 5.1.17. *Let X be a Banach space. The following assertions are equivalent:*

(i) X has the uniform ∞-WBS;

(ii) there exists $C > 0$ such that for all $(x_n)_n \subset B_X$ such that $x_n \xrightarrow{w} 0$, there exists a subsequence $(x'_n)_n$ of $(x_n)_n$ such that for every subsequence $(x''_n)_n$ of $(x'_n)_n$

$$\left\| \sum_{k=1}^{n} x''_k \right\| \leq C$$

for all $n \geq 1$;

(iii) there exists $M > 0$ such that any normalized weakly null sequence $(x_n)_n$ contains a subsequence $(x'_n)_n$ equivalent to the canonical basis of c_0 such that

$$\left\| \sum_{n=1}^{\infty} a_n x'_n \right\| \leq M \sup_{n \geq 1} |a_n|$$

for all $(a_n)_n \in c_{00}$.

Proof. $(i) \implies (ii)$ Let $C > 0$ given by the definition of uniform ∞-WBS. Let $(x_n)_n \subset B_X$ such that $x_n \xrightarrow{w} 0$. Define

$$\mathcal{S} = \left\{ M = (m_k)_{k\in\mathbb{N}} \in \mathcal{P}_\infty(\mathbb{N}) \mid \ \forall n \geq 1 \quad \left\| \sum_{k=1}^{n} x_{m_k} \right\| \leq C \right\}.$$

Note that $\mathcal{S}$ is closed in $\mathcal{P}_\infty(\mathbb{N})$ (in fact, it is closed in the product topology). In particular, $\mathcal{S}$ is Ramsey. It follows that there exists $M \in \mathcal{P}_\infty(\mathbb{N})$ such that $\mathcal{P}_\infty(M) \subset \mathcal{S}$ or $\mathcal{P}_\infty(M) \subset \mathcal{P}_\infty(\mathbb{N}) \setminus \mathcal{S}$. However, the second case is impossible since it would contradict the fact that X has the uniform ∞-WBS.
$(ii) \implies (iii)$ Note that (ii) implies easily that if $(x_n)_n$ is a weakly null sequence in S_X then there exists a subsequence $(x'_n)_n$ which is basic and such that

$$\left\| \sum_{k=1}^{n} \varepsilon_k x'_k \right\| \leq M := 2C$$

for all $k \geq 1$ and all signs $\varepsilon_1, ..., \varepsilon_k$. It follows that $\sum_n x'_n$ is a weakly unconditionally Cauchy series such that $\sum_{n=1}^{\infty} |x^*(x'_n)| \leq M$ for all $x^* \in B_{X^*}$. The operator T defined by $T : c_0 \to X$ by $T((a_n)_n) = \sum_{n=1}^{\infty} a_n x_n$ is bounded and verifies that $\|T\| \leq M$, from what (iii) follows.
$(iii) \implies (i)$ is obvious. □

Property (iii) of the previous proposition has been called property $(\star)$ in [45] and property (US) in [117]. In [45], Cembranos asked if this property is equivalent to the hereditary Dunford-Pettis property. Knaust and Odell answered this question in the affirmative showing that property (US) (or $(\star)$) is equivalent to (iii) in Proposition 5.1.15 (see Theorem 3.1 in [117]). Then the two previous propositions imply immediately the following result:

Theorem 5.1.18. *The ∞-WBS and the uniform ∞-WBS are equivalent.*

We conclude this part with some remarks about the spreading-(s) property introduced by Farmaki in [73].

Definition 5.1.19. A Banach space has the *spreading-(s) property* if every normalized weakly null sequence has a good subsequence such that the fundamental sequence of the associated spreading model is equivalent to the canonical basis of c_0.

If $(x_n)_n$ is a good weakly null sequence, then it is not difficul to see that the fundamental sequence $(e_n)_n$ of the associated spreading model is equivalent to the canonical basis of c_0 if and only if $\sup_n \|\sum_{k=1}^{n} e_k\| < \infty$.

Proposition 5.1.20. *The ∞-WBS implies the spreading-(s) property.*

Proof. Suppose that X has the ∞-WBS. Let $(x_n)_n$ be a weakly null normalized sequence. Without loss of generality, we can suppose that $(x_n)_n$ is a good sequence which generates a spreading model with fundamental basis (e_n). By Lemma 5.1.13, we have that $\sup_n \|\sum_{k=1}^{n} e_k\| < \infty$, i.e. $(e_n)_n$ is equivalent to the canonical basis of c_0. □

Summing up the previous results and those of Farmaki, we have the following graph of implications (where HDPP holds for hereditary Dunford-Pettis property). For the definition of weak HDPP, we refer the reader to [73] (definition 1.14).

$$\begin{array}{ccccc} \infty\text{-WBS} & \Longrightarrow & \text{spreading-(s) property} & \Longrightarrow & \text{WBS} \\ \Updownarrow & & \Updownarrow & & \\ \text{HDPP} & \Longrightarrow & \text{weak HDPP} & & \end{array}$$

5.1.4 The UWBS-index

In this subsection, we wish to determine the optimal index p such that X has the (strong) p-WBS.

Definition 5.1.21. Let X be a Banach space. For all $m \geq 1$, we define $\phi_X(m)$ as the infimum of the b such that for all $(x_n)_n \subset B_X$ with $x_n \xrightarrow{w} 0$, there exist $p_1 < p_2 < ... < p_m$ such that

$$\frac{1}{m}\left\|\sum_{i=1}^{m} x_{p_i}\right\| \leq b.$$

Proposition 5.1.22. *Let X be a Banach space. Then*

(a) X has the weak-(A_m) if and only if $\phi_X(m) < 1$.

(b) X has the uniform WBS if and only if $\phi_X(m) < 1$ for some $m \geq 2$.

Proof. (a) is obvious and (b) follows from the equivalence between the uniform WBS and the weak-(A_∞). □

Now we give some arithmetical properties of the sequence $(\phi_X(n))_n$.

Proposition 5.1.23. *Let X be a Banach space. Then*

(a) $(\phi_X(n))_n$ is a submultiplicative sequence, that is for all $m, n \geq 1$

$$\phi_X(nm) \leq \phi_X(n)\phi_X(m).$$

(b) $(n\phi_X(n))_n$ is a subadditive sequence, that is for all $m, n \geq 1$

$$(n+m)\phi_X(n+m) \leq n\phi_X(n) + m\phi_X(m).$$

(c) $(n\phi_X(n))_n$ is an increasing sequence.

(d) If $\phi_X(m) < 1$ for some $m \geq 1$, then $\phi_X(n) \to 0$.

Proof. (a) Let $n, m \geq 1$, $\varepsilon > 0$. Let $(x_k)_k \subset B_X$ such that $x_k \xrightarrow{w} 0$. There exists $p_1^1 < ... < p_n^1$ such that

$$\frac{1}{n}\left\|\sum_{i=1}^{n} x_{p_i^1}\right\| \leq \phi_X(n) + \varepsilon.$$

Then, there exist $p_1^2 < ... < p_n^2$ with $p_1^2 > p_n^1$ such that

$$\frac{1}{n}\left\|\sum_{i=1}^{n} x_{p_i^2}\right\| \leq \phi_X(n) + \varepsilon.$$

Following by induction, we construct sequences $p_1^k < ... < p_n^k$ such that $p_1^{k+1} > p_n^k$ and

$$\frac{1}{n}\left\|\sum_{i=1}^{n} x_{p_i^k}\right\| \leq \phi_X(n) + \varepsilon.$$

For all $k \geq 1$, define

$$y_k = \frac{1}{n(\phi_X(n)+\varepsilon)} \sum_{i=1}^{n} x_{p_i^k}.$$

Note that $y_k \in B_X$ and $y_k \xrightarrow{w} 0$. So, there exist $p_1 < ... < p_m$ such that

$$\frac{1}{m}\left\|\sum_{i=1}^{m} y_{p_i}\right\| \leq \phi_X(m) + \varepsilon.$$

But $\sum_{i=1}^{m} y_{p_i}$ is a sum of nm distincts terms of $(x_k)_k$, say $x'_1, ..., x'_{nm}$, so it follows that

$$\frac{1}{mn}\left\|\sum_{i=1}^{nm} x'_i\right\| \leq (\phi_X(n) + \varepsilon)(\phi_X(m) + \varepsilon).$$

We deduce that

$$\phi_X(nm) \leq (\phi_X(n) + \varepsilon)(\phi_X(m) + \varepsilon)$$

for all $\varepsilon > 0$ and then

$$\phi_X(nm) \leq \phi_X(n)\phi_X(m).$$

(b) Let $\varepsilon > 0$. If $(x_k)_k \subset B_X$ is such that $x_k \xrightarrow{w} 0$, just take $p_1 < ... < p_{n+m}$ such that

$$\frac{1}{n}\left\|\sum_{i=1}^{n} x_{p_i}\right\| \leq \phi_X(n) + \varepsilon \quad \text{and} \quad \frac{1}{n}\left\|\sum_{i=n+1}^{n+m} x_{p_i}\right\| \leq \phi_X(m) + \varepsilon.$$

It follows that

$$\frac{1}{n+m}\left\|\sum_{i=1}^{n+m} x_{p_i}\right\| \leq \frac{n\phi_X(n) + m\phi_X(m)}{m+n} + \frac{2\varepsilon}{n+m},$$

which implies that

$$\phi_X(n+m) \leq \frac{n\phi_X(n) + m\phi_X(m)}{m+n}.$$

(c) Let $n \geq 1$ and $\varepsilon > 0$. Let $(x_k)_k \subset B_X$ is such that $x_k \xrightarrow{w} 0$. Let show that there exist $p_1 < p_2 < ... < p_n$ such that

$$\frac{1}{n}\left\|\sum_{k=1}^{n} x_{p_k}\right\| \leq \frac{(1+\varepsilon)(n+1)}{n}\phi_X(n+1) + \varepsilon(1+\varepsilon)\frac{n+1}{n},$$

which easily implies the result letting $\varepsilon \to 0^+$. If $(x_k)_k$ has a convergent subsequence then we are done. Else we can suppose that $\inf_k \|x_k\| > 0$. It follows that $(x_k)_k$ admits a $(1+\varepsilon)$-basic subsequence. For simplicity, we still denote it by $(x_k)_k$. By definition of $\phi_X(n+1)$, there exists $p_1 < p_2 < ... < p_{n+1}$ such that

$$\frac{1}{n+1}\left\|\sum_{k=1}^{n+1} x_{p_k}\right\| \leq \phi_X(n+1) + \varepsilon.$$

It follows that

$$\begin{aligned}
\frac{1}{n}\left\|\sum_{k=1}^{n} x_{p_k}\right\| &\leq \frac{1+\varepsilon}{n}\left\|\sum_{k=1}^{n+1} x_{p_k}\right\| \\
&= \frac{(1+\varepsilon)(n+1)}{n}\frac{1}{n+1}\left\|\sum_{k=1}^{n+1} x_{p_k}\right\| \\
&\leq \frac{(1+\varepsilon)(n+1)}{n}\phi_X(n+1) + \varepsilon(1+\varepsilon)\frac{n+1}{n},
\end{aligned}$$

and the proof is complete.

(d) Using (b) and Fekete's lemma (see [74]), it follows that

$$\phi_X(n) = \frac{n\phi_X(n)}{n} \to \inf_k \frac{k\phi_X(k)}{k} = \inf_k \phi_X(k).$$

Take $m \geq 2$ such that $\phi_X(m) < 1$. By (i), it follows that

$$0 \leq \phi_X(m^k) \leq (\phi_X(m))^k$$

for all $k \geq 1$. Taking limit, we deduce that $\inf_k \phi_X(k) = 0$. □

Definition 5.1.24. Let X be a Banach space. We define the *uniform WBS-index* UWBS(X) of X as the supremum of the $p > 1$ such that X has the (strong) p-WBS. If X does not have the (strong) p-WBS for any $p > 1$, we set UWBS(X) = 1.

We are ready to prove the main result of this part:

Theorem 5.1.25. *Let X be a Banach space.*

(a) If $\phi_X(n) > \frac{1}{n}$ for all $n \geq 2$, then

$$UWBS(X) = \sup_{n\geq 2} \frac{\ln(n)}{\ln(n\phi_X(n))}.$$

(b) If $\phi_X(n) \leq \frac{1}{n}$ for some $n \geq 2$, then

$$UWBS(X) = +\infty.$$

Proof. (a) Suppose that $\phi_X(n) > \frac{1}{n}$ for all $n \geq 2$. If X does not have the uniform WBS, then $\phi_X(n) = 1$ for all $n \geq 2$ and the equality is true. Now suppose that X has the uniform WBS. The inequality

$$\mathrm{UWBS}(X) \geq \sup_{n\geq 2} \frac{1}{1 + \frac{\ln(\phi_X(n))}{\ln(n)}} = \sup_{n\geq 2} \frac{\ln(n)}{\ln(n\phi_X(n))}$$

is included in the proof of Theorem 5.1.9. Now let $p > 1$ such that X has the p-WBS. There exists $C > 0$ such that for all $(x_n)_n \subset B_X$ such that $x_n \xrightarrow{w} 0$, there exists a subsequence $(x'_n)_n$ of $(x_n)_n$ such that

$$\left\| \sum_{k=1}^n x'_k \right\| \leq C n^{\frac{1}{p}}$$

for all $n \geq 1$. Let $q \in (1, p)$. There exists n_q such that

$$C n^{\frac{1}{p}-1} \leq n^{\frac{1}{q}-1}$$

for all $n \geq n_q$. It follows that for all $(x_n)_n \subset B_X$ such that $x_n \xrightarrow{w} 0$, there exists a subsequence $(x'_n)_n$ of $(x_n)_n$ such that

$$\frac{1}{n} \left\| \sum_{k=1}^n x'_k \right\| \leq n^{\frac{1}{q}-1}$$

for all $n \geq n_q$. This implies in particular that

$$0 < \phi_X(n) \leq n^{\frac{1}{q}-1},$$

or equivalently,

$$q \leq \frac{1}{1 + \frac{\ln(\phi_X(n))}{\ln(n)}}$$

for all $q \in (1, p)$ and all $n \geq n_q$. It follows that

$$q \leq \sup_{n \geq 2} \frac{1}{1 + \frac{\ln(\phi_X(n))}{\ln(n)}}$$

for all $q \in (1, p)$ and then $p \leq \sup_{n \geq 2} \frac{1}{1+\frac{\ln(\phi_X(n))}{\ln(n)}}$, which implies that

$$\mathrm{UWBS}(X) \leq \sup_{n \geq 2} \frac{1}{1 + \frac{\ln(\phi_X(n))}{\ln(n)}}.$$

(b) Let $n \geq 2$ such that $\phi_X(n) \leq \frac{1}{n}$. By the proof Theorem 5.1.9, we have that

$$\mathrm{UWBS}(X) \geq \frac{1}{1 + \frac{\ln\left(\frac{1}{n}+\varepsilon\right)}{\ln(n)}} - \varepsilon = \frac{\ln(n)}{\ln(1 + n\varepsilon)} - \varepsilon$$

for all $\varepsilon > 0$. Taking $\varepsilon \to 0^+$, we conclude the proof. □

Remark 5.1.26. c_0 appears to be a borderline case in the previous theorem. In fact, let us show that $\phi_{c_0}(n) = \frac{1}{n}$ for all $n \geq 1$. Considering the canonical basis (or using the next proposition), it is easily seen that $\phi_{c_0}(n) \geq \frac{1}{n}$ for all $n \geq 1$. Now let $\varepsilon > 0$ and let $(x_k)_k \subset B_X$ be a weakly null sequence. Fix $n \geq 1$. Let show that there exist $p_1 < ... < p_n$ such that

$$\left\| \sum_{k=1}^{n} x_{p_i} \right\| \leq (1+\varepsilon)^2,$$

it will imply that $\phi_{c_0}(n) \leq \frac{(1+\varepsilon)^2}{n}$ and then $\phi_{c_0}(n) \leq \frac{1}{n}$. If $(x_n)_n$ admits a convergent subsequence, it is obvious. So let suppose that $\inf_n \|x_n\| > 0$. By Bessaga–Pełczynski selection Principle (for example, see corollary 4.27 in [67]), there exist a subsequence $(x'_n)_n$ of $(x_n)_n$ and a block basis sequence $(e'_n)_n$ of the canonical basis of c_0 such that $x'_n - e'_n \to 0$ and

$$\left\| \sum_{k=1}^{p} a_k x'_k \right\| \leq (1+\varepsilon) \left\| \sum_{k=1}^{p} a_k e'_k \right\|$$

for all $p \geq 1$ and all $a_1, ..., a_p \in \mathbb{R}$. Since $x'_n - e'_n \to 0$, there exists n_0 such that $\|e'_n\| \leq 1 + \varepsilon$ for all $n \geq n_0$. It follows that

$$\left\| \sum_{k=n_0}^{n_0+n-1} x'_k \right\| \leq (1+\varepsilon) \left\| \sum_{k=n_0}^{n_0+n-1} e'_k \right\| = (1+\varepsilon) \max_{n_0 \leq k \leq n_0+n-1} \|e'_k\| \leq (1+\varepsilon)^2$$

and the proof is done. In particular, it implies that $\mathrm{UWBS}(c_0) = \infty$. In fact, more is true: using the same argument, it is easy to see that c_0 has the uniform ∞-WBS.

The formula given by (i) in the previous theorem can be found in [148] without any hypothesis on ϕ_X. However, if we do not suppose that $\phi_X(n) > \frac{1}{n}$ for all $n \geq 2$, then the formula is easily seen to be false by considering a Banach space with the Schur property for example:

Proposition 5.1.27. *Let X be a Banach space. The following assertions are equivalent:*

(i) X has the Schur property;

(ii) $\phi_X(n) = 0$ for all $n \geq 1$;

(iii) $\phi_X(n) = o\left(\frac{1}{n}\right)$;

(iv) there exists $n \geq 2$ such that $\phi_X(n) < \frac{1}{n}$.

Proof. $(i) \implies (ii) \implies (iii)$ is easy. Suppose that (iii) holds and that X does not have the Schur property. There exists $(x_n)_n \subset B_X$ such that $x_n \xrightarrow{w} 0$ and $(x_n)_n$ does not converge to 0. By taking a subsequence, we can suppose that $\alpha := \inf_n \|x_n\| > 0$. Then, taking a further subsequence if necessary, we can suppose that $(x_n)_n$ is a 2-basic sequence. Let $n \geq 2$. By definition of $\phi_X(n)$, there exist $p_1 < p_2 < ... < p_n$ such that

$$\frac{1}{n}\left\|\sum_{k=1}^{n} x_{p_i}\right\| \leq \phi_X(n) + \frac{\alpha}{4n}.$$

It follows that

$$\alpha \leq \|x_{p_1}\| \leq 2\left\|\sum_{k=1}^{n} x_{p_i}\right\| \leq 2n\phi_X(n) + \frac{\alpha}{2}.$$

We deduce that $\alpha \leq 4n\phi_X(n) \to 0$, which is a contradiction since $\alpha > 0$.

$(iii) \implies (iv)$ is obvious. Suppose that (iv) holds. Let $n \geq 2$ such that $\phi_X(n) < \frac{1}{n}$. By submultiplicativity of ϕ_X, one has that $\phi_X(n^k) < \frac{1}{n^k}$ for all $k \geq 1$. By (c) of Proposition 5.1.23, it follows that $(n^k\phi_X(n^k))_k$ is an increasing bounded sequence and then converges to some $a \in [0, 1]$. For all $k \geq 1$, we have that

$$n^k\phi_X(n^k) \leq (n\phi_X(n))^k$$

and since $n\phi_X(n) < 1$, we deduce that $a = 0$. However $(n^k\phi_X(n^k))_k$ is positive and increasing and then it follows that $\phi_X(n^k) = 0$ for all $k \geq 1$. Using (c) of Proposition 5.1.23 again, it follows that $\phi_X(p) = 0$ for all $p \geq 1$, i.e. (ii) holds.

5.2 The uniform BS

Definition 5.2.1. Let X be a Banach space and let $k \geq 2$. We say that X has (A_k) if X is reflexive and has the weak-(A_k). We say that X has (A_∞) if X has (A_k) for some $k \geq 2$.

Definition 5.2.2. Let X be a Banach space. We say that X has the uniform Banach-Saks (in short uniform BS) if there exists a sequence $(a_n)_n$ of real positive numbers such that $a_n \to 0$ and for all $(x_n)_n \subset B_X$, there exist $x \in B_X$ and a subsequence $(x'_n)_n$ of $(x_n)_n$ such that

$$\left\|\frac{1}{m}\sum_{k=1}^{m} x'_k - x\right\| \leq a_m$$

for all $m \geq 1$.

Definition 5.2.3. Let X be a Banach space and $p > 1$. We say that X has the strong p-BS if there exists $C > 0$ such that for all $(x_n)_n \subset B_X$, there exist $x \in B_X$ and a subsequence $(x'_n)_n$ of $(x_n)_n$ such that

$$\left\|\sum_{k=1}^{n} x'_k - x\right\| \leq Cn^{\frac{1}{p}}$$

for all $n \geq 1$.

Definition 5.2.4. Let X be a Banach space and $p>1$. We say that X has the p-BS if for all $(x_n)_n \subset B_X$, there exist $C>0$, $x \in B_X$ and a subsequence $(x'_n)_n$ of $(x_n)_n$ such that

$$\left\|\sum_{k=1}^{n} x'_k - x\right\| \leq C n^{\frac{1}{p}}$$

for all $n \geq 1$.

We start with a similar version of Proposition 5.1.4:

Proposition 5.2.5. *Let X a Banach space. The following assertions are equivalent:*

(i) X has the uniform BS;

(ii) there exists a sequence $(b_n)_n$ of real positive numbers such that $b_n \to 0$ and for all $(x_n)_n \subset B_X$, there exist $x \in B_X$ and a subsequence $(x'_n)_n$ of $(x_n)_n$ such that for every subsequence $(x''_n)_n$ of $(x'_n)_n$

$$\left\|\frac{1}{m}\sum_{k=1}^{m} x''_k - x\right\| \leq b_m$$

for all $m \geq 1$;

(iii) there exists a sequence $(c_n)_n$ of real positive numbers such that $c_n \to 0$ and for all $(x_n)_n \subset B_X$, there exists $x \in B_X$ such that all $m,n \geq 1$, there exists $n \leq p_1 < ... < p_m$ such that

$$\left\|\frac{1}{m}\sum_{i=1}^{m} x_{p_i} - x\right\| \leq c_m.$$

Proof. $(ii) \implies (i) \implies (iii)$ is obvious. Suppose that (iii) holds. Let $(c_n)_n$ as in (iii). Define

$$b_n = \frac{6q_n}{n} + 3c_n$$

where $q_n = E\left(\frac{\ln(n)}{\ln(2)}\right)$ for all $n \in \mathbb{N}$. Let $(x_n)_n \subset B_X$. Let x given by (iii). Define $y_n = \frac{x_n - x}{2} \in B_X$ for all $n \in \mathbb{N}$. If $(y_n)_n$ admits a convergent subsequence to 0, we conclude as in Proposition 5.1.4 using Lemma 5.1.3. By taking a subsequence if necessary, we can suppose that $(y_n)_n$ is a good sequence generating a spreading model Z with fundamental basis $(e_n)_n$. For all $k \in \mathbb{N}$, define $N_k = 2^k$. By Propsition 1.4.3, there exists a strictly increasing sequence $(p_k)_k$ such that for all $n_1 < ... < n_{N_k}$ with $n_1 \geq p_k$ and all $a_1, ..., a_{N_k}$

$$\frac{1}{2}\left\|\sum_{i=1}^{N_k} a_i y_{n_i}\right\| \leq \left\|\sum_{i=1}^{N_k} a_i e_i\right\| \leq \frac{3}{2}\left\|\sum_{i=1}^{N_k} a_i y_{n_i}\right\|.$$

For all $k \in \mathbb{N}$, we define $y'_k = y_{p_k}$. Let $m \in \mathbb{N}$ and take k such that $m \leq N_k$. By (iii), there exists $p_k \leq n_1 < ... < n_m$ such that

$$\frac{2}{m}\left\|\sum_{i=1}^{m} y_{n_i}\right\| = \left\|\frac{1}{m}\sum_{i=1}^{m} x_{n_i} - x\right\| \leq c_m.$$

Following the proof of Proposition 5.1.4, we obtain that for all subsequences $(y''_n)_n$ of $(y'_n)_n$ and for all $n \geq 1$ that

$$\frac{1}{n}\left\|\sum_{k=1}^{n} y''_i\right\| \leq \frac{3q_n}{n} + \frac{3}{2}c_n,$$

that is

$$\left\| \frac{1}{n} \sum_{k=1}^{n} x_i'' - x \right\| \leq \frac{6q_n}{n} + 3c_n = b_n$$

and the proof is complete. □

The results of the previous section and the fact that the BS property implies reflexivity give directly the following result:

Proposition 5.2.6. *Let X be a Banach space. The following assertions are equivalent:*

(i) X has the uniform BS;

(ii) X is reflexive and has the uniform WBS;

(iii) X has the (strong) p-BS for some $p \in (1, +\infty)$;

(iv) X has (A_∞).

Remark 5.2.7. The previous results imply that weak-(A_∞) and (A_∞) are preserved under isomorphism.

For the definitions of *nearly uniformly convex* (NUC) and *nearly uniformly smooth* (NUS), we refer the reader to [145]. We also refer the reader to [119] for the definition of the property *k-NUCε*. By Theorem 1 in [53], if X is a Banach space with the *(β)-property* (introduced by Rolewicz in [150]) then X and X^* have the BS. A careful study of the proof shows that any NUS space or any space with the (β)-property property has (A_2). We then have the two following propositions:

Proposition 5.2.8. *Let X be a Banach space such that one of the following assertions holds:*

(a) X is k-NUCε for some $k \geq 2$ and $\varepsilon \in (0, 1)$;

(b) X is NUS.

Then X has the uniform BS.

Proof. If X is k-NUCε, X has (A_k) (Proposition 10 in [119]) and then X has the uniform BS. A NUS space has (A_2) by the above discussion. □

Thanks to the equivalence between (A_∞) and the uniform BS, we can improve Theorem 1 in [53]. Recall that X is a *(β)-space* if it can be renormed to have the (β)-property.

Proposition 5.2.9. *Let X be a (β)-space. Then X and X^* have the uniform BS.*

Proof. Since the uniform BS is invariant by isomorphism, we can already suppose that the norm on X has the (β)-property. Since the (β)-property implies (A_2) by the above discussion, X has the uniform BS. Moreover, the (β)-property implies that X is NUC (see Proposition 6 in [150]). Thus X^* is NUS (by Theorem 2.4 in [145]) and we conclude by the previous proposition. □

Corollary 5.2.10. *Any superreflexive Banach space has the uniform BS.*

Proof. Since a superreflexive space is a (β)-space, we can conclude thanks to the previous proposition. However, we give a more direct proof (contained in Kakutani's proof in [111]). Let X a superreflexive Banach space. Enflo's theorem (see Theorem 2.5.1) implies that X admits a equivalent uniformly convex norm. It is easily seen that X endowed with this new norm has (A_2). In fact, define $\theta = \max\left\{1 - \delta_X\left(\frac{1}{2}\right), \frac{3}{4}\right\}$ where δ_X is the modulus of convexity of this new norm. Let $(x_n)_n \subset B_X$ such that $x_n \xrightarrow{w} 0$. If $\|x_1\| \leq \frac{1}{2}$, then

$$\left\|\frac{x_1 + x_2}{2}\right\| \leq \frac{3}{4} \leq \theta.$$

So, we can suppose that $\|x_1\| > \frac{1}{2}$. It follows that there exists $n > 1$ such that $\|x_1 - x_n\| \geq \frac{1}{2}$. In fact, if $\|x_1 - x_n\| < \frac{1}{2}$ for all n, then

$$|x^*(x_1)| = \lim_n |x^*(x_1) - x^*(x_n)| \leq \limsup_n \|x_1 - x_n\| \leq \frac{1}{2}$$

for all $x^* \in B_{X^*}$. That implies that $\|x_1\| \leq \frac{1}{2}$, which is a contradiction. Then let $n > 1$ such that $\|x_1 - x_n\| \geq \frac{1}{2}$. By uniform convexity, it follows that

$$\left\|\frac{x_1 + x_n}{2}\right\| \leq 1 - \delta_X\left(\frac{1}{2}\right) \leq \theta$$

and the proof is complete. □

Remark 5.2.11. By Theorem 1.3.6, a superreflexive Banach space has non-trivial type. So, we could also have invoqued Proposition 5.3.10 to prove the previous corollary.

5.3 The uniform WABS

Definition 5.3.1. Let X be a Banach space and let $k \geq 2$. We say that X has the *weak alternate-*(A_k) if there exists $\theta \in (0,1)$ such that for all $(x_n)_n \subset B_X$ such that $x_n \xrightarrow{w} 0$, there exist $p_1 < p_2 < ... < p_k$ such that

$$\frac{1}{k}\left\|\sum_{i=1}^{k}(-1)^i x_{p_i}\right\| \leq \theta.$$

We say that X has the weak alternate-(A_∞) if X has the weak alternate-(A_k) for some $k \geq 2$.

Definition 5.3.2. Let X be a Banach space. We say that X has the *uniform weak alternate Banach-Saks property* (in short uniform WABS) if there exists a sequence $(a_n)_n$ of real positive numbers such that $a_n \to 0$ and for all $(x_n)_n \subset B_X$ such that $x_n \xrightarrow{w} 0$, there is a subsequence $(x'_n)_n$ of $(x_n)_n$ such that

$$\frac{1}{m}\left\|\sum_{k=1}^{m}(-1)^k x'_k\right\| \leq a_m$$

for all $m \geq 1$.

Proposition 5.3.3. *Let X a Banach space. The following assertions are equivalent:*

(i) X has the uniform WABS;

(ii) *there exists a sequence $(b_n)_n$ of real positive numbers such that $b_n \to 0$ and for all $(x_n)_n \subset B_X$ such that $x_n \xrightarrow{w} 0$, there is a subsequence $(x'_n)_n$ of $(x_n)_n$ such that for every subsequence $(x''_n)_n$ of $(x'_n)_n$*

$$\frac{1}{m}\left\|\sum_{k=1}^{m}(-1)^k x''_k\right\| \leq b_m$$

for all $m \geq 1$;

(iii) *there exists a sequence $(c_n)_n$ of real positive numbers such that $c_n \to 0$ and for all $(x_n)_n \subset B_X$ such that $x_n \xrightarrow{w} 0$ and for all $m \geq 1$, there exists $p_1 < \ldots < p_m$ such that*

$$\frac{1}{m}\left\|\sum_{i=1}^{m}(-1)^i x_{p_i}\right\| \leq c_m.$$

Proof. The proof is similar to the proof of Proposition 5.1.4 and is left to the reader. □

Remark 5.3.4. We could have given the following definition of the uniform WABS: X has the uniform WABS if there exists a sequence $(a_n)_n$ of real positive numbers such that $a_n \to 0$ and for all $(x_n)_n \subset B_X$ such that $x_n \xrightarrow{w} 0$, there exist $x \in B_X$ and a subsequence $(x'_n)_n$ of $(x_n)_n$ such that

$$\left\|\frac{1}{m}\sum_{k=1}^{m}(-1)^k x'_k - x\right\| \leq a_m$$

for all $m \geq 1$. Using the same tools as in the previous proposition, we would be able to construct a sequence $(b_n)_n$ of real positive numbers such that $b_n \to 0$ and for all $(x_n)_n \subset B_X$ such that $x_n \xrightarrow{w} 0$, there exist $x \in B_X$ and a subsequence $(x'_n)_n$ of $(x_n)_n$ such that for every subsequence $(x''_n)_n$ of $(x'_n)_n$

$$\left\|\frac{1}{m}\sum_{k=1}^{m}(-1)^k x''_k - x\right\| \leq b_m \tag{5.7}$$

for all $m \geq 1$. Let $(x_n)_n \in B_X$, consider a subsequence $(x'_n)_n$ of $(x_n)_n$ and $x \in B_X$ with the previous property. Consider the two following subsequences of $(x'_n)_n$

$$x'_1, x'_4, x'_7, x'_{10}, x'_{13}, x'_{16} \ldots$$

$$x'_2, x'_3, x'_8, x'_9, x'_{14}, x'_{15} \ldots$$

Then the subsequence $(x''_n)_n$ of $(x'_n)_n$ obtained by ordering the previous terms by increasing index fullfils that

$$\frac{1}{m}\left\|\sum_{k=1}^{m}(-1)^k x''_k\right\| \leq 2c_m$$

for all $m \geq 1$ where c_m can be expressed in terms of b_m and $c_m \to 0$. In fact, for simplicity, let show it for $m = 8$:

$$\begin{aligned}\frac{1}{8}\left\|\sum_{k=1}^{8}(-1)^k x''_k\right\| &= \frac{1}{8}\|(-x'_1 + x'_4 - x'_7 + x'_{10}) + (x'_2 - x'_3 + x'_8 - x'_9)\| \\ &\leq \frac{1}{8}(\| - x'_1 + x'_4 - x'_7 + x'_{10} - x\| + \| - x'_2 + x'_3 - x'_8 + x'_9 - x\|) \\ &\leq \frac{1}{8}(b_4 + b_4) = \frac{b_4}{4}.\end{aligned}$$

It follows that we can always consider that $x = 0$ in (5.7).

Theorem 5.3.5. *The uniform WABS and the weak alternate-(A_∞) are equivalent.*

Proof. Same proof as in Theorem 5.1.5. □

Theorem 5.3.6. *The uniform WBS and the uniform WABS are equivalent.*

Proof. Let X be a Banach space with the uniform WBS. Let $(x_n)_n \subset B_X$ such that $x_n \xrightarrow{w} 0$. Let $(b_n)_n$ as in (ii) of proposition 5.1.4. Take $(x'_n)_n$ a subsequence of $(x_n)_n$ given by (ii) of proposition 5.1.4. We have that $\frac{1}{n}\left\|\sum_{k=1}^n x'_{2k}\right\| \leq b_n$ and $\frac{1}{n}\left\|\sum_{k=1}^n x'_{2k+1}\right\| \leq b_n$ for all $n \in \mathbb{N}$. It follows that for all $n \geq 1$

$$\frac{1}{2n}\left\|\sum_{k=1}^{2n}(-1)^k x'_k\right\| = \left\|\frac{1}{2n}\sum_{k=1}^{n} x'_{2k} - \frac{1}{2n}\sum_{k=0}^{n-1} x'_{2k+1}\right\| \leq \frac{1}{2}b_n + \frac{1}{2}b_n = b_n$$

and

$$\begin{aligned}\frac{1}{2n+1}\left\|\sum_{k=1}^{2n+1}(-1)^k x'_k\right\| &= \left\|\frac{1}{2n+1}\sum_{k=0}^{n} x'_{2k+1} - \frac{1}{2n+1}\sum_{k=1}^{n} x'_{2k}\right\| \\ &\leq \frac{n+1}{2n+1}b_{n+1} + \frac{n}{2n+1}b_n,\end{aligned}$$

that implies that X has the uniform WABS since the right-hand terms tend to 0.

Now suppose that X has the uniform WABS. Take $(b_n)_n$ as in (ii) in Proposition 5.3.3. Define $a_n = \frac{3q_n}{n} + 6b_n$ where $q_n = E\left(\frac{\ln(n)}{\ln(2)}\right)$ for all $n \in \mathbb{N}$. Let $(x_n)_n \subset B_X$ such that $x_n \xrightarrow{w} 0$. If $(x_n)_n$ as a subsequence that converges to 0, then there exists a further subsequence $(x'_n)_n$ such that

$$\frac{1}{m}\left\|\sum_{k=1}^{m} x'_k\right\| \leq a_m$$

for all $m \geq 1$. So, let suppose that $(x_n)_n$ does not have any convergent subsequence. Let $(x'_n)_n$ given by (ii) in Proposition 5.3.3. We can suppose that this subsequence defines a spreading model Z with fundamental basis $(e_n)_n$. Let $n \geq 1$. By Proposition 1.4.3, there exists p_n such that

$$\frac{1}{n}\left\|\sum_{i=1}^{n}(-1)^i e_i\right\| \leq \frac{3}{2n}\left\|\sum_{i=1}^{n}(-1)^i x_{p_n+i}\right\| \leq \frac{3}{2}b_n.$$

It follows that $\left(\frac{1}{n}\sum_{i=1}^n(-1)^i e_i\right)_n$ converges to 0. Since $x'_n \xrightarrow{w} 0$, it follows that $(e_n)_n$ is a basic monotone unconditionnal sequence. Then we have for all $n \geq 1$

$$\begin{aligned}\frac{1}{n}\left\|\sum_{i=1}^{n} e_i\right\| &= \frac{1}{n}\left\|\sum_{\substack{i=1\\ i \text{ even}}}^{n}(-1)^i e_i + \sum_{\substack{i=1\\ i \text{ odd}}}^{n}(-1)^{i+1} e_i\right\| \\ &\leq \frac{1}{n}\left\|\sum_{\substack{i=1\\ i \text{ even}}}^{n}(-1)^i e_i\right\| + \frac{1}{n}\left\|\sum_{\substack{i=1\\ i \text{ odd}}}^{n}(-1)^{i+1} e_i\right\| \\ &\leq \frac{1}{n}\left\|\sum_{i=1}^{n}(-1)^i e_i\right\| + \frac{1}{n}\left\|\sum_{i=1}^{n}(-1)^{i+1} e_i\right\| \\ &\leq 3b_n.\end{aligned}$$

For $k \geq 1$, define $N_k = 2^k$. By Proposition 1.4.3 again, there exists a strictly increasing sequence $(p_k)_k$ such that for all $n_1 < ... < n_{N_k}$ with $n_1 \geq p_k$ and all $a_1, ..., a_{N_k}$

$$\frac{1}{2}\left\|\sum_{i=1}^{N_k} a_i x'_{n_i}\right\| \leq \left\|\sum_{i=1}^{N_k} a_i e_i\right\| \leq \frac{3}{2}\left\|\sum_{i=1}^{N_k} a_i x'_{n_i}\right\|.$$

For all $k \in \mathbb{N}$, we define $x''_k = x'_{p_k}$. Let $n \geq 1$. We have that

$$\begin{aligned}\frac{1}{n}\left\|\sum_{k=1}^{n} x''_i\right\| &\leq \frac{1}{n}\left\|\sum_{k=1}^{q_n} x''_i\right\| + \frac{1}{n}\left\|\sum_{k=q_n+1}^{n} x''_i\right\| \\ &\leq \frac{q_n}{n} + \frac{2}{n}\left\|\sum_{k=q_n+1}^{n} e_i\right\| \\ &\leq \frac{q_n}{n} + \frac{2}{n}\left(\left\|\sum_{k=1}^{n} e_i\right\| + \left\|\sum_{k=1}^{q_n} e_i\right\|\right) \\ &\leq \frac{3q_n}{n} + 6b_n = a_n,\end{aligned}$$

and we conclude that X has the uniform WBS. $\square$

Remark 5.3.7. The previous proof gives an alternative proof of the equivalence between WBS and WABS (see [51]). In fact, the proof is even easier since we do not have to evaluate the speed of convergence.

Remark 5.3.8. We could have defined the strong p-WABS in a obvious way. The proof of the last theorem shows that X has the strong p-WABS if and only if X has the strong p-WBS. In particular, X has the uniform WABS if and only if X has the strong p-WABS for some $p > 1$.

Theorem 5.3.9. *Let X a Banach space. The following assertions are equivalent:*

(i) X has the uniform WABS;

(ii) there exists a sequence $(a_n)_n$ of real positive numbers such that $a_n \to 0$ and for all $(x_n)_n \subset B_X$ such that $x_n \xrightarrow{w} 0$, there exist a sequence of signs $(\varepsilon_n)_n$ and a subsequence $(x'_n)_n$ of $(x_n)_n$ such that

$$\frac{1}{m}\left\|\sum_{k=1}^{m} \varepsilon_k x'_k\right\| \leq a_m$$

for all $m \geq 1$;

(iii) there exists a sequence $(b_n)_n$ of real positive numbers such that $b_n \to 0$ and for all $(x_n)_n \subset B_X$ such that $x_n \xrightarrow{w} 0$, there exist a sequence of signs $(\varepsilon_n)_n$ and a subsequence $(x'_n)_n$ of $(x_n)_n$ such that for every subsequence $(x''_n)_n$ of $(x'_n)_n$

$$\frac{1}{m}\left\|\sum_{k=1}^{m} \varepsilon_k x''_k\right\| \leq b_m$$

for all $m \geq 1$;

(iv) *there exists a sequence $(c_n)_n$ of real positive numbers such that $c_n \to 0$ and for all $(x_n)_n \subset B_X$ such that $x_n \xrightarrow{w} 0$ and for all $m \geq 1$, there exist $p_1 < ... < p_m$ and signs $\varepsilon_1, ..., \varepsilon_m$ such that*

$$\frac{1}{m}\left\|\sum_{i=1}^{m} \varepsilon_i x_{p_i}\right\| \leq c_m;$$

(v) *there exists a sequence $(b'_n)_n$ of real positive numbers such that $b'_n \to 0$ and for all $(x_n)_n \subset B_X$ such that $x_n \xrightarrow{w} 0$, there is a subsequence $(x'_n)_n$ of $(x_n)_n$ such that for every subsequence $(x''_n)_n$ of $(x'_n)_n$ and all sequences of signs $(\varepsilon_n)_n$*

$$\frac{1}{m}\left\|\sum_{k=1}^{m} \varepsilon_k x''_k\right\| \leq b'_m$$

for all $m \geq 1$;

(vi) *there exists a sequence $(c'_n)_n$ of real positive numbers such that $c'_n \to 0$ and for all $(x_n)_n \subset B_X$ such that $x_n \xrightarrow{w} 0$ and for all $m \geq 1$, there exist $p_1 < ... < p_m$ such that for all signs $\varepsilon_1, ..., \varepsilon_m$*

$$\frac{1}{m}\left\|\sum_{i=1}^{m} \varepsilon_i x_{p_i}\right\| \leq c'_m;$$

(vii) *there exist $k \geq 2$ and $\theta \in (0,1)$ such that for all $(x_n)_n \subset B_X$ such that $x_n \xrightarrow{w} 0$, there exist $p_1 < p_2 < ... < p_k$ and signs $\varepsilon_1, ...\varepsilon_k$ such that*

$$\frac{1}{k}\left\|\sum_{i=1}^{k} \varepsilon_i x_{p_i}\right\| \leq \theta;$$

(viii) *there exist $k \geq 2$ and $\theta \in (0,1)$ such that for all $(x_n)_n \subset B_X$ such that $x_n \xrightarrow{w} 0$, there exist $p_1 < p_2 < ... < p_k$ such that for all signs $\varepsilon_1, ..., \varepsilon_k$*

$$\frac{1}{k}\left\|\sum_{i=1}^{k} \varepsilon_i x_{p_i}\right\| \leq \theta.$$

Proof. $(i) \implies (iii)$ is obvious $(iii) \implies (i)$ We will show that X has the uniform WBS. Define $a_n = \frac{3q_n}{n} + 6b_n$ where $q_n = E\left(\frac{\ln(n)}{\ln(2)}\right)$ for all $n \in \mathbb{N}$. Let $(x_n)_n \subset B_X$ such that $x_n \xrightarrow{w} 0$. We suppose that $(x_n)_n$ does not have any convergent subsequence and is a good sequence. Take $(x'_n)_n$ and $(\varepsilon_n)_n$ as in (ii). Let Z the spreading model generating by $(x'_n)_n$ with fundamental sequence $(e_n)_n$. By Proposition 1.4.3, there exists p_n such that

$$\frac{1}{n}\left\|\sum_{i=1}^{n} \varepsilon_{p_n+i} e_i\right\| \leq \frac{3}{2n}\left\|\sum_{i=1}^{n} \varepsilon_{p_n+i} x_{p_n+i}\right\| \leq \frac{3}{2} b_n.$$

Using the fact that $(e_n)_n$ is a basic unconditionnal monotone sequence, we obtain that

$$\frac{1}{n}\left\|\sum_{i=1}^{n} e_i\right\| \leq 3b_n$$

for all $n \geq 1$ and we conclude as in the previous proof.

$(iii) \implies (ii) \implies (iv)$ is direct. Suppose that (iv) holds. Let $(c_n)_n$ as in (iv). We can suppose that $(x_n)_n$ does not admit any convergent subsequence. By taking a subsequence if necessary, we can suppose that $(x_n)_n$ is a good sequence generating a spreading model Z with fundamental basis $(e_n)_n$. For all $k \in \mathbb{N}$, define $N_k = 2^k$. Define $q_n = E\left(\frac{\ln(n)}{\ln(2)}\right)$. By Proposition 1.4.3, there exists a strictly increasing sequence $(p_k)_k$ such that for all $n_1 < ... < n_{N_k}$ with $n_1 \geq p_k$ and all $a_1, ..., a_{N_k}$

$$\frac{1}{2}\left\|\sum_{i=1}^{N_k} a_i x_{n_i}\right\| \leq \left\|\sum_{i=1}^{N_k} a_i e_i\right\| \leq \frac{3}{2}\left\|\sum_{i=1}^{N_k} a_i x_{n_i}\right\|.$$

For all $k \in \mathbb{N}$, we define $x'_k = x_{p_k}$. Let $m \in \mathbb{N}$ and take k such that $m \leq N_k$. By considering the sequence $(x_n)_{n \geq p_k}$ and by (iv), there exists $p_k \leq n_1 < ... < n_m$ and signs $\varepsilon_1, ..., \varepsilon_m$ such that

$$\frac{1}{m}\left\|\sum_{i=1}^{m} \varepsilon_i x_{n_i}\right\| \leq c_m.$$

It follows that

$$\frac{1}{m}\left\|\sum_{i=1}^{m} \varepsilon_i e_i\right\| \leq \frac{3}{2m}\left\|\sum_{i=1}^{m} \varepsilon_i x_{n_i}\right\| \leq \frac{3}{2}c_m$$

and since $(e_n)_n$ is a basic unconditionnal monotone sequence

$$\frac{1}{m}\left\|\sum_{i=1}^{m} e_i\right\| \leq 3c_m$$

for all $m \geq 1$. Now, let $(x''_n)_n$ be a subsequence of $(x'_n)_n$. Let $n \geq 1$. We have that

$$\begin{aligned}
\frac{1}{n}\left\|\sum_{k=1}^{n} x''_i\right\| &\leq \frac{1}{n}\left\|\sum_{k=1}^{q_n} x''_i\right\| + \frac{1}{n}\left\|\sum_{k=q_n+1}^{n} x''_i\right\| \\
&\leq \frac{q_n}{n} + \frac{2}{n}\left\|\sum_{k=q_n+1}^{n} e_i\right\| \\
&\leq \frac{q_n}{n} + \frac{2}{n}\left(\left\|\sum_{k=1}^{n} e_i\right\| + \left\|\sum_{k=1}^{q_n} e_i\right\|\right) \\
&\leq \frac{3q_n}{n} + 6c_n := b_n
\end{aligned}$$

and then X has the uniform WBS, i.e (i) holds.

$(v) \implies (iii)$ is obvious. Suppose that (iii) holds. Take $(b_n)_n$ as in (iii). Let $(x_n)_n \subset B_X$ such that $x_n \xrightarrow{w} 0$. We suppose that $(x_n)_n$ does not have any convergent subsequence and is a good sequence generating a spreading model Z with fundamental basis $(e_n)_n$. Let $(x'_n)_n$ given by (iii). By using the same techniques as previously, one has that

$$\frac{1}{m}\left\|\sum_{i=1}^{m} (-1)^i e_i\right\| \leq \frac{3}{2}c_m$$

and then

$$\frac{1}{m}\left\|\sum_{i=1}^{m} \varepsilon_i e_i\right\| \leq 3c_m$$

for all sequences of signs $(\varepsilon_n)_n$ and all $m \geq 1$. We conclude this implication with the same arguments used to show that $(iv) \implies (i)$.

$(vi) \implies (i)$ is direct by Proposition 5.3.3. $(v) \implies (vi)$ is obvious. $(v) \implies (viii) \implies (vii)$ is also obvious. $(vii) \implies (ii)$ can be proved following the proof of $(ii) \implies (i)$ in Theorem 5.1.5. □

The following result has been proved by Rakov in [149] with a different proof. However, his proof uses a non-trivial result about the existence of unconditionnal sequences. Here we give an alternative proof.

Corollary 5.3.10. *Any Banach space with non-trivial type p has the (strong) p-WBS.*

Proof. There exists $C > 0$ such that for all $n \geq 1$ and all $x_1, ..., x_n \in X$, one has that

$$\int_0^1 \left\| \sum_{k=1}^n x_k r_k(t) \right\| \mathrm{d}t \leq C \left(\sum_{k=1}^n \|x_k\|^p \right)^{\frac{1}{p}}$$

where $(r_k)_k$ is the sequence of Rademacher functions. Let $n \geq 1$ and take $x_1, ..., x_n \in B_X$. We can deduce that

$$\frac{1}{n} \int_0^1 \left\| \sum_{k=1}^n x_k r_k(t) \right\| \mathrm{d}t \leq C n^{\frac{1}{p}-1}$$

and then there exist signs $\varepsilon_1, ..., \varepsilon_n$ such that

$$\frac{1}{n} \left\| \sum_{k=1}^n \varepsilon_k x_k \right\| \leq C n^{\frac{1}{p}-1}.$$

So (iv) of the previous theorem holds with $c_m = C m^{\frac{1}{p}-1}$. Using the same techniques as in the previous proofs, we can deduce that (vi) of the previous theorem holds with $c'_m \leq B m^{\frac{1}{p}-1}$ for some $B > 0$. It follows that (iii) of Proposition 5.3.3 holds with $c_m = c'_m \leq B m^{\frac{1}{p}-1}$. The proof of Proposition 5.3.3 allows us to conclude that X has the strong p-WABS and then the strong p-WBS (see remark 5.3.8). □

5.4 The uniform ABS

Definition 5.4.1. Let X be a Banach space and let $k \geq 2$. We say that X has the *alternate-(A_k)* if there exists $\theta \in (0, 1)$ such that for all $(x_n)_n \subset B_X$, there exist $p_1 < p_2 < ... < p_k$ such that

$$\frac{1}{k} \left\| \sum_{i=1}^k (-1)^i x_{p_i} \right\| \leq \theta.$$

We say that X has the alternate-(A_∞) if X has the alternate-(A_k) for some $k \geq 2$.

Definition 5.4.2. Let X be a Banach space. We say that X has the *uniform alternate Banach-Saks property* (in short uniform ABS) if there exists a sequence $(a_n)_n$ of real positive numbers such that $a_n \to 0$ and for all $(x_n)_n \subset B_X$, there is a subsequence $(x'_n)_n$ of $(x_n)_n$ such that

$$\frac{1}{m} \left\| \sum_{k=1}^m (-1)^k x'_k \right\| \leq a_m$$

for all $m \geq 1$.

Remark 5.4.3. The same observation as in Remark 5.3.4 holds for the uniform ABS.

It is clear that a reflexive space with the uniform WABS has the uniform ABS. However, we do not have an answer to the following question.

Problem 5.4.4. Does the uniform ABS imply reflexivity?

Proposition 5.4.5. *The uniform BS implies the uniform ABS.*

Proof. Let X be a Banach space with uniform BS. It follows that X is reflexive and has the uniform WABS. By Theorem 5.3.6, it follows that X has the uniform WABS. Since X is also reflexive, it has the uniform ABS. $\square$

Proposition 5.4.6. *Let X a Banach space. The following assertions are equivalent:*

(i) X has the uniform ABS;

(ii) there exists a sequence $(b_n)_n$ of real positive numbers such that $b_n \to 0$ and for all $(x_n)_n \subset B_X$, there is a subsequence $(x'_n)_n$ of $(x_n)_n$ such that for every subsequence $(x''_n)_n$ of $(x'_n)_n$;

$$\frac{1}{m}\left\|\sum_{k=1}^{m}(-1)^k x''_k\right\| \leq b_m$$

for all $m \geq 1$.

(iii) there exists a sequence $(c_n)_n$ of real positive numbers such that $c_n \to 0$ and for all $(x_n)_n \subset B_X$ and all $m \geq 1$, there exists $p_1 < \dots < p_m$ such that

$$\frac{1}{m}\left\|\sum_{i=1}^{m}(-1)^i x_{p_i}\right\| \leq c_m.$$

Proof. The proof is similar to that of Proposition 5.2.5, even easier since we do not have to introduce y_n. $\square$

Theorem 5.4.7. *The uniform ABS and the alternate-(A_∞) are equivalent.*

Proof. We only prove the non-trivial implication. Suppose that there exists $N \geq 2$ such that X has the alternate-(A_N). Take $\delta > 0$ as in the definition of alternate-(A_N). We will follow Kakutani's method. Let $(x_n)_n \subset B_X$. Let $(m_n)_n$ be a increasing sequence such that $m_1 \geq 2$ and for $n \geq 0$

$$\frac{1}{N}\left\|\sum_{i=1}^{N}(-1)^i x_{m_{nN+i}}\right\| \leq \delta.$$

For $n \geq 0$, define $x_n^1 = \frac{1}{N}\sum_{i=1}^{N}(-1)^i x_{m_{nN+i}}$. Since $\|x_n^1\| \leq \delta$, there exists a increasing sequence $(m_n(1))_n$ with $m_1(1) \geq 2$ such that for $n \geq 0$

$$\frac{1}{N}\left\|\sum_{i=1}^{N}(-1)^i x_{m_{nN+i}(1)}\right\| \leq \delta^2.$$

For $n \geq 0$, define $x_n^2 = \frac{1}{N}\sum_{i=1}^{N}(-1)^i x_{m_{nN+i}(1)}$. Following by induction, we construct sequence $(m_n(p))_n$ and $(x_n^p)_n$ for all p such that $m_1(p) \geq 2$, $\|x_n^p\| \leq \delta^p$ and such that

$$x_n^{p+1} = \frac{1}{N}\sum_{i=1}^{N}(-1)^i x^p_{m_{nN+i}(p)}.$$

By construction, the sequence $(x_1^p)_p$ is a bloc sequence of $(x_n)_n$ such that each term x_1^p is the arithmetic mean of N^p terms $\pm x_n$. It follows that x_1^p can be written

$$x_p^1 = \frac{1}{N^p}\sum_{i=1}^{N^p}(-1)^i x_{l_i(p)}$$

such that

$$1 < l_1(1) < ... < l_N(1) < l_1(2) < ... < l_{N^2}(2) < l_1(3) < ... < l_{N^3}(3) < ...$$

We define $n_1 = 1$, $n_2 = l_1(1)$, ... , $n_{N+1} = l_N(1)$, $n_{N+2} = l_1(2)$ and so on. Let $k \geq 1$ and define $q = E\left(\frac{\ln(k)}{\ln(N)}\right)$ and $r = E\left(\frac{k}{N^q}\right)$. Note that $rN^q \leq k \leq (r+1)N^q$. It follows that

$$\begin{aligned}\left\|\sum_{i=1}^{k}(-1)^i x_{n_i}\right\| &\leq \|(-1)^1 x_{n_1} + ... + (-1)^{N^q-1} x_{n_{N^q-1}}\| \\ &+ \sum_{j=2}^{r}\|(-1)^{(j-1)N^q} x_{n_{(j-1)N^q}} + ... + (-1)^{jN^q} x_{n_{jN^q}}\| \\ &+ \|(-1)^{rN^q} x_{n_{rN^q}} + ... + (-1)^k x_{n_k}\| \\ &\leq (N^q - 1) + (r-1)N^q\delta^q + N^q.\end{aligned}$$

It follows that for all $k \geq 1$

$$\frac{1}{k}\left\|\sum_{i=1}^{k}(-1)^i x_{n_i}\right\| \leq a_k := \frac{N^{q_k}}{k} - \frac{1}{k} + (r_k - 1)\frac{N^{q_k}\delta^{q_k}}{k} + \frac{N^{q_k}}{k}$$

where $q_k = E\left(\frac{\ln(k)}{\ln(N)}\right)$ and $r_k = E\left(\frac{k}{N^q}\right)$. Note that

$$0 \leq \frac{N^{q_k}}{k} \leq \frac{N^{\frac{\ln(k)}{\ln(N)}}}{k} = \frac{\ln(k)}{k} \to 0$$

and

$$0 \leq r_k\frac{N^{q_k}\delta^{q_k}}{k} \leq \delta^{q_k} \to 0.$$

It follows that $a_k \to 0$ and the proof is complete since a_k only depends on k, N and δ. □

Remark 5.4.8. The previous results imply that weak alternate-(A_∞) and alternate-(A_∞) are preserved preserved under isomorphism.

We conclude this section with an application to the symmetric Kottman constant. We recall the following definition:

Definition 5.4.9. Let X be a Banach space. The *symmetric Kottman constant* of X is defined by

$$K^s(X) = \sup\{\delta > 0 \mid \exists (x_n)_n \subset S_X \ \|x_n \pm x_m\| \geq \delta \ \ \textit{whenever } n \neq m\}.$$

We also define the *isomorphic symmetric Kottman constant* by

$$\overline{K^s(X)} = \{K^s(Y) \mid Y \textit{ is isomorphic to } X\}.$$

Corollary 5.4.10. *Let X be a Banach space without the uniform WABS. Then $\overline{K^s(X)} = \{2\}$.*

Proof. Since the uniform WBS is stable under isomorphism, it is enough to prove that $K^s(X) = 2$. By (vii) of the previous theorem, for all $\theta \in (0,1)$, there exists $(x_n)_n \subset B_X$ such that $x_n \xrightarrow{w} 0$ and for all $n < m$

$$\left\| \frac{x_n \pm x_m}{2} \right\| > \theta.$$

We deduce that $K^s(X) = 2$. □

In [90], the authors proved that if X fails to have ABS, then $\overline{K^s(X)} = \{2\}$ (see Corollary 5.6 in [90]). The previous corollary generalizes this result. In [130], the authors asked if any reflexive space admits a renorming such that $K_s(X) < 2$. In [90], the authors answered this question negatively by proving that the Tsirelson space T fulfills that $\overline{K^s(T)} = \{2\}$. Thanks to the previous corollary, we obtain the following generalization:

Corollary 5.4.11. *Let X be a reflexive Banach space without the uniform ABS. Then* $\overline{K^s(X)} = \{2\}$.

Chapter 6

Extremal structure in ultraproducts

Before presenting the main results, we start recalling some definitions. Let C be a bounded convex subset of a Banach space X. The set of extreme points of C is denoted by $\mathrm{ext}(C)$. Recall that a point $x \in C$ is *strongly extreme* if for all sequences $(y_n)_n, (z_n)_n \subset C$ such that

$$\left\| x - \frac{y_n + z_n}{2} \right\| \underset{n}{\to} 0,$$

one has that $\|y_n - z_n\| \underset{n}{\to} 0$. The set of strongly extreme points of C is denoted by $\text{str-ext}(C)$.

A *slice* of C is a subset of C defined by

$$S(C, x^*, \alpha) = \{x \in C \mid x^*(x) > \sup_C x^* - \alpha\}$$

where $x^* \in X^*$ and $\alpha > 0$.

Let Z be a subspace of X^*. A point $x \in C$ is a *Z-denting point* if for all $\varepsilon > 0$, there exist $x^* \in Z$ and $\alpha > 0$ such that $x \in S(C, x^*, \alpha)$ and $\mathrm{diam}(S(C, x^*, \alpha)) < \varepsilon$. We denote it by $x \in \mathrm{dent}_Z(C)$. A X^*-denting point is simply called denting and we write $\mathrm{dent}(C) = \mathrm{dent}_{X^*}(C)$.

A point $x \in C$ is a *Z-exposed point* if there exists $x^* \in Z$ such that $x^*(x) > x^*(y)$ for all $y \in C \setminus \{x\}$. We also said that x^* exposes x in C. The set of Z-exposed point of C is denoted by $\exp_Z(C)$. A point $x \in C$ is said *Z-strongly exposed* if there exists $x^* \in Z$ exposing x and such that for all sequences $(x_n)_n \subset C$ such that $x^*(x_n) \underset{n}{\to} x^*(x)$, it follows that $x_n \underset{n}{\to} x$. In this case, we write $x \in \text{str-exp}_Z(C)$. It is easy to show that $x \in C$ is Z-strongly exposed if there exists $x^* \in Z$ such that $x \in S(C, x^*, \alpha)$ for all $\alpha > 0$ and $\lim_{\alpha \to 0^+} \mathrm{diam}(S(C, x^*, \alpha)) = 0$. As before, an X^*-(strongly) exposed point is said (strongly) exposed and we write $\exp(C) = \exp_{X^*}(C)$ and $\text{str-exp}(C) = \text{str-exp}_{X^*}(C)$. Obviously,

$$\text{str-exp}(C) \subset \mathrm{dent}(C) \subset \text{str-ext}(C) \subset \mathrm{ext}(C).$$

6.1 Extreme and strongly extreme points

We will begin by exploring the extreme points of a set $C_{\mathcal{U}}$ for a given bounded closed and convex subset C of X. Let us start with the following characterization of extreme points of $C_{\mathcal{U}}$.

Theorem 6.1.1. *Let C be a bounded closed convex subset of a Banach space X and $\mathcal{U}$ be a free ultrafilter on an infinite set I. Let $(x_i)_{\mathcal{U}} \in C_{\mathcal{U}}$. The following assertions are equivalent:*

(i) $(x_i)_\mathcal{U} \in \mathrm{ext}(C_\mathcal{U})$;

(ii) *for any* $(y_i)_\mathcal{U}, (z_i)_\mathcal{U} \in C_\mathcal{U}$ *so that* $\lim_\mathcal{U} \|x_i - \frac{y_i+z_i}{2}\| = 0$, *it follows that* $\lim_\mathcal{U} \|x_i - y_i\| = 0$ *and* $\lim_\mathcal{U} \|x_i - z_i\| = 0$.

Proof. Given $(y_i)_\mathcal{U}, (z_i)_\mathcal{U} \in C_\mathcal{U}$, notice that $\lim_\mathcal{U} \|x_i - \frac{y_i+z_i}{2}\| = 0$ if, and only if, $(x_i)_\mathcal{U} = \frac{1}{2}((y_i)_\mathcal{U} + (z_i)_\mathcal{U})$ in $C_\mathcal{U}$. On the other hand,

$$(x_i)_\mathcal{U} = (y_i)_\mathcal{U} \Leftrightarrow (x_i - y_i)_\mathcal{U} = (0)_\mathcal{U} \Leftrightarrow \lim_{\mathcal{U}} \|x_i - y_i\| = 0.$$

This gives the characterisation. □

Given $(x_i)_{i\in I} \in C^I$, it is not difficult to realise that being a (strongly) extreme point of $C_\mathcal{U}$ depends on the considered ultrafilter $\mathcal{U}$ on I. For instance, just take $C = [-1, 1]^2 \subset \mathbb{R}^2$ and $x_n = (1, 0)$ if n is odd and $x_n = (1, 1)$ if n is even. It is easy to find free ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on $\mathbb{N}$ such that $(x_n)_\mathcal{U} \in \mathrm{ext}(C_\mathcal{U})$ but $(x_n)_\mathcal{V} \notin \mathrm{ext}(C_\mathcal{V})$. Our next goal is to characterise when $(x_i)_\mathcal{U}$ is a strongly extreme point *for every* free ultrafilter $\mathcal{U}$ in terms of the space $c_0(I)$. Note that this result will be improved in Theorem 6.1.7 below.

Recall that $c_0(I)$ is the set of those bounded functions $f\colon I \longrightarrow \mathbb{R}$ for which $\{i \in I : |f(i)| \geq \varepsilon\}$ is finite for every $\varepsilon > 0$.

Lemma 6.1.2. *Let I be an infinite set. Let $f : I \to \mathbb{R}$ be a bounded function. The following assertions are equivalent:*

(i) $f \in c_0(I)$;

(ii) $\lim_\mathcal{U} f(i) = 0$ *for all free ultrafilter* $\mathcal{U}$ *on* I.

Proof. Suppose that (i) holds and let $\varepsilon > 0$. Let $\mathcal{U}$ be any free ultrafilter on I, thus it contains all the sets with finite complement. It follows that $\{i \in I : |f(i)| \leq \varepsilon\} \in \mathcal{U}$, which proves that $\lim_\mathcal{U} f(i) = 0$.

Now suppose that $f \notin c_0(I)$. Then there exists $\varepsilon > 0$ such that $J := \{i \in I : |f(i)| > \varepsilon\}$ is infinite. Let $\mathcal{U}$ be a free ultrafilter on I with $J \in \mathcal{U}$. Since $J \in \mathcal{U}$, it is clear that $\lim_\mathcal{U} f(i) \geq \varepsilon > 0$. □

Theorem 6.1.3. *Let C be a bounded convex subset of a Banach space X. Let I be a infinite set and $(x_i)_{i\in I} \in C^I$. The following assertions are equivalent:*

(i) $(x_i)_\mathcal{U} \in \mathrm{ext}(C_\mathcal{U})$ *for every free ultrafilter* $\mathcal{U}$ *on* I;

(ii) *for any* $(y_i)_{i\in I}, (z_i)_{i\in I} \in C^I$ *so that* $\left(\left\|x_i - \frac{y_i+z_i}{2}\right\|\right)_{i\in I} \in c_0(I)$, *it follows that* $(\|x_i - y_i\|)_{i\in I} \in c_0(I)$ *and* $(\|x_i - z_i\|)_{i\in I} \in c_0(I)$.

Proof. For (i)⇒(ii) assume that (ii) does not hold. Then there exist $\varepsilon_0 > 0$ and elements $(y_i)_{i\in I}, (z_i)_{i\in I} \in C^I$ so that $(\|x_i - \frac{y_i+z_i}{2}\|)_{i\in I} \in c_0(I)$ and the set $J := \{i \in I : \|x_i - y_i\| \geq \varepsilon_0\}$ is infinite. Let $\mathcal{U}$ be a free ultrafilter on I with $J \in \mathcal{U}$. Note that $\lim_\mathcal{U} \|x_i - \frac{y_i+z_i}{2}\| = 0$ by Lemma 6.1.2, that is, $(x_i)_\mathcal{U} = \frac{(y_i)_\mathcal{U}+(z_i)_\mathcal{U}}{2}$. Moreover, it is not difficult to prove that $\lim_\mathcal{U} \|x_i - y_i\| \geq \varepsilon_0$, so by Theorem 6.1.1 we have that $(x_i)_\mathcal{U}$ is not an extreme point of $C_\mathcal{U}$.

For (ii) $\Longrightarrow$ (i) assume that (i) does not hold, that is, there exists a free ultrafilter $\mathcal{U}$ over I so that $(x_i)_\mathcal{U}$ is not an extreme point of $C_\mathcal{U}$. By Theorem 6.1.1 there are $(y_i)_{i\in I}, (z_i)_{i\in I} \in C^I$ so that $\lim_\mathcal{U} \|x_i - \frac{y_i+z_i}{2}\| = 0$ but $\lim_\mathcal{U} \|x_i - y_i\| > \varepsilon_0$ for certain $\varepsilon_0 > 0$. This implies that the set $B := \{i \in I : \|x_i - y_i\| > \varepsilon_0\}$ belongs to $\mathcal{U}$. Now, construct inductively a sequence

$$i_n \in \left\{i \in I : \left\|x_i - \frac{y_i + z_i}{2}\right\| < \frac{1}{n}\right\} \cap B \setminus \{i_1, \ldots, i_{n-1}\}$$

(the intersection is non-emtpy because the previous set actually belongs to $\mathcal{U}$). Define $(y_i')_{i\in I}$ and $(z_i')_{i\in I}$ by $y_i' = z_i' = x_i$ if $i \notin \{i_n : n \in \mathbb{N}\}$ and $y_{i_n}' = y_{i_n}$ and $z_{i_n}' = z_{i_n}$ for every $n \in \mathbb{N}$. We have that $(\|x_i - \frac{y_i'+z_i'}{2}\|)_{i\in I} \in c_0(I)$ because clearly

$$\left\{ i \in I : \left\| x_i - \frac{y_i' + z_i'}{2} \right\| \geq \frac{1}{j} \right\} \subseteq \{i_1, \ldots, i_{j-1}\} \quad \forall j \in \mathbb{N}.$$

On the other hand, we clearly have $\{i_n : n \in \mathbb{N}\} \subseteq \{i \in I : \|x_i' - y_i'\| \geq \varepsilon_0\}$, which implies that $(\|x_i' - y_i'\|)_{i\in I} \notin c_0(I)$, and then (ii) does not hold. This completes the proof of $(ii) \implies (i)$. □

Our aim is now to study when $(x_i)_{\mathcal{U}}$ is a strongly extreme point of $C_{\mathcal{U}}$. Let us start with the following result.

Proposition 6.1.4. *Let C be a bounded convex set of a Banach space X and $\mathcal{U}$ be a free ultrafilter on an infinite set I. Let $(x_i)_{i\in I} \in C^I$.*

(a) Assume that for every countable subset $J = \{i_n : n \in \mathbb{N}\} \subset I$ there is a free ultrafilter $\mathcal{V}$ on J such that $(x_{i_n})_{\mathcal{V}} \in \operatorname{ext}(C^{\mathcal{V}})$. Then $(x_i)_{\mathcal{U}} \in \operatorname{str-ext}(C_{\mathcal{U}})$.

(b) Assume that $I = \mathbb{N}$ and $(x_n)_n \subset C$ is such that for all subsequences $(x_n')_n$ of $(x_n)_n$ there is a free ultrafilter $\mathcal{V}$ on $\mathbb{N}$ such that $(x_n')_{\mathcal{V}} \in \operatorname{ext}(C_{\mathcal{V}})$. Then $(x_n)_{\mathcal{U}} \in \operatorname{str-ext}(C_{\mathcal{U}})$.

Proof. (a) Suppose that $(x_i)_{\mathcal{U}} \notin \operatorname{str-ext}(C_{\mathcal{U}})$. Then there exist two sequences $((y_i^n)_{\mathcal{U},i})_n, ((z_i^n)_{\mathcal{U},i})_n \subset C_{\mathcal{U}}$ such that

$$\frac{(y_i^n)_{\mathcal{U},i} + (z_i^n)_{\mathcal{U},i}}{2} \xrightarrow[n]{} (x_i)_{\mathcal{U}}$$

but $\|(y_i^n)_{\mathcal{U},n} - (z_i^n)_{\mathcal{U},i}\| > \varepsilon$ for all $n \in \mathbb{N}$ and for some $\varepsilon > 0$. Without loss of generality, we can suppose that $\|y_i^n - z_i^n\| > \varepsilon$ for all $i \in I$ and $n \in \mathbb{N}$.

This means that for every $k \in \mathbb{N}$ there exists n_k such that for all $n \geq n_k$ we have

$$\|(x_i)_{\mathcal{U}} - \frac{(y_i^n)_{\mathcal{U}} + (z_i^n)_{\mathcal{U}}}{2}\| < \frac{1}{k},$$

that is, the set

$$A_{n,k} := \left\{ i \in I : \|x_i - \frac{y_i^n + z_i^n}{2}\| < \frac{1}{k} \right\}$$

belongs to $\mathcal{U}$ (in particular, it is infinite) for every k and every $n \geq n_k$.

Thus, we may choose $i_k \in A_{n_k,k} \setminus \{i_1, \ldots, i_{k-1}\}$ inductively. Then $J = \{i_k : k \in \mathbb{N}\}$ is countable and

$$\|x_{i_k} - \frac{y_{i_k}^{n_k} + z_{i_k}^{n_k}}{2}\| < \frac{1}{k} \qquad \forall k \in \mathbb{N}.$$

This implies that $\left(\|x_{i_k} - \frac{y_{i_k}^{n_k} + z_{i_k}^{n_k}}{2}\| \right)_{n\in\mathbb{N}} \in c_0(\mathbb{N})$. So, given any free ultrafilter $\mathcal{V}$ on $\mathbb{N}$, we have

$$(x_{i_k})_{\mathcal{V}} = \frac{(y_{i_k}^{n_k})_{\mathcal{V}} + (z_{i_k}^{n_k})_{\mathcal{V}}}{2}$$

by Lemma 6.1.2. Since $\|y_{i_k}^{n_k} - z_{i_k}^{n_k}\| \geq \varepsilon$ for all k, it follows that $(x_{i_k})_{\mathcal{V}} \notin \operatorname{ext}(C_{\mathcal{V}})$, which is a contradiction.

(b) Just mimic the proof of (a) noting that the sequence $(i_k)_k$ can be chosen to be strictly increasing. □

In [159, Theorem 2.1] it was proved that, given $x \in S_X$ and a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$, $\mathcal{J}(x)$ is an extreme point of $B_{X_{\mathcal{U}}}$ if, and only if, x is strongly extreme. We can now improve that result.

Theorem 6.1.5. *Let C be a bounded convex subset of a Banach space X, $\mathcal{U}$ be a CI ultrafilter on an infinite set I and let $x \in C$. The following assertions are equivalent:*

(i) $\mathcal{J}(x) \in \text{str-ext}(C_{\mathcal{U}})$;

(ii) $\mathcal{J}(x) \in \text{ext}(C_{\mathcal{U}})$;

(iii) $x \in \text{str-ext}(C)$.

Proof. Obviously, we have that $(i) \implies (ii)$. Assume now that $x \in \text{str-ext}(C)$. Given a countable subset $J = \{i_n : n \in \mathbb{N}\} \subset I$ and a free ultrafilter $\mathcal{V}$ on $\mathbb{N}$, Theorem 2.1 in [159] shows that $j_{\mathcal{V}}(x) \in \text{ext}(C^{\mathcal{V}})$ (here $j_{\mathcal{V}}$ denotes the canonical embedding of C into $C^{\mathcal{V}}$). Thus, the hypotheses of Proposition 6.1.4 are satisfied, so we get that $\mathcal{J}(x) \in \text{str-ext}(C_{\mathcal{U}})$. That is, (iii)$\Rightarrow$(i).

Finally, assume that $\mathcal{J}(x) \in \text{ext}(C_{\mathcal{U}})$. If $x \notin \text{str-ext}(C)$, then there sequences $(y_n)_n$, $(z_n)_n \subset C$ and a number $\varepsilon > 0$ such that

$$\lim_{n\to\infty} \|x - \frac{y_n + z_n}{2}\| = 0$$

and $\|x - y_n\| \geq \varepsilon$ for all $n \in \mathbb{N}$. By extracting a subsequence, we may assume that $\|x - \frac{y_n+z_n}{2}\| < \frac{1}{n}$ for all $n \in \mathbb{N}$. Now, let $(I_n)_n \subset \mathcal{U}$ be a sequence of sets with $I_1 = I$, $I_{n+1} \subsetneq I_n$ for all n and $\bigcap_{n\in\mathbb{N}} I_n = \emptyset$, and define $(y_i')_{i\in I}$, $(z_i')_{i\in I}$ by $y_i' = y_n$ and $z_i' = z_n$ if $i \in I_n \setminus I_{n+1}$. Note that

$$I_n \subset \left\{ i \in I : \|x - \frac{y_i' + z_i'}{2}\| \leq \frac{1}{n} \right\} \qquad \forall n \in \mathbb{N}.$$

Since $I_n \in \mathcal{U}$, we get

$$\mathcal{J}(x) = \frac{(y_i')_{\mathcal{U}} + (z_i')_{\mathcal{U}}}{2}.$$

As $\mathcal{J}(x)$ is extreme, this implies $\mathcal{J}(x) = (y_i')_{\mathcal{U}}$. Since $\|x - y_i'\| \geq \varepsilon$ for all $i \in I$, this is a contradiction. Therefore, $x \in \text{str-ext}(C)$, as desired. This shows that (ii)$\Rightarrow$(iii). $\square$

Our aim will be to determine when $(x_i)_{\mathcal{U}}$ is a strongly extreme point of $C_{\mathcal{U}}$ which, thanks to the previous theorem, is equivalent to the fact that $j_{\mathcal{V}}((x_i)_{\mathcal{U}})$ is an extreme point of $(C_{\mathcal{U}})_{\mathcal{V}}$ in the space $(X_{\mathcal{U}})_{\mathcal{V}}$. Remember that $(X_{\mathcal{U}})_{\mathcal{V}}$ is isometric to $X_{\mathcal{U}\times\mathcal{V}}$ (where $\mathcal{U}\times\mathcal{V}$ is the product ultrafilter defined in Section 1.1). The isometry is $T\colon (X_{\mathcal{U}})_{\mathcal{V}} \to X_{\mathcal{U}\times\mathcal{V}}$ defined by $T((x_{i,j})_{\mathcal{U},i})_{\mathcal{V},j}) = (x_{i,j})_{\mathcal{U}\times\mathcal{V}}$ (see Proposition 1.2.7). Moreover, it is clear that $T((A_{\mathcal{U}})_{\mathcal{V}}) = A_{\mathcal{U}\times\mathcal{V}}$ for all bounded sets $A \subset X$.

Theorem 6.1.6. *Let C be a bounded convex subset of a Banach space X, $\mathcal{U}$ be a CI ultrafilter on an infinite set I. Then* $\text{ext}(C_{\mathcal{U}}) = \text{str-ext}(C_{\mathcal{U}})$.

Proof. Let $(U_n)_n \subset \mathcal{U}$ be a strictly decreasing sequence of sets such that $\bigcap_{n>0} U_n = \emptyset$. Let $(x_i)_{\mathcal{U}} \in \text{ext}(C_{\mathcal{U}})$. We need to show that $(x_i)_{\mathcal{U}} \in \text{str-ext}(C_{\mathcal{U}})$. By Theorem 6.1.5, it is enough to prove that $\mathcal{J}((x_i)_{\mathcal{U}}) \in \text{ext}(C_{\mathcal{U}\times\mathcal{U}})$. Suppose that it is not true, then there exist $(y_{i,j})_{\mathcal{U}\times\mathcal{U}}, (z_{i,j})_{\mathcal{U}\times\mathcal{U}}$ and $\varepsilon_0 > 0$ such that $\|(y_{i,j}) - (z_{i,j})\| = \lim_{\mathcal{U}\times\mathcal{U}} \|y_{i,j} - z_{i,j}\| > \varepsilon_0$ and

$$\mathcal{J}((x_i)_{\mathcal{U}}) = \frac{(y_{i,j})_{\mathcal{U}\times\mathcal{U}} + (z_{i,j})_{\mathcal{U}\times\mathcal{U}}}{2}.$$

Up to changing the definition of $(y_{i,j})$ and $(z_{i,j})$ out of the set $\{(i,j) \in I^2 \mid \|y_{i,j} - z_{i,j}\| > \varepsilon_0\} \in \mathcal{U}\times\mathcal{U}$, we can assume that $\|y_{i,j} - z_{i,j}\| > \varepsilon_0$ holds for every $i, j \in I$.

It follows that

$$\left\{ (i,j) \in I^2 \mid \left\| x_i - \frac{y_{i,j} + z_{i,j}}{2} \right\| < \frac{1}{n} \right\} \in \mathcal{U}\times\mathcal{U}$$

for all $n > 0$, that is

$$J_n := \left\{ j \in I \mid \left\{ i \in I \mid \left\| x_i - \frac{y_{i,j} + z_{i,j}}{2} \right\| < \frac{1}{n} \right\} \in \mathcal{U} \right\} \in \mathcal{U}.$$

For $j \in I$ and $n > 0$, define $I_{n,j} = \left\{ i \in I \mid \left\| x_i - \frac{y_{i,j}+z_{i,j}}{2} \right\| < \frac{1}{n} \right\}$ and note that $I_{n,j} \in \mathcal{U}$ if $j \in J_n$. Since $J_1 \in \mathcal{U}$, we have that $J_1 \neq \emptyset$ and then let $j_1 \in J_1$. Define $I'_1 = I_{1,j_1} \cap U_1 \in \mathcal{U}$. Now choose $j_2 \in J_2$ and define $I'_2 = I_{2,j_2} \cap I'_1 \cap U_2 \in \mathcal{U}$. Following by induction, we define $I'_n = I_{n,j_n} \cap I'_{n-1} \cap U_n \in \mathcal{U}$ where $j_n \in J_n$. For $i \in I$, define $y_i = y_{i,j_n}$ if $i \in I'_n \setminus I'_{n+1}$ for some $n > 0$ and $y_i = x$ otherwise where x is an arbitrary point of C. We define z_i in the same way. Note that $\|y_i - z_i\| > \varepsilon_0$ for all $i \in I'_1$. In fact, if $i \in I'_1$, then there exists $n \geq 0$ such that $i \in I'_n \setminus I'_{n+1}$ (since $\bigcap_{n>0} I'_n = \emptyset$) and $\|y_i - z_i\| = \|y_{i,j_n} - z_{i,j_n}\| > \varepsilon_0$. We deduce that $\|(y_i)_\mathcal{U} - (z_i)_\mathcal{U}\| \geq \varepsilon_0$. Let's show that $(x_i)_\mathcal{U} = \frac{(y_i)_\mathcal{U} + (z_i)_\mathcal{U}}{2}$, which will contradict the extremality of $(x_i)_\mathcal{U}$ and will conclude the proof. Let $\varepsilon > 0$ and take n_0 such that $\frac{1}{n_0} < \varepsilon$. We are going to show that $I'_{n_0} \subset \left\{ i \in I \mid \left\| x_i - \frac{y_i+z_i}{2} \right\| < \varepsilon \right\}$, which implies that the last set belongs to $\mathcal{U}$. So let $i \in I'_{n_0}$. There exists $n \geq n_0$ such that $i \in I'_n \setminus I'_{n+1}$. Then we have that

$$\left\| x_i - \frac{y_i + z_i}{2} \right\| = \left\| x_i - \frac{y_{i,j_n} + z_{i,j_n}}{2} \right\| < \frac{1}{n} \leq \frac{1}{n_0} < \varepsilon$$

and the proof is complete. □

Now, we are able to extend Theorem 6.1.3 including strongly extreme points.

Theorem 6.1.7. *Let C be a bounded convex set of a Banach space X and I be an infinite set. Let $(x_i)_{i\in I} \in C^I$. The following assertions are equivalent:*

(i) $(x_i)_\mathcal{U} \in \text{str-ext}(C_\mathcal{U})$ *for every free ultrafilter $\mathcal{U}$ on I;*

(ii) $(x_i)_\mathcal{U} \in \text{ext}(C_\mathcal{U})$ *for every free ultrafilter $\mathcal{U}$ on I;*

(iii) for any $(y_i)_{i\in I}, (z_i)_{i\in I} \in C^I$ so that $\left(\left\| x_i - \frac{y_i+z_i}{2} \right\|\right)_{i\in I} \in c_0(I)$, it follows that $(\|x_i - y_i\|)_{i\in I} \in c_0(I)$ and $(\|x_i - z_i\|)_{i\in I} \in c_0(I)$;

(iv) $(x_j)_\mathcal{V} \in \text{ext}(C_\mathcal{V})$ *holds for every countable subset $J \subseteq I$ and every free ultrafilter $\mathcal{V}$ over J.*

Proof. (iv) $\Longrightarrow$ (i) follows from (a) of Proposition 6.1.4. Moreover, (i) $\Longrightarrow$ (ii) is obvious, whereas (ii)$\Leftrightarrow$(iii) is Theorem 6.1.3.

Finally, let us prove (iii)$\Rightarrow$(iv), for which we take a countable subset $J \subseteq I$ and, in order to prove (iv), by Theorem 6.1.3, take $(y_j), (z_j) \in C^J$ so that $(\|x_j - \frac{y_j+z_j}{2}\|) \in c_0(J)$, and let us prove that $(\|x_j - y_j\|) \in c_0(J)$. Define $y_i := x_i$ and $z_i := x_i$ if $i \in I \setminus J$, and it is obvious that $(\|x_i - \frac{y_i+z_i}{2}\|) \in c_0(I)$ since $(\|x_j - \frac{y_j+z_j}{2}\|) \in c_0(J)$. Using (iii) we get that $(\|x_i - y_i\|) \in c_0(I)$ (analogously $(\|x_i - z_i\|) \in c_0(I)$). From here it is obvious that $(\|x_j - y_j\|) \in c_0(J)$ as desired. □

Remark 6.1.8. Note that we have proved in Theorem 6.1.6 that $\text{ext}(C_\mathcal{U}) = \text{str-ext}(C_\mathcal{U})$ holds when $\mathcal{U}$ is a CI ultrafilter. The previous theorem shows that the hypothesis of countably incompleteness can be removed if we require $(x_i)_\mathcal{V}$ being extreme for every free ultrafilter $\mathcal{V}$.

If $I = \mathbb{N}$, the previous theorem can be stated in terms of convergent sequences:

Corollary 6.1.9. *Let C be a bounded convex subset of a Banach space X. Let $(x_n)_n \subset C$. The following assertions are equivalent:*

(i) $(x_n)_\mathcal{U} \in \text{str-ext}(C_\mathcal{U})$ *for every free ultrafilter $\mathcal{U}$ on $\mathbb{N}$;*

(ii) $(x_n)_{\mathcal{U}} \in \operatorname{ext}(C_{\mathcal{U}})$ *for every free ultrafilter* $\mathcal{U}$ *on* $\mathbb{N}$*;*

(iii) For every pair of sequences $(y_n)_n, (z_n)_n$ *in* C*, if* $\|x_n - \frac{y_n+z_n}{2}\| \to 0$ *then* $\|x_n - y_n\| \to 0$ *and* $\|x_n - z_n\| \to 0$*;*

(iv) $(x'_n)_n \in \operatorname{ext}(C_{\mathcal{U}})$ *for every subsequence* $(x'_n)_n$ *of* $(x_n)_n$ *and every free ultrafilter* $\mathcal{U}$ *on* $\mathbb{N}$*.*

Proof. $(i) \iff (ii) \iff (iii) \implies (iv)$ follows directly from the previous theorem. $(iv) \implies (i)$ is easily deduced from (b) of Proposition 6.1.4. □

6.2 Denting points

In this subsection we will study the denting points in ultraproducts. To this end, let us consider the following notion.

Definition 6.2.1. Let C be a bounded convex subset of a Banach space X. A subset $\{x_i\}_{i\in I} \subset C$ is said to be a *uniformly denting set* if for every $\varepsilon > 0$ there exists $\alpha_\varepsilon > 0$ with the following property: for every $i \in I$ there exists $x_i^* \in S_{X^*}$ so that

$$x_i \in S(C, x_i^*, \alpha_\varepsilon) \quad \text{and} \quad \operatorname{diam}(S(C, x_i^*, \alpha_\varepsilon)) < \varepsilon.$$

This definition should be compared with that of [160] and [159, P. 4] of a uniform notion of dentable set.

Now we have the following result.

Theorem 6.2.2. *Let* C *be a bounded convex subset of a Banach space* X*,* $\mathcal{U}$ *be a free ultrafilter on an infinite set* I *and* $\{x_i\}_{i\in I}$ *be a uniformly denting set in* C*. Then* $(x_i)_{\mathcal{U}} \in \operatorname{dent}_{(X^*)_{\mathcal{U}}}(C_{\mathcal{U}})$*.*

For the proof we need the following lemma.

Lemma 6.2.3. *Let* C *be a bounded convex subset of a Banach space* X*. Let* $x^* \in S_{X^*}$ *and* $\alpha > 0$*. Then* $\operatorname{diam}(S(C, x^*, \frac{3}{2}\alpha)) \leq 2 \operatorname{diam}(S(C, x^*, \alpha))$*.*

Proof. Let $y, z \in S(C, x^*, \frac{3}{2}\alpha)$, and let us estimate $\|y - z\|$. To this end, pick $x \in S(C, x^*, \alpha/2)$ and notice that, if we call $\lambda := \sup_C x^*$, we get

$$x^*\left(\frac{x+y}{2}\right) > \frac{\lambda - \frac{\alpha}{2} + \lambda - \frac{3\alpha}{2}}{2} = \lambda - \alpha,$$

so $\frac{x+y}{2} \in S(C, x^*, \alpha)$. Similarly $\frac{x+z}{2} \in S(C, x^*, \alpha)$. Consequently

$$\operatorname{diam}(S(C, x^*, \alpha)) \geq \left\| \frac{x+z}{2} - \frac{x+y}{2} \right\| = \frac{\|y - z\|}{2},$$

from where the result follows by the arbitrariness of y, z. □

Proof of Theorem 6.2.2. Pick $\varepsilon > 0$, and let us find a slice S of $C_{\mathcal{U}}$ containing $(x_i)_{\mathcal{U}}$ whose diameter is smaller than or equal to 2ε. To this end, since $\{x_i\}_{i\in I}$ is uniformly dentable we can find $\alpha > 0$ and $\{x_i^*\}_{i\in I} \subset S_{X^*}$ so that $x_i \in S(C, x_i^*, \alpha)$ and $\operatorname{diam}(S(C, x_i^*, \alpha)) < \varepsilon$. Note that

$$\langle (x_i^*)_{\mathcal{U}}, (x_i)_{\mathcal{U}} \rangle = \lim_{\mathcal{U}} x_i^*(x_i) \geq \lim_{\mathcal{U}} \sup_C x_i^* - \alpha,$$

so $(x_i)_\mathcal{U} \in S = S(C_\mathcal{U}, (x_i^*)_\mathcal{U}, \frac{3\alpha}{2})$. Now, in view of Lemma 6.2.3, it is enough to prove that $\operatorname{diam}(S(C_\mathcal{U}, (x_i^*)_\mathcal{U}, \alpha)) \leq \varepsilon$. In order to do so, pick $(y_i)_\mathcal{U}, (z_i)_\mathcal{U} \in S(C_\mathcal{U}, (x_i)_\mathcal{U}, \alpha)$. Since $\langle (x_i^*)_\mathcal{U}, (y_i)_\mathcal{U}\rangle > \sup_{C_\mathcal{U}} (x_i^*)_\mathcal{U} - \alpha$ and $\langle (x_i^*)_\mathcal{U}, (z_i)_\mathcal{U}\rangle > \sup_{C_\mathcal{U}(x_i^*)_\mathcal{U}} -\alpha$, we get that

$$\lim_\mathcal{U}(x_i^*(y_i) - \sup_C x_i^*) + \alpha > 0, \text{ and } \quad \lim_\mathcal{U}(x_i^*(z_i) - \sup_C x_i^*) + \alpha > 0.$$

Thus,

$$J := \{i \in I : \min\{x_i^*(y_i), x_i^*(z_i)\} > \sup_C x_i^* - \alpha\} \in \mathcal{U}.$$

Given $i \in J$, we get that $y_i, z_i \in S(C, x_i^*, \alpha)$, and so $\|y_i - z_i\| < \varepsilon$. Consequently,

$$J \subseteq L := \{i \in I : \|y_i - z_i\| < \varepsilon\},$$

so $L \in \mathcal{U}$. It is immediate to obtain that $\|(y_i)_\mathcal{U} - (z_i)_\mathcal{U}\| = \lim_\mathcal{U} \|y_i - z_i\| \leq \varepsilon$, from where $\operatorname{diam}(S(C_\mathcal{U}, (x_i^*)_\mathcal{U}, \alpha)) \leq \varepsilon$ and the proof is finished. □

In the particular case of points of the form $\mathcal{J}(x)$, we can say more.

Theorem 6.2.4. *Let C be a bounded convex subset of a Banach space X, $x \in C$ and $\mathcal{U}$ be a free ultrafilter on an infinite set I. Then $x \in \operatorname{dent}(C)$ if and only if $\mathcal{J}(x) \in \operatorname{dent}_{(X^*)_\mathcal{U}}(C_\mathcal{U})$.*

Proof. If $x \in \operatorname{dent}(A)$, then $\mathcal{J}(x) \in \operatorname{dent}_{(X^*)_\mathcal{U}}(C_\mathcal{U})$ by Theorem 6.2.2.

Now suppose $\mathcal{J}(x) \in \operatorname{dent}_{(X^*)_\mathcal{U}}(A)$. Let $\varepsilon > 0$. There exist $(x_i^*)_\mathcal{U} \in (X^*)_\mathcal{U}$ and $\alpha > 0$ such that $\operatorname{diam}(S(C_\mathcal{U}, (x_i^*)_\mathcal{U}, \alpha)) \leq \varepsilon$ and $\mathcal{J}(x) \in S(C_\mathcal{U}, (x_i^*)_\mathcal{U}, \alpha)$, i.e. $\lim_\mathcal{U} x_i^*(x) > \sup_{C_\mathcal{U}} (x_i^*)_\mathcal{U} - \alpha$. Let $\eta \in (0, \alpha)$ such that

$$\lim_\mathcal{U} x_i^*(x) > \sup_{C_\mathcal{U}} (x_i^*)_\mathcal{U} - \alpha + 2\eta.$$

Define

$$J_1 = \{j \in I \mid x_j^*(x) > \sup_{C_\mathcal{U}} (x_i^*)_\mathcal{U} - \alpha + 2\eta\} \in \mathcal{U}$$

and

$$J_2 = \{j \in I \mid \sup_{C_\mathcal{U}} (x_i^*)_\mathcal{U} > \sup_C x_j^* - \eta\} \in \mathcal{U}.$$

Note that $J_2 \in \mathcal{U}$ by Lemma 1.2.6. Furthermore, for all $j \in J := J_1 \cap J_2$, we have that $x \in S(C, x_j^*, \alpha - \eta)$. In fact, if $j \in J$, we have that

$$x_j^*(x) > \sup_{C_\mathcal{U}} (x_i^*)_\mathcal{U} - \alpha + 2\eta > \sup_C x_j^* - \alpha + \eta.$$

Now let us show that there exists $j \in J$ such that $\operatorname{diam}(S(C, x_j^*, \alpha - \eta)) \leq 2\varepsilon$. Suppose by contradiction that it is not true. Then for all $j \in J$ there exist $y_j, z_j \in S(C, x_j^*, \alpha - \eta)$ such that $\|y_j - z_j\| > 2\varepsilon$. We have that

$$x_j^*(y_j) > \sup_C x_j^* - \alpha + \eta$$

for all $j \in J$. It follows that

$$\langle (x_i^*)_\mathcal{U}, (y_i)_\mathcal{U}\rangle \geq \lim_\mathcal{U} \sup_C x_i^* - \alpha + \eta = \sup_{C_\mathcal{U}} (x_i^*)_\mathcal{U} - \alpha + \eta > \sup_{C_\mathcal{U}} (x_i^*)_\mathcal{U} - \alpha$$

by Lemma 1.2.6. This proves that $(y_i)_\mathcal{U} \in S(C_\mathcal{U}, (x_i^*)_\mathcal{U}, \alpha)$. In a similar way, we have that $(z_i)_\mathcal{U} \in S(C_\mathcal{U}, (x_i^*)_\mathcal{U}, \alpha)$. We deduce that $\|(y_i)_\mathcal{U} - (z_i)_\mathcal{U}\| \leq \varepsilon$, which contradicts the choice of y_j and z_j. □

6.3 Exposed and strongly exposed points

Let us conclude the section of general results with an analysis of strongly exposed points. As it is done in the previous subsection, we will begin by considering a uniform notion.

Definition 6.3.1. Let C be a bounded convex subset of a Banach space X. A set $\{x_i\}_{i\in I} \subset C$ is said to be a *uniformly strongly exposed set* if there exists $\{x_i^*\}_{i\in I} \subset S_{X^*}$ such that for every $\varepsilon > 0$ there exists $\alpha_\varepsilon > 0$ satisfying that

$$x_i \in S(C, x_i^*, \alpha_\varepsilon) \quad \text{and} \quad \operatorname{diam}(S(C, x_i^*, \alpha_\varepsilon)) < \varepsilon \quad \forall i \in I.$$

This definition was probably introduced in the celebrated paper of J. Lindenstrauss [126], where it is proved that if a Banach space X satisfies that its unit ball is the closed convex hull of a strongly exposed set then X has Lindenstrauss property A, i.e. the set of norm-attaining operators $NA(X,Y)$ is dense in $L(X,Y)$ for every Banach space Y [126, Proposition 1]. See [41, Section 3] for a number of examples of Banach spaces where the previous condition holds.

Anyway, our interest in uniformly strongly exposed sets comes from the following result.

Theorem 6.3.2. *Let C be a bounded convex subset of a Banach space X and I be an infinite set. Let $\{x_i\}_{i\in I}$ be a family of points exposed in C by $\{x_i^*\}_{i\in I} \subset B_{X^*}$. The following are equivalent:*

(i) $\{x_i^*\}_{i\in I}$ uniformly strongly exposes $\{x_i\}_{i\in I}$;

(ii) $(x_i^*)_\mathcal{U}$ strongly exposes $(x_i)_\mathcal{U}$ in $C_\mathcal{U}$ for every free ultrafilter $\mathcal{U}$ on I;

(iii) $(x_i^*)_\mathcal{U}$ exposes $(x_i)_\mathcal{U}$ in $C_\mathcal{U}$ for every free ultrafilter $\mathcal{U}$ on I.

Proof. (i)⇒(ii). Note that, by Lemma 1.2.6,

$$\langle (x_i^*)_\mathcal{U}, (x_i)_\mathcal{U}\rangle = \lim_\mathcal{U} x_i^*(x_i) = \lim_\mathcal{U} \sup_C x_i^* = \sup_{C_\mathcal{U}} (x_i^*)_\mathcal{U}.$$

Now let $\varepsilon > 0$ and take $\alpha > 0$ given by the definition of uniformly strongly exposed set. Suppose that $(y_i)_\mathcal{U} \in S(C_\mathcal{U}, (x_i^*)_\mathcal{U}, \alpha)$. By the previous equalities, it means that $\lim_\mathcal{U} x_i^*(y_i) > \lim_\mathcal{U} x_i^*(x_i) - \alpha$. In particular,

$$J := \{i \in I \mid x_i^*(y_i) > x_i^*(x_i) - \alpha\} \in \mathcal{U}.$$

Then $\|y_i - x_i\| < \varepsilon$ for all $i \in J$. We conclude that $\|(x_i)_\mathcal{U} - (y_i)_\mathcal{U}\| \leq \varepsilon$, thus $(x_i)_\mathcal{U} \in \text{str-exp}_{(X^*)_\mathcal{U}}(C_\mathcal{U})$.

(ii)⇒(iii) is obvious, so let us prove (iii)⇒(i). Assume that $\{x_i\}_{i\in I}$ is not uniformly strongly exposed by $\{x_i^*\}_{i\in I}$, and let us find a free ultrafilter $\mathcal{U}$ on I so that $(x_i)_\mathcal{U}$ is not exposed by $(x_i^*)_\mathcal{U}$.

Since $\{x_i\}_{i\in I}$ is not uniformly strongly exposed by $\{x_i^*\}_{i\in I}$, there exists $\varepsilon_0 > 0$ so that, for every $n \in \mathbb{N}$, there exists $i_n \in I$ and $y_{i_n} \in C$ so that $x_{i_n}^*(y_{i_n}) > \sup_C x_i^* - \frac{1}{n}$ but $\|x_{i_n} - y_{i_n}\| \geq \varepsilon_0$. Define the set $L := \{i_n : n \in \mathbb{N}\}$ and note that L is infinite. Otherwise, there is n_0 such that $i_n = i_{n_0}$ for $n \geq n_0$. We have that

$$x_{i_{n_0}}^*(y_{i_{n_0}}) = x_{i_n}^*(y_{i_n}) > \sup_C x_{i_n}^* - \frac{1}{n} = \sup_C x_{i_{n_0}}^* - \frac{1}{n} \quad \forall n \geq n_0,$$

so taking limit we deduce that $x_{i_{n_0}}^*(y_{i_{n_0}}) = \sup_C x_{i_{n_0}}^*$. Since $\|x_{i_{n_0}} - y_{i_{n_0}}\| \geq \varepsilon_0$, we derive a contradiction with the fact that $x_{i_{n_0}}^*$ exposes $x_{i_{n_0}}$. Consequently, L is infinite.

Now, let $\mathcal{U}$ be a free ultrafilter on I with $L \in \mathcal{U}$. Define $y_i := y_{i_n}$ if $i = i_n$ and $y_i = 0$ otherwise. First, note that

$$\|(x_i)_\mathcal{U} - (y_i)_\mathcal{U}\| = \lim_\mathcal{U} \|x_i - y_i\| \geq \varepsilon_0$$

since the set $\{i \in I : \|x_i - y_i\| \geq \varepsilon_0\}$ belongs to $\mathcal{U}$.

On the other hand, we claim that $\langle (x_i^*)_\mathcal{U}, (y_i)_\mathcal{U}\rangle = \sup_{C_\mathcal{U}} (x_i^*)_\mathcal{U}$. Indeed, Lemma 1.2.6 implies that

$$\sup_{C_\mathcal{U}} (x_i^*)_\mathcal{U} = \lim_\mathcal{U} \sup_C x_i^* = \lim_\mathcal{U} x_i^*(x_i),$$

so let us prove that $\langle (x_i^*)_\mathcal{U}, (y_i)_\mathcal{U}\rangle \geq \lim_\mathcal{U} x_i^*(x_i)$. To this end, pick $\varepsilon > 0$. The set

$$B := \{j \in I : |x_j^*(y_j) - \lim_\mathcal{U} x_i^*(y_i)| < \varepsilon\}$$

belongs to $\mathcal{U}$. Now, given $p \in \mathbb{N}$, find

$$j \in B \cap L \cap \{k \in I : |x_k^*(x_k) - \lim_\mathcal{U} x_i^*(x_i)| < \varepsilon\} \setminus \{i_1, \ldots, i_p\}$$

(the previous set is non-empty because, actually, it belongs to $\mathcal{U}$). Now we have

$$\lim_\mathcal{U} x_i^*(y_i) > x_j^*(y_j) - \varepsilon > x_j^*(x_j) - \frac{1}{p} - \varepsilon > \lim_\mathcal{U} x_i^*(x_i) - \frac{1}{p} - 2\varepsilon.$$

The arbitrariness of p and ε conclude that $\langle (x_i^*)_\mathcal{U}, (y_i)_\mathcal{U}\rangle = \sup_{C_\mathcal{U}} (x_i)_\mathcal{U}$. This shows that $(x_i)_\mathcal{U}$ is not exposed by $(x_i^*)_\mathcal{U}$ and finishes the proof. $\square$

Now, we focus on the case of elements of the form $\mathcal{J}(x)$ for $x \in C$.

Corollary 6.3.3. *Let C be a bounded convex subset of a Banach space X and $\mathcal{U}$ be a free ultrafilter on an infinite set I. Let $x \in C$. Then*

(a) if $\mathcal{J}(x) \in \exp_{(X^*)_\mathcal{U}}(C_\mathcal{U})$, then $x \in \exp(C)$;

(b) $\mathcal{J}(x) \in \text{str-exp}_{(X^*)_\mathcal{U}}(C_\mathcal{U})$ if and only if $x \in \text{str-exp}(C)$.

Proof. (a) Suppose that $\mathcal{J}(x)$ is exposed by $(x_i^*)_\mathcal{U}$. By weak*-compactness of B_{X^*}, define $x^* = w^*\text{-}\lim_\mathcal{U} x_i^*$. Let $y \in A$ such that $y \neq x$. We have

$$x^*(x) = \lim_\mathcal{U} x_i^*(x) = \langle (x_i^*)_\mathcal{U}, \mathcal{J}(x)\rangle > \langle (x_i^*)_\mathcal{U}, \mathcal{J}(y)\rangle = \lim_\mathcal{U} x_i^*(y) = x^*(y),$$

i.e. x^* exposes x.

(b) Follows directly from Theorem 6.3.2. $\square$

We will obtain in Corollary 6.3.5 a strenghtening of Corollary 6.3.3 under the assumption of CI ultrafilters. This strengthening will come from the following result.

Theorem 6.3.4. *Let C be a bounded convex subset of a Banach space X, and $\mathcal{U}$ be a CI ultrafilter on an infinite set I. Then $\exp_{(X^*)_\mathcal{U}}(C_\mathcal{U}) = \text{str-exp}_{(X^*)_\mathcal{U}}(C_\mathcal{U})$.*

Proof. Since the inclusion $\supseteq$ is immediate, let us prove $\subseteq$. Suppose that $f := (x_i^*)_\mathcal{U} \in (X^*)_\mathcal{U}$ exposes $z := (x_i)_\mathcal{U} \in C_\mathcal{U}$. Let us prove that f actually strongly exposes z. Indeed, consider a sequence $(z_m)_m := ((x_i^{(m)})_{\mathcal{U},i})_m \subseteq C_\mathcal{U}$ such that $\langle f, z_m\rangle \to \langle f, z\rangle$ as $m \to \infty$, and let us prove that $\|z_m - z\| \to 0$. Assume on the contrary that $\|z_m - z\| \not\to 0$. Up to passing to a subsequence, we can assume that $|\langle f, z_m - z\rangle| < \frac{1}{m}$ and $\|z_m - z\| > \beta > 0$ holds for every $m \in \mathbb{N}$ and for some $\beta > 0$. Pick, for every $k \in \mathbb{N}$, $A_k \in \mathcal{U}$ satisfying that $A_1 \supset A_2 \supset \ldots$ such that, for every $k \in \mathbb{N}$, we have $|x_i^*(x_i^{(m)}) - x_i^*(x_i)| < \frac{1}{m}$ and $\|x_i^{(m)} - x_i\| > \beta$ for all $m \leq k$ and every $i \in A_k$. Since $\mathcal{U}$ is CI we can also assume that $\bigcap_{k\in\mathbb{N}} A_k = \emptyset$.

Now define $(y_i)_{i\in I} \in C_{\mathcal{U}}$ be such that $y_i := x_i^{(m)}$ if $i \in A_m \setminus A_{m+1}$ and for the negligible case $i \in I \setminus A_1$ we set $y_i = a$ for some $a \in C$, so $(y_i)_{\mathcal{U}} \in C_{\mathcal{U}}$. Note that

$$\langle f, (y_i)_{\mathcal{U}} \rangle = \lim_{\mathcal{U}} x_i^*(y_i) = \lim_{\mathcal{U}} x_i^*(x_i) = \langle f, z \rangle.$$

Since z is exposed by f we conclude that

$$\lim_{\mathcal{U}} \|x_i - y_i\| = \|(x_i)_{\mathcal{U}} - (y_i)_{\mathcal{U}}\| = 0,$$

but this is a contradiction because clearly $\lim_{\mathcal{U}} \|x_i - y_i\| \geq \beta > 0$ by the construction. This contradiction implies that f actually strongly exposes z, as desired. □

If $\mathcal{U}$ is supposed to be CI, we obtain the following improvement of Corollary 6.3.3:

Corollary 6.3.5. *Let C be a bounded convex subset of a Banach space X, $\mathcal{U}$ be a CI ultrafilter on an infinite set I and let $x \in C$. The following assertions are equivalent:*

(i) $\mathcal{J}(x) \in \text{str-exp}_{(X^*)_{\mathcal{U}}}(C_{\mathcal{U}})$;

(ii) $\mathcal{J}(x) \in \exp_{(X^*)_{\mathcal{U}}}(C_{\mathcal{U}})$;

(iii) $x \in \text{str-exp}(C)$.

Proof. The equivalence between (i) and (ii) follows from the previous theorem. Corollary 6.3.3 gives the equivalence between (i) and (iii). □

If C is a bounded convex set, we sum up the properties linking x and $\mathcal{J}(x)$ in the following graph of implications:

$$\begin{array}{ccccccc}
x \in \text{str-exp}(C) & \Longrightarrow & x \in \text{dent}(C) & \Longrightarrow & x \in \text{str-ext}(C) & \Longrightarrow & x \in \text{ext}(C) \\
\Updownarrow & & \Updownarrow & & \Updownarrow & & \Uparrow \\
\mathcal{J}(x) \in \text{str-exp}_{(X^*)_{\mathcal{U}}}(C_{\mathcal{U}}) & \Longrightarrow & \mathcal{J}(x) \in \text{dent}_{(X^*)_{\mathcal{U}}}(C_{\mathcal{U}}) & \Longrightarrow & \mathcal{J}(x) \in \text{str-ext}(C_{\mathcal{U}}) & \Longleftrightarrow & \mathcal{J}(x) \in \text{ext}(C_{\mathcal{U}})
\end{array}$$

Note that none of the previous implications can be reversed in the general case (since there exist extreme points which are not strongly extreme, strongly extreme points which are not denting and denting points which are not strongly exposed).

6.4 Application to super weak compactness

In this section we will study the extremality under compactness assumptions. To be more precise, let X be a Banach space and $K \subseteq B_X$ be a convex bounded subset. We will deal with the assumption that K is SWC. Before we enter in details, let us explain why this context, though very restrictive, is interesting for us: looking to our results for denting points and strongly exposed points, we have not been able to completely characterise when a point $(x_i)_{\mathcal{U}}$ is a denting (respectively strongly exposed) point because we do not have a good access to the space $(X_{\mathcal{U}})^*$, which differs from $(X^*)_{\mathcal{U}}$ if X is not superreflexive.

However, in the particular case of $K_{\mathcal{U}}$ being weakly compact this difficulty is overcome thanks to the following lemma applied to $Z = (X^*)_{\mathcal{U}}$:

Lemma 6.4.1. *Let C be a weakly compact convex subset of a Banach space X. Let Z be a subspace of X^* such that (X, Z) is a dual pair. Let x be a point of a slice S of C. Then there exists a slice S' of C defined by an element of Z such that $x \in S' \subset S$. In particular, if $x \in C$ is denting then x is Z-denting.*

Proof. Note that $C \setminus S$ is a weakly compact set which does not contain x. In particular, $C \setminus S$ is $\sigma(X, Z)$-compact. Since Z separates points of X, the topology $\sigma(X, Z)$ is Hausdorff. By the Hahn-Banach theorem, there exists $x^* \in (X, \sigma(X, Z))^* = Z$ and a slice S' of C defined by x^* such that $(C \setminus S) \cap S' = \emptyset$ and $x \in S'$. It follows that $x \in S' \subset S$. $\square$

For instance, here we obtain that a point $(x_i)_\mathcal{U} \in K_\mathcal{U}$ is denting if, and only if, $(x_i)_\mathcal{U}$ belongs to a sequence of slices of diameter as small as desired where the slices are defined by elements of $(X^*)_\mathcal{U}$. This difficulty will be overcome in the context of super weakly compact subsets.

Recall that every weakly compact convex set is the closed convex hull of its strongly exposed points (see Theorem 8.13 in [67] for example). In the case of ultraproducts we can say a bit more.

Theorem 6.4.2. *Let K be a relatively SWC convex subset of a Banach space X and $\mathcal{U}$ be a CI ultrafilter on an infinite set I. Then*

$$K_\mathcal{U} = \overline{\text{conv}}(\text{str-exp}_{(X^*)_\mathcal{U}}(K_\mathcal{U})).$$

This is just a particular case of the following result:

Lemma 6.4.3. *Let K be weakly compact convex subset of a Banach space X and Z be a subspace of X^* such that (X, Z) is a dual pair. Then*

$$K = \overline{\text{conv}}(\text{str-exp}_Z(K)).$$

Proof. First, let's show that every subset C of K is Z-dentable, that is, there are slices of C given by elements of Z with arbitrarily small diameter. If $\overline{\text{conv}}(C)$ is Z-dentable then C is dentable too, so we can suppose that C is closed and convex. Since K is weakly compact, so is C. In particular, C is dentable and then Z-dentable by Lemma 6.4.1. A slight modification of Theorem 8 in [31] allows us to conclude that the subset of Z that strongly exposes elements of K is dense in Z. Now, suppose by contradiction that $K \neq \overline{\text{conv}}(\text{str-exp}_Z(K)$. Since $\overline{\text{conv}}(\text{str-exp}_Z(K))$ is weakly compact and then $\sigma(X, Z)$-compact, there exists $x^* \in Z$ such that

$$\sup_K x^* > \sup_{\text{str-exp}_Z(K)} x^*.$$

By density, we deduce that there exists $y^* \in Z$ strongly exposing an element $x \in K$ such that

$$y^*(x) = \sup_K y^* > \sup_{\text{str-exp}_Z(K)} y^*,$$

which is a contradiction. $\square$

Proof of Theorem 6.4.2. Note that $K_\mathcal{U}$ is weakly compact thanks to Proposition 1.2.5. Now, apply Lemma 6.4.3 taking $Z = (X_\mathcal{U})^* \subset (X_\mathcal{U})^*$. $\square$

Using Tu's result (see Theorem 1.5.2), it is possible to localise the set of extreme points of a super weakly compact set:

Proposition 6.4.4. *Let K be a SWC convex subset of a Banach space X and $\mathcal{U}$ be a CI ultrafilter on an infinite set I. Then*

$$\text{ext}(K_\mathcal{U}) \subset \overline{(\text{str-exp}(K))_\mathcal{U}}^w \quad \text{and} \quad \text{dent}(K_\mathcal{U}) \subset (\text{str-exp}\, K)_\mathcal{U}.$$

Proof. Since K is weakly compact, we have that $K = \overline{\text{conv}}(\text{str-exp}(K))$. Thus, by Theorem 1.5.2, it follows that

$$K_{\mathcal{U}} = (\overline{\text{conv}}(\text{str-exp}(K)))_{\mathcal{U}} = \overline{(\text{conv}(\text{str-exp}(K)))_{\mathcal{U}}} = \overline{\text{conv}}((\text{str-exp}(K))_{\mathcal{U}}).$$

so any slice of $K_{\mathcal{U}}$ has non-empty intersection with $(\text{str-exp}(K))_{\mathcal{U}}$. Since the slices are a neighbourhood basis for the extreme (resp. denting) points of $K_{\mathcal{U}}$ in the weak (resp. norm) topology, we have $\text{ext}(K_{\mathcal{U}}) \subset \overline{(\text{str-exp}(K))_{\mathcal{U}}}^w$ and $\text{dent}(K_{\mathcal{U}}) \subset \overline{(\text{str-exp}\, K)_{\mathcal{U}}} = (\text{str-exp}\, K)_{\mathcal{U}}$, where the last equality follows from Lemma 1.2.5. $\square$

Proposition 6.4.5. *Let K be a SWC subset of a Banach space X, $\mathcal{U}$ be a CI ultrafilter on an infinite set I and let $x \in K$. Then $\mathcal{J}(x) \in \text{dent}(K_{\mathcal{U}})$ if and only if $x \in \text{dent}(K)$.*

Proof. Since K is SWC, it follows that $\mathcal{J}(x)$ is $(X^*)_{\mathcal{U}}$-denting if and only if $\mathcal{J}(x) \in \text{dent}(K_{\mathcal{U}})$ (by Lemma 6.4.1). We conclude by Theorem 6.2.4. $\square$

Note that Theorem 6.4.2 is a useful tool in the search of a characterisation of when $(x_i)_{\mathcal{U}}$ is a denting point in $K_{\mathcal{U}}$ if K is SWC. However, in order to get a complete characterisation in terms of a condition on the points x_i's, we will consider a notion which is stronger than super weak compactness: the one of *uniform convexity* (see Definition 6.4.6). This geometric property on K will allow us to characterise the denting points of uniformly convex subsets of a Banach space (see Theorem 6.4.19).

Let us now consider the formal definition of uniformly convex set.

Definition 6.4.6. A symmetric bounded closed convex set C of a Banach space X is said to be *uniformly convex* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\forall x, y \in C, \quad \|x - y\| > \varepsilon \implies \frac{x+y}{2} \in (1-\delta)C.$$

In such case, we define the convexity modulus of C by

$$\delta_C(\varepsilon) = \inf \left\{ 1 - \left| \frac{x+y}{2} \right|_C \ : \ x, y \in C, \|x - y\| > \varepsilon \right\}$$

where $|\cdot|_C$ is the Minkowski functional of C. By convention, $\inf \emptyset = 1$.

By [122, Proposition 4.2], any uniformly convex set is SWC. We include the proof for the lack of completness:

Proposition 6.4.7. *Let C be a uniformly convex subset of a Banach space X. Then C is SWC.*

Proof. Fix $\varepsilon > 0$. Is is clear that $[C]'_\varepsilon \subset (1 - \delta_C(\varepsilon))C$. An homogeneity argument implies that $[C]^n_\varepsilon \subset (1 - \delta_C(\varepsilon))^n C$. It follows that there exists $n \in \mathbb{N}$ such that $\text{diam}([C]^n_\varepsilon) < \varepsilon$ and then $[C]^{n+1}_\varepsilon = \emptyset$. Then C is finitely-dentable and we conclude by Corollar 2.4.12. $\square$

One can think that there is a big difference between SWC sets and uniformly convex sets. However, thanks to a result of Raja and Lancien (see [122, Proposition 4.3]), we see that from a topological point of view this is not the case. Indeed, given a symmetric super weakly compact subset K and $\varepsilon > 0$, there exists a uniformly convex set C_ε so that $C_\varepsilon \subseteq K \subseteq (1+\varepsilon)C_\varepsilon$.

We will also consider the following weakening of uniform convexity.

Definition 6.4.8. A symmetric bounded closed convex set C of a Banach space X is said to be strictly convex if for all $x, y \in C$ such that $x \neq y$ and $|x|_C = |y|_C = 1$, one has that $\left|\frac{x+y}{2}\right|_C < 1$.

In general, every extreme point x of a symmetric bounded closed convex set $C \neq \{0\}$ satisfies $|x|_C = 1$. In the case C is strictly convex, one can easily check that indeed $\mathrm{ext}(C) = |\cdot|_C^{-1}(\{1\})$. We will use this fact in the sequel.

The following result generalizes the fact that a Banach space is uniformly convex if and only if its ultraproduct is strictly (or uniformly) convex.

Proposition 6.4.9. *Let K be a symmetric bounded convex subset of a Banach space X. Let $\mathcal{U}$ be a CI ultrafilter on an infinite set I. The following assertions are equivalent:*

(i) K is uniformly convex;

(ii) $K_{\mathcal{U}}$ is uniformly convex.

In that case, we have that $\delta_K = \delta_{K_{\mathcal{U}}}$. Moreover, if 0 is an interior point of K, then the previous statements are equivalent to:

(iii) $K_{\mathcal{U}}$ is strictly convex.

For the proof we will need the following result.

Lemma 6.4.10. *Let C be a symmetric bounded convex subset of a Banach space X, $\mathcal{U}$ be a CI ultrafilter on an infinite set I and $(x_i)_{\mathcal{U}} \in C_{\mathcal{U}}$. Then $|(x_i)_{\mathcal{U}}|_{C_{\mathcal{U}}} \leq \lim_{\mathcal{U}} |x_i|_C$. Moreover, if 0 is an interior point of C, the reverse equality also holds.*

Proof. Define $l := \lim_{\mathcal{U}} |x_i|_C$. Let $\varepsilon > 0$. For all $i \in I$, we have that $x_i \in (|x_i|_C + \varepsilon)C$. Define $J := \{i \in I : ||x_i|_C - l| < \varepsilon\} \in \mathcal{U}$. For all $i \in J$, it follows that $x_i \in (l + 2\varepsilon)C$. Then $(x_i)_{\mathcal{U}} \in (l+2\varepsilon)C_{\mathcal{U}}$ for all $\varepsilon > 0$. Since $C_{\mathcal{U}}$ is closed, we deduce that $(x_i)_{\mathcal{U}} \in lC_{\mathcal{U}}$. We conclude that $|(x_i)_{\mathcal{U}}|_{C_{\mathcal{U}}} \leq l$.

Now, assume that 0 is an interior point of C. Let $\lambda := |(x_i)_{\mathcal{U}}|_{C_{\mathcal{U}}}$ and notice that $(x_i)_{\mathcal{U}} \in \lambda C_{\mathcal{U}}$, so there exists $(y_i)_{i \in I} \in C^I$ so that $(x_i)_{\mathcal{U}} = (\lambda y_i)_{\mathcal{U}}$. Note that

$$\lim_{\mathcal{U}} \|x_i - \lambda y_i\| = 0 \Leftrightarrow \lim_{\mathcal{U}} |x_i - \lambda y_i|_C = 0$$

since $\|\cdot\|$ and $|\cdot|_C$ are equivalent norms on $\mathrm{span}(C)$. This implies that

$$\lim_{\mathcal{U}} |x_i|_C = \lim_{\mathcal{U}} |\lambda y_i|_C = \lambda \lim_{\mathcal{U}} |y_i|_C \leq \lambda$$

where the last inequality holds since $y_i \in C$ holds for every i. This proves the equality in such case. $\square$

Proof of Proposition 6.4.9. $(i) \implies (ii)$ Suppose that K is uniformly convex. Note that $K_{\mathcal{U}}$ is closed since $\mathcal{U}$ is CI. Let $\varepsilon > 0$. Take $(x_i)_{\mathcal{U}}, (y_i)_{\mathcal{U}} \in K_{\mathcal{U}}$ such that $\|(x_i)_{\mathcal{U}} - (y_i)_{\mathcal{U}}\| > \varepsilon$. Then, we can suppose (changing some coordinates if necessary) that $\|x_i - y_i\| > \varepsilon$ for all $i \in I$. Let $\eta \in (0, \delta_K(\varepsilon))$ arbitrary. It follows that $\frac{x_i+y_i}{2} \in (1-\eta)K$ for all $i \in I$ and then

$$\frac{(x_i)_{\mathcal{U}} + (y_i)_{\mathcal{U}}}{2} = \left(\frac{x_i + y_i}{2}\right)_{\mathcal{U}} \in (1-\eta)K_{\mathcal{U}}.$$

Since η was arbitrary, we conclude that $0 < \delta_K(\varepsilon) \leq \delta_{K_{\mathcal{U}}}(\varepsilon)$, i.e. $K_{\mathcal{U}}$ is uniformly convex.

$(ii) \implies (i)$ Suppose that $K_{\mathcal{U}}$ is uniformly convex. Let $\varepsilon > 0$. Let $x, y \in K$ such that $\|x - y\| > \varepsilon$. Let $\eta \in (0, \delta_{K_{\mathcal{U}}}(\varepsilon))$ arbitrary. It follows that $\|\mathcal{J}(x) - \mathcal{J}(y)\| > \varepsilon$ and then $\mathcal{J}\left(\frac{x+y}{2}\right) = \frac{\mathcal{J}(x)+\mathcal{J}(y)}{2} \in (1-\eta)K_{\mathcal{U}}$. Let $(z_i)_{i\in I} \in K^I$ such that $\mathcal{J}\left(\frac{x+y}{2}\right) = (1-\eta)(z_i)_{\mathcal{U}}$. Since $\lim_{\mathcal{U}} \left\|\frac{x+y}{2} - (1-\eta)z_i\right\| = 0$, it follows that $\frac{x+y}{2} \in \overline{(1-\eta)K} = (1-\eta)K$. Since η was arbitrary, we conclude that $0 < \delta_{K_{\mathcal{U}}}(\varepsilon) \leq \delta_K(\varepsilon)$, i.e. K is uniformly convex.

Now suppose that 0 is an interior point of K and that $\mathcal{U}$ is a CI ultrafilter. $(ii) \implies (iii)$ is obvious. We will show the implication $(iii) \implies (i)$. Let suppose that K is not uniformly convex. Then there exists $\varepsilon > 0$ such that for all $n \in \mathbb{N}$, there exist $x_n, y_n \in K$ with $\|x_n - y_n\| > \varepsilon$ and $\left|\frac{x_n+y_n}{2}\right|_K \to 1$. Let $(I_n)_n \subset \mathcal{U}$ be a sequence of distinct sets such that $\bigcap_n I_n = \emptyset$, $I_0 = I$ and $I_{n+1} \subset I_n$ for all $n \in \mathbb{N}$. Define $x_i' = x_n$ if $i \in I_n \setminus I_{n+1}$ for some $n \in \mathbb{N}$. Define y_i' in the same way. It is clear that $\|(x_i')_{\mathcal{U}} - (y_i')_{\mathcal{U}}\| \geq \varepsilon$. Moreover, it is easy to show that $\lim_{\mathcal{U}} \left|\frac{x_i'+y_i'}{2}\right|_K = 1$. The previous lemma implies that

$$\left|\frac{(x_i')_{\mathcal{U}} + (y_i')_{\mathcal{U}}}{2}\right|_{K_{\mathcal{U}}} = 1.$$

By triangle inequality, we also have that $|(x_i')_{\mathcal{U}}|_{K_{\mathcal{U}}} = |(y_i')_{\mathcal{U}}|_{K_{\mathcal{U}}} = 1$. So $K_{\mathcal{U}}$ cannot be strictly convex. □

Remark 6.4.11. Proposition 6.4.9 reproves the very well-known result that a Banach space X is uniformly convex if, and only if, $X_{\mathcal{U}}$ is strictly convex, where $\mathcal{U}$ is a CI ultrafilter.

In the sequel we aim to give a characterisation of the extreme points of a uniformly convex set. In order to do so, we need a preliminary result.

Lemma 6.4.12. *Let C be symmetric bounded convex subset of a Banach space X and $\mathcal{U}$ be a CI ultrafilter on an infinite set I. If $(x_i)_{\mathcal{U}} \in \operatorname{ext}(C_{\mathcal{U}})$, then $\lim_{\mathcal{U}} |x_i|_C = 1$.*

Proof. Since $x_i \in C$ for all $i \in I$, we have that $\lim_{\mathcal{U}} |x_i|_C \leq 1$. Moreover we have that $(x_i)_{\mathcal{U}} \in \operatorname{ext}(C_{\mathcal{U}})$ so $|(x_i)_{\mathcal{U}}|_{C_{\mathcal{U}}} = 1$. We conclude by Lemma 6.4.10. □

Theorem 6.4.13. *Let K be a uniformly convex subset of a Banach space X and $\mathcal{U}$ be a CI ultrafilter on an infinite set I. Let $(x_i)_{\mathcal{U}} \in K_{\mathcal{U}}$. The following assertions are equivalent:*

(i) $(x_i)_{\mathcal{U}} \in \operatorname{ext}(K_{\mathcal{U}})$*;*

(ii) for any $(y_i)_{i\in I} \in K^I$ such that $(y_i)_{\mathcal{U}} = (x_i)_{\mathcal{U}}$, it follows $\lim_{\mathcal{U}} |y_i|_K = 1$.

If 0 is an interior point of C, then they are also equivalent to:

(iii) $\lim_{\mathcal{U}} |x_i|_K = 1$.

Proof. Suppose that $(x_i)_{\mathcal{U}} \in \operatorname{ext}(K_{\mathcal{U}})$. Let $(y_i)_{i\in I} \in K^I$ such that $(y_i)_{\mathcal{U}} = (x_i)_{\mathcal{U}}$. Obviously, we have that $(y_i)_{\mathcal{U}} \in \operatorname{ext}(K_{\mathcal{U}})$ and we conclude by the previous lemma.

Now suppose that $(x_i)_{\mathcal{U}} \notin \operatorname{ext}(K_{\mathcal{U}})$. Then there exist $(y_i)_{\mathcal{U}}, (z_i)_{\mathcal{U}}$ such that $(x_i)_{\mathcal{U}} = \frac{(y_i)_{\mathcal{U}}+(z_i)_{\mathcal{U}}}{2}$ and $\|(y_i)_{\mathcal{U}} - (z_i)_{\mathcal{U}}\| > \varepsilon$ for some $\varepsilon > 0$. Let δ associated to ε given by the uniform convexity of $K_{\mathcal{U}}$. It follows that $(x_i)_{\mathcal{U}} = \frac{(y_i)_{\mathcal{U}}+(z_i)_{\mathcal{U}}}{2} \in (1-\delta)K_{\mathcal{U}}$. So there exists $(y_i')_{i\in I} \in (1-\delta)K^I \subset K^I$ such that $(x_i)_{\mathcal{U}} = (y_i')_{\mathcal{U}}$. Since $|y_i'|_K \leq 1-\delta$ for all $i \in I$, it follows that $\lim_{\mathcal{U}} |y_i'|_K < 1$.

Finally, assume that 0 is an interior point of C. Clearly (ii)⇒(iii), and (iii)⇒(i) by Lemmas 6.4.10 and the comment following the definition of strict convexity. □

The next proposition shows the extremal structure of a uniformly convex set has great properties.

Proposition 6.4.14. *Let $K \subset X$ be a uniformly convex set of a Banach space X and Z be a subspace of X^*. Then*

(a) $\operatorname{ext}(K) = \operatorname{dent}(K)$.

(b) $\exp_Z(K) = \text{str-exp}_Z(K) = \{x \in K \mid \exists x^* \in Z \ : \ \sup\limits_K x^* = x^*(x) > 0\}$.

Proof. (a) Let $x \in \text{ext}(K)$ and let $\varepsilon > 0$. Take δ associated to ε given by the definition of uniform convexity. By the Hahn-Banach theorem, there exists $x^* \in X^* \setminus \{0\}$ such that

$$x^*(x) > \sup_{(1-\delta)K} x^*.$$

Since K is symmetric, we can suppose that $\sup_K x^* = 1$. Define a slice of K by $S := S(x^*, K, \delta)$. We have that $x^*(x) > \sup_{(1-\delta)K} x^* = 1 - \delta$, so $x \in S$ and $S \cap (1-\delta)K = \emptyset$. Let's show that $\text{diam}(S) \leq \varepsilon$. Suppose on the contrary that there exist $y, z \in S$ such that $\|y - z\| > \varepsilon$. By uniform convexity, it follows that $\frac{y+z}{2} \in (1-\delta)K$. This is a contradiction since $\frac{y+z}{2} \in S$.

(b) Clearly $\text{str-exp}_Z(K) \subset \exp_Z(K)$. Now, assume that $x^* \in Z$ exposes x. Then $0 = x^*(0) < x^*(x)$, so we get that x^* satisfies our purposes.

Finally, that $x^* \in Z$ satisfies that $\sup_K x^* = x^*(x) > 0$, and let us prove that $x^* \in Z$ strongly exposes x. To this end, pick $\varepsilon > 0$, and let us find a slice of K determined by x^* with diameter smaller than ε. Let $\delta > 0$ be associated to ε in the definition of uniformly convex set. Without loss of generality, we can suppose that $x^*(x) = \sup_K x^* = 1$. Let $y \in K$ so that $\|x - y\| \geq \varepsilon$. Then $\frac{x+y}{2} \in (1-\delta)K$. Then

$$\frac{x^*(x) + x^*(y)}{2} \leq 1 - \delta,$$

from where

$$x^*(y) \leq 1 - 2\delta.$$

Summarising we have proved that if $\|x - y\| \geq \varepsilon$ then $y \notin S := \{z \in K : x^*(z) > 1 - 2\delta\}$, which is a slice of K since $\alpha > 0$. This is equivalent to the following: if $x^*(y) \geq 1 - 2\delta = x^*(x) - 2\delta$ then $\|x - y\| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we get that x^* strongly exposes x and the proof is complete.

$\square$

Corollary 6.4.15. *Let K be a uniformly convex subset of a Banach space X and $\mathcal{U}$ be a CI ultrafilter on an infinite set I be a Banach space. Then* $\text{ext}(K_{\mathcal{U}}) = \text{dent}(K_{\mathcal{U}})$ *and* $\exp(K_{\mathcal{U}}) = \text{str-exp}(K_{\mathcal{U}})$*.*

Proof. By Proposition 6.4.9, $K_{\mathcal{U}}$ is uniformly convex. The result follows from the previous proposition. $\square$

Corollary 6.4.16. *Let K be a uniformly convex subset of a Banach space X, $\mathcal{U}$ be a CI ultrafilter on an infinite set I and $x \in K$. Then $x \in \text{ext}(K)$ if and only if $\mathcal{J}(x) \in \text{ext}(K_{\mathcal{U}})$.*

Proof. It follows from Proposition 6.4.14 and Theorem 6.1.5. $\square$

The following diagram summarises the implications between the properties of $x \in K$ and the ones of $\mathcal{J}(x) \in K_{\mathcal{U}}$ for a uniformly convex set K. The situation is much simpler than in the general case:

$$\begin{array}{ccccccccc}
x \in \exp(K) & \Longleftrightarrow & x \in \text{str-exp}(K) & \Longrightarrow & x \in \text{dent}(K) & \Longleftrightarrow & x \in \text{str-ext}(K) & \Longleftrightarrow & x \in \text{ext}(K) \\
\Updownarrow & & \Updownarrow & & \Updownarrow & & \Updownarrow & & \Updownarrow \\
\mathcal{J}(x) \in \exp_{(X^*)_{\mathcal{U}}}(K_{\mathcal{U}}) & \Longleftrightarrow & \mathcal{J}(x) \in \text{str-exp}_{(X^*)_{\mathcal{U}}}(K_{\mathcal{U}}) & \Longrightarrow & \mathcal{J}(x) \in \text{dent}_{(X^*)_{\mathcal{U}}}(K_{\mathcal{U}}) & \Longleftrightarrow & \mathcal{J}(x) \in \text{str-ext}(K_{\mathcal{U}}) & \Longleftrightarrow & \mathcal{J}(x) \in \text{ext}(K_{\mathcal{U}})
\end{array}$$

Remark 6.4.17. In general we do not know whether every extreme point is a strongly exposed point in a uniformly convex set K. However, it turns out that every extreme point is strongly exposed in a sense which depends on the Minkowski functional $|\cdot|_K$. We say that a point is *intrinsically strongly exposed* if there exists a linear functional $f\colon X \longrightarrow \mathbb{R}$ (not necessarily bounded) so that $f(x) = \sup_K f = 1$ and that, for every $\varepsilon > 0$, there exists $\delta > 0$ so that $f(y) > 1-\delta \Rightarrow \|x-y\| < \varepsilon$.

Let us prove that if $x \in K$ satisfies that $|x|_K = 1$ then it is intrinsically strongly exposed. Define $Y = (\text{span}(K), |\cdot|_K)$. It is well known that Y is a Banach space such that $B_Y = K$. By the Hahn-Banach theorem, there exists $f \in S_{Y^*}$ such that $f(x) = |x|_K = 1$. Let us prove that f strongly exposes x in the above sense. To do so, pick $\varepsilon > 0$ and take δ of the definition of uniformly convex set. Now if $y \in K$ satisfies that $f(y) > 1-\delta$ we get

$$1 - \frac{\delta}{2} < \frac{f(x+y)}{2} \leq \frac{|x+y|_K}{2} = \left|\frac{x+y}{2}\right|_K,$$

which means that $\frac{x+y}{2} \notin (1-\delta)K$. This implies that $\|x-y\| < \varepsilon$. The arbitrariness of ε implies that every point of $\{x \in K : |x|_K = 1\}$ is strongly exposed (actually, it is uniformly strongly exposed).

Note that f is $|\cdot|_K$-continuous. However, in the general case, f is not necessarily $\|\cdot\|$-continuous (unless span(K) is closed since in that case the norm induced by X on Y and $|\cdot|_K$ are equivalent by the open mapping theorem).

Remark 6.4.18. In spite of the previous remark, we can at least prove that str-exp(K) is dense in ext(K) in a uniformly convex set K. Indeed, since K is weakly compact, $K = \overline{\text{conv}}(\text{str-exp}(K))$ and so $\text{dent}(K) \subset \overline{\text{str-exp}(K)}$. By Proposition 6.4.14, $\text{ext}(K) = \text{dent}(K)$ provided that K is uniformly convex.

Note that, by applying Corollary 6.4.15 and Theorem 6.1.1, we can now give a characterisation of the denting points of a uniformly convex set.

Theorem 6.4.19. *Let K be a uniformly convex subset of a Banach space X and $\mathcal{U}$ be a free ultrafilter over I. Let $(x_i)_\mathcal{U} \in C_\mathcal{U}$. The following assertions are equivalent:*

(i) $(x_i)_\mathcal{U} \in \text{dent}(K_\mathcal{U})$,

(ii) *for any $(y_i)_\mathcal{U}, (z_i)_\mathcal{U} \in C_\mathcal{U}$ so that $\lim_\mathcal{U} \|x_i - \frac{y_i+z_i}{2}\| = 0$, it follows that $\lim_\mathcal{U} \|x_i - y_i\| = 0$ and $\lim_\mathcal{U} \|x_i - z_i\| = 0$.*

Recall that F is a face of a convex set C if for any $x, y \in C$ such that $(x,y) \cap F \neq \emptyset$, one has that $[x,y] \subset F$. A face F of C is proper if $F \neq \emptyset$ and $F \neq C$.

Lemma 6.4.20. *Let C be a symmetric bounded closed convex subset of a Banach space X. If F is a proper face of C then $F \subset |\cdot|_C^{-1}(\{1\})$. If moreover C is strictly convex, then F is a singleton.*

Proof. Note first that $0 \notin F$, otherwise it follows easily that $C = F$, a contradiction. Now, suppose there exists $x \in F$ such that $|x|_C < 1$. Thus x belongs to the non-trivial segment $(0, x/|x|_C)$ in C. It follows that $0 \in F$, so we obtain again that $C = F$, a contradiction.

Finally, assume that C is strictly convex and take $x, y \in F$. We have that $|x|_C = |y|_C = 1$ and, since $\frac{x+y}{2} \in F$, we also have that $\left|\frac{x+y}{2}\right|_C = 1$. Then $x = y$ by strict convexity, so F is a singleton. $\square$

Theorem 6.4.21. *Let $\mathcal{U}$ be a free ultrafilter on an infinite set I. Let K be a uniformly convex subset of a Banach space X such that $K_\mathcal{U}$ separates points of $(X^*)_\mathcal{U}$. Then*

$$(\exp(K))_\mathcal{U} \subset \exp_{(X^*)_\mathcal{U}}(K_\mathcal{U}) = \text{str-exp}_{(X^*)_\mathcal{U}}(K_\mathcal{U}).$$

Proof. First, note that the equality follows from Proposition 6.4.14. Let $(x_i)_{\mathcal{U}} \in (\exp(K))_{\mathcal{U}}$. We can obviously suppose that $x_i \in \exp(K)$ for all $i \in I$. Let $x_i^* \in S_{X^*}$ that exposes x_i. Define a face of $K_{\mathcal{U}}$ by

$$F = \left\{ (y_i)_{\mathcal{U}} \in K_{\mathcal{U}} \mid \langle (x_i^*)_{\mathcal{U}}, (y_i)_{\mathcal{U}} \rangle = \sup_{K_{\mathcal{U}}} (x_i^*)_{\mathcal{U}} \right\}.$$

It is clear that $(x_i)_{\mathcal{U}} \in F$ and, in particular, $F \neq \emptyset$. Since $K_{\mathcal{U}}$ is strictly convex, in order to apply the previous lemma, we need to show that F is a proper face. If $F = K$, then $(x_i^*)_{\mathcal{U}|K_{\mathcal{U}}} = 0$ since $0 \in K$. By hypothesis, it follows that $(x_i^*)_{\mathcal{U}} = 0$, which is a contradiction since $\|(x_i^*)_{\mathcal{U}}\| = 1$. By Lemma 6.4.20, we conclude that $F = \{(x_i)_{\mathcal{U}}\}$, i.e. $(x_i)_{\mathcal{U}}$ is exposed by $(x_i^*)_{\mathcal{U}}$. $\square$

Chapter 7

Ultraproducts in Lipschitz-free spaces

7.1 Definitions and basic properties

7.1.1 Ultraproduct of metric spaces

An excellent reference on this topic is a revised unpublished version of [153]. Since that version might not be available to the reader, we have chosen to include here the necessary definitions and properties. Let I be any infinite set. From now on, $\mathcal{U}$ will denote a nonprincipal ultrafilter on I. Let $\{(M_i, d_i)\}_{i \in I}$ be a family of metric spaces and fix a distinguished point $0_i \in M_i$ for every $i \in I$. Let us consider the set

$$\ell_\infty(M_i) = \left\{ (x_i)_{i \in I} \in \prod_{i \in I} M_i : \sup_{i \in I} d_i(x_i, 0_i) < \infty \right\}.$$

Notice that for every $(x_i)_{i \in I}, (y_i)_{i \in I} \in \prod_{i \in I} M_i$ we have $\sup_{i \in I} d_i(x_i, y_i) < \infty$. Therefore, one can consider

$$d((x_i)_{i \in I}, (y_i)_{i \in I}) := \lim_{\mathcal{U}, i} d_i(x_i, y_i).$$

It is clear that d is a pseudometric on $\ell_\infty(M_i)$. We consider the equivalence relation given by $(x_i)_{i \in I} \sim (y_i)_{i \in I}$ if and only if $d((x_i)_{i \in I}, (y_i)_{i \in I}) = 0$. We denote

$$(M_i)_{\mathcal{U}} = \ell_\infty(M_i) / \sim$$

and $\pi \colon \ell_\infty(M_i) \to (M_i)_{\mathcal{U}}$ the canonical projection. Then the expression

$$d_{\mathcal{U}}(\bar{x}, \bar{y}) := d((x_i)_{i \in I}, (y_i)_{i \in I}),$$

for $\bar{x}, \bar{y} \in (M_i)_{\mathcal{U}}$ and $\pi((x_i)_{i \in I}) = \bar{x}$, $\pi((y_i)_{i \in I}) = \bar{y}$, defines a metric on $(M_i)_{\mathcal{U}}$. For simplicity, we usually omit π and we write $(x_i)_{\mathcal{U}} = \pi((x_i)_{i \in I}$. The metric space $((M_i)_{\mathcal{U}}, d_{\mathcal{U}})$ is called the *ultraproduct of the metric spaces* $(M_i)_{i \in I}$. Moreover, if $M_i = M$ and $0_i = 0 \in M$ for every $i \in I$ then the space $(M_i)_{\mathcal{U}}$ is called the *ultraproduct of the metric space* M and denoted $M_{\mathcal{U}}$. If the context is clear, we simply write d instead of $d_{\mathcal{U}}$.

Remark 7.1.1. Although the concepts are different, we use the same notation as in Definition 1.2.4. In this chapter, only this new notion is used and there is no possible confusion.

Let us notice that if the spaces M_i are uniformly bounded then the definition of $(M_i)_\mathcal{U}$ does not depend on the choice of the distinguished points. In the case that the M_i are normed spaces, we will always consider that the distinguished point is $0 \in M_i$ for every $i \in I$ and we recover then the usual definition of ultraproduct for Banach spaces.

The following result summarises several known properties of the ultraproduct of metric spaces. We include the proofs (analogous to the Banach case ones) for completeness.

Fact 7.1.2. (a) If $0_i \in N_i \subset M_i$ for each $i \in I$, then $(N_i)_\mathcal{U}$ embeds isometrically in $(M_i)_\mathcal{U}$. Moreover, if $\mathcal{U}$ is CI and N_i is dense in M_i for every $i \in I$ then $(N_i)_\mathcal{U}$ is isometric to $(M_i)_\mathcal{U}$.

(b) If $\mathcal{U}$ is CI, then $(M_i)_\mathcal{U}$ is a complete metric space.

(c) If the M_i are normed spaces, then $(M_i)_\mathcal{U}$ is a Banach space.

(d) M embeds isometrically in $M_\mathcal{U}$. Moreover, if M is a normed space then there exists a linear isometry from M into $M_\mathcal{U}$.

(e) If M is a proper metric space (that is, closed balls in M are compact sets) then $M_\mathcal{U}$ is isometric to M.

Proof. (*a*) Consider $f_i \colon N_i \to M_i$ the canonical inclusion for each $i \in I$. Then it is straightforward that $(f_i)_{i\in I}$ defines an isometry from $(N_i)_\mathcal{U}$ into a subset of $(M_i)_\mathcal{U}$. In other to prove the second statement, take $\bar{x} \in (M_i)_{i\in I}$ and fix $(x_i)_{i\in I}$ with $\bar{x} = (x_i)_{i\in I}$. Take a decreasing sequence $(I_n)_{n\in\mathbb{N}} \subset \mathcal{U}$ such that $\bigcap_{n=1}^\infty I_n = \emptyset$. We will define $y_i \in N_i$ for each $i \in I$ satisfying that $\lim_{i,\mathcal{U}} d_i(x_i, y_i) = 0$, so $\bar{x} = (y_i)_{i\in I}$. If $i \notin I_1$, take $y_i \in M_i$ arbitrary. If $i \in I_n \setminus I_{n+1}$, take $y_i \in N_i$ so that $d_i(x_i, y_i) < 1/n$. Since $\bigcap_{n=1}^\infty I_n = \emptyset$, this defines y_i for every $i \in I$. Now notice that

$$\{i \in I : d_i(x_i, y_i) < 1/n\} \supset I_n \in \mathcal{U}$$

and so $\lim_{i,\mathcal{U}} d_i(x_i, y_i) = 0$. This shows that $(f_i)_{i\in I}$ is onto, as desired.

(*b*) By the previous property, we may assume that M_i is complete for each $i \in I$. Notice that $\pi \colon (\ell_\infty(M_i), d_\infty) \to ((M_i)_\mathcal{U}, d_\mathcal{U})$ is 1-Lipschitz and onto. Since completeness is preserved by uniformly continuous surjections, we only need to check that $(\ell_\infty(M_i), d_\infty)$ is complete. For that, mimic the proof of the completeness of ℓ_∞.

(*c*) It is clear that $d_\mathcal{U}$ is a norm on $(M_i)_\mathcal{U}$ whenever the M_i are normed spaces. Moreover, $\ell_\infty(M_i)$ is a Banach space and $N_\mathcal{U} = \{(x_i)_{i\in I} \mid \lim_\mathcal{U} \|x_i\| = 0\}$ is a closed subspace. So $(M_i)_\mathcal{U} = \ell_\infty(M_i)/N_\mathcal{U}$ is a Banach space.

(*d*) Given $x \in M$, take $x_i = x$ for every $i \in I$. Then $(x_i)_{i\in I} \in \ell_\infty(M)$. Thus $\phi(x) := (x_i)_{i\in I}$ defines an isometry from M into a subset of $M_\mathcal{U}$. For the last statement, notice that the map ψ is a linear operator whenever M is a normed space.

(*e*) Given $\bar{x} \in M_\mathcal{U}$, take $(x_i)_{i\in I}$ such that $\bar{x} = (x_i)_{i\in I}$. Then $R = \sup\{d(x_i, 0) : i \in I\} < \infty$ and so $\{x_i : i \in I\}$ is contained in the compact set $\overline{B}(0, R)$. Therefore, there exists $\psi(\bar{x}) := \lim_{\mathcal{U},i} x_i$. Notice that

$$d(\psi(\bar{x}), \psi(\bar{y})) = d\left(\lim_{\mathcal{U},i} x_i, \lim_{\mathcal{U},i} y_i\right) = \lim_{\mathcal{U},i} d(x_i, y_i) = d_\mathcal{U}(\bar{x}, \bar{y}),$$

so ψ defines an isometry from $M_\mathcal{U}$ into M. Moreover, given $x \in M$ we have $x = \psi(\phi(x))$ and therefore ψ is onto.

□

Note in passing that the ultraproduct of metric spaces is closely related to the Gromov-Hausdorff limit. Indeed, if M is the Gromov-Hausdorff limit of a sequence of pointed proper metric spaces M_n, then M is isometric to the ultraproduct $(M_n)_\mathcal{U}$ (see e.g. [38]).

7.1.2 Lipschitz-free spaces

Let M be a pointed metric space M, that is, a metric space with a distinguished point denoted 0. We will denote by $\mathrm{Lip}_0(M)$ the Banach space of all real-valued Lipschitz functions on M vanishing at 0, endowed with the norm given by the Lipschitz constant:

$$\|f\| = \mathrm{Lip}(f) = \sup\left\{\frac{|f(x)-f(y)|}{d(x,y)} : x, y \in M, x \neq y\right\}$$

For $x \in M$, the linear map $\delta(x)\colon \mathrm{Lip}_0(M) \to \mathbb{R}$ given by $\langle f, \delta(x)\rangle = f(x)$ defines an element of $\mathrm{Lip}_0(M)^*$. The *Lipschitz-free space over* M (also called Arens-Eells space and transportation cost space) is defined as the closed subspace of $\mathrm{Lip}_0(M)^*$ generated by these evaluation functionals, that is,

$$\mathcal{F}(M) := \overline{\mathrm{span}}^{\|\cdot\|}\{\delta(x) : x \in M\} \subset \mathrm{Lip}_0(M)^*.$$

The map δ defines an isometric embedding of M into $\mathcal{F}(M)$ such that the following fundamental property holds: for every Banach space X and every Lipschitz function $f\colon M \to X$ with $f(0) = 0$, there is a unique bounded linear operator $T_f\colon \mathcal{F}(M) \to X$ such that $T_f \circ \delta = f$ and $\|T_f\| = \mathrm{Lip}(f)$. It follows in particular that $\mathcal{F}(M)^* = \mathrm{Lip}_0(M)$. We refer the reader to the monographs [138, 171] and the survey [80] for more properties and applications of these spaces.

7.2 Ultraproduct of $\mathcal{F}(M)$ and $\mathrm{Lip}_0(M)$

Recall that, for a Banach space X, the ultraproduct $(X^*)_{\mathcal{U}}$ embeds isometrically into $(X_{\mathcal{U}})^*$ and it is norming for $X_{\mathcal{U}}$. The following result provides an analogous for metric spaces (with Lipschitz functions playing the role of linear functionals). We just need to recall that, given $\lambda \geq 1$, a set $A \subset X^*$ is *λ-norming for X* if $\sup_{x^* \in A \cap B_{X^*}} |x^*(x)| \geq \frac{1}{\lambda}\|x\|$ for every $x \in X$.

Theorem 7.2.1. *Let $\mathcal{U}$ be an ultrafilter on a set I and let $(M_i)_{i\in I}$ be a family of metric spaces. Define an operator $T\colon \mathrm{Lip}_0(M_i)_{\mathcal{U}} \to \mathrm{Lip}_0((M_i)_{\mathcal{U}})$ where $T((f_i)_{\mathcal{U}}) \in \mathrm{Lip}_0((M_i)_{\mathcal{U}})$ is the function given by*

$$T((f_i)_{\mathcal{U}})((x_i)_{\mathcal{U}}) = \lim_{\mathcal{U},i} f_i(x_i).$$

Then T is a well-defined linear operator with $\|T\| \leq 1$ and $T(B_{\mathrm{Lip}_0(M_i)_{\mathcal{U}}})$ is a 1-norming set for $\mathcal{F}((M_i)_{\mathcal{U}})$.

Proof. First notice that if $(x_i)_{i\in I} = (y_i)_{i\in I}$ in $(M_i)_{\mathcal{U}}$ and $(f_i)_{i\in I} = (g_i)_{i\in I}$ in $(\mathrm{Lip}_0(M_i))_{\mathcal{U}}$, then

$$\lim_{\mathcal{U},i} |f_i(x_i) - g_i(y_i)| \leq \lim_{\mathcal{U},i}(\|f_i - g_i\| d(x_i, 0) + \|g_i\| d(x_i, y_i)) = 0.$$

So the formula does not depend on the chosen representations. Moreover, if $\bar{f} = (f_i)_{\mathcal{U}} \in \mathrm{Lip}_0(M_i)_{\mathcal{U}}$ we have that

$$|T\bar{f}(\bar{x}) - T\bar{f}(\bar{y})| \leq \|\bar{f}\| d(\bar{x}, \bar{y}) \quad \forall \bar{x}, \bar{y} \in (M_i)_{\mathcal{U}}$$

so $T\bar{f} \in \mathrm{Lip}_0((M_i)_{\mathcal{U}})$. Therefore T is well-defined and we have that $\|T\bar{f}\| \leq \|\bar{f}\|$ for each $\bar{f} \in \mathrm{Lip}_0(M_i)_{\mathcal{U}}$.

Now, we will prove that $T(B_{\mathrm{Lip}_0(M_i)_{\mathcal{U}}})$ is a 1-norming set. It is well known (see e.g. Proposition 3.3 in [112]) that for that if suffices to check that given $\varepsilon > 0$, a finite subset $A \subset (M_i)_{\mathcal{U}}$, $\varepsilon > 0$ and $f \in \mathrm{Lip}_0((M_i)_{\mathcal{U}})$, there exists $(f_i)_{\mathcal{U}} \in \mathrm{Lip}_0(M_i)_{\mathcal{U}}$ such that $\|(f_i)_{\mathcal{U}}\| \leq (1+\varepsilon)\|f\|$ and $T((f_i)_{\mathcal{U}})|_A = f|_A$.

Let $A = \{(x_i^j)_{\mathcal{U}}\}_{1\leq j\leq n}$ be a finite set, we may assume that $0 = (0_i)_{\mathcal{U}} \in A$ and that the $(x_i^j)_{\mathcal{U}}$ are all different. Fix $\varepsilon > 0$ and $f \in \mathrm{Lip}_0((M_i)_{\mathcal{U}})$. For $j, j' \in \{1, \dots, n\}$ distinct, we have that

$$I_{j,j'} = \left\{ i \in I \mid d(x_i^j, x_i^{j'}) > \frac{1}{1+\varepsilon} d((x_i^j)_{\mathcal{U}}, (x_i^{j'})_{\mathcal{U}}) \right\} \in \mathcal{U}.$$

It follows that $J = \bigcap_{j\neq j'} I_{j,j'} \in \mathcal{U}$ and then we can assume that

$$d(x_i^j, x_i^{j'}) > \frac{1}{1+\varepsilon} d((x_i^j)_{\mathcal{U}}, (x_i^{j'})_{\mathcal{U}})$$

for all $j \neq j'$ and all $i \in I$. For $i \in I$, define a function $f_i\colon \{x_i^1, \dots, x_i^n\} \to \mathbb{R}$ by $f_i(x_i^j) = f((x_k^j)_{\mathcal{U},k})$ for all $j \in \{1, \dots, n\}$, note that $f_i(0_i) = 0$. If $j \neq j'$, we have that

$$\begin{aligned} |f_i(x_i^j) - f_i(x_i^{j'})| &= |f((x_k^j)_{\mathcal{U},k}) - f((x_k^{j'})_{\mathcal{U},k})| \\ &\leq \|f\| d((x_k^j)_{\mathcal{U},k}, (x_k^{j'})_{\mathcal{U},k}) \leq (1+\varepsilon)\|f\| d(x_i^j, x_i^{j'}) \end{aligned}$$

proving that f_i is $(1+\varepsilon)\|f\|$-Lipschitz and belongs to $\mathrm{Lip}_0(\{x_i^1, \dots, x_i^n\})$. Now we extend f_i to a $(1+\varepsilon)\|f\|$-Lipschitz function on M_i and we still denote it by f_i. We have that $\|(f_i)_{\mathcal{U}}\| \leq (1+\varepsilon)\|f\|$ and

$$T((f_i)_{\mathcal{U}})((x_i^j)_{\mathcal{U}}) = \lim_{\mathcal{U},i} f_i(x_i^j) = \lim_{\mathcal{U},i} f((x_k^j)_{\mathcal{U},k}) = f((x_k^j)_{\mathcal{U},k})$$

for all $j \in \{1, ..., n\}$, proving that $T((f_i)_{\mathcal{U}})$ and f coincide on A. A standard argument using the denseness of finitely supported elements in $\mathcal{F}((M_i)_{\mathcal{U}})$ gives that $T(B_{\mathrm{Lip}_0(M_i)_{\mathcal{U}}})$ is 1-norming for $\mathcal{F}((M_i)_{\mathcal{U}})$, as desired. □

The operator T defined in the previous theorem is not injective, in general. Indeed, we have the following characterization.

Proposition 7.2.2. *Let M be a metric space and let $\mathcal{U}$ be a CI ultrafilter on a set I. Let $T\colon \mathrm{Lip}_0(M)_{\mathcal{U}} \to \mathrm{Lip}_0(M_{\mathcal{U}})$ defined in Theorem 7.2.1. The following assertions are equivalent:*

- *(i) M is uniformly discrete and bounded;*
- *(ii) T is injective;*
- *(iii) T is an isometry.*

Proof. $(iii) \implies (ii)$ is obvious.

$(ii) \implies (i)$ Let $(I_n)_n \subset \mathcal{U}$ be decreasing sequence of sets having empty intersection. Suppose by contradiction that M is unbounded. Given $i \in I_n \setminus I_{n+1}$, consider the function f_i given by $f_i(x) = d(x, B(0, n))$. It is easy to check that $\|f_i\| = 1$. Let $\bar{f} = (f_i)_{\mathcal{U}}$. Then it follows $\|\bar{f}\| = 1$ and $T\bar{f} = 0$, which is a contradiction. It follows that M is bounded.

Now suppose that M is not uniformly discrete. Then there exist two sequences $(x_n)_n$ and $(y_n)_n$ in M such that $x_n \neq y_n$ for all $n \in \mathbb{N}$ and $d_n := d(x_n, y_n) \to 0$. We have that $x_n \neq 0$ or $y_n \neq 0$, so by taking a subsequence if necessary we can suppose without loss of generality that $y_n \neq 0$ for all $n \in \mathbb{N}$. For all $n \in \mathbb{N}$, we define a 1-Lipschitz function $g_n\colon M \to [0, d_n]$ by

$$g_n(x) = \max\{d_n - d(y_n, x), 0\}$$

and let $h_n \in \mathrm{Lip}_0(M)$ given by $h_n(x) = g_n(x) - g_n(0)$. It is clear that $\|h_n\|$=1 and $\|h_n\|_\infty \leq d_n$. Now let $i \in I$ and define $f_i = h_n$ where n is such that $i \in I_n \setminus I_{n+1}$. Let $\bar{f} = (f_i)_{\mathcal{U}} \in \mathrm{Lip}_0(M)_{\mathcal{U}}$.

We have that $\|\bar{f}\| = \lim_{\mathcal{U}} \|f_i\| = 1$, so $\bar{f} \neq 0$. However we have that $T\bar{f} = 0$ since $\|h_n\|_\infty \leq d_n$ for $n \in \mathbb{N}$, which is again a contradiction. So M is uniformly discrete.

$(i) \implies (iii)$ Let $\theta = \inf\{d(x,y) : x, y \in M, x \neq y\} > 0$. Take $\bar{f} \in \mathrm{Lip}_0(M)_{\mathcal{U}}$ and $(f_i)_{i\in I}$ with $\bar{f} = (f_i)_{\mathcal{U}}$. By Theorem 7.2.1, we just need to prove that $\|T\bar{f}\| \geq \|\bar{f}\|$. Let $\varepsilon > 0$. For all $i \in I$, pick $x_i, y_i \in M$ two distinct points of M such that $|f_i(x_i) - f_i(y_i)| \geq (1-\varepsilon)\|f_i\| d(x_i, y_i)$. Since M is bounded, we may consider $\bar{x} = (x_i)_{\mathcal{U}}$ and $\bar{y} = (y_i)_{\mathcal{U}}$. Moreover, we have that $d(\bar{x}, \bar{y}) = \lim_{\mathcal{U}} d(x_i, y_i) \geq \theta$, so $\bar{x} \neq \bar{y}$. Taking limit on $\mathcal{U}$, it follows that $|T\bar{f}(\bar{x}) - T\bar{f}(\bar{y})| \geq (1-\varepsilon)\|\bar{f}\| d(\bar{x}, \bar{y})$ and then $\|T\bar{f}\| \geq (1-\varepsilon)\|\bar{f}\|$. Since this is true for all $\varepsilon > 0$, we obtain that $\|T\bar{f}\| \geq \|\bar{f}\|$. □

Note that the implication $(i) \Rightarrow (iii)$ works for any ultrafilter (not necessarily CI).

Remark 7.2.3. In general the operator T is not onto. In fact, let M be a bounded infinite uniformly discrete set. Suppose also that $\mathcal{U}$ is CI and let $(I_n)_n$ be a decreasing sequence in $\mathcal{U}$ with empty intersection. Let $f = T((f_i)_{\mathcal{U}})$. Given $i \in I_n \setminus I_{n+1}$, take $x_i, y_i \in M$ two distinct points with $f_i(x_i) - f_i(y_i) \geq (1 - 1/n)\|f_i\| d(x_i, y_i)$. Let $\bar{x} = (x_i)_{\mathcal{U}}$ and $\bar{y} = (y_i)_{\mathcal{U}}$ in $M_{\mathcal{U}}$ and note that these two elements are distinct since M is uniformly discrete. Then clearly $\bar{f}(\bar{x}) - \bar{f}(\bar{y}) = d(\bar{x}, \bar{y})$, that is,

$$T((\mathrm{Lip}_0(M))_{\mathcal{U}}) \subset \mathrm{SNA}(M_{\mathcal{U}})$$

where $\mathrm{SNA}(M_{\mathcal{U}})$ denotes the set of Lipschitz functions on N attaining their Lipschitz constant at a pair of points of $M_{\mathcal{U}}$. However, whenever the metric space M is infinite, there are Lipschitz functions on $M_{\mathcal{U}}$ which do not attain the Lipschitz constant (otherwise, every linear functional on $\mathcal{F}(M_{\mathcal{U}})$ attains its norm, and then $\mathcal{F}(M_{\mathcal{U}})$ is reflexive).

Theorem 7.2.4. *Let $\mathcal{U}$ be an ultrafilter on a set I and let $(M_i)_{i\in I}$ be a family of metric spaces. Then $\mathcal{F}((M_i)_{\mathcal{U}})$ is linearly isometric to $\overline{\mathrm{span}(\delta(M_i)_{\mathcal{U}})} \subset \mathcal{F}(M_i)_{\mathcal{U}}$.*

Proof. Let $s\colon (M_i)_{\mathcal{U}} \to \mathcal{F}(M_i)_{\mathcal{U}}$ defined by $s((x_i)_{\mathcal{U}}) = (\delta_{x_i})_{\mathcal{U}}$. Note that s is an isometry since

$$\begin{aligned} d((x_i)_{\mathcal{U}}, (y_i)_{\mathcal{U}}) &= \lim_{\mathcal{U}} d(x_i, y_i) = \lim_{\mathcal{U}} \|\delta_{x_i} - \delta_{y_i}\| = \|(\delta_{x_i})_{\mathcal{U}} - (\delta_{x_i})_{\mathcal{U}}\| \\ &= \|s((x_i)_{\mathcal{U}}) - s((y_i)_{\mathcal{U}})\| \end{aligned}$$

for all $(x_i)_{\mathcal{U}}, (x_i)_{\mathcal{U}} \in (M_i)_{\mathcal{U}}$. By the linearization property of Lipschitz-free spaces, s extends to a continuous linear operator $S\colon \mathcal{F}((M_i)_{\mathcal{U}}) \to \mathcal{F}(M_i)_{\mathcal{U}}$ such that $\|S\| = 1$. Let $\varepsilon > 0$ and fix $\mu = \sum_{j=1}^n a_j \delta_{(x_i^j)_{\mathcal{U}}} \in \mathcal{F}((M_i)_{\mathcal{U}})$. Let $T\colon \mathrm{Lip}_0(M_i)_{\mathcal{U}} \to \mathrm{Lip}_0((M_i)_{\mathcal{U}})$ be the operator defined in Theorem 7.2.1. Since $T(B_{\mathrm{Lip}_0(M_i)_{\mathcal{U}}})$ is 1-norming, there exists $(f_i)_{\mathcal{U}} \in \mathrm{Lip}_0(M_i)_{\mathcal{U}}$ such that $\|\mu\| = \langle T((f_i)_{\mathcal{U}}), \mu\rangle$ and $\|(f_i)_{\mathcal{U}}\| \leq 1 + \varepsilon$. It follows that

$$\begin{aligned} \|\mu\| = \langle T((f_i)_{\mathcal{U}}), \mu\rangle &= \sum_{j=1}^n a_j \langle T((f_i)_{\mathcal{U}}), \delta_{(x_i^j)_{\mathcal{U}}}\rangle = \sum_{j=1}^n a_j \lim_{\mathcal{U},i} f_i(x_i^j) \\ &= \sum_{j=1}^n a_j \langle (f_i)_{\mathcal{U}}, (\delta_{x_i^j})_{\mathcal{U}}\rangle = \langle (f_i)_{\mathcal{U}}, S(\mu)\rangle \leq (1+\varepsilon)\|S(\mu)\|, \end{aligned}$$

and we deduce that $\|\mu\| \leq \|S(\mu)\|$ since ε was arbitrary. By density of the measures with finite support, it follows that S is an isometry.

□

Remark 7.2.5. The previous proof gives that $\langle T((f_i)_{\mathcal{U}}), \mu\rangle = \langle (f_i)_{\mathcal{U}}, S(\mu)\rangle$ for all $\mu \in \mathcal{F}((M_i)_{\mathcal{U}})$ and all $(f_i)_{\mathcal{U}} \in \mathrm{Lip}_0(M_i)_{\mathcal{U}}$. In other words, $S^*|_{\mathrm{Lip}_0(M_i)_{\mathcal{U}}} = T$.

7.3 Finite representability of metric spaces

We immediately obtain the following consequence of Theorem 7.2.4.

Theorem 7.3.1. *Let M be a metric space and let $\mathcal{U}$ be an ultrafilter. Then $\mathcal{F}(M_{\mathcal{U}})$ is finitely representable in $\mathcal{F}(M)$.*

We will deal with a related notion for metric spaces introduced by Lee, Naor and Peres in [123]. We take the terminology from [15]. For a biLipschitz embedding ϕ, $\operatorname{dist}(\phi) = \operatorname{Lip}(f)\operatorname{Lip}(f^{-1})$ denotes its *distortion*.

Definition 7.3.2. Let $\lambda \geq 1$ and M, N be metric spaces. We say that M is *finitely λ-Lipschitz representable* into N if for every finite subset F in M and every $\varepsilon > 0$ there is a map $\phi\colon F \to N$ such that $\operatorname{dist}(\phi) \leq \lambda + \varepsilon$.

Moreover, we will consider the following notions.

Definition 7.3.3. Let M and N be metric spaces. If M is finitely λ-Lipschitz representable in N for some $\lambda \geq 1$, we say that M is *crudely finitely Lipschitz representable* in N. If M is finitely 1-Lipschitz representable in N, we say that M is *finitely representable* in N.

In the case of Banach spaces, this notion coincides with the usual finite representability. Indeed, the following is a consequence of Theorem 13 in [144].

Proposition 7.3.4. *Let X and Y be two Banach spaces and let $\lambda \geq 1$. Then X is finitely λ-Lipschitz representable in Y if and only if X is λ-finitely representable in Y.*

Our first goal is to show that the finite Lipschitz representability admits a characterization in terms of ultraproducts which is analogous to the corresponding result for Banach spaces (see Theorem 1.3.2).

Proposition 7.3.5. *Let M, N be metric spaces. The following assertions are equivalent:*

(i) M is finitely λ-Lipschitz representable into N;

(ii) there exist an ultrafilter $\mathcal{U}$ on a set I, scaling factors $r_i > 0$, points $0_i \in N$ and a λ-biLipschitz embedding of M into $(N, 0_i, r_i d)_{\mathcal{U}}$.

In that case, if moreover M is separable, then for any CI ultrafilter $\mathcal{U}$ there are $r_i > 0$, points $0_i \in N$, and a λ-biLipschitz embedding of M into $(N, 0_i, r_i d)_{\mathcal{U}}$.

Proof. Suppose that (i) holds. Fix a point $0 \in M$ and define

$$I := \{(A, \varepsilon) \ : \ 0 \in A \subset M, \ |A| < \infty, \ 0 < \varepsilon < 1\}$$

with the partial order defined by $(A_1, \varepsilon_1) \preceq (A_2, \varepsilon_2)$ if and only if $A_1 \subset A_2$ and $\varepsilon_1 \geq \varepsilon_2$. Since any pair of element of I has an infimum, it is easy to show that

$$\beta := \{\{i \in I : \ i_0 \preceq i\} \mid i_0 \in I\}$$

is a filter basis. Then let $\mathcal{U}$ be any ultrafilter containing the filter generated by β. For all $i = (A_i, \varepsilon_i) \in I$, there exists a one-to-one function $\phi_i\colon A_i \to N$ such that $\operatorname{dist}(\phi_i) \leq \lambda + \varepsilon_i$. Consider the metric space $(N_i, 0_i, d_i)$ where $N_i = N$, $d_i = \|\phi_i^{-1}\| d$ and $0_i = \phi_i(0)$. Given $x \in M$, let $y_i = \phi_i(x)$ if $x \in A_i$ with $i = (A_i, \varepsilon_i)$ and $y_i = 0_i$ if not. Note that

$$d_i(\phi_i(x), 0_i) \leq \|\phi_i\| d_i(x, 0) \leq (\lambda + 1) d(x, 0)$$

and so $(\phi_i(x))_{i\in I}$ gives an element of $(N_i)_{\mathcal{U}}$. This means that $x \mapsto (y_i)_{\mathcal{U}}$ defines a map $\phi\colon M \to (N_i)_{\mathcal{U}}$.

Now, let $\varepsilon_0 > 0$ arbitrary and take $x, x' \in M$. Note that $I_0 := \{(A,\varepsilon) \in I \mid x, x' \in A,\ \varepsilon \leq \varepsilon_0\}$ belongs to $\mathcal{U}$. For $i \in I_0$, we have that

$$d(x,x') \leq \|\phi_i^{-1}\| d(y_i, y_i') \leq \|\phi_i^{-1}\|\|\phi_i\| d(x,x') \leq (\lambda + \varepsilon_0) d(x,x').$$

Letting $r_i = \|\phi_i^{-1}\|$ and taking limit on $\mathcal{U}$, we obtain that

$$d(x,y) \leq d(\phi(x),\phi(y)) \leq (\lambda+\varepsilon_0)d(x,y).$$

Since ε_0 was arbitrary, we conclude that ϕ is a λ-biLipschitz embedding.

For the other implication, suppose that there exists $\phi\colon M \to (N, 0_i, r_i d)_{\mathcal{U}}$ with $\operatorname{dist}(\phi) \leq \lambda$ for some ultrafilter $\mathcal{U}$ on a set I and numbers $r_i > 0$. Let $A = \{x^1, \ldots, x^p\}$ be a finite subset of different elements of M and fix $\varepsilon > 0$. Each $\phi(x^k)$ can be written $\phi(x^k) = (y_i^k)_{\mathcal{U}}$. For $i \in I$, define a function $\phi_i\colon A \to N$ by $\phi_i(x^k) = y_i^k$. Note that for $k, l \in \{1, \ldots, p\}$, we have that

$$\|\phi^{-1}\|^{-1} d(x^k, x^l) \leq d(\phi(x^k), \phi(x^l)) \leq \|\phi\| d(x^k, x^l)$$

and

$$d(\phi(x^k),\phi(x^l)) = \lim_{i,\mathcal{U}} r_i d(\phi_i(x^k), \phi_i(x^l)).$$

It follows that

$$\left\{ i \in I \mid (1-\varepsilon)\|\phi^{-1}\|^{-1} d(x^k,x^l) \leq r_i d(\phi_i(x^k),\phi_i(x^l)) \leq (1+\varepsilon)\|\phi\| d(x^k,x^l)\ \forall k,l \right\}$$

belongs to $\mathcal{U}$ and so it is not empty. Taking i in this set we have that

$$(1-\varepsilon) r_i^{-1} \|\phi^{-1}\| d(a,b) \leq d(\phi_i(a),\phi_i(b)) \leq (1+\varepsilon) r_i^{-1} \|\phi\| d(a,b)$$

for all $a, b \in A$. That is,

$$\operatorname{dist}(\phi_i) \leq \frac{1+\varepsilon}{1-\varepsilon} \operatorname{dist}(\phi) \leq \frac{1+\varepsilon}{1-\varepsilon}\lambda$$

and so (i) holds.

Now suppose that M is separable and that (i) holds. Let $\mathcal{U}$ be any CI ultrafiltrer over a set I and let $(I_n)_n \subset \mathcal{U}$ be a decreasing sequence with empty intersection. Let $\{x_n\}_n$ be a countable dense subset of M. For all $n \in \mathbb{N}$, there exists a function $\phi_n\colon \{x_k\}_{1\leq k\leq n} \to N$ such that $\operatorname{dist}(\phi_n) \leq (1 + 1/n)\lambda$. Given $i \in \bigcup_n I_n$, let n_i be such that $i \in I_{n_i} \setminus I_{n_i+1}$, and consider the metric space $(N_i, 0_i, d_i)$ where $N_i = N$, $d_i = \|\phi_{n_i}^{-1}\| d$ and $0_i = \phi_{n_i}(x_1)$. If $i \in I \setminus I_1$, define $r_i > 0$ arbitrarily. Note that, given $m \in \mathbb{N}$,

$$d_i(\phi_{n_i}(x_m), 0_i) \leq \operatorname{dist}(\phi_{n_i}) d(x_m, x_1) \leq 2\lambda d(x_m, x_1),$$

and so we may consider the element $(\phi_{n_i}(x_m))_{\mathcal{U},i}$.

Now, define a function $\phi\colon \{x_n\}_n \to (N_i)_{\mathcal{U}}$ by $\phi(x_m) = (\phi_{n_i}(x_m))_{\mathcal{U},i}$. We will prove that ϕ is an isometry and then will extend to a unique isometry defined on M. Let $\varepsilon > 0$ and $p_0 \in \mathbb{N}$ such that $\frac{1}{p_0} < \varepsilon$. Let $p < q$ and define $\tilde{q} = \max\{p_0, q\}$. Let $I_0 = \bigcup_{n \geq \tilde{q}} I_n \in \mathcal{U}$ and take $i \in I_0$. It is clear that $y_i^p = \phi_{n_i}(x_p)$ and $y_i^q = \phi_{n_i}(x_q)$. It follows that

$$\begin{aligned} d(x_p,x_q) &\leq \|\phi_{n_i}^{-1}\| d(y_i^p, y_i^q) \leq \|\phi_{n_i}\| \|\phi_{n_i}^{-1}\| d(x_p,x_q) \\ &\leq (1+1/n_i)\lambda d(x_p,x_q) < (1+\varepsilon)\lambda d(x_p,x_q) \end{aligned}$$

Taking limit on $\mathcal{U}$, we deduce that

$$d(x_p,x_q) \leq d(\phi(x_p),\phi(x_q)) \leq (1+\varepsilon)\lambda d(x_p,x_q).$$

Since ε was arbitrary, we conclude that ϕ is an isometry and the proof is complete. □

Note that in the case $N = X$ is a Banach space clearly one may assume that $\phi_i(0) = 0$ and $\|\phi_i^{-1}\| = 1$ (and then $r_i = 1$) in the proof of $(i) \Rightarrow (ii)$ above, so we get:

Theorem 7.3.6. *Let M be a metric space and X be a Banach space. The following assertions are equivalent:*

(i) M is finitely λ-Lipschitz representable in X;

(ii) there exists an ultrafilter $\mathcal{U}$ such that M is λ-biLipschitz equivalent to a subset of $X_{\mathcal{U}}$.

In that case, if moreover M is separable and $\mathcal{U}$ is a CI ultrafilter, then M is λ-biLipschitz equivalent to a subset of $X_{\mathcal{U}}$.

Theorem 7.3.7. *Let M be a metric space and X be a Banach space. Assume that M is finitely λ-Lipschitz representable in X. Then $\mathcal{F}(M)$ is λ-finitely representable in $\mathcal{F}(X)$.*

Proof. Assume M is finitely λ-Lipschitz representable in X. By Theorem 7.3.6, there exists an ultrafilter $\mathcal{U}$ such that M λ-biLipschitz embeds in $X_{\mathcal{U}}$. It follows that $\mathcal{F}(M)$ is λ-isomorphic to a subspace of $\mathcal{F}(X_{\mathcal{U}})$. By Theorem 7.2.4, we deduce that $\mathcal{F}(M)$ is λ-isomorphic to a subspace of $\mathcal{F}(X)_{\mathcal{U}}$. This means exactly that $\mathcal{F}(M)$ is λ-finitely representable in $\mathcal{F}(X)$. □

Remark 7.3.8. Note that if M and N are bounded metric spaces satisfying that for every finite subset $F \subset M$ and every $\varepsilon > 0$ there exists a function $f \colon F \to N$ such that

$$(1+\varepsilon)^{-1} d(x,y) \leq d(\phi(x), \phi(y)) \leq (1+\varepsilon) d(x,y) \quad \forall x, y \in F,$$

then a similar argument shows that $\mathcal{F}(M)$ is finitely representable in $\mathcal{F}(N)$.

We obtain some immediate consequences:

Corollary 7.3.9. *Let X and Y be Banach spaces. Then $\mathcal{F}(X)$ is finitely representable in $\mathcal{F}(Y)$ in any of the following cases:*

(a) $X = \ell_2$ and Y is any infinite-dimensional Banach space.

(b) $X = Y^{**}$ and Y is any Banach space.

(c) $X = L_p([0,1])$ and $Y = \ell_p$, where $1 \leq p < \infty$.

Proof. In each of the cases, we have that X is finitely representable in Y. In fact, for (a) it is a consequence of Dvoretzky's theorem (see Theorem 1.3.4). For (b), it is the principle of local reflexivity (see Theorem 1.3.5) and (c) is part of Theorem 6.2 in [67]. □

Corollary 7.3.10. *Let X and Y be Banach spaces such that X coarsely Lipschitz embeds into Y. Then $\mathcal{F}(X)$ is crudely finitely representable in $\mathcal{F}(Y)$.*

Proof. That follows from Ribe's theorem (see Theorem 14.2.27 in [1]). □

7.4 Some remarks on the cotype of Lipschitz-free spaces

Not much is known about the Rademacher cotype of Lipschitz-free spaces. Bourgain proved ([33], see also Theorem 10.16 in [138]) that $\mathcal{F}(\ell_1)$ has trivial cotype, but whether $\mathcal{F}(\mathbb{R}^n)$ has a nontrivial cotype is a long-standing open problem. Note that as a consequence of Corollary 7.3.9 the following dichotomy holds:

(a) $\mathcal{F}(\ell_2)$ has cotype; or

(b) $\mathcal{F}(X)$ does not have cotype for any infinite-dimensional Banach space X.

We obtain now some remarks concerning the cotype of $\mathcal{F}(M)$. Recall that the notion of metric cotype was introduced by Mendel and Naor in [133]. Note that if M is a metric space such that $\mathcal{F}(M)$ has Rademacher cotype q, then M also has metric cotype q. In particular, if $M = X$ is a Banach space then X has Rademacher cotype q (this follows directly from the fact that the metric cotype passes to subspaces and is equivalent to the usual cotype for Banach spaces).

On the other hand, the cotype of $\mathcal{F}(M)$ is related to the metric type introduced by Bourgain, Milman and Wolfson in [34].

Proposition 7.4.1. *Let M be a metric space such that $\mathcal{F}(M)$ has Rademacher cotype. Then M has BMW type. In particular, if $M = X$ is a Banach space then X has Rademacher type.*

Proof. Suppose that M does not have BMW type. By Theorem 2.6 in [34], M contains uniformly biLipschitz copies of the Hamming cubes $\mathbb{F}_2^n$. Bourgain's result mentioned earlier provides a constant $C \geq 1$ such that for all m there exists n such that $\mathcal{F}(\mathbb{F}_2^n)$ contains a C-isomorphic copy of ℓ_∞^m. Since the space $\mathcal{F}(M)$ contains D-isomorphic copies of the spaces $\mathcal{F}(\mathbb{F}_2^n)$ for some $D \geq 1$, it follows that $\mathcal{F}(M)$ contains CD-isomorphic copies of the spaces ℓ_∞^m. In particular, $\mathcal{F}(M)$ can not have cotype. If M is Banach space then M has BMW type if and only if M has Rademacher type by Corollary 5.9 in [34]. $\square$

Remark 7.4.2. If X is a Banach space such that $\mathcal{F}(X)$ has Rademacher cotype, then we can deduce easily from Theorem 7.3.7 that X has Rademacher type. In fact, if X does not have Rademacher type then ℓ_1 is finitely representable in X (see Theorem 1.3.6) and then $\mathcal{F}(\ell_1)$ is finitely representable in $\mathcal{F}(X)$. This is a contradiction since $\mathcal{F}(\ell_1)$ does not have Rademacher cotype.

It is not known which metric spaces M satisfy that $\mathcal{F}(M)$ and $\mathcal{F}(\mathcal{F}(M))$ are isomorphic (one example is Pełczyński universal space, see [81]). The next result shows in particular that if $\mathcal{F}(M)$ has cotype then $\mathcal{F}(M)$ and $\mathcal{F}(\mathcal{F}(M))$ cannot be isomorphic.

Corollary 7.4.3. *Let M be an infinite metric space. Then $\mathcal{F}(\mathcal{F}(M))$ does not have Rademacher cotype.*

Proof. Suppose that $\mathcal{F}(\mathcal{F}(M))$ has cotype. It follows from the previous result that $\mathcal{F}(M)$ has type. This is impossible since $\mathcal{F}(M)$ contains an isomorphic copy of ℓ_1. $\square$

Aliaga, Noûs, Petitjean and Procházka have proved recently in [4] that several isomorphic properties of $\mathcal{F}(X)$ (such as the Schur property and weak sequential completeness) are compactly determined. We finish the section by showing that this is also the case of the cotype. The proof adapts some ideas from [140].

Proposition 7.4.4. *Let X be a Banach space and let $q \geq 2$. The following assertions are equivalent:*

(i) $\mathcal{F}(X)$ has Rademacher cotype (resp. cotype q);

(ii) $\mathcal{F}(K)$ *has Rademacher cotype (resp. cotype q) for any (countable) compact set $K \subset X$;*

(iii) $\mathcal{F}(\{x_n\}_n)$ *has Rademacher cotype (resp. cotype q) for any null sequence $(x_n)_n \subset X$.*

Proof. The implications $(i) \implies (ii) \implies (iii)$ are trivial. Suppose that $\mathcal{F}(X)$ does not have Rademacher cotype (resp. cotype q). It follows that $\mathcal{F}(2^{-n}B_X)$ does not have cotype (resp. cotype q) for all $n \in \mathbb{N}$. In particular, for all $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ and $\mu_1^n, \dots, \mu_m^n \in \mathcal{F}(2^{-n}B_X)$ such that

$$\left(\sum_{k=1}^{m} \|\mu_k^n\|^n\right)^{\frac{1}{n}} > n \int_0^1 \left\|\sum_{k=1}^{m} \mu_k^n r_k(t)\right\| dt$$

$$\left(\text{resp. } \left(\sum_{k=1}^{m} \|\mu_k^n\|^q\right)^{\frac{1}{q}} > n \int_0^1 \left\|\sum_{k=1}^{m} \mu_k^n r_k(t)\right\| dt\right).$$

Since the measures with finite support are dense in a Lipschitz-free space, we can and do suppose that $\mu_1^n, \dots, \mu_m^n \in \mathcal{F}(K_n)$ where K_n is a finite subset of $2^{-n}B_X$. Define $K = \bigcup_n K_n \cup \{0\}$. Then K is a null sequence such that $\mathcal{F}(K)$ does not have Rademacher cotype (resp. cotype q). ☐

Remark 7.4.5. Since $\mathcal{F}(\ell_1)$ does not have Rademacher cotype, the previous theorem implies that there exists a null sequence $(x_n)_n$ in ℓ_1 such that $\mathcal{F}(\{x_n\}_n)$ does not have cotype. Moreover, it is possible to explicite such a sequence. For $n \geq 1$, define

$$x_n = \frac{1}{k^2}\left(r_1\left(\frac{m}{2^k}\right), ..., r_k\left(\frac{m}{2^k}\right), 0, 0, ...\right)$$

where k and m are such that $2^k - 1 \leq n < 2^{k+1} - 1$ and $n = 2^k - 1 + m$ with $0 \leq m \leq 2^k - 1$. Note that $\mathcal{F}(\mathbb{F}_2^k) = \mathcal{F}(\{x_n\}_{2^k-1 \leq n < 2^{k+1}-1})$ isometrically since the metric space $\{x_n\}_{2^k-1 \leq n < 2^{k+1}-1}$ is obtained by scaling the distance on $\mathbb{F}_2^k$. It follows that $\mathcal{F}(\mathbb{F}_2^k)$ is an isometric subspace of $\mathcal{F}(\{x_n\}_n)$ for all $k \geq 1$. So $\mathcal{F}(\{x_n\}_n)$ does not have cotype.

7.5 Stability of $\mathcal{F}(M)$ and $\mathrm{Lip}_0(M)$ under ultraproducts

Several classes of Banach spaces, as Banach lattices, C*-algebras and $C(K)$ spaces, are stable under ultraproducts [97]. Given a metric space M and an ultrafilter $\mathcal{U}$, it is natural to ask if $\mathcal{F}(M)_{\mathcal{U}}$ is isomorphic to $\mathcal{F}(M_{\mathcal{U}})$ or more generally if there exists a metric space N such that $\mathcal{F}(M)_{\mathcal{U}}$ is isomorphic to $\mathcal{F}(N)$. The first question is easily seen to be false with the following example:

Example 7.5.1. Let M be an infinite proper metric space. Then $M_{\mathcal{U}} = M$ isometrically by Fact 7.1.2.e) whereas $\mathcal{F}(M)_{\mathcal{U}}$ is not separable. Thus, $\mathcal{F}(M_{\mathcal{U}})$ is not isomorphic to $\mathcal{F}(M)_{\mathcal{U}}$.

In the first version of our preprint, we provided some examples of metric spaces (as $M = [0, 1]$ and $M = \mathbb{N}$) such that $\mathcal{F}(M)_{\mathcal{U}}$ is not isomorphic to a Lipschitz-free space, and we asked whether an analogous statement holds for every metric space. T. Kania has kindly provided an answer for a general metric space by strengthening our previous result.

Proposition 7.5.2. *Let $\mathcal{U}$ be a CI ultrafilter on an infinite set I, M be a metric space and X be an infinite-dimensional Banach space. Then $X_{\mathcal{U}}$ is not isomorphic to a subspace of $\mathcal{F}(M)$.*

Proof. Since $\mathcal{U}$ is CI, there exists a strictly decreasing sequence $(I_n)_n$ such that $\bigcap_{n\in\mathbb{N}} I_n = \emptyset$. Define $(a_i)_{i\in I}$ by $a_i = \frac{1}{n}$ and $n_i = n$ if $i \in I_n \setminus I_{n+1}$. By Dvoretzky's theorem, for all $i \in I$ there exists a subspace X_i of X and an isomorphism $T_i \colon \ell_2^{n_i} \to X_i$ such that

$$\|x\| \leq \|T_i(x)\| \leq (1 + a_i)\|x\|$$

for all $x \in \ell_2^{n_i}$. Now we define $T \colon (\ell_2^{n_i})_{\mathcal{U}} \to X_{\mathcal{U}}$ by $T((x_i)_{\mathcal{U}}) = (T_i(x))_{\mathcal{U}}$. Since $\lim_{\mathcal{U}} a_i = 0$, the previous inequality implies that T is an isometry. We have that the ultraproduct of Hilbert spaces $(\ell_2^{n_i})_{\mathcal{U}}$ is also a Hilbert space and it is non-separable (see Theorem 3.1 in [36]). The conclusion follows from the fact that a Lipschitz-free space does not contain a non-separable weakly compact set [113]. $\square$

Corollary 7.5.3. *Let $\mathcal{U}$ be a CI ultrafilter and M be a infinite metric space. Then $\mathcal{F}(M)_{\mathcal{U}}$ is not isomorphic to a subspace of a Lipschitz-free space.*

Thanks to Gelfand-Naimark theorem, the ultraproduct of $\mathcal{C}(K)$-spaces is still a $\mathcal{C}(K)$-space, i.e. if $\mathcal{U}$ is an ultrafilter on a set I and if $(K_i)_{i\in I}$ is a family of compact spaces, then there exists a compact space K such that $(\mathcal{C}(K_i))_{\mathcal{U}} = \mathcal{C}(K)$ isometrically. Moreover, if there is an algebra isomorphism between $(\mathcal{C}(K_i))_{\mathcal{U}}$ and $\mathcal{C}(K)$ then $(K_i)_{\mathcal{U}}$ is homeomorphic to a dense subset of K [97]. The following result is the analogue for $\mathrm{Lip}_0(K)$.

Proposition 7.5.4. *Let K be a compact metric space. Let $\mathcal{U}$ be an ultrafilter on a set I and let $(M_i)_{i\in I}$ be a family of uniformly bounded metric spaces. If there exists an algebra isomorphism between $(\mathrm{Lip}_0(M_i))_{\mathcal{U}}$ and $\mathrm{Lip}_0(K)$, then $(M_i)_{\mathcal{U}}$ is biLipschitz equivalent to a subset of K.*

Proof. Let $R \colon \mathrm{Lip}_0(K) \to (\mathrm{Lip}_0(M_i))_{\mathcal{U}}$ be an algebra isomorphism. If $(x_i)_{\mathcal{U}} \in (M_i)_{\mathcal{U}}$, we can define a functional $F_{(x_i)_{\mathcal{U}}} \in \mathrm{Lip}_0(K)^*$ by

$$F_{(x_i)_{\mathcal{U}}}(f) = \lim_{\mathcal{U}} f_i(x_i)$$

for all $f \in \mathrm{Lip}_0(K)$ where $(f_i)_{\mathcal{U}} = R(f)$. In other words, we have $F_{(x_i)_{\mathcal{U}}}(f) = TR(f)((x_i)_{\mathcal{U}})$ where $T \colon (\mathrm{Lip}_0(M_i))_{\mathcal{U}} \to (\mathrm{Lip}_0(M_i)_{\mathcal{U}})$ is the operator defined in Theorem 7.2.1. It is clear that $F_{(x_i)_{\mathcal{U}}}$ is also multiplicative. By Lemma 7.28 in [171], $F_{(x_i)_{\mathcal{U}}}$ is an evaluation, that is there exists a unique $h((x_i)_{\mathcal{U}}) \in K$ such that $F_{(x_i)_{\mathcal{U}}} = \delta_{h((x_i)_{\mathcal{U}})}$. This allows to define a map $h \colon (M_i)_{\mathcal{U}} \to K$, we will show this is the biLipschitz map we are looking for.

Let $(x_i)_{\mathcal{U}}, (y_i)_{\mathcal{U}} \in (M_i)_{\mathcal{U}}$. We have that

$$\begin{aligned} d(h((x_i)_{\mathcal{U}}), h((y_i)_{\mathcal{U}})) &= \|\delta_{h((x_i)_{\mathcal{U}})} - \delta_{h((y_i)_{\mathcal{U}})}\| \\ &= \|F_{(x_i)_{\mathcal{U}}} - F_{(y_i)_{\mathcal{U}}}\| \\ &= \sup_{f\in B_{\mathrm{Lip}_0(K)}} |F_{(x_i)_{\mathcal{U}}}(f) - F_{(y_i)_{\mathcal{U}}}(f)| \\ &= \sup_{f\in B_{\mathrm{Lip}_0(K)}} |TR(f)((x_i)_{\mathcal{U}}) - TR(f)((y_i)_{\mathcal{U}})| \end{aligned}$$

It follows that on the one hand:

$$d(h(x_i)_{\mathcal{U}}, h(y_i)_{\mathcal{U}}) \leq \sup_{f\in B_{\mathrm{Lip}_0(K)}} \|TR(f)\| d((x_i)_{\mathcal{U}}, (y_i)_{\mathcal{U}}) \leq \|R\| d((x_i)_{\mathcal{U}}, (y_i)_{\mathcal{U}}).$$

On the other hand, taking $\varepsilon > 0$, there exists $(f_i)_{\mathcal{U}} \in \mathrm{Lip}_0(M_i)_{\mathcal{U}}$ such that $\|\delta_{(x_i)_{\mathcal{U}}} - \delta_{(y_i)_{\mathcal{U}}}\| = \langle T((f_i)_{\mathcal{U}}), \delta_{(x_i)_{\mathcal{U}}} - \delta_{(y_i)_{\mathcal{U}}}\rangle$ and $\|(f_i)_{\mathcal{U}}\| \leq 1 + \varepsilon$ by Theorem 7.2.1. Let $g \in \mathrm{Lip}_0(K)$ such that

$R(g) = (f_i)_{\mathcal{U}}$ and note that $\|g\| \leq (1+\varepsilon)\|R^{-1}\|$. It follows that

$$\begin{aligned}
d((x_i)_{\mathcal{U}}, (y_i)_{\mathcal{U}}) &= \|\delta_{(x_i)_{\mathcal{U}}} - \delta_{(y_i)_{\mathcal{U}}}\| \\
&= \langle TR(g), \delta_{(x_i)_{\mathcal{U}}} - \delta_{(y_i)_{\mathcal{U}}} \rangle \\
&= (1+\varepsilon)\|R^{-1}\| \left\langle TR\left(\frac{g}{(1+\varepsilon)\|R^{-1}\|}\right), \delta_{(x_i)_{\mathcal{U}}} - \delta_{(y_i)_{\mathcal{U}}} \right\rangle \\
&\leq (1+\varepsilon)\|R^{-1}\| \sup_{f \in B_{\mathrm{Lip}_0(K)}} |\langle TR(f), \delta_{(x_i)_{\mathcal{U}}} - \delta_{(y_i)_{\mathcal{U}}} \rangle| \\
&= (1+\varepsilon)\|R^{-1}\| \sup_{f \in B_{\mathrm{Lip}_0(K)}} |TR(f)((x_i)_{\mathcal{U}}) - TR(f)((y_i)_{\mathcal{U}})| \\
&= (1+\varepsilon)\|R^{-1}\| d(h(x_i)_{\mathcal{U}}, h(y_i)_{\mathcal{U}}),
\end{aligned}$$

and since ε was arbitrary, we obtain that $d((x_i)_{\mathcal{U}}, (y_i)_{\mathcal{U}}) \leq \|R^{-1}\| d(h(x_i)_{\mathcal{U}}, h(y_i)_{\mathcal{U}})$. Then we deduce that h is biLipschitz. □

We finish the paper remarking that the analogy with the case of ultraproducts $C(K)$-spaces is not complete. Indeed, the map h constructed in the proof above does not have dense range, in general. For instance, assume $M_i = M$ is a compact metric space. Then we have $T \circ R(f) = f \circ h$ for each $f \in \mathrm{Lip}_0(K)$, that is, $T \circ R$ is the composition operator $C_h \colon \mathrm{Lip}_0(K) \to \mathrm{Lip}_0(M)$. Since R is an isomorphism and T is not injective (by Proposition 7.2.2) we get that C_h is not injective. It follows (see Proposition 2.25 in [171]) that $h(M) = \overline{h(M)}$ is properly contained in K.

Chapter 8

A note on non-separable Lipschitz-free spaces

8.1 Equivalent properties to separability

We denote by dens(M) the density character of a metric space M, that is, the minimum cardinality of a dense subset of M. The first uncountanle ordinal is denoted by ω_1.

Given a family $\{(M_\gamma, d_\gamma) : \gamma \in \Gamma\}$ of pointed metric spaces, their *metric sum* is defined as the metric space $M = \{0\} \cup \bigcup_{\gamma\in\Gamma}(M_\gamma \setminus \{0\})$ with the metric given by $d(x, x') = d_\gamma(x, x')$ for $x, x' \in M_\gamma$ and $d(x, x') = d_\gamma(x, 0) + d_{\gamma'}(0, x')$ for $x \in M_\gamma$, $x' \in M_{\gamma'}$. That is, M is obtained by taking the disjoint union of the M_γ's, identifying their base points, and having all paths between different M_γ's go through the base point. $\mathcal{F}(M)$ is then linearly isometric to the ℓ_1-sum of the spaces $\mathcal{F}(M_\gamma)$ (see e.g. [171, Proposition 3.9]). In particular, when all M_γ are two-point spaces, we get $\mathcal{F}(M) = \ell_1(\Gamma)$ and $\mathrm{Lip}(M) = \ell_\infty(\Gamma)$ for the metric space $M = \Gamma \cup \{0\}$ with the metric given by $d(\gamma, 0) = 1$ and $d(\gamma, \gamma') = 2$ for all $\gamma \neq \gamma' \in \Gamma$.

Differentiability properties. Our first result characterizes the differentiability of convex functions in Lipschitz free spaces in terms of separability.

We recall that a Banach space X is *weak Asplund* if every continuous convex function defined on an open convex subset U of X is Gâteaux differentiable on a G_δ dense subset of U. We recall that a Banach space X is a *Gâteaux differentiability space* (in short, GDS) if every continuous convex function defined on an open convex subset U of X is Gâteaux differentiable on a dense subset of U. It has been shown in [135] that the class of Gâteaux differentiability spaces is strictly larger than the class of weak Asplund spaces.

Theorem 8.1.1. *Let M be a metric space. The following assertions are equivalent:*

(i) $\mathcal{F}(M)$ is separable;

(ii) $\mathcal{F}(M)$ is weakly Asplund;

(iii) $\mathcal{F}(M)$ is a GDS.

Proof. The implications (i)⇒(ii)⇒(iii) are true for any Banach space.

We thus only need to prove that (iii) implies (i). Suppose that $\mathcal{F}(M)$ or, equivalently, M is not separable. By [92, Proposition 3], $\mathcal{F}(M)$ contains a complemented subspace X isomorphic to $l_1(\omega_1)$. Since the norm of $l_1(\omega_1)$ is nowhere Gâteaux differentiable (see [57, Example 1.6.c]), it follows that $l_1(\omega_1)$ is not a GDS and then neither is X. Since the property of being a GDS is preserved by passing to complemented subspaces (see for example [142, Proposition 6.8]), we conclude that $\mathcal{F}(M)$ is not a GDS. $\square$

Remark 8.1.2. From the previous theorem, one can also see that $\mathcal{F}(M)$ is separable if and only if $\mathcal{F}(M)$ is WCG space or a subspace of WCG space as these two classes contain all separable Banach spaces and are contained in the class of weakly Asplund spaces.

On the other hand, note that a Lipschitz-free space $\mathcal{F}(M)$ is Asplund if and only if it is finite-dimensional, that is, if M is finite. In fact, any infinite dimensional Lipschitz-free space contains an isomorphic copy of l_1, which is not Asplund. In particular, $\mathrm{Lip}_0(M)$ is WCG if and only if M is finite (see Theorem 2.43 in [142]).

Finally, the previous theorem also shows that $\mathcal{F}(M)$ is separable if and only if it admits an equivalent ((strongly) uniformly) Gâteaux differentiable norm. Indeed, existence of such norm on a space X implies that X is weak Asplund.

Weak* sequential compactness of the dual ball. Since every GDS has weak* sequentially compact dual ball, it is natural to ask whether $\mathcal{F}(M)$ has to be separable if $(B_{\mathrm{Lip}_0(M)}, w^*)$ is weakly* sequentially compact. It turns out that this question is undecidable in ZFC.

We denote by $[\mathbb{N}]^\omega$ the family of all infinite subsets of $\mathbb{N}$. We write $\aleph_1$ for the cardinality of the set of all countable ordinal numbers and $\mathfrak{c}$ for the cardinality of the real numbers. Using the notation from [165], we say that a subset $\mathscr{S} \subset [\mathbb{N}]^\omega$ is a *splitting family* if it satisfies the following: for any $A \in [\mathbb{N}]^\omega$, there is $S \in \mathscr{S}$ such that $A \cap S$ and $A \setminus S$ are both infinite. The *splitting cardinal* $\mathfrak{s}$ is defined as the smallest possible cardinality of a splitting family in $[\mathbb{N}]^\omega$. It is easy to check that $\aleph_1 \leq \mathfrak{s} \leq \mathfrak{c}$, and the four possibilities $\aleph_1 = \mathfrak{s} = \mathfrak{c}$ (i.e. the Continuum Hypothesis), $\aleph_1 = \mathfrak{s} < \mathfrak{c}$, $\aleph_1 < \mathfrak{s} = \mathfrak{c}$, $\aleph_1 < \mathfrak{s} < \mathfrak{c}$ are all known to be consistent in ZFC.

Theorem 8.1.3. *Let M be a complete metric space. Then the following assertions are equivalent:*

(i) $\mathrm{dens}(M) \geq \mathfrak{s}$,

(ii) M contains a uniformly discrete subset of cardinality $\mathfrak{s}$,

(iii) $B_{\mathrm{Lip}_0(M)}$ is not w^*-sequentially compact.

In particular, $B_{\ell_\infty(\Gamma)}$ is w^*-sequentially compact if and only if $|\Gamma| < \mathfrak{s}$. Therefore it is undecidable whether $B_{\ell_\infty(\omega_1)}$ is w^*-sequentially compact.

For the proof of $(i) \implies (ii)$ we will use the following stronger observation of independent interest.

Lemma 8.1.4. *Let κ be a cardinal of uncountable cofinality. Then* $\mathrm{dens}(M) \geq \kappa$ *if and only if M contains a uniformly discrete subset of cardinality κ.*

Proof. For each $n \in \mathbb{N}$, let A_n be a maximal $\frac{1}{n}$-separated subset of M. Then $\bigcup_{n\in\mathbb{N}} A_n$ is dense in M, hence it has cardinality at least κ. Since κ has uncountable cofinality, this implies that $|A_n| \geq \kappa$ for some n.

For the converse, let $A \subset M$ be a ε-separated family of cardinality κ and let D be a dense set in M. For every $x \in A$ let $\psi(x) \in D \cap B(x, \varepsilon/3)$. Then ψ is injective and $|D| \geq \kappa$ $\square$

Remark 8.1.5. Note that the direct implication of the previous lemma fails if κ has countable cofinality: suppose that $\kappa = \sup_n \kappa_n$ with $\kappa > \kappa_n$ for all n, let M_n be a metric space of cardinality κ_n where all non-zero distances are $\frac{1}{n}$, and construct M as a separated union of the spaces M_n. Then $\mathrm{dens}(M) = \kappa$ but M contains no uniformly discrete subset of cardinality κ.

Proof. $(i) \implies (ii)$: It is easy to prove that the cofinality of $\mathfrak{s}$ is not countable (see e.g. [60, Proposition 1.1]), so Lemma 8.1.4 yields the conclusion.

$(ii) \implies (iii)$: Fix a set $\{x_\gamma : \gamma < \mathfrak{s}\} \subset M$ such that any pair of points are at distance at least $r > 0$, and define functions $f_\gamma \in B_{\mathrm{Lip}_0(M)}$ by

$$f_\gamma(x) = \max\{r - d(x, x_\gamma), 0\}$$

for $x \in M$, so that $f_\gamma(x_\gamma) = r$ and $f_\gamma(x_\lambda) = 0$ for $\lambda < \mathfrak{s}$, $\lambda \neq \gamma$. Now fix a splitting family $\mathscr{S} = \{S_\gamma : \gamma < \mathfrak{s}\}$ in $[\mathbb{N}]^\omega$ and define a sequence $(g_n)_n$ in $B_{\mathrm{Lip}_0(M)}$ by $g_n = \sup\{f_\gamma : n \in S_\gamma\}$ so that

$$g_n(x_\gamma) = \begin{cases} r & \text{, if } n \in S_\gamma \\ 0 & \text{, if } n \notin S_\gamma \end{cases}.$$

Let us see that no subsequence $(g_{n_k})_{k\in\mathbb{N}}$ of (g_n) converges weakly*, i.e. pointwise. Let $A = \{n_k : k \in \mathbb{N}\} \in [\mathbb{N}]^\omega$, then there exists $\gamma < \mathfrak{s}$ such that $A \cap S_\gamma$ and $A \setminus S_\gamma$ are both infinite, thus $g_{n_k}(x_\gamma)$ takes both values 0 and r for infinitely many values of k so it cannot converge. We conclude that $B_{\mathrm{Lip}(M)}$ is not w^*-sequentially compact.

$(iii) \implies (i)$: We may assume that M is infinite, as the implication is trivial otherwise. Suppose that there is a sequence $(f_n)_n$ in $B_{\mathrm{Lip}_0(M)}$ with no pointwise convergent subsequence. Let D be a dense subset of M and $\mathscr{S}$ be the family of sets $S_{x,q} \subset \mathbb{N}$, for $x \in D$ and $q \in \mathbb{Q}$, given by

$$S_{x,q} = \{n \in \mathbb{N} : f_n(x) > q\}.$$

Fix an arbitrary $A \in [\mathbb{N}]^\omega$ and order it as $A = \{n_k : k \in \mathbb{N}\}$ with $n_k < n_{k+1}$ for all k. Since $(f_{n_k})_k$ is equi-Lipschitz, there is $x \in D$ such that $(f_{n_k}(x))_k$ does not converge (here we use the easily provable and well known fact that if a sequence of equi-Lipschitz functions converges pointwise on a dense set, then it converges pointwise everywhere). Since $(f_{n_k}(x))_k$ is bounded by $d(x,0)$, there must be 2 different cluster points $a < b$. If $q \in \mathbb{Q}$ is such that $a < q < b$ then the set $S_{x,q}$ splits the set A. This shows that $\mathscr{S}$ contains a splitting family in $[\mathbb{N}]^\omega$ and therefore $|D| = |D \times \mathbb{Q}| \geq |\mathscr{S}| \geq \mathfrak{s}$.

The last assertion follows from the fact that $\mathrm{Lip}_0(M) = \ell_\infty(\Gamma)$ when M is the space of infinite cardinality Γ where $d(x,0) = 1$ and $d(x,y) = 2$ for all $x, y \neq 0$. $\square$

Under the Continuum Hypothesis, condition (i) in Theorem 8.1.3 is equivalent to the non-separability of M, thus also of $\mathcal{F}(M)$, so we get the following consequence:

Corollary 8.1.6. *Let M be a metric space. Under the Continuum Hypothesis, the following assertions are equivalent:*

(i) $\mathcal{F}(M)$ is separable;

(ii) $B_{\mathrm{Lip}_0(M)}$ is w^*-sequentially compact.

Topological properties of the dual ball. Recall that a Banach space X is said to have *Corson's property* $(\mathcal{C})$ if for every family $\mathcal{A}$ of closed convex sets in X with empty intersection there is a countable subfamily $\mathcal{B}$ of $\mathcal{A}$ with empty intersection.

Theorem 8.1.7. *Let M be a complete metric space. Then the following assertions are equivalent:*

(i) $\mathcal{F}(M)$ is separable;

(ii) $\mathcal{F}(M)$ has Corson's property $(\mathcal{C})$;

Proof. The implication $(i) \implies (ii)$ is standard for any Banach space X in place of $\mathcal{F}(M)$.

Now we show that $(ii) \implies (i)$. Assume that $\mathcal{F}(M)$, and therefore M, is non-separable. Then $\ell_1(\omega_1)$ is isomorphic to a subspace of $\mathcal{F}(M)$. Now $\mathcal{F}(M)$ cannot have Corson's property $(\mathcal{C})$ as this property passes to subspaces and it is known that $\ell_1(\omega_1)$ does not have this property (See [67, Exercise 14.46]; alternatively, since $\text{dens}(C[0,\omega_1]) = \aleph_1$ there is a quotient map from $\ell_1(\omega_1)$ onto $C[0,\omega_1]$. Now this last space is known to fail Corson's property $(\mathcal{C})$ (see [67, Theorem 14.36]) and this property is stable by quotients.)

Remark 8.1.8. In fact, when combined with previously known implications, Theorem 8.1.7 characterizes some well-known Banach space properties in Lipschitz-free spaces $\mathcal{F}(M)$. These include $\mathcal{F}(M)$ being (in order of increasing generality, see [109, Zizler's article, Theorem 3.8]) weakly countably determined, weakly Lindelöf determined, and weakly Lindelöf.

Similarly, by combining Theorem 8.1.7 and well known implications, separability of $\mathcal{F}(M)$ is equivalent to any of the following properties of $(B_{\text{Lip}_0(M)}, w^*)$: uniform Eberlein, Eberlein and angelic.

Weak* separability of the dual. In [158], Talponen introduced the following property: a Banach space X has the *Countable Separation Property* (CSP) if any set $A \subset X^*$ that separates points of X has a countable subset $B \subset A$ that also separates points of X. It is obvious that separable Banach spaces have the CSP, and that X having the CSP implies that B_{X^*} is w^*-separable.

Proposition 8.1.9. *A Lipschitz-free space $\mathcal{F}(M)$ has the CSP if and only if M is separable.*

Proof. Suppose that M is not separable and let $(B_\gamma)_{\gamma<\omega_1}$ be a family of closed balls in M with $\inf_{\gamma\neq\gamma'} d(B_\gamma, B_{\gamma'}) > 0$. Define $A_\gamma := \{0\} \cup \bigcup_{\gamma\leq\lambda<\omega_1} B_\lambda$. Then $(A_\gamma)_{\gamma<\omega_1}$ is a strictly decreasing family of closed subsets of M whose intersection is $\{0\}$. Therefore $(\mathcal{F}(A_\gamma))_{\gamma<\omega_1}$ is a strictly decreasing family of closed subspaces of $\mathcal{F}(M)$ whose intersection is

$$\bigcap_{\gamma<\omega_1} \mathcal{F}(A_\gamma) = \mathcal{F}\left(\bigcap_{\gamma<\omega_1} A_\gamma\right) = \mathcal{F}(\{0\}) = \{0\}$$

by [5, Theorem 2.1]. By [159, Theorem 4.1], $\mathcal{F}(M)$ does not have the CSP. $\square$

We conclude this section by noting that the w^*-separability of $B_{\text{Lip}_0(M)}$ does not imply the separability of $\mathcal{F}(M)$. In fact, $B_{\text{Lip}(\ell_\infty)}$ is w^*-separable whereas ℓ_∞ is non-separable. More precisely, we have:

Proposition 8.1.10. *Let M be a pointed metric space. The following assertions are equivalent:*

(i) M isometrically embeds into ℓ_∞;

(ii) $B_{\text{Lip}_0(M)}$ is w^*-separable;

(iii) $\mathcal{F}(M)$ is isometric to a subspace of ℓ_∞.

Proof. The equivalence between (ii) and (iii) is true for any Banach space and can be found in [55]. Let us now show that $(i) \implies (iii)$. Recall that Kalton in [113, Proposition 5.1] has proved that $\mathcal{F}(\ell_\infty)$ is isometric to a subspace of ℓ_∞. Since (i) implies that $\mathcal{F}(M) \subseteq \mathcal{F}(\ell_\infty)$ the conclusion follows. Finally, since M is isometric to a subset of $\mathcal{F}(M)$, it is clear that $(iii) \implies (i)$, and the proof is complete. $\square$

8.2 A non-separable example on dual Lipschitz-free spaces

The characterization of those metric spaces M for which $\mathcal{F}(M)$ is (isometrically) a dual Banach space remains an important open problem in Lipschitz-free space theory. This issue has been treated e.g. in [170, 112, 78], and has only recently been solved for compact M in [2]. As part of the effort to find a complete solution, several less far-reaching questions have arisen about the nature of Lipschitz-free spaces $\mathcal{F}(M)$ admitting a predual, including:

(a) Is duality a hereditary property? That is: if $N \subset M$, is then $\mathcal{F}(N)$ a dual space as well?

(b) Does there always exist a predual of $\mathcal{F}(M)$ that is made up exclusively of locally flat functions?

(c) Must $\mathcal{F}(M)$ have the Radon-Nikodým property? Equivalently by [2, Theorem C], must M be purely 1-unrectifiable?

Question (a) is motivated by the fact that all known restrictions to duality stem from the metric space M being "too big" and containing specific metric structures, such as nontrivial geodesics which cause $\mathcal{F}(M)$ to contain L_1 (and thus prevent duality in the separable case), or the construction from [78, Example 5.8]. Their non-containment is of course hereditary.

Regarding question (b), we recall that a function $f \in \mathrm{Lip}(M)$ is *locally flat* if we have $\lim_{r\to 0} \|f|_{B(x,r)}\| = 0$ for all $x \in M$. It was shown in [2] that the space of all locally flat functions in $\mathrm{Lip}(M)$ is always an isometric predual of $\mathcal{F}(M)$ when M is compact. In [112] and later [78], the non-compact case was studied and sufficient conditions were found under which there is a predual consisting of all locally flat functions that are moreover continuous with respect to a different topology on M (which plays the role of the relative weak* topology). It is a natural suspicion that this may be the general behavior.

Finally, question (c) is motivated by the fact that separable dual spaces have the RNP; in fact, the RNP is equivalent to duality for $\mathcal{F}(M)$ if M is compact [2]. There are known examples of nonseparable dual Banach spaces without the RNP, but not within the class of Lipschitz-free spaces so far.

The goal of this section is to provide an example of a nonseparable dual Lipschitz-free space that gives a negative answer to all three questions above. Our example is a well-known mathematical object: the space $\mathcal{M}(K)$ of Radon measures on a metrizable compact space K. It is not difficult to show that, more generally, $\mathcal{M}(S)$ is isometrically a Lipschitz-free space for every Polish (i.e. separable and completely metrizable) space S. If two measures μ and ν are singular, we write $\mu \perp \nu$.

Proposition 8.2.1. *Let S be a Polish space. Then $\mathcal{M}(S)$ is linearly isometric to $\mathcal{F}(M)$ for some metric space M. Specifically*

$$\mathcal{M}(S) \cong \ell_1(|S|) \oplus_1 \left(\bigoplus_{\kappa} L_1 \right)_1 \cong \mathcal{F}(M)$$

for some cardinal $\kappa \leq \mathfrak{c}$, and M is the metric sum of $|S|$ two-point spaces and κ copies of $[0,1]$.

Proof. The second isometry is clear since $\mathcal{F}([0,1]) = L_1$, so we focus on the first one. If S is finite or countable then the statement is clear (with $\kappa = \emptyset$) so we may assume that $|S| = \mathfrak{c}$ by the Cantor-Bendixson theorem (see e.g. [115, Corollary 6.5]). We follow the argument in [1, Proposition 4.3.8(iii)], with slight changes and additional details in order to obtain the exact form of $\mathcal{M}(S)$.

Using Zorn's lemma, find a maximal family $(\mu_i)_{i\in I}$ of Borel probability measures on S that contains all Dirac measures δ_x, $x \in S$, and such that $\mu_i \perp \mu_j$ for $i \neq j \in I$. Now define a mapping

$$T : \mathcal{M}(S) \to \left(\bigoplus_{i\in I} L_1(\mu_i) \right)_1$$

by letting $(T\lambda)_i$ be the Radon-Nikodým derivative of λ with respect to μ_i, i.e. the unique $f_i \in L_1(\mu_i)$ such that $d\lambda = f_i\, d\mu_i + d\nu_i$ where $\nu_i \perp \mu_i$. T is clearly linear, we will check that it is an onto isometry.

Let $F \subset I$ be finite. Since μ_i, $i \in F$ are pairwise mutually singular, there are pairwise disjoint Borel sets $E_i \subset S$, $i \in F$ such that μ_i is concentrated on E_i. Then

$$\sum_{i\in F} \|f_i\|_{L_1(\mu_i)} = \sum_{i\in F} \int_{E_i} |f_i|\, d\mu_i = \sum_{i\in F} |\lambda|(E_i) \leq \|\lambda\|.$$

It follows that $\|T\lambda\| = \sum_{i\in I} \|f_i\|_{L_1(\mu_i)} \leq \|\lambda\|$. Now let $\lambda' \in \mathcal{M}(S)$ be given by $d\lambda' = \sum_{i\in I} f_i\, d\mu_i$, which we now know to converge absolutely, and notice that the Radon-Nikodým derivative of $\lambda - \lambda'$ with respect to any μ_i is $f_i - f_i = 0$, i.e. $\lambda - \lambda' \perp \mu_i$. By the maximality of (μ_i) we get $\lambda - \lambda' = 0$ and therefore $\|\lambda\| = \|\lambda'\| = \sum_{i\in I} \|f_i\|_{L_1(\mu_i)}$, using again the fact that $\mu_i \perp \mu_j$ for all $i \neq j$. This proves that T is an onto isometry.

By Theorem 4.13 in [37] and the Example immediately preceding it, every space $L_1(\mu_i)$ is separable. Note that $L_1(\delta_x) \cong \mathbb{R}$ for each $x \in S$, and every other μ_i is purely nonatomic by construction, therefore $L_1(\mu_i) \cong L_1$ by Theorem 14.9 in [120] and its Corollary. Thus we get the desired isometric identification, where κ is the cardinality of $\{\mu_i : i \in I\} \setminus \{\delta_x : x \in S\}$; the fact that $\kappa \leq \mathfrak{c}$ follows e.g. from a density argument. □

Corollary 8.2.2. *There is a nonseparable Lipschitz-free space $\mathcal{F}(M)$ with the following properties:*

(a) $\mathcal{F}(M)$ is a dual space;

(b) $\mathcal{F}(M)$ admits both separable and nonseparable isometric preduals;

(c) there is $N \subset M$ such that $\mathcal{F}(N)$ is not isomorphic to a dual space;

(d) $\mathcal{F}(M)$ does not have the Radon-Nikodým property;

(e) there is no predual of $\mathcal{F}(M)$ that consists of locally flat functions.

Proof. Let M be such that $\mathcal{F}(M) = \mathcal{M}([0,1])$ as given by Proposition 8.2.1. That space is clearly the dual of $C([0,1])$, but M contains a nontrivial geodesic N so $\mathcal{F}(N) = L_1$ is not isomorphic to a dual space, and $\mathcal{F}(M)$ cannot have the RNP because it contains L_1 (alternatively, the failure of the RNP for $\mathcal{F}(M)$ follows also from the fact that it is a non-separable Banach space with a separable predual). Notice also that

$$\big(\big(C([0,1]) \oplus_\infty c_0(\mathfrak{c})\big)\big)^* = \mathcal{M}([0,1]) \oplus_1 \ell_1(\mathfrak{c}) = \mathcal{M}([0,1])$$

so $\mathcal{F}(M)$ also has a nonseparable isometric predual. For the last statement, note that any locally flat function is constant on geodesics and therefore locally flat functions cannot separate evaluation functionals on different points of N. □

Corollary 8.2.2 provides negative answers to all three questions posed at the beginning of this section. However, the counterexample is strongly dependent on non-separability. The answer to question (c) is positive in the separable case, so it is natural to ask whether the other questions also have a positive answer for separable M:

Problem 8.2.3. Suppose that M is separable and $\mathcal{F}(M)$ is a dual space.

(a) Is $\mathcal{F}(N)$ also a dual space for every $N \subset M$?

(b) Does there exist a subspace $X \subset \mathrm{Lip}_0(M)$ such that $X^* = \mathcal{F}(M)$ that contains only locally flat functions?

Chapter 9

Proximinality and uniformly approximable sets in L^p

Consider $(\Omega, \mathcal{F}, \mu)$ a measure space. For any $k \geq 1$, we denote by $\mathscr{G}_{p,k}(\Omega, \mathcal{F}, \mu)$, or simply $\mathscr{G}_{p,k}$ when the measure space $(\Omega, \mathcal{F}, \mu)$ is clear from the context, the set of simple functions given by

$$\mathscr{G}_{p,k} = \left\{ \sum_{i=1}^{l} a_i \mathbb{1}_{A_i} \in L^p(\Omega, \mathcal{F}, \mu) : \ \{A_i\}_{1 \leq i \leq l} \text{ measurable partition of } \Omega, \ a_i \in \mathbb{R} \text{ for all } i, \, l \leq k \right\}.$$

So $\mathscr{G}_{p,k}$ is the set of simpe functions in $L^p(\Omega, \mathcal{F}, \mu)$ taking less than k different values.

We recall some notions from approximation theory. Let X be a Banach space and let K be a closed subset of X. The *metric projection* on K is the multi-valued mapping $P_K : X \rightrightarrows K$ defined by $P_K(x) = \{y \in K \ : \ \|x - y\| = d(x, K)\}$ (where $d(A, B)$ is the distance between two subsets A and B of X). If $P_K(x)$ is not empty for all $x \in X$, we say that K is *proximinal*. If $P_K(x)$ is a singleton for all $x \in X$, we say that K is *Chebyshev*.

In some of the results we will need to consider diffuse and atomic measures. For that reason we fix some notations at this respect. We recall that an *atom* in a measure space $(\Omega, \mathcal{F}, \mu)$ is a measurable set A that satisfies: $\mu(A) > 0$ and if $B \subset A$ is a measurable set such that $\mu(B) < \mu(A)$ then $\mu(B) = 0$. Notice that if A_1, A_2 are two atoms with finite measure, then either $\mu(A_1 \cap A_2) = 0$ or they differ on a set of measure 0, that is, $\mu(A_1 \Delta A_2) = 0$ (where Δ is the symmetric difference). A measurable space is said to be *atomic* if every measurable set of positive measure contains an atom. An atomic space is said to have a finite number of atoms of finite measure, up to measure 0, if there exists a finite collection (eventually empty) $\mathcal{A}$ of atoms of finite measure such that for any atom B either $\mu(B) = \infty$ or there exists $A \in \mathcal{A}$ such that $\mu(A \Delta B) = 0$. A *diffuse* measure, is a measure that has no atoms. Notice that the measure $\mu \equiv 0$ is by definition diffuse, and we refer to this case as the trivial one. More information about measure theory can be found in [26].

9.1 Minimizing the distance to the sets $\mathscr{G}_{p,k}$

The main objective of this section is to prove that $\mathscr{G}_{p,k}$ proximinal, i.e. given some $f \in L^p(\Omega, \mathcal{F}, \mu)$, the distance from f to $\mathscr{G}_{p,k}$ is reached at some function $g \in \mathscr{G}_{p,k}$ (see Theorem 0.0.34). We denote by

$$\mathscr{D}_{p,k}(f) = \inf\{\|f - h\|_p : \ h \in \mathscr{G}_{p,k}\}.$$

for all $p \in [1, \infty]$, that is the distance between f and $\mathscr{G}_{p,k}$. A function $g \in P_{\mathscr{G}_{p,k}}$ will be called a *minimizer*. As we mentioned in the introduction, the classical results of optimization do not apply in this case since $\mathscr{G}_{p,k}$ is not convex nor compact. Even in the reflexive case (that is $1 < p < \infty$), it is not clear if the problem admits a solution. However, if $1 < p < \infty$ and $\mathscr{G}_{p,k}$ is weakly closed, it is easy to see that there exists a minimizer. In fact, let $(g_n)_n \subset \mathscr{G}_{p,k}$ such that $\|g_n - f\| \to \mathscr{D}_{p,k}(f)$. In particular, $(g_n)_n$ is bounded and then admits a subsequence $(g_{n'})_{n'}$ that weakly converges to some $g \in \mathscr{G}_{p,k}$. Since the norm is weakly lower semicontinuous, we obtain that

$$\mathscr{D}_{p,k}(f) \leq \|f - g\|_p \leq \lim_{n'} \|f - g_{n'}\|_p = \mathscr{D}_{p,k}(f),$$

implying that $\mathscr{D}_{p,k}(f) = \|f - g\|_p$. Unfortunately, as the following discussion will show, $\mathscr{G}_{p,k}$ is not weakly closed in general, a fact that depends strongly on the measure space. On the one hand, in the case of the ℓ_p spaces for $1 \leq p < \infty$, every $\mathscr{G}_{p,k}$ is closed under the weak topology. This follows directly from the fact that if $(f_n)_n \subset \mathscr{G}_{p,k}$ converges to f weakly in ℓ_p, then $(f_n)_n$ converges pointwise to f. From this fact it follows that $f(\mathbb{N})$ is a finite set with cardinality at most k and therefore $f \in \mathscr{G}_{p,k}$. On the other extreme we have the following result:

Proposition 9.1.1. *Consider $([0, 1], \mathcal{L}, dx)$ the Lebesgue measure and let $p \in [1, \infty)$. Then $\mathscr{G}_{p,k}$ is weakly dense in $L^p([0, 1], \mathcal{L}, dx)$ for all $k \geq 2$.*

Proof. It is enough to prove the case $k = 2$. Consider an integer $r \geq 2$. Every $x \in [0, 1)$ has a unique expansion

$$x = \sum_{n=1}^{\infty} \zeta_n^r(x) r^{-n}.$$

where $\zeta_n^r(x) \in \{0, ..., r - 1\}$ and $(\zeta_n^r(x))_n$ is not eventually constant $r - 1$. For $x = 1$, we define $\zeta_n^r(x) = r - 1$ for all $n \geq 1$.

Let us prove that for every $A \in \mathcal{L}$, the sequence $(\mathbb{1}_{A\cap\{\zeta_n^2=1\}})_n$ converges weakly to the function $f = \frac{1}{2}\mathbb{1}_A$. Indeed, assume first that $A = [0, 1]$. For $n \geq 1$, let $\Psi_n : [0, 1] \to [0, 1]$ be the bi-measurable and measure preserving transformation which flips the n-th binary digit. Then for all continuous functions $g : [0, 1] \to \mathbb{R}$ it holds

$$\int_{\{\zeta_n^2=1\}} g(x)\ dx = \int_{\{\zeta_n^2=0\}} g(x)\ dx + R_n,$$

where $R_n = \int_{\{\zeta_n^2=1\}} g(x) - g(\Psi_n(x))\ dx$. The continuity of g, allow us to prove that R_n converges to zero. This shows $\mathbb{1}_{\{\zeta_n^2=1\}}$ converges weakly to $\frac{1}{2}\mathbb{1}_{[0,1]}$. Thus, for all $h \in L^q$, where q is the conjugated index of p, and all $A \in \mathcal{L}$ we have

$$\lim_{n\to\infty} \int \mathbb{1}_{\{\zeta_n^2=1\}}(x)\mathbb{1}_A(x)h(x)\ dx = \frac{1}{2} \int \mathbb{1}_A(x)h(x)\ dx,$$

showing that $(\mathbb{1}_{A\cap\{\zeta_n^2=1\}})_n$ converges weakly to $\frac{1}{2}\mathbb{1}_A$.

In a similar way, it is shown that for all $A \in \mathcal{L}$, any integer number $r \geq 2$, any $m \in \{1, ..., r\}$ and all $0 \leq t_1 < t_2 ... < t_m \leq r - 1$, the sequence

$$f_n = \mathbb{1}_{A\cap\cup_{j=1}^m\{\zeta_n^r=t_j\}} = \sum_{j=1}^m \mathbb{1}_{A\cap\{\zeta_n^r=t_j\}} \in \mathscr{G}_{p,2},$$

converges weakly to $\frac{m}{r}\mathbb{1}_A$.

Now, for any $\ell \geq 1$, any partition $\{A_j\}_{1\leq j\leq \ell}$ of measurable sets, any collection $\{r_j\}_{1\leq j\leq \ell}$ of integer numbers greater or equal than 2, any collection $\{m_j\}_{1\leq j\leq \ell}$ such that $m_j \in \{1, ..., r_j\}$

and any collection of integer numbers $\{t_{j,i} \ : \ 1 \leq i \leq m_j, \ 1 \leq j \leq \ell\}$ such that $0 \leq t_{j,1} < ... < t_{j,m_j} \leq r_j - 1$, we obtain that the sequence

$$f_n = \sum_{j=1}^{\ell} \mathbb{1}_{A_j \cap \cup_{i=1}^{m_j} \{\zeta_n^{r_j} = t_{j,i}\}} = \sum_{j=1}^{\ell} \sum_{i=1}^{m_j} \mathbb{1}_{A_j \cap \{\zeta_n^{r_j} = t_{j,i}\}},$$

converges weakly to $\sum\limits_{j=1}^{\ell} \frac{m_j}{r_j} \mathbb{1}_{A_j}$. We notice that $f_n = \mathbb{1}_{B_n}$, where

$$B_n = \bigcup_{j=1}^{\ell} \bigcup_{i=1}^{m_j} A_j \cap \{\zeta_n^{r_j} = t_{j,i}\},$$

so $f_n \in \mathscr{G}_{p,2}$. This shows that the weak closure of $\mathscr{G}_{p,2}$ contains all the simple functions of the form $f = \sum\limits_{j=1}^{\ell} \alpha_j \mathbb{1}_{A_j}$, where $\ell \geq 1$, $\{A_j\}_{1 \leq j \leq \ell}$ is any finite measurable partition and $\alpha_j \in [0, 1]$ for all $j \in \{1, ..., \ell\}$. Moreover, any such simple function is the weak limit of a sequence $(\mathbb{1}_{F_n})_n$ for some sequence $(F_n)_n$ of measurable sets. From here it follows that the weak closure of $\mathscr{G}_{p,2}$ contains all the simple functions. Indeed, consider a simple function $f = \sum\limits_{j=1}^{\ell} a_j \mathbb{1}_{A_j}$, with $\ell \geq 1$ and $a_j \in \mathbb{R}$ for all $j \in \{1, ..., \ell\}$. By adding a large constant C, we have $f + C = \sum\limits_{j=1}^{\ell} b_j \mathbb{1}_{A_j}$, where $b_j = a_j + C > 0$ for all $j \in \{1, ..., \ell\}$. Letting $D = \max_{1 \leq j \leq l} b_j$, we deduce that $\frac{1}{D}(f + C)$ is the weak limit of a sequence $(\mathbb{1}_{F_n})_n$ for some sequence of measurable sets $(F_n)_n$. Then

$$f_n := D\mathbb{1}_{F_n} - C = (D - C)\mathbb{1}_{F_n} - C\mathbb{1}_{F_n^c} \in \mathscr{G}_{p,2},$$

converges weakly to f. The density of the simple functions in L^p, in the strong topology, shows the result. □

The previous result implies obviously that $\mathscr{G}_{p,k}$ is not weakly closed in general, and the usual optimization methods do not work in this context, we have to find a minimizer by a more constructive way.

Definition 9.1.2. In what follows, for a measurable set A of positive and finite measure, we consider $\mathcal{M}_p(f, A)$ as one of the *p-th means* of f on A where $p \in [1, \infty)$. The function

$$a \mapsto \int_A |f(x) - a|^p \, d\mu(x)$$

is convex, nonnegative and finite on $\mathbb{R}$, which converges to ∞ as $a \to \pm\infty$. Therefore, this function has at least one global minimum. For $p = 1$, the set of minima is a bounded interval with extremes a^* and b^* and it is customary to take, the median, as

$$\mathcal{M}_1(f, A) = \frac{a^* + b^*}{2}.$$

For $p > 1$ the minimum is unique due to strict convexity and we denote it by $\mathcal{M}_p(f, A)$. For example, for $p = 2$

$$\mathcal{M}_2(f, A) = \frac{1}{\mu(A)} \int_A f(x) \, d\mu(x),$$

is the mean of f over the set A. If a set has measure 0, we simply put $\mathcal{M}_p(f, A) = 0$.

The next concept will play an important role in what follows.

Definition 9.1.3. Assume $f \in L^p(\Omega, \mathcal{F}, \mu)$, $p \in [1, \infty)$. A function $g \in \mathscr{G}_{p,k}$

$$g = \sum_{i=1}^{q} a_i \mathbb{1}_{C_i},$$

with $1 \leq q \leq k$, is said in *f-special form* if there exist $-\infty \leq r_1 < ... < r_k < r_{k+1} \leq \infty$ such that

- $C_i = f^{-1}([r_i, r_{i+1}))$ for all $i \in \{1, ..., q-1\}$, $C_q = f^{-1}([r_q, r_{q+1}])$ and $\{C_i\}_{1 \leq i \leq q}$ is a partition of Ω;
- $-\infty < a_1 < ... < a_q < \infty$;
- for all $i \in \{1, ..., q\}$ such that $\mu(C_i) < \infty$, it holds a_i is a p-th mean of f on C_i.

Suppose that $g = \sum\limits_{i=1}^{q} a_i \mathbb{1}_{C_i}$ is in f-special form. Note that if μ is an infinite measure there exists a unique $1 \leq s \leq q$ such that $a_s = 0$ and $\mu(C_i) < \infty$ for all $i \neq s$. We also have that $a_i = \mathcal{M}_p(f, C_i)$ for all $i \in \{1, ..., q\}$ if $p > 1$. Moreover notice that $g = h \circ f$, where $h = \sum_{i=1}^{q-1} a_i \mathbb{1}_{[r_i, r_{i+1})} + a_q \mathbb{1}[r_q, r_{q+1}]$ is a Borel function and g is f-measurable, that is, g is measurable with respect $\sigma(f) = f^{-1}(\mathscr{B})$, where $\mathscr{B}$ is the Borel σ-field in $\mathbb{R}$.

9.1.1 The case of a finite measure, $p \in [1, \infty)$

If the measure if finite, we start by proving that there exists an approximation sequence which is uniformly bounded:

Lemma 9.1.4. *Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space and $p \in [1, \infty)$. Let $f \in L^p(\Omega, \mathcal{F}, \mu)$ and $k \geq 1$. Then there exists a uniformly bounded sequence $(g_n)_n \subset \mathscr{G}_{p,k}$ such that*

$$\|f - g_n\|_p \to \mathscr{D}_{p,k}(f).$$

Proof. Let $(h_n)_n \in \mathscr{G}_{p,k}$ be a sequence such that $\|f - h_n\|_p \to \mathscr{D}_{p,k}(f)$. Assume that $h_n = \sum_{i=1}^{m(n)} c_{i,n} \mathbb{1}_{A_{i,n}}$, where $(c_{i,n})_{1 \leq i \leq m(n)}$ are all different, $\{A_{i,n}\}_{1 \leq i \leq m(n)}$ is a measurable partition with sets of positive measure and $m(n) \leq k$. We assume that $m(n) = m$ is constant by passing to a subsequence if necessary. We modify this approximating sequence by considering $a_{i,n} = \mathcal{M}_p(f, A_{i,n})$ any of the p-th means of f in $A_{i,n}$. By definition of the p-th means we have, for all $i \in \{1, ..., m\}$

$$\int_{A_{i,n}} |f(x) - a_{i,n}| \, d\mu(x) \leq \int_{A_{i,n}} |f(x) - c_{i,n}| \, d\mu(x),$$

showing that $\tilde{h}_n = \sum_{i=1}^{m} a_{i,n} \mathbb{1}_{A_{i,n}} \in \mathscr{G}_k$ is a minimizing sequence since

$$\mathscr{D}_{p,k}(f) \leq \|f - \tilde{h}_n\|_p \leq \|f - h_n\| \to \mathscr{D}_{p,k}(f)$$

If $m < k$, we define $a_{i,n} = 0$ and $A_{i,n} = \emptyset$ for $i \in \{m+1, ..., k\}$. We assume that $\{A_{i,n}\}_{1 \leq i \leq k}$ are ordered in decreasing order according to their measure

$$\mu(A_{1,n}) \geq \mu(A_{2,n}) \geq ... \geq \mu(A_{k,n}) \geq 0.$$

In this way, the vector $v_n = (\mu(A_{1,n}), \mu(A_{2,n}), ..., \mu(A_{k,n}))$ belongs to the compact set in $\mathbb{R}^k$

$$\Delta = \left\{ x \in \mathbb{R}^k : \ x_1 \geq x_2 \geq ... \geq x_k \geq 0, \ \sum_i x_i = \mu(\Omega) \right\}$$

By passing to a subsequence if necessary, we can assume that $(v_n)_n$ converges to some vector $v = (v_1, v_2, ..., v_k) \in \Delta$. If q is the largest index such that $v_q > 0$ (q could be exactly k) then, we have $q \geq 1$ and $v_1 \geq ... \geq v_q > 0 = v_{q+1} = ... = v_k$. We notice that $q \leq m$. Now, define $B_n = \bigcup_{i=q+1}^{k} A_{i,n}$ for all $n \in \mathbb{N}$, that we take as the empty set if $q = k$, so

$$\lim_{n\to\infty} \mu(B_n) = \lim_{n\to\infty} \sum_{i=q+1}^{k} \mu(A_{i,n}) = 0.$$

On the other hand, for all $i \in \{1, ..., q\}$ we have

$$\lim_{n\to\infty} \mu(A_{i,n}) = v_i > 0,$$

and so, passing to a further subsequence we can assume there exists a finite constant Γ such that for all n and all $i \in \{1, ..., q\}$ it holds

$$\frac{1}{\mu(A_{i,n})} \leq \Gamma \tag{9.1}$$

The finite measure ν defined by

$$\nu(A) = \int_A |f(x)|^p \ d\mu(x),$$

is absolutely continuous with respect to μ, which means that, for all $\rho > 0$ there exists a $\delta > 0$ such that, for any measurable set A if $\mu(A) \leq \delta$ then $\nu(A) = \int_A |f(x)|^p \ d\mu(x) \leq \rho$. This property shows that

$$\lim_{n\to\infty} \int_{B_n} |f(x)|^p \ d\mu(x) = 0.$$

Now, we modify further the approximation sequence by defining

$$b_{i,n} = \begin{cases} a_{i,n} & \text{for } i \in \{1, ..., q\} \\ 0 & \text{for } i \in \{q+1, ..., k\} \end{cases}, \tag{9.2}$$

and define

$$g_n = \sum_{i=1}^{k} b_{i,n} \mathbb{1}_{A_{i,n}} = \sum_{i=1}^{q} \mathcal{M}_p(f, A_{i,n}) \mathbb{1}_{A_{i,n}} + 0 \mathbb{1}_{B_n} \in \mathscr{G}_{p,k}. \tag{9.3}$$

We need to show that $(g_n)_n$ is a good approximation sequence and it is uniformly bounded. For the first claim notice that for $i \in \{q+1, ..., k\}$, we have

$$\int_{A_{i,n}} |f(x) - \tilde{h}_n(x)|^p \ d\mu(x) = \int_{A_{i,n}} |f(x) - \mathcal{M}_p(f, A_{i,n})|^p \ d\mu(x) \leq \int_{A_{i,n}} |f(x)|^p \ d\mu(x),$$

where we have used the optimality of $\mathcal{M}_p(f, A_{i,n})$ in the last inequality. This shows that

$$\begin{aligned}
\mathscr{D}_{p,k}(f)^p &\leq \|f - \tilde{h}_n\|_p^p = \sum_{i=1}^{m} \int_{A_{i,n}} |f(x) - \mathcal{M}_p(f, A_{i,n})|^p \, d\mu(x) \\
&\leq \sum_{i=1}^{q} \int_{A_{i,n}} |f(x) - \mathcal{M}_p(f, A_{i,n})|^p \, d\mu(x) + \int_{B_n} |f(x)|^p \, d\mu(x) = \|f - g_n\|_p^p \\
&\leq \sum_{i=1}^{m} \int_{A_{i,n}} |f(x) - \mathcal{M}_p(f, A_{i,n})|^p \, d\mu(x) + \int_{B_n} |f(x)|^p \, d\mu(x) \\
&\leq \|f - \tilde{h}_n\|_p^p + \int_{B_n} |f(x)|^p \, d\mu(x) \to \mathscr{D}_{p,k}(f)^p
\end{aligned}$$

Now, we prove that $(g_n)_n$ is uniformly bounded. We notice that $g_n = 0$ on B_n, so we must study g_n on B_n^c. For $i \in \{1, ..., q\}$ and $x \in A_{i,n}$ we have $g_n(x) = \mathcal{M}_p(f, A_{i,n})$ and so

$$\|\mathcal{M}_p(f, A_{i,n})\mathbb{1}_{A_{i,n}}\|_p \leq \|(f - \mathcal{M}_p(f, A_{i,n}))\mathbb{1}_{A_{i,n}}\|_p + \|f\mathbb{1}_{A_{i,n}}\|_p \leq 2\|f\mathbb{1}_{A_{i,n}}\|_p \leq 2\|f\|_p,$$

where we have used again the optimality of $\mathcal{M}_p(f, A_{i,n})$. This shows that

$$|\mathcal{M}_p(f, A_{i,n})| \leq 2\frac{\|f\|_p}{\mu(A_{i,n})^{\frac{1}{p}}} \leq 2\|f\|_p \Gamma^{\frac{1}{p}},$$

where Γ is the constant obtained in (9.1). □

The next result proves that $\mathscr{G}_{p,k}$ is proximinal in case of finite measure spaces. Remember that P_K is the metric projection over K.

Theorem 9.1.5. *Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space, $p \in [1, \infty)$ and $k \geq 1$. Then $\mathscr{G}_{p,k}$ is proximinal.*

Moreover, if $f \in L^p(\Omega, \mathcal{F}, \mu)$ and $g = \sum_{i=1}^{q} b_i \mathbb{1}_{A_i} \in P_{\mathscr{G}_{p,k}}(f)$ is a minimizer with $q \leq k$, $-\infty < b_1 < ... < b_q < \infty$ and $\{A_i\}_{1 \leq i \leq q}$ a partition of Ω with sets of positive measure, there exists a minimizer $\widetilde{g} \in P_{\mathscr{G}_{p,q}}(f)$ in f-special form:

$$\widetilde{g} = \sum_{i=1}^{q} \mathcal{M}_p(f, f^{-1}(C_i)) \; \mathbb{1}_{f^{-1}(C_i)}$$

where

- $r_1 = -\infty, r_{q+1} = \infty$ *and* $r_i = \frac{b_{i-1}+b_i}{2}$ *for all* $i \in \{2, ..., q\}$;
- $C_i = f^{-1}([r_i, r_{i+1}))$ *for all* $i \in \{1, ..., q-1\}$ *and* $C_q = f^{-1}([r_q, r_{q+1}])$;
- b_i *is a p-th mean of f on $f^{-1}(C_i)$ for all $i \in \{1, ..., q\}$ such that $\mu(f^{-1}(C_i)) > 0$.*

If q is the smallest among all minimizers, then $\mu(C_i) > 0$ for all $i \in \{1, ..., q\}$.

Proof. By Lemma 9.1.4, let $(g_n)_n \subset \mathscr{G}_{p,k}$ be a uniformly bounded sequence such that $\|f - g_n\|_p \to \mathscr{D}_{p,k}(f)$. Let $C > 0$ such that $|g_n| < C$ for all $n \in \mathbb{N}$. We write $g_n = \sum_{i=1}^{k} b_{i,n} \mathbb{1}_{A_{i,n}}$ where $\{A_{i,n}\}_{1 \leq i \leq k}$ is a partition of Ω and $-C \leq b_{1,n} \leq ... \leq b_{k,n} \leq C$. The vector $u_n = (b_{1,n}, ..., b_{k,n})$ belongs to the compact set $[-C, C]^k$ and therefore, by taking a subsequence if necessary, we can assume that $(u_n)_n$ converges to some $u = (b_1, ..., b_k) \in [-C, C]^k$ with $b_1 \leq ... \leq b_k$. Some of the entries in u can be equal, for that we consider $z_1 < ... < z_l$ the distinct entries in u where $1 \leq l \leq k$. We define $r_1 = -\infty, r_{l+1} = \infty$ and $r_j = \frac{z_{j-1}+z_j}{2}$ for $j \in \{2, ..., l\}$. Consider the intervals $I_j = [r_j, r_{j+1})$ for $j \in \{1, ..., l-1\}$ and $I_l = [r_l, r_{l+1}]$. For $j \in \{1, ..., l\}$, we also define

$L_j = \{i \in \{1,...,k\} : \ b_i = z_j\}$, which is a partition of $\{1,...,k\}$. For all $n \in \mathbb{N}$, consider the function

$$\tilde{g}_n = \sum_{i=1}^{k} b_i \mathbb{1}_{A_{i,n}}.$$

Then, we have

$$\|f - \tilde{g}_n\|_p \le \|f - g_n\|_p + \|g_n - \tilde{g}_n\|_p \le \|f - g_n\|_p + \max_{1\le i\le k} |b_{i,n} - b_i| \mu(\Omega)^{\frac{1}{p}} \to \mathscr{D}_{p,k}(f),$$

proving that $(\tilde{g}_n)_n$ is also a minimizing sequence. Finally, our candidate for minimizer is the function $g = \sum_{j=1}^{l} z_j \mathbb{1}_{f^{-1}(I_j)} \in \mathscr{G}_{p,k}$. For all $i \in \{1,...,k\}$, all $j \in \{1,...,l\}$ and all n, we have

$$\int_{f^{-1}(I_j)\cap A_{i,n}} |f(x) - z_j|^p \ d\mu(x) \le \int_{f^{-1}(I_j)\cap A_{i,n}} |f(x) - b_i|^p \ d\mu(x).$$

This is clear if $i \in L_j$ because in that case $z_j = b_i$. Now, if $i \in L_{j'}$ with $j' \neq j$, we have $b_i = z_{j'}$ and for all $x \in f^{-1}(I_j)$ it holds $|f(x) - z_j| \le |f(x) - z_{j'}| = |f(x) - b_i|$. Now, summing over i, j we get for all n that

$$\begin{aligned}\mathscr{D}_{p,k}(f)^p \le \|f - g\|_p^p &= \sum_{i,j} \int_{f^{-1}(I_j)\cap A_{i,n}} |f(x) - z_j|^p \ d\mu(x) \le \sum_{i,j} \int_{f^{-1}(I_j)\cap A_{i,n}} |f(x) - b_i|^p \ d\mu(x) \\ &\le \|f - \tilde{g}_n\|_p^p \to \mathscr{D}_{p,k}(f)^p,\end{aligned}$$

proving that $g \in P_{\mathscr{G}_{p,k}}(f)$.

Now, we prove the last part of the Theorem. Assume that $g = \sum_{i=1}^{q} b_i \mathbb{1}_{A_i} \in P_{\mathscr{G}_{p,k}}(f)$ is a minimizer, with $b_1 < ... < b_q$, $\{A_i\}_{1\le i\le q}$ a partition of Ω where all the sets A_i have positive measure and $q \le k$. Let $r_1 = -\infty, r_{q+1} = \infty, r_i = \frac{b_{i-1}+b_i}{2}$ for $i = \{2,...,q\}$ and

$$C_i = f^{-1}([r_i, r_{i+1})) \text{ for } i \in \{1,...q-1\}, \ C_q = f^{-1}([r_i, r_{i+1}]).$$

For all $i \in \{1,...,q\}$, we modify the sets A_i as

$$\widetilde{A}_i = \big(A_i \cup f^{-1}(\{r_i\})\big) \setminus f^{-1}(\{r_{i+1}\}). \tag{9.4}$$

Let us prove that $\mu(\widetilde{A}_1 \Delta C_1) = \mu(A_j \cap E_2) = 0$ for all $j > 2$ where $E_2 = f^{-1}(\{r_2\})$. Define

$$g' = b_1 \mathbb{1}_{\widetilde{A}_1} + \sum_{i=3}^{q} b_i \mathbb{1}_{A_i \setminus E_2} + b_2 \mathbb{1}_{A_2 \cup E_2} \in \mathscr{G}_{p,k}$$

and note that $\{\widetilde{A}_1, \{A_j \setminus E_2\}_{j>2}, A_2 \cup E_2\}$ is a partition of Ω. Consider the following decomposition

$$\begin{aligned}&\|f - g\|_p^p = \sum_j \int_{A_j} |f(x) - b_j|^p \ d\mu(x) = \int_{\widetilde{A}_1} |f(x) - b_1|^p \ d\mu(x) + \sum_{j>2} \int_{A_j\setminus E_2} |f(x) - b_j|^p \ d\mu(x) \\ &+ \sum_{j>2} \int_{A_j\cap E_2} |f(x) - b_j|^p \ d\mu(x) + \int_{A_1\cap E_2} |f(x) - b_1|^p \ d\mu(x) + \int_{A_2} |f(x) - b_2|^p \ d\mu(x) \\ &\ge \int_{\widetilde{A}_1} |f(x) - b_1|^p \ d\mu(x) + \sum_{j>2} \int_{A_j\setminus E_2} |f(x) - b_j|^p \ d\mu(x) + \int_{A_2\cup E_2} |f(x) - b_2|^p \ d\mu(x) \\ &+ (b_3 - b_2)^p \sum_{j>2} \mu(A_j \cap E_2) \\ &= \|f - g'\|_p^p + (b_3 - b_2)^p \sum_{j>2} \mu(A_j \cap E_2),\end{aligned}$$

where the second equality follows from the fact that $\widetilde{A}_1 = A_1 \setminus E_2$ (up to a set of measure zero). The inequality is proved noting that, for $x \in A_j \cap E_2$ with $j > 2$ it holds $|f(x) - b_2| =$

$b_2 - r_2 < b_3 - r_2 \leq b_j - r_2 = |f(x) - b_j|$, which implies $|f(x) - b_j| \geq |f(x) - b_2| + b_3 - b_2$, and for $x \in A_1 \cap E_2$ it holds $|f(x) - b_1| = |f(x) - b_2|$. So, since g is a minimizer we deduce that $\mu(A_j \cap E_2) = 0$ for all $j > 2$. Thus, we get

$$\|f-g\|_p^p = \int_{\widetilde{A}_1} |f(x)-b_1|^p \; d\mu(x) + \sum_{j>2} \int_{A_j} |f(x)-b_j|^p \; d\mu(x) + \int_{A_2 \cup f^{-1}(\{r_2\})} |f(x)-b_2|^p \; d\mu(x),$$

showing that

$$b_1 \mathbb{1}_{\widetilde{A}_1} + b_2 \mathbb{1}_{A_2 \cup f^{-1}(\{r_2\})} + \sum_{j>2} b_j \mathbb{1}_{A_j} \in P_{\mathscr{G}_{p,k}}(f).$$

A similar argument shows that $\mu(A_j \cap C_1) = 0$ and $\mu(\widetilde{A}_1 \cap C_j) = 0$ for all $j \geq 2$. Since $\{A_i\}_{1 \leq i \leq q}$ is a partition we conclude that $\mu(C_1) = \sum_i \mu(A_i \cap C_1) = \mu(C_1 \cap A_1) = \mu(C_1 \cap \widetilde{A}_1)$, proving that $C_1 \subset \widetilde{A}_1$ except for a set of measure 0. On the other hand, using again that $\{\widetilde{A}_1, A_2 \cup f^{-1}(\{r_2\}), \{A_j \setminus f^{-1}(\{r_2\})\}_{j>2}\}$ is also a partition, we conclude that $\widetilde{A}_1 \subset C_1$ except for a set of measure 0. In a similar way, we prove $\mu(\widetilde{A}_i \cap C_j) = \mu(\widetilde{A}_i \Delta C_i) = 0$, for all $i \neq j$.

At this point we should mention that some of the $\tilde{A}_i$ could have measure 0. For example this occurs if $A_1 = f^{-1}(\{r_2\})$. In any case, we have

$$\|f-g\|_p^p = \sum_{i=1}^q \int_{\widetilde{A}_i} |f(x)-b_i|^p \; d\mu(x) = \sum_{i=1}^q \int_{C_i} |f(x)-b_i|^p \; d\mu(x),$$

showing that

$$\hat{g} = \sum_{i=1}^q b_i \mathbb{1}_{C_i} \in P_{\mathscr{G}_{p,k}}(f),$$

is a minimizer. On the other hand, if $\mu(C_i) > 0$ we have $\int_{C_i} |f(x) - \mathcal{M}_p(f, C_i)|^p \; d\mu(x) \leq \int_{C_i} |f(x) - b_i|^p \; d\mu(x)$. The inequality cannot be strict, otherwise we contradict the minimality of $\hat{g}$, showing that b_i is a p-th means of f on C_i, and therefore,

$$\widetilde{g} = \sum_{i=1}^q \mathcal{M}_p(f, C_i) \mathbb{1}_{C_i} \in P_{\mathscr{G}_{p,k}}(f),$$

is a minimizer in f-special form, as we wanted to prove. In case that q is the minimal among all minimizers, we conclude that $\mu(C_i) > 0$ for all i. $\square$

Remark 9.1.6. In the last part of the theorem, for any minimizer g, we have constructed a minimizer $\widetilde{g}$ in f-special form, but it may happens that some of the sets $(C_i)_i$ have measure 0, which can be discarded to get a minimizer with fewer terms. An interesting question is if this procedure applied to any minimizer gives always a minimizer with the smallest possible number of terms (see Proposition 9.1.17).

Recall that given $f \in L^p(\Omega, \mathcal{F}, \mu)$, the distribution of f is the measure μ_f defined on $(\mathbb{R}, \mathscr{B})$ given by, for all $B \in \mathscr{B}$

$$\mu_f(B) = \mu(f^{-1}(B)).$$

Let g be a minimizer of f in $\mathscr{G}_{p,k}$ in f-special form provided by Theorem 9.1.5

$$g = \sum_{i=1}^q a_i \mathbb{1}_{f^{-1}([r_i, r_{i+1}))},$$

So, $g = \ell \circ f$ with

$$\ell = \sum_{i=1}^{q} a_i \mathbb{1}_{[r_i, r_{i+1})},$$

and

$$\begin{aligned}\|f-g\|_p^p &= \int_\Omega |f(x)-g(x)|^p \, d\mu(x) = \int_\Omega |f(x)-\ell(f(x))|^p \, d\mu(x) = \int_{\mathbb{R}} |y-\ell(y)|^p \, d\mu_f(y) \\ &= \|\mathrm{id}-\ell\|^p_{L^p(\mathbb{R},\mathscr{B},\mu_f)}.\end{aligned}$$

Thus, the problem of finding a minimizer for f is equivalent to find a minimizer for the identity function id in $\mathscr{G}_{p,k}(\mathbb{R},\mathscr{B},\mu_f)$. The following result shows that when μ_f is continuous, this search can be done over the subclass of simple functions in f-special form. Before stating the result, let us fix some notations. The cumulative distribution associated to μ_f is the function $F_f(x) = \mu_f((-\infty, x])$ Notice that $F_f(-\infty) = 0$ and $F_f(\infty) = \mu(\Omega)$. The convex support of μ_f is the interval $[\mathfrak{a}_f, \mathfrak{b}_f]$, where

$$\mathfrak{a}_f = \sup\{z : \ F_f(z) = 0\}, \mathfrak{b}_f = \inf\{z : \ F_f(z) = F_f(\infty)\}.$$

The following lemma is needed to study the uniqueness of minimizers, where p-th means are characterized as roots of certain equations, suitable for our purposes. We include a proof, inspired by exercise 1.4.23 in [157], for the sake of completeness.

Lemma 9.1.7. *Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space and $f \in L^p(\Omega, \mathcal{F}, \mu)$, for $p \in [1, \infty)$. For $p = 1$, we also assume that F_f is continuous and strictly increasing on $[\mathfrak{a}_f, \mathfrak{b}_f]$. Let $I \subset \mathbb{R}$ be an interval with extremities $c, d \in \overline{\mathbb{R}}$ such that $\mu_f(I) > 0$. Then, the p-th mean $m = \mathcal{M}_p(f, f^{-1}(I), \mu) = \mathcal{M}_p(id, I, \mu_f)$ is characterized as the unique solution of the equation*

$$\int_{I\cap(-\infty,m]} (m-x)^{p-1} \, d\mu_f(x) = \int_{I\cap(m,\infty)} (x-m)^{p-1} \, d\mu_f(x), \tag{9.5}$$

which for $p = 1$ is equivalent to

$$F_f(m) - F_f(c) = \frac{1}{2}(F_f(d) - F_f(c)). \tag{9.6}$$

Proof. The proof is based on the following equality for all $x, b \in \mathbb{R}$

$$|x-b|^p = p\int_{-\infty}^{b} (t-x)^{p-1}\mathbb{1}_{\{x\le t\}} \, dt + p\int_{b}^{\infty} (x-t)^{p-1}\mathbb{1}_{\{t<x\}} \, dt,$$

which implies that

$$|x-b|^p - |x-a|^p = p\int_a^b \left((t-x)^{p-1}\mathbb{1}_{\{x\le t\}} - (x-t)^{p-1}\mathbb{1}_{\{t<x\}}\right) \, dt.$$

Fix $m \in \mathbb{R}$. Define a function $L : \mathbb{R} \to \mathbb{R}$ by

$$L(b) = \int_I |x-b|^p \, d\mu_f(x) - \int_I |x-m|^p \, d\mu_f(x).$$

It is clear that m is a minimum of L if and only if m is a p-th mean. Using Fubini's Theorem and the previous equality, we obtain that for all $b \in \mathbb{R}$

$$L(b) = p\int_m^b \int_{I\cap(-\infty,t]} (t-x)^{p-1} \, d\mu_f(x)dt - p\int_m^b \int_{I\cap(t,\infty)} (x-t)^{p-1} d\mu_f(x)dt$$

Note that L is convex, coercive and continuous and then reaches a minimum.

Suppose $p > 1$. The functions $t \mapsto \int_{I\cap[-\infty,t]}(t-x)^{p-1}\, d\mu_f(x)$ and $t \mapsto \int_{I\cap(t,\infty)}(x-t)^{p-1}\, d\mu_f(x)$ are continuous, and therefore L is strictly convex and continuously differentiable, which proves that $L'(b)=0$ is the equation for the unique minima, that is,

$$L'(b) = p\int_{I\cap(-\infty,b]}(b-x)^{p-1}\, d\mu_f(x) - p\int_{I\cap(b,\infty)}(x-b)^{p-1}\, d\mu_f(x) = 0.$$

It follows that m is the p-th mean if and only if m fulfills (9.5).

For $p=1$, using that F_f is continuous, we have

$$L(b) = \int_m^b F_f(t) - F_f(c) - (F_f(d) - F_f(t))\, dt.$$

Again, since F_f is continuous we obtain that L is continuously differentiable. Then, if b is any minima for L, it holds that $L'(b)=0$, that is, $F_f(b) - F_f(c) = \frac{1}{2}(F_f(d) - F_f(c))$. Since F_f is assumed to be strictly increasing, this equation has a unique solution, and then L has exactly one minimum. Then, m is a 1-th if and only if $F_f(m) - F_f(c) = \frac{1}{2}(F_f(d)-F_f(c))$. □

Notice that in the previous lemma we can replace $(-\infty, m]$ by $(-\infty, m)$ and (m,∞) by $[m,\infty)$ in 9.5, because $x=m$ does not add to the integrals. In the case F_f is just increasing, $I=(c,d]$ and $p=1$, all the 1-th means satisfy the equations $L^{'+}(m) \geq 0$ and $L^{'-}(m) \leq 0$, which are equivalent to

$$F_f(m) - F_f(c) \geq \frac{1}{2}(F_f(d) - F_f(c)),\ F_f(m-) - F_f(c) \leq \frac{1}{2}(F_f(d) - F_f(c)),$$

and the solution set is, in general, an interval.

The next result shows that when F_f is continuous all minimizers are in f-special form.

Corollary 9.1.8. *Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space, $p \in [1,\infty)$ and $k \geq 1$. Let $f \in L^p(\Omega,\mathcal{F},\mu)$ and assume that F_f is continuous. Then any minimizer $g \in P_{\mathscr{G}_{p,k}}(f)$ is of the f-special form*

$$g = \sum_{i=1}^k a_i \mathbb{1}_{f^{-1}([r_i, r_{i+1}))}, \tag{9.7}$$

where

- *$[r_i, r_{i+1})$ has positive μ_f-measure for all $i \in \{1,...,k\}$;*
- *$\mathfrak{a}_f = r_1 < ... < r_k < r_{k+1} = \mathfrak{b}_f$ and $r_i = \frac{a_i + a_{i+1}}{2}$ for all $i \in \{2,...,k\}$;*
- *a_i is a p-th mean of id on $[r_i, r_{i+1})$ under μ_f for all $i \in \{1,...,k\}$. Moreover, if F_f is strictly increasing on $[\mathfrak{a}_f, \mathfrak{b}_f]$, then $a_i = \mathcal{M}_p(id, [r_i, r_{i+1}), \mu_f)$ for all $i \in \{1,...,k\}$.*

Proof. Notice first that $f \notin \mathscr{G}_{p,k}$, because the image of f cannot be a finite set a.e., since μ_f is not atomic. This implies that there is no minimizer in $\mathscr{G}_{p,q}$, with $q < k$ (see Proposition 9.1.17 below). So, any minimizer has the structure

$$g = \sum_{i=1}^k a_i \mathbb{1}_{A_i}$$

where $a_1 < ... < a_k$, $\{A_i\}_{1\leq i\leq k}$ is a partition with sets of positive measure and a_i is a p-th mean of f in A_i, for all $i \in \{1,...,k\}$. In the previous proof, we then modify this minimizer to get one in f-special form. If one goes over that proof and using the fact that F_f is continuous, one realizes that in equation (9.4), we get $\widetilde{A}_i = A_i$ a.e. and then $A_i = C_i$ a.e., proving that $A_i = f^{-1}([r_i, r_{i+1}))$ a.e.

The fact that a_i is a p-th mean is just the fact that g is a minimizer. For $p > 1$, the uniqueness of the p-th mean shows that $a_i = \mathcal{M}_p(\mathrm{id}, [r_i, r_{i+1}), \mu_f)$. This is also true for $p = 1$, when F_f is continuous and strictly increasing in $[\mathfrak{a}_f, \mathfrak{b}_f]$ (see Lemma 9.1.7). □

Remark 9.1.9. The previous result could be used as the basis of an algorithm to approximate a minimizer. Assume that μ_f is a continuous distribution. For any $s \in \mathbb{R}$, were s plays the role of r_2 in the representation (9.7), we define $r_1(s) = -\infty, r_2(s) = s$ and $a_1 = a_1(s) = \mathcal{M}_p(\mathrm{id}, (-\infty, s), \mu_f)$. Then, we define $a_2(s) = 2r_2(s) - a_1(s)$, which is a relation that should satisfy any minimizer. Then, compute $r_3(s)$ so that

$$a_2(s) = \mathcal{M}_p(\mathrm{id}, [r_2(s), r_3(s)), \mu_f).$$

and continue in this way defining $a_3(s), r_4(s), ..., a_k(s), r_{k+1}(s)$. It may happens that at some iteration $r_{i+1}(s)$ is not well defined for some $i \leq k-1$ because, for all $t \in [r_i(s), \infty]$

$$a_i(s) > \mathcal{M}_p(\mathrm{id}, [r_i(s), t), \mu_f),$$

which shows that there is no minimizer starting with $r_2 = s$. So, we say s is admissible if $r_{k+1}(s)$ is well defined. For every admissible s we have a candidate

$$\ell_s = \sum_{i=1}^{k-1} a_i(s) \mathbb{1}_{[r_i(s), r_{i+1}(s))} + \mathcal{M}_p(\mathrm{id}, [r_k(s), \infty), \mu_f) \mathbb{1}_{[r_k(s), \infty)}$$

and we can compute $R(s) = \int |x - \ell_s(x)|^p \, d\mu_f(x)$. For s which is not admissible put $R(s) = \infty$. Then a minimizer of R gives a minimizer for f. One expects that the set of admissible values of s is an interval. We shall work on this algorithm in a forcoming paper.

For example, if μ_f is a normal $\mathcal{N}(0,1)$, $p = 2$ and $k = 3$, this algorithm gives the following approximation

$$h \approx -1.2 \ \mathbb{1}_{(-\infty, -0.6)} + 0 \ \mathbb{1}_{[-0.6, 0.6)} + 1.2 \ \mathbb{1}_{[0.6, \infty)}.$$

Notice that $\|\mathrm{id}\|_2 = 1$ and $(\mathscr{D}_{2,3}(\mathrm{id}))^2 \approx 0.18$, which means that, in the language of statistics, 82% of the variance of f is explained by a simple function taking 3 values.

Uniqueness of minimizers is a much harder problem. Here, we present a partial result in case μ_f satisfies a certain monotone likelihood ratio property.

Theorem 9.1.10. *Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space, $p \in [1, \infty)$ and $f \in L^p(\Omega, \mathcal{F}, \mu)$. Assume μ_f has a density with respect to the Lebesgue measure $\Psi : (\mathfrak{a}_f, \mathfrak{b}_f) \to (0, \infty)$, which we extend by* 0 *outside this interval. Consider for $s \in (0, \mathfrak{b}_f - \mathfrak{a}_f)$ the function $G_s : (\mathfrak{a}_f, \mathfrak{b}_f) \to [0, \infty)$ given by $G_s(y) = \frac{\Psi(y+s)}{\Psi(y)}$ and assume that G_s is decreasing. Moreover, we suppose that one of the following hypotheses hold:*

(H1) either $\mathfrak{a}_f$ or $\mathfrak{b}_f$ is finite;

(H2) $G_s((\mathfrak{a}_f, \mathfrak{b}_f))$ is an infinite set;

(H3) Ψ is continuous.

Then there exists a unique minimizer for f in $\mathscr{G}_{p,k}$ for all $k \geq 1$.

Proof. The case $k = 1$ is direct from the fact that F_f is is strictly increasing (because Ψ is strictly positive on $(\mathfrak{a}_f, \mathfrak{b}_f)$) and therefore the p-th means are unique. So, we assume that $k \geq 2$.

According to Corollary 9.1.8, and since F_f is strictly increasing all minimizers for f have the f-special form given in (9.7). Fix one of them $g = \ell \circ f$, where

$$\ell = \sum_{i=1}^{k} a_i \mathbb{1}_{I_i}$$

with $r_1 = \mathfrak{a}_f, r_{k+1} = \mathfrak{b}_f, r_i = \frac{a_{i-1}+a_i}{2}$ for $i \in \{2, ..., k\}$, $I_i = [r_i, r_{i+1})$ for $i \in \{1, ..., k-1\}$, $I_k = [r_k, r_{k+1}]$, $\mu_f(I_i) > 0$ for $i \in \{1, ..., k\}$ and $a_i = \mathcal{M}_p(\text{id}, I_i, \mu_f)$ for $i \in \{1, ..., k\}$. Assume there exists another minimizer $\widetilde{g} = \widetilde{\ell} \circ f$ with

$$\widetilde{\ell} = \sum_{i=1}^{k} \widetilde{a}_i \mathbb{1}_{\widetilde{I}_i}$$

where $\widetilde{r}_1 = \mathfrak{a}_f, \widetilde{r}_{k+1} = \mathfrak{b}_f, \widetilde{r}_i = \frac{\widetilde{a}_{i-1}+\widetilde{a}_i}{2}$ for $i \in \{2, ..., k\}$, $\widetilde{I}_i = [\widetilde{r}_i, \widetilde{r}_{i+1})$ for $i \in \{1, ..., k-1\}$, $\widetilde{I}_k = [\widetilde{r}_k, \widetilde{r}_{k+1}]$, $\mu_f(\widetilde{I}_i) > 0$ for $i \in \{1, ..., k\}$ and $\widetilde{a}_i = \mathcal{M}_p(\text{id}, \widetilde{I}_i, \mu_f)$ for $i \in \{1, ..., k\}$. We need to prove that $\ell = \widetilde{\ell}$. Consider $s = \widetilde{r}_2 - r_2$. Switching g and $\widetilde{g}$ if necessary, we can suppose without loss of generality that $s \geq 0$, and since $r_2, \widetilde{r}_2 \in (\mathfrak{a}_f, \mathfrak{b}_f)$ then $s < \mathfrak{b}_f - \mathfrak{a}_f$. Define $\delta_i = \widetilde{a}_i - a_i$ for all $i \in \{1, ..., k\}$ and $\eta_i = \widetilde{r}_i - r_i$ for all $i \in \{2, ..., k\}$.

Case 1: Suppose that $s = 0$. We shall prove that $\ell = \widetilde{\ell}$. Notice that $I_1 = \widetilde{I}_1$ and $\widetilde{a}_1 = \mathcal{M}_p(\text{id}, I_1, \mu_f)$ and by uniqueness of the p-th mean we deduce that $a_1 = \widetilde{a}_1$. But $\widetilde{a}_1$, $\widetilde{a}_2$ and $\widetilde{r}_2$ are related by $\widetilde{a}_2 = 2\widetilde{r}_2 - \widetilde{a}_1 = 2r_2 - a_1 = a_2$, showing that $\widetilde{a}_2 = a_2$. Using the fact that $a_2 = \mathcal{M}_p(\text{id}, I_2, \mu_f) = \mathcal{M}_p(\text{id}, \widetilde{I}_2, \mu_f)$ and Lemma 9.1.7, we have that

$$\begin{aligned}\int_{a_2}^{\widetilde{r}_3} (x - a_2)^{p-1}\Psi(x)dx &= \int_{\widetilde{a}_2}^{\widetilde{r}_3} (x - a_2)^{p-1}\Psi(x)dx = \int_{\widetilde{r}_2}^{\widetilde{a}_2} (a_2 - x)^{p-1}\Psi(x)dx \\ &= \int_{r_2}^{a_2} (a_2 - x)^{p-1}\Psi(x)dx = \int_{a_2}^{r_3} (x - a_2)^{p-1}\Psi(x)dx.\end{aligned}$$

Since $\widetilde{r}_3 \leq \mathfrak{b}_f$ and since Ψ is strictly positive, we conclude that $\widetilde{r}_3 = r_3$. Repeating this argument we conclude that $\widetilde{r}_i = r_i$ for all $i \in \{2, ..., k\}$ and $\widetilde{a}_i = a_i$ for all $i \in \{1, ..., k\}$. Thus $\widetilde{\ell} = \ell$.

Case 2: Suppose that $s > 0$ and let us arrive to a contradiction if we suppose that (H1), (H2) or (H3) holds. This part will be divided in several steps.

Step 1: We are going to show that the following properties hold:

(a) $\delta_k \geq \eta_k \geq \delta_{k-1} \geq \eta_{k-1} \geq ... \geq \eta_2 \geq \delta_1$;

(b) if one of these inequalities is strict then all the inequalities on the left are also strict;

(c) all of these inequalities are in fact equalities if and only if $\mathfrak{a}_f = -\infty$ and for all $i \in \{1, ..., k-1\}$ it holds

$$\forall y \in (r_i, r_{i+1}) \quad \frac{\Psi(y+s)}{\Psi(y)} = \frac{\Psi(a_i + s)}{\Psi(a_i)}.$$

Define a function $\phi : z \mapsto \int_{\mathfrak{a}_f}^{a_1+z} (z + a_1 - x)^{p-1}\Psi(x)dx - \int_{a_1+z}^{\widetilde{r}_2} (x - a_1 - z)^{p-1}\Psi(x)dx$. It is clear that ϕ is strictly increasing. Recall that $\widetilde{a}_1 = \mathcal{M}_p(\text{id}, [\mathfrak{a}_f, \widetilde{r}_2), \mu_f)$, which is characterized by

$$\int_{\mathfrak{a}_f}^{\widetilde{a}_1} (\widetilde{a}_1 - x)^{p-1}\Psi(x)\ dx = \int_{\widetilde{a}_1}^{\widetilde{r}_2} (x - \widetilde{a}_1)^{p-1}\Psi(x)\ dx,$$

so $\phi(\delta_1) = 0$. Note also that

$$\begin{aligned}\phi(0) &= \int_{\mathfrak{a}_f}^{a_1} (a_1 - x)^{p-1}\Psi(x)dx - \int_{a_1}^{\widetilde{r}_2} (x - a_1)^{p-1}\Psi(x)dx \\ &= \int_{a_1}^{r_2} (x - a_1)^{p-1}\Psi(x)dx - \int_{a_1}^{\widetilde{r}_2} (x - a_1)^{p-1}\Psi(x)dx < 0,\end{aligned}$$

since $\widetilde{r}_2 - r_2 = s > 0$. Moreover, we have

$$\begin{aligned}\int_{\mathfrak{a}_f}^{a_1+s} (a_1 + s - x)^{p-1}\Psi(x)dx &\geq \int_{\mathfrak{a}_f+s}^{a_1+s} (a_1 + s - x)^{p-1}\Psi(x)dx = \int_{\mathfrak{a}_f}^{a_1} (a_1 - x)^{p-1}\frac{\Psi(x+s)}{\Psi(x)}\Psi(x)dx \\ &\geq \frac{\Psi(a_1+s)}{\Psi(a_1)}\int_{\mathfrak{a}_f}^{a_1} (a_1 - x)^{p-1}\Psi(x)dx = \frac{\Psi(a_1+s)}{\Psi(a_1)}\int_{a_1}^{r_2} (x - a_1)^{p-1}\Psi(x)dx \\ &\geq \int_{a_1}^{r_2} (x - a_1)^{p-1}\frac{\Psi(x+s)}{\Psi(x)}\Psi(x)\ dx = \int_{a_1+s}^{\widetilde{r}_2} (x - a_1 - s)^{p-1}\Psi(x)dx\end{aligned}$$

proving that $\phi(s) \geq 0$. It follows that $0 < \delta_1 \leq s = \eta_2$. The only way that $\delta_1 = \eta_2 = s$ is that the previous inequalities are only equalities, which means that $\mathfrak{a}_f = -\infty$ and $\frac{\Psi(y+s)}{\Psi(y)} = \frac{\Psi(a_1+s)}{\Psi(a_1)}$ holds for all $y \in (\mathfrak{a}_f, r_2)$ dy-a.e., but since G_s is decreasing this property holds for all $y \in (\mathfrak{a}_f, r_2)$. We summarize this condition for future reference

$$\mathfrak{a}_f = -\infty \text{ and } \forall y \in (\mathfrak{a}_f, r_2) \quad \frac{\Psi(y+s)}{\Psi(y)} = \frac{\Psi(a_1+s)}{\Psi(a_1)}. \tag{9.8}$$

On the other hand, since $\widetilde{r}_2 = \frac{\widetilde{a}_1+\widetilde{a}_2}{2}$, we deduce

$$\widetilde{a}_2 = 2\widetilde{r}_2 - \widetilde{a}_1 = 2r_2 - a_1 + 2\eta_2 - \delta_1 = a_2 + 2\eta_2 - \delta_1,$$

from where we deduce that $\delta_2 = 2\eta_2 - \delta_1 \geq \eta_2$, with equality $\delta_2 = \eta_2$ if and only if $\delta_2 = \eta_2 = \delta_1 = s$. Now, if there exists $\widetilde{r}_3 \leq \mathfrak{b}_f$ such that

$$\mathcal{M}_p(\mathrm{id}, [\widetilde{r}_2, \widetilde{r}_3), \mu_f) = \widetilde{a}_2,$$

we deduce that $\eta_3 \geq \delta_2$. Indeed, this follows from the inequalities

$$\begin{aligned}\int_{\widetilde{a}_2}^{\widetilde{r}_3} (x - \widetilde{a}_2)^{p-1}\Psi(x)dx &= \int_{\widetilde{r}_2}^{\widetilde{a}_2} (\widetilde{a}_2 - x)^{p-1}\Psi(x)dx = \int_{r_2+\eta_2}^{a_2+\delta_2} (\widetilde{a}_2 - x)^{p-1}\Psi(x)\ dx \\ &\geq \int_{r_2+\delta_2}^{a_2+\delta_2} (\widetilde{a}_2 - x)^{p-1}\Psi(x)dx = \int_{r_2}^{a_2} (a_2 - x)^{p-1}\Psi(x+\delta_2)dx \\ &= \int_{r_2}^{a_2} (a_2 - x)^{p-1}\frac{\Psi(x+\delta_2)}{\Psi(x)}\Psi(x)dx \geq \frac{\Psi(a_2+\delta_2)}{\Psi(a_2)}\int_{r_2}^{a_2} (a_2 - x)^{p-1}\Psi(x)dx \\ &= \frac{\Psi(a_2+\delta_2)}{\Psi(a_2)}\int_{a_2}^{r_3} (x - a_2)^{p-1}\Psi(x)dx \geq \int_{a_2}^{r_3} (x - a_2)^{p-1}\frac{\Psi(x+\delta_2)}{\Psi(x)}\Psi(x)dx \\ &= \int_{a_2}^{r_3} (x - a_2)^{p-1}\Psi(x+\delta_2)dx = \int_{a_2+\delta_2}^{r_3+\delta_2} (x - \widetilde{a}_2)^{p-1}\Psi(x)dx \\ &= \int_{\widetilde{a}_2}^{r_3+\delta_2} (x - \widetilde{a}_2)^{p-1}\Psi(x)dx\end{aligned}$$

proving that $\widetilde{r}_3 \geq r_3 + \delta_2$, i.e. $\eta_3 \geq \delta_2$. Also, we notice that $\eta_3 = \delta_2$ if and only if

$$\eta_2 = \delta_2 \text{ and } \forall y \in (r_2, r_3) \quad \frac{\Psi(y+\delta_2)}{\Psi(y)} = \frac{\Psi(a_2+\delta_2)}{\Psi(a_2)}, \tag{9.9}$$

which in particular implies that $\eta_3 = \eta_2 = \delta_2 = \delta_1 = s$ and (9.8) holds. Iterating this idea, we complete Step 1.

Step 2: Since $\widetilde{g}$ has f-special form, we have that $\widetilde{a}_k = \mathcal{M}_1(\mathrm{id}, [\widetilde{r}_k, \mathfrak{b}_f], \mu_f)$. Since $\delta_k \geq \eta_k$ by (a) in Step 1, we obtain that

$$\begin{aligned}
\int_{\widetilde{a}_k}^{\mathfrak{b}_f} (x-\widetilde{a}_k)^{p-1}\Psi(x)dx &= \int_{\widetilde{r}_k}^{\widetilde{a}_k} (\widetilde{a}_k - x)^{p-1}\Psi(x)dx = \int_{r_k+\eta_k}^{a_k+\delta_k} (\widetilde{a}_k - x)^{p-1}\Psi(x)\ dx \\
&\geq \int_{r_k+\delta_k}^{a_k+\delta_k} (\widetilde{a}_k - x)^{p-1}\Psi(x)dx = \int_{r_k}^{a_k} (a_k - x)^{p-1}\Psi(x+\delta_k)dx \\
&= \int_{r_k}^{a_k} (a_k - x)^{p-1}\frac{\Psi(x+\delta_k)}{\Psi(x)}\Psi(x)dx \geq \frac{\Psi(a_k+\delta_k)}{\Psi(a_k)}\int_{r_k}^{a_k} (a_k - x)^{p-1}\Psi(x)dx \\
&= \frac{\Psi(a_k+\delta_k)}{\Psi(a_k)}\int_{a_k}^{\mathfrak{b}_f} (x-a_k)^{p-1}\Psi(x)dx \geq \frac{\Psi(a_k+\delta_k)}{\Psi(a_k)}\int_{a_k}^{\mathfrak{b}_f-\delta_k} (x-a_k)^{p-1}\Psi(x)dx \\
&\geq \int_{a_k}^{\mathfrak{b}_f-\delta_k} (x-a_k)^{p-1}\frac{\Psi(x+\delta_k)}{\Psi(x)}\Psi(x)dx = \int_{a_k+\delta_k}^{\mathfrak{b}_f} (x-\widetilde{a}_k)^{p-1}\Psi(x)dx \\
&= \int_{\widetilde{a}_k}^{\mathfrak{b}_f} (x-\widetilde{a}_k)^{p-1}\Psi(x)dx
\end{aligned}$$

It follows that all the inequalities are in fact equalities and then the following properties hold:

(d) $\mathfrak{b}_f = \infty$;

(e) $\eta_k = \delta_k$;

(f) $\forall y \in (r_k, \infty)\ \ \frac{\Psi(y+\delta_k)}{\Psi(y)} = \frac{\Psi(a_k+\delta_k)}{\Psi(a_k)}$.

We notice that (e) implies that all inequalities in (a) are equalities and then (c) holds. This together with (d) and (f) implies that if $\widetilde{g}$ is a minimizer then

(g) $\delta_k = \eta_k = \delta_{k-1} = \eta_{k-1} = ... = \eta_2 = \delta_1 = s$;

(h) $\mathfrak{a}_f = -\infty, \mathfrak{b}_f = \infty$ and for all $i \in \{1, ..., k\}$ it holds

$$\forall y \in (r_i, r_{i+1})\ \ \frac{\Psi(y+s)}{\Psi(y)} = \frac{\Psi(a_i+s)}{\Psi(a_i)}.$$

Step 3: Conclusion. Clearly under (H1) or (H2) the function $\widetilde{g}$ cannot be a minimizer. It remains to consider that (H3) holds. From (h) and the continuity of Ψ it holds that

$$\forall y \in (-\infty, \infty)\ \ \frac{\Psi(y+s)}{\Psi(y)} = \frac{\Psi(a_1+s)}{\Psi(a_1)} = C \in (0, \infty)$$

Iterating this equality we have $\Psi(y+2s) = \frac{\Psi(y+2s)}{\Psi(y+s)}\frac{\Psi(y+s)}{\Psi(y)}\Psi(y) = C^2\Psi(y)$, and then for all $n \in \mathbb{Z}$ and all y

$$\Psi(y+ns) = C^n\Psi(y).$$

Then, if $C \geq 1$, we have

$$C^n\int_0^1 \Psi(y)\ dy = \int_0^1 \Psi(y+ns)\ dy = \int_{ns}^{ns+1} \Psi(y)\ dy \underset{n\to\infty}{\longrightarrow} 0,$$

which is a contradiction. A similar contradiction is obtained if $C < 1$, because $\Psi(y - ns) = C^{-n}\Psi(y)$ and then

$$C^{-n}\int_{-1}^{0} \Psi(y)\ dy = \int_{-1}^{0} \Psi(y - ns)\ dy = \int_{-(1+ns)}^{-ns} \Psi(y)\ dy \underset{n\to\infty}{\longrightarrow} 0,$$

proving that $\widetilde{g}$ cannot be a minimizer, and the result is shown.

□

Remark 9.1.11. Examples of distributions that satisfie the hypothesis of the previous proposition are the exponential distribution $\mu_f(dx) = e^{-x}\ dx$ for $x \geq 0$, the normal distribution $\mathcal{N}(0,1)$ and the uniform distribution $\mu_f(dx) = dx$ for $x \in [0,1]$. In the uniform case, we obtain an explicit solution for the minimizer of $f \in L^p(\Omega, \mathcal{F}, \mu)$. For all $k \geq 1$ this unique minimizer is $g = \ell \circ f$, where

$$\ell = \sum_{i=1}^{k} \frac{2i-1}{2k}\ \mathbb{1}_{[\frac{i-1}{k}, \frac{i}{k})},$$

independently of $p \in [1, \infty)$.

9.1.2 The case of an infinite measure, $p \in [1, \infty)$

The case of infinite measure needs an extra work and use some ideas already developed in the finite measure case.

Theorem 9.1.12. *Let $(\Omega, \mathcal{F}, \mu)$ be an infinite measure space, $p \in [1, \infty)$ and $k \geq 1$. Then $\mathscr{G}_{p,k}$ is proximinal.*

Moreover, if $f \in L^p(\Omega, \mathcal{F}, \mu)$ and $g = \sum\limits_{i=1}^{q} b_i \mathbb{1}_{A_i} \in P_{\mathscr{G}_{p,k}}(f)$ is a minimizer, with $q \leq k$, $-\infty < b_1 < ... < b_q < \infty$, $\{A_i\}_{1\leq i\leq q}$ a partition of Ω such that $\mu(A_i) > 0$ for all $i \in \{1, ..., q\}$ and a unique $1 \leq s \leq q$ such that $b_s = 0$. Then, there exists a minimizer $\widetilde{g} \in P_{\mathscr{G}_{p,q}}(f)$ in f-special form

$$\widetilde{g} = \sum_{i=1, i\neq s}^{q} \mathcal{M}_p(f, f^{-1}(C_i))\mathbb{1}_{f^{-1}(C_i)} + 0\mathbb{1}_{f^{-1}(C_s)},$$

where

- *$r_1 = -\infty, r_{q+1} = \infty$ and $r_i = \frac{b_{i-1}+b_i}{2}$ for all $i \in \{2, ..., q\}$ (notice that $r_s < 0 < r_{s+1}$);*
- *$C_i = f^{-1}([r_i, r_{i+1}))$ for $i \in \{1, ..., q-1\}$ and $C_q = f^{-1}([r_q, r_{q+1}])$;*
- *if $\mu(f^{-1}(C_i)) > 0$ and $i \neq s$, then b_i is a p-th mean of f on $f^{-1}(C_i)$;*

If q is the smallest among all minimizers, then $\mu(C_i) > 0$ for all i.

Proof. For $k = 1$ the result is obvious since $\mathscr{G}_{p,1} = \{0\}$. So for the rest of the proof we assume that $k \geq 2$.

Let $f \in L^p(\Omega, \mathcal{F}, \mu)$ and consider a sequence $(g_n)_n \in \mathscr{G}_{p,k}$ such that $g_n = \sum_{i=1}^{q(n)} a_{i,n}\mathbb{1}_{A_{i,n}}$, where $a_{1,n} < ... < a_{q(n),n} \in \mathbb{R}$, $\{A_{i,n}\}_{1\leq i\leq q(n)}$ is a measurable partition with sets of positive measure and $q(n) \leq k$ for all $n \in \mathbb{N}$, and such that

$$\|f - g_n\|_p \to \mathscr{D}_{p,k}(f).$$

Since $g_n \in L^p(\Omega, \mathcal{F}, \mu)$ there exists a unique $1 \leq s(n) \leq q(n)$ such that $a_{s(n),n} = 0$ and we have that $\mu(A_{i,n}) < \infty$ for all $i \neq s(n)$. Passing to a subsequence, we can assume that

$1 \le s(n) = s \le q(n) = q \le k$. Define $r_1 = r_{1,n} = -\infty$, $r_{q+1,n} = \infty$ and $r_{i,n} = \frac{a_{i-1,n}+a_{i,n}}{2}$ for $i \in \{2, ..., q\}$. We point out that if $q = 1$, then $g_n = 0$, for all n and so $h = 0$ is a minimizer. Then for the rest of the proof, we assume $q \ge 2$.

Now, consider $I_{i,n} = [r_{i,n}, r_{i+1,n})$ and the corresponding $C_{i,n} = f^{-1}(I_{i,n})$ for all $i \in \{1, ..., q\}$. For all $n \in \mathbb{N}$, define

$$\widetilde{g}_n = \sum_{i=1}^{q} a_{i,n} \mathbb{1}_{f^{-1}(I_{i,n})}.$$

If $i, j \in \{1, ..., q\}$, we have that $|f(x) - a_{i,n}| \ge |f(x) - a_{j,n}|$ for all $x \in C_j$. It follows that for all $n \in \mathbb{N}$

$$\begin{aligned}\|f - g_n\|_p^p &= \sum_{i=1}^{q} \int_{A_i} |f(x) - a_{i,n}|^p d\mu(x) = \sum_{j=1}^{q}\sum_{i=1}^{q} \int_{A_i \cap C_j} |f(x) - a_{i,n}|^p d\mu(x) \\ &\ge \sum_{j=1}^{q}\sum_{i=1}^{q} \int_{A_i \cap C_j} |f(x) - a_{j,n}|^p d\mu(x) \\ &= \sum_{j=1}^{q} \int_{C_j} |f(x) - a_{j,n}|^p d\mu(x) = \|f - \widetilde{g}_n\|_p^p\end{aligned}$$

proving that $(\widetilde{g}_n)_n$ is also a minimizing sequence.

For all $i \in \{1, ..., q\}$, the sequence $(a_{i,n})_n$ has a convergent subsequence in $\overline{\mathbb{R}}$. Then we can also assume that $a_{i,n} \to a_i \in \overline{\mathbb{R}}$ for all $i \in \{1, ..., q\}$. We denote by $z_1 < ... < z_\ell$ the different values in $\{a_1, ..., a_q\}$, where $\ell \le q$. We point out that $z_t = 0$ for some $1 \le t \le \ell$. For each $1 \le m \le \ell$, we denote $L_m = \{i : \ 1 \le i \le q \text{ and } a_i = z_m\}$. Each L_m is an interval in $\mathbb{N}$, because we have assumed $a_{1,n} < ... < a_{i,n} < ... < a_{q,n}$, for each n. We define $i_m^- = \min\{L_m\}$ and $i_m^+ = \max\{L_m\}$ for all $m \in \{1, ..., \ell\}$ and also $i_{\ell+1}^- = \ell + 1$. Note that $L_m = \{i_m^-, ..., i_m^+\}$ for all $m \in \{1, ..., \ell\}$.

Assume that $z_\ell = \infty$ or $z_1 = -\infty$. In this situation $\ell \ge 2$, because $z_t = 0$. As in the case of finite measure we can modify $(\widetilde{g}_n)_n$ to get a uniformly bounded minimizing sequence. Consider first the case $z_\ell = \infty$ and recall that $i_\ell^- = \min\{L_\ell\} \in \{s+1, ..., q\}$. Then, we have

$$r_{i_\ell^-,n} = \frac{a_{i_\ell^- -1,n} + a_{i_\ell^-,n}}{2} \to \infty,$$

because $a_{i_\ell^- -1,n} \ge a_{s,n} = 0$ and then $a_{i_\ell^- -1,n} \to a_{i_\ell^- -1} = z_{\ell-1} \in [0, \infty)$. Consider

$$\widehat{g}_n = \sum_{i<i_\ell^-} a_{i,n}\mathbb{1}_{f^{-1}(I_{i,n})} + a_{i_\ell^- -1,n}\mathbb{1}_{f^{-1}([r_{i_\ell^-,n},\infty))} = \sum_{i<i_\ell^- -1} a_{i,n}\mathbb{1}_{f^{-1}(I_{i,n})} + a_{i_\ell^- -1,n}\mathbb{1}_{f^{-1}([r_{i_\ell^- -1,n},\infty))}$$

An important fact is that $\int_{\{f \ge r_{i_\ell^-,n}\}} |f(x) - a_{i_\ell^- -1,n}|^p \, d\mu(x) \to 0$, because $(a_{i_\ell^- -1,n})_n$ is a bounded sequence and $r_{i_\ell^-,n} \to \infty$. Then

$$\|f - \widehat{g}_n\|_p^p \le \|f - \widetilde{g}_n\|_p^p + \int_{\{f > r_{i_\ell^-,n}\}} |f(x) - a_{i_\ell^- -1,n}|^p \, d\mu(x) \to (\mathscr{D}_{p,k}(f))^p.$$

Then, the sequence $(\widehat{g}_n)_n$ is a minimizing sequence, which is uniformly upper bounded. Similarly, we can modify this sequence to get a minimizing sequence, which is uniformly bounded. Then, in what follows we assume $(\widetilde{g}_n)_n$ is uniformly bounded and $-\infty < z_1, z_\ell < \infty$.

Now we consider 2 different cases.

Case 1: $\ell = 1$. In this situation $a_1 = ... = a_q = 0$. Notice that $r_{q,n} = \frac{a_{q-1,n}+a_{q,n}}{2} \to 0$, so if $0 < f(x) < \infty$, then $\widetilde{g}_n(x) = a_{q,n}$ for all large n and $\widetilde{g}_n(x) \to 0$. In the same way, if $f(x) < 0$, then $\widetilde{g}_n(x) = a_{1,n}$ for all large n and $\widetilde{g}_n(x) \to 0$. On the other hand, if $f(x) = 0$, then $\widetilde{g}_n(x) = 0$. Then by Fatou's Lemma we conclude

$$\liminf_n \|f - \widetilde{g}_n\|_p^p \geq \int \liminf_n |f(x) - \widetilde{g}_n|^p \; d\mu(x) = \|f\|_p^p,$$

and we obtain $\mathscr{D}_{p,k}(f) \geq \|f\|_p$, showing that $h = 0$ is a minimizer.

Case 2: $\ell \geq 2$. For all $m \in \{1, ..., \ell\}$ recall that $i_m^- = \min\{L_m\}$ and $i_{\ell+1}^- = \ell + 1$. Then, for all $2 \leq m \leq \ell$

$$r_{i_m^-,n} \to r_m := \frac{z_{m-1} + z_m}{2}.$$

and $r_1 = -\infty < r_2 < ... < r_\ell < r_{\ell+1} := \infty$. Now, we choose a particular subsequence $(n')_{n'}$. We start with $(r_{i_2^-,n})_n$. If there exist an increasing subsequence of $(r_{i_2^-,n})_n$, we fix one of these subsequences as $(n^{(2)})$ and we put $T(2) = \text{in}$, for increasing. Otherwise we take $(n^{(2)})$ so that $(r_{i_2^-,n^{(2)}})_{n^{(2)}}$ is strictly decreasing, and we put $T(2) = \text{sd}$, for strictly decreasing. We repeat this procedure for $(r_{i_3^-,n^{(2)}})_{n^{(2)}}$, to obtain, if possible, $(n^{(3)})$ a subsequence of $(n^{(2)})$ so $(r_{i_3^-,n^{(3)}})_{n^{(3)}}$ is increasing, and put $T(3) = \text{in}$. Otherwise we take $(n^{(3)})$ a subsequence of $(n^{(2)})$ so that $(r_{i_3^-,n^{(3)}})_{n^{(3)}}$ is strictly decreasing, and we put $T(3) = \text{sd}$. We continue until $m = \ell$. We also put $T(1) = \text{in}$ and $T(\ell + 1) = \text{in}$. Denote by $(n') = (n^{(\ell)})$.

Now, we define the intervals that give a minimizer. For all $m \in \{1, ..., \ell\}$ let

$$I_m = \begin{cases} [r_m, r_{m+1}) & \text{if } T(m) = \text{in}, \; T(m+1) = \text{in} \\ [r_m, r_{m+1}] & \text{if } T(m) = \text{in}, \; T(m+1) = \text{sd} \\ (r_m, r_{m+1}) & \text{if } T(m) = \text{sd}, \; T(m+1) = \text{in} \\ (r_m, r_{m+1}] & \text{if } T(m) = \text{sd}, \; T(m+1) = \text{sd} \end{cases} \tag{9.10}$$

We notice that $\cup_{m=1}^{\ell} I_m = [-\infty, \infty)$, and for all $m \in \{1, ..., \ell\}$ and all n', we define $J_{m,n'} = \cup_{i \in L_m} [r_{i,n'}, r_{i+1,n'}) = [r_{i_m^-,n'}, r_{i_{m+1}^-,n'})$. Then, it holds

$$\mathbb{1}_{f^{-1}(J_{m,n'})} \to \mathbb{1}_{f^{-1}(I_m)} \; a.e.$$

The last piece of information we need is that the set $\cup_{m \neq t} f^{-1}(J_{m,n'})$ is contained in a fixed set of finite measure $\widetilde{A}$ for large n'. If $t = \ell$, then $\cup_{m \neq t} f^{-1}(J_{m,n'}) \subset f^{-1}((-\infty, r_{t,n'}]) \subset \widetilde{A} = f^{-1}((-\infty, \frac{r_t}{2}])$, for large n', because $r_{t,n'} \to r_t = \frac{z_{t-1}}{2} < z_t = 0$, and then $\widetilde{A}$ has finite measure. Similarly, if $t = 1$, then $\cup_{m \neq t} f^{-1}(J_{m,n'}) \subset f^{-1}([r_{2,n'}, \infty]) \subset \widetilde{A} = f^{-1}([\frac{r_2}{2}, \infty))$, for large n'. This set has finite measure because $r_2 > 0$. In the general case, $1 < t < q$, we have for large n'

$$\cup_{m \neq t} f^{-1}(J_{m,n'}) \subset \widetilde{A} = f^{-1}((-\infty, r_t/2]) \cup f^{-1}([r_{t+1}/2, \infty)),$$

which has finite measure because $r_t < 0 < r_{t+1}$.

Now, consider the decomposition

$$\|f - \widetilde{g}_{n'}\|_p^p = \int |f(x) - \widetilde{g}_{n'}(x)|^p \mathbb{1}_{f^{-1}(J_{t,n'})} \; d\mu(x) + \sum_{m \neq t} \int_{f^{-1}(J_{m,n'})} |f(x) - \widetilde{g}_{n'}(x)|^p \; d\mu(x).$$

We use now Fatou's Lemma for the first term and the Dominated Convergence Theorem for the second term. In the first term, we have the a.e. convergence

$$|f - \widetilde{g}_{n'}| \mathbb{1}_{f^{-1}(J_{t,n'})} \to |f| \mathbb{1}_{f^{-1}(I_t)}$$

With respect to the second term, for large n', we have $\max_{1\leq i\leq q}|a_{i,n'}| \leq \max\{|z_1|, z_\ell\}+1 := C$, also

$$|f(x) - \widetilde{g}_{n'}(x)|^p \, \mathbb{1}_{\cup_{m\neq t} f^{-1}(J_{m,n'})} \leq 2^{p-1}\left(|f(x)|^p + C^p\right)\mathbb{1}_{\widetilde{A}} \in L^1(\Omega,\mathcal{F},\mu),$$

$\widetilde{g}_{n'}\mathbb{1}_{\cup_{m\neq t} f^{-1}(J_{m,n'})} \to \sum_{m\neq t} z_m \mathbb{1}_{f^{-1}(I_m)}$ a.e. and $f\mathbb{1}_{\cup_{m\neq t} f^{-1}(J_{m,n'})} \to f\mathbb{1}_{f^{-1}(\cup_{m\neq t} I_m)}$ a.e. So, we get

$$\liminf_{n'} \|f - \widetilde{g}_n\|_p^p \geq \int_{f^{-1}(I_t)} |f(x)|^p \, d\mu(x) + \sum_{m\neq t}\int_{f^{-1}(I_m)} |f(x) - z_m|^p \, d\mu(x)$$

and then $h = \sum_{m=1}^{\ell} z_m \mathbb{1}_{f^{-1}(I_m)}$ is a minimizer, where the intervals $\{I_m\}_{1\leq m\leq t}$ are either open, closed or semi-closed, they are disjoint and $\cup_{m=1}^{\ell} I_m = \mathbb{R}$ (see (9.10)).

From here it is clear that a minimizer exists in f-special form as we have done in the finite measure case. Also notice that if $m \neq t$ and $0 < \mu(f^{-1}(I_m))$, we must have z_m is a p-th mean for f in $f^{-1}(I_m)$, since h is a minimizer. $\square$

9.1.3 The case $p = \infty$

In this section we shall prove that $\mathscr{G}_{\infty,k}$ is proximinal. We start with a lemma.

Lemma 9.1.13. *Let $(\Omega,\mathcal{F},\mu)$ be a measure space and $f \in L^\infty(\Omega,\mathcal{F},\mu)$. Then, for all $k \geq 1$, we have that $\mathscr{D}_{\infty,k}(f) = \eta_k(f)$ where*

$$\eta_k(f) = \inf_{h=f \text{ a.e}} \inf\{\alpha > 0 \mid h(\Omega) \text{ can be covered by at most } k \text{ closed balls of radius } \alpha\}.$$

Proof. Let $\varepsilon > 0$ and let $g \in \mathscr{G}_{\infty,k}$ such that $\|f - g\|_\infty \leq \mathscr{D}_{\infty,k}(f) + \varepsilon$. Write $g = \sum_{i=1}^k a_i \mathbb{1}_{A_i}$ where $\{A_i\}_{1\leq i\leq k}$ is a partition of Ω. For every i, the set $C_i = \{x \in A_i : |f(x) - a_i| > \|(f-g)\mathbb{1}_{A_i}\|_\infty\}$ has measure 0 and therefore $h = f\mathbb{1}_{\Omega\setminus\cup_j C_j} + \sum_{j=1}^{k} a_j \mathbb{1}_{C_j}$ satisfies $h = f$ a.e. and

$$h(\Omega) \subset \bigcup_{i=1}^{k} [a_i - \mathscr{D}_{\infty,k}(f) - \varepsilon, a_i + \mathscr{D}_{\infty,k}(f) + \varepsilon].$$

It follows that $\eta_k(f) \leq \mathscr{D}_{\infty,k}(f) + \varepsilon$ and since ε is arbitrary, we obtain that $\eta_k(f) \leq \mathscr{D}_{\infty,k}(f)$. To prove the other inequality, let again $\varepsilon > 0$ and pick $l \leq k$, $a_1, ..., a_l \in \mathbb{R}$ and $h = f$ a.e. such that

$$h(\Omega) \subset \bigcup_{i=1}^{l} [a_i - \eta_k(f) - \varepsilon, a_i + \eta_k(f) + \varepsilon].$$

For $1 \leq i \leq l$, define $A_i = h^{-1}([a_i - \eta_k(f) - \varepsilon, a_i + \eta_k(f) + \varepsilon]) \in \mathcal{F}$. Now define $B_1 = A_1$ and $B_i = A_i \setminus \bigcup_{j=1}^{i-1} A_i$ for $i \in \{2, ..., l\}$. Then $\{B_i\}_{1\leq i\leq l}$ is a partition of Ω. Defining $g = \sum_{i=1}^{l} a_i \mathbb{1}_{B_i} \in \mathscr{G}_{\infty,k}$, it is clear that $\|f - g\|_\infty \leq \eta_k(f) + \varepsilon$. It follows that $\mathscr{D}_{\infty,k}(f) \leq \eta_k(f) + \varepsilon$ and then $\mathscr{D}_{\infty,k}(f) \leq \eta_k(f)$. $\square$

Proposition 9.1.14. *Let $(\Omega,\mathcal{F},\mu)$ be a measure space. Then $\mathscr{G}_{\infty,k}$ is proximinal for all $k \geq 1$.*

Proof. For all $n \in \mathbb{N}$, let $\alpha_n = \eta_k(f) + \frac{1}{n}$. So, for all $n \in \mathbb{N}$, there exist $a_1^n, ..., a_{l_n}^n \in \mathbb{R}$ with $1 \leq l_n \leq k$ and $h_n = f$ a.e. such that $h_n(\Omega) \subset \bigcup_{i=1}^{l_n}[a_i^n - \alpha_n, a_i^n + \alpha_n]$. Of course there exists i_0 such that $\mu(h_n^{-1}([a_{i_0}^n - \alpha_n, a_{i_0}^n + \alpha_n])) > 0$. If for some i it holds $\mu(h^{-1}([a_i^n - \alpha_n, a_i^n + \alpha_n])) = 0$, we can redefine h_n on a set of measure 0, to have $h_n(w) = a_{i_0}$ for all $w \in h_n^{-1}([a_i^n - \alpha_n, a_i^n + \alpha_n])$. So, we can assume for all i it holds $\mu(h_n^{-1}([a_i^n - \alpha_n, a_i^n + \alpha_n])) > 0$. Consider

$$t_{i,n} = \frac{1}{\mu(h_n^{-1}([a_i^n - \alpha_n, a_i^n + \alpha_n]))}\int_{h_n^{-1}([a_i^n - \alpha_n, a_i^n + \alpha_n])} h_n(x) \, d\mu(x) \in [a_i^n - \alpha_n, a_i^n + \alpha_n],$$

which obviously satisfies $|t_{i,n}| \leq \|h_n\|_\infty = \|f\|_\infty$. Then, for all i, n, it holds

$$|a_i^n| \leq |t_{i,n}| + |a_i^n - t_{i,n}| \leq \|f\|_\infty + \alpha_n \leq \|f\|_\infty + \eta_k(f) + 1,$$

which implies that the set $\{a_i^n \ : \ 1 \leq i \leq l_n \ n \in \mathbb{N}\}$ is bounded.

Considering a subsequence if necessary, we can suppose that $l_n = l \in \{1, ..., k\}$ for all $n \in \mathbb{N}$. By compactness and taking a further subsequence, we can also assume that $a_i^n \to a_i$ for all $i \in \{1, ..., l\}$. Define

$$C = \{\omega \in \Omega \mid \forall n \in \mathbb{N} \ f(\omega) = h_n(\omega)\} \in \mathcal{F}$$

and note that $\mu(C^c) = 0$. Let us show that $f(C) \subset \bigcup_{i=1}^l [a_i - \eta_k(f), a_i + \eta_k(f)]$. In fact, if $\omega \in C$ then for all $n \in \mathbb{N}$ there exists $i(\omega, n) \in \{1, ..., l\}$ such that $f(\omega) = h_n(\omega) \in [a_{i(\omega,n)}^n - \alpha_n, a_{i(\omega,n)}^n + \alpha_n]$. There exists a subsequence $\phi(n) = \phi(n)(\omega)$ such that the sequence $(i(\omega, \phi(n)))_n$ is constant and equal to some $i_0(\omega) \in \{1, ..., l\}$. It follows that $f(\omega) \in [a_{i_0(\omega)} - \eta_k(f), a_{i_0(\omega)} + \eta_k(f)] \subset \bigcup_{i=1}^l [a_i - \eta_k(f), a_i + \eta_k(f)]$. Define $h = f\mathbb{1}_C + t\mathbb{1}_{C^c}$ where t is any real belonging to $\bigcup_{i=1}^l [a_i - \eta_k(f), a_i + \eta_k(f)]$. We have that $f = h$ a.e. and $h(\Omega) \subset \bigcup_{i=1}^l [a_i - \eta_k(f), a_i + \eta_k(f)]$. For $1 \leq i \leq l$, define $A_i = h^{-1}([a_i - \eta_k(f), a_i + \eta_k(f)]) \in \mathcal{F}$. Now define $B_1 = A_1$ and $B_i = A_i \setminus \bigcup_{j=1}^{i-1} A_i$ for $i \in \{2, ..., l\}$. Then $\{B_i\}_{1 \leq i \leq l}$ is a partition of Ω. Defining $g = \sum_{i=1}^l a_i \mathbb{1}_{B_i} \in \mathscr{G}_{\infty,k}$, it is clear that $\|f - g\|_\infty = \|h - g\|_\infty \leq \eta_k(f)$. Moreover, we have that $\mathscr{D}_{\infty,k}(f) = \eta_k(f)$ by the previous lemma and so we conclude that $\|f - g\|_\infty = \mathscr{D}_{\infty,k}(f)$.

□

9.1.4 Extra properties of minimizers and the sets $(\mathscr{G}_{p,k})_{p,k}$

In this section we include some extra properties of the sets $(\mathscr{G}_{p,k})_{p,k}$ as well as some natural questions like uniqueness of minimizers and the existence of a continuous selection for $P_{\mathscr{G}_{p,k}}$.

Let us start by proving that $\mathscr{G}_{p,k}$ is a closed set, for all $p \geq 1, k \geq 1$, something that it is not straightforward to do. Nevertheless, this is a direct consequence of the previous results.

Corollary 9.1.15. *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, $p \in [1, \infty]$ and $k \geq 1$. Then $\mathscr{G}_{p,k}$ is closed.*

Proof. Assume $(g_n)_n \subset \mathscr{G}_{p,k}$ converges in $L^p(\Omega, \mathcal{F}, \mu)$ to g. Then

$$\inf\{\|g - h\|_p : \ h \in \mathscr{G}_{p,k}\} = 0.$$

From the previous results, there exists a minimizer $\bar{h} \in \mathscr{G}_{p,k}$, that is $g = \bar{h}$ a.e. and the result is shown. □

A question that appears when proving the existence of minimizers is the following. Assume there exists a best approximation of f by an element of $\mathscr{G}_{p,k}$ which is in fact an element of $\mathscr{G}_{p,m}$ for some $m < k$, then it is natural to think that f should belong to $\mathscr{G}_{p,m}$. This is true when $p \in [1, \infty)$ and it is not true for $p = \infty$. Before doing that we require the following lemma.

Lemma 9.1.16. *Assume that $f \in L^p(\Omega, \mathcal{F}, \mu)$, for $1 \leq p < \infty$, and $A = f^{-1}(I)$ is a set of positive and finite measure, where I is an interval. Assume b is a p-th mean of f on A, then $b \in \bar{I}$.*

Proof. Assume the interval $\bar{I} = [c, d]$, where $c, d \in \overline{\mathbb{R}}$ and let us prove that $b \geq c$. If $c = -\infty$ it is clear that $c < b$. So assume c is finite. By contradiction, if $b < c$ we have $|f(x) - b| = f(x) - c + (c - b) > f(x) - c = |f(x) - c|$, for all $x \in f^{-1}(I)$ and then, since $\mu(f^{-1}(I)) > 0$, we get

$$\int_{f^{-1}(I)} |f(x) - b|^p \ d\mu(x) > \int_{f^{-1}(I)} |f(x) - c|^p \ d\mu(x),$$

which is contradiction. Similarly, it is shown that $b \leq d$. □

Proposition 9.1.17. *Assume $f \in L^p(\Omega, \mathcal{F}, \mu)$ with $p \in [1, \infty)$. Let $m, k \in \mathbb{N}$ such that $1 \leq m < k$. Suppose that there exists $g \in \mathscr{G}_{p,m} \cap P_{\mathscr{G}_{p,k}}(f)$. Then $f \in \mathscr{G}_{p,m}$.*

Proof. Suppose that the measure is finite. We can assume that

$$g = \sum_{i=1}^{r} b_i \mathbb{1}_{f^{-1}(I_i)},$$

where $r \leq m$, $\{I_i\}_{1 \leq i \leq r}$ is a family of disjoint intervals such that $\{f^{-1}(I_i)\}_{1 \leq i \leq r}$ is a partition of Ω and $b_1 < ... < b_r$. Suppose by contradiction that $f \notin \mathscr{G}_{p,m}$. Then in particular it holds that $\mu(f^{-1}(\{b_1, ..., b_r\}^c) > 0$. Since $\Theta = \{b_1, ..., b_r\}^c$ is open, it is a countable union of open intervals $(J_n)_n$ and therefore for some n_0 we should have $\mu(f^{-1}(J_{n_0})) > 0$. By the continuity of the measure, there exists a closed bounded interval $J \subset J_{n_0}$ such that $\mu(f^{-1}(J)) > 0$, and therefore $\mu(f^{-1}(J \cap I_{i_0})) > 0$, for some i_0. Hence, we obtain

$$\int_{f^{-1}(I_{i_0} \cap J)} |f - \mathcal{M}_p(f, f^{-1}(I_{i_0} \cap J))|^p \, d\mu(x) < \int_{f^{-1}(I_{i_0} \cap J)} |f - b_{i_0}|^p \, d\mu(x),$$

since an equality in the previous formula would imply that $b_{i_0} \in \overline{I_{i_0} \cap J} \subset J \subset \{b_1, ..., b_r\}^c$, by the previous lemma. If we define

$$h = \sum_{i=1, i \neq i_0}^{r} b_i \mathbb{1}_{f^{-1}(I_i)} + b_{i_0} \mathbb{1}_{f^{-1}(I_{i_0} \cap J^c)} + \mathcal{M}_p(f, f^{-1}(I_{i_0} \cap J)) \mathbb{1}_{f^{-1}(I_{i_0} \cap J)} \in \mathscr{G}_{p,r+1} \subset \mathscr{G}_{p,k},$$

we have that $\|f - h\|_p < \|f - g\|_p$ which contradicts the minimality of g. We conclude that $f \in \mathscr{G}_{p,m}$.

In case the measure is infinite, with the same notation as above, we know that $b_{i_1} = 0$ for some i_1. As above there exists a closed and bounded interval $J \subset \{b_1, ..., b_r\}^c \subset \{0\}^c$, such that $\mu(f^{-1}(J)) > 0$. Without loss of generality we can assume that $J \subset [a, \infty)$, for some $a > 0$. Then

$$\mu(f^{-1}(J)) a^p \leq \|f\|^p,$$

proving that $f^{-1}(J)$ has finite and positive measure. The argument now goes as in the case of finite measure. ☐

The following result shows that, for $p \in [1, \infty)$, the error in the approximation by functions in $\mathscr{G}_{p,k}$ decreases strictly with k until eventually reaching zero.

Corollary 9.1.18. *Assume that $f \in L^p(\Omega, \mathcal{F}, \mu)$ with $p \in [1, \infty)$ and consider $\mathscr{D}_{p,\infty}(f) = 0$. Define $k^* = \min\{k : \mathscr{D}_{p,k}(f) = 0\} \in [1, \infty]$. Then, $(\mathscr{D}_{p,k}(f))_{k \leq k^*}$ is strictly decreasing and $\mathscr{D}_{p,k}(f) = 0$ for all $k \geq k^*$, that is*

$$k^* = \min\{k : \mathscr{D}_{p,k+1}(f) = \mathscr{D}_{p,k}(f)\} = \min\{k : \mathscr{D}_{p,k}(f) = 0\}.$$

The previous results are not true for $p = \infty$. In fact, we have the following example:

Example 9.1.19. Consider the Lebesgue measure in $[0, 1]$, the function

$$f(x) = \begin{cases} x & \text{for } x \notin (\frac{1}{3}, \frac{2}{3}) \\ \frac{1}{3} & \text{for } x \in (\frac{1}{3}, \frac{2}{3}) \end{cases}$$

and $k = 3$. It is not difficult to show that $\mathscr{D}_{\infty,3}(f) = \frac{1}{6}$, where there are multiple minimizers, for example

$$h = \frac{1}{6} \mathbb{1}_{[0, \frac{1}{3}]} + \frac{1}{2} \mathbb{1}_{(\frac{1}{3}, \frac{2}{3})} + \frac{5}{6} \mathbb{1}_{[\frac{2}{3}, 1]}$$

is a minimizer, but also

$$g = \frac{1}{6}\,\mathbb{1}_{[0,\frac{2}{3})} + \frac{5}{6}\,\mathbb{1}_{[\frac{2}{3},1]} \in \mathscr{G}_{\infty,2}$$

is a minimizer, in particular $\mathscr{D}_{\infty,3}(f) = \mathscr{D}_{\infty,2}(f) > 0$. Nevertheless, $f \notin \mathscr{G}_{\infty,k}$ for all k. This also shows that $\mathscr{G}_{\infty,3}$ is not Chebyschev.

In Proposition 9.1.10, we have shown that under certain conditions on f, there exists a unique minimizer. An important question then is if $\mathscr{G}_{p,k}$ is Chebyschev, that is, if there is a unique minimizer for all f. As we have seen in the previous example this is not true for $p = \infty$, and we complement this for all p.

Example 9.1.20. Consider again the Lebesgue measure in $[0, 1]$. Then $\mathscr{G}_{p,2}$ is not Chebyshev for any $p \in [1, \infty]$. To see that, let $f = -\mathbb{1}_{[0,\frac{1}{3})} + 0\mathbb{1}_{[\frac{1}{3},\frac{2}{3})} + \mathbb{1}_{[\frac{2}{3},1]}$. A possible minimizer in $\mathscr{G}_{2,2}$ has the form $g = a\mathbb{1}_{f^{-1}((-\infty,r_2))} + b\mathbb{1}_{f^{-1}([r_2,\infty])}$, for suitable a, b, r_2 (see Theorem 9.1.5). If $r_2 \le -1$ or $r_2 > 1$, a candidate to be a minimizer is $g_1 = 0$. For $-1 < r_2 \le 0$ the candidate is $g_2 = -\mathbb{1}_{[0,\frac{1}{3})} + \frac{1}{2}\mathbb{1}_{[\frac{1}{3},1]}$. Finally, for $0 < r_2 < 1$ the candidate is $g_3 = -\frac{1}{2}\mathbb{1}_{[0,\frac{2}{3})} + \mathbb{1}_{[\frac{2}{3},1]}$. The corresponding errors are

$$\|f - g_1\|_2^2 = \frac{2}{3},\ \|f - g_2\|_2^2 = \|f - g_3\|_2^2 = \frac{1}{6},$$

showing that g_2 and g_3 are two minimizers and then $\mathscr{G}_{2,2}$ is not Chebyschev. Finally, for every $p \in [1, \infty]$ both g_2 and g_3 are minimizers in $\mathscr{G}_{p,2}$, showing that this set is not Chebyschev for any p. Moreover, for $1 < p < \infty$, it can be proved that g_2, g_3 are the only minimizers. For $p = 1$, there is a continuum of minimizers since

$$g_a = -\mathbb{1}_{[0,\frac{1}{3})} + a\mathbb{1}_{[\frac{1}{3},1]}$$

is a minimizer for all $a \in [0, 1]$. For $p = \infty$, there is also a continuum of minimizers since

$$h_b = b\mathbb{1}_{[0,\frac{1}{3})} + \frac{1}{2}\mathbb{1}_{[\frac{1}{3},1]}$$

is a minimizer for all $b \in [\frac{-3}{2}, \frac{-1}{2}]$.

Remark 9.1.21. We have proved that $\mathscr{G}_{p,k}$ is proximinal and closed for all $k \ge 1$ and $p \in [1, \infty]$. However, $\mathscr{G}_{p,k}$ is not Chebyshev in general as we have shown in the previous examples. Then, it is natural to ask if $P_{\mathscr{G}_{p,k}}$ admits a continuous selection. If such continuous selection exists, then $\mathscr{G}_{p,k}$ has to be almost-convex (see Lemma 5 in [168]). Remember that a subset K of a Banach space is said to be *almost-convex* (see [168]) if for every closed ball B such that $K \cap B = \emptyset$, there exists a closed ball B' of arbitrary large radius such that $K \cap B' = \emptyset$ and $B \subset B'$. If $p \in (1, \infty)$, a subset K is almost-convex if and only if K is convex (see Lemma 2 in [168]). So, the question is if $\mathscr{G}_{p,k}$ can be convex. For $k \ge 2$ and $p < \infty$, $\mathscr{G}_{p,k}$ is convex if and only if $L^p(\Omega, \mathcal{F}, \mu)$ is finite dimensional and $L^p(\Omega, \mathcal{F}, \mu)) = \mathscr{G}_{p,k}$. Indeed, assume $k \ge 2$ and that $\mathscr{G}_{p,k}$ is convex. Then it is direct to show that $\mathscr{G}_{p,k}$ is a vector space, because it is homogeneous. Then $\mathscr{G}_{p,\ell} = \mathscr{G}_{p,k}$, for all $\ell \ge k$. This is done by induction, so the only interesting case is $\ell = k+1$. Take $g = \sum_{i=1}^{k+1} a_i\mathbb{1}_{A_i}$, which can be seen as the sum of three elements $g_1, g_2, g_3 \in \mathscr{G}_{p,k}$

$$g_1 = \sum_{i=1}^{k-1} a_i\mathbb{1}_{A_i} + 0\mathbb{1}_{A_k\cup A_{k+1}},\ g_2 = a_k\mathbb{1}_{A_k} + 0\mathbb{1}_{\cup_{j\neq k}A_j},\ g_3 = a_{k+1}\mathbb{1}_{A_{k+1}} + 0\mathbb{1}_{\cup_{j\neq k+1}A_j}.$$

Therefore, $\mathscr{G}_{p,k} = \cup_\ell \mathscr{G}_{p,l}$ is dense and closed in $L^p(\Omega, \mathcal{F}, \mu)$, which implies $\mathscr{G}_{p,k} = L^p(\Omega, \mathcal{F}, \mu)$. The conclusion is that the unit ball of $L^p(\Omega, \mathcal{F}, \mu)$ is UA and then $L^p(\Omega, \mathcal{F}, \mu)$ is finite dimensional (see Theorem 9.3.10 in Section 9.3.3).

9.2 The p-variation

In this part we introduce a new notion of variation for functions in $L^p(\Omega, \mathcal{F}, \mu)$. There are several notions of variation or oscillation for functions. Our notion notion could be contrasted with the definition of oscillation given in [26] (p.296), which helps to characterize compact sets in L^1. However, both concepts are not comparable, in general.

Definition 9.2.1. Let $p \in [1, \infty)$. For $f \in L^p(\Omega, \mathcal{F}, \mu)$ and A a measurable set of finite measure, we define $var_p(f, A)$, the *p-variation* of f in A, as

$$var_p(f, A)^p = \begin{cases} \frac{1}{\mu(A)} \int_{A \times A} |f(x) - f(y)|^p \, d\mu(x) d\mu(y) & \text{if } \mu(A) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Given $\mathcal{P} = (A_i)_i$, a finite collection of disjoint measurable sets each one of finite measure, which we also assume it contains at least one set of positive measure, we define the total p-variation of f in $\mathcal{P}$ as

$$var_p(f, \mathcal{P}) = \left(\sum_i var_p(f, A_i)^p \right)^{1/p} = \left(\sum_{i:\, \mu(A_i) > 0} \frac{1}{\mu(A_i)} \int_{A_i \times A_i} |f(x) - f(y)|^p \, d\mu(x) d\mu(y) \right)^{1/p}.$$

For a measurable set A of finite measure, we define the *k-th total p-variation* of f in A as

$$\mathrm{Var}_{p,k}(f, A) = \inf \Big\{ var_p(f, \mathcal{P}) : \mathcal{P} \text{ is a partition of } A, \, |\mathcal{P}| \leq k \Big\}$$

where the infimum is taken over the set of finite measurable partitions of A consisting of at most k measurable sets. Finally, we define the *total k-th variation* of f as

$$\mathrm{Var}_{p,k}(f) = \sup_{\substack{A \in \mathcal{F} \\ \mu(A) < \infty}} \mathrm{Var}_{p,k}(f, A)$$

Note that if μ is finite then $\mathrm{Var}_{p,k}(f, \Omega) \leq \mathrm{Var}_{p,k}(f)$, and it is not clear if both measures of total variation are equivalent, something that we study below (see Proposition 9.2.6).

Remark 9.2.2. Notice that the sets in $\mathcal{P}$ that have measure 0 can be removed by gluing them to an element of $\mathcal{P}$ with positive measure. We redefine a new collection $\widetilde{\mathcal{P}}$, which has fewer elements and $var_p(f, \mathcal{P}) = var_p(f, \widetilde{\mathcal{P}})$. So, in what follows, we can always assume that $\mathcal{P}$ is a collection with sets of positive and finite measure.

We compile some basic properties of $\mathrm{Var}_{p,k}(\bullet, \Omega)$ and $\mathrm{Var}_{p,k}(\bullet)$ in the next result:

Proposition 9.2.3. *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $p \in [1, \infty)$. Then $(Var_{p,k})_{k \geq 1}$ is a decreasing family of continuous semi-norms on $L^p(\Omega, \mathcal{F}, \mu)$ such that $Var_{p,k}(\bullet) \leq 2\| \bullet \|_p$ for all $k \geq 1$. The same properties hold for $(Var_{p,k}(\bullet, \Omega))_{k \geq 1}$, in the case μ is a finite measure.*

Proof. The fact that $\mathrm{Var}_{p,k}$ is a semi-norm is easy and is left to the reader. The monotony of $(\mathrm{Var}_{p,k})_{k \geq 1}$ follows directly from the definition. Let $f \in L^p(\Omega, \mathcal{F}, \mu)$ and A be a measurable set of finite and positive measure. First note that $var_p(f, A) \leq 2\|f \mathbb{1}_A\|_p$. In fact, using that $(a + b)^p \leq 2^{p-1}(a^p + b^p)$ holds for all nonnegative numbers a, b and Fubini's theorem, we have that

$$\begin{aligned} var_p(f, A)^p &= \frac{1}{\mu(A)} \int_{A \times A} |f(x) - f(y)|^p \, d\mu(x) d\mu(y) \\ &\leq \frac{2^{p-1}}{\mu(A)} \int_{A \times A} |f(x)|^p + |f(y)|^p d\mu(x) d\mu(y) \qquad (9.11) \\ &= 2^p \|f \mathbb{1}_A\|_p^p. \end{aligned}$$

It follows that if $\mathcal{P}$ is a finite measurable partition of A then $\mathrm{Var}_p(f,\mathcal{P}) \le 2\|f\mathbb{1}_A\|_p$ and therefore, we deduce that $\mathrm{Var}_{p,k}(f) \le 2\|f\|_p$. In particular, $\mathrm{Var}_{p,k}$ is continuous. In case the measure is finite we have

$$\mathrm{Var}_{p,k}(f,\Omega) \le \mathrm{Var}_{p,k}(f) \le 2\|f\|_p.$$

□

Remark 9.2.4. Assume that μ is a finite measure. We notice that for a fixed function $f \in L^p(\Omega,\mathcal{F},\mu)$, we have $\lim_{k\to\infty} \mathrm{Var}_{p,k}(f,\Omega) = 0$. Indeed, let $k \in \mathbb{N}$ and define the sets

$$A_i = \left\{x : \frac{i}{k} \le f(x) < \frac{i+1}{k}\right\}, \text{ for } i \in \{-k^2, ..., k^2-2\},$$

$A_{k^2-1} = \left\{x : k - \frac{1}{k} \le f(x) \le k\right\}$ and $A_{k^2} = \{x : |f(x)| > k\}$. Then, we have, for all $i \in \{-k^2, ..., k^2-1\}$

$$var_p(f, A_i) \le \frac{\mu(A_i)^{\frac{1}{p}}}{k}$$

and for $i = k^2$

$$var_p(f, A_{k^2}) \le 2\|f\mathbb{1}_{A_{k^2}}\|_p.$$

Thus,

$$\mathrm{Var}_{p,k}(f,\Omega)^p \le \frac{1}{k^p}\mu(\Omega) + 2^p \int_{|f|>k} |f(x)|^p \, d\mu(x),$$

and then $\lim_k \mathrm{Var}_{p,k}(f,\Omega) = 0$.

We also notice that the same property holds for $(\mathrm{Var}_{p,k}(f))_k$, in general measure spaces, but its proof is more involved and we postponed to Corollary 9.2.8.

The following lemma proves that the variation of a function can always be computed on a σ-finite set if the measure has no atoms of infinite mass.

Lemma 9.2.5. *Assume $(\Omega,\mathcal{F},\mu)$ is a measurable space such that μ has no atoms of infinite mass and $p \in [1,\infty)$. Let $f \in L^p(\Omega,\mathcal{F},\mu)$ and fix $k \ge 1$. Then there exists an increasing sequence of finite measure sets $(\Omega_n^*)_n \subset \mathcal{F}$ such that*

$$Var_{p,k}(f) = \lim_n Var_{p,k,n}(f) = Var_{p,k,*}(f),$$

where

- *$Var_{p,k,*}(f)$ is the total variation of $f|_{\Omega^*}$ computed in $(\Omega^*, \mathcal{F}|_{\Omega^*}, \mu|_{\Omega^*})$ with $\Omega^* = \bigcup_n \Omega_n^*$;*
- *$Var_{p,k,n}(f)$ is the total variation of $f|_{\Omega_n^*}$ computed in $(\Omega_n^*, \mathcal{F}|_{\Omega_n^*}, \mu|_{\Omega_n^*})$.*

Proof. We can obviously suppose that μ is infinite. Define $F = \{x : f(x) = 0\}$. In case F has finite measure, we define $\widetilde{D} = F$. If F has infinite measure, we consider a subset $\widetilde{D} \subset F$ which is σ-finite and of infinite measure. Note that such a set exists. Indeed, take

$$a = \sup_{D\in\mathcal{F}, D\subset F, \mu(D)<\infty} \mu(D).$$

Let us prove that $a = \infty$. Consider a sequence $(D_l)_l$ of subsets of F, each one of finite measure such that $\lim_l \mu(D_l) = a$. It is clear that $\widetilde{D}_l = \cup_{i\le l} D_i$ is an increasing sequence of sets of finite measure, included in F which satisfies $\mu(D_l) \le \mu(\widetilde{D}_l)$, proving that $\mu(\widetilde{D}_l) \uparrow a$ and $\widetilde{D} = \cup_l \widetilde{D}_l$ satisfies $\mu(\widetilde{D}) = a$. If a is finite then, $F \setminus \widetilde{D}$ has infinite measure. By hypothesis this set contains a set H of finite and positive measure. Then $\mu(\widetilde{D} \cup H) = \mu(\widetilde{D}) + \mu(H) > a$, which is a contradiction.

Now consider a sequence of sets of finite measure $(A_n)_n$ such that

$$\mathrm{Var}_{p,k}(f, A_n) \geq \mathrm{Var}_{p,k}(f) - \frac{1}{n}.$$

For every $m \geq 1$ the set $C_m = \left\{x : |f(x)| > \frac{1}{m}\right\}$ has finite measure. The set $\Omega^* = \cup_n A_n \cup \cup_m C_m \cup \widetilde{D}$ is σ-finite and it has infinite measure, because $\mu(\bigcup_m C_m \cup F) = \mu(\Omega) = \infty$. We consider

$$\Omega_n^* = \begin{cases} \cup_{i \leq n} A_i \cup C_i \cup \widetilde{D} & \text{if } \mu(\widetilde{D}) < \infty \\ \cup_{i \leq n} A_i \cup C_i \cup \widetilde{D}_i & \text{if } \mu(\widetilde{D}) = \infty \end{cases},$$

which is an increasing sequence of sets of finite and positive measure, such that $\Omega_n^* \uparrow \Omega^*$. Define $\mathrm{Var}_{p,k,n}(f)$ the total variation of $f|_{\Omega_n^*}$ computed in $(\Omega_n^*, \mathcal{F}|_{\Omega_n^*}, \mu|_{\Omega_n^*})$, that is

$$\mathrm{Var}_{p,k,n}(f) = \sup_{A \in \mathcal{F}, A \subset \Omega_n^*} \mathrm{Var}_{p,k}(f|_{\Omega_n^*}, A) = \sup_{A \in \mathcal{F}, A \subset \Omega_n^*} \mathrm{Var}_{p,k}(f, A) \leq \mathrm{Var}_{p,k}(f).$$

Similarly, we define $\mathrm{Var}_{p,k,*}(f)$, which is the total variation of $f|_{\Omega^*}$ computed in $(\Omega^*, \mathcal{F}|_{\Omega^*}, \mu|_{\Omega^*})$. It is clear that for every n, by construction,

$$\mathrm{Var}_{p,k}(f) - \frac{1}{n} \leq \mathrm{Var}_{p,k}(f, A_n) \leq \mathrm{Var}_{p,k,n}(f) \leq \mathrm{Var}_{p,k,*}(f) \leq \mathrm{Var}_{p,k}(f),$$

and also that $(\mathrm{Var}_{p,k,n}(f))_n$ is increasing, showing that

$$\mathrm{Var}_{p,k,n}(f) \uparrow \mathrm{Var}_{p,k}(f)$$

and $\mathrm{Var}_{p,k,*}(f) = \mathrm{Var}_{p,k}(f)$. □

The next proposition shows that the variation and $\mathscr{D}_{p,k}$ have the same behaviour. This will be a fundamental tool to caracterize the uniform approximability of sets.

Proposition 9.2.6. *Assume $(\Omega, \mathcal{F}, \mu)$ is a measurable space and $p \in [1, \infty)$. For any $k \geq 1$ and any $f \in L^p(\Omega, \mathcal{F}, \mu)$, we have*

(i)

$$\mathscr{D}_{p,k+1}(f) \leq \mathit{Var}_{p,k}(f) \leq 2\mathscr{D}_{p,k}(f).$$

(ii) If the measure μ is finite, it holds

$$\mathscr{D}_{p,k}(f) \leq \mathit{Var}_{p,k}(f, \Omega) \leq \mathit{Var}_{p,k}(f) \leq 2\mathscr{D}_{p,k}(f) \leq 2\,\mathit{Var}_{p,k}(f, \Omega).$$

(iii) If μ has no atoms of infinite mass, we have that

$$\mathscr{D}_{p,k}(f) \leq \mathit{Var}_{p,k}(f) \leq 2\mathscr{D}_{p,k}(f).$$

Proof. (i) For the upper bound, consider $g \in \mathscr{G}_k \cap L^p(\Omega, \mathcal{F}, \mu)$ a function such that

$$\mathscr{D}_{p,k}(f)^p = \|f - g\|_p^p.$$

Assume that $g = \sum_{i=1}^k c_i \mathbb{1}_{A_i}$, where $\{A_i\}_{1 \leq i \leq k}$ is a finite partition of Ω. Clearly if $\mu(A_i) = \infty$, then $c_i = 0$. For A a set of finite measure define a partition of A by $\mathcal{P} = \{A \cap A_i\}_{1 \leq i \leq k}$.

Using that $(a+b)^p \le 2^{p-1}(a^p+b^p)$ for all positive numbers a and b, we get for all i such that $\mu(A\cap A_i)>0$

$$var_p(f,A\cap A_i)^p = \frac{1}{\mu(A\cap A_i)}\int_{(A\cap A_i)\times(A\cap A_i)}|f(x)-f(y)|^p d\mu(x)d\mu(y) \le 2^p\int_{A\cap A_i}|f(z)-c_i|^p d\mu(z).$$

Then,

$$var_p(f,\mathcal{P})^p \le 2^p\sum_i\int_{A\cap A_i}|f(z)-c_i|^p d\mu(z) \le 2^p\|f-g\|_p^p.$$

Therefore, we get

$$\mathrm{Var}_{p,k}(f,A) \le 2\|f-g\|_p = 2\mathscr{D}_{p,k}(f).$$

For the lower bound let $\varepsilon>0$ and take a set A of finite measure such that $\|f\mathbb{1}_{A^c}\|_p^p<\varepsilon$. By definition of $\mathrm{Var}_{p,k}(f,A)$, there exists a finite partition $\mathcal{P}=\{A_i\}_{1\le i\le n}$ of A, with $n\le k$, such that (we assume all the sets in $\mathcal{P}$ has positive measure)

$$\begin{aligned}var_p(f,\mathcal{P})^p &= \sum_i\frac{1}{\mu(A_i)}\int_{A_i\times A_i}|f(x)-f(y)|^p d\mu(x)d\mu(y)\\ &\le (\mathrm{Var}_{p,k}(f,A))^p+\varepsilon \le (\mathrm{Var}_{p,k}(f))^p+\varepsilon.\end{aligned}$$

For every $i\le n$ by the definition of $\mathcal{M}_p(f,A_i)$, we have

$$\mathbb{1}_{A_i}(y)\int_{A_i}|f(x)-\mathcal{M}_p(f,A_i)|^p\,d\mu(x) \le \mathbb{1}_{A_i}(y)\int_{A_i}|f(x)-f(y)|^p\,d\mu(x),$$

and therefore, integrating over y we get

$$\int_{A_i}|f(x)-\mathcal{M}_p(f,A_i)|^p\,d\mu(x) \le \frac{1}{\mu(A_i)}\int_{A_i\times A_i}|f(x)-f(y)|^p\,d\mu(x),$$

and then

$$\sum_i\int_{A_i}|f(x)-\mathcal{M}_p(f,A_i)|^p\,d\mu(x) \le var_p(f,\mathcal{P})^p.$$

Finally, define $g=\sum_i\mathcal{M}_p(h,A_i)\mathbb{1}_{A_i}+0\mathbb{1}_{A^c}\in\mathscr{G}_{p,k+1}\cap L^p(\Omega,\mathcal{F},\mu)$ to obtain that

$$\|f\mathbb{1}_A-g\|_p^p \le var_p(f,\mathcal{P})^p \le (\mathrm{Var}_{p,k}(f))^p+\varepsilon$$

To finish this part, notice that

$$\begin{aligned}\|f-g\|_p^p &= \|f\mathbb{1}_A-g\mathbb{1}_A\|_p^p+\|f\mathbb{1}_{A^c}-g\mathbb{1}_{A^c}\|_p^p\\ &= \|f\mathbb{1}_A-g\|_p^p+\|f\mathbb{1}_{A^c}\|_p^p \le (\mathrm{Var}_{p,k}(f))^p+2\varepsilon\end{aligned}$$

which implies that $\mathscr{D}_{p,k+1}(f)\le\mathrm{Var}_{p,k}(f)$.

(ii) The proof is similar to (i). The upper bound follows immediately from the lower bound to be proved. For the lower estimate, in the above proof we can take $A=\Omega$.

(iii) Let $(\Omega_n^*)_n\subset\mathcal{F}$ and $\Omega^*=\bigcup_n\Omega_n^*$ given by Lemma 9.2.5, such that

$$\mathrm{Var}_{p,k}(f)=\lim_n\mathrm{Var}_{p,k,n}(f)=\mathrm{Var}_{p,k,*}(f).$$

We first assume that f is bounded by some constant $C>0$. Then, using the result we have shown for the finite measure case, we have on Ω_n^*

$$\inf\{\|f|_{\Omega_n^*}-g\|_p:\ g\in\mathscr{G}_{p,k}(\Omega_n^*)\}\le\mathrm{Var}_{p,k,n}(f).$$

By Theorem 9.1.5, the left hand side is attained at some function g_n defined in Ω_n^*, which is also bounded by C. We can assume this minimizer has the following form

$$g_n = \sum_{i=1}^{q(n)} b_{i,n} \mathbb{1}_{B_{i,n}},$$

where $\{B_{i,n}\}_{1 \le i \le q(n)}$ is a partition of sets of positive measure of Ω_n^* and

$$-C \le b_{1,n} < ... < b_{q(n),n} \le C,$$

$$r_{1,n} = -C - 1, \; r_{q(n)+1,n} = C + 1, \; r_{i,n} = \frac{b_{i-1,n} + b_{i,n}}{2} \text{ for } i \in \{2, ..., q(n)\},$$

$$B_{i,n} = f^{-1}([r_{i,n}, r_{i+1,n})) \cap \Omega_n^* \text{ for } i \in \{1, ..., q(n)-1\} \text{ and } B_{q(n),n} = f^{-1}([r_{q(n),n}, r_{q(n)+1,n}]) \cap \Omega_n^*,$$

$$b_{i,n} = \mathcal{M}_p(f, B_{i,n}) \text{ for } i \in \{1, ..., q(n)\}$$

and $q(n) \le k$. As before, we can assume by passing to a subsequence that $q(n) = q$ is constant and the vector $v_n = (r_{1,n}, b_{1,n}, r_{2,n}, ..., r_{q,n}, b_{q,n}, r_{q+1,n})$ converges to a vector in $[-C-1, C+1]^{3q}$, which we denote by $v = (r_1, b_1, r_2, ..., r_q, b_q, r_{q+1})$. Also we denote by $L = (b_1, ..., b_q)$ and $w_1 < ... < w_m$ the different values in L, where $m \le q$.

Let us show that $w_{t^*} = 0$ for some t^*. For that, remark that $\Omega_n^* = \cup_{i=1}^q B_{i,n}$ and therefore, there exists an index $i(n)$, such that

$$\mu(B_{i(n),n}) \ge \frac{1}{q} \mu(\Omega_n^*),$$

showing that $\lim_n \mu(B_{i(n),n}) = \infty$. We can assume that $i(n) = i$ is constant, by passing to a subsequence if necessary. Using the optimality of $b_{i,n} = \mathcal{M}_p(f, B_{i,n})$, we get

$$|b_{i,n}|^p \mu(B_{i,n}) \le 2^{p-1} \left(\int_{B_{i,n}} |f(x) - \mathcal{M}_p(f, B_{i,n})|^p \, d\mu(x) + \int_{B_{i,n}} |f(x)|^p \, d\mu(x) \right) \le 2^p \|f\|_p^p.$$

This shows that $b_{i,n} \to b_i = 0$, and the claim holds by taking t^* such that $w_{t^*} = b_i = 0$.

Consider $I_t = \{j \in \{1, ...q\} : \; b_j = w_t\}$ for $t \in \{1, ..., m\}$. Notice that each I_t is a nonempty interval of $I = \{1, ..., q\}$. Assume that $I_t = \{l(t), ..., u(t)\}$, then we have $r_{l(t)+1,n} \to w_t, ..., r_{u(t),n} \to w_t, b_{l(t),n} \to w_t, ..., b_{u(t),n} \to w_t$ and

$$\lim_n r_{l(t),n} = r_{l(t)} = \frac{b_{l(t)-1} + w_t}{2} < w_t < \frac{b_{u(t)+1} + w_t}{2} = r_{u(t)+1} = \lim_n r_{u(t)+1,n},$$

with the obvious modifications in the case $l(t) = 1$ or $u(t) = q$. By construction we have for all $i < l(t^*)$ it holds $r_{i+1,n} \le r_{l(t^*),n} < \frac{r_{l(t^*)}}{2} = r_-$, for all large n, because $r_{l(t^*)} < 0$. Similarly, for all $i \ge u(t^*)$ we have $r_{i+1,n} \ge r_{u(t^*)+1,n} > \frac{r_{u(t^*)+1}}{2} = r_+ > 0$, for all large n. This implies that, for large n

$$\bigcup_{i < l(t^*)} B_{i,n} \subset f^{-1}([-C-1, r_-]),$$

which is a set of finite measure: $\mu(f^{-1}([-C-1, r_-])) < \infty$. Consider a modification of g_n given by

$$\ell_n = \sum_{i \notin [l(t^*), u(t^*)]} b_i \mathbb{1}_{B_{i,n}} + \sum_{l(t^*) \le i \le u(t^*)} b_{i,n} \mathbb{1}_{B_{i,n}}.$$

We have $\|g_n - \ell_n\|_p$ converges to zero. Indeed, this follows from the inequality

$$\begin{aligned} \|g_n - \ell_n\|_p^p &= \sum_{i \notin \{l(t^*), ..., u(t^*)\}} |b_i - b_{i,n}|^p \mu(B_{i,n}) \\ &\le \max_j |b_j - b_{j,n}|^p \mu \left(f^{-1}([-C-1, r_-]) \cup f^{-1}([r_+, C+1]) \right) \to 0. \end{aligned}$$

Using the triangular inequality and the optimality of g_n, we get

$$\|f|_{\Omega_n^*} - g_n\|_p \leq \|f|_{\Omega_n^*} - \ell_n\|_p \leq \|f|_{\Omega_n^*} - g_n\|_p + \|g_n - \ell_n\|_p,$$

and we plan to use Fatou's Lemma. Before doing that, we will fix a subsequence with certain monotonic properties. Since $r_{1,n}, r_{q+1,n}$ are constant, there is no restriction here. For $i \in \{2,...,q\}$ we choose a subsequence in the following order. If $(r_{2,n})_n$ has an strictly decreasing subsequence, we consider this as $n^{(2)}$ and define $T(2) = \text{sd}$ (for strictly decreasing) otherwise, we consider $n^{(2)}$ so that $(r_{2,n})_n$ is increasing along this subsequence and $T(2) = \text{in}$ (for increasing). Now, we construct $n^{(3)}$. If $(r_{3,n^{(2)}})_{n^{(2)}}$ has an strictly decreasing subsequence we take this as $n^{(3)}$ and $T(3) = \text{sd}$, otherwise we take $n^{(3)}$ so that $(r_{3,n^{(3)}})_{n^{(3)}}$ is increasing, and $T(3) = \text{in}$. We continue in this way until we define $n^{(q)}$. We put $T(1) = \text{in}$ and $T(q+1) = \text{sd}$. We call $n' = n^{(q)}$. In this way we have the a.e. convergence

$$\mathbb{1}_{f^{-1}([r_{i,n'}, r_{i+1,n'}))\cap\Omega^*_{n'}} \to \begin{cases} \mathbb{1}_{f^{-1}([r_i, r_{i+1}])\cap\Omega^*} & \text{if } T(i) = \text{in}, T(i+1) = \text{sd} \\ \mathbb{1}_{f^{-1}([r_i, r_{i+1}))\cap\Omega^*} & \text{if } T(i) = \text{in}, T(i+1) = \text{in} \\ \mathbb{1}_{f^{-1}((r_i, r_{i+1}])\cap\Omega^*} & \text{if } T(i) = \text{sd}, T(i+1) = \text{sd} \\ \mathbb{1}_{f^{-1}((r_i, r_{i+1}))\cap\Omega^*} & \text{if } T(i) = \text{sd}, T(i+1) = \text{in} \end{cases}$$

We call $\mathscr{J}_i$ the interval, with extremes r_i, r_{i+1}, according to the above classification. An important remark is that $\cup_i \mathscr{J}_i = [-C-1, C+1]$.

Using the Dominated Convergence Theorem we conclude that

$$\sum_{i \notin \{l(t^*),...,u(t^*)\}} \int_{B_{i,n'}} |f(x) - b_i|^p \; d\mu(x) \to \sum_{i \notin \{l(t^*),...,u(t^*)\}} \int_{f^{-1}(\mathscr{J}_i)} |f(x) - b_i|^p \; d\mu(x).$$

On the other hand, using Fatou's Lemma we conclude

$$\begin{aligned} \liminf_{n'} \sum_{i \in \{l(t^*),...,u(t^*)\}} \int_{B_{i,n'}} |f(x) - b_{i,n'}|^p \; d\mu(x) &\geq \int \liminf_{n'} |f(x) - g_n(x)|^p \mathbb{1}_{B_{n'}} \; d\mu(x) \\ &\geq \int |f(x)|^p \mathbb{1}_{f^{-1}(\tilde{\mathscr{J}})\cap\Omega^*} \; d\mu(x). \end{aligned}$$

where $B_{n'} = \bigcup_{i \in \{l(t^*),...,u(t^*)\}} B_{i,n'}$ and $\widetilde{\mathscr{J}} = \bigcup_{i \in \{l(t^*),...,u(t^*)\}} \mathscr{J}_i$. Here, we have used that for all $x \in B_{n'}$ we have

$$|g_{n'}(x)| \leq \max_{i \in \{l(t^*),...,u(t^*)\}} |b_{i,n'}| \to 0.$$

Hence,

$$\begin{aligned} |f(x)|\mathbb{1}_{B_{i,n'}} &\leq |f(x) - g_{n'}(x)|\mathbb{1}_{B_{i,n'}} + \max\nolimits_{i \in \{l(t^*),...,u(t^*)\}} |b_{i,n'}|\mathbb{1}_{B_{i,n'}} \\ |f(x) - g_{n'}(x)|\mathbb{1}_{B_{i,n'}} &\leq |f(x)|\mathbb{1}_{B_{i,n'}} + \max\nolimits_{i \in \{l(t^*),...,u(t^*)\}} |b_{i,n'}|\mathbb{1}_{B_{i,n'}}, \end{aligned}$$

showing that

$$\liminf_{n'} |f(x) - g_{n'}(x)|\mathbb{1}_{B_{i,n'}} = |f(x)| \liminf_{n'} \mathbb{1}_{B_{i,n'}}.$$

Putting all together, we conclude that

$$\mathrm{Var}_{p,k}(f) \geq \liminf_{n'} \|f|_{\Omega^*_{n'}} - g_{n'}\|_p \geq \|f|_{\Omega^*} - \ell\|_p,$$

where the function $\ell \in \mathscr{G}_{p,q}(\Omega^*)$ is defined on Ω^* as

$$\ell = \sum_{i \notin \{l(t^*),...,u(t^*)\}} b_i \mathbb{1}_{f^{-1}(\mathscr{J}_i)\cap\Omega^*} + 0\mathbb{1}_{f^{-1}(\widetilde{\mathscr{J}})\cap\Omega^*}.$$

Notice that for $i \notin \{l(t^*), ..., u(t^*)\}$, we have $\mu(f^{-1}(\mathcal{J}_i) \cap \Omega^*) \leq \mu\left(f^{-1}([-C-1, r_-] \cup f^{-1}([r_+, C+1])\right) < \infty$ and so $\mu(f^{-1}(\widetilde{\mathcal{J}}) \cap \Omega^*) = \infty$.

Since $b_i = 0$ for some i and $f = 0$ outside Ω^*, we can extend ℓ by 0 outside Ω^* and still this extension $\bar{\ell}$ belongs to $\mathscr{G}_{p,q} \subset \mathscr{G}_{p,k}$. So, we get that

$$\mathscr{D}_{p,k}(f) \leq \|f - \bar{\ell}\|_p \leq \mathrm{Var}_{p,k}(f).$$

and the result is shown in the case f is bounded.

Now, for the general case, consider $\varepsilon > 0$ and a large $C > 0$, such that $\|f\mathbb{1}_{|f|>C}\|_p \leq \varepsilon$. From the domination $\mathrm{Var}_{p,k}(\bullet) \leq 2\|f\|_p$, and the seminorm property of $\mathrm{Var}_{p,k}$ we conclude

$$\mathrm{Var}_{p,k}(f\mathbb{1}_{|f|\leq C}) \leq \mathrm{Var}_{p,k}(f) + \mathrm{Var}_{p,k}(f - f\mathbb{1}_{|f|\leq C}) \leq \mathrm{Var}_{p,k}(f) + 2\varepsilon.$$

Using what we have shown, we get there exists and $\ell \in \mathscr{G}_{p,k}$ such that

$$\|f\mathbb{1}_{|f|\leq C} - \ell\|_p \leq \mathrm{Var}_{p,k}(f\mathbb{1}_{|f|\leq C}) \leq \mathrm{Var}_{p,k}(f) + 2\varepsilon.$$

On the other hand, we have

$$\|f - \ell\|_p \leq \|f\mathbb{1}_{|f|\leq C} - \ell\|_p + \|f\mathbb{1}_{|f|\leq C} - f\|_p \leq \|f\mathbb{1}_{|f|\leq C} - \ell\|_p + \varepsilon,$$

which shows that

$$\mathscr{D}_{p,k}(f) \leq \mathrm{Var}_{p,k}(f) + 3\varepsilon,$$

and the result is shown. ☐

Remark 9.2.7. Examples that satisfies (iii) in the previous proposition are the σ-finite measures. In particular, it can be applied to $\ell^p = L^p(\mathbb{N}, \mathcal{P}(\mathbb{N}), \delta)$, where δ is the counting measure. But, there are non σ-finite measures that satisfies that hypothesis as well, the counting measures on any uncountable space.

Corollary 9.2.8. *Assume $(\Omega, \mathcal{F}, \mu)$ is a measurable space and $p \in [1, \infty)$. For all $f \in L^p(\Omega, \mathcal{F}, \mu)$ it holds that $\lim\limits_{k\to\infty} Var_{p,k}(f) = 0$.*

Proof. This follows directly from the previous proposition since $\lim_k \mathscr{D}_{p,k}(f) = 0$ by density of the simple functions. ☐

A question of some interest is when $\mathrm{Var}_{p,k}(f) = 0$, for a function $f \in L^p(\Omega, \mathcal{F}, \mu)$. Clearly, if $f \in \mathscr{G}_{p,k}$ then $\mathrm{Var}_{p,k}(f) = 0$. The next result answers the converse.

Proposition 9.2.9. *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $p \in [1, \infty)$. Let $k \geq 1$. We have*

(a) if μ is a general measure, then $\mathscr{G}_{p,k} \subset Var_{p,k}^{-1}(\{0\}) \subset \mathscr{G}_{p,k+1}$.

(b) if μ has no atoms of infinite mass, then $Var_{p,k}^{-1}(\{0\}) = \mathscr{G}_{p,k}$.

Proof. (a). Let $f \in L^p(\Omega, \mathcal{F}, \mu)$ satisfying $\mathrm{Var}_{p,k}(f) = 0$. From (i) of Proposition 9.2.6, we have

$$\mathscr{D}_{p,k+1}(f) \leq \mathrm{Var}_{p,k}(f) = 0,$$

which implies that $f = g$ a.e. for some $g \in \mathscr{G}_{p,k+1}$ (see Theorems 9.1.5 and 9.1.12). The other inclusion is obvious.

(b) The proof is similar to (i) and uses (iii) in Proposition 9.2.6. ☐

Remark 9.2.10. Notice that if μ has an atom of infinite mass it may happens that $\mathrm{Var}_{p,k}(f) = 0$, but $f \in \mathscr{G}_{p,k+1} \setminus \mathscr{G}_{p,k}$. Indeed, assume $\Omega = \{1, ..., k+1\}$, where the mass of each atom in $\{1, ..., k\}$ is one and the mass at atom $\{k+1\}$ is infinite. Every function in $L^p(\Omega, \mathcal{F}, \mu)$ for $p \in [1, \infty)$ satisfies $f(k+1) = 0$. The function f given by $f(i) = i$ for $i \in \{1, ..., k\}$ and $f(k+1) = 0$ belongs to $\mathscr{G}_{p,k+1} \setminus \mathscr{G}_{p,k}$. Nevertheless, $\mathrm{Var}_{p,k}(f) = 0$, which is exactly the case (a) in Proposition 9.2.9. Also, this example explains why the lower bound in Proposition 9.2.6 (i) is computed over $\mathscr{G}_{p,k+1}$ and not over $\mathscr{G}_{p,k}$, in general.

9.3 Uniform approximability

In this section, we investigate some properties of uniformly approximable sets, whose definition is the following:

Definition 9.3.1. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $p \in [1, +\infty]$. Let $\mathscr{A} \subset L^p(\Omega, \mathcal{F}, \mu)$. For $\varepsilon > 0$, we define

$$N_{p,\varepsilon}(\mathscr{A}) = \inf\{k \geq 1 \ : \ \forall f \in \mathscr{A}, \ \exists h \in \mathscr{G}_{p,k} \ \ \|f - h\|_p \leq \varepsilon\}.$$

As usual if the set where the infimum is taken is empty we set $N_{p,\varepsilon}(\mathscr{A}) = \infty$. We say that $\mathscr{A}$ is *uniformly approximable* (in short UA) in $L^p(\Omega, \mathcal{F}, \mu)$ if $N_{p,\varepsilon}(\mathscr{A}) < \infty$ for any $\varepsilon > 0$.

Concretely a set $\mathscr{A}$ is UA in $L^p(\Omega, \mathcal{F}, \mu)$ if for any $\varepsilon > 0$ there exists $k \geq 1$ such that any function in $\mathscr{A}$ can be ε-approximated in $L^p(\Omega, \mathcal{F}, \mu)$ by simple functions taking less than k different values.

9.3.1 Uniform integrability

In this subsection, we prove that the class of uniform approximable sets is strictly larger than the class of uniform integrable sets. Assume that $(\Omega, \mathcal{F}, \mu)$ is a measure space and let $p \in [1, \infty)$. Remember that a subset $\mathscr{A} \subset L^p(\Omega, \mathcal{F}, \mu)$ is *uniformly integrable* (in short, UI) if

$$\inf_{g \in L^p_+(\Omega, \mathcal{F}, \mu)} \sup_{f \in \mathscr{A}} \int_{|f| > g} |f(x)|^p \ d\mu(x) = 0,$$

where $L^p_+(\Omega, \mathcal{F}, \mu)$ is the set of nonnegative functions in $L^p(\Omega, \mathcal{F}, \mu)$. Note that if μ is a finite measure, then this definition coincides with the usual one, that is $\mathscr{A}$ is UI in $L^p(\Omega, \mathcal{F}, \mu)$ if and only if (see [101], page 254)

$$\lim_{a \to \infty} \sup_{f \in \mathscr{A}} \int\limits_{|f(x)| \geq a} |f^p(x)| \, d\mu(x) = 0.$$

Proposition 9.3.2. *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $p \in [1, +\infty)$. Assume $\mathscr{A} \subset L^p(\Omega, \mathcal{F}, \mu)$ is UI. Then, $\mathscr{A}$ is UA.*

Proof. Consider $\varepsilon > 0$, and take $g \in L^p_+(\Omega, \mathcal{F}, \mu)$, such that

$$\sup_{f \in \mathscr{A}} \int_{|f| > g} |f(x)|^p \ d\mu(x) \leq \frac{\varepsilon^p}{3}.$$

Fix $f \in \mathscr{A}$. Consider $n \in \mathbb{N}$, large enough such that $\int_{g > n} g^p(x) \ d\mu(x) + \int_{g < \frac{1}{n}} g^p(x) \ d\mu(x) \leq \frac{\varepsilon^p}{3}$. The set $B_n = \left\{x : \ \frac{1}{n} \leq g(x) \leq n\right\} \cap \{x : |f(x)| \leq g(x)\} \subset C_n = \left\{x : \ \frac{1}{n} \leq g(x) \leq n\right\}$ has finite measure. Notice that over B_n we have $|f| \leq n$. Take now $k \geq 2$ such that $\left(\frac{n}{k}\right)^p \mu(C_n) \leq \frac{\varepsilon^p}{3}$ and define

$$A_i = B_n \cap \left\{x : \frac{ni}{k} \leq f(x) < \frac{n(i+1)}{k}\right\}$$

for $i \in \{-k, ..., k-2\}$, $A_{k-1} = B_n \cap \left\{x : \frac{n(k-1)}{k} \leq f(x) \leq n\right\}$ and

$$h = \sum_{i=-k}^{k-1} \frac{in}{k} \mathbb{1}_{A_i} + 0 \mathbb{1}_{B_n^c} \in \mathscr{G}_{p,2k+1}.$$

Then, we have

$$\begin{aligned}\int_{B_n} |f(x)-h(x)|^p \, d\mu(x) &= \sum_{i=-k}^{k-1} \int_{A_i} |f(x)-h(x)|^p \, d\mu(x) \leq \left(\tfrac{n}{k}\right)^p \sum_{i=-k}^{k-1} \mu(A_i) \\ &\leq \left(\tfrac{n}{k}\right)^p \mu(B_n) \leq \left(\tfrac{n}{k}\right)^p \mu(C_n) \leq \tfrac{\varepsilon^p}{3}.\end{aligned}$$

On the other hand, $B_n^c = (C_n^c \cap \{x : |f(x)| \leq g(x)\}) \cup \{x : |f(x)| > g(x)\}$ and so

$$\begin{aligned}\int_{B_n^c} |f(x)-h(x)|^p \, d\mu(x) &= \int_{B_n^c} |f(x)|^p \, d\mu(x) = \int_{C_n^c \cap |f| \leq g} |f(x)|^p \, d\mu(x) + \int_{|f|>g} |f(x)|^p \, d\mu(x) \\ &\leq \int_{C_n^c} g^p(x) \, d\mu(x) + \int_{|f|>g} |f(x)|^p \, d\mu(x) \leq \tfrac{2\varepsilon^p}{3}\end{aligned}$$

Finally, we have $\|f-h\|_p \leq \varepsilon$, and the result is shown. □

Remark 9.3.3. Note that the converse of Proposition 9.3.2 is not true in general. In fact $\mathscr{G}_{p,2}$ is UA, but this set is not UI in $L^p(\Omega, \mathcal{F}, \mu)$ in general. Indeed, assume the space has finite measure and there exists a sequence $(B_n)_n$ of measurable sets with positive measure such that $\mu(B_n) \to 0$. Then $f_n = \mu(B_n)^{-1/p} \mathbb{1}_{B_n}$ belongs to $\mathscr{G}_{p,2}$, each one has norm 1 and the subfamily $(f_n)_n$ is not UI, since for all $a \geq 0$, we have

$$\sup_n \int\limits_{f_n > a} f_n^p \, d\mu = 1.$$

The conclusion is that UA is weaker than UI.

Remark 9.3.4. If $(\Omega, \mathcal{F}, \mu)$ is a finite measure space, the following examples are UI in $L^p(\Omega, \mathcal{F}, \mu)$, for $p \in [1, \infty)$,

- $\mathscr{A}$ is bounded in $L^q(\Omega, \mathcal{F}, \mu)$ for some $q > p$;
- $\mathscr{A}$ is bounded by a fixed function $g \in L^p(\Omega, \mathcal{F}, \mu)$.

The following result can be prove using the fact that totally boundedness implies UI in $L^p(\Omega, \mathcal{F}, \mu)$ for $p \in [1, \infty)$. However, since the case $p = \infty$ needs a proof, we give a more direct argument:

Proposition 9.3.5. *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $p \in [1, +\infty]$. If $\mathscr{A}$ is totally bounded in $L^p(\Omega, \mathcal{F}, \mu)$ then $\mathscr{A}$ is UA.*

Proof. Let $\varepsilon > 0$. There exist finitely many functions $f_1, ..., f_n$ such that $\mathscr{A} \subset \bigcup_{j=1}^n B(f_j, \varepsilon)$. By density of the simple functions, there exist $k \in \mathbb{N}$ and $g_j \in \mathscr{G}_{p,k}$ such that $\|f_j - g_j\|_p \leq \varepsilon$ for all $j \in \{1, ..., n\}$. Now if $f \in \mathscr{A}$ then, there exists $j_0 = j_0(f) \in \{1, ..., n\}$ such that $\|f - f_{j_0}\|_p \leq \varepsilon$. It follows that

$$\|f - g_{j_0}\|_p \leq \|f - f_{j_0}\|_p + \|f_{j_0} - g_{j_0}\|_p \leq 2\varepsilon$$

and the proof is complete. □

9.3.2 Characterization of the uniform approximability

If M is a metric space, we recall that the covering numbers of M are defined for every $\varepsilon > 0$ by

$$\mathcal{N}(M, \varepsilon) = \inf \{N \geq 1 \ : \ M \text{ can be covered by } N \text{ closed balls of radius } \varepsilon\}.$$

For more informations about covering numbers and its applications to Machine Learning, we refer the reader to [169] and [8].

If $(\Omega, \mathcal{F}, \mu)$ is a measure space, we define the covering numbers of a measurable function f by

$$\mathcal{N}(f,\varepsilon) = \inf\{\mathcal{N}(g(\Omega),\varepsilon) \ : \ g \text{ measurable function such that } f = g \text{ a.e.}\}.$$

This notion allows us to caracterize the uniform approximability in $L^\infty(\Omega, \mathcal{F}, \mu)$ in terms of uniformly bounded covering numbers. Before doing that, we notice that if $f \in L^\infty(\Omega, \mathcal{F}, \mu)$, then $\mathcal{N}(f,\varepsilon) < \infty$. Indeed, we know that $|f| \le \|f\|_\infty$ holds a.e., so by considering $g = f\mathbb{1}_{\{|f| \le \|f\|_\infty\}}$, we have $g = f$ a.e. and

$$\mathcal{N}(f,\varepsilon) \le \frac{2}{\varepsilon}\|f\|_\infty + 1.$$

If f is a measurable function and $\mathcal{N}(f,\varepsilon) < \infty$ then $f \in L^\infty(\Omega, \mathcal{F}, \mu)$. On the other hand, by definition of infimum, there exists a measurable function g such that $f = g$ a.e., and

$$\mathcal{N}(f,\varepsilon) \le \mathcal{N}(g(\Omega),\varepsilon) \le \mathcal{N}(f,\varepsilon) + \frac{1}{2},$$

showing that $\mathcal{N}(f,\varepsilon) = \mathcal{N}(g(\Omega),\varepsilon)$.

Theorem 9.3.6. *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $\mathscr{A} \subset L^\infty(\Omega, \mathcal{F}, \mu)$. The following assertions are equivalent:*

(i) $\mathscr{A}$ is UA;

(ii) $\sup_{f \in \mathscr{A}} \mathcal{N}(f,\varepsilon) < \infty$ for all $\varepsilon > 0$.

In this case, we have that $N_{\infty,\varepsilon}(\mathscr{A}) = \sup_{f \in \mathscr{A}} \mathcal{N}(f,\varepsilon)$ for all $\varepsilon > 0$.

Proof. Let $\varepsilon > 0$ and suppose $\sup_{f \in \mathscr{A}} \mathcal{N}(f,\varepsilon) = \infty$. Fix $k \ge 1$ and choose $f \in \mathscr{A}$ such that $\mathcal{N}(f,\varepsilon) \ge 10(k+1)$. Changing the representant of f is necessary, we can suppose that $m_\varepsilon := \mathcal{N}(f,\varepsilon) = \mathcal{N}(f(\Omega),\varepsilon)$. So there exists a collection $\mathcal{J}$ of closed balls $\{I_i = [a_i, b_i]\}_{1 \le i \le m_\varepsilon}$, of radius ε, such that

$$f(\Omega) \subset \bigcup_{i=1}^{m_\varepsilon} I_i$$

Using the minimality of this covering each interval cannot be covered by the other intervals, so for $i \neq j$ we have $I_i \setminus I_j \neq \emptyset$. Consider the measurable sets $A_i = f^{-1}(I_i)$. We shall prove that the minimality of $\mathcal{J}$ implies that $\mu(A_i) > 0$. Indeed, assume that for some i we have $\mu(A_i) = 0$. Take any $j \neq i$ (notice that we have assumed that m_ε is at least $10(k+1) > 2$) and $a \in I_j \setminus I_i$. The measurable function

$$h = f\mathbb{1}_{A_i^c} + a\mathbb{1}_{A_i}$$

coincides with f up to measure 0 and $h(\Omega) \subset \bigcup_{r \neq i} I_r$, so $\mathcal{N}(h(\Omega),\varepsilon) \le m_\varepsilon - 1$, which is a contradiction.

We say that a subcollection $\mathcal{C} \subset \mathcal{J}$ is ε-separated if for two different intervals $I, J \in \mathcal{C}$, we have the distance between them $d(I,J)$ is greater than ε. Notice that a collection with only one interval from $\mathcal{J}$ is ε-separated. Take $\mathcal{C}^*$ a maximal ε-separated subcollection with respect to inclusion. Now, if $I \in \mathcal{J} \setminus \mathcal{C}^*$ there exists an interval $L = [a,b] \in \mathcal{C}^*$ such that $d(I,L) \le \varepsilon$, otherwise the maximality of $\mathcal{C}^*$ is contradicted. Then,

$$I \subset [a-4\varepsilon, a-2\varepsilon] \cup [a-2\varepsilon, a] \cup [a,b] \cup [b, b+2\varepsilon] \cup [b+2\varepsilon, b+4\varepsilon],$$

showing that the collection

$$\mathcal{D} = \{[a_i-4\varepsilon, a_i-2\varepsilon], [a_i-2\varepsilon, a_i], [a_i,b_i], [b_i, b_i+2\varepsilon], [b_i+2\varepsilon, b_i+4\varepsilon] : [a_i,b_i] \in \mathcal{C}^*\},$$

is a covering of $f(\Omega)$ with closed balls of radius ε. Therefore

$$m_\varepsilon \leq |\mathcal{D}| \leq 5|\mathcal{C}^*|,$$

showing that $n = |\mathcal{C}^*| \geq \frac{1}{5}m_\varepsilon$ (here $|\mathcal{C}^*|$ is the cardinal of $\mathcal{C}^*$).

Consider now $g \in \mathscr{G}_{\infty,k}$. We say that an interval $I \in \mathcal{C}^*$ is unmarked if $d(g(\Omega), I) > \varepsilon$. There are at least $n - 2k \geq \frac{1}{5}m_\varepsilon - 2k = 2(\frac{1}{10}m_\varepsilon - k) > 1$ unmarked intervals in $\mathcal{C}^*$. Consider $I_i \in \mathcal{C}^*$ any unmarked interval, then for all $x \in A_i = f^{-1}(I_i)$, we have

$$|f(x) - g(x)| > \varepsilon.$$

Since $\mu(A_i) > 0$, we conclude that $\|f - g\|_\infty > \varepsilon$ and therefore $N_{\infty,\varepsilon}(\mathscr{A}) > \frac{m_\varepsilon}{10} - 1$, showing that $N_{\infty,\varepsilon}(\mathscr{A}) = \infty$. So we have proved that (i) implies (ii).

Now, let us show that (ii) implies (i). So, we are assuming that $M_\varepsilon = \sup_{f\in\mathscr{A}} \mathcal{N}(f,\varepsilon) < \infty$ for all $\varepsilon > 0$. Fix $\varepsilon > 0$ and let $f \in \mathscr{A}$. Suppose that $m_\varepsilon := \mathcal{N}(f,\varepsilon) = \mathcal{N}(f(\Omega),\varepsilon) \leq M_\varepsilon$. Again we can write

$$f(\Omega) \subset \bigcup_{i=1}^{m_\varepsilon} I_i$$

where $I_i = [a_i, b_i]$ are closed balls of radius ε. We assume that the left extremes are ordered increasingly: $a_1 < a_2 < ... < a_{m_\varepsilon}$. We define recursively $\tilde{a}_1 = a_1, \tilde{b}_1 = b_1$ and for $i \geq 2$

$$\tilde{a}_i = \max\{b_{i-1}, a_i\},\ \tilde{b}_i = b_i.$$

Define $\tilde{I}_i = [\tilde{a}_i, \tilde{b}_i]$ for $i \in \{1, ..., m_\varepsilon\}$. The fact that every interval I_i cannot be covered by the intervals $\{I_j\}_{j\neq i}$ allows us to show the following facts about the new intervals $\{\tilde{I}_i\}_{1\leq i\leq m_\varepsilon}$

$$\forall i\ \ \tilde{I}_i = I_i \setminus \bigcup_{j=1}^{i-1}[a_j, b_j),\ \forall i\ \bigcup_{j=1}^{i} I_j = \bigcup_{j=1}^{i} \tilde{I}_j,$$

$$\operatorname{int}(\tilde{I}_i) = (\tilde{a}_i, \tilde{b}_i) \neq \emptyset,$$

$$\forall i < j:\ \ \tilde{I}_i \cap \tilde{I}_j \subset \begin{cases} \emptyset & \text{if } j - i \geq 2 \\ \\ \{\tilde{b}_i\} & \text{if } j = i+1 \end{cases}.$$

Thus, $\{\tilde{I}_i\}_{1\leq i\leq m_\varepsilon}$ is a collection of closed balls of radii at most ε, that covers $f(\Omega)$, which are disjoint except for consecutive intervals that can intersects at one extreme.

With this new intervals we can produce a partition of $f(\Omega)$, by choosing $\widehat{I}_1 = \tilde{I}_1$ and for $i \geq 2$

$$\widehat{I}_i = \begin{cases} (\tilde{a}_i, \tilde{b}_i] & \text{if } \tilde{I}_i \cap \tilde{I}_{i-1} \neq \emptyset \\ \\ \tilde{I}_i & \text{otherwise} \end{cases}$$

We now define $\widehat{A}_i = f^{-1}(\widehat{I}_i)$, which is a partition of Ω (maybe some of them are empty). If $x \in \widehat{A}_i$ then $f(x) \in \widehat{I}_i \subset I_i = [a_i, b_i]$ and therefore $|f(x) - \frac{a_i+b_i}{2}| \leq \varepsilon$. Define the simple function

$$g(x) = \sum_{i=1}^{m_\varepsilon} \frac{a_i + b_i}{2} \mathbb{1}_{\widehat{A}_i},$$

that belongs to $\mathscr{G}_{\infty,m_\varepsilon}$ and satisfies for all $x \in \Omega$

$$|f(x) - g(x)| \leq \varepsilon,$$

showing that $\|f - g\|_\infty \leq \varepsilon$. We conclude that $N_{\infty,\varepsilon}(\mathscr{A}) \leq \sup_{f\in\mathscr{A}} \mathcal{N}(f, \varepsilon)$.

To finish, we prove that $N_{\infty,\varepsilon}(\mathscr{A}) = \sup_{f\in\mathscr{A}} \mathcal{N}(f, \varepsilon)$. For that purpose consider $k = N_{\infty,\varepsilon}(\mathscr{A})$, which means that for all $f \in \mathscr{A}$, there exists $g \in \mathscr{G}_{\infty,k}$, such that $\|f - g\|_\infty \leq \varepsilon$. We assume that $g = \sum_{i=1}^k c_i \mathbb{1}_{B_i}$, where $(B_i)_{i=1}^k$ is a partition of Ω. For any $i \in \{1, ..., k\}$ we have

$$\|(f - c_i)\mathbb{1}_{B_i}\|_\infty \leq \|f - g\|_\infty \leq \varepsilon,$$

which means that $A_i = \{x \in B_i : |f(x) - c_i| > \varepsilon\}$ is a measurable set of measure 0. Since $\mu(\Omega) > 0$, not all the sets B_j can have measure 0, so we assume without loss of generality that $\mu(B_1) > 0$. Consider $B = \Omega \setminus \bigcup\limits_{i=1}^{k} A_i$, $h = f\mathbb{1}_B + c_1\mathbb{1}_{B^c}$ and $\widetilde{g} = c_1\mathbb{1}_{B_1\cup B^c} + \sum_{i=2}^k c_i\mathbb{1}_{B_i\setminus A_i}$. We notice that $f = h$ a.e. and $\widetilde{g} = g$ a.e. On the other hand, $B_1 \cup B^c, B_2 \setminus A_2, ..., B_k \setminus A_k$ is a partition and $\widetilde{g} \in \mathscr{G}_{\infty,k}$. Also, it is clear that $B_1 \setminus A_1, ..., B_k \setminus A_k, B^c$ is a partition and

$$\widetilde{g} = g\mathbb{1}_B + c_1\mathbb{1}_{B^c}.$$

With these modifications, we have for all $x \in \Omega$

$$|h(x) - \widetilde{g}(x)| \leq \varepsilon.$$

This is clear for $x \in B$. For $x \in B^c$, we have $h(x) = c_1 = \widetilde{g}(x)$ and the claim is shown. Finally, the collection of closed ball of radius ε given by: $\{[c_i - \varepsilon, c_i + \varepsilon]\}_{1\leq i\leq k}$ is an ε-cover of $h(\Omega)$, showing that $\mathcal{N}(f, \varepsilon) \leq k$. The conclusion is that

$$\sup_{f\in\mathscr{A}} \mathcal{N}(f, \varepsilon) \leq k = N_{\infty,\varepsilon}(\mathscr{A}),$$

and the result is shown. □

Corollary 9.3.7. *Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space and $\mathscr{A}$ be a set of measurable functions. Assume that $\mathscr{A}$ is UA in $L^q(\Omega, \mathcal{F}, \mu)$ for some $q \in [1, \infty]$, then $\mathscr{A}$ is UA in $L^p(\Omega, \mathcal{F}, \mu)$ for all $p \in [1, q]$ and for all $\varepsilon > 0$ it holds*

$$N_{p,\varepsilon}(\mathscr{A}) \leq N_{q,\varepsilon\mu(\Omega)^{-r}}(\mathscr{A}),$$

where $r = \frac{1}{p} - \frac{1}{q}$.

In particular if $\sup_{f\in\mathscr{A}} \mathcal{N}(f, \varepsilon) < \infty$ for all $\varepsilon > 0$, then $\mathscr{A}$ is UA in $L^p(\Omega, \mathcal{F}, \mu)$ for all $p \in [1, \infty]$ and for all $\varepsilon > 0$ it holds $N_{p,\varepsilon}(\mathscr{A}) \leq \sup_{f\in\mathscr{A}} \mathcal{N}(f, \varepsilon\mu(\Omega)^{\frac{-1}{p}})$.

Proof. This is a direct consequence of Hölder's inequality. In fact, assume that $p \leq q$ and consider $g \in \mathscr{G}_{p,k}, f \in \mathscr{A}$ then, we have

$$\|f - g\|_p \leq \|f - g\|_q(\mu(\Omega))^r$$

where $r = \frac{1}{p} - \frac{1}{q}$. From this it follows that $\mathscr{A}$ is UA in $L^p(\Omega, \mathcal{F}, \mu)$ and

$$N_{p,\varepsilon}(\mathscr{A}) \leq N_{q,\varepsilon\mu(\Omega)^{-r}}(\mathscr{A}).$$

The second assertion follows from Theorem 9.3.6 □

The previous result gives a large class of UA sets when the measure is finite. For example suppose that Ω is a bounded metric space, $\mathcal{F}$ is the Borel σ-algebra and μ is a finite measure on $\mathcal{F}$. Then the set of 1-Lipschitz functions is UA in $L^p(\Omega,\mathcal{F},\mu)$ for any $p \in [1,+\infty]$.

The following result is a characterization of UA in L^p for $p \in [1,\infty)$, where we shall prove that a class is UA if and only $\mathrm{Var}_{p,k}(f)$ converges toward 0, when $k \to \infty$, uniformly in the class.

Theorem 9.3.8. *Let $(\Omega,\mathcal{F},\mu)$ be a measure space, $p \in [1,\infty)$ and let $\mathscr{A} \subset L^p(\Omega,\mathcal{F},\mu)$. Then, the following assertions are equivalent:*

(i) $\mathscr{A}$ is UA in $L^p(\Omega,\mathcal{F},\mu)$;

(ii) $\lim_{k\to\infty} \sup_{f\in\mathscr{A}} Var_{p,k}(f) = 0$.

In this case if we define $r_\varepsilon(\mathscr{A}) = \min\{k \in \mathbb{N} \ : \ \sup_{f\in\mathscr{A}} Var_{p,k}(f) \le \varepsilon\}$, we have that for all $\varepsilon > 0$

$$r_{2\varepsilon}(\mathscr{A}) \le N_{p,\varepsilon}(\mathscr{A}) \le r_\varepsilon(\mathscr{A}) + 1.$$

Moreover, if the measure μ is finite both properties (i), (ii) are equivalent to

(iii) $\lim_{k\to\infty} \sup_{f\in\mathscr{A}} Var_{p,k}(f,\Omega) = 0$.

In this case if we define $m_\varepsilon(\mathscr{A}) = \min\{k \in \mathbb{N} \ : \ \sup_{f\in\mathscr{A}} Var_{p,k}(f,\Omega) \le \varepsilon\}$, we have that for all $\varepsilon > 0$

$$m_{2\varepsilon}(\mathscr{A}) \le r_{2\varepsilon}(\mathscr{A}) \le N_{p,\varepsilon}(\mathscr{A}) \le m_\varepsilon(\mathscr{A}) \le r_\varepsilon(\mathscr{A})$$

Proof. Suppose that $\mathscr{A}$ is UA and fix $\varepsilon > 0$. Then we have that $\mathscr{D}_{p,k}(f) \le \varepsilon$ for all $f \in \mathscr{A}$, where $k = N_{p,\varepsilon}(\mathscr{A})$. By Proposition 9.2.6, we deduce that $\mathrm{Var}_{p,k}(f) \le 2\varepsilon$ for all $f \in \mathscr{A}$. It follows that $r_{2\varepsilon}(\mathscr{A}) \le N_{p,\varepsilon}(\mathscr{A})$, implying that (ii) holds. Now suppose that (ii) holds. Using Proposition 9.2.6 again, it is easy to see that $N_{p,\varepsilon}(\mathscr{A}) \le r_\varepsilon(\mathscr{A})+1$, from what we deduce that (i) is true. In the case that μ is finite, the equivalence between (ii) and (iii) and the last assertion of the theorem follow directly from Proposition 9.2.6. □

9.3.3 The unit ball of L^p

Now we investigate when the unit ball of $L^p(\Omega,\mathcal{F},\mu)$ is UA. The case $p = \infty$ is simple:

Proposition 9.3.9. *Let $(\Omega,\mathcal{F},\mu)$ be a measure space. Then $B_{L^\infty(\Omega,\mathcal{F},\mu)}$ is UA. More precisely we have that $N_{\infty,\varepsilon}(B_{L^\infty(\Omega,\mathcal{F},\mu)}) \le \left[\frac{2}{\varepsilon}\right] + 1$ (where $[.]$ is the integer part) for all $\varepsilon > 0$.*

Proof. It is a direct consequence of theorem 9.3.6. □

The main objective of this section is to prove the following result:

Theorem 9.3.10. *Let $(\Omega,\mathcal{F},\mu)$ be a measure space and $p \in [1,+\infty)$. The following assertions are equivalent:*

(i) $B_{L^p(\Omega,\mathcal{F},\mu)}$ is UA;

(ii) $L^p(\Omega,\mathcal{F},\mu)$ is finite dimensional;

(iii) μ is atomic and has only a finite number of atoms with finite measure, up to measure 0.

More precisely, if the previous assertions are false then $N_{p,\varepsilon}(B_{L^p(\Omega,\mathcal{F},\mu)}) = \infty$ for all $\varepsilon \in (0,1)$.

This theorem will be proved thanks to several intermediary results. We start with the following result:

Proposition 9.3.11. *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $p \in [1, +\infty)$. Suppose that there exists a sequence of disjoint measurable sets $(A_n)_n$ of positive measure such that $\mu(A_n) \to 0$. Then $B_{L^p(\Omega,\mathcal{F},\mu)}$ is not UA. More precisely, we have that $N_{p,\varepsilon}(B_{L^p(\Omega,\mathcal{F},\mu)}) = \infty$ for all $\varepsilon \in (0, 1)$.*

Proof. We are going to prove that for all $k \geq 1$

$$\sup_{f \in L^p(\Omega,\mathcal{F},\mu), f \neq 0} \inf_{h \in \mathscr{G}_{p,k}} \frac{\|f - h\|_p^p}{\|f\|_p^p} = 1. \tag{9.12}$$

Note that this equality implies easily that $N_{p,\varepsilon}(B_{L^p(\Omega,\mathcal{F},\mu)}) = \infty$ for all $\varepsilon \in (0, 1)$. Consider $r \geq 2$ and consider a subsequence $(n_k)_k$ such that $\mu(A_{n_k}) \in (r^{-n_k-1}, r^{-n_k}]$. We further assume that $n_{k+1} - n_k \geq 2$. With this sequence we consider

$$f_N(x) = \sum_{j=1}^{N} \mu(A_{n_j})^{\frac{-1}{p}} \mathbb{1}_{A_{n_j}}(x).$$

for all $N \geq 1$ and note that $\|f_N\|_p^p = N$. Let $h \in \mathscr{G}_{p,k}$ and $N > 2 + 2k$. We say that an index $1 < j < N$ is unmarked if $\mathrm{Im}(h) \cap \left(r^{\frac{n_{j-1}}{p}}, r^{\frac{n_{j+1}}{p}}\right) = \emptyset$. Note that there are at least $N - 2 - 2k$ unmarked indexes. For such unmarked index j, we have for $x \in A_{n_j}$

$$f_N(x) = \mu(A_{n_j})^{\frac{-1}{p}} \geq r^{\frac{n_j}{p}} > r^{\frac{n_j-1}{p}} > r^{\frac{n_{j-1}}{p}} \geq h(x), \text{ or}$$

$$f_N(x) = \mu(A_{n_j})^{\frac{-1}{p}} < r^{\frac{n_j+1}{p}} < r^{\frac{n_{j+1}}{p}} \leq h(x).$$

In the first case we have

$$f_N(x) - h(x) \geq f_N(x) - r^{\frac{n_j-1}{p}} \geq f_N(x) - r^{\frac{-1}{p}} f_N(x) = \left(1 - r^{\frac{-1}{p}}\right) f_N(x).$$

In the second case we get

$$h(x) - f_N(x) \geq r^{\frac{n_{j+1}}{p}} - f_N(x) \geq r^{\frac{n_j+2}{p}} - f_N(x) = r^{\frac{1}{p}} r^{\frac{n_j+1}{p}} - f_N(x) > \left(r^{\frac{1}{p}} - 1\right) f_N(x).$$

Notice that

$$\theta = 1 - r^{\frac{-1}{p}} = \frac{r^{\frac{1}{p}} - 1}{r^{\frac{1}{p}}} < r^{\frac{1}{p}} - 1.$$

So, we have $|f_N(x) - h(x)| \geq \theta f_N(x)$ on A_{n_j}. Then, we conclude that

$$\|f_N - h\|_p^p \geq \sum_{j:\text{unmarked}} \int_{A_{n_j}} |f_N(x) - h(x)|^p \, dx \geq \theta^p \sum_{j:\text{unmarked}} \int_{A_{n_j}} |f_N(x)|^p \, dx \geq \theta^p (N - 2 - 2k).$$

Hence, we have

$$\frac{\|f_N - h\|_p^p}{\|f_N\|_p^p} \geq \theta^p \frac{N - 2 - 2K}{N},$$

and we get

$$\sup_{f \in L^p(\Omega,\mathcal{F},\mu), f \neq 0} \inf_{h \in \mathscr{G}_{p,k}} \frac{\|f - h\|_p^p}{\|f\|_p^p} \geq \theta^p = (1 - r^{\frac{-1}{p}})^p.$$

Now, it is enough to make $r \uparrow \infty$. □

An inmediate corollary is obtained for diffuse measures.

Corollary 9.3.12. *Assume that μ is a non trivial diffuse measure, then for all $p \in [1,\infty)$ the unit ball $B_{L^p(\Omega,\mathcal{F},\mu)}$ is not UA and $N_{p,\varepsilon}(B_{L^p(\Omega,\mathcal{F},\mu)}) = \infty$, for all $\varepsilon \in (0,1)$.*

Proof. This follows directly from Sierpiński's theorem (see [152]). In fact, consider a measurable set B_0 such that $0 < \mu(B_0) = a < \infty$ (if such set does not exists then Ω is an atom of μ). Then, there exists $B_1 \subset B_0$ such that $\mu(B_1) = \frac{a}{2}$. Applying the same idea to $B_0 \setminus B_1$, there exists $B_2 \subset (B_0 \setminus B_1)$ such that $\mu(B_2) = \frac{\mu(B_0\setminus B_1)}{2} = \frac{a}{4}$. Inductively, we construct a sequence of disjoint subsets $(B_k)_k$ such that

$$B_{k+1} \subset B_0 \setminus \bigcup_{i=1}^{k} B_i,$$

and

$$\mu(B_{k+1}) = \frac{\mu\left(B_0 \setminus \bigcup\limits_{i=1}^{k} B_i\right)}{2} = \frac{a}{2^{k+1}}$$

for all $k \in \mathbb{N}$. The result follows from the previous proposition. □

Proposition 9.3.13. *Assume that $(\Omega,\mathcal{F},\mu)$ is an atomic measure space and $p \in [1,\infty)$. Then the following assertions are equivalent:*

(i) μ has a finite number of atoms of finite measure, up to measure 0;

(ii) The space $L^p(\Omega,\mathcal{F},\mu)$ is finite dimensional;

(iii) The unit ball $B_{L^p(\Omega,\mathcal{F},\mu)}$ is UA.

Moreover, if the previous assertions are false then $N_{p,\varepsilon}(B_{L^p(\Omega,\mathcal{F},\mu)}) = \infty$ for all $\varepsilon \in (0,1)$.

Proof. Assume (i) holds. Denote by $\{A_k\}_{1\le k\le n}$ a finite collection of atoms of finite measure, such that all other atom C of finite measure coincides with some of them up to measure 0. Take $B = \Omega \setminus \cup_{k=1}^n A_k$. If $\mu(B) > 0$ there there exists an atom $C \subset B$. This atom C satisfies that $\mu(C \setminus A_k) = \mu(C) > 0$ and it cannot coincide with A_k up to measure 0. Then C has infinite measure. Then either $\mu(B) = 0$ or $\mu(B) = \infty$ and contains no measurable subset of positive finite measure. Then, $L^p(\Omega,\mathcal{F},\mu)$ is generated by the finite collection $\{\mathbb{1}_{A_k}\}_{1\le k\le n}$, so (ii) holds. Clearly $(ii) \Rightarrow (iii)$.

So, for the rest of the proof we assume that there exists a countable collection of disjoint atoms $(A_n)_n$ each one of finite positive measure. Here there are two different situations. The first one is the existence of an infinite subsequence of atoms $(A_{n_k})_k$ such that $\mu(A_{n_k}) \to 0$. Then, we can apply Theorem 9.3.11, to conclude that the unit ball $B_{L^p(\Omega,\mathcal{F},\mu)}$ is not UA.

The second possibility is the existence of a constant $a > 0$ such that $\mu(A_n) \ge a$, for all n. We now procede to prove that $B_{L^p(\Omega,\mathcal{F},\mu)}$ is not UA. We do it for $p = 1$, the other cases are treated similary.

In what follows we fix $k \ge 2$ and $R > 1$, and we consider the partial sums

$$S_i = \sum_{2^{i-1}\le j<2^i} \mu(A_j) \ge a2^{i-1},$$

for $i \ge 1$, and we construct a strictly increasing sequence of integers $(t_q)_q$ such that the interval $[R^{t_q}, R^{t_q+1})$ contains at least one of these partial sums. We call S_{i_q} any such partial sums, for example the smallest one, that is, for q such that $[R^{t_q}, R^{t_q+1}) \cap \{S_i\}_{i\ge1} \neq \emptyset$, we take

$$i_q = \min\{r \in \mathbb{N} : R^{t_q} \le S_r < R^{t_q+1}\}.$$

We also define

$$B_q = \bigcup_{j=2^{i_q-1}}^{2^{i_q}-1} A_j,$$

the union of the atoms that has mass S_{i_q}. We consider the function

$$f = \sum_{q=3}^{M+2} R^{-t_q} \, \mathbb{1}_{B_q},$$

where M is a large integer. For the moment we choose $M > 2kR$. Take $h \in \mathscr{G}_{1,k}$ and as before we say that $3 \le q \le M+2$ is an unmarked index if

$$\operatorname{Im}(h) \cap (R^{-t_q-1}, R^{-t_q+1}) = \emptyset.$$

There are at least $M - 2k$ unmarked indexes. For an unmarked index q and $x \in B_q$, we either have

$$f(x) - h(x) \ge R^{-t_q} - R^{-t_q-1} \ge f(x)\left(1 - \tfrac{1}{R}\right) = f(x)\tfrac{R-1}{R}, \text{ or}$$

$$h(x) - f(x) \ge R^{-t_q+1} - R^{-t_q} \ge f(x)(R-1).$$

In any case, we have for $x \in B_q$

$$|f(x) - h(x)| \ge f(x)\frac{R-1}{R},$$

and then

$$\begin{aligned} \|f-h\|_1 &\ge \tfrac{R-1}{R} \sum_{q\text{: unmarked}} R^{-t_q}\mu(B_q) = \tfrac{R-1}{R}\left(\sum_q R^{-t_q}\mu(B_q) - \sum_{q\text{:marked}} R^{-t_q}\mu(B_q)\right) \\ &\ge \tfrac{R-1}{R}(\|f\|_1 - 2kR) = \|f\|_1 \tfrac{R-1}{R}\left(1 - \tfrac{2kR}{\|f\|_1}\right) \end{aligned}$$

Now, we estimate the norm of f. Clearly, we have $\|f\|_1 = \sum_q R^{-t_q}\mu(B_q)$, which gives the lower estimate

$$M \le \|f\|_1, \tag{9.13}$$

and then the lower bound

$$\|f-h\|_1 \ge \|f\|_1 \frac{R-1}{R}\left(1 - \frac{2kR}{M}\right). \tag{9.14}$$

So, we conclude that for $\tilde{f} = f/\|f\|$

$$\inf\{\|\tilde{f} - g\|_1 : g \in \mathscr{G}_{1,k}\} \ge \frac{R-1}{R}\left(1 - \frac{2kR}{M}\right),$$

and therefore

$$\sup_{f \in B_{L^1(\Omega,\mathcal{F},\mu)}} \inf\{\|f-g\|_1 : g \in \mathscr{G}_{1,k}\} \ge \frac{R-1}{R}\left(1 - \frac{2kR}{M}\right)$$

Taking $M \uparrow \infty$, we conclude that

$$\sup_{f \in B_{L^1(\Omega,\mathcal{F},\mu)}} \inf\{\|f-g\|_1 : g \in \mathscr{G}_{1,k}\} \ge \frac{R-1}{R}.$$

Now we take $R \uparrow \infty$, to get finally that

$$\sup_{f \in B_{L^1(\Omega,\mathcal{F},\mu)}} \inf\{\|f-g\|_1 : g \in \mathscr{G}_{1,k}\} \geq 1$$

independently of $k \geq 2$. For $k=1$, we point out that $\mathscr{G}_{1,1} = \{0\}$ and so

$$\sup_{f \in B_{L^1(\Omega,\mathcal{F},\mu)}} \inf\{\|f-g\|_1 : g \in \mathscr{G}_{1,1}\} = 1.$$

Hence, $N_{1,\varepsilon}(B_{L^1(\Omega,\mathcal{F},\mu)}) = \infty$, for all $\varepsilon < 1$. □

In order to prove Theorem 9.3.10, we shall use a result in [106], where the notion of atomic and nonatomic are different from the (standard) notions we are using. In this discussion we add an $*$ to distinguish the notions we are using and the corresponding in [106]. According to [106] a measurable set A is an $*$-*atom* if $\mu(A) > 0$ and for all $E \in \mathcal{F}$ either $\mu(A \cap E) = 0$ or $\mu(A \setminus E) = 0$. It is direct to show that if A is an $*$-atom for μ, then it is an atom for μ. Indeed, assume that $B \subset A$ satisfies $\mu(B) < \mu(A)$, then $\mu(A \setminus B) = \mu(A) - \mu(B) > 0$ and we conclude that $0 = \mu(A \cap B) = \mu(B)$, proving that A is an atom for μ. The converse is not always true (see the example below). It is true if A has finite measure. In fact, suppose that A is an atom of finite measure and let E be a measurable set. If $\mu(A \cap E) > 0$ then $\mu(A) = \mu(A \cap E)$, showing that $\mu(A \setminus E) = 0$ since $E \cap A$ has finite measure, and therefore A is an $*$-atom.

A measure is $*$-atomic if every measurable set A of positive measure contains an $*$-atom. A measure that has no $*$-atoms is said $*$-nonatomic. Here is an example of a $*$-nonatomic measure which is atomic in the standard sense. Consider $(\mathbb{R}, \mathcal{P}(\mathbb{R}))$ as a measurable space and

$$\mu(A) = \begin{cases} \infty & \text{if } A \text{ is uncountable} \\ 0 & \text{otherwise} \end{cases}$$

If $\mu(A) > 0$, then A is uncountable and can be splitted into two uncountable disjoint sets B and C. Then $\mu(A \cap B) = \infty$ and $\mu(A \setminus B) = \infty$. So, there are no $*$-atoms and then according to the above definition μ is $*$-nonatomic.

The other concept we need is the notion of $*$-singular. Two measures ν and λ are said $*$-singular if for all measurable sets E, there exist two measurable sets F and G contained in E such that

$$\nu(F) = \nu(E),\ \lambda(F) = 0,\ \text{and}\ \lambda(G) = \lambda(E),\ \nu(G) = 0.$$

The main theorem we need is the following.

Theorem 2.1 in [106]. *Assume $(\Omega, \mathcal{F}, \mu)$ is a measure space. Then μ can be decomposed as $\mu = \nu + \lambda$, where ν is $*$-atomic and λ is $*$-nonatomic. We can assume that ν, λ are $*$-singular, in which case the decomposition is unique.*

We are now ready to prove the main result of this subsection:

Proof of Theorem 9.3.10. It is clear that $(iii) \implies (ii) \implies (i)$. Now suppose that (i) holds, that is, $B_{L^p(\Omega,\mathcal{F},\mu)}$ is UA. By Theorem 2.1 in [106], there is a unique decomposition $\mu = \nu + \lambda$ where ν is $*$-atomic measure, λ is $*$-nonatomic and ν and λ are $*$-singular. Consider

$$\mathcal{C} = \{[A] : \ A \text{ is an } *\text{-atom for } \nu \text{ of finite } \nu\text{-measure }\}$$

where $[A]$ is the equivalence class of measurable sets B such that $\nu(A \Delta B) = 0$. Notice that $[A] \in \mathcal{C}$ if and only if A is an atom of finite ν-measure. Therefore, if $[A] \neq [B] \in \mathcal{C}$ then $\nu(A \cap B) = 0$, that is, A and B are disjoint up to ν-measure 0.

If $\mathcal{C}$ is infinite, we take a countable collection $(E_n)_n$ of atoms for ν, which are disjoint up to ν-measure zero, and each one has finite and positive ν-measure. For every n there exists $F_n \subset E_n$, such that $\nu(F_n) = \nu(E_n)$ and $\lambda(F_n) = 0$. Clearly, $(F_n)_n$ is a countable class of disjoint atoms for ν, which have positive and finite measure. The measurable set $A = \cup_{n=1}^{\infty} F_n$ satisfies $\lambda(A) = 0$. This shows that $\mu|_A = \nu|_A$, so $L^p(A, \mathcal{F}|_A, \mu|_A)$ and $L^p(A, \mathcal{F}|_A, \nu|_A)$ can be identified.

On the other hand, the measure $\nu|_A$ is atomic. Indeed, assume that $D \subset A$ has positive measure. Then for some n it holds $\nu(D \cap F_n) > 0$ and then $D \cap F_n$ contains an $*$-atom H of ν, which has finite measure, and therefore it is an atom for ν. We can apply Proposition 9.3.13 to conclude that $B_{L^p(A,\mathcal{F}|_A,\nu|_A)}$ is not UA, and a fortiori $B_{L^p(\Omega,\mathcal{F},\mu)}$ is not UA, which is a contradiction.

The conclusion is that ν has a finite number of atoms $(A_n)_{n\in J}$ of finite measure, up to measure 0, where J is a finite (eventually empty) set. Therefore, if $B = \Omega \setminus \cup_{n\in J} A_n$, then any measurable $C \subset B$ has 0 or infinite ν-measure.

On the other hand, there exists $G \subset B$ such that $\nu(G) = 0$ and $\lambda(G) = \lambda(B)$. If there exists $H \subset B$ a measurable set such that $0 < \lambda(H) < \infty$, then we arrive to a contradiction. Indeed, consider $K \subset H$ such that $\nu(K) = 0$ and $\lambda(K) = \lambda(H)$. Since $\lambda(H)$ is finite, this means that $\lambda(H \setminus K) = 0$. Now, $\lambda|_K$ is a diffuse measure, because if there exists $L \subset K$ an atom for λ, then this atom has finite measure and therefore it is an $*$-atom for λ, which is not possible. The contradiction is obtained because $B_{L^p(K,\mathcal{F}|_K,\mu|_K)}$ and $B_{L^p(K,\mathcal{F}|_K,\lambda|_K)}$ can be identified and the latter is not UA, according to Corollary 9.3.12.

The conclusion is that $\lambda(H)$ is 0 or infinite for every $H \subset B$. Since $\lambda(B^c) = 0$, we conclude that $\lambda(H)$ is either 0 or infinite for every measurable set H. Also, $\mu(H)$ is 0 or infinite, for any $H \subset B$ and $\mu = \nu$ on $A = B^c$. Therefore, μ is an atomic measure and it has a finite collection of disjoint atoms with finite measure, up to measure zero.

The last part of the Theorem follows from either Corollary 9.3.12 or Proposition 9.3.13. $\square$

9.3.4 Stability of the class of UA sets

In this subsection, we study the image of a UA set under classical operations. We start with the following easy proposition:

Proposition 9.3.14. *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $p \in [1, +\infty]$. Let $\mathscr{A}, \mathscr{B} \subset L^p(\Omega, \mathcal{F}, \mu)$ and $\varepsilon > 0$. Then:*

(a) if $\mathscr{A} \subset \mathscr{B}$ then $N_{p,\varepsilon}(\mathscr{A}) \le N_{p,\varepsilon}(\mathscr{B})$;

(b) $N_{p,\varepsilon}(\mathscr{A}) = N_{p,\varepsilon}(\overline{\mathscr{A}})$;

(c) $N_{p,|\lambda|\varepsilon}(\lambda\mathscr{A}) = N_{p,\varepsilon}(\mathscr{A})$ for all $\lambda \in \mathbb{R}$;

(d) $N_{p,\varepsilon}(\mathscr{A} + \mathscr{B}) \le \min_{t,s>0,t+s\le\varepsilon} N_{p,t}(\mathscr{A}) N_{p,s}(\mathscr{B})$;

In particular if $\mathscr{A}$ and $\mathscr{B}$ are UA then $\overline{\mathscr{A}}$, $\lambda\mathscr{A}$ and $\mathscr{A} + \mathscr{B}$ are UA.

Proof. The proof is left to the reader. $\square$

In the next result, we prove that the closed convex hull of a bounded UA set is still UA.

Theorem 9.3.15. *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $p \in (1, +\infty)$. If $\mathscr{A} \subset L^p(\Omega, \mathcal{F}, \mu)$ is a UA set, then $\mathscr{A}_K = \{f \in \overline{co}(\mathscr{A}) \mid \forall g \in \mathscr{A} \ \|f - g\|_p \le K\}$ is also UA for all $K \ge 0$. More precisely, we have that*

$$N_{p,\varepsilon}(\mathscr{A}_K) \le \min_{\eta\in(0,1)} \left(N_{p,(1-\eta)\varepsilon}(\mathscr{A})\right)^{s(\eta)}$$

for all $\varepsilon > 0$*, where* $s(\eta) = \left[\frac{CK}{\eta\varepsilon}\right]^{\frac{\min\{p,2\}}{\min\{p,2\}-1}} + 1$ *and* C *is a constant depending on* $(\Omega, \mathcal{F}, \mu)$ *and* p*. In particular, if* $\mathscr{A}$ *is bounded then* $\overline{co}(A)$ *is UA.*

Proof. Fix $K \geq 0$. For $n \in \mathbb{N}$, define

$$\mathrm{co}_n(\mathscr{A}) = \left\{ \sum_{i=1}^{n} a_i f_i \mid a_i \geq 0,\ \sum_{i=1}^{n} a_i = 1,\ f_i \in \mathscr{A} \right\}.$$

Remember that $L^p(\Omega, \mathcal{F}, \mu)$ has non-trivial Rademacher type $r = \min\{p, 2\}$ (see Theorem 6.2.14 in [1]). By Theorem 2.5 of [59], one has that

$$d(\mathrm{co}_n(\mathscr{A}), f) \leq \frac{CK}{n^{1-\frac{1}{r}}}$$

for all $f \in \mathscr{A}_K$ and $n \in \mathbb{N}$ where C is a constant depending on $(\Omega, \mathcal{F}, \mu)$ and p. Therefore, if we take $\varepsilon > 0, \eta \in (0, 1)$ and $n_0 = \left[\frac{CK}{\eta\varepsilon}\right]^{\frac{r}{r-1}} + 1$, we will have that $d(\mathrm{co}_{n_0}(\mathscr{A}), f) < \eta\varepsilon$, for all $f \in \mathscr{A}_K$.

Thus, if $f_0 \in \mathscr{A}_K$ there exists $g_0 = \sum_{i=1}^{n_0} a_i f_i \in \mathrm{co}_{n_0}(\mathscr{A})$ such that $\|f_0 - g_0\|_p < \eta\varepsilon$. On the other hand, since $\mathscr{A}$ is UA, there exists $h_i \in \mathscr{G}_{p,k}$ where $k = N_{p,(1-\eta)\varepsilon}(\mathscr{A})$ such that

$$\|f_i - h_i\|_p \leq (1 - \eta)\varepsilon$$

for all $i \in \{1, ..., n_0\}$. One can deduce that

$$\left\| f - \sum_{i=1}^{n_0} a_i h_i \right\|_p \leq \|f_0 - g_0\|_p + \left\| \sum_{i=1}^{n_0} a_i f_i - \sum_{i=1}^{n_0} a_i h_i \right\|_p \leq \eta\varepsilon + \sum_{i=1}^{n_0} a_i (1 - \eta)\varepsilon = \varepsilon$$

with $\sum_{i=1}^{n_0} a_i h_i \in \mathscr{G}_{p,k^{n_0}}$. We conclude that $N_{p,\varepsilon}(\mathscr{A}_K) \leq k^{n_0}$. $\square$

Remark 9.3.16. Note that if $\mathscr{A}$ is an unbounded UA set then $\overline{\mathrm{co}}(\mathscr{A})$ may not be UA. In fact, $\mathscr{G}_{p,2}$ is UA but $\overline{\mathrm{co}}(\mathscr{G}_{p,2}) = L^p(\Omega, \mathcal{F}, \mu)$ (since $\mathrm{co}(\mathscr{G}_{p,2})$ is the set of simple functions) is not UA in general for any $p \in [1, \infty]$. Remark that the previous theorem is not interesting if $p = \infty$ since any bounded set is UA by Proposition 9.3.9.

Remark 9.3.17. The previous theorem is false if $p = 1$. In fact remember that $B_{\ell_1} = \overline{co}(\mathrm{ext}(B_{\ell_1}))$ and $\mathrm{ext}(B_{\ell_1}) = \{\pm\delta_n\}_{n\in\mathbb{N}}$. It follows that $\mathrm{ext}(B_{\ell_1})$ is UA but we have seen that B_{ℓ_1} is not UA (see Theorem 9.3.10). More generally, using the previous result, it is easy to show that there exists a UA set $\mathscr{A} \subset \ell_p$ such that $B_{\ell_p} = \overline{co}(\mathscr{A})$ if and only if $p \in \{1, \infty\}$.

In the next result we study stability properties of UA classes under Hölder transformations. Recall that a real function Ψ is uniformly α-Hölder if there exists a constant K, such that

$$|\Psi(x) - \Psi(y)| \leq K|x - y|^\alpha.$$

With this definition, the identity function is not uniformly α-Hölder for $\alpha < 1$. To enlarge the class of uniformly α-Hölder functions we consider the following classes of Hölder functions, denoted $\mathbb{H}(K, \alpha)$ for $0 < \alpha \leq 1$, which consists of real functions Ψ such that for all x, y, it holds

$$|\Psi(x) - \Psi(y)| \leq K(|x| + |y| + 1)^{1-\alpha}|x - y|^\alpha.$$

We can assume without loss of generality that $K \geq 1$. We notice that $\mathbb{H}(K, 1)$ is the set of K-Lipschitz functions. If $0 < \beta \leq \alpha$, then $\mathbb{H}(K, \alpha) \subset \mathbb{H}(K, \beta)$. Also $\mathbb{H}(K, \alpha)$ contains the class of uniformly Hölder functions.

Proposition 9.3.18. *Assume that $(\Omega, \mathcal{F}, \mu)$ is a finite measure space and $\mathscr{A}$ is UA in $L^q(\Omega, \mathcal{F}, \mu)$, for some $q \in [1, \infty]$. Consider $\alpha \in (0,1]$, and we assume further that $\mathscr{A}$ is bounded in $L^q(\Omega, \mathcal{F}, \mu)$ when $\alpha < 1$. Then, the $\mathbb{H}(K, \alpha)$-transform of $\mathscr{A}$ given by*

$$\mathbb{H}(K,\alpha)(\mathscr{A}) = \{\Psi(f) : \, f \in \mathscr{A}, \, \Psi \in \mathbb{H}(K,\alpha)\}$$

is UA in $L^p(\Omega, \mathcal{F}, \mu)$ for any $1 \le p \le q$. Moreover, for $\varepsilon \in (0,1]$

$$N_{p,\varepsilon}(\mathbb{H}(K,\alpha)(\mathscr{A})) \le N_{q,(\varepsilon/\Gamma)^{\frac{1}{\alpha}}}(\mathscr{A}),$$

where

$$\Gamma = \Gamma(\alpha, p, q) = K\mu(\Omega)^{-r} \begin{cases} \left(2B + 1 + \mu(\Omega)^{\frac{1}{q}}\right)^{1-\alpha} & \text{if } \alpha < 1 \\ 1 & \text{if } \alpha = 1 \end{cases}$$

with B a bound for $\mathscr{A}$ in $L^q(\Omega, \mathcal{F}, \mu)$ and $r = \frac{1}{p} - \frac{1}{q}$.

Proof. The case $\alpha = 1$ is straightforward so, we assume $\alpha < 1$. We assume first that $p = q$. Consider $\varepsilon \in (0,1]$, $k = N_{q,\varepsilon}(\mathscr{A})$, $f \in \mathscr{A}$, $g \in \mathscr{G}_{q,k}$ such that $\|f-g\|_q \le \varepsilon$ and $\Psi \in \mathbb{H}(K,\alpha)$. We have

$$\int |\Psi(f(x)) - \Psi(g(x))|^q d\mu(x) \le K^q \int (|f(x)| + |g(x)| + 1)^{q(1-\alpha)} \, |f(x) - g(x)|^{q\alpha} d\mu(x).$$

Now, we apply Hölder's inequality for $s = \frac{1}{\alpha}$ and its conjugated index $t = \frac{1}{1-\alpha}$ to get

$$\int |\Psi(f(x)) - \Psi(g(x))|^q d\mu(x) \le K^q \left(\int (|f(x)| + |g(x)| + 1)^q d\mu(x) \right)^{1-\alpha} \left(\int |f(x) - g(x)|^q d\mu(x) \right)^{\alpha}$$

which implies

$$\|\Psi(f) - \Psi(g)\|_q \le K \left(\|f\|_q + \|g\|_q + \|1\|_q\right)^{(1-\alpha)} \|f-g\|_q^{\alpha}.$$

If B is a bound for $\mathscr{A}$, we conclude that $\|g\|_q \le B+1$, which shows

$$\|\Psi(f) - \Psi(g)\|_q \le K \left(2B + 1 + \mu(\Omega)^{\frac{1}{q}}\right)^{(1-\alpha)} \varepsilon^{\alpha} = \Gamma \varepsilon^{\alpha}.$$

Since $\Psi(g) \in \mathscr{G}_{q,k}$, we deduce that

$$N_{q,\Gamma\varepsilon^{\alpha}}(\mathbb{H}(K,\alpha)(\mathscr{A})) \le N_{q,\varepsilon}(\mathscr{A}),$$

and the result is shown in this case. The case $p < q$ follows from Corollary 9.3.7. ∎

We point out that under the hypothesis of the Theorem, we have $\mathscr{A} \subset \mathbb{H}(K,\alpha)(\mathscr{A})$.

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List of symbols

$\mathbb{1}_A$	characteristic function of A
A^c	complementary of A
$A^{(n)}$	family of all subsets of A with cardinal n
$A_{\mathcal{U}}$	ultraproduct of a set
$A \Delta B$	symmetric difference between A and B
$\aleph_1$	cardinal of the set of all countable ordinal numbers
$\mathrm{bc}(x_n)_n$	basic constant of a basic sequence $(x_n)_n$
$\mathscr{B}$	family of Borel subsets of $\mathbb{R}$
B_X	closed unit ball of X
$B_X(x_0, \alpha)$	closed ball of center $x_0 \in X$ and radius α
$\mathfrak{c}$	cardinality of the continuum
$\mathrm{Card}(A)$ or $\lvert A \rvert$	cardinal of A
$[C]'_\varepsilon$	derivation of C
$[C]'_{F,\varepsilon}$	derivation of C with respect to F
$\mathrm{conv}(A)$	convex hull of A
$\mathscr{D}_{p,k}(f)$	distance from f to $\mathscr{G}_{p,k}(\Omega, \mathcal{F}, \mu)$
δ	isometric embedding of a metric space M into $\mathcal{F}(M)$
δ_f	modulus of convexity of f
δ_X	modulus of convexity of a Banach space X
δ_C	modulus of convexity of a uniformly convex set C
$\Delta(C)$	measure of the Radon-Nikodym property
$\mathscr{D}_{p,k}(f)$	distance between f and $\mathscr{G}_{p,k}$
$\mathrm{dens}(M)$	the density character of a metric space M
$\mathrm{diam}(A)$	diameter of A
$\mathrm{dom}(f)$	domain of f
Δ_Φ	convex difference of Φ
$\mathrm{Dz}(C, \varepsilon)$	dentability index of a set C
$\mathrm{Dz}(F, \varepsilon)$	dentability index of a function F
$\mathrm{epi}(f)$	epigraph of f
$\exp_Z(C)$	set of Z-exposed points of C
$\mathrm{ext}(C)$	set of Z-exposed points of C
$\mathbb{F}_2^n$	Hamming cubes
F_f	cumulative distribution associated to the distribution μ_f of f

$\mathcal{F}(M)$	Lipschitz-free space over M
$\gamma(A)$	measure of weak nomcompactness of A
$\Gamma(A)$	measure of super weak noncompactness of A
$\Gamma(X)$	class of lower semicontinuous convex proper functions on X
$\mathbb{H}(K,\alpha)$	particular class of Hölder functions
$\mathcal{J}$	canonical injection of X into its ultapower $X_{\mathcal{U}}$
$(f \,\square\, g)$	infimal convolution of f and g
$\breve{f}$	closed convex hull of f
f^*	Fenchel transform of f
$\mathscr{G}_{p,k}$	set of simple functions in L^p taking less than k different values
$K^s(X)$	symmetric Kottman constant of X
$\overline{K^s(X)}$	isomorphic symmetric Kottman constant of X
$\mathcal{L}$	family of Lebesgue measurable subsets of $\mathbb{R}$
$\lim_{\mathcal{U},i} x_i$	limit of $(x_i)_{i\in I}$ with respect to $\mathcal{U}$
$\mathrm{Lip}_0(M)$	Banach space of Lipschitz functions on M vanishing at 0
$\mathcal{M}_p(f,A)$	p-th mean of f on A
$\mathcal{M}(S)$	space of Radon measures on a Polish space S
$\mathfrak{W}^{super}$	ideal of SWC operators
μ_f	distribution of f
$N_{p,\varepsilon}(\mathcal{A})$	measure of UA of A
$\mathcal{N}(M,\varepsilon)$	covering numbers of M
$\mathcal{N}(f,\varepsilon)$	covering numbers of f
p_f	gage of uniform convexity of f
$\mathcal{P}(I)$	power set of I
P_K	metric projection on K
$\mathcal{P}_\infty(I)$ or $[I]^\omega$	family of all infinite subsets of I
ϕ_X	measure of weak-(A_k)
ρ_f	modulus of smoothness of f
$(r_n)_n$	sequence of Rademacher functions
$\lvert s\rvert$	length of a finite sequence s of $\{0,1\}$
$s \frown s'$	concatenation of two finite sequences s and s' of $\{0,1\}$
$S(C,x^*,\alpha)$	slice of a set C
$\mathrm{span}(A)$	vector space generated by A
$S(A,x^*,\alpha)$	slice of A with parameters $x^*\in X^*$ and $\alpha>0$
$\text{str-exp}_Z(C)$	set of Z-strongly exposed points of C
$\text{str-ext}(C)$	set of strongly extreme points of C
$\mathfrak{s}$	splitting cardinal
T^*	adjoint operator of T
$T_{\mathcal{U}}$	ultraproduct of an operator T with respect to $\mathcal{U}$
$\mathrm{UWBS}(X)$	UWBS-index of X
$var_p(f,A)$	p-variation of f in A
$var_p(f,\mathcal{P})$	total p-variation of f in $\mathcal{P}$
$\mathrm{Var}_{p,k}(f,A)$	k-th variation of f in A

$\mathrm{Var}_{p,k}(f)$	total k-th variation of f in A
ω	first infinite countable ordinal
$\omega(A)$	De Blasi's measure of weak noncompactness of A
ω_1	first uncountable ordinal
$X_{\mathcal{U}}$	ultraproduct of a Banach space X with respect to $\mathcal{U}$
$[x]$	integer part of x