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Inequalities for the lattice point enumerator

Desigualdades para el enumerador de puntos del retículo

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Contents

Resumen

Este trabajo está dedicado al estudio de desigualdades geométricas discretas y, en particular, a la discretización de desigualdades en el campo de la Geometría Convexa. En este contexto, por discretización hacemos referencia al proceso de obtención de análogos de desigualdades ya conocidas en Geometría Convexa, preservando la estructura y cualidades originales en la medida de lo posible, en un marco en el que el espacio ambiente, las medidas involucradas, los objetos de estudio, los funcionales implicados, o cualquier combinación de éstos, son discretos. Conviene aclarar que, en el campo de la Geometría Discreta, "discreto" se entiende a menudo en un sentido más restrictivo que el topológico habitual. En concreto, diremos que un conjunto es discreto si existe una distancia mínima universal entre cualquier pareja de puntos del conjunto.

Por razones operacionales, habitualmente es necesario asumir una mayor estructura aparte de la discreción del espacio ambiente. En particular, es común considerar que el espacio sea también un grupo aditivo. Un conjunto verificando las dos propiedades anteriores es un "retículo". Trabajaremos con retículos embebidos en $\mathbb{R}^n$, y como dichos retículos son precisamente las imágenes lineales del retículo entero $\mathbb{Z}^n$, la mayor parte del análisis en este trabajo se centrará en el mismo, indicando ocasionalmente cómo adaptar los resultados al caso general de retículos arbitrarios.

De todos los posibles marcos de trabajo en el contexto anterior, quizá los dos más comunes sean el estudio de subconjuntos finitos de $\mathbb{Z}^n$ con la medida de cardinalidad $|\cdot|$, y el estudio de subconjuntos acotados de $\mathbb{R}^n$ con la medida dada por el "enumerador de puntos del retículo" $\mathrm{G_n}(\cdot) = |\cdot \cap \mathbb{Z}^n|$. Nuestro trabajo se centra principalmente en éste último, aunque el primero cobrará especial relevancia en el capítulo 2.

Además de preservar la estructura general de las desigualdades durante el proceso de discretización, procuraremos a menudo obtener la propiedad adicional de que las nuevas desigualdades permitan recuperar las originales en el contexto continuo. Esto, aparte de proveer un indicador general de la calidad de la discretización, proporciona un potencial método alternativo de progresar en el contexto continuo. Para este propósito el enumerador de puntos del retículo es particularmente apropiado, ya que tiene un mejor comportamiento con respecto a la dilatación de conjuntos, y buena parte de los métodos para obtener versiones continuas de desigualdades a partir de análogos discretos derivan de argumentos estándar de integración Riemann, o de otras aproximaciones de teoría de la medida, que pasan precisamente por

dilatar los conjuntos implicados y tomar límites. En cualquier caso, en ocasiones esto también puede llevarse a cabo únicamente con la cardinalidad.

La estructura general de la tesis es la siguiente. Por un lado, los primeros tres capítulos se ocupan de la discretización de tres conocidas desigualdades en Geometría Convexa, a saber, la desigualdad de Brunn-Minkowski, la desigualdad isoperimétrica y la desigualdad de Rogers-Shephard, respectivamente. El último capítulo, por otro lado, parte ya de un origen discreto, el segundo teorema de Minkowski, aunque también puede entenderse como un proyecto de discretización, dado que está dedicado a conectar el funcional volumen con el enumerador de puntos del retículo mediante los mínimos sucesivos de Minkowski (véase la definición 4.1), así como a obtener un análogo del segundo teorema de Minkowski para este último funcional.

La estrategia del proceso de discretización empleado en los tres primeros capítulos fue introducida en varios artículos recientes, incluyendo [63, 69], dedicados a la desigualdad de Brunn-Minkowski. Esta desigualdad afirma que para cualesquiera conjuntos compactos no vacíos $K,L\subset\mathbb{R}^n$ (aunque el resultado es cierto para familias más generales de conjuntos medibles) se tiene que

$$\mathrm{vol}\big((1-\lambda)K+\lambda L\big)^{1/n}\geq(1-\lambda)\,\mathrm{vol}(K)^{1/n}+\lambda\,\mathrm{vol}(L)^{1/n}.$$

Es fácil ver que los evidentes candidatos discretos

$$\big|(1-\lambda)A+\lambda B\big|^{1/n}\geq|A|^{1/n}+|B|^{1/n},$$

para conjuntos finitos $A,B\subset\mathbb{Z}^n$, y

$$\mathrm{G_n}\big((1-\lambda)K+\lambda L\big)^{1/n}\geq(1-\lambda)\,\mathrm{G_n}(K)^{1/n}+\lambda\,\mathrm{G_n}(L)^{1/n},$$

para conjuntos acotados no vacíos $K,L\subset\mathbb{R}^n$, no son ciertos en general. Por tanto, si uno quiere mantener intacta la estructura de la desigualdad, es preciso alterar los conjuntos implicados. Los autores de los artículos anteriormente mencionados mostraron que, extendiendo los conjuntos de la cota superior de forma apropiada, las desigualdades podían cumplirse en general. En particular, en [69] se demostró que para todo par de conjuntos acotados no vacíos $K,L\subset\mathbb{R}^n$, y cualquier $\lambda\in(0,1)$, se tiene

$$\mathrm{G_n}\big((1-\lambda)K+\lambda L+(-1,1)^n\big)^{1/n}\geq(1-\lambda)\,\mathrm{G_n}(K)^{1/n}+\lambda\,\mathrm{G_n}(L)^{1/n}.$$

La desigualdad es ajustada, y el cubo sumado no puede reducirse. Asimismo, implica la desigualdad de Brunn-Minkowski clásica. Esta idea, y algunos de los principios subyacentes, constituyen los cimientos de gran parte de los resultados de discretización contenidos en los tres primeros capítulos.

En el capítulo 1 comenzamos obteniendo una versión de la última desigualdad para coeficientes positivos arbitrarios. En concreto, en el teorema 1.2 mostramos que

$$\mathrm{G_n}\big(tK+sL+(-1,\lceil t+s\rceil)^n\big)^{1/n}\geq t\,\mathrm{G_n}(K)^{1/n}+s\,\mathrm{G_n}(L)^{1/n}$$

para todo $t,s\geq0$ y cualesquiera conjuntos acotados no vacíos $K,L\subset\mathbb{R}^n$ con $\mathrm{G_n}(K)\,\mathrm{G_n}(L)>0$. Esta extensión, que en el contexto continuo se sigue de forma trivial de la homogeneidad de orden n del funcional volumen, ha de ser probada independientemente en el caso discreto, ya que el enumerador de puntos del retículo no es homogéneo. La prueba, no obstante, se basa en las mismas ideas. En concreto, primero se obtiene una versión discreta de una desigualdad funcional más general, conocida como la "desigualdad de Borell-Brascamp-Lieb" (véase el teorema 1.3). Particularizando esta desigualdad con las funciones apropiadas (específicamente, con las funciones características de los conjuntos involucrados), se desprende el resultado buscado. La extensión de conjuntos mediante la suma de cubos en el marco geométrico se traslada de forma natural al marco funcional definiendo una extensión particular de funciones por medio del producto de Asplund (véase (4)).

Tras esto, extendemos los análogos discretos anteriores al marco L_p, para un parámetro $p \geq 1$. Este marco, en el campo de la Geometría Convexa, fue introducido por Firey en los años 60 al definir una extensión apropiada de la suma de Minkowski, la p-suma $+_p$ (así como el producto p-escalar $\cdot$). Esta operación estaba originalmente restringida a los conjuntos compactos y convexos conteniendo al origen, sin embargo, más tarde fue generalizada por diversos autores a conjuntos acotados no vacíos arbitrarios. En este contexto, se verifica que para toda pareja de conjuntos compactos no vacíos $K, L \subset \mathbb{R}^n$, todo $\lambda \in (0,1)$ y cualquier $p \geq 1$,

$$\mathrm{vol}\big((1-\lambda)\cdot K +_p \lambda \cdot L\big)^{p/n} \geq (1-\lambda)\,\mathrm{vol}(K)^{p/n} + \lambda\,\mathrm{vol}(L)^{p/n}.$$

Aquí demostramos en el teorema 1.10 que, de modo análogo, para todo λ y p en las mismas condiciones, y para cualesquiera conjuntos acotados no vacíos $K, L \subset \mathbb{R}^n$ con $\mathrm{G_n}(K)\,\mathrm{G_n}(L) > 0$,

$$\mathrm{G_n}\big((1-\lambda)\cdot K +_p \lambda \cdot L + (-1,1)^n\big)^{p/n} \geq (1-\lambda)\,\mathrm{G_n}(K)^{p/n} + \lambda\,\mathrm{G_n}(L)^{p/n}.$$

Como antes, el cubo no puede ser reducido (ni tampoco p-sumado, véase la observación 1.15), la desigualdad es óptima, e implica su análogo continuo para el volumen (véase el teorema 1.19). El enfoque consiste nuevamente en obtener primero una versión funcional discreta más fuerte, en concreto, una desigualdad discreta de tipo L_p Borell-Brascamp-Lieb (véase el teorema 1.12), la cual implica su correspondiente versión continua (véase el teorema 1.18), para luego particularizarla.

Concluimos el capítulo estudiando el marco anterior para valores del parámetro p entre 0 y 1. En este contexto, la definición de la p-suma ha de ser nuevamente ajustada, esta vez, por medio de la llamada "forma de Wulff" de una función (véase (3)). Para $p = 0$ en particular, esto da lugar a un problema conocido habitualmente como la "conjetura log-Brunn-Minkowski" (véase [26]), que afirma que para todo $\lambda \in (0,1)$ y para cualesquiera conjuntos compactos, convexos y simétricos respecto del origen $K, L \subset \mathbb{R}^n$ se tiene que

$$\mathrm{vol}\big((1-\lambda)\cdot K +_0 \lambda \cdot L\big) \geq \mathrm{vol}(K)^{1-\lambda}\,\mathrm{vol}(L)^{\lambda}.$$

La conjetura ha sido demostrada en varios escenarios, incluyendo el caso plano en el artículo original [26], y el caso en el que los conjuntos implicados son incondicionales (véase [95], así como [78] para el resultado general con $0 < p < 1$). Aquí probamos que, en los mismos escenarios y con las mismas hipótesis (véase el teorema 1.20),

$$\mathrm{G_n}\left((1-\lambda)\cdot\big(K + C_n\big) +_0 \lambda \cdot \big(L + C_n\big) + \left(-\frac{1}{2}, \frac{1}{2}\right)^n\right) \geq \mathrm{G_n}(K)^{1-\lambda}\,\mathrm{G_n}(L)^{\lambda},$$

donde $C_n = [-1/2, 1/2]^n$. En esta ocasión la aproximación se desvía de la explotada en secciones anteriores, y en su lugar, surge directamente de relaciones entre el volumen y el enumerador de puntos del retículo. En cualquier caso, los cubos involucrados nuevamente no pueden reducirse (véase la observación 1.22), y la desigualdad implica su versión continua por argumentos similares a los anteriores.

El caso más general con $0 < p < 1$ también se estudia en está sección, obteniendo resultados análogos (véase el teorema 1.23). Finalmente, consideramos enumeradores de puntos y medidas alternativas para las cuales pueden obtenerse resultados de similar índole, en el espíritu de la desigualdad discreta de log-Brunn-Minkowski (véanse la proposición 1.25, el corolario 1.27 y los teoremas 1.28 y 1.29).

En el capítulo 2 nos centramos en la desigualdad isoperimétrica, quizás uno de los resultados geométricos más clásicos, y adicionalmente hoy día, una de las primeras consecuencias de la desigualdad de Brunn-Minkowski. Este resultado, en el contexto continuo habitual, sostiene que el área de superficie $\mathrm{S}(\cdot)$ de un conjunto compacto y convexo de $\mathbb{R}^n$ se minimiza, entre aquéllos con igual volumen, cuando el conjunto es una bola euclídea. Alternativamente, el funcional $\mathrm{S}(\cdot)^n/\,\mathrm{vol}(\cdot)^{n-1}$ es minimizado por las bolas euclídeas.

Una formulación equivalente de la desigualdad isoperimétrica es su versión para paralelos exteriores, a saber, que las bolas euclídeas minimizan el funcional $\mathrm{vol}(K+t\mathrm{B}_n)$ para todo $t \geq 0$ entre los conjuntos con igual volumen, donde B_n es la bola unidad euclídea cerrada. Esta versión tiene la ventaja de involucrar únicamente una medida, y en consecuencia, es más apropiada a la hora de extender la desigualdad isoperimétrica a otros espacios. En el contexto discreto, Radcliffe y Veomett demostraron en [86] que el funcional $\left|\cdot+\{-1,0,1\}^n\right|$, en la familia de subconjuntos de $\mathbb{Z}^n$ con cardinalidad prescrita, es minimizado por los segmentos iniciales de un orden de $\mathbb{Z}^n$ adecuadamente definido, es decir, por los conjuntos que consisten en los primeros r puntos de $\mathbb{Z}^n$ en este orden, para $r \in \mathbb{N}\setminus\{0\}$. También extendieron este resultado a $\mathbb{N}^n$. Los autores ya observaron, no obstante, que estos segmentos iniciales no son los únicos conjuntos minimizantes de dicho funcional, y es de hecho sencillo encontrar contraejemplos.

Aquí extendemos este estudio en dos direcciones. En primer lugar, obtenemos una caracterización de la igualdad bajo condiciones específicas. En concreto, demostramos que cuando la cardinalidad fijada es de la forma ρ^n para algún $\rho \in \mathbb{N}\setminus\{0\}$, los correspondientes segmentos iniciales, que son cubos reticulares estándar, son de hecho los únicos conjuntos que minimizan el funcional $\left|\cdot+\{0,\dots,s\}^n\right|$ para todo $s \in \mathbb{N}\setminus\{0\}$ (véase el teorema 2.16), y por tanto, en particular, de $\left|\cdot+\{-1,0,1\}^n\right|$ (debido a la invarianza por traslaciones enteras, véanse los teoremas 2.14 y 2.15). Este resultado de caracterización se obtiene mediante un proceso de reducción que llamamos “normalización” (véase la definición 2.27), ya que, como mostraremos, es suficiente probar la caracterización para conjuntos que son invariantes bajo este proceso (véanse el corolario 2.32 y el lema 2.33). Esta caracterización nos permitirá adicionalmente caracterizar el caso de igualdad en otras desigualdades discretas preestablecidas, incluyendo una desigualdad de tipo Brunn-Minkowski probada en [69] (véase el teorema 2.34).

En segundo lugar, obtenemos una desigualdad en el mismo espíritu para el enumerador de puntos del retículo. En particular, encontramos conjuntos que minimizan el funcional $\mathrm{G_n}\big(K+t[0,1]^n\big)$ para todo $t>0$ en la familia de conjuntos acotados no vacíos con $\mathrm{G_n}(K)>0$ fijado (véase el teorema 2.37). Asimismo, obtenemos un resultado de caracterización de la igualdad cuando $\mathrm{G_n}(K)=(\rho+1)^n$ para algún $\rho\in\mathbb{N}$. Esta última propiedad se extiende a retículos arbitrarios (véase la observación 2.40), lo cual resulta útil para recuperar la correspondiente versión para paralelos exteriores de la desigualdad isoperimétrica para el volumen (véase el corolario 2.44).

En el capítulo 3 estudiamos la desigualdad de Rogers-Shephard y otras relacionadas. Esta desigualdad provee de una cota superior para el volumen del cuerpo diferencia $K-K$ de un conjunto compacto y convexo $K\subset\mathbb{R}^n$ (la correspondiente cota inferior se deriva de la desigualdad de Brunn-Minkowski). En concreto, se tiene que

$$\mathrm{vol}(K-K)\leq\binom{2n}{n}\mathrm{vol}(K).$$

Esta desigualdad puede extenderse con el fin de involucrar a dos conjuntos del siguiente modo:

$$\mathrm{vol}(K+L)\,\mathrm{vol}\big(K\cap(-L)\big)\leq\binom{2n}{n}\mathrm{vol}(K)\,\mathrm{vol}(L).$$

Aquí obtenemos análogos discretos de las desigualdades anteriores para el enumerador de puntos del retículo mediante diversas aproximaciones. Por un lado, relacionando el volumen con el enumerador de puntos del retículo directamente podemos obtener las cotas

$$\mathrm{G}_n(K-K)\leq\binom{2n}{n}\mathrm{G}_n\left(K+\left(-\frac{3}{4},\frac{3}{4}\right)^n\right)$$

y

$$\mathrm{G}_n(K+L)\,\mathrm{G}_n\big(K\cap(-L)\big)\leq\binom{2n}{n}\mathrm{G}_n\big(K+(-1,1)^n\big)\,\mathrm{G}_n\big(L+(-1,1)^n\big)$$

(véanse los teoremas 3.6 y 3.7). Por otro lado, una versión discreta del enfoque original de Rogers y Shephard basado en el “covariograma” nos permite obtener unas desigualdades adicionales de tipo Rogers-Shephard (véase el teorema 3.5) que, de hecho, no son comparables con las anteriores.

Análogamente, estudiamos también una versión discreta de una conocida desigualdad de tipo “proyección-sección” obtenida igualmente por Rogers y Shephard en [89]. Esta desigualdad proporciona una cota superior para el producto de los volúmenes de una sección y una proyección de un conjunto compacto y convexo con respecto a subespacios lineales de $\mathbb{R}^n$. En concreto, demostraron que para cualquier subespacio lineal $H \subset \mathbb{R}^n$ de dimensión $k \in \{1,\dots,n-1\}$, y para todo conjunto compacto y convexo $K \subset \mathbb{R}^n$, se tiene

$$\mathrm{vol}_{n-k}(P_{H^\perp}K)\,\mathrm{vol}_k(K\cap H) \leq \binom{n}{k}\mathrm{vol}(K),$$

donde P_H denota la proyección ortogonal sobre H y vol_k es el volumen k-dimensional. En este trabajo probamos que, bajo las mismas hipótesis,

$$\mathrm{G}_{n-k}(P_{H^\perp}K)\,\mathrm{G}_k(K\cap H) \leq \binom{n}{k}\mathrm{G}_n\big(K+(-1,1)^n\big)$$

(véase el teorema 3.1), donde G_k es el enumerador de puntos del retículo restringido a un subespacio lineal de dimensión k. Este resultado requiere la asunción adicional de que el subespacio lineal sea coordenado.

Finalmente, también obtenemos un análogo discreto de la desigualdad de Berwald (véase el teorema 3.10), que, en el contexto continuo, proporciona una relación de monotonía entre ciertos momentos normalizados de funciones cóncavas (véase el teorema L). Este resultado es particularmente relevante en nuestro estudio porque procura una demostración unificada de las dos desigualdades para el volumen anteriormente mencionadas, simplemente particularizándola con funciones específicas (véase la discusión de la página 64). En el contexto discreto, no obstante, esta desigualdad no permitirá recuperar las nuevas desigualdades mencionadas arriba, y en su lugar, obtendremos otras alternativas que combinan el volumen con el enumerador de puntos del retículo (véanse los corolarios 3.14 y 3.15).

Al igual que en los capítulos anteriores, probaremos que las nuevas desigualdades discretas obtenidas permiten recuperar las correspondientes desigualdades para el volumen, tanto en el caso de las desigualdades de Rogers-Shephard y de tipo “proyección-sección” (véase el teorema 3.17) como en el caso de la desigualdad de Berwald (véase el teorema 3.19).

Para concluir, el capítulo 4 está dedicado a estudiar relaciones directas entre el volumen y el enumerador de puntos del retículo. La motivación principal es obtener un análogo discreto del segundo teorema de Minkowski (véase el teorema N), que proporciona las cotas

$$\frac{1}{n!}\prod_{i=1}^{n}\frac{2}{\lambda_i(K)} \leq \mathrm{vol}(K) \leq \prod_{i=1}^{n}\frac{2}{\lambda_i(K)},$$

donde $K \subset \mathbb{R}^n$ es un conjunto compacto y convexo y $\lambda_i(K)$ es el i-ésimo mínimo sucesivo de K, $i=1,\dots,n$, es decir, el menor número positivo tal que la correspondiente dilatación del simetrizado central $(K-K)/2$ contiene al menos i puntos linealmente independientes de $\mathbb{Z}^n$.

El enfoque deriva de una serie de conjeturas planteadas por Betke, Henk y Wills (véanse las conjeturas 4.2 y 4.3) que, de ser ciertas, implicarían una versión de la desigualdad anteriormente mencionada para el enumerador de puntos del retículo. En este trabajo demostramos una versión algo más débil de esta última conjetura, en concreto, probamos las cotas

$$\mathrm{G_n}(K) \leq \mathrm{vol}(K)\prod_{i=1}^{n}\left(1+\frac{n\lambda_i(K)}{2}\right)$$

y, si $\lambda_n(K) \leq 2/n$,

$$\mathrm{G_n}(\mathrm{int}\,K) \geq \mathrm{vol}(K)\prod_{i=1}^{n}\left(1-\frac{n\lambda_i(K)}{2}\right),$$

(véase el teorema 4.11). En el teorema 4.15 mejoramos estas cotas en el plano, confirmando (al menos asintóticamente) la conjetura original. Estas desigualdades, además de recuperar cotas clásicas de Blichfeldt, van der Corput, Minkowski y otros, permiten también obtener que

$$\mathrm{G_n}(K) \leq \prod_{i=1}^{n} \left(\frac{2}{\lambda_i(K)} + n \right)$$

(véase el corolario 4.12), es decir, un análogo discreto de la cota superior del segundo teorema de Minkowski (una cota inferior óptima ya fue probada por Betke, Henk y Wills).

La estrategia fundamental consiste en reducir el estudio a una clase mucho más limitada de conjuntos, los llamados "cuerpos convexos anti-blocking" (véanse la definición 4.8 y la proposición 4.9). Esta reducción se lleva a cabo mediante una transformación conocida como el "Blaschke shaking" (véase la definición 4.6), un proceso que guarda semejanza con la simetrización de Steiner y que pertenece a una clase más amplia de transformaciones denominadas "shakings". A fin de demostrar los resultados más fuertes en el plano, generalizamos este procedimiento al caso no ortogonal (véase la definición 4.17). Otra herramienta importante que desarrollamos es una desigualdad de Davenport-Schymura inversa (véase el teorema 4.21).

Los resultados originales de esta tesis están recogidos en los trabajos [9, 43, 64, 65, 66, 67].

Abstract

This work is concerned with the study of discrete geometric inequalities and, in particular, with the discretization of inequalities in the field of Convex Geometry. In this context, by discretization we refer to the process of obtaining analogues of already known inequalities in Convex Geometry, preserving as much of the original structure and qualities as possible, in a setting where the ambient space, the measures involved, the objects of study, the implicated functionals, or any combination of these, are discrete. It is worth clarifying that, in the field of Discrete Geometry, "discrete" is often understood in a sense stronger than the standard topological one. Namely, we will say that a set is discrete if there exists a minimum universal distance between any two points of the set.

For operational purposes it is often necessary to enforce additional structure on top of the discreteness of the space, and in particular, it is rather common to consider the space to also be an additive group. A set verifying the two properties above is called a "lattice". We will consider lattices embedded in $\mathbb{R}^n$, and since any such lattice is just a linear image of the integer lattice $\mathbb{Z}^n$ and viceversa, most of the analysis in this work will be restricted to $\mathbb{Z}^n$, with eventual indications of how the results can be adapted to the setting of arbitrary lattices.

Among all the possible frameworks of choice in the above setting, perhaps two of the most common ones are the study of finite subsets of $\mathbb{Z}^n$ together with the cardinality measure $|\cdot|$, and the study of bounded subsets of $\mathbb{R}^n$ together with the "lattice point enumerator" measure $\mathrm{G_n}(\cdot) = |\cdot \cap \mathbb{Z}^n|$. Our work is primarily focused on the latter, although the former will also play an important role, specifically, in Chapter 2.

On top of preserving the general structure of the inequalities during the process of discretization, we will often strive for the additional property that the new inequalities allow one to retrieve the original ones in the continuous setting. This, apart from serving as an overall indicator of how fit the discrete version is, also provides a potential new method for furthering progress in the continuous field. For this purpose the lattice point enumerator is particularly well suited, since it is better behaved than the cardinality with respect to set dilations, and a vast number of methods to retrieve the continuous inequalities from the discrete ones are derived from standard Riemann integration arguments, or other similar measure-theoretic approaches that involve progressively dilating sets and taking limits. Nonetheless, this can at times be achieved with the cardinality alone as well.

The overall structure of the thesis is the following one. On the one hand, the first three chapters deal with the discretization of three well-known inequalities in Convex Geometry, namely, the Brunn-Minkowski inequality, the isoperimetric inequality, and the Rogers-Shephard inequality, respectively. The fourth chapter, on the other hand, already starts from a discrete source, since it stems from Minkowski's Second Theorem, although it can also be understood as a discretization effort, since it deals with connecting the volume functional with the lattice point enumerator via Minkowski's successive minima (see Definition 4.1), and obtaining an analogue of Minkowski's Second Theorem for the latter functional.

The strategy for the discretization processes employed in the first three chapters was introduced in several recent papers, including [63, 69], that dealt with the Brunn-Minkowski inequality. This classical inequality states that for any non-empty compact sets $K,L\subset\mathbb{R}^n$ (although the result holds true for more general measurable sets) one has

$$\mathrm{vol}\big((1-\lambda)K+\lambda L\big)^{1/n}\geq(1-\lambda)\,\mathrm{vol}(K)^{1/n}+\lambda\,\mathrm{vol}(L)^{1/n}.$$

It is easy to see that the obvious potential discrete analogues

$$\big|(1-\lambda)A+\lambda B\big|^{1/n}\geq|A|^{1/n}+|B|^{1/n},$$

for finite sets $A,B\subset\mathbb{Z}^n$, and

$$\mathrm{G_n}\big((1-\lambda)K+\lambda L\big)^{1/n}\geq(1-\lambda)\,\mathrm{G_n}(K)^{1/n}+\lambda\,\mathrm{G_n}(L)^{1/n},$$

for non-empty bounded sets $K,L\subset\mathbb{R}^n$, do not hold in general. Therefore, should one want to maintain the structure of the inequality untouched, the sets involved would have to be altered. The authors of the aforementioned papers showed that, by suitably extending the sets on the left-hand side, the inequalities could be made to hold. In particular, in [69] it was proved that for any pair of non-empty bounded sets $K,L\subset\mathbb{R}^n$, and any $\lambda\in(0,1)$, one has

$$\mathrm{G_n}\big((1-\lambda)K+\lambda L+(-1,1)^n\big)^{1/n}\geq(1-\lambda)\,\mathrm{G_n}(K)^{1/n}+\lambda\,\mathrm{G_n}(L)^{1/n}.$$

This inequality is tight, and the cube being added cannot be reduced. Furthermore, it implies the classical Brunn-Minkowski inequality. This idea, and some of the underlying principles, are the foundation of many of the discretization results of the first three chapters.

In Chapter 1 we begin by obtaining a version of the last inequality for arbitrary positive coefficients. Namely, we show in Theorem 1.2 that

$$\mathrm{G_n}\big(tK+sL+(-1,\lceil t+s\rceil)^n\big)^{1/n}\geq t\,\mathrm{G_n}(K)^{1/n}+s\,\mathrm{G_n}(L)^{1/n}$$

for any $t,s\geq0$ and any non-empty bounded sets $K,L\subset\mathbb{R}^n$ with $\mathrm{G_n}(K)\,\mathrm{G_n}(L)>0$. This, which in the continuous setting follows trivially from the homogeneity of degree n of the volume functional, needs to be proved independently in the discrete setting, since the lattice point enumerator is not homogeneous. The proof, nonetheless, follows the same ideas. In particular, a discrete version of a more general inequality, known as the "Borell-Brascamp-Lieb" inequality, is obtained first (see Theorem 1.3). Particularizing this inequality for suitable functions (specifically, for the characteristic function of the sets involved), then yields the desired result. The extension of the sets by adding cubes in the geometric inequalities gets translated in a natural way to this functional setting by defining a particular extension of a function via the Asplund product (see (4)).

We follow this with an extension of the above discrete analogues to the L_p setting, for a parameter $p\geq1$. This setting, in the field of Convex Geometry, was introduced by Firey in the 60's by defining a suitable extension of the Minkowski addition, the p-addition $+_p$ (as well as a p-scalar product $\cdot$). This was originally restricted to the setting of compact convex sets containing the origin, although it was later extended by various authors to arbitrary non-empty bounded sets. In this context, it holds that for any pair of non-empty compact sets $K,L\subset\mathbb{R}^n$, any $\lambda\in(0,1)$ and any $p\geq 1$,

$$\mathrm{vol}\big((1-\lambda)\cdot K+_p\lambda\cdot L\big)^{p/n}\geq(1-\lambda)\,\mathrm{vol}(K)^{p/n}+\lambda\,\mathrm{vol}(L)^{p/n}.$$

Here, we show in Theorem 1.10 that, accordingly, for any such λ and p, and for any non-empty bounded sets $K,L\subset\mathbb{R}^n$ with $\mathrm{G_n}(K)\,\mathrm{G_n}(L)>0$,

$$\mathrm{G_n}\big((1-\lambda)\cdot K+_p\lambda\cdot L+(-1,1)^n\big)^{p/n}\geq(1-\lambda)\,\mathrm{G_n}(K)^{p/n}+\lambda\,\mathrm{G_n}(L)^{p/n}.$$

As before, the cube cannot be reduced (nor substituted by its p-addition, see Remark 1.15), the inequality is sharp, and it implies the continuous one for the volume (see Theorem 1.19). The approach is again to obtain a stronger discrete functional inequality first, an L_p Borell-Brascamp-Lieb type inequality (see Theorem 1.12), which implies its corresponding continuous version (see Theorem 1.18), and then particularize it.

We finish the chapter by studying the setting above for values of the parameter p between 0 and 1. In this setting, the definition of the p-addition needs to be adjusted again, by means of the "Wulff shape" of a function (see (3)). For $p=0$ in particular, this leads to a problem often referred to as the "log-Brunn-Minkowski conjecture" (see [26]), which states that for any $\lambda\in(0,1)$ and any origin-symmetric compact convex sets $K,L\subset\mathbb{R}^n$,

$$\mathrm{vol}\big((1-\lambda)\cdot K+_0\lambda\cdot L\big)\geq\mathrm{vol}(K)^{1-\lambda}\,\mathrm{vol}(L)^{\lambda}.$$

This conjecture has been proved in several scenarios, including the planar case in the original paper [26], as well as when the sets involved are unconditional (see [95], as well as [78] for the general $0<p<1$ result). Here, we show that, in the same scenarios and with the same hypothesis (see Theorem 1.20),

$$\mathrm{G_n}\left((1-\lambda)\cdot\big(K+C_n\big)+_0\lambda\cdot\big(L+C_n\big)+\Big(-\frac{1}{2},\frac{1}{2}\Big)^n\right)\geq\mathrm{G_n}(K)^{1-\lambda}\,\mathrm{G_n}(L)^{\lambda},$$

where $C_n=[-1/2,1/2]^n$. The approach this time diverts from the one exploited in the previous sections, and instead arises directly from relations between the volume and the lattice point enumerator. Nevertheless, the cubes involved again cannot be reduced (see Remark 1.22), and the inequality implies the continuous one by similar arguments as before.

The more general $0<p<1$ setting is also studied, obtaining analogous results (see Theorem 1.23). Finally, we consider some alternative point enumerators and discrete measures for which results in a similar spirit to the discrete log-Brunn-Minkowski one can also be shown (see Proposition 1.25, Corollary 1.27, Theorem 1.28 and Theorem 1.29).

In Chapter 2 we shift our focus to the isoperimetric inequality, possibly one of the most classical geometric results, and additionally nowadays, one of the first consequences of the Brunn-Minkowski inequality. This result, in the continuous setting, states that the surface area $\mathrm{S}(\cdot)$ of a compact convex set of $\mathbb{R}^n$ is minimized, among the family of those with the same volume, when the set is an Euclidean ball. Alternatively, the functional $\mathrm{S}(\cdot)^n/\mathrm{vol}(\cdot)^{n-1}$ is minimized by the Euclidean balls.

An equivalent formulation of the isoperimetric inequality is the so-called "neighbourhood form", which states that Euclidean balls minimize the functional $\mathrm{vol}(K+t\mathrm{B}_n)$ for all $t>0$ among the family of sets with prescribed volume, where B_n is the closed unit Euclidean ball. This form has the advantage of only involving a single measure, and is thus more suitable to be extended to other spaces. In the discrete setting, Radcliffe and Veomett showed in [86] that the functional $\left|\cdot+\{-1,0,1\}^n\right|$ is minimized, among the subsets of $\mathbb{Z}^n$ with prescribed cardinality, by the initial segments in a suitable order, i.e., by the sets consisting of the first r points of $\mathbb{Z}^n$ in this order, for $r\in\mathbb{N}\setminus\{0\}$. They also extended this result to $\mathbb{N}^n$. The authors already pointed out, however, that these initial segments are not the only minimizing sets, and the counterexamples are indeed easy to find.

Here, we further this study in two directions. First, we obtain a characterization of the equality under special conditions. Namely, we show that when the cardinality is ρ^n, for some $\rho \in \mathbb{N} \setminus \{0\}$, then the corresponding initial segments, which are standard lattice cubes, are in fact characterized as being the only minimizers of the functional $\left|\cdot + \{0,\dots,s\}^n\right|$ for any $s \in \mathbb{N}\setminus\{0\}$ (see Theorem 2.16), and thus, in particular, of $\left|\cdot + \{-1,0,1\}^n\right|$ (due to the invariance by integer translations, see Theorems 2.14 and 2.15). This characterization result is obtained via a reduction process we denote by "normalization" (see Definition 2.27), since, as we will show, it suffices to obtain the characterization for sets that are invariant under this process (see Corollary 2.32 and Lemma 2.33). The characterization result will further allow us to characterize the equality case in other pre-established discrete inequalities, including a Brunn-Minkowski type one proved in [69] (see Theorem 2.34).

Second, we obtain an inequality in the same spirit for the lattice point enumerator. Namely, we obtain minimizers for the functional $\mathrm{G_n}\big(K+t[0,1]^n\big)$ for all $t>0$ among the family of non-empty bounded sets $K\subset\mathbb{R}^n$ with $\mathrm{G_n}(K)>0$ fixed (see Theorem 2.37), and also show a characterization result when $\mathrm{G_n}(K)=(\rho+1)^n$ for some $\rho\in\mathbb{N}$. This latter result can again be extended to arbitrary lattices (see Remark 2.40), and is then utilized to retrieve the corresponding neighbourhood form of the isoperimetric inequality for the volume (see Corollary 2.44).

In Chapter 3 we study the Rogers-Shephard inequality. This inequality provides an upper bound for the volume of the difference body $K-K$ of a compact convex set $K\subset\mathbb{R}^n$ (a corresponding lower bound is provided by the Brunn-Minkowski inequality). Namely, one has that

$$\mathrm{vol}(K-K)\leq\binom{2n}{n}\mathrm{vol}(K).$$

This inequality can be extended to involve two different sets as follows:

$$\mathrm{vol}(K+L)\,\mathrm{vol}\big(K\cap(-L)\big)\leq\binom{2n}{n}\mathrm{vol}(K)\,\mathrm{vol}(L).$$

Here, we obtain discrete analogues of the inequalities above for the lattice point enumerator using several different approaches. On the one hand, relating the volume with the lattice point enumerator directly allows us to obtain the bounds

$$\mathrm{G}_n(K-K)\leq\binom{2n}{n}\mathrm{G}_n\left(K+\left(-\frac{3}{4},\frac{3}{4}\right)^n\right)$$

and

$$\mathrm{G}_n(K+L)\,\mathrm{G}_n\big(K\cap(-L)\big)\leq\binom{2n}{n}\mathrm{G}_n\big(K+(-1,1)^n\big)\,\mathrm{G}_n\big(L+(-1,1)^n\big)$$

(see Theorems 3.6 and 3.7). On the other hand, a discrete version of Rogers and Shephard's original "covariogram" approach allows us to get further discrete Rogers-Shephard type inequalities (see Theorem 3.5) which, in fact, are not comparable with the previous ones.

Similarly, we also study a discrete version of a well-known "projection-section" type inequality obtained by Rogers and Shephard in [89]. This inequality provided an upper bound for the product of the volumes of a section and a projection of a compact convex set with respect to a linear subspace of $\mathbb{R}^n$. Namely, they proved that for any linear subspace $H\subset\mathbb{R}^n$ of dimension $k\in\{1,\dots,n-1\}$, and any compact and convex set $K\subset\mathbb{R}^n$, one has

$$\mathrm{vol}_{n-k}(P_{H^\perp}K)\,\mathrm{vol}_k(K\cap H)\leq\binom{n}{k}\mathrm{vol}(K),$$

where P_H denotes the orthogonal projection onto H and vol_k is the k-dimensional volume. Here we show, under the same hypothesis, that

$$\mathrm{G}_{n-k}(P_{H^\perp}K)\,\mathrm{G}_k(K\cap H)\leq\binom{n}{k}\mathrm{G}_n\big(K+(-1,1)^n\big)$$

(see Theorem 3.1), where G_k is the lattice point enumerator restricted to a linear subspace of dimension k. This result requires the additional assumption that the linear subspace be a coordinate one.

Finally, we also obtain a discrete analogue of Berwald's inequality (see Theorem 3.10), which, in the continuous setting, provides a monotonicity relation between certain normalized moments of a function (see Theorem L). This result is particularly relevant in this context since it also yields a unified proof for both of the inequalities for the volume discussed above, by particularizing it with specific functions (see Section 3.2, page 64). In the discrete setting, however, our inequality will not retrieve the discrete ones mentioned above, but rather, some alternative ones (see Corollaries 3.14 and 3.15).

As in the previous chapters, we show that the discrete inequalities proved here allow one to retrieve the corresponding continuous ones for the volume, both in the case of the Rogers-Shephard and the projection-section inequalities (see Theorem 3.17) and in the case of Berwald's inequality (see Theorem 3.19).

To conclude, Chapter 4 is devoted to study direct relations between the volume and the lattice point enumerator. The motivation is to obtain a discrete analogue of Minkowski's Second Theorem (see Theorem N), which provides the bounds

$$\frac{1}{n!}\prod_{i=1}^{n}\frac{2}{\lambda_i(K)} \leq \mathrm{vol}(K) \leq \prod_{i=1}^{n}\frac{2}{\lambda_i(K)},$$

where K is a compact convex set and $\lambda_i(K)$ is the i-th successive minima of K, $i = 1,\dots,n$, i.e., the minimum positive number such that the corresponding dilation of the central symmetral $(K-K)/2$ contains i linearly independent points of $\mathbb{Z}^n$.

The approach stems from a series of conjectures posed by Betke, Henk and Wills (see Conjectures 4.2 and 4.3) which, if proved true, would imply a version of the aforementioned inequality for the lattice point enumerator. We show a weakened version of the latter conjecture, namely, we prove in Theorem 4.11 the bounds

$$\mathrm{G_n}(K) \leq \mathrm{vol}(K)\prod_{i=1}^{n}\left(1+\frac{n\lambda_i(K)}{2}\right)$$

and, if $\lambda_n(K) \leq 2/n$,

$$\mathrm{G_n}(\mathrm{int}\,K) \geq \mathrm{vol}(K)\prod_{i=1}^{n}\left(1-\frac{n\lambda_i(K)}{2}\right).$$

These bounds are improved in the planar case (see Theorem 4.15), confirming (at least asymptotically) the original conjecture. The inequalities, apart from retrieving classical bounds by Blichfeldt, van der Corput, Minkowski and others, also allow us to obtain

$$\mathrm{G_n}(K) \leq \prod_{i=1}^{n}\left(\frac{2}{\lambda_i(K)}+n\right)$$

(see Corollary 4.12), i.e., a discrete analogue of the upper bound in Minkowski's Second Theorem (a sharp lower bound was already shown by Betke, Henk and Wills).

The main strategy is to reduce the study to a much narrower class of sets, the so-called "anti-blocking convex bodies" (see Definition 4.8 and Proposition 4.9). This reduction is done via a transformation known as the "Blaschke shaking" (see Definition 4.6), a process which bares resemblance to Steiner's symmetrization and which belongs to a broader class of transformations called "shakings". In order to prove the stronger planar results we generalize this process to the non-orthogonal setting (see Definition 4.17). Another important tool is a reversal of an inequality of Davenport-Schymura (see Theorem 4.21).

The original results of this thesis are collected in [9, 43, 64, 65, 66, 67].

Background

We will denote the non-negative integers by $\mathbb{N}$. As usual, we will write $\mathbb{R}^n$ to represent the n-dimensional Euclidean space endowed with the standard inner product $\langle\cdot,\cdot\rangle$ and the Euclidean norm $\|\cdot\|$. The n-dimensional integer lattice will be denoted by $\mathbb{Z}^n$. We will further write $\mathbb{N}^n$ to refer to the points in $\mathbb{Z}^n$ with coordinates in $\mathbb{N}$. More generally, an n-dimensional lattice $\Lambda \subset \mathbb{R}^n$ is the set of all integer combinations of n linearly independent vectors $v_1,\dots,v_n$, with the set $\mathcal{B} = \{v_1,\dots,v_n\}$ being called a basis of Λ.

For any set $A \subset \mathbb{R}^n$, the smallest convex set containing A is called the *convex hull* of A, and is denoted by $\operatorname{conv} A$. Similarly, $\operatorname{lin} A$ (resp. $\operatorname{aff} A$) is the *linear* (resp. *affine*) hull of A, i.e., the smallest linear (resp. affine) subspace of $\mathbb{R}^n$ containing A. As usual, $\dim A = \dim(\operatorname{aff} A)$ is the dimension of A. Furthermore, $\operatorname{int} A$, $\operatorname{cl} A$ and $\operatorname{bd} A$ are, respectively, the interior, closure and boundary of A with respect to the Euclidean topology. Additionally, we will denote the set of all k-dimensional linear subspaces of $\mathbb{R}^n$ by $\mathcal{L}^n_k$, $k = 0,\dots,n$. For any $H \in \mathcal{L}^n_k$, the set $H^\perp \in \mathcal{L}^n_{n-k}$ is the orthogonal complement of H, and the operator P_H is the orthogonal projection onto H. For the sake of simplicity, for $u \in \mathbb{R}^n$ we will just write $u^\perp := \big(\operatorname{lin}\{u\}\big)^\perp$.

The Euclidean closed unit ball will be represented by B_n. We will also work with *lattice cubes*, i.e., intersections of cubes with sides parallel to the coordinate hyperplanes and $\mathbb{Z}^n$. For all $i = 1,\dots,n$, the i-th canonical unit vector will by denoted by e_i, and, for the sake of brevity, we will write $\ell_i = \operatorname{lin}\{\mathrm{e}_i\}$ to denote the i-th coordinate line. As usual, $\lfloor x \rfloor$ will represent the floor function of $x \in \mathbb{R}$, i.e., the greatest integer less than or equal to x, and $\lceil x \rceil$ will denote the ceiling function of x, i.e., the least integer greater than or equal to x. For $x, y \in \mathbb{R}^n$, we will write $[x,y]$ to represent the closed segment with endpoints x and y, i.e.,

$$[x,y] = \big\{(1-\lambda)x + \lambda y : 0 \le \lambda \le 1\big\},$$

and analogously, we will set

$$(x,y) = \big\{(1-\lambda)x + \lambda y : 0 < \lambda < 1\big\}$$

for the open segment with endpoints x, y. The semi-open and semi-closed intervals will be denoted accordingly. Moreover, given $v_1,\dots,v_n \in \mathbb{R}^n$, we will write $(v_1 \cdots v_n)$ to represent the square matrix having the v_i's as columns.

For a measurable set $M \subset \mathbb{R}^n$ with $\dim M = k$ for some $k = 0, \dots, n$, $\mathrm{vol}_k(M)$ signifies its k-dimensional Lebesgue measure. In particular, $\mathrm{vol}(M) := \mathrm{vol}_n(M)$ is its standard volume, and when integrating, as usual, $\mathrm{d}x$ will stand for $\mathrm{d}\,\mathrm{vol}(x)$. As discrete counterparts we use $|X|$ to denote the cardinality of any finite set $X \subset \mathbb{R}^n$, together with the lattice point enumerator $\mathrm{G_n}(K) = |K \cap \mathbb{Z}^n|$ for any bounded set $K \subset \mathbb{R}^n$. The next fact will turn out to be crucial throughout the dissertation: it follows from standard measure theory arguments (see Figure 1) that, for any compact convex set $K \subset \mathbb{R}^n$ (and, in fact, for any Jordan measurable set, see e.g. [53, (3), page 120]), the volume and the lattice point enumerator are equivalent "on a big scale", in the sense that

$$\lim_{r \to \infty} \frac{\mathrm{G_n}(rK)}{r^n} = \mathrm{vol}(K). \tag{1}$$

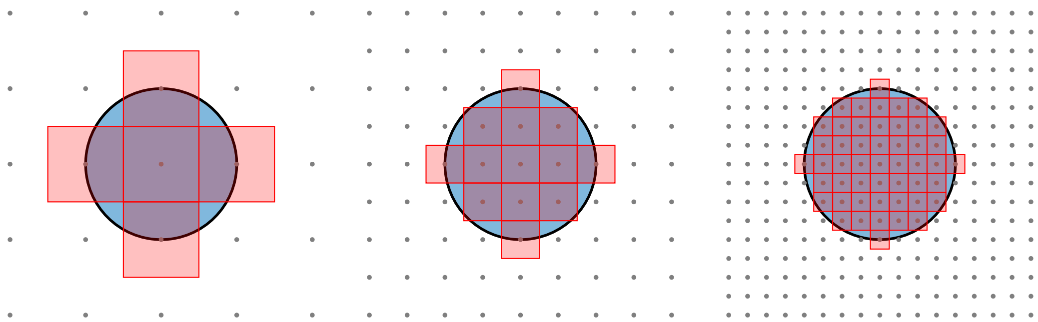

Figure 1: Illustration of (1).

Additionally, $\mathrm{S}(M)$ represents the *surface area* (Minkowski content) of a non-empty compact convex set $M \subset \mathbb{R}^n$, i.e.,

$$\mathrm{S}(M) = \lim_{\varepsilon \to 0^+} \frac{\mathrm{vol}(M + \varepsilon \mathrm{B}_n) - \mathrm{vol}(M)}{\varepsilon}. \tag{2}$$

Given any non-empty set $M \subset \mathbb{R}^n$, any $t \in \mathbb{R}$ and any $i \in \{1, \dots, n\}$, we shall denote by $M^i(t)$ the section of M at height t orthogonal to e_i, i.e.,

$$M^i(t) = \left\{ (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1} : (x_1, \dots, x_{i-1}, t, x_i, \dots, x_{n-1}) \in M \right\}.$$

For the sake of brevity, we will also just use $M(t) = M^n(t)$. Furthermore, for any $t \in \mathbb{R}$ we will write

$$M_{\geq t} = \left\{ (x_1, \dots, x_n) \in M : x_i \geq t,\ i = 1, \dots, n \right\},$$

a notation which will be adapted in the natural sense for other inequality signs.

The *Minkowski addition* of two non-empty sets $A, B \subset \mathbb{R}^n$ is defined as $A + B = \{a + b : a \in A : b \in B\}$. Moreover, λA represents the set $\{\lambda a : a \in A\}$ for any $\lambda \in \mathbb{R}$, and as usual, we write $-A = (-1)A$, that is, the reflection of A at the origin. We will similarly denote by $A - B = A + (-B)$. Furthermore, A is *origin-symmetric* if $-A = A$. In this setting, we have the following standard properties, which are straightforward to verify.

Proposition 1 Let $A, B, C \subset \mathbb{R}^n$ and $\lambda, \mu > 0$. Then:

- $(A \cup B) + C = (A + C) \cup (B + C)$,
- $(A \cap B) + C \subset (A + C) \cap (B + C)$,
- $(A \cup B) + (A \cap B) \subset A + B$,
- $\lambda A + \lambda B = \lambda(A + B)$,
- $\lambda A + \mu A \supset (\lambda + \mu)A$.

A *convex body* is a non-empty compact and convex set $K \subset \mathbb{R}^n$. More generally, a set $A \subset \mathbb{R}^n$ is *star-shaped* if there exists a point $x_0 \in A$ such that $[x_0, x] \subset A$ for all $x \in A$. In the setting of convex bodies, all the inclusions in Proposition 1 are equalities, provided that $A \cup B$ is also a convex body (see, e.g., [97, Section 3.1]).

The *support function* of a convex body K is given by $h_K(u) = \max_{x\in K}\langle x,u\rangle$ for all $u \in \mathbb{R}^n$. It is well-known that the support functions are exactly the sublinear forms of $\mathbb{R}$, and that they characterize the convex bodies (see, e.g., [97, Theorem 1.7.1]). This, together with the fact that the unit balls of the real normed spaces are precisely the origin-symmetric convex bodies, yields a strong and fruitful bond between the fields of Convex Geometry and Functional Analysis.

A useful way to construct convex bodies is the so-called *Wulff shape* of a function (see, e.g., [97, Section 7.5]). Given a positive continuous function $f : \mathbb{S}^{n-1} \longrightarrow \mathbb{R}_{>0}$, the set

$$\mathcal{W}(f) = \bigcap_{u\in\mathbb{S}^{n-1}} \{x \in \mathbb{R}^n : \langle x,u\rangle \le f(u)\} \tag{3}$$

is clearly a convex body containing the origin in its interior, known as the Wulff shape of f (here, $\mathbb{S}^{n-1}$ denotes the unit sphere of $\mathbb{R}^n$). We certainly also have $h_{\mathcal{W}(f)} \le f$, with equality if and only if f is sublinear. Moreover, the following property holds:

Lemma 2 [97, Lemma 7.5.2] If $\{f_j : \mathbb{S}^{n-1} \longrightarrow \mathbb{R}_{>0}\}_{j\in\mathbb{N}}$ converges uniformly to f, then $\{\mathcal{W}(f_j)\}_{j\in\mathbb{N}}$ converges to $\mathcal{W}(f)$ in the Hausdorff metric.

We will denote the *characteristic function* of a set $A \subset \mathbb{R}^n$ by $\chi_{_A}$, i.e.,

$$\chi_{_A} = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Additionally, for a function $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}_{\ge 0}$ we write $\phi^\diamond : \mathbb{R}^n \longrightarrow \mathbb{R}_{\ge 0}$ to represent the function defined by

$$\phi^\diamond(z) = \sup_{u\in(-1,1)^n} \phi(z+u) \quad \text{for all } z \in \mathbb{R}^n, \tag{4}$$

unless otherwise stated. Such an extension of ϕ is just the *Asplund product* $\star$ of the functions ϕ and $\chi_{(-1,1)^n}$, which can be seen as the functional analogue of the Minkowski sum of sets in the setting of log-concave functions. Indeed,

$$\begin{aligned} \phi^\diamond(z) &= \sup_{u\in(-1,1)^n} \phi(z+u) = \sup_{u\in(-1,1)^n} \phi(z-u) = \sup_{u\in\mathbb{R}^n} \phi(z-u)\chi_{(-1,1)^n}(u) \\ &= \sup_{u_1+u_2=z} \phi(u_1)\chi_{(-1,1)^n}(u_2) = \big(\phi \star \chi_{(-1,1)^n}\big)(z). \end{aligned} \tag{5}$$

For more information on the Asplund product, also known as the *sup-convolution*, we refer the reader to [97, Section 9.5] and the references therein. In addition, as usual, $|\phi|_\infty = \sup_{x\in\mathbb{R}^n} \phi(x)$.

We will need to utilize the inequality between the arithmetic and geometric means. We recall that given $\{a_i\}_{i=1}^r \subset \mathbb{R}_{\ge 0}$ for some $r \in \mathbb{N}\setminus\{0\}$, this inequality states that

$$\frac{1}{r}\sum_{i=1}^r a_i \ge \left(\prod_{i=1}^r a_i\right)^{1/r}, \tag{6}$$

with equality if and only if $a_1 = \cdots = a_n$.

Finally, we will make extensive use of *Hölder's inequality*. In one of its general forms (see, e.g., [56, Theorem 11]), it states that given some $r \in \mathbb{N}\setminus\{0\}$, if we have $\{a_i = (a_{i1},\dots,a_{in})\}_{i=1}^r \subset \mathbb{R}^n_{\ge 0}$ and $\{\lambda_i\}_{i=1}^r \subset \mathbb{R}_{\ge 1}$ with $\sum_{i=1}^r 1/\lambda_i = 1$, then

$$\sum_{j=1}^n \prod_{i=1}^r a_{ij} \le \prod_{i=1}^r \left(\sum_{j=1}^n a_{ij}^{\lambda_i}\right)^{1/\lambda_i}, \tag{7}$$

with equality if and only if the points $\{a_i\}_{i=1}^r$ are linearly dependent. For $r = 2$, when $\lambda_1, \lambda_2 \in \mathbb{R}_{\leq 1}$ with $1/\lambda_1 + 1/\lambda_2 = 1$, the reverse inequality holds with the same equality case:

$$\sum_{j=1}^{n} a_{1j} a_{2j} \geq \left(\sum_{j=1}^{n} a_{1j}^{\lambda_1} \right)^{1/\lambda_1} \left(\sum_{j=1}^{n} a_{2j}^{\lambda_2} \right)^{1/\lambda_2}, \tag{8}$$

(see, e.g., [56, Theorem 13]).

1

Brunn-Minkowski type inequalities

The Brunn-Minkowski theorem is a core foundation of the Brunn-Minkowski theory, and its impact far outreaches this field, yielding results in many related mathematical disciplines. It states that the functional $\mathrm{vol}\big((1-\lambda)K+\lambda L\big)^{1/n}$, for $K,L\subset\mathbb{R}^n$ convex bodies, is concave in $\lambda\in[0,1]$. Generalizations and analogues of this result have proved to be a fruitful field of study, involving other operations, spaces and measures, as well as obtaining related inequalities and simpler proofs of already known ones, being the isoperimetric inequality one of the most remarkable examples in this respect. We refer the reader to [14, 45] for extensive survey articles on the topic, as well as to the updated monograph [97, Chapter 9] and the references therein.

In this chapter we will obtain several discrete analogues of the Brunn-Minkowski inequality for the lattice point enumerator $\mathrm{G_n}(\cdot)$. The common approach that we will follow in almost all scenarios is to first obtain stronger functional versions, then particularize these, and finally discuss some consequences and applications.

1.1 The Brunn-Minkowski inequality: different versions and extensions

A classical result connecting the notions of volume and Minkowski addition is the *Brunn-Minkowski* inequality (see, e.g., [45]).

Theorem A — Brunn-Minkowski inequality. Let $K,L\subset\mathbb{R}^n$ be non-empty compact sets. Then

$$\mathrm{vol}(K+L)^{1/n}\geq\mathrm{vol}(K)^{1/n}+\mathrm{vol}(L)^{1/n}. \tag{1.1}$$

If K and L are also convex, equality holds if and only if the sets are either homothetic or they lie in parallel hyperplanes.

Furthermore, for any $\lambda\in(0,1)$,

$$\mathrm{vol}\big((1-\lambda)K+\lambda L\big)\geq\mathrm{vol}(K)^{1-\lambda}\,\mathrm{vol}(L)^{\lambda}. \tag{1.2}$$

This inequality is known as the *multiplicative* Brunn-Minkowski inequality, and is a simple consequence of (1.1) and the arithmetic-geometric mean inequality. Indeed, both inequalities (1.1) and (1.2) are equivalent.

Due to the homogeneity of degree n of the volume (i.e., $\mathrm{vol}(\lambda A) = \lambda^n \mathrm{vol}(A)$, for $A \subset \mathbb{R}^n$ and $\lambda \geq 0$), (1.1) is equivalent to the fact that $\mathrm{vol}(\cdot)^{1/n}$ is a concave functional. The original hypothesis can be significantly relaxed, as the Brunn-Minkowski inequality holds for arbitrary measurable sets.

One of the first, and most well-known, functional equivalent versions of the Brunn-Minkowski inequality is the *Prékopa-Leindler* inequality ([83, 84]).

Theorem B — Prékopa-Leindler inequality. Let $\lambda \in (0,1)$ and let $f,g,h : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ be Lebesgue integrable functions verifying

$$h\big((1-\lambda)x+\lambda y\big) \geq f(x)^{1-\lambda} g(y)^{\lambda}$$

for all $x,y \in \mathbb{R}^n$. Then

$$\int_{\mathbb{R}^n} h(x)\,\mathrm{d}x \geq \left(\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g(x)\,\mathrm{d}x\right)^{\lambda}. \tag{1.3}$$

To see that this inequality implies (1.2), it suffices to consider $f = \chi_K$, $g = \chi_L$ and $h = \chi_{(1-\lambda)K+\lambda L}$. For the converse, one can apply (1.2) to the superlevel sets of f, g, and h.

A powerful generalization of Prékopa-Leindler's inequality is the *Borell-Brascamp-Lieb* inequality ([23, 27]), which provides a uniparametric family of analogues.

Before stating it, we recall the notions of α-sum and α-mean, $\alpha \in \mathbb{R} \cup \{\pm\infty\}$. For any $t,s > 0$ and any $a,b > 0$ we will write

$$\mathcal{S}_{\alpha}^{t,s}(a,b) = \begin{cases} \min\{a,b\} & \text{if } \alpha = -\infty, \\ \left(ta^{\alpha} + sb^{\alpha}\right)^{1/\alpha} & \text{if } \alpha \in \mathbb{R} \setminus \{0\}, \\ \max\{a,b\} & \text{if } \alpha = \infty, \end{cases}$$

and

$$\mathcal{M}_{\alpha}^{\lambda}(a,b) = \begin{cases} \mathcal{S}_{\alpha}^{1-\lambda,\lambda}(a,b) & \text{if } \alpha \neq 0, \\ a^{1-\lambda} b^{\lambda} & \text{if } \alpha = 0. \end{cases}$$

If $ab = 0$, we set $\mathcal{S}_{\alpha}^{t,s}(a,b) = \mathcal{M}_{\alpha}^{\lambda}(a,b) = 0$. For a general reference for α-means of non-negative numbers, we refer the reader to the classic text of Hardy, Littlewood and Pólya [56] and to the handbook [28]).

Theorem C — Borell-Brascamp-Lieb inequality. Let $\lambda \in (0,1)$ and let $-1/n \leq p \leq \infty$. Consider Lebesgue integrable functions $f,g,h : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ verifying

$$h\big((1-\lambda)x+\lambda y\big) \geq \mathcal{M}_p^{\lambda}\big(f(x),g(y)\big) \tag{1.4}$$

for all $x,y \in \mathbb{R}^n$. Then

$$\int_{\mathbb{R}^n} h(x)\,\mathrm{d}x \geq \mathcal{M}^{\lambda}_{\frac{p}{np+1}} \left(\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x, \int_{\mathbb{R}^n} g(x)\,\mathrm{d}x\right). \tag{1.5}$$

The case $p = 0$ recovers the Prékopa-Leindler inequality (1.3), whereas taking $p = \infty$ and $f = \chi_K$, $g = \chi_L$ and $h = \chi_{(1-\lambda)K+\lambda L}$ retrieves the additive form of the Brunn-Minkowski inequality (1.1).

In subsequent sections we will prove discrete analogues of the above result, and this will enable us to obtain multiple consequences, including the announced discrete versions of the Brunn-Minkowski inequality.

1.1.1 The L_p Brunn-Minkowski inequality

The L_p *Brunn-Minkowski theory* is a recent and remarkable extension of the classical Brunn-Minkowski theory which groundworks were laid by Lutwak after initiating a deep and systematic study of p-additions and their consequences in [74, 75]. This is not only a very active area of research nowadays, but it has further supposed to be the starting point for new developments and generalizations. An example of the latter can be seen in [48, 49, 80] and the references therein, where the authors perform a thorough investigation into the fundamental characteristics of operations between sets and provide an elegant construction that allows one to define a general pointwise operation between sets. For more information on the L_p *Brunn-Minkowski theory* and its consequences we refer the reader to [97, Section 9.1].

The L_p sum $K +_p L$, $p \geq 1$, of two convex bodies $K, L \subset \mathbb{R}^n$ containing the origin was originally defined by Firey in [41] as the only convex body whose support function is given by

$$h_{K+_pL}(\cdot) = \big(h_K(\cdot)^p + h_L(\cdot)^p\big)^{1/p}. \tag{1.6}$$

Naturally associated with the p-addition one finds a p-scalar product: $r \cdot K = r^{1/p}K$ for any $r > 0$. This p-scalar multiplication depends on p, but for the sake of simplicity we will just write "$\cdot$" instead of "$\cdot_p$".

It is easy to see that $+_1$ is the standard Minkowski sum, and thus, that this notion provides a uniparametric generalization of the Brunn-Minkowski inequality. Additionally, it can be seen that $K +_\infty L = \mathrm{conv}(K \cup L)$.

This operation is order reversing in the parameter p, that is, if $1 \leq p \leq q$, then $K +_q L \subset K +_p L$. On the contrary, it is order preserving for p-convex combinations, i.e., $(1-\lambda) \cdot K +_p \lambda \cdot L \subset (1-\lambda) \cdot K +_q \lambda \cdot L$, for $\lambda \in (0,1)$.

Of particular interest for us is the L_p version of the Brunn-Minkowski inequality. This was proved by Firey in [41], and states that for any two convex bodies $K, L \subset \mathbb{R}^n$ containing the origin, any $\lambda \in (0,1)$, and all $p \geq 1$, one has

$$\mathrm{vol}\big((1-\lambda) \cdot K +_p \lambda \cdot L\big)^{p/n} \geq (1-\lambda)\,\mathrm{vol}(K)^{p/n} + \lambda\,\mathrm{vol}(L)^{p/n}. \tag{1.7}$$

We note that the hypothesis for the sets in the previous definition, unlike in the standard Brunn-Minkowski inequality, cannot be relaxed. Indeed, compactness and convexity are required so that the support functions characterize the sets, and the bodies need to contain the origin for the support function to be non-negative.

In order to elude this inconvenience, Lutwak, Yang and Zhang (see [76]) introduced an alternative pointwise definition which is valid for arbitrary sets. Specifically, for any two non-empty bounded sets $K, L \subset \mathbb{R}^n$ and any $p \geq 1$, they defined

$$K +_p L = \Big\{(1-\mu)^{1/q}x + \mu^{1/q}y : x \in K, y \in L, \mu \in [0,1]\Big\}, \tag{1.8}$$

where $q \in [1,\infty]$ is the Hölder conjugate of p, i.e., such that $1/p + 1/q = 1$, and showed that when K and L are convex bodies containing the origin, the definition coincides with the one of Firey. We note that if $p = 1$ then $q = \infty$, and thus the above notion again reduces to the standard Minkowski addition. The authors also proved the corresponding L_p Brunn-Minkowski inequality (1.7) in this general setting.

It is desirable to extend the aforementioned notions to the case $0 \leq p < 1$, and, in particular, to the case $p = 0$. A strong reason for this is that the corresponding and recently conjectured Brunn-Minkowski inequality in this setting for symmetric convex bodies, known in the literature as the *log-Brunn-Minkowski* inequality, would be stronger than all other L_p versions of the form

$$\mathrm{vol}\big((1-\lambda)\cdot K +_p \lambda\cdot L\big) \geq \mathrm{vol}(K)^{1-\lambda}\,\mathrm{vol}(L)^{\lambda} \tag{1.9}$$

for any $p>0$ (see [26]).

Conjecture 1.1 — log-Brunn-Minkowski inequality. Let $K,L\subset\mathbb{R}^n$ be origin-symmetric convex bodies, and let $\lambda\in(0,1)$. Then

$$\mathrm{vol}\big((1-\lambda)\cdot K +_0 \lambda\cdot L\big) \geq \mathrm{vol}(K)^{1-\lambda}\,\mathrm{vol}(L)^{\lambda}. \tag{1.10}$$

However, it is easy to see that both definitions for $K+_p L$ given above can be problematic when $p<1$. Indeed, the former fails since the p-sum of support functions is no longer sublinear when $p<1$, and thus, is not the support function of any convex body; whereas for the latter, difficulties may arise due to the fact that q would be negative. Therefore, the extension is obtained, in the setting of convex bodies, by means of the Wulff shape (see (3)) determined by the support functions of the sets. In particular, given two convex bodies $K,L\subset\mathbb{R}^n$ containing the origin and a fixed $\lambda\in(0,1)$,

$$(1-\lambda)\cdot K +_p \lambda\cdot L := \mathcal{W}\Big(\big((1-\lambda)h_K(u)^p+\lambda h_L(u)^p\big)^{1/p}\Big).$$

It can be seen that this definition coincides with the one of Firey when $p\geq 1$. In the case $p=0$, the previous notion translates into the limit case

$$(1-\lambda)\cdot K +_0 \lambda\cdot L = \mathcal{W}\Big(h_K(u)^{1-\lambda}h_L(u)^{\lambda}\Big).$$

Conjecture 1.1 was solved in the plane already in [26], and the corresponding equality cases, both for (1.9) and (1.10) were characterized. The authors also noted that the central symmetry hypothesis cannot be removed. The conjecture can be solved in the complex case as a consequence of a generalization of the Blaschke-Santaló inequality due to Cordero-Erausquin, as shown by Rotem in [90].

Since then, other symmetric scenarios have been considered. For the case of unconditional convex bodies (i.e., convex bodies that have orthogonal symmetry with respect to all the canonical hyperplanes) in general dimension, Conjecture 1.1 was solved by Saroglou in [95]. In [78], Marsiglietti proved the more general inequality

$$\mathrm{vol}\big((1-\lambda)\cdot K +_p \lambda\cdot L\big)^{p/n} \geq (1-\lambda)\,\mathrm{vol}(K)^{p/n}+\lambda\,\mathrm{vol}(L)^{p/n} \tag{1.11}$$

for all $0<p<1$ and $\lambda\in(0,1)$, when K and L are unconditional convex bodies. These results were generalized by Böröczky and Kalantzopoulos in [25] to the setting of convex bodies which have linear symmetry (not necessarily orthogonal) with respect to n hyperplanes with linearly independent normal vectors. In a more functional setting, Saroglou also showed in [96] that the conjecture implies the corresponding inequality for any log-concave measure.

Furthermore, the question of the stability of Brunn-Minkowski type inequalities of this form has also been studied, and local results (with respect to the Hausdorff topology) have been obtained, for instance, in [33, 35, 73, 85].

For further information on the log-Brunn-Minkowski, we refer the reader to the previous manuscripts and the references therein.

1.1.2 Discrete versions of the Brunn-Minkowski inequality

As already stated, another common approach to extend the Brunn-Minkowski inequality is to consider alternative spaces and measures. In this regard, the integer lattice $\mathbb{Z}^n$ endowed with the cardinality measure has been extensively studied. Perhaps the first naive inequality in this context is the classic one given by $|X+Y| \geq |X|+|Y|-1$ for any non-empty finite sets $X,Y \subset \mathbb{Z}^n$.

Ruzsa obtained in [93, 94] some of the first strengthenings of this inequality. In particular, he showed that if also $|Y| \leq |X|$ and $\dim(X+Y)=n$ then

$$|X+Y| \geq |X|+n|Y|-\frac{n(n+1)}{2}. \tag{1.12}$$

In [46] Gardner and Gronchi obtained a powerful discrete analogue of the following form of the Brunn-Minkowski inequality, in the setting of $\mathbb{Z}^n$ with the cardinality: $\mathrm{vol}(K+L) \geq \mathrm{vol}(B_K+B_L)$, where B_K and B_L denote the centered Euclidean balls of the same volume as K and L, respectively. More precisely, they proved that if X,Y are non-empty finite subsets of the integer lattice $\mathbb{Z}^n$, with dimension $\dim Y=n$, then

$$|X+Y| \geq \left|D^Y_{|X|}+D^Y_{|Y|}\right|. \tag{1.13}$$

Here $D^Y_{|X|}, D^Y_{|Y|}$ are Y-*initial segments*: for $m \in \mathbb{N}\setminus\{0\}$, D^Y_m is the set of the first m points of $\mathbb{Z}^n_{\geq 0}$ in the so-called "Y-order", which is a particular order defined on $\mathbb{Z}^n_{\geq 0}$ depending only on the cardinality of Y. For both a proper definition and a deep study of it we refer the reader to [46]. As consequences of (1.13), they also got two additional engaging discrete Brunn-Minkowski type inequalities improving Ruzsa's inequality (1.12):

$$|X+Y|^{1/n} \geq |X|^{1/n}+\frac{1}{(n!)^{1/n}}\big(|Y|-n\big)^{1/n}$$

and, if $|Y| \leq |X|$, then

$$|X+Y| \geq |X|+(n-1)|Y|+\big(|X|-n\big)^{(n-1)/n}\big(|Y|-n\big)^{1/n}-\frac{n(n-1)}{2}.$$

In [63], a discrete Brunn-Minkowski type inequality in the classical form (1.1) was proved: for $X,Y \subset \mathbb{Z}^n$ non-empty finite sets,

$$|\overline{X}+Y|^{1/n} \geq |X|^{1/n}+|Y|^{1/n},$$

where $\overline{X}$ is a suitable extension of X (not depending on Y).

Another way of extending the set in the left-hand side in order to obtain a discrete Brunn-Minkowski type inequality is to sum the discrete unit cube $\{0,1\}^n$. This has been performed, e.g., in [50, 69]. In particular, we have the following result:

Theorem D [69, Theorem 3.2] Let $X,Y \subset \mathbb{Z}^n$ be non-empty finite sets. Then

$$\left|X+Y+\{0,1\}^n\right|^{1/n} \geq |X|^{1/n}+|Y|^{1/n}. \tag{1.14}$$

Equality holds if both X and Y are lattice cubes.

Recently, other discrete analogues of the Brunn-Minkowski inequality have been obtained for the cardinality [50, 63, 69], functional extensions of it [55, 64, 65, 68, 69, 71, 100] and versions for the lattice point enumerator $\mathrm{G_n}(\cdot)$ [55, 64, 65, 67, 69], some of which we will discuss in subsequent sections. In this respect, in [69] the authors show the necessity of extending $(1-\lambda)K+\lambda L$ to $(1-\lambda)K+\lambda L+(-1,1)^n$ in order to get a discrete analogue of (1.1) for all $\lambda \in (0,1)$, as follows:

Theorem E Let $\lambda \in (0,1)$ and let $K,L \subset \mathbb{R}^n$ be non-empty bounded sets. Then

$$\mathrm{G_n}\big((1-\lambda)K+\lambda L+(-1,1)^n\big)^{1/n} \geq (1-\lambda)\,\mathrm{G_n}(K)^{1/n}+\lambda\,\mathrm{G_n}(L)^{1/n}. \tag{1.15}$$

The inequality is sharp.

Apart from the above-mentioned discrete Brunn-Minkowski type inequalities, various discrete counterparts, for the lattice point enumerator $\mathrm{G_n}(\cdot)$, of classical results in Convex Geometry have been recently proven. Some examples of such results are *Koldobsky's slicing inequality* [2], *Meyer's inequality* [42], an isoperimetric type inequality [67] (which we will discuss in Chapter 2), or a Rogers-Shephard type inequality [9] (which we will discuss in Chapter 3). We refer the reader to these articles and the references therein for other connected problems, questions and results.

1.2 The Brunn-Minkowski inequality for positive combinations of sets

In the continuous setting, due to the homogeneity of the volume, (1.1) is equivalent to

$$\mathrm{vol}(tK+sL)^{1/n} \geq t\,\mathrm{vol}(K)^{1/n} + s\,\mathrm{vol}(L)^{1/n} \tag{1.16}$$

for any $t,s \geq 0$ and any non-empty compact sets $K,L \subset \mathbb{R}^n$. However, since $\mathrm{G_n}(\cdot)$ is not homogeneous, it is not possible to directly obtain this generalization from (1.15). Nonetheless, the proof itself can be adapted. The results of this section are contained in [67] unless otherwise stated. In particular, in this section we will prove:

Theorem 1.2 [67, Theorem 1.1] Let $t,s \geq 0$ and let $K,L \subset \mathbb{R}^n$ be non-empty bounded sets such that $\mathrm{G_n}(K)\,\mathrm{G_n}(L) > 0$. Then

$$\mathrm{G_n}\left(tK+sL+(-1,\lceil t+s\rceil)^n\right)^{1/n} \geq t\,\mathrm{G_n}(K)^{1/n} + s\,\mathrm{G_n}(L)^{1/n}. \tag{1.17}$$

Equality is attained, when $t+s \in \mathbb{Z}$, if $K=[0,a]^n$ and $L=[0,b]^n$ are cubes with $a,b,ta+sb \in \mathbb{Z}$.

We will also show that (1.17) implies (1.16) for non-empty compact sets (see Theorem 1.9).

In [67] a direct geometrical proof is provided. Here, we will use a different approach: we will show Theorem 1.2 by proving a more general functional result, from which Theorem 1.2 will be derived as an immediate consequence. Indeed, we will show the following result:

Theorem 1.3 Let $t,s > 0$ and let $K,L \subset \mathbb{R}^n$ be non-empty bounded sets. Let $-1/n \leq \alpha \leq \infty$, $\alpha \neq 0$, and let $f,g,h : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ be non-negative functions such that

$$h(tx+sy) \geq \left[tf(x)^\alpha + sg(y)^\alpha\right]^{1/\alpha} \tag{1.18}$$

for all $x \in K$, $y \in L$ with $f(x)g(y) > 0$. Then

$$\sum_{z\in\left(M+(-1,\lceil t+s\rceil)^n\right)\cap\mathbb{Z}^n} h^{\bar{\diamond}}(z) \geq \mathcal{S}^{t,s}_{\frac{\alpha}{n\alpha+1}}\left(\sum_{x\in K\cap\mathbb{Z}^n} f(x), \sum_{y\in L\cap\mathbb{Z}^n} g(y)\right), \tag{1.19}$$

where $M = tK+sL$ and $h^{\bar{\diamond}}(z) = \sup_{u\in(-1,\lceil t+s\rceil)^n} h(z-u)$.

The proof follows the same steps to those of the proof of [69, Theorem 2.2], replacing convex combinations $(1-\lambda)x+\lambda y$, for $\lambda \in (0,1)$, by linear combinations $tx+sy$, with $t,s>0$, and making the suitable adaptations. We nonetheless include it here for the sake of completeness. We observe that the adjustment in the definition of $h^{\bar{\diamond}}(\cdot)$ (compared to $h^{\diamond}(\cdot)$, see (5)) stems from the lack of symmetry of the cube $\left(-1,\lceil t+s\rceil\right)^n$ when $t+s \neq 1$.

We start by proving the following result, which will be used to obtain the 1-dimensional case of Theorem 1.2. It again turns out to be the corresponding analogue of [69, Lemma 2.1] (by suitably replacing the parameters $1-\lambda$ and λ by t and s, respectively).

Lemma 1.4 Let $t,s \geq 0$ and let $K,L,M \subset \mathbb{R}$ be non-empty sets verifying that $tK+sL \subset M$. If $M = \bigcup_{i=1}^{r}[a_i,b_i]$, with $a_i,b_i \in \mathbb{Z}$ for all $i=1,\dots,r$, is a finite union of pairwise disjoint compact intervals with integer extremes then

$$\mathrm{G}_1(M) + r(t+s-1) \geq t\,\mathrm{G}_1(K) + s\,\mathrm{G}_1(L).$$

Proof. We prove the result by induction on r. For the case $r=1$, i.e., when $M=[a_1,b_1]$ is a (non-empty) compact interval (with $a_1,b_1 \in \mathbb{Z}$), we have on the one hand that $\mathrm{G}_1(M) = b_1 - a_1 + 1$. Moreover, denoting by $a = \inf K$, $b = \sup K$, $c = \inf L$ and $d = \sup L$, we clearly get $\mathrm{G}_1(K) \leq \mathrm{G}_1([a,b]) = \lfloor b \rfloor - \lceil a \rceil + 1$ and $\mathrm{G}_1(L) \leq \mathrm{G}_1([c,d]) = \lfloor d \rfloor - \lceil c \rceil + 1$. On the other hand, the inclusion $tK+sL \subset M$ implies that $b_1 \geq t\lfloor b \rfloor + s\lfloor d \rfloor$ and $a_1 \leq t\lceil a \rceil + s\lceil c \rceil$, and thus $b_1 - a_1 \geq t\left(\lfloor b \rfloor - \lceil a \rceil\right) + s\left(\lfloor d \rfloor - \lceil c \rceil\right)$. Altogether, we get $\mathrm{G}_1(M) - 1 \geq t(\mathrm{G}_1(K)-1) + s(\mathrm{G}_1(L)-1)$, showing the case $r=1$.

So, we suppose that the inequality is true for $r \geq 1$ and assume that $M = \bigcup_{i=1}^{r+1}[a_i,b_i]$, where $b_i < a_{i+1}$ for all $1 \leq i \leq r$.

Denoting by $M_1 = [a_1,b_1]$ and $M_2 = \bigcup_{i=2}^{r+1}[a_i,b_i]$, we may assume, without loss of generality, that $M_1 \cap (tK+sL) \neq \emptyset$. Hence, we may define $m = \sup\bigl(M_1 \cap (tK+sL)\bigr)$ and then, since K and L are bounded (because $tK+sL \subset M$), there exist $k \in \mathrm{cl}\,K$ and $l \in \mathrm{cl}\,L$ (where $\mathrm{cl}\,\cdot$ denotes the topological closure) such that $tk + sl = m$. Thus, considering the sets $K_1 = \{x \in K : x \leq k\}$, $K_2 = K \setminus K_1$, $L_1 = \{x \in L : x \leq l\}$ and $L_2 = L \setminus L_1$, we have that $tK_1 + sL_1 \subset M_1$ and $tK_2 + sL_2 \subset M_2$. Therefore, applying the induction hypothesis (and taking into account that M_1 are M_2 are disjoint), we finish the proof:

$$\begin{aligned}\mathrm{G}_1(M) + (r+1)(t+s-1) &= \mathrm{G}_1(M_1) + (t+s-1) + \mathrm{G}_1(M_2) + r(t+s-1)\\ &\geq t\,\mathrm{G}_1(K_1) + s\,\mathrm{G}_1(L_1) + t\,\mathrm{G}_1(K_2) + s\,\mathrm{G}_1(L_2) = t\,\mathrm{G}_1(K) + s\,\mathrm{G}_1(L). \quad \blacksquare\end{aligned}$$

Now we prove the case $n=1$ of Theorem 1.2, which will be necessary in the proof of Theorem 1.3.

Lemma 1.5 Let $t,s \geq 0$ and let $K,L \subset \mathbb{R}$ be non-empty bounded sets. Then

$$\mathrm{G}_1\Bigl(tK+sL+\bigl(-1,\lceil t+s \rceil\bigr)\Bigr) \geq t\,\mathrm{G}_1(K) + s\,\mathrm{G}_1(L). \tag{1.20}$$

Equality is attained, when $t+s \in \mathbb{Z}$, if $K=[0,a]$ and $L=[0,b]$ are intervals with $a,b,ta+sb \in \mathbb{Z}$.

Proof. Let $M = \bigcup_{x \in tK+sL}\bigl[\lfloor x \rfloor, \lceil x \rceil\bigr]$. Clearly, since K and L are bounded, M is a finite union of compact disjoint intervals, say $M = \bigcup_{i=1}^{r}[a_i,b_i]$ for some $a_i,b_i \in \mathbb{Z}$, $i=1,\dots,r$.

For $I = \bigl\{1 \leq i < r : a_{i+1} - b_i \leq \lceil t+s-1 \rceil\bigr\}$, let $M' = M \cup \left(\bigcup_{i \in I}[b_i,a_{i+1}]\right)$ and let

$$M'' = M' + \bigl[0, \lceil t+s-1 \rceil\bigr] = M + \bigl[0, \lceil t+s-1 \rceil\bigr].$$

From Lemma 1.4 we obtain

$$\mathrm{G}_1(M'') = \mathrm{G}_1(M') + \bigl(r-|I|\bigr)\bigl(\lceil t+s-1 \rceil\bigr) \geq \mathrm{G}_1(M') + \bigl(r-|I|\bigr)(t+s-1) \geq t\,\mathrm{G}_1(K) + s\,\mathrm{G}_1(L).$$

This yields (1.20) since $M \cap \mathbb{Z} = \bigl(tK+sL+(-1,1)\bigr) \cap \mathbb{Z}$ and

$$M'' \cap \mathbb{Z} = \Bigl(tK+sL+\bigl(-1,\lceil t+s-1 \rceil + 1\bigr)\Bigr) \cap \mathbb{Z}.$$

Finally, in order to show that equality may be attained (for some $t,s \geq 0$), it is enough to consider $a,b,t,s > 0$ such that $a,b,t+s,ta+sb \in \mathbb{Z}$, and take $K=[0,a]$ and $L=[0,b]$, for which we have

$$tK+sL+\bigl(-1,\lceil t+s \rceil\bigr) = (-1, ta+sb+t+s),$$

and thus

$$\mathrm{G}_1\Bigl(tK+sL+\bigl(-1,\lceil t+s \rceil\bigr)\Bigr) = t(a+1) + s(b+1) = t\,\mathrm{G}_1(K) + s\,\mathrm{G}_1(L). \quad \blacksquare$$

Before proving Theorem 1.3 we need to state an auxiliary result, which can be regarded as a discrete counterpart of the well-known *Cavalieri Principle* for the lattice point enumerator (see [69] and the references therein).

Lemma 1.6 [69, Corollary 2.1] Let $\Omega \subset \mathbb{R}^n$ be a bounded set, let $f : \mathbb{R}^n \longrightarrow [0,\infty)$ and consider a set $\{k_i\}_{i=0}^r$ such that $0 = k_0 < k_1 < \cdots < k_r$ and $f(\Omega\cap\mathbb{Z}^n) \subset \{k_i\}_{i=0}^r$. Then

$$\sum_{x\in\Omega\cap\mathbb{Z}^n} f(x) = \sum_{i=1}^{r}(k_i - k_{i-1})\,\mathrm{G_n}\big(\{x\in\Omega : f(x)\geq k_i\}\big).$$

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. We may assume that $\big(\sum_{x\in K\cap\mathbb{Z}^n} f(x)\big)\big(\sum_{y\in L\cap\mathbb{Z}^n} g(y)\big) > 0$, since otherwise the result is trivial. We will prove it by induction on the dimension. So, we let $n = 1$, and we write

$$a = \max_{x\in K\cap\mathbb{Z}} f(x) > 0, \quad b = \max_{y\in L\cap\mathbb{Z}} g(y) > 0 \quad \text{and} \quad c = (ta^\alpha + sb^\alpha)^{1/\alpha} > 0.$$

We also consider the normalized functions $F, G, H, H^{\bar{\diamond}} : \mathbb{R} \longrightarrow \mathbb{R}_{\geq 0}$ given by

$$F(x) = \frac{f(x)}{a}, \quad G(y) = \frac{g(y)}{b}, \quad H(z) = \frac{h(z)}{c} \quad \text{and} \quad H^{\bar{\diamond}}(z) = \frac{h^{\bar{\diamond}}(z)}{c}.$$

We will first show that, for all $x \in K$ and all $y \in L$, we have

$$H(tx+sy) \geq \min\big\{F(x), G(y)\big\}. \tag{1.21}$$

Without loss of generality we can assume that $f(x)g(y) > 0$. Then, setting $\theta = sb^\alpha/c^\alpha$, we have on the one hand, for any $\alpha \neq \infty$,

$$\begin{aligned} h(tx+sy) &\geq \big(tf(x)^\alpha + sg(y)^\alpha\big)^{1/\alpha} = c\left(\frac{ta^\alpha F(x)^\alpha + sb^\alpha G(y)^\alpha}{c^\alpha}\right)^{1/\alpha} \\ &= \big((1-\theta)F(x)^\alpha + \theta G(y)^\alpha\big)^{1/\alpha} \geq c\cdot\min\big\{F(x), G(y)\big\}, \end{aligned}$$

where in the last step we have used that $\min\{\cdot,\cdot\} = \mathcal{M}^\lambda_{-\infty}(\cdot,\cdot)$ and that $\mathcal{M}^\lambda_\alpha(\cdot,\cdot)$ is increasing in $\alpha \in \mathbb{R}$. On the other hand, if $\alpha = \infty$, we clearly have

$$h(tx+sy) \geq \max\big\{f(x), g(y)\big\} = c\cdot\max\left\{\frac{a}{c}F(x), \frac{b}{c}G(y)\right\} \geq c\cdot\min\big\{F(x), G(y)\big\},$$

since $c = \max\{a,b\}$ in this case. This shows (1.21).

Let us now define, for any $\tau \in \mathbb{R}_{\geq 0}$, the level sets

$$\begin{aligned} K_\tau &= \big\{x\in K : F(x)\geq\tau\big\}, \quad L_\tau = \big\{y\in L : G(y)\geq\tau\big\}, \\ M_\tau &= \big\{z\in M : H(z)\geq\tau\big\}, \quad N_\tau = \big\{z\in M + \big(-1,\lceil t+s\rceil\big) : H^{\bar{\diamond}}(z)\geq\tau\big\}. \end{aligned}$$

By construction, these sets are non-empty for all $\tau \in [0,1]$. Then, on the one hand, (1.18) implies that $tK_\tau + sL_\tau \subset M_\tau$ for every $\tau \in [0,1]$, and thus, Lemma 1.5 yields that

$$\mathrm{G}_1\Big(M_\tau + \big(-1,\lceil t+s\rceil\big)\Big) \geq t\,\mathrm{G}_1(K_\tau) + s\,\mathrm{G}_1(L_\tau), \tag{1.22}$$

for all $\tau \in [0,1]$. On the other hand, it is clear by definition that $H^{\bar{\diamond}}(z+u) \geq H(z)$ for any $u \in \big(-1,\lceil t+s\rceil\big)$, and this implies that

$$M_\tau + \big(-1,\lceil t+s\rceil\big) \subset N_\tau, \tag{1.23}$$

for every $\tau \in [0,1]$.

Next we consider, for some $r \in \mathbb{N} \setminus \{0\}$, points $\{k_i\}_{i=0}^r \subset \mathbb{R}_{\geq 0}$ with $0 = k_0 < k_1 < \cdots < k_r$ such that

$$\{k_i\}_{i=0}^r \supset F(K \cap \mathbb{Z}) \cup G(L \cap \mathbb{Z}) \cup H^{\bar{\diamond}}\Big(\big(M + (-1, \lceil t+s \rceil)\big) \cap \mathbb{Z}\Big).$$

By construction, there exists $\sigma \in \{0, \dots, r\}$ with $k_\sigma = \max_{x \in K \cap \mathbb{Z}} F(x) = \max_{y \in L \cap \mathbb{Z}} G(y) = 1$. Then, by applying Lemma 1.6, together with (1.22) and (1.23), we get

$$\begin{aligned}
\sum_{z \in (M+(-1,\lceil t+s \rceil)) \cap \mathbb{Z}} h^{\bar{\diamond}}(z) &= \sum_{z \in (M+(-1,\lceil t+s \rceil)) \cap \mathbb{Z}} c H^{\bar{\diamond}}(z) = c \sum_{i=1}^{r} (k_i - k_{i-1}) \, \mathrm{G}_1(N_{k_i}) \\
&\geq c \sum_{i=1}^{\sigma} (k_i - k_{i-1}) \, \mathrm{G}_1(N_{k_i}) \geq c \sum_{i=1}^{\sigma} (k_i - k_{i-1}) \, \mathrm{G}_1\Big(M_{k_i} + \big(-1, \lceil t+s \rceil\big)\Big) \\
&\geq c \sum_{i=1}^{\sigma} (k_i - k_{i-1}) \big(t \, \mathrm{G}_1(K_{k_i}) + s \, \mathrm{G}_1(L_{k_i})\big) \\
&= c \left(t \sum_{x \in K \cap \mathbb{Z}} F(x) + s \sum_{y \in L \cap \mathbb{Z}} G(y) \right) = c \left(\frac{t}{a} \sum_{x \in K \cap \mathbb{Z}} f(x) + \frac{s}{b} \sum_{y \in L \cap \mathbb{Z}} g(y) \right).
\end{aligned} \tag{1.24}$$

To finish, we observe that if $\alpha \neq \infty$, then by setting

$$\lambda_1 = -\alpha, \quad \lambda_2 = \frac{\alpha}{\alpha+1}, \quad a_1 = \Big(\big(t^{1/\alpha} a\big)^{-1}, \, \big(s^{1/\alpha} b\big)^{-1} \Big) \quad \text{and}$$
$$a_2 = \left(t^{(\alpha+1)/\alpha} \sum_{x \in K \cap \mathbb{Z}} f(x), \; s^{(\alpha+1)/\alpha} \sum_{y \in L \cap \mathbb{Z}} g(y) \right),$$

and applying the reverse Hölder inequality (8) to the last term in (1.24) yields (1.19) for $n = 1$, as desired. If $\alpha = \infty$ this last inequality is trivial, since in that case $\alpha/(\alpha+1) = 1$.

Now, we assume the result holds for all dimensions up to $n-1$. The n-dimensional case will follow from the successive application of the $(n-1)$-dimensional and 1-dimensional cases.

First, we fix points $\tau_K \in \pi(K)$, $\tau_L \in \pi(L)$ and $\tau = t\tau_K + s\tau_L$, where $\pi(M) = \big\{ r \in \mathbb{R} : M(r) \neq \emptyset \big\}$ for $M \subset \mathbb{R}^n$. We also consider the functions $f_1, g_1, h_1 : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}_{\geq 0}$ given by

$$f_1(x) = f(x, \tau_K), \quad g_1(y) = g(y, \tau_L) \quad \text{and} \quad h_1(z) = h(z, \tau).$$

For any $x \in K(\tau_K)$ and any $y \in L(\tau_L)$, (1.18) implies that $h_1(tx+sy) \geq \big(t f_1(x)^\alpha + s g_1(y)^\alpha\big)^{1/\alpha}$. Therefore, the induction hypothesis (i.e. (1.19) for dimension $n-1$) yields

$$\sum_{z \in \Omega_1 \cap \mathbb{Z}^{n-1}} h_1{}^{\bar{\diamond}}(z) \geq \left(t \left(\sum_{x \in K(\tau_K) \cap \mathbb{Z}^{n-1}} f_1(x) \right)^{\beta} + s \left(\sum_{y \in L(\tau_L) \cap \mathbb{Z}^{n-1}} g_1(y) \right)^{\beta} \right)^{1/\beta}, \tag{1.25}$$

where $\Omega_1 = tK(\tau_K) + sL(\tau_L) + \big(-1, \lceil t+s \rceil\big)^{n-1}$ and $\beta = \alpha/\big((n-1)\alpha+1\big)$. By convexity, it is clear that $\Omega_1 \subset \Omega_2(\tau)$, where $\Omega_2 = tK + sL + \big(-1, \lceil t+s \rceil\big)^{n-1} \times \{0\}$. If we write $h'(z,r) = \sup_{u \in (-1,\lceil t+s \rceil)^{n-1}} h(z-u, r)$ for any $z \in \mathbb{R}^{n-1}$ and any $r \in \mathbb{R}$, then (1.25) implies

$$\sum_{z \in \Omega_2(\tau) \cap \mathbb{Z}^{n-1}} h'(z, \tau) \geq \left(t \left(\sum_{x \in K(\tau_K) \cap \mathbb{Z}^{n-1}} f_1(x) \right)^{\beta} + s \left(\sum_{y \in L(\tau_L) \cap \mathbb{Z}^{n-1}} g_1(y) \right)^{\beta} \right)^{1/\beta}. \tag{1.26}$$

Now, if we consider the functions $f_2, g_2, h_2 : \mathbb{R} \longrightarrow \mathbb{R}_{\geq 0}$ given by

$$f_2(r) = \sum_{x \in K(r) \cap \mathbb{Z}^{n-1}} f(x, r), \quad g_2(r) = \sum_{y \in L(r) \cap \mathbb{Z}^{n-1}} g(y, r) \quad \text{and} \quad h_2(r) = \sum_{z \in \Omega_2(r) \cap \mathbb{Z}^{n-1}} h'(z, r),$$

then (1.26) translates into $h_2(t\tau_K+s\tau_L)\geq\big(tf_2(\tau_K)^\beta+sg_2(\tau_L)^\beta\big)^{1/\beta}$. Since this reasoning is valid for any $\tau_K\in\pi(K)$ and any $\tau_L\in\pi(L)$, we can apply the 1-dimensional result to the sets $\pi(K),\pi(L)$ and the functions f_2,g_2,h_2 to obtain

$$\sum_{r\in\Omega_3\cap\mathbb{Z}}h_2{}^{\bar{\diamond}}(r)\geq\left(t\left(\sum_{\tau_K\in\pi(K)\cap\mathbb{Z}}f_2(\tau_K)\right)^{\gamma}+s\left(\sum_{\tau_L\in\pi(L)\cap\mathbb{Z}}g_2(\tau_L)\right)^{\gamma}\right)^{1/\gamma},\tag{1.27}$$

where $\Omega_3=t\pi(K)+s\pi(L)+\big(-1,\lceil t+s\rceil\big)=\pi\Big(tK+sL+\big(-1,\lceil t+s\rceil\big)^n\Big)$ and $\gamma=\beta/(\beta+1)=\alpha/(n\alpha+1)$.

On the one hand, it is clear that

$$\sum_{\tau_K\in\pi(K)\cap\mathbb{Z}}f_2(\tau_K)=\sum_{x\in K\cap\mathbb{Z}^n}f(x)\quad\text{and}\quad\sum_{\tau_L\in\pi(L)\cap\mathbb{Z}}g_2(\tau_L)=\sum_{y\in L\cap\mathbb{Z}^n}g(y).\tag{1.28}$$

On the other hand, we get

$$\sum_{r\in\Omega_3\cap\mathbb{Z}}h_2{}^{\bar{\diamond}}(r)\leq\sum_{z\in\left(M+(-1,\lceil t+s\rceil)^n\right)\cap\mathbb{Z}^n}h^{\bar{\diamond}}(z).$$

Indeed, since clearly

$$\Omega_2(r-w)=\Big(M+\big(-1,\lceil t+s\rceil\big)^{n-1}\times\{0\}\Big)(r-w)\subset\Big(M+\big(-1,\lceil t+s\rceil\big)^n\Big)(r)$$

for all $w\in\big(-1,\lceil t+s\rceil\big)$, we have that

$$\begin{aligned}\sum_{r\in\Omega_3\cap\mathbb{Z}}h_2{}^{\bar{\diamond}}(r)&=\sum_{r\in\Omega_3\cap\mathbb{Z}}\sup_{w\in(-1,\lceil t+s\rceil)}\sum_{z\in\Omega_2(r-w)\cap\mathbb{Z}^{n-1}}h'(z,r-w)\\&\leq\sum_{r\in\Omega_3\cap\mathbb{Z}}\sum_{z\in\left(M+(-1,\lceil t+s\rceil)^n\right)(r)\cap\mathbb{Z}^{n-1}}\sup_{w\in(-1,\lceil t+s\rceil)}h'(z,r-w).\end{aligned}$$

A straightforward computation shows that the above expression is equal to $\sum_{z\in\left(M+(-1,\lceil t+s\rceil)^n\right)\cap\mathbb{Z}^n}h^{\bar{\diamond}}(z)$. This, together with (1.27) and (1.28), finishes the proof. ■

Finally, we can easily derive Theorem 1.2 from Theorem 1.3:

Proof of Theorem 1.2. It suffices to apply Theorem 1.3 with $\alpha=\infty$, $f=\chi_{_K}$, $g=\chi_{_L}$ and $h=\chi_{_{tK+sL}}$, for which we clearly have that $h^{\bar{\diamond}}=\chi_{_{tK+sL+(-1,\lceil t+s\rceil)^n}}$. The equality check is trivial. ■

Remark 1.7 Theorem 1.2 can be extended to the setting of an arbitrary n-dimensional lattice $\Lambda\subset\mathbb{R}^n$. Indeed, if $\mathcal{B}=\{v_1,\dots,v_n\}$ is a basis of Λ, we may consider $\varphi:\mathbb{R}^n\longrightarrow\mathbb{R}^n$ the linear bijection given by $\varphi(x)=\sum_{i=1}^n x_iv_i$ for any $x=(x_1,\dots,x_n)\in\mathbb{R}^n$. Then, denoting by $\mathrm{G}_\Lambda(M)=|M\cap\Lambda|$, Theorem 1.2 yields

$$\mathrm{G}_\Lambda\Big(tK+sL+\varphi\big((-1,\lceil t+s\rceil)^n\big)\Big)^{1/n}\geq t\,\mathrm{G}_\Lambda(K)^{1/n}+s\,\mathrm{G}_\Lambda(L)^{1/n}$$

for any non-empty bounded sets $K,L\subset\mathbb{R}^n$ with $\mathrm{G}_\Lambda(K)\mathrm{G}_\Lambda(L)>0$ and all $t,s\geq0$. ♦

Theorem 1.2 allows us to obtain a property for the lattice point enumerator that resembles the homogeneity of the volume:

Corollary 1.8 Let $t\geq0$ and let $K\subset\mathbb{R}^n$ be a non-empty bounded set. Then

$$\mathrm{G_n}\Big(tK+\big(-1,\lceil t\rceil\big)^n\Big)\geq t^n\,\mathrm{G_n}(K).$$

Proof. We may assume, without loss of generality, that $\mathrm{G_n}(K)>0$. Then the result follows from Theorem 1.2 for $s=0$ and $L=\{0\}$. ■

To conclude the section, we point out that in [69, Corollary 2.3] it was shown that the discrete inequality (1.17) with $t=1-\lambda$ and $s=\lambda$ (for $\lambda\in(0,1)$) implies the corresponding form of the classical Brunn-Minkowski inequality (i.e., for such values of t and s). Since the latter is equivalent, by homogeneity, to (1.16) for arbitrary $t,s\geq0$, we immediately get the following result.

Theorem 1.9 [67, Theorem 1.3] The discrete Brunn-Minkowski inequality (1.17) implies the classical Brunn-Minkowski inequality (1.16) for non-empty compact sets.

Furthermore, a direct proof of this result can be given in a way similar to what is performed in Section 2.5 within the proof of Theorem 2.43, by directly approximating the volume by means of the lattice point enumerator, as the lattice shrinks.

1.3 A discrete L_p Brunn-Minkowski inequality for $p \geq 1$

In this section we will prove a discrete analogue of the L_p Brunn-Minkowski inequality (1.7) for the lattice point enumerator (see Theorem 1.10). We will also show that this new discrete inequality implies its continuous analogue (1.7) (see Theorem 1.19). In both cases, the approach will be to first obtain a more general functional version of the result, namely, an appropriate discrete Borell-Brascamp-Lieb type theorem (see Theorems 1.12 and 1.18). The results are collected in [65]. The main theorem of this section is the following one:

Theorem 1.10 [65, Theorem 1.1] Let $\lambda \in (0,1)$ and $p \geq 1$, and let $K, L \subset \mathbb{R}^n$ be bounded sets with $\mathrm{G_n}(K)\,\mathrm{G_n}(L) > 0$. Then

$$\mathrm{G_n}\big((1-\lambda)\cdot K +_p \lambda \cdot L + (-1,1)^n\big)^{p/n} \geq (1-\lambda)\,\mathrm{G_n}(K)^{p/n} + \lambda\,\mathrm{G_n}(L)^{p/n}. \tag{1.29}$$

The inequality is sharp.

For any fixed $p \geq 1$, the Minkowski addition of the cube $(-1,1)^n$ on the left-hand side of the latter inequality cannot be, in general, neither reduced (by means of a smaller cube) nor substituted by its p-sum (see Remark 1.15). And again, as in the classical framework, the case $p = 1$ of this result recovers (1.15).

1.3.1 Functional results: Background and main theorems

Following the spirit of [69], we will prove Theorem 1.10 for the lattice point enumerator $\mathrm{G_n}(\cdot)$ by showing (the more general version of) its functional counterpart (see Theorem 1.12). In particular, in [69], Theorem E was obtained by first proving the following discrete analogue of Borell-Brascamp-Lieb's inequality (Theorem C).

Theorem F Let $\lambda \in (0,1)$ and let $K, L \subset \mathbb{R}^n$ be non-empty bounded sets. Let $-1/n \leq \alpha \leq \infty$ and let $f, g, h : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ be non-negative functions such that

$$h\big((1-\lambda)x + \lambda y\big) \geq \big[(1-\lambda)f(x)^\alpha + \lambda g(y)^\alpha\big]^{1/\alpha}$$

for all $x \in K$, $y \in L$ with $f(x)g(y) > 0$. Then

$$\sum_{z \in (M+(-1,1)^n)\cap\mathbb{Z}^n} h^\diamond(z) \geq \mathcal{M}^{\lambda}_{\frac{\alpha}{n\alpha+1}}\left(\sum_{x\in K\cap\mathbb{Z}^n} f(x), \sum_{y\in L\cap\mathbb{Z}^n} g(y)\right),$$

where $M = (1-\lambda)K + \lambda L$.

Taking into account the definition of p-sum given by (1.8), it is natural to wonder about the possibility of extending Theorem C to the L_p setting by suitably modifying the condition on the functions there involved (cf. (1.4)). Such an expected L_p version of the Borell-Brascamp-Lieb inequality has been very recently obtained in [92] (shown independently, for the case $\alpha > 0$, in [103]):

Theorem G Let $\lambda\in(0,1)$ and $p\geq 1$. Let $-1/n\leq\alpha\leq\infty$ and let $f,g,h:\mathbb{R}^n\longrightarrow\mathbb{R}_{\geq 0}$ be integrable functions such that

$$h\Big((1-\lambda)^{1/p}(1-\mu)^{1/q}x+\lambda^{1/p}\mu^{1/q}y\Big)\geq\Big[(1-\lambda)^{1/p}(1-\mu)^{1/q}f(x)^\alpha+\lambda^{1/p}\mu^{1/q}g(y)^\alpha\Big]^{1/\alpha} \quad (1.30)$$

for all $x,y\in\mathbb{R}^n$ with $f(x)g(y)>0$ and all $\mu\in[0,1]$. Then

$$\int_{\mathbb{R}^n}h(x)\,\mathrm{d}x\geq\mathcal{M}^{\lambda}_{\frac{p\alpha}{n\alpha+1}}\left(\int_{\mathbb{R}^n}f(x)\,\mathrm{d}x,\int_{\mathbb{R}^n}g(x)\,\mathrm{d}x\right).$$

Before continuing, we would like to clarify the special case of $\alpha=0$ in condition (1.30) of the previous result:

Remark 1.11 On the one hand, it is clear that

$$\lim_{\alpha\to 0^+}\left(sa^\alpha+rb^\alpha\right)^{1/\alpha}=\begin{cases} a^{1-r}b^r & \text{if } s+r=1,\\ 0 & \text{if } s+r<1.\end{cases}$$

On the other hand, we have

$$(1-\lambda)^{1/p}(1-\mu)^{1/q}+\lambda^{1/p}\mu^{1/q}=1$$

if $\mu=\lambda$, and

$$(1-\lambda)^{1/p}(1-\mu)^{1/q}+\lambda^{1/p}\mu^{1/q}<1$$

for all $\mu\in[0,1]$ with $\mu\neq\lambda$, by Hölder's inequality (7) (jointly with its equality case). Then, by convention, the case $\alpha=0$ in (1.30) will be understood as

$$h\big((1-\lambda)x+\lambda y\big)\geq f(x)^{1-\lambda}g(y)^\lambda$$

for all $x,y\in\mathbb{R}^n$. In other words, the case $\alpha=0$ in Theorem G is the same to the one in Theorem C, i.e., the classical *Prékopa-Leindler inequality* (see Theorem B). ♦

We will now show the corresponding L_p version of Theorem F. In other words, we prove a discrete analogue of Theorem G, which, in particular, will imply Theorem 1.10. We follow here the underlying idea of the original proof of (1.7) given in [76].

Theorem 1.12 [65, Theorem 2.1] Let $\lambda\in(0,1)$ and $p\geq 1$, and let $K,L\subset\mathbb{R}^n$ be non-empty bounded sets. Let $-1/n\leq\alpha\leq\infty$ and let $f,g,h:\mathbb{R}^n\longrightarrow\mathbb{R}_{\geq 0}$ be non-negative functions such that

$$h\Big((1-\lambda)^{1/p}(1-\mu)^{1/q}x+\lambda^{1/p}\mu^{1/q}y\Big)\geq\Big[(1-\lambda)^{1/p}(1-\mu)^{1/q}f(x)^\alpha+\lambda^{1/p}\mu^{1/q}g(y)^\alpha\Big]^{1/\alpha} \quad (1.31)$$

for all $x\in K$, $y\in L$ with $f(x)g(y)>0$ and all $\mu\in[0,1]$. Then

$$\sum_{z\in(M_p+(-1,1)^n)\cap\mathbb{Z}^n}h^\diamond(z)\geq\mathcal{M}^{\lambda}_{\frac{p\alpha}{n\alpha+1}}\left(\sum_{x\in K\cap\mathbb{Z}^n}f(x),\sum_{y\in L\cap\mathbb{Z}^n}g(y)\right), \quad (1.32)$$

where $M_p=(1-\lambda)\cdot K+_p\lambda\cdot L$.

As in the classical framework, the case $\alpha=0$ in this result is that of Theorem F (see Remark 1.11).
Proof. Along the proof, we will assume that

$$\left(\sum_{x\in K\cap\mathbb{Z}^n}f(x)\right)\left(\sum_{y\in L\cap\mathbb{Z}^n}g(y)\right)>0,$$

since the result is trivial otherwise. Now we set, for any given $\mu_0 \in [0,1]$ (to be suitably chosen later),

$$t = t(\mu_0) := (1-\lambda)^{1/p}(1-\mu_0)^{1/q} \quad \text{and} \quad s = s(\mu_0) := \lambda^{1/p}\mu_0^{1/q},$$

for which one has, by Hölder's inequality (7), that $t+s \leq 1$. Notice that the assumption (1.31) can be then rewritten, in terms of t,s, as

$$h(tx+sy) \geq \left[tf(x)^\alpha + sg(y)^\alpha\right]^{1/\alpha}$$

for all $x \in K$ and $y \in L$ with $f(x)g(y) > 0$, and thus Theorem 1.3 yields

$$\sum_{z\in\left(tK+sL+(-1,\lceil t+s\rceil)^n\right)\cap\mathbb{Z}^n} h^\diamond(z) \geq \mathcal{S}^{t,s}_{\frac{\alpha}{n\alpha+1}}\left(\sum_{x\in K\cap\mathbb{Z}^n} f(x), \sum_{y\in L\cap\mathbb{Z}^n} g(y)\right). \tag{1.33}$$

Moreover, from (1.8) we clearly have

$$M_p = (1-\lambda)\cdot K +_p \lambda\cdot L \supset (1-\lambda)^{1/p}(1-\mu_0)^{1/q}K + \lambda^{1/p}\mu_0^{1/q}L = tK+sL.$$

This, together with (1.33) and the fact that $(-1,1)^n \supset \left(-1,\lceil t+s\rceil\right)^n$, allows us to conclude that

$$\sum_{z\in\left(M_p+(-1,1)^n\right)\cap\mathbb{Z}^n} h^\diamond(z) \geq \mathcal{S}^{t,s}_{\frac{\alpha}{n\alpha+1}}\left(\sum_{x\in K\cap\mathbb{Z}^n} f(x), \sum_{y\in L\cap\mathbb{Z}^n} g(y)\right).$$

Notice also that if $\alpha = -1/n$ then $\alpha/(n\alpha+1) = -\infty$ and hence we are done. Then, in the following we may assume that $\alpha \neq 0, -1/n$ (cf. Remark 1.11). For the sake of brevity we write

$$\beta := \frac{\alpha}{n\alpha+1} \in (-\infty,0)\cup\left(0,\frac{1}{n}\right], \quad F := \sum_{x\in K\cap\mathbb{Z}^n} f(x) \quad \text{and} \quad G := \sum_{y\in L\cap\mathbb{Z}^n} g(y).$$

Thus, taking

$$\mu_0 := \frac{\lambda\left(\sum_{y\in L\cap\mathbb{Z}^n} g(y)\right)^{p\beta}}{(1-\lambda)\left(\sum_{x\in K\cap\mathbb{Z}^n} f(x)\right)^{p\beta} + \lambda\left(\sum_{y\in L\cap\mathbb{Z}^n} g(y)\right)^{p\beta}} = \frac{\lambda G^{p\beta}}{(1-\lambda)F^{p\beta}+\lambda G^{p\beta}},$$

a straightforward computation yields

$$\begin{aligned}
\mathcal{S}^{t,s}_\beta(F,G) &= \left((1-\lambda)^{1/p}(1-\mu_0)^{1/q}F^\beta + \lambda^{1/p}\mu_0^{1/q}G^\beta\right)^{1/\beta} \\
&= \left((1-\lambda)^{1/p}\left[\frac{(1-\lambda)F^{p\beta}}{(1-\lambda)F^{p\beta}+\lambda G^{p\beta}}\right]^{1/q} F^\beta + \lambda^{1/p}\left[\frac{\lambda G^{p\beta}}{(1-\lambda)F^{p\beta}+\lambda G^{p\beta}}\right]^{1/q} G^\beta\right)^{1/\beta} \\
&= \left(\frac{(1-\lambda)F^{\beta(1+p/q)}+\lambda G^{\beta(1+p/q)}}{\left((1-\lambda)F^{p\beta}+\lambda G^{p\beta}\right)^{1/q}}\right)^{1/\beta} = \left((1-\lambda)F^{p\beta}+\lambda G^{p\beta}\right)^{1/(p\beta)} = \mathcal{M}^\lambda_{p\beta}(F,G).
\end{aligned}$$

This concludes the proof. ■

Remark 1.13 Following the same approach as the one in the proof of Theorem 1.12, just replacing the sums of the functions f, g and h by their integrals on $\mathbb{R}^n$, one may also derive Theorem G. For this, one can similarly exploit the suitable version of Theorem 1.5 for linear combinations $tx+sy$ instead of the one for means $(1-\lambda)x+\lambda y$ (see [23]). ♦

An analogous result for arbitrary lattices can be obtained. Given an n-dimensional lattice Λ with basis $\mathcal{B}=\{v_1,\dots,v_n\}$, let $\varphi:\mathbb{R}^n\longrightarrow\mathbb{R}^n$ be the linear (bijective) map defined by $\varphi(x)=\sum_{i=1}^n x_iv_i$ for each $x=(x_1,\dots,x_n)\in\mathbb{R}^n$. Taking into account the pointwise definition of the p-sum given in (1.8), we have

$$\varphi\big((1-\lambda)\cdot\varphi^{-1}(K)+_p\lambda\cdot\varphi^{-1}(L)\big)=(1-\lambda)\cdot K+_p\lambda\cdot L.$$

This allows us to extend the statement of Theorem 1.12 to the setting of an n-dimensional lattice $\Lambda\subset\mathbb{R}^n$, by considering the auxiliary functions $f_{\mathcal{B}},g_{\mathcal{B}},h_{\mathcal{B}}:\mathbb{R}^n\longrightarrow\mathbb{R}_{\geq 0}$ given by

$$f_{\mathcal{B}}(x)=f\big(\varphi(x)\big),\quad g_{\mathcal{B}}(x)=g\big(\varphi(x)\big)\ \text{ and }\ h_{\mathcal{B}}(x)=h\big(\varphi(x)\big)$$

for any $x\in\mathbb{R}^n$, as follows:

Corollary 1.14 Let $\lambda\in(0,1)$, $-1/n\leq\alpha\leq\infty$, and $p\geq 1$. Consider non-empty bounded sets $K,L\subset\mathbb{R}^n$. Let $f,g,h:\mathbb{R}^n\longrightarrow\mathbb{R}_{\geq 0}$ be non-negative functions such that

$$h\Big((1-\lambda)^{1/p}(1-\mu)^{1/q}x+\lambda^{1/p}\mu^{1/q}y\Big)\geq\Big[(1-\lambda)^{1/p}(1-\mu)^{1/q}f(x)^\alpha+\lambda^{1/p}\mu^{1/q}g(y)^\alpha\Big]^{1/\alpha}$$

for all $x\in K$, $y\in L$ with $f(x)g(y)>0$ and all $\mu\in[0,1]$. Let $\Lambda\subset\mathbb{R}^n$ be an n-dimensional lattice with basis $\mathcal{B}=\{v_1,\dots,v_n\}$ and let $\varphi(x)=\sum_{i=1}^n x_iv_i$ for $x\in\mathbb{R}^n$. Then

$$\sum_{z\in\left(M_p+\varphi((-1,1)^n)\right)\cap\Lambda}h^{\diamond_{\mathcal{B}}}(z)\geq\mathcal{M}^{\lambda}_{\frac{p\alpha}{n\alpha+1}}\left(\sum_{x\in K\cap\Lambda}f(x),\ \sum_{y\in L\cap\Lambda}g(y)\right),$$

where $M_p=(1-\lambda)\cdot K+_p\lambda\cdot L$ and $h^{\diamond_{\mathcal{B}}}(z)=\sup_{u\in\varphi((-1,1)^n)}h(z+u)$ for all $z\in\mathbb{R}^n$.

1.3.2 Some geometric consequences

We start by showing, as previously announced, that, as in the classical setting, the geometric inequality (1.29) can be derived from the functional one (1.32):

Proof of Theorem 1.10. By applying (1.32) with $\alpha=\infty$ to the characteristic functions $f=\chi_K$, $g=\chi_L$ and $h=\chi_{(1-\lambda)\cdot K+_p\lambda\cdot L}$, for which $h^\diamond=\chi_{(1-\lambda)\cdot K+_p\lambda\cdot L+(-1,1)^n}$, one immediately gets (1.29).

Finally, to show that the equality can be attained, it is enough to consider $K=L=[0,m]^n$ with $m\in\mathbb{N}$, for which $\mathrm{G_n}\big((1-\lambda)\cdot K+_p\lambda L+(-1,1)^n\big)=(m+1)^n=\mathrm{G_n}(K)=\mathrm{G_n}(L)$. ■

Now, for bounded sets $K,L\subset\mathbb{R}^n$ with $\mathrm{G_n}(K)\mathrm{G_n}(L)>0$, it was shown in [69] that

$$\mathrm{G_n}\left(\frac{K+L}{2}+[0,1]^n\right)^{1/n}\geq\frac{\mathrm{G_n}(K)^{1/n}+\mathrm{G_n}(L)^{1/n}}{2},$$

i.e., that (1.15) for $\lambda=1/2$ also holds by replacing the cube $(-1,1)^n$ by $[0,1]^n$. However, the latter inequality is in general not true for any $\lambda\in(0,1)$. Thus, and regarding (1.29), it is a natural question to wonder whether $(-1,1)^n$ might be similarly reduced to a smaller cube.

Remark 1.15 We notice on the one hand that the set $(-1,1)^n$ cannot be reduced to a strictly smaller cube of the form $(-1,a]^n$ (or $[-a,1)^n$) with $a\in(0,1)$, for any fixed value of $p\geq 1$. Indeed, it is enough to consider, as an example, the sets $K=[0,1]$, $L=[0,2]$ in dimension $n=1$ and the combination

$$M=(1-\lambda)\cdot K+_p\lambda\cdot L+(-1,a]=\Big(-1,\mathcal{M}_p^\lambda(1,2)+a\Big]=\Big(-1,(1-\lambda+\lambda 2^p)^{1/p}+a\Big]$$

(observe that $K,L\subset\mathbb{R}^n$ are n-dimensional convex bodies containing the origin and hence, as mentioned in the introduction, the p-sum defined by (1.8) agrees with the classical definition given by (1.6)).

Then, since $\mathrm{G}_1(M) = \lfloor \mathcal{M}_p^\lambda(1,2) + a \rfloor + 1$ and $\mathcal{M}_p^\lambda(1,2) \in [1,2]$, it is enough to find $\lambda > 0$ such that $\mathcal{M}_p^\lambda(1,2) + a < 2$. But this is always possible because

$$\lim_{\lambda \to 0^+} \mathcal{M}_p^\lambda(1,2) = \lim_{\lambda \to 0^+} (1 - \lambda + \lambda 2^p)^{1/p} = 1,$$

and therefore we have $\mathrm{G}_1(M) = 2$ for $\lambda > 0$ small enough. However, for the right-hand side of (1.29) we have $\mathrm{G}_1(K) = 2$ and $\mathrm{G}_1(L) = 3$, and thus

$$\mathcal{M}_p^\lambda\left(\mathrm{G}_1(K), \mathrm{G}_1(L)\right) = \mathcal{M}_p^\lambda(2,3) \in [2,3].$$

Since $\lambda > 0$, we know that $\mathcal{M}_p^\lambda\left(\mathrm{G}_1(K), \mathrm{G}_1(L)\right) > 2$, which shows that

$$\mathrm{G}_1\big((1-\lambda) \cdot K +_p \lambda \cdot L + (-1, a]\big) < \mathcal{M}_p^\lambda\left(\mathrm{G}_1(K), \mathrm{G}_1(L)\right).$$

On the other hand, taking a look at (1.15), one could think that its natural L_p version could be given by considering the p-sum of the cube $(-1,1)^n$ on the left-hand side of (1.29) (instead of its Minkowski addition). In fact, when dealing with n-dimensional convex bodies $K, L \subset \mathbb{R}^n$ containing the origin, one has that

$$(1-\lambda) \cdot K +_p \lambda \cdot L +_p [-1,1]^n \subset (1-\lambda) \cdot K +_p \lambda \cdot L + [-1,1]^n$$

for any $p \geq 1$ (see [41]). So, p-summing the cube $(-1,1)^n$ on the left-hand side of (1.29) would be, sometimes, tighter than (Minkowski) adding it. Nevertheless, this is not possible either. Indeed, by again considering the sets $K = [0,1]$, $L = [0,2]$ in dimension $n = 1$ and $p = 2$, for which we have by (1.6) that

$$\frac{1}{2} \cdot K +_2 \frac{1}{2} \cdot L = \left[0, \sqrt{2.5}\,\right],$$

we get, now using (1.8),

$$\mathrm{G}_1\!\left(\frac{1}{2} \cdot K +_2 \frac{1}{2} \cdot L +_2 (-1,1)\right) \leq \mathrm{G}_1\left(\left(-1, \sqrt{3.5}\right)\right) = 2 < \sqrt{6.5} = \mathcal{M}_2^{1/2}(2,3) = \mathcal{M}_2^{1/2}\left(\mathrm{G}_1(K), \mathrm{G}_1(L)\right),$$

as desired. ♦

We observe now that Theorem 1.10 holds also true for arbitrary non-negative (L_p) linear combinations of K and L, albeit with a suitable modification of the cube. More precisely, we have:

Corollary 1.16 Let $t, s \geq 0$ and $p \geq 1$, and let $K, L \subset \mathbb{R}^n$ be bounded sets such that $\mathrm{G_n}(K)\,\mathrm{G_n}(L) > 0$. Then

$$\mathrm{G_n}\left(t \cdot K +_p s \cdot L + \left(-1, \left\lceil (t+s)^{1/p} \right\rceil\right)^n\right)^{p/n} \geq t\,\mathrm{G_n}(K)^{p/n} + s\,\mathrm{G_n}(L)^{p/n}.$$

Proof. The proof follows the same argument to that of Theorem 1.12, by replacing $1-\lambda$ and λ by t and s, respectively, for the characteristic functions $f = \chi_K$, $g = \chi_L$ and $h = \chi_{t \cdot K +_p s \cdot L}$. So, in this case, it is enough to set

$$\bar{t} = \bar{t}(\mu_0) := t^{1/p}(1-\mu_0)^{1/q} \quad \text{and} \quad \bar{s} = \bar{s}(\mu_0) := s^{1/p}\mu_0^{1/q},$$

for which $\bar{t} + \bar{s} \leq (t+s)^{1/p}$ by Hölder's inequality (7), and then

$$\left(-1, \left\lceil (t+s)^{1/p} \right\rceil\right)^n \supset \left(-1, \left\lceil \bar{t} + \bar{s} \right\rceil\right)^n.$$

The proof is now concluded as the one of Theorem 1.12. ■

Regarding Theorem D, it makes no sense to wonder about an L_p version of inequality (1.14), by just replacing $A + B$ by $A +_p B$ on the left-hand side, since $A +_p B$ is no longer finite (see (1.8)), for $p > 1$. However, from Corollary 1.16 for $K = A$, $L = B$ and $t = s = 1$ we get the following result:

Corollary 1.17 Let $A,B \subset \mathbb{Z}^n$ be finite, $A,B \neq \emptyset$. Then

$$\mathrm{G_n}\big(A+_p B+(-1,2)^n\big)^{p/n} \geq |A|^{p/n}+|B|^{p/n}. \tag{1.34}$$

Clearly, for $p=1$, the latter inequality is exactly (1.14), since $A+B\subset\mathbb{Z}^n$ and the sole integer points in $(-1,2)^n$ are those in $\{0,1\}^n$.

We would further like to note that unlike in the linear case ($p=1$), the cube $(-1,2)^n$ on the left-hand side of (1.34) cannot be, in general, reduced to $\{0,1\}^n$ or even to $[0,1]^n$. To see this, it is enough to consider $n=1$, $A=\{0,\dots,a\}$ and $B=\{0,\dots,b\}$ for some $a,b\in\mathbb{N}$ with $0<a\leq b$. Indeed, on the one hand, taking into account that $\mathcal{S}_p(\cdot,\cdot)$ is decreasing in p, we have (see e.g. [56, Theorem 19])

$$\mathcal{S}_p\big(|A|,|B|\big)=\mathcal{S}_p(a+1,b+1)\in[b+1,a+b+2],$$

and moreover, $\mathcal{S}_p(a+1,b+1)>a+b+1$ for $p>1$ small enough. On the other hand, if we denote by $K=[0,a]$ and $L=[0,b]$, then $K+_pL=\big[0,\mathcal{S}_p(a,b)\big]$ since K and L are 1-dimensional convex bodies containing the origin (and thus their p-sum is also given by (1.6)). Furthermore, due to the fact that $\mathcal{S}_p(a,b)<a+b$ for all $p>1$, we obtain $\big\lfloor\mathcal{S}_p(a,b)\big\rfloor+1\leq a+b$. Therefore, altogether we get

$$\begin{aligned}\mathrm{G_1}\big(A+_pB+[0,1]\big)&\leq \mathrm{G_1}\big(K+_pL+[0,1]\big)=\big\lfloor\mathcal{S}_p(a,b)\big\rfloor+2\\&\leq a+b+1<\mathcal{S}_p(a+1,b+1)=\mathcal{S}_p\big(|A|,|B|\big)\end{aligned}$$

for any $p>1$ small enough. In fact, taking for instance $a=b=1$ and $p=3/2$, the latter inequality holds, which shows that $[0,1]^n$ cannot replace $(-1,2)^n$ on the left-hand side of (1.34).

1.3.3 From the discrete setting to the continuous one

Next we show that our discrete counterpart, Theorem 1.12, implies the continuous result collected in Theorem G, under mild assumptions for the functions there involved, in the spirit of what happens for $p=1$ (see [69, Theorem 2.4]).

Theorem 1.18 [65, Theorem 2.2] The discrete L_p Borell-Brascamp-Lieb type inequality (Theorem 1.12) implies the (continuous) L_p Borell-Brascamp-Lieb inequality (Theorem G), provided that the functions f,g are Riemann integrable and h is upper semicontinuous.

Proof. Let $f,g,h:\mathbb{R}^n\longrightarrow\mathbb{R}_{\geq 0}$ be functions in the conditions of Theorem G, namely, verifying (1.30) for all $x,y\in\mathbb{R}^n$ with $f(x)g(y)>0$ and all $\mu\in[0,1]$, for some fixed $p\geq1$, $\lambda\in(0,1)$ and $-1/n\leq\alpha\leq\infty$.

We will first prove that, given $k\in\mathbb{N}\setminus\{0\}$ and $C=[-k,k]^n$, we have

$$\int_C h(z)\,\mathrm{d}z\geq\mathcal{M}^{\lambda}_{\frac{p\alpha}{n\alpha+1}}\left(\int_C f(x)\,\mathrm{d}x,\int_C g(x)\,\mathrm{d}x\right). \tag{1.35}$$

Theorem G will then follow simply by taking limits as $k\to\infty$. To this aim, we may assume that the functions f, g and h vanish outside C (multiplying them by the characteristic functions of C, if necessary). We shall also write $C_0=[-k,k)^n$.

For each $m\in\mathbb{N}$, we let $\mathrm{R}^m=[0,2^{-m})$ and $\mathrm{O}^m=\mathrm{R}^m-\mathrm{R}^m=(-2^{-m},2^{-m})$, and we define the functions $f_m,g_m,h_m:\mathbb{R}^n\longrightarrow\mathbb{R}_{\geq0}$ given by

$$f_m(x)=\sup_{z\in x+\mathrm{R}^m}f(z),\quad g_m(x)=\sup_{z\in x+\mathrm{R}^m}g(z)\quad\text{and}\quad h_m(x)=\sup_{z\in x+\mathrm{R}^m}h(z).$$

Moreover, for the sake of simplicity, we set $t:=(1-\lambda)^{1/p}(1-\mu)^{1/q}$ and $s:=\lambda^{1/p}\mu^{1/q}$ for any given $\mu\in[0,1]$, for which we get, as a consequence of Hölder's inequality (7), that $t+s\leq1$. Again, condition

(1.30) can be rewritten in terms of t, s as

$$h(tz_1 + sz_2) \geq \left[tf(z_1)^\alpha + sg(z_2)^\alpha\right]^{1/\alpha}$$

for all $z_1, z_2 \in \mathbb{R}^n$ with $f(z_1)g(z_2) > 0$. Thus, since $(t+s)\mathrm{R}^m \subset \mathrm{R}^m$, we have

$$\begin{aligned}
h_m(tx+sy) &= \sup_{z \in tx+sy+\mathrm{R}^m} h(z) \geq \sup_{z \in t(x+\mathrm{R}^m)+s(y+\mathrm{R}^m)} h(z) = \sup_{z_1 \in x+\mathrm{R}^m, z_2 \in y+\mathrm{R}^m} h(tz_1+sz_2) \\
&\geq \sup_{z_1 \in x+\mathrm{R}^m, z_2 \in y+\mathrm{R}^m} \left[tf(z_1)^\alpha + sg(z_2)^\alpha\right]^{1/\alpha} = \left[t\left(\sup_{z_1 \in x+\mathrm{R}^m} f(z_1)\right)^\alpha + s\left(\sup_{z_2 \in y+\mathrm{R}^m} g(z_2)\right)^\alpha\right]^{1/\alpha} \\
&= \left[tf_m(x)^\alpha + sg_m(y)^\alpha\right]^{1/\alpha}
\end{aligned}$$

for all $x, y \in C$ (and so, in particular, for all $x, y \in C_0$) with $f_m(x)g_m(y) > 0$ and all $\mu \in [0,1]$. Hence, the functions f_m, g_m, h_m are in the conditions of Corollary 1.14 and we may apply it for the sets $K = L = C_0$ and the lattice $2^{-m}\mathbb{Z}^n$. Note that in this case $\varphi\big((-1,1)^n\big) = \mathrm{O}^m$ and thus we obtain

$$\sum_{z \in [M_p + \mathrm{O}^m] \cap 2^{-m}\mathbb{Z}^n} h_m^{\diamond_m}(z) \geq \mathcal{M}^{\lambda}_{\frac{p\alpha}{n\alpha+1}}\left(\sum_{x \in C_0 \cap 2^{-m}\mathbb{Z}^n} f_m(x), \sum_{y \in C_0 \cap 2^{-m}\mathbb{Z}^n} g_m(y)\right), \tag{1.36}$$

where $M_p = (1-\lambda)\cdot C_0 +_p \lambda \cdot C_0$ and $h_m^{\diamond_m}(z) = \sup_{u \in \mathrm{O}^m} h_m(z+u)$. Now, since C is an n-dimensional convex body containing the origin, from (1.6) we get

$$(1-\lambda)\cdot C_0 +_p \lambda \cdot C_0 \subset (1-\lambda)\cdot C +_p \lambda \cdot C = C,$$

which, jointly with the fact that $(C+\mathrm{O}^m) \cap 2^{-m}\mathbb{Z}^n = C \cap 2^{-m}\mathbb{Z}^n$, allows us to deduce (from (1.36)) that

$$\sum_{z \in C \cap 2^{-m}\mathbb{Z}^n} h_m^{\diamond_m}(z) \geq \mathcal{M}^{\lambda}_{\frac{p\alpha}{n\alpha+1}}\left(\sum_{x \in C_0 \cap 2^{-m}\mathbb{Z}^n} f_m(x), \sum_{y \in C_0 \cap 2^{-m}\mathbb{Z}^n} g_m(y)\right). \tag{1.37}$$

We now consider the function $\overline{h} : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ given by $\overline{h}(x) = \sup_{\theta \in 3\mathrm{O}^m} h(x+\theta)$, and show that, for every fixed $z \in \mathbb{R}^n$ and any $x \in z + \mathrm{O}^m$, we have $\overline{h}(x) \geq h_m^{\diamond_m}(z)$. Indeed,

$$\begin{aligned}
\overline{h}(x) &= \sup_{\theta \in 3\mathrm{O}^m} h(x+\theta) = \sup_{w \in \mathrm{O}^m} \sup_{v \in \mathrm{O}^m} \sup_{u \in \mathrm{O}^m} h(x+u+v+w) \geq \sup_{w \in \mathrm{O}^m} \sup_{v \in \mathrm{O}^m} \sup_{u \in \mathrm{R}^m} h(x+u+v+w) \\
&= \sup_{w \in \mathrm{O}^m} \sup_{v \in \mathrm{O}^m} h_m(x+v+w) = \sup_{w \in \mathrm{O}^m} h_m^{\diamond_m}(x+w) \geq h_m^{\diamond_m}(z).
\end{aligned} \tag{1.38}$$

Furthermore, for any $r > 0$ let

$$C_r = \big\{x \in C : h(x) \geq r\big\} \quad \text{and} \quad \overline{C}_r = \big\{x \in C + \mathrm{R}^m : \overline{h}(x) > r\big\}.$$

Notice that the superlevel sets C_r are compact, since h is upper semicontinuous and C is compact (see, e.g., [87, Theorem 1.6]), and then we clearly have $C_r = \bigcap_{m=1}^{\infty}(C_r + 3\mathrm{O}^m)$. Moreover, since h vanishes outside C, from the definition of $\overline{h}$ we get $\overline{C}_r \subset C_r + 3\mathrm{O}^m$ for all $r > 0$. Thus, by Fubini's theorem and the monotone convergence theorem, we obtain

$$\begin{aligned}
\int_C h(x)\,\mathrm{d}x &= \int_0^\infty \mathrm{vol}(C_r)\,\mathrm{d}r = \int_0^\infty \mathrm{vol}\left(\bigcap_{m=1}^{\infty}(C_r + 3\mathrm{O}^m)\right)\mathrm{d}r = \int_0^\infty \lim_{m\to\infty} \mathrm{vol}(C_r + 3\mathrm{O}^m)\,\mathrm{d}r \\
&= \lim_{m\to\infty} \int_0^\infty \mathrm{vol}(C_r + 3\mathrm{O}^m)\,\mathrm{d}r \geq \lim_{m\to\infty} \int_0^\infty \mathrm{vol}(\overline{C}_r)\,\mathrm{d}r = \lim_{m\to\infty} \int_{C+\mathrm{R}^m} \overline{h}(x)\,\mathrm{d}x.
\end{aligned}$$

This, together with (1.38) and the fact that $C + \mathrm{R}^m = C \cap 2^{-m}\mathbb{Z}^n + \mathrm{R}^m$, implies that

$$\int_C h(x)\,\mathrm{d}x \geq \lim_{m\to\infty} \int_{C+\mathrm{R}^m} \overline{h}(x)\,\mathrm{d}x \geq \lim_{m\to\infty} 2^{-mn} \sum_{z \in C \cap 2^{-m}\mathbb{Z}^n} h_m^{\diamond_m}(z).$$

Finally, since f is Riemann integrable and

$$2^{-mn}\sum_{x\in C_0\cap 2^{-m}\mathbb{Z}^n} f_m(x)$$

is an *upper Riemann sum* of f for the partition

$$\{x+\mathrm{R}^m : x\in C_0\cap 2^{-m}\mathbb{Z}^n\}$$

of C, we clearly have

$$\lim_{m\to\infty} 2^{-mn}\sum_{x\in C_0\cap 2^{-m}\mathbb{Z}^n} f_m(x)=\int_C f(x)\,\mathrm{d}x.$$

The same holds for the function g and then, taking limits on both sides of (1.37), we get (1.35). This finishes the proof. ∎

We conclude this section by showing that the L_p Brunn-Minkowski inequality (1.7), in the setting of n-dimensional convex bodies, can be derived as a consequence of the new discrete inequality (1.29) for the lattice point enumerator $\mathrm{G_n}(\cdot)$:

Theorem 1.19 [65, Theorem 1.2] The discrete L_p Brunn-Minkowski type inequality (1.29) implies the L_p Brunn-Minkowski inequality (1.7) for n-dimensional convex bodies K and L.

Proof. Due to the well-known fact that a function is Riemann integrable if and only if it is continuous almost everywhere, since the boundary of a convex set has null measure, and also taking into account the characterization of the upper semicontinuity in terms of the level sets, we directly get the desired result as a consequence of Theorem 1.18. ∎

We emphasize the necessity of assuming convexity in Theorem 1.19: if one considers, for instance, bounded measurable sets $K,L\subset\mathbb{R}^n$ of positive volume containing no rational points, one clearly cannot expect to recover the L_p Brunn-Minkowski inequality (1.7) by shrinking the lattice $\mathbb{Z}^n$ by means of successively considering $2^{-m}\mathbb{Z}^n$, $m\in\mathbb{N}$.

1.4 A discrete log-Brunn-Minkowski inequality. The L_p case for $0<p<1$.

In this section we will obtain discrete versions of the log-Brunn-Minkowski inequality (1.10) for various point enumerators (including the lattice point enumerator) in some special cases, namely, when the convex bodies involved are unconditional, or simply origin-symmetric in the plane (see Theorem 1.20 and Corollary 1.26). The methods developed will also be extended to the L_p setting for $0<p<1$, thus yielding discrete analogues of (1.11) (see Theorem 1.23). Finally, inequalities for some alternatives measures will also be obtained using a similar technique to the one employed in the previous section, i.e., functionalization via Borell-Brascamp-Lieb type inequalities (see Theorems 1.28 and 1.29). The results contained in this section have been collected in [64].

1.4.1 Preliminaries and notation

To begin with, let us establish some notation, for the sake of simplicity. Given $x,y\in\mathbb{R}^n$, we will write $xy\in\mathbb{R}^n$ to denote the point with coordinates $(xy)_i=x_iy_i$ for all $i=1,\dots,n$, while, if $x\in\mathbb{R}^n_{\geq 0}$, x^λ will be the point such that $(x^\lambda)_i=x_i^\lambda$ for any given $\lambda>0$, $i=1,\dots,n$. Analogously, we will denote by $AB=\{ab\in\mathbb{R}^n : a\in A, b\in B\}$ for any $A,B\subset\mathbb{R}^n$, as well as $A^\lambda=\{a^\lambda : a\in A\}$ for any set $A\subset\mathbb{R}^n_{\geq 0}$ and any scalar $\lambda>0$. Finally, for any bounded set $K\subset\mathbb{R}^n$ and any discrete set Λ, we will use the functional $\mathrm{G}_\Lambda(K)=|K\cap\Lambda|$.

Next we introduce an operation closely related to the standard p-sum of convex bodies, which was utilized in [95] and [78] for the L_p Brunn-Minkowski inequalities discussed in Section 1.1.1. Given two non-empty sets $K,L\subset\mathbb{R}^n_{\geq 0}$ and $\lambda\geq 0$,

$$\begin{aligned}(1-\lambda)\cdot K\oplus_p\lambda\cdot L=\Big\{\Big(\big((1-\lambda)x_1^p+\lambda y_1^p\big)^{1/p},\dots,\big((1-\lambda)x_n^p+\lambda y_n^p\big)^{1/p}\Big):\\ (x_1,\dots,x_n)\in K,(y_1,\dots,y_n)\in L\Big\}.\end{aligned}\tag{1.39}$$

Again, the case $p=0$ must be understood as its limit case, and thus,

$$(1-\lambda)\cdot K\oplus_0\lambda\cdot L=K^{1-\lambda}L^{\lambda}.$$

It was proved in [78] that $(1-\lambda)\cdot K\oplus_p\lambda\cdot L\subset(1-\lambda)\cdot K+_p\lambda\cdot L$ for any $p\in[0,1]$, which implies that, in order to obtain an L_p Brunn-Minkowski type inequality for $p\in[0,1]$, it suffices to consider the set $(1-\lambda)\cdot K\oplus_p\lambda\cdot L$ in the left-hand side (cf. (1.11)). Using this approach, Marsiglietti proved that if $K,L\subset\mathbb{R}^n$ are unconditional convex bodies, $\lambda\in(0,1)$ and $0<p<1$, then

$$\mathrm{vol}\big((1-\lambda)\cdot K\oplus_p\lambda\cdot L\big)\geq\mathcal{M}^{\lambda}_{p/n}\big(\mathrm{vol}(K),\mathrm{vol}(L)\big).\tag{1.40}$$

Furthermore, in order to obtain additional inequalities for different point enumerators, we shall consider, for any $p>0$, the change of variables $\varphi_p:\mathbb{R}^n_{\geq 0}\longrightarrow\mathbb{R}^n_{\geq 0}$ given by $\varphi_p(x)_i=x_i^{1/p}$, $i=1,\dots,n$. Analogously, we will denote by $\psi_a:\mathbb{R}^n_{\geq 0}\longrightarrow\mathbb{R}^n_{\geq 1}$ the change of variables given by $\psi_a(x)_i=a^{x_i}$, for any $a>1$ (so that ψ_a is bijective). These changes of variables will allow us to establish the spaces and functionals with which we will obtain our results. We will write $\Gamma_p=\varphi_p(\mathbb{Z}^n_{\geq 0})$ and $\Lambda_a=\psi_a(\mathbb{Z}^n_{\geq 0})$.

1.4.2 Log-Brunn-Minkowski type inequalities for different point enumerators

Our initial result provides a discretization of Saroglou's result (see Section 1.1.1), i.e., a discrete version of the conjectured inequality (1.10) for the lattice point enumerator of unconditional convex bodies, as well as of (1.10) for origin-symmetric planar convex bodies. For the sake of brevity, the (closed) origin-symmetric unit cube will be denoted as $C_n:=[-1/2,1/2]^n$.

Theorem 1.20 [64, Theorem 2.1] Let $K,L\subset\mathbb{R}^n$ be origin-symmetric convex bodies and let $\lambda\in(0,1)$. If either K,L are unconditional convex bodies or $n=2$, then

$$\mathrm{G_n}\left((1-\lambda)\cdot(K+C_n)+_0\lambda\cdot(L+C_n)+\left(-\frac{1}{2},\frac{1}{2}\right)^n\right)\geq\mathrm{G_n}(K)^{1-\lambda}\,\mathrm{G_n}(L)^{\lambda}.\tag{1.41}$$

Furthermore, it is equivalent to the log-Brunn-Minkowski inequality (1.10) both for unconditional convex bodies or when $n=2$.

The proof relies on several relations between the volume and the lattice point enumerator of a convex bounded set $K\subset\mathbb{R}^n$. On the one hand, it is well-known that

$$\mathrm{G_n}(K)\leq\mathrm{vol}\left(K+\left(-\frac{1}{2},\frac{1}{2}\right)^n\right)\quad\text{and}\quad\mathrm{vol}(K)\leq\mathrm{G_n}\left(K+\left(-\frac{1}{2},\frac{1}{2}\right)^n\right).\tag{1.42}$$

The first inequality can be found in [51, (3.3)]. The second one is gathered in [52, page 877]. On the other hand, we recall the remarkable fact that, roughly speaking, the volume and the lattice point enumerator of a convex body $K\subset\mathbb{R}^n$ are equivalent when K is large enough, i.e.,

$$\lim_{r\to\infty}\frac{\mathrm{G_n}(rK)}{r^n}=\mathrm{vol}(K)$$

(see (1)). We will also need the following more general version of this relation (whose proof can be found in [9, (3.14)]):

Lemma 1.21 Let $K \subset \mathbb{R}^n$ be a convex body and let $M \subset \mathbb{R}^n$ be a bounded set containing the origin. Then

$$\lim_{r\to\infty} \frac{\mathrm{G_n}(rK+M)}{r^n} = \mathrm{vol}(K). \tag{1.43}$$

Proof. Given $\varepsilon > 0$ and a large enough $r > 0$, it follows that $M \subset (r\varepsilon K) + z_r$ for some $z_r \in \mathbb{Z}^n$, and thus

$$\begin{aligned}\mathrm{vol}(K) = \lim_{r\to\infty} \frac{\mathrm{G}_n(rK)}{r^n} &\le \liminf_{r\to\infty} \frac{\mathrm{G}_n(rK+M)}{r^n} \le \limsup_{r\to\infty} \frac{\mathrm{G}_n(rK+M)}{r^n} \\ &\le \lim_{r\to\infty} \frac{\mathrm{G}_n\big(r(K+\varepsilon K)\big)}{r^n} = \mathrm{vol}(K+\varepsilon K) = (1+\varepsilon)^n \,\mathrm{vol}(K).\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, (1.43) holds. ∎

Now we are in the conditions to prove Theorem 1.20.

Proof of Theorem 1.20. Clearly, if $K \subset \mathbb{R}^n$ is an unconditional convex body (or just origin-symmetric), so is $K + C_n$. Thus, using (1.42) and Saroglou's result (inequality (1.10) for unconditional convex bodies) we obtain

$$\begin{aligned}\mathrm{G_n}(K)^{1-\lambda}\,\mathrm{G_n}(L)^{\lambda} &\le \mathrm{vol}(K+C_n)^{1-\lambda}\,\mathrm{vol}(L+C_n)^{\lambda} \le \mathrm{vol}\Big((1-\lambda)\cdot\big(K+C_n\big) +_0 \lambda\cdot\big(L+C_n\big)\Big) \\ &\le \mathrm{G_n}\left((1-\lambda)\cdot\big(K+C_n\big) +_0 \lambda\cdot\big(L+C_n\big) + \Big(-\frac{1}{2},\frac{1}{2}\Big)^n\right),\end{aligned}$$

as required. The case of $n = 2$ is analogous but using the known log-Brunn-Minkowski inequality (1.10) for origin-symmetric planar convex bodies. This shows, moreover, that the log-Brunn-Minkowski inequality yields the discrete version (1.41).

In order to conclude the proof, we show that (1.41) implies (1.10) when K and L are unconditional sets (respectively, when $n = 2$). We observe that, for any convex bodies $K, L \subset \mathbb{R}^n$ and any $r > 0$,

$$(1-\lambda)\cdot(rK) +_0 \lambda\cdot(rL) = r\big((1-\lambda)\cdot K +_0 \lambda\cdot L\big).$$

Now, let $K, L \subset \mathbb{R}^n$ be unconditional convex bodies (respectively, let $n = 2$), and fix $\varepsilon > 0$. Then, using (1), (1.41) and (1.43) we get, on one hand,

$$\begin{aligned}\mathrm{vol}(K)^{1-\lambda}\,\mathrm{vol}(L)^{\lambda} &= \lim_{r\to\infty} \frac{\mathrm{G_n}(rK)^{1-\lambda}\,\mathrm{G_n}(rL)^{\lambda}}{r^n} \\ &\le \lim_{r\to\infty} \frac{\mathrm{G_n}\Big((1-\lambda)\cdot(rK+C_n) +_0 \lambda\cdot(rL+C_n) + \big(-\frac{1}{2},\frac{1}{2}\big)^n\Big)}{r^n} \\ &= \lim_{r\to\infty} \frac{\mathrm{G_n}\left(r\Big((1-\lambda)\cdot\big(K+\frac{1}{r}C_n\big) +_0 \lambda\cdot\big(L+\frac{1}{r}C_n\big)\Big) + \big(-\frac{1}{2},\frac{1}{2}\big)^n\right)}{r^n} \\ &\le \lim_{r\to\infty} \frac{\mathrm{G_n}\Big(r\big((1-\lambda)\cdot(K+\varepsilon \mathrm{B}_n) +_0 \lambda\cdot(L+\varepsilon \mathrm{B}_n)\big) + \big(-\frac{1}{2},\frac{1}{2}\big)^n\Big)}{r^n} \\ &= \mathrm{vol}\big((1-\lambda)\cdot(K+\varepsilon \mathrm{B}_n) +_0 \lambda\cdot(L+\varepsilon \mathrm{B}_n)\big),\end{aligned} \tag{1.44}$$

which is valid for all $\varepsilon > 0$. On the other hand, one clearly has that

$$h_{K+\varepsilon \mathrm{B}_n}(u)^{1-\lambda} h_{L+\varepsilon \mathrm{B}_n}(u)^{\lambda} = \big(h_K(u)+\varepsilon\big)^{1-\lambda}\big(h_L(u)+\varepsilon\big)^{\lambda}$$

for any $\varepsilon > 0$ (for K and L fixed). Furthermore, since each sequence of functions $\{h_K+\varepsilon\}_\varepsilon$ and $\{h_L+\varepsilon\}_\varepsilon$ converges uniformly to h_K and h_L, respectively, and the function $(x,y) \mapsto x^{1-\lambda}y^{\lambda}$ is uniformly continuous in any closed rectangle $[0,a]\times[0,b]$, we get that $\{(h_K+\varepsilon)^{1-\lambda}(h_L+\varepsilon)^{\lambda}\}_\varepsilon$, and so also $\{h_{K+\varepsilon \mathrm{B}_n}^{1-\lambda} h_{L+\varepsilon \mathrm{B}_n}^{\lambda}\}_\varepsilon$, converges uniformly to $h_K^{1-\lambda} h_L^{\lambda}$. Then Lemma 2 ensures that the sequence of Wulff shapes associated to the functions $h_{K+\varepsilon \mathrm{B}_n}^{1-\lambda} h_{L+\varepsilon \mathrm{B}_n}^{\lambda}$, namely,

$$(1-\lambda)\cdot(K+\varepsilon \mathrm{B}_n) +_0 \lambda \cdot (L+\varepsilon \mathrm{B}_n),$$

converges to $(1-\lambda)\cdot K +_0 \lambda \cdot L$ in the Hausdorff metric. Finally, the continuity of the volume yields

$$\lim_{\varepsilon\to 0} \mathrm{vol}\big((1-\lambda)\cdot(K+\varepsilon \mathrm{B}_n) +_0 \lambda \cdot (L+\varepsilon \mathrm{B}_n)\big) = \mathrm{vol}\big((1-\lambda)\cdot K +_0 \lambda \cdot L\big).$$

This, together with (1.44), shows (1.10) and concludes the proof. ■

Remark 1.22 We note that the cube C_n cannot be removed in the left-hand side of (1.20), not even summing up a bigger cube instead of $(-1/2,1/2)^n$; i.e., an inequality of the form

$$\mathrm{G_n}\Big(\big((1-\lambda)\cdot K +_0 \lambda \cdot L\big) + (-\beta,\beta)^n\Big) \geq \mathrm{G_n}(K)^{1-\lambda}\, \mathrm{G_n}(L)^{\lambda} \tag{1.45}$$

for all convex bodies $K, L \subset \mathbb{R}^n$ does not hold for any constant $\beta > 0$. Indeed, it suffices to consider the sets $K = [-a,a]$ and $L = [-b,b]$ in $\mathbb{R}$, where $0 < b < 1/2$ and $a \in \mathbb{N}$ is large enough in order for the inequality

$$a^{1-\lambda}\left(\left(1+\frac{1}{2a}\right)^{1-\lambda}\frac{1}{2^{\lambda}} - b^{\lambda}\right) > \beta + \frac{1}{2}$$

to hold. Then, the above expression rewrites as

$$2a^{1-\lambda} b^{\lambda} + 2\beta + 1 < (2a+1)^{1-\lambda}, \tag{1.46}$$

and since $\big((1-\lambda)\cdot K +_0 \lambda \cdot L\big) + (-\beta,\beta) = (-a^{1-\lambda}b^{\lambda} - \beta, a^{1-\lambda}b^{\lambda} + \beta)$, we have

$$\mathrm{G_1}\Big(\big((1-\lambda)\cdot K +_0 \lambda \cdot L\big) + (-\beta,\beta)\Big) \leq 2(a^{1-\lambda}b^{\lambda} + \beta) + 1.$$

Furthermore, $\mathrm{G_1}(K) = 2a+1$ and $\mathrm{G_1}(L) \geq 1$ and, consequently, (1.46) contradicts (1.45), as desired.

Finally, the Minkowski addition of $(-1/2,1/2)^n$ is also necessary i.e., an inequality of the form

$$\mathrm{G_n}\big((1-\lambda)\cdot(K+C_n) +_0 \lambda \cdot (L+C_n) + (-\beta,\beta)^n\big) \geq \mathrm{G_n}(K)^{1-\lambda}\, \mathrm{G_n}(L)^{\lambda} \tag{1.47}$$

does not hold, in general, if $0 \leq \beta < 1/2$. To show it, we consider the sets $K = [-a,a]$ and $L = [-b,b]$ in $\mathbb{R}$, for fixed $a, b > 0$. Then, it is clear that

$$\mathrm{G_1}\big((1-\lambda)\cdot(K+C_1) +_0 \lambda \cdot (L+C_1) + (-\beta,\beta)\big) \leq 2\left\lfloor \left(a+\frac{1}{2}\right)^{1-\lambda}\left(b+\frac{1}{2}\right)^{\lambda} + \beta \right\rfloor + 1.$$

Note that, if $\beta < 1/2$, we may choose $0 < \lambda < 1$ such that $(2b+1)^{\lambda} < 2(1-\beta)$, because $2(1-\beta) > 1$ and $\lim_{\lambda\to 0^+}(2b+1)^{\lambda} = 1$. This condition is equivalent to $(1/2^{1-\lambda})(b+1/2)^{\lambda} + \beta < 1$, and a simple continuity argument then shows that, for sufficiently small values of a, we also have the inequality $(a+1/2)^{1-\lambda}(b+1/2)^{\lambda} + \beta < 1$. Consequently, in this case we have

$$\mathrm{G_1}\big((1-\lambda)\cdot(K+C_1) +_0 \lambda \cdot (L+C_1) + (-\beta,\beta)\big) = 1,$$

which contradicts (1.47) when $b > 1$ because, in that case, $\mathrm{G_1}(K) \geq 1$ and $\mathrm{G_1}(L) > 1$. ♦

Following the same argument as the one in the proof of Theorem 1.20, but now using (1.40), one can get the L_p version of that theorem when $0 < p < 1$, i.e., a discrete version of Marsiglietti's result. Again, neither C_n nor $(-1/2,1/2)^n$ can be removed from the inequality.

Theorem 1.23 [64, Theorem 3.1] Let $K,L\subset\mathbb{R}^n$ be two unconditional convex bodies and let $\lambda\in(0,1)$. Then, for any $0<p<1$,

$$\mathrm{G_n}\left((1-\lambda)\cdot(K+C_n)\oplus_p\lambda\cdot(L+C_n)+\left(-\frac{1}{2},\frac{1}{2}\right)^n\right)\geq\mathcal{M}^{\lambda}_{p/n}\big(\mathrm{G_n}(K),\mathrm{G_n}(L)\big). \tag{1.48}$$

Furthermore, it is equivalent to the L_p Brunn-Minkowski inequality (1.40) for unconditional convex bodies.

Next we will deal with the point enumerator $\mathrm{G}_{\Lambda_a}(\cdot)$ (and $\mathrm{G}_{\Gamma_p}(\cdot)$). First, we prove the following simple properties of the functions ψ_a and φ_p, which will be useful throughout the rest of the section.

Lemma 1.24 Let $K,L\subset\mathbb{R}^n_{\geq1}$ be non-empty bounded sets and let $\lambda\in(0,1)$. Then

i) $\mathrm{G}_{\Lambda_a}(K)=\left|\psi_a^{-1}(K)\cap\mathbb{Z}^n\right|$ and
ii) $\psi_a^{-1}\left(K^{1-\lambda}L^{\lambda}\right)=(1-\lambda)\psi_a^{-1}(K)+\lambda\psi_a^{-1}(L)$.

Furthermore, if $K,L\subset\mathbb{R}^n_{\geq0}$ then, for any $0<p<1$,

iii) $\mathrm{G}_{\Gamma_p}(K)=\left|\varphi_p^{-1}(K)\cap\mathbb{Z}^n\right|$ and
iv) $\varphi_p^{-1}\big((1-\lambda)\cdot K\oplus_p\lambda\cdot L\big)=(1-\lambda)\varphi_p^{-1}(K)+\lambda\varphi_p^{-1}(L)$.

Proof. On the one hand

$$\mathrm{G}_{\Lambda_a}(K)=|K\cap\Lambda_a|=\left|K\cap\psi_a\left(\mathbb{Z}^n_{\geq0}\right)\right|=\left|\psi_a^{-1}(K)\cap\mathbb{Z}^n_{\geq0}\right|=\left|\psi_a^{-1}(K)\cap\mathbb{Z}^n\right|.$$

On the other hand

$$\psi_a^{-1}\left(x^{1-\lambda}y^{\lambda}\right)_i=\log_a\left(x_i^{1-\lambda}y_i^{\lambda}\right)=(1-\lambda)\log_a x_i+\lambda\log_a y_i=(1-\lambda)\psi_a^{-1}(x)_i+\lambda\psi_a^{-1}(y)_i$$

for all $x\in K, y\in L$ and all $i=1,\dots,n$. Completely analogous arguments yield properties iii) and iv). ■

The following result for G_{Λ_a} can now be shown.

Proposition 1.25 Let $a>1$ and $\lambda\in(0,1)$, and let $K,L\subset\mathbb{R}^n_{\geq1}$ be non-empty bounded sets with $\mathrm{G}_{\Lambda_a}(K)\mathrm{G}_{\Lambda_a}(L)>0$. Then

$$\mathrm{G}_{\Lambda_a}\left((1,a^2)^nK^{1-\lambda}L^{\lambda}\right)^{1/n}\geq(1-\lambda)\mathrm{G}_{\Lambda_a}(K)^{1/n}+\lambda\mathrm{G}_{\Lambda_a}(L)^{1/n}, \tag{1.49}$$

and the inequality is sharp.

Proof. We observe that the cube $(-1,1)^n$ in inequality (1.15) can be replaced by $(0,2)^n$ due to the invariance by integer translations, and so, we may apply it to the sets $\psi_a^{-1}(K)$ and $\psi_a^{-1}(L)$ to obtain

$$\left|\left[(1-\lambda)\psi_a^{-1}(K)+\lambda\psi_a^{-1}(L)+(0,2)^n\right]\cap\mathbb{Z}^n\right|^{1/n}\geq(1-\lambda)\left|\psi_a^{-1}(K)\cap\mathbb{Z}^n\right|^{1/n}+\lambda\left|\psi_a^{-1}(L)\cap\mathbb{Z}^n\right|^{1/n}.$$

Now, using both items i) and ii) of Lemma 1.24, and taking into account that $\psi_a\big((0,2)^n\big)=(1,a^2)^n$, we get (1.49).

To see that equality may be attained, we consider $K=L=[1,a^m]^n$ for any $m\in\mathbb{N}$, for which one has $(1,a^2)^nK^{1-\lambda}L^{\lambda}=(1,a^{m+2})^n$ and

$$\mathrm{G}_{\Lambda_a}\big([1,a^m]^n\big)=\mathrm{G}_{\Lambda_a}\big((1,a^{m+2})^n\big)=(m+1)^n.$$ ■

In particular, using the arithmetic-geometric mean inequality, a discrete log-Brunn-Minkowski type inequality for $\mathrm{G}_{\Lambda_a}(\cdot)$ is obtained as a direct consequence:

Corollary 1.26 Let $a>1$ and $\lambda\in(0,1)$, and let $K,L\subset\mathbb{R}^n_{\geq 1}$ be non-empty bounded sets. Then

$$\mathrm{G}_{\Lambda_a}\left((1,a^2)^n K^{1-\lambda}L^{\lambda}\right)\geq \mathrm{G}_{\Lambda_a}(K)^{1-\lambda}\mathrm{G}_{\Lambda_a}(L)^{\lambda},$$

and the inequality is sharp.

Proposition 1.25 can be adapted to the $0<p<1$ setting using items iii) and iv) of Lemma 1.24:

Corollary 1.27 Let $0<p<1$ and $\lambda\in(0,1)$, and let $K,L\subset\mathbb{R}^n_{\geq 0}$ be non-empty bounded sets with $\mathrm{G}_{\Gamma_p}(K)\mathrm{G}_{\Gamma_p}(L)>0$. Then

$$\mathrm{G}_{\Gamma_p}\left((1-\lambda)\cdot K\oplus_p\lambda\cdot L\oplus_p\left(0,2^{1/p}\right)^n\right)^{1/n}\geq(1-\lambda)\mathrm{G}_{\Gamma_p}(K)^{1/n}+\lambda\mathrm{G}_{\Gamma_p}(L)^{1/n}.$$

1.4.3 A log-Brunn-Minkowski type inequality for an alternative discrete measure

Next, we observe that the proofs of the log-Brunn-Minkowski inequality for unconditional convex bodies by Saroglou [95] and of the $0<p<1$ case by Marsiglietti [78] strongly rely on the Prékopa-Leindler inequality and on the more general Borell-Brascamp-Lieb inequality, respectively. Since discrete versions of these inequalities have been recently obtained (see Theorem F), we wonder whether a similar approach could yield new discrete versions of the log-Brunn-Minkowski inequality (as well as of (1.40) in the case $0<p<1$). To this end, we define an alternative (discrete) measure for which a result in the same spirit as the ones in the previous section can also be proved. For any $a>1$ and any bounded set $M\subset\mathbb{R}^n$, let

$$\mu_a(M)=\sum_{z\in M\cap\Lambda_a}\phi(z),\tag{1.50}$$

where the density function $\phi:\mathbb{R}^n\longrightarrow\mathbb{R}$ is given by

$$\phi(z)=\prod_{i=1}^n z_i.$$

We note that μ_a coincides with G_{Λ_a} when the density function $\phi\equiv 1$. We will use a similar technique as the one used in [95] to approach the problem in the discrete setting.

Theorem 1.28 [64, Theorem 2.2] Let $K,L\subset\mathbb{R}^n_{\geq 1}$ be non-empty bounded sets, and let $a>1$ and $\lambda\in(0,1)$. Then

$$a^n\mu_a\left(\left(a^{-1},a\right)^n K^{1-\lambda}L^{\lambda}\right)\geq\mu_a(K)^{1-\lambda}\mu_a(L)^{\lambda}.$$

Proof. To begin with, we clearly have

$$a^n\mu_a\left(\left(a^{-1},a\right)^n K^{1-\lambda}L^{\lambda}\right)=a^n\sum_{w\in[(a^{-1},a)^nK^{1-\lambda}L^{\lambda}]\cap\Lambda_a}\phi(w)\geq\sum_{w\in[(a^{-1},a)^nK^{1-\lambda}L^{\lambda}]\cap\Lambda_a}\ \sup_{v\in(a^{-1},a)^n}\phi(v)\phi(w).$$

Applying the change of variables defined by ψ_a, and using the fact that $\psi_a^{-1}\left(\left(a^{-1},a\right)^n\right)=(-1,1)^n$ and the symmetry of $(-1,1)^n$, the above expression rewrites into

$$\sum_{z\in\psi_a^{-1}((a^{-1},a)^nK^{1-\lambda}L^{\lambda})\cap\mathbb{Z}^n}\ \sup_{u\in(-1,1)^n}a^{\sum_{i=1}^n u_i}a^{\sum_{i=1}^n z_i}=\sum_{z\in\psi_a^{-1}((a^{-1},a)^nK^{1-\lambda}L^{\lambda})\cap\mathbb{Z}^n}\ \sup_{u\in(-1,1)^n}a^{\sum_{i=1}^n u_i+z_i}.$$

Next, we denote by $h(z)=a^{\sum_{i=1}^n z_i}$, and use Lemma 1.24 to get that the last sum equals to

$$\begin{aligned}\sum_{z\in\psi_a^{-1}((a^{-1},a)^nK^{1-\lambda}L^{\lambda})\cap\mathbb{Z}^n}\ \sup_{u\in(-1,1)^n}h(u+z)&=\sum_{z\in\psi_a^{-1}((a^{-1},a)^nK^{1-\lambda}L^{\lambda})\cap\mathbb{Z}^n}h^{\diamond}(z)\\&=\sum_{z\in[(1-\lambda)\psi_a^{-1}(K)+\lambda\psi_a^{-1}(L)+(-1,1)^n]\cap\mathbb{Z}^n}h^{\diamond}(z).\end{aligned}$$

Now, if we consider the functions $f = g = h$, it is straightforward to verify that they are under the conditions of the discrete Prékopa-Leindler inequality (Theorem F for $p = 0$), that is,

$$h\big((1-\lambda)x+\lambda y\big) \geq f(x)^{1-\lambda} g(y)^{\lambda}$$

for all $x \in \psi_a^{-1}(K)$ and $y \in \psi_a^{-1}(L)$, which yields

$$\sum_{z\in[(1-\lambda)\psi_a^{-1}(K)+\lambda\psi_a^{-1}(L)+(-1,1)^n]\cap\mathbb{Z}^n} h^{\diamond}(z) \geq \left(\sum_{x\in\psi_a^{-1}(K)\cap\mathbb{Z}^n} f(x)\right)^{1-\lambda} \left(\sum_{y\in\psi_a^{-1}(L)\cap\mathbb{Z}^n} g(x)\right)^{\lambda}.$$

Finally, performing the change of variables to f and g similarly to how we did it for h, and putting it all together, we can conclude the result:

$$a^n \mu_a\left(\left(a^{-1},a\right)^n K^{1-\lambda} L^{\lambda}\right) \geq \left(\sum_{x\in K\cap\Lambda_a} \phi(x)\right)^{1-\lambda} \left(\sum_{y\in L\cap\Lambda_a} \phi(y)\right)^{\lambda} = \mu_a(K)^{1-\lambda}\mu_a(L)^{\lambda}. \qquad \blacksquare$$

In order to extend the previous result to the $0 < p < 1$ setting, we need to consider the density function $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ given by

$$\phi(z) = \left(\sum_{i=1}^n z_i^p\right)^{1/p},$$

and the measure ν_p defined as

$$\nu_p(A) = \sum_{z\in A\cap\Gamma_p} \phi(z),$$

for any non-empty bounded set $A \subset \mathbb{R}^n$.

Additionally, since φ_p can only be defined for points with non-negative coordinates, the definition of $\phi^{\diamond}$ must be adapted (for this result only) to

$$\phi^{\diamond}(z) = \sup_{u\in(0,2)^n} \phi(z-u),$$

which, due to the invariance by integer translations of the standard lattice point enumerator, still allows one to apply results like Theorem F or Theorem 1.12.

With these ingredients, and using a similar argument to the one employed in the proof of Theorem 1.28, we can show the following result.

Theorem 1.29 [64, Theorem 4.1] Let $K, L \subset \mathbb{R}^n_{\geq 0}$ be non-empty bounded sets and let $\lambda \in (0,1)$. Then, for any $0 < p < 1$,

$$\nu_p\left((1-\lambda)\cdot K \oplus_p \lambda \cdot L \oplus_p \left(0,4^{1/p}\right)^n\right) \geq \mathcal{M}^{\lambda}_{\frac{p}{np+1}}\left(\nu_p(K),\nu_p(L)\right).$$

Proof. By definition we have

$$\nu_p\left((1-\lambda)\cdot K \oplus_p \lambda \cdot L \oplus_p \left(0,4^{1/p}\right)^n\right) = \sum_{z\in\left[(1-\lambda)\cdot K\oplus_p\lambda\cdot L\oplus_p(0,4^{1/p})^n\right]\cap\Gamma_p} \left(\sum_{i=1}^n z_i^p\right)^{1/p},$$

and since clearly

$$\left(0,2^{1/p}\right)^n \oplus_p \left\{\left(2^{1/p},\dots,2^{1/p}\right)\right\} \subset \left(0,4^{1/p}\right)^n,$$

the above expression can be bounded by

$$\begin{aligned} \nu_p\Big((1-\lambda)\cdot K\oplus_p\lambda\cdot L\oplus_p\big(0,4^{1/p}\big)^n\Big) &\geq \sum_{z\in\left[(1-\lambda)\cdot K\oplus_p\lambda\cdot L\oplus_p(0,2^{1/p})^n\right]\cap\Gamma_p}\left(\sum_{i=1}^n z_i^p+2\right)^{1/p} \\ &\geq \sum_{z\in\left[(1-\lambda)\cdot K\oplus_p\lambda\cdot L\oplus_p(0,2^{1/p})^n\right]\cap\Gamma_p}\sup_{u\in(0,2^{1/p})^n}\left(\sum_{i=1}^n z_i^p+u_i^p\right)^{1/p}. \end{aligned}$$

Next, applying the change of variables defined by φ_p, and denoting by $h(z)=\left(\sum_{i=1}^n z_i\right)^{1/p}$, the last term rewrites into

$$\begin{aligned} &\sum_{z\in\left[\varphi_p^{-1}\left((1-\lambda)\cdot K\oplus_p\lambda\cdot L\oplus_p(0,2^{1/p})^n\right)\right]\cap\mathbb{Z}^n}\sup_{u\in(0,2)^n}\left(\sum_{i=1}^n z_i+u_i\right)^{1/p} \\ &\qquad= \sum_{z\in\left[\varphi_p^{-1}\left((1-\lambda)\cdot K\oplus_p\lambda\cdot L\oplus_p(0,2^{1/p})^n\right)\right]\cap\mathbb{Z}^n}\sup_{u\in(0,2)^n} h(z+u) \\ &\qquad= \sum_{z\in\left[\varphi_p^{-1}\left((1-\lambda)\cdot K\oplus_p\lambda\cdot L\oplus_p(0,2^{1/p})^n\right)\right]\cap\mathbb{Z}^n} h^\diamond(z) \\ &\qquad= \sum_{z\in\left[(1-\lambda)\varphi_p^{-1}(K)+\lambda\varphi_p^{-1}(L)+(0,2)^n\right]\cap\mathbb{Z}^n} h^\diamond(z), \end{aligned}$$

where the last identity arises from Lemma 1.24 iv).

Now, if we consider the functions $f=g=h$, it is immediate that the condition

$$h\big((1-\lambda)x+\lambda y\big)=\mathcal{M}_p^\lambda\big(f(x),g(y)\big)$$

holds, and thus, Theorem F yields

$$\sum_{z\in\left[(1-\lambda)\varphi_p^{-1}(K)+\lambda\varphi_p^{-1}(L)+(0,2)^n\right]\cap\mathbb{Z}^n} h^\diamond(z)\geq\mathcal{M}^\lambda_{\frac{p}{np+1}}\left(\sum_{x\in\varphi_p^{-1}(K)\cap\mathbb{Z}^n} f(x),\sum_{y\in\varphi_p^{-1}(L)\cap\mathbb{Z}^n} g(y)\right).$$

Finally, performing the change of variables defined by φ_p, it is easy to check that the above expression is equal to

$$\mathcal{M}^\lambda_{\frac{p}{np+1}}\big(\nu_p(K),\nu_p(L)\big),$$

and so we can conclude that

$$\nu_p\Big((1-\lambda)\cdot K\oplus_p\lambda\cdot L\oplus_p\big(0,4^{1/p}\big)^n\Big)\geq\mathcal{M}^\lambda_{\frac{p}{np+1}}\big(\nu_p(K),\nu_p(L)\big),$$

as desired. ■

2 Isoperimetric type inequalities

The isoperimetric inequality is one of the most renowned classical inequalities in mathematics, dating back to the hellenistic period. As a result, it is perhaps one of the most outstanding and striking consequences of the Brunn-Minkowski inequality (1.1). Its form for convex bodies in $\mathbb{R}^n$ states that the volume $\mathrm{vol}(\cdot)$ and the surface area $\mathrm{S}(\cdot)$ of any n-dimensional convex body K satisfy

$$\left(\frac{\mathrm{S}(K)}{\mathrm{S}(\mathrm{B}_n)}\right)^n \geq \left(\frac{\mathrm{vol}(K)}{\mathrm{vol}(\mathrm{B}_n)}\right)^{n-1}. \tag{2.1}$$

In other words, Euclidean balls minimize the surface area among those convex bodies with prescribed positive volume.

There exist various facets of the isoperimetric inequality (see e.g. [97, Section 7.2] and the references therein), having different ramifications into other settings such as its versions in the spherical and hyperbolic spaces (see e.g. [29]), or its version for *mixed volumes* known as *Minkowski's first inequality* (cf. [97, Theorem 7.2.1]). The isoperimetric inequality has been the starting point for new engaging related results, such as a reverse isoperimetric inequality (see [12]), and it has led to various remarkable consequences not only in geometry but also in analysis (see e.g. [32]), such as an equivalent analytic version due to Sobolev (see e.g. [45, Section 5]). Other related inequalities, which can be consulted in [97, Section 7.2], include *Diskant's inequality* or the *Bonnesen-type inequalities* in the plane. For an extensive survey article on this inequality we refer the reader to [82].

The isoperimetric inequality (2.1) admits the following (slightly more general) "*neighbourhood* form" (see e.g. [79, Proposition 14.2.1]), which is a direct consequence of the Brunn-Minkowski inequality: for any n-dimensional compact set $K \subset \mathbb{R}^n$, and all $t \geq 0$, we have

$$\mathrm{vol}(K + t\mathrm{B}_n) \geq \mathrm{vol}(r\mathrm{B}_n + t\mathrm{B}_n), \tag{2.2}$$

where $r\mathrm{B}_n$, $r > 0$, is a ball of the same volume as K. In fact, if K is in addition convex, by subtracting $\mathrm{vol}(K) = \mathrm{vol}(r\mathrm{B}_n)$, dividing both sides of (2.2) by t, and taking limits as $t \to 0^+$, one immediately gets (2.1) from (2.2) (see (2)).

The neighbourhood $K + t\mathrm{B}_n$, $t \geq 0$, of an n-dimensional compact set K coincides with the set of all points of $\mathbb{R}^n$ having (Euclidean) distance from K at most t. Exchanging the role of the unit ball B_n in (2.2) by another (n-dimensional) convex body $E \subset \mathbb{R}^n$, i.e., changing the involved "distance", one is naturally led to the fact that

$$\mathrm{vol}(K+tE) \geq \mathrm{vol}(rE+tE) \tag{2.3}$$

for all $t \geq 0$, where again $r > 0$ is such that rE has the same volume as K. Thus, the advantage of using the volume of a neighbourhood of K, instead of its surface area, is that it can be extended to non-convex compact sets, and, moreover, to other spaces in which the latter notion makes no sense; it just suffices to consider a metric and a measure on the given space.

A brief survey on the neighborhood form of the isoperimetric inequality can be found in [79, Section 14.2], where different spaces are considered (e.g. the *Gauss space* and the n-dimensional discrete unit cube $\{0,1\}^n$). In [104] and [37], this type of inequalities are studied in $\mathbb{Z}^n$ endowed with the L_1 norm, characterizing the equality in some particular cases. Similar inequalities also hold in other discrete metric spaces, in the settings of combinatorics and graph theory (for which we refer the reader to [57]).

In this chapter we will present several new discrete analogues of the isoperimetric inequality, both for the cardinality (see Theorem 2.9 and Corollary 2.11) and for the lattice point enumerator (see Theorems 2.37 and 2.41). We will further obtain some characterizations of the equality in special cases (see Theorems 2.14, 2.15, 2.16, 2.37 and Corollary 2.34), and finally, we will also show that these inequalities imply the classical versions for the volume (see Theorem 2.43 and Corollary 2.44). The results collected in this chapter can be found in [67, 66].

2.1 Discrete isoperimetric inequalities. Preliminaries

In [86], Radcliffe and Veomett proved an exceptional discrete isoperimetric inequality in the spirit of (2.3) for the integer lattice $\mathbb{Z}^n$ endowed with the L_∞ norm considering the cardinality as the measure. To this end, the authors defined the following complete order in $\mathbb{Z}^n$.

Definition 2.1 If $n = 1$, the order $\prec$ in $\mathbb{Z}$ is defined by

$$0 \prec 1 \prec -1 \prec 2 \prec -2 \prec \cdots \prec m \prec -m \prec \ldots$$

For $n \geq 2$ and for any vector $w = (w_1,\ldots,w_n) \in \mathbb{Z}^n$, let

$$m_w = \max_{\prec}\{w_i : i = 1,\ldots,n\}, \quad i_w = \min\{i : w_i = m_w\}$$

and

$$w' = (w_1,\ldots,w_{i_w-1},w_{i_w+1},\ldots,w_n) \in \mathbb{Z}^{n-1}.$$

Then, $\prec$ is defined recursively as follows: for any $u, v \in \mathbb{Z}^n$ with $u \neq v$, one has $u \prec v$ if either

1. $m_u \prec m_v$ or
2. $m_u = m_v$ and then either $i_v < i_u$ or ($i_v = i_u$ and) $u' \prec v'$.

Moreover, we write $u \preceq v$ if either $u \prec v$ or $u = v$.

We note that, in order to define $\prec$ in $\mathbb{N}^n$, one could see that order as the restriction of the order in $\mathbb{Z}^n$ to the subset $\mathbb{N}^n$, or as the generalization of the usual order in $\mathbb{N}$, to which one applies the same process described in Definition 2.1.

For any $r \in \mathbb{N} \setminus \{0\}$, we denote by $\mathcal{I}_r$ (resp., $\mathcal{J}_r$) the initial segment in $\mathbb{Z}^n$ (resp., $\mathbb{N}^n$) of cardinality r, that is, the set of the first r points with respect to the order $\prec$ of $\mathbb{Z}^n$ (resp., $\mathbb{N}^n$). The following figure shows examples of initial segments, both in $\mathbb{Z}^n$ and in $\mathbb{N}^n$.

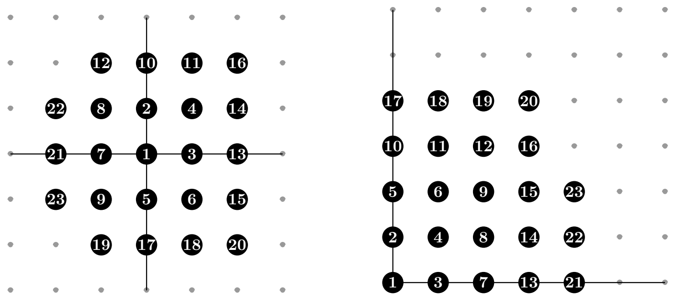

Figure 2.1: The initial segments $\mathcal{I}_{23}$ (left) and $\mathcal{J}_{23}$ (right) for $n = 2$.

The authors then showed that the initial segments $\mathcal{I}_r \subset \mathbb{Z}^n$ minimize the functional $\left|X + \{-1,0,1\}^n\right|$, among all sets $X \subset \mathbb{Z}^n$ with $|X| = r$:

Theorem H [86, Theorem 1] Let $X \subset \mathbb{Z}^n$ with $r = |X| \in \mathbb{N} \setminus \{0\}$. Then

$$\left|X + \{-1,0,1\}^n\right| \geq \left|\mathcal{I}_r + \{-1,0,1\}^n\right|. \tag{2.4}$$

Radcliffe and Veomett also considered the restriction of the order to $\mathbb{N}^n$ to show an analogous result for the corresponding initial segments $\mathcal{J}_r \subset \mathbb{N}^n$.

Theorem I [86, Corollary 1] Let $X \subset \mathbb{N}^n$ with $r = |X| \in \mathbb{N} \setminus \{0\}$. Then

$$\left|\left(X + \{-1,0,1\}^n\right) \cap \mathbb{N}^n\right| \geq \left|\left(\mathcal{J}_r + \{-1,0,1\}^n\right) \cap \mathbb{N}^n\right|. \tag{2.5}$$

Remark 2.2 It can be easily verified from the definition of $\prec$ that for $r = (\rho+1)^n$, with $\rho \in \mathbb{N}$, the initial segments $\mathcal{I}_r \subset \mathbb{Z}^n$ and $\mathcal{J}_r \subset \mathbb{N}^n$ are both lattice cubes. More precisely, $\mathcal{I}_r = \{-\rho/2, \ldots, \rho/2\}^n$ for ρ even and $\mathcal{I}_r = \left\{-(\rho+1)/2+1, \ldots, (\rho+1)/2\right\}^n$ for ρ odd, whereas $\mathcal{J}_r = \{0, \ldots, \rho\}^n$ for all $\rho \in \mathbb{N}$. See Remark 2.6 for a more precise description of the structure of the initial segments. ♦

For any given $x \in \mathbb{Z}^n$ we denote its *rank*, i.e., its position with respect to the order $\prec$ in $\mathbb{Z}^n$, by $r(x) \in \mathbb{N} \setminus \{0\}$. Furthermore, for any non-empty finite set $X \subset \mathbb{Z}^n$, the rank of X is defined as

$$r(X) = \sum_{x \in X} r(x).$$

We will use the same notation when working with $\prec$ in $\mathbb{N}^n$, without specifying if there is no ambiguity.

Definition 2.3 [86, Definition 2] A non-empty set $X \subset \mathbb{N}^n$, $n > 1$, is said to be downward compressed in the i-th coordinate, $i = 1, \ldots, n$, with respect to $x = (x_1, \ldots, x_{n-1}) \in \mathbb{N}^{n-1}$ if the set

$$\left\{y \in \mathbb{N} : (x_1, \ldots, x_{i-1}, y, x_i, \ldots, x_{n-1}) \in X\right\}$$

is either empty or of the form $\{y \in \mathbb{N} : 0 \leq y \leq a\}$ for some $a \in \mathbb{N}$.

Moreover, we say that $X \subset \mathbb{N}^n$ is downward compressed in the i-th coordinate if it is downward compressed in the i-th coordinate with respect to all $x \in \mathbb{N}^{n-1}$.

Finally, we say that $X \subset \mathbb{N}^n$ is downward compressed if it is downward compressed in the i-th coordinate for all $i = 1, \ldots, n$ (see Figure 2.2).

2.1.1 Comparing the initial segments in $\mathbb{Z}^n$ and $\mathbb{N}^n$

Let us obtain in this section several useful properties of the order from Definition 2.1 and the corresponding initial segments. These will be useful throughout the rest of the chapter. First, we make a few observations.

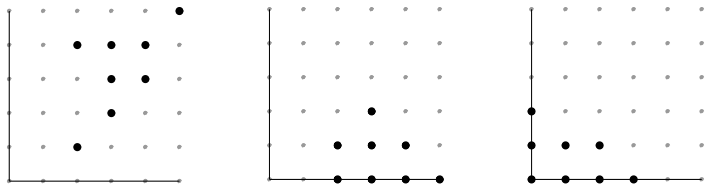

Figure 2.2: From left to right: a finite set, a downward compressed set in the 2nd coordinate and a downward compressed set.

Remark 2.4 Let $x,y,z \in \mathbb{Z}^n$ (resp. $\mathbb{N}^n$). Then:

1. If for some $i \in \{1,\dots,n\}$ we have $x_i \prec y_i$ and $x_j = y_j$ for all $j \neq i$, then $x \prec y$.
2. In particular, if $x_i \preceq y_i$ for all $i = 1,\dots,n$, then $x \preceq y$.
3. If $x \prec y$, then $x+z \prec y+z$. ♦

On the one hand, a straightforward consequence of the above observation is that every initial segment $\mathcal{J}_r \subset \mathbb{N}^n$ is downward compressed. And since clearly every downward compressed set $X \subset \mathbb{N}^n$ verifies

$$\left(X+\{-1,0,1\}^n\right)\cap\mathbb{N}^n = X+\{0,1\}^n, \tag{2.6}$$

then so does $\mathcal{J}_r$ for all $r \in \mathbb{N}$.

On the other hand, in [86, page 11], the authors show that

$$\left|\left(\mathcal{J}_r+\{-1,0,1\}^n\right)\cap\mathbb{N}^n\right| + 2^{\mathrm{l}(x)} = \left|\left(\mathcal{J}_{r+1}+\{-1,0,1\}^n\right)\cap\mathbb{N}^n\right|,$$

where $x \in \mathbb{N}^n$ satisfies $\mathcal{J}_r \cup \{x\} = \mathcal{J}_{r+1}$ and $\mathrm{l}(x) \in \{0,\dots,n\}$ is the number of coordinates equal to zero in x. Putting all this together immediately yields the following result.

Lemma 2.5 Let $r \in \mathbb{N}$ and let $x \in \mathbb{N}^n$ be such that $\mathcal{J}_r \cup \{x\} = \mathcal{J}_{r+1}$. Then

$$\left|\mathcal{J}_r+\{0,1\}^n\right| + 2^{\mathrm{l}(x)} = \left|\mathcal{J}_{r+1}+\{0,1\}^n\right|.$$

We will now analyze the $(n-1)$-dimensional sections of the initial segments, providing a description which will become crucial in subsequent sections. We note that, for any $x \in \mathbb{Z}^n$ (resp. $\mathbb{N}^n$), all but the "last" section of $\mathcal{I}_{r(x)}$ (resp. $\mathcal{J}_{r(x)}$) are uniquely determined by m_x and i_x. Indeed:

Remark 2.6 For $x \in \mathbb{Z}^n$, let $r = r(x)$. Then, from the fact that $\mathcal{I}_r = \{z \in \mathbb{Z}^n : z \preceq x\}$ we get that the only non-empty $(n-1)$-dimensional sections of $\mathcal{I}_r$ (with respect to e_{i_x}) are

$$(\mathcal{I}_r)^{i_x}(m) = \{t \in \mathbb{Z} : t \prec m_x\}^{i_x-1} \times \{t \in \mathbb{Z} : t \preceq m_x\}^{n-i_x} \tag{2.7}$$

for all $m \prec m_x$, and

$$(\mathcal{I}_r)^{i_x}(m_x) = \{z \in \mathbb{Z}^{n-1} : z \preceq x'\} \subset \{t \in \mathbb{Z} : t \prec m_x\}^{i_x-1} \times \{t \in \mathbb{Z} : t \preceq m_x\}^{n-i_x}. \tag{2.8}$$

Now, let $y \in \mathbb{N}^n$ and $r = r(y)$. Then, since $\mathcal{J}_r = \{z \in \mathbb{N}^n : z \preceq y\} \subset \mathbb{N}^n$ and $\{t \in \mathbb{N} : t \prec m_y\} = \{0,\dots,m_y-1\}$, the prior relations translate into

$$(\mathcal{J}_r)^{i_y}(m) = \{0,\dots,m_y-1\}^{i_y-1} \times \{0,\dots,m_y\}^{n-i_y} \tag{2.9}$$

for all $0 \leq m < m_y$, and

$$(\mathcal{J}_r)^{i_y}(m_y) = \{z \in \mathbb{N}^{n-1} : z \preceq y'\} \subset \{0,\dots,m_y-1\}^{i_y-1} \times \{0,\dots,m_y\}^{n-i_y} \tag{2.10}$$

(see Figure 2.3). ♦

(a) The initial segment $\mathcal{I}_{44} \subset \mathbb{Z}^3$, whose last point is $x = (-1,2,1)$ with $m_x = 2$ and $i_x = 2$.

(b) The initial segment $\mathcal{J}_{44} \subset \mathbb{N}^3$, whose last point is $y = (2,3,1)$ with $m_y = 3$ and $i_y = 2$.

Figure 2.3: The sections of the initial segments $\mathcal{I}_{44}$ and $\mathcal{J}_{44}$.

Before proceeding it is convenient to make the following useful observation.

Remark 2.7 We note that the sequence

$$\begin{array}{ccccc} 1, & 2, & \dots, & 2^{n-2}, & 2^{n-1}, \\ 2^n, & 2^{n-1}\cdot 3, & \dots, & 2^2\cdot 3^{n-2}, & 2\cdot 3^{n-1}, \\ 3^n, & 3^{n-1}\cdot 4, & \dots, & 3^2\cdot 4^{n-2}, & 3\cdot 4^{n-1}, \\ & & \vdots & & \end{array}$$

is strictly increasing, and therefore,

$$\left\{ \left[s^i(s+1)^{n-i}, s^{i-1}(s+1)^{n-i+1}\right) \cap \mathbb{N} : s \in \mathbb{N}\setminus\{0\}, i = 1,\dots,n \right\}$$

is a partition of $\mathbb{N}\setminus\{0\}$. ♦

This allows us to obtain a fundamental connection between the initial segments in $\mathbb{Z}^n$ and $\mathbb{N}^n$:

Lemma 2.8 Let $r \in \mathbb{N}$, $r > 0$. Then $\left|\mathcal{I}_r + \{0,1\}^n\right| = \left|\mathcal{J}_r + \{0,1\}^n\right|$.

Proof. We proceed by induction on the dimension n. The case $n = 1$ is immediate since we have $\left|\mathcal{I}_r + \{0,1\}\right| = r + 1 = \left|\mathcal{J}_r + \{0,1\}\right|$.

Now, let $n > 1$, and assume that the $(n-1)$-dimensional case is already proved. Let $x \in \mathbb{Z}^n$, $y \in \mathbb{N}^n$ be the last points in the order $\prec$ of $\mathcal{I}_r$ and $\mathcal{J}_r$, respectively (so, $r(x) = r = r(y)$), and let

$$s = \left|\{m \in \mathbb{Z} : m \prec m_x\}\right|,$$

i.e., the number of sections of $\mathcal{I}_r$ of the form (2.7). Then, using (2.7) and (2.8), we have

$$s^{i_x}(s+1)^{n-i_x} < r \le s^{i_x-1}(s+1)^{n-i_x+1}. \tag{2.11}$$

Analogously, from (2.9) and (2.10), we get

$$m_y^{i_y}(m_y+1)^{n-i_y} < r \le m_y^{i_y-1}(m_y+1)^{n-i_y+1}. \tag{2.12}$$

Therefore, using Remark 2.7, (2.11) and (2.12) imply that $s = m_y$ and $i_x = i_y$ and, consequently,

$$\left|\{z \in \mathbb{Z}^{n-1} : z \preceq x'\}\right| = r - s^{i_x}(s+1)^{n-i_x} = \left|\{z \in \mathbb{N}^{n-1} : z \preceq y'\}\right|.$$

Remark 2.6 also yields that both initial segments are the union of a lattice box of cardinality $s^{i_x}(s+1)^{n-i_x}$ with an $(n-1)$-dimensional initial segment of cardinality $r - s^{i_x}(s+1)^{n-i_x}$, in their respective orders. Moreover, we have

$$\left|\mathcal{I}_r + \{0,1\}^n\right| = (s+1)^{i_x}(s+2)^{n-i_x} + \left|\{z \in \mathbb{Z}^{n-1} : z \preceq x'\} + \{0,1\}^{n-1}\right|$$

and

$$\left|\mathcal{J}_r+\{0,1\}^n\right| = (s+1)^{i_x}(s+2)^{n-i_x} + \left|\{z\in\mathbb{N}^{n-1} : z\preceq y'\}+\{0,1\}^{n-1}\right|.$$

This concludes the proof since the induction hypothesis implies that

$$\left|\{z\in\mathbb{Z}^{n-1} : z\preceq x'\}+\{0,1\}^{n-1}\right| = \left|\{z\in\mathbb{N}^{n-1} : z\preceq y'\}+\{0,1\}^{n-1}\right|. \qquad \blacksquare$$

2.2 New discrete isoperimetric inequalities for the cardinality

As suggested by the work in the previous section (see (2.6) or Lemma 2.8), and as we will further see in this section (see Corollary 2.13), in order to develop a framework that allows us to obtain results that can be applied to the settings of $\mathbb{Z}^n$ and $\mathbb{N}^n$ simultaneously, it will be convenient to work with the functional $\left|X+\{0,1\}^n\right|$, $X\subset\mathbb{Z}^n$.

Therefore, following the ideas from [104, 86], we now prove the following new discrete isoperimetric type inequality. We will later show that it is, in fact, equivalent to Theorem I (see Proposition 2.12). In Section 2.4 we will see that this result for the cardinality can be used to obtain discrete isoperimetric-type inequalities in the setting of compact sets of $\mathbb{R}^n$ for the lattice point enumerator (see Theorems 2.37 and 2.41), which will in turn enable us to recover the original continuous versions of the isoperimetric inequality for the volume (see Theorem 2.43).

Theorem 2.9 [66, Lemma 2.6] Let $X\subset\mathbb{N}^n$ be a non-empty finite set with $|X|=r$. Then

$$\left|X+\{0,1\}^n\right| \geq \left|\mathcal{J}_r+\{0,1\}^n\right|. \tag{2.13}$$

Proof. If $n=1$, since $\mathcal{J}_r=\{0,\dots,r-1\}\subset\mathbb{N}$, by applying Theorem D to the sets $X,\{0\}\subset\mathbb{N}$ we immediately get

$$\left|X+\{0,1\}\right| \geq r+1 = \left|\mathcal{J}_r+\{0,1\}\right|. \tag{2.14}$$

Let $n>1$. If X is downward compressed then the result is a direct consequence of Theorem I, together with the fact that $\mathcal{J}_r$ is also downward compressed and (2.6).

If X is not downward compressed, it is enough to show that we can find a downward compressed set $Z\subset\mathbb{N}^n$ such that $|X|=|Z|$ and $\left|X+\{0,1\}^n\right|\geq\left|Z+\{0,1\}^n\right|$, and apply the previous case.

So, we assume that X is not downward compressed in the i-th coordinate, for some $i\in\{1,\dots,n\}$, and we define the set $Y\subset\mathbb{N}^n$ as

$$Y=\bigcup_{x\in X}\left\{(x_1,\dots,x_{i-1},t,x_{i+1},\dots,x_n)\in\mathbb{N}^n : t<\left|(x+\ell_i)\cap X\right|\right\}.$$

The set Y is downward compressed in the i-th coordinate and satisfies $\left|(x+\ell_i)\cap X\right|=\left|(x+\ell_i)\cap Y\right|$ for all $x\in\mathbb{N}^n$. Therefore, $|Y|=|X|$. Furthermore, since Y is downward compressed in the i-th coordinate, then for all $x\in\mathbb{N}^n$ one has that $(x+\ell_i)\cap Y$ has no "holes", i.e., it is formed by consecutive points of $\mathbb{N}^n$ in $x+\ell_i$. Hence, $\left|(x+\ell_i)\cap\left(Y+\{0,1\}^n\right)\right|\leq\left|(x+\ell_i)\cap\left(X+\{0,1\}^n\right)\right|$, and therefore, $\left|Y+\{0,1\}^n\right|\leq\left|X+\{0,1\}^n\right|$.

We also note that, by repeatedly "compressing" the set X with respect to different coordinates as many times as necessary, we eventually get a downward compressed set $Z\subset\mathbb{N}^n$ after a finite number of steps. Indeed, by looking at the ranks of X and Y, we note that $r(X)\geq r(Y)$ with an strict inequality if $X\neq Y$ (cf. Remark 2.4), and so it is a consequence of the fact that $r(X)$ is bounded from below. ■

Remark 2.10 We note that $\mathcal{J}_r+\{0,1\}^n$ is an initial segment, which follows from (2.6) and the fact that $\left(\mathcal{J}_r+\{-1,0,1\}^n\right)\cap\mathbb{N}^n$ is an initial segment (cf. [86, page 11]). ♦

As a consequence of the previous remark, by iterating Theorem 2.9, one gets the following corollary.

Corollary 2.11 Let $X \subset \mathbb{N}^n$ be a non-empty finite set with $|X| = r$. Then, for all $s \in \mathbb{N}$,

$$\left|X + \{0,\dots,s\}^n\right| \geq \left|\mathcal{J}_r + \{0,\dots,s\}^n\right|. \tag{2.15}$$

Since for every $r \in \mathbb{N} \setminus \{0\}$ the set $\mathcal{I}_r + \{-1,0,1\}^n$ is an initial segment in $\mathbb{Z}^n$ (see [86, Lemma 1]), iterating Theorem H similarly yields that

$$\left|X + \{-s,\dots,s\}^n\right| \geq \left|\mathcal{I}_r + \{-s,\dots,s\}^n\right| \tag{2.16}$$

for any $X \subset \mathbb{Z}^n$ and all $s \in \mathbb{N}$, which is equivalent to Theorem H. It is also easy to show that Corollary 2.11 and Theorem I are equivalent.

Proposition 2.12 The discrete isoperimetric inequalities (2.5) and (2.15) are equivalent.

Proof. Since the proof of Corollary 2.11 uses Theorem 2.9 (and thus, Theorem I), we only need to show that (2.15) implies (2.5). But this is a direct consequence of the fact that

$$\left|\left(X + \{-1,0,1\}^n\right) \cap \mathbb{N}^n\right| \geq \left|X + \{0,1\}^n\right| \geq \left|\mathcal{J}_r + \{0,1\}^n\right| = \left|\left(\mathcal{J}_r + \{-1,0,1\}^n\right) \cap \mathbb{N}^n\right|,$$

for all finite sets $X \subset \mathbb{N}^n$ with $|X| \geq r$ (cf. (2.6)). ■

We finish the section by noting the strong connection between the initial segments in $\mathbb{Z}^n$ and $\mathbb{N}^n$ that was already pointed out in the previous section.

On the one hand, just like $\mathcal{I}_r$, the initial segments $\mathcal{J}_r$ also give equality in (2.16) for any $s \in \mathbb{N}$, and thus, also in (2.4). Indeed, it suffices to apply Corollary 2.11 with the cube $\{0,\dots,2s\}^n$, together with the translation invariance of the cardinality.

On the other hand, the initial segments $\mathcal{I}_r \subset \mathbb{Z}^n$ also attain the equality in (2.15) (and thus in (2.5)). In fact, since for every $r \in \mathbb{N} \setminus \{0\}$ the set $\mathcal{I}_r + \{-1,0,1\}^n$ is an initial segment in $\mathbb{Z}^n$, and likewise, $\mathcal{J}_r + \{0,1\}^n$ is an initial segment in $\mathbb{N}^n$ (see Remark 2.10), by iterating these properties, using (2.15), (2.16) and Lemma 2.8, and due to the translation invariance of the cardinality, we have the following result which generalizes Lemma 2.8:

Corollary 2.13 Let $r \in \mathbb{N}$, $r > 0$. Then $\left|\mathcal{I}_r + \{0,\dots,s\}^n\right| = \left|\mathcal{J}_r + \{0,\dots,s\}^n\right|$ for all $s \in \mathbb{N}$.

2.3 Characterization of the equality cases for the cardinality

Radcliffe and Veomett already observed in [86] that it is not possible to fully characterize (in general) the equality case in (2.4) and (2.5), since there are examples of sets that reach the equality and are not isomorphic to any initial segment.

Nevertheless, here we show, on the one hand, that lattice cubes can be characterized as the only sets (of the appropriate cardinality) attaining equality in Theorem H and Theorem I:

Theorem 2.14 [66, Theorem 1.1] Let $X \subset \mathbb{Z}^n$ with $|X| = (\rho+1)^n$ for some $\rho \in \mathbb{N}$. Then equality holds in (2.4) if and only if X is a lattice cube.

Theorem 2.15 [66, Theorem 1.2] Let $X \subset \mathbb{N}^n$ with $|X| = (\rho+1)^n$ for some $\rho \in \mathbb{N}$. Then equality holds in (2.5) if and only if $X = \{0,\dots,\rho\}^n$.

We will do this by showing a stronger characterization of lattice cubes as the only minimizers of the functional $\left|X+\{0,\dots,s\}^n\right|$ for all $s\in\mathbb{N}\setminus\{0\}$:

Theorem 2.16 [66, Theorem 3.1] Let $X\subset\mathbb{N}^n$ with $|X|=(\rho+1)^n$ for some $\rho\in\mathbb{N}$ and let $s\in\mathbb{N}$ with $s>0$. If

$$\left|X+\{0,\dots,s\}^n\right|=\left|\mathcal{J}_{(\rho+1)^n}+\{0,\dots,s\}^n\right|,$$

then X is a lattice cube.

This will further allow us to characterize lattice cubes in a wider family of discrete inequalities, for instance, in the Brunn-Minkowski type inequality given in Theorem D (see Corollary 2.34), or in the inequalities for the lattice point enumerator that we will prove in Section 2.4.

We first set further new definitions and get some initial results. The next subsections address separately the 2-dimensional and the general case of Theorem 2.16 for $s=1$. An additional inductive argument then shows Theorem 2.16 in its full generality. Finally, as a consequence, we obtain Theorems 2.14 and 2.15. We refer the reader to [104] and [37] for similar studies with other norms.

Definition 2.17 We say that a non-empty finite set $X\subset\mathbb{N}^n$ is optimal if for all $A\subset\mathbb{N}^n$ with $|A|=|X|$ we have $\left|A+\{0,1\}^n\right|\geq\left|X+\{0,1\}^n\right|$.

Definition 2.18 Given a finite set $X\subset\mathbb{N}^n$, we define the (n-dimensional) neighborhood of X as $\mathrm{N}_X^n=\left(X+\{0,1\}^n\right)\setminus X$ if $X\neq\emptyset$, and $\mathrm{N}_X^n=\emptyset$ if $X=\emptyset$. Moreover, its cardinality will be denoted by $\mathrm{n}(X)=\left|\mathrm{N}_X^n\right|$.

We note that the optimality of a finite set can be defined in terms of the functional $\mathrm{n}(\cdot)$, since any set $X\subset\mathbb{N}^n$ is optimal if and only if $\mathrm{n}(A)\geq\mathrm{n}(X)$ for each $A\subset\mathbb{N}^n$ with $|A|=|X|$.

Lemma 2.19 Let $n>1$ and let $X\subset\mathbb{N}^n$ be a non-empty finite set. If $|X|>(\rho+1)^n$ for some $\rho\in\mathbb{N}$, then $\left|X+\{0,1\}^n\right|>(\rho+2)^n$ and $\mathrm{n}(X)>(\rho+2)^n-(\rho+1)^n$.

Proof. Let $\mathcal{J}_a\subset\mathcal{J}_b\subset\mathcal{J}_c\subset\mathbb{N}^n$ with $a=(\rho+1)^n$, $b=(\rho+1)^n+1$ and $c=|X|$. Then, $\mathcal{J}_a=\{0,\dots,\rho\}^n$ and $\mathcal{J}_b=\mathcal{J}_a\cup(0,\dots,0,\rho+1)$ (see Remark 2.2), and Theorem 2.9 yields

$$\left|X+\{0,1\}^n\right|\geq\left|\mathcal{J}_c+\{0,1\}^n\right|>\left|\mathcal{J}_a+\{0,1\}^n\right|=(\rho+2)^n.$$

In the following, we show that for any $r\in\mathbb{N}$, $r>0$, we have $\mathrm{n}(\mathcal{J}_{r+1})\geq\mathrm{n}(\mathcal{J}_r)$. Let $x_0\in\mathbb{N}^n$ be the last point in the order $\prec$ of $\mathcal{J}_{r+1}$. Then, using Remark 2.4, and since $x\preceq(1,\dots,1)$ for all $x\in\{0,1\}^n$, we get

$$z+x\preceq z+(1,\dots,1)\prec x_0+(1,\dots,1),$$

for all $z\prec x_0$. This implies that $x_0+(1,\dots,1)\in\left(\mathcal{J}_{r+1}+\{0,1\}^n\right)\setminus\left(\mathcal{J}_r+\{0,1\}^n\right)$. Consequently,

$$\left|\mathcal{J}_{r+1}+\{0,1\}^n\right|\geq\left|\mathcal{J}_r+\{0,1\}^n\right|+1,$$

and, since $|\mathcal{J}_{r+1}|=|\mathcal{J}_r|+1$, we deduce that

$$\mathrm{n}(\mathcal{J}_{r+1})=\left|\mathcal{J}_{r+1}+\{0,1\}^n\right|-|\mathcal{J}_{r+1}|\geq\left|\mathcal{J}_r+\{0,1\}^n\right|-|\mathcal{J}_r|=\mathrm{n}(\mathcal{J}_r),$$

as desired. This concludes the proof, since we also have

$$\mathrm{n}(\mathcal{J}_a)=(\rho+2)^n-(\rho+1)^n\quad\text{and}\quad\mathrm{n}(\mathcal{J}_b)=(\rho+2)^n+2^{n-1}-\left((\rho+1)^n+1\right)$$

(see, e.g., Lemma 2.5), and therefore,

$$\mathrm{n}(X)=\left|X+\{0,1\}^n\right|-|X|\geq\left|\mathcal{J}_c+\{0,1\}^n\right|-c=\mathrm{n}(\mathcal{J}_c)\geq\mathrm{n}(\mathcal{J}_b)>(\rho+2)^n-(\rho+1)^n. \qquad\blacksquare$$

Definition 2.20 We say that a non-empty finite set $X \subset \mathbb{N}^n$ is connected if for each $x, y \in X$, each $i \in \{1, \dots, n\}$, and any $m \in \mathbb{N}$ such that $x_i < m < y_i$, there exists $z \in X$ satisfying $z_i = m$.

An important observation is that any optimal set (see Definition 2.17) is connected. Indeed, a similar argument to the one in [37, Proposition 1.4], translating one connected component next to the boundary of another one (without overlapping) to strictly decrease the functional $\mathrm{n}(\cdot)$, shows the following result:

Proposition 2.21 If $X \subset \mathbb{N}^n$ is optimal, then X is connected.

2.3.1 Characterization in dimension 2

The 2-dimensional case of Theorem 2.16 is based on the fact that any connected set X can be enlarged up to a suitable lattice box without increasing the functional $\mathrm{n}(X)$.

Thus, for a finite non-empty set $X \subset \mathbb{N}^2$, we denote by $\mathcal{B}(X) \subset \mathbb{N}^2$ the smallest lattice box (with respect to set inclusion) such that $X \subset \mathcal{B}(X)$, i.e., $\mathcal{B}(X) = \big([a_1, b_1] \times [a_2, b_2]\big) \cap \mathbb{N}^2$, where, for $i = 1, 2$,

$$a_i = \min\big\{x_i : (x_1, x_2) \in X\big\} \quad \text{and} \quad b_i = \max\big\{x_i : (x_1, x_2) \in X\big\}.$$

Lemma 2.22 Let $X \subset \mathbb{N}^2$ be a non-empty connected finite set. Then

$$\mathrm{n}\big(\mathcal{B}(X)\big) \le \mathrm{n}(X).$$

Proof. We may assume, by applying a translation to X if necessary, that $\mathcal{B}(X) = \big([1, b_1] \times [1, b_2]\big) \cap \mathbb{N}^2$ for some $b = (b_1, b_2) \in \mathbb{N}^2$. Then,

$$|\mathcal{B}(X)| = b_1 b_2, \qquad \big|\mathcal{B}(X) + \{0,1\}^n\big| = (b_1 + 1)(b_2 + 1)$$

and therefore $\mathrm{n}\big(\mathcal{B}(X)\big) = (b_1 + 1)(b_2 + 1) - b_1 b_2 = b_1 + b_2 + 1$.

Let $X_1, X_2 \subset \big(X + \{0,1\}^2\big) \setminus X$ be defined as

$$X_1 = \big\{(x_1, x_2) \in X : \nexists (m, x_2) \in X \text{ with } m > x_1\big\} + \mathrm{e}_1 \quad \text{and}$$
$$X_2 = \big\{(x_1, x_2) \in X \cup X_1 : \nexists (x_1, m) \in X \cup X_1 \text{ with } m > x_2\big\} + \mathrm{e}_2.$$

We note that $X_1 \subset (X + \mathrm{e}_1) \setminus X$ and $X_2 \subset \big(X + \{0,1\}^2\big) \setminus (X + \mathrm{e}_1)$. Therefore $X_1 \cap X_2 = \emptyset$ and, since X is connected, $|X_1| = b_2$ and $|X_2| = b_1 + 1$. Altogether we conclude the proof since we have

$$\mathrm{n}\big(\mathcal{B}(X)\big) = b_1 + b_2 + 1 = |X_1| + |X_2| \le \mathrm{n}(X). \qquad \blacksquare$$

We are now under the conditions to prove the following lemma, which corresponds to the 2-dimensional case of Theorem 2.16 for $s = 1$.

Lemma 2.23 Let $X \subset \mathbb{N}^2$ be a non-empty finite set with $|X| = (\rho + 1)^2$ for some $\rho \in \mathbb{N}$. If

$$\big|X + \{0,1\}^2\big| = (\rho + 2)^2,$$

then X is a lattice cube.

Proof. By the hypothesis on X, we deduce that X is optimal and, consequently, by Proposition 2.21, it is connected. Furthermore, it must satisfy $X = \mathcal{B}(X)$. Otherwise we would have $\big|\mathcal{B}(X)\big| > |X| = (\rho + 1)^2$ and, by Lemma 2.22, $\mathrm{n}\big(\mathcal{B}(X)\big) \le \mathrm{n}(X) = (\rho + 2)^2 - (\rho + 1)^2$, which would contradict Lemma 2.19.

We assume, by applying a translation to X if necessary, that $X = \mathcal{B}(X) = \big([0,b_1]\times[0,b_2]\big)\cap\mathbb{N}^2$ for some $b=(b_1,b_2)\in\mathbb{N}^2$. Then, the equality condition in the well-known arithmetic-geometric mean inequality (see, e.g., [28, page 71]) shows that the equations

$$(\rho+1)^2 = |X| = (b_1+1)(b_2+1) \quad \text{and} \quad 2(\rho+1)+1 = \mathrm{n}(X) = b_1+b_2+3$$

imply $\rho = b_1 = b_2$, as desired. ■

2.3.2 Characterization in general dimension

The proof when $n>2$ in Theorem 2.16 is based on a process that we call "normalization" (see Definition 2.27). It extends the process of normalization introduced in [104, Section 4] and adapts it to the L_∞ setting. The next lemma shows that the functional $\mathrm{n}(\cdot)$ can be estimated in terms of the sections of the set.

Lemma 2.24 Let $n>1$ and $i\in\{1,\dots,n\}$, and let $X\subset\mathbb{N}^n$ be a non-empty finite set. Then

$$\mathrm{n}(X) \geq \left|\Big(\bigcup_{m\in\mathbb{N}} X^i(m)\Big)+\{0,1\}^{n-1}\right| + \sum_{m\in\mathbb{N}}\left|\mathrm{N}^{n-1}_{X^i(m)}\right|. \tag{2.17}$$

Furthermore, if the sections $X^i(m)$ form a decreasing sequence, namely, $X^i(0)\supset X^i(1)\supset\dots$, then equality holds in (2.17).

Proof. In order to prove (2.17), we consider the sets

$$Y_m = \{y\in X : y_i = m\} + \big(\{0,1\}^{i-1}\times\{0\}\times\{0,1\}^{n-i}\big)$$

for all $m\in\mathbb{N}$ such that $\{y\in X: y_i=m\}\neq\emptyset$, and

$$Y = \big\{y\in X+\{0,1\}^n : y+k\mathrm{e}_i\notin X+\{0,1\}^n \text{ for all } k>0\big\}.$$

Clearly, the sets Y_m are pairwise disjoint and do not intersect with Y. Furthermore, $Y_m\subset X+\{0,1\}^n$ and $|Y_m| = \left|X^i(m)+\{0,1\}^{n-1}\right|$ for all $m\in\mathbb{N}$ with $X^i(m)\neq\emptyset$. Moreover, observe that $|Y| = \left|P_{\mathrm{e}_i^\perp}Y\right|$ and

$$P_{\mathrm{e}_i^\perp}Y = P_{\mathrm{e}_i^\perp}\big(X+\{0,1\}^n\big) = \Big(\bigcup_{m\in\mathbb{N}} X^i(m)\Big)+\{0,1\}^{n-1}.$$

Therefore,

$$\left|X+\{0,1\}^n\right| \geq |Y| + \sum_{m\in\mathbb{N}}|Y_m| = \left|\Big(\bigcup_{m\in\mathbb{N}} X^i(m)\Big)+\{0,1\}^{n-1}\right| + \sum_{\substack{m\in\mathbb{N}\\ X^i(m)\neq\emptyset}}\left|X^i(m)+\{0,1\}^{n-1}\right|.$$

By subtracting $|X| = \sum_{m\in\mathbb{N}}|X^i(m)|$ we conclude the proof of (2.17). Now, if we have $m_0\in\mathbb{N}$ such that $X^i(m)=\emptyset$ for all $m>m_0$ and

$$X^i(0)\supset X^i(1)\supset X^i(2)\supset\cdots\supset X^i(m_0)\neq\emptyset,$$

then

$$\left|\big\{y\in X+\{0,1\}^n : y_i=0\big\}\right| = \left|X^i(0)+\{0,1\}^{n-1}\right|$$

and

$$\left|\big\{y\in X+\{0,1\}^n : y_i=m+1\big\}\right| = \left|X^i(m)+\{0,1\}^{n-1}\right|$$

for all $0\leq m\leq m_0$. With this we can conclude the proof:

$$\begin{aligned}\left|X+\{0,1\}^n\right| &= \sum_{m=0}^{m_0+1}\left|\left\{y\in X+\{0,1\}^n : y_i=m\right\}\right| = \left|X^i(0)+\{0,1\}^{n-1}\right| + \sum_{m=0}^{m_0}\left|X^i(m)+\{0,1\}^{n-1}\right| \\ &= \left|\Big(\bigcup_{m\in\mathbb{N}} X^i(m)\Big)+\{0,1\}^{n-1}\right| + \sum_{\substack{m\in\mathbb{N}\\ X^i(m)\neq\emptyset}}\left|X^i(m)+\{0,1\}^{n-1}\right|. \qquad \blacksquare\end{aligned}$$

We note that any optimal set $X\subset\mathbb{N}^n$ must reach equality in Lemma 2.24: indeed, simply by changing each section $X^i(m)$ by an initial segment in $\mathbb{N}^{n-1}$ of the same cardinality, and then rearranging the sections in decreasing order, we get a new set $Z\subset\mathbb{N}^n$ that gives equality in Lemma 2.24. Therefore

$$\begin{aligned}\mathrm{n}(Z) &= \left|\Big(\bigcup_{m\in\mathbb{N}} Z^i(m)\Big)+\{0,1\}^{n-1}\right| + \sum_{m\in\mathbb{N}}\left|\mathrm{N}^{n-1}_{Z^i(m)}\right| = \left|Z^i(0)+\{0,1\}^{n-1}\right| + \sum_{m\in\mathbb{N}}\left|\mathrm{N}^{n-1}_{Z^i(m)}\right| \\ &\le \left|X^i(0)+\{0,1\}^{n-1}\right| + \sum_{m\in\mathbb{N}}\left|\mathrm{N}^{n-1}_{X^i(m)}\right| \le \left|\Big(\bigcup_{m\in\mathbb{N}} X^i(m)\Big)+\{0,1\}^{n-1}\right| + \sum_{m\in\mathbb{N}}\left|\mathrm{N}^{n-1}_{X^i(m)}\right| \le \mathrm{n}(X),\end{aligned} \tag{2.18}$$

and thus $\mathrm{n}(Z)=\mathrm{n}(X)$ due to the optimality of X. This allows us to deduce the following result.

Corollary 2.25 Let $n>1$ and $i\in\{1,\dots,n\}$, and let $X\subset\mathbb{N}^n$ be an optimal set. Then the sections $X^i(m)$ are optimal (as $(n-1)$-dimensional sets) and satisfy

$$\mathrm{n}(X) = \left|\Big(\bigcup_{m\in\mathbb{N}} X^i(m)\Big)+\{0,1\}^{n-1}\right| + \sum_{m\in\mathbb{N}}\left|\mathrm{N}^{n-1}_{X^i(m)}\right|. \tag{2.19}$$

We note that the converse is not true: there are examples of non-optimal sets satisfying (2.19) for all $i=1,\dots,n$, and having all $(n-1)$-dimensional sections optimal (see Figure 2.4).

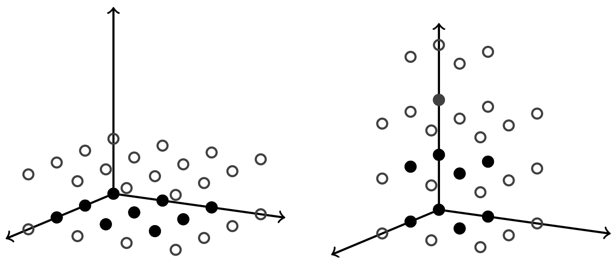

Figure 2.4: Left: A set $X\subset\mathbb{N}^3$ (in black) and $X+\{0,1\}^3$ (in white). Right: $\mathcal{J}_9\subset\mathbb{N}^3$ (in black) and $\mathcal{J}_9+\{0,1\}^3$ (in white). X satisfies (2.19) and its 2-dimensional sections are optimal, but $\mathrm{n}(X)=23>22=\mathrm{n}(\mathcal{J}_9)$.

The following result shows, roughly speaking, that in order to minimize the expression $\mathrm{n}(\mathcal{J}_a)+\mathrm{n}(\mathcal{J}_b)$ for $a,b\in\mathbb{N}$ with $a+b$ fixed, one may begin by choosing a,b such that one of the resulting initial segments is the largest possible lattice box of the form $\{0,\dots,\rho-1\}^j\times\{0,\dots,\rho\}^{n-j}$ for some $\rho\in\mathbb{N}$ and $j\in\{1,\dots,n\}$. Furthermore, it shows that a single initial segment $\mathcal{J}_{a+b}$ does never exceed this minimum.

Lemma 2.26 Let $a,b,c\in\mathbb{N}\setminus\{0\}$ with $\max\{a,b\}<c<a+b$ and such that $c=\rho^j(\rho+1)^{n-j}$ for some $\rho\in\mathbb{N}$ and $j\in\{1,\dots,n\}$. Then

$$\mathrm{n}(\mathcal{J}_a)+\mathrm{n}(\mathcal{J}_b)\ge\mathrm{n}(\mathcal{J}_{a+b-c})+\mathrm{n}(\mathcal{J}_c). \tag{2.20}$$

Moreover,

$$\mathrm{n}(\mathcal{J}_a)+\mathrm{n}(\mathcal{J}_b)>\mathrm{n}(\mathcal{J}_{a+b}). \tag{2.21}$$

Proof. We proceed by induction on the dimension n. Both inequalities are clear for $n = 1$ since for every initial segment $\mathcal{J}_r \subset \mathbb{N}$ we have $\mathrm{n}(\mathcal{J}_r) = 1$.

Assume now that $n > 1$ and that the lemma holds for every value of the dimension up to $n-1$. It suffices to show that if $1 < a \le b < c = \rho^j(\rho+1)^{n-j}$, then it is possible to find $d \in \mathbb{N}$ with $0 < d < a$ and $d \le c - b$ such that

$$\mathrm{n}(\mathcal{J}_a) + \mathrm{n}(\mathcal{J}_b) \ge \mathrm{n}(\mathcal{J}_{a-d}) + \mathrm{n}(\mathcal{J}_{b+d}), \tag{2.22}$$

and iterating this process will prove the lemma. Indeed, notice on the one hand that the conditions above imply that $1 \le a - d < b + d \le c$, and if $c < a + b$, it is easy to check that the process will necessarily conclude when the upper bound is reached, i.e., (2.20). On the other hand, if $c \ge a + b$, then it will necessarily conclude when the lower bound is reached, i.e.,

$$\mathrm{n}(\mathcal{J}_a) + \mathrm{n}(\mathcal{J}_b) \ge \mathrm{n}(\mathcal{J}_1) + \mathrm{n}(\mathcal{J}_{a+b-1}),$$

which implies (2.21) since Lemma 2.5 yields $2^n - 1 + \mathrm{n}(\mathcal{J}_{a+b-1}) > \mathrm{n}(\mathcal{J}_{a+b})$.

Now, in other to prove (2.22), we let $x, y \in \mathbb{N}^n$ be the last points with respect to $\prec$ in $\mathcal{J}_a$, $\mathcal{J}_b$, respectively. Also, for the sake of brevity, we denote by $G, H \subset \mathbb{N}^{n-1}$ the last non-empty sections of $\mathcal{J}_a, \mathcal{J}_b \subset \mathbb{N}^n$, i.e., $G = (\mathcal{J}_a)^{i_x}(m_x)$ and $H = (\mathcal{J}_b)^{i_y}(m_y)$.

We note that since $1 < a$, $\mathcal{J}_a$ has at least two non-empty sections (with respect to the direction e_{i_x}), and therefore $|G| < a$. Using Remark 2.6 (in particular, (2.9) and (2.10)), we know that

$$a = m_x^{i_x}(m_x+1)^{n-i_x} + |G| \quad \text{and} \quad b = m_y^{i_y}(m_y+1)^{n-i_y} + |H| \tag{2.23}$$

with

$$|G| \le m_x^{i_x-1}(m_x+1)^{n-i_x} \quad \text{and} \quad |H| \le m_y^{i_y-1}(m_y+1)^{n-i_y}. \tag{2.24}$$

Also, since $a \le b$ then $x \preceq y$, and thus either $m_x < m_y$, or $m_x = m_y$ with $i_x \ge i_y$. This implies (see Remark 2.7) that

$$|G| \le m_x^{i_x-1}(m_x+1)^{n-i_x} \le m_y^{i_y-1}(m_y+1)^{n-i_y}. \tag{2.25}$$

Likewise, since

$$m_y^{i_y}(m_y+1)^{n-i_y} < b < c = \rho^j(\rho+1)^{n-j},$$

then Remark 2.7 implies that $c \ge m_y^{i_y-1}(m_y+1)^{n-i_y+1}$. This, together with (2.23), shows that

$$m_y^{i_y-1}(m_y+1)^{n-i_y} - |H| \le c - b. \tag{2.26}$$

Now, we consider the following cases, which are exhaustive as a consequence of (2.24):

(i) $|G| > |H|$.
(ii) $|G| \le |H| < m_y^{i_y-1}(m_y+1)^{n-i_y}$.
(iii) $|G| \le |H| = m_y^{i_y-1}(m_y+1)^{n-i_y}$ and $i_y > 1$.
(iv) $|G| \le |H| = m_y^{i_y-1}(m_y+1)^{n-i_y}$ and $i_y = 1$.

In case (i) we choose $d = |G| - |H|$, and so we may, roughly speaking, interchange the last sections G and H, i.e., we have $H = (\mathcal{J}_{a-d})^{i_x}(m_x)$ and $G = (\mathcal{J}_{b+d})^{i_y}(m_y)$. The rest of the sections (and their union) remain the same, i.e., $(\mathcal{J}_{a-d})^{i_x}(m) = (\mathcal{J}_a)^{i_x}(m)$ for all $0 \le m < m_x$ and $(\mathcal{J}_{b+d})^{i_y}(m) = (\mathcal{J}_b)^{i_y}(m)$ for all $0 \le m < m_y$. Therefore, by using Corollary 2.25, we get (2.22) with equality. Clearly $0 < d \le |G| < a$, and $d \le c - b$ follows from (2.25) and (2.26).

In case (ii) we set $\bar{a} = |G|$, $\bar{b} = |H|$ and $\bar{c} = m_y^{i_y-1}(m_y+1)^{n-i_y}$, and we use the induction hypothesis (in dimension $n-1$). We choose $d = \min(\bar{a}, \bar{c} - \bar{b})$ and so, we get

$$\left|\mathrm{N}^{n-1}_{\mathcal{J}_{\bar{a}}}\right| + \left|\mathrm{N}^{n-1}_{\mathcal{J}_{\bar{b}}}\right| \ge \left|\mathrm{N}^{n-1}_{\mathcal{J}_{\bar{a}+\bar{b}-\bar{c}}}\right| + \left|\mathrm{N}^{n-1}_{\mathcal{J}_{\bar{c}}}\right|$$

if $\bar{c} < \bar{a} + \bar{b}$, and

$$\left|\mathrm{N}^{n-1}_{\mathcal{J}_{\bar{a}}}\right| + \left|\mathrm{N}^{n-1}_{\mathcal{J}_{\bar{b}}}\right| > \left|\mathrm{N}^{n-1}_{\mathcal{J}_{\bar{a}+\bar{b}}}\right|$$

if $\bar{c} \geq \bar{a} + \bar{b}$. Again, from Remark 2.6 we get that $(\mathcal{J}_{a-d})^{i_x}(m) = (\mathcal{J}_a)^{i_x}(m)$ for all $0 \leq m < m_x$ and $(\mathcal{J}_{b+d})^{i_y}(m) = (\mathcal{J}_b)^{i_y}(m)$ for all $0 \leq m < m_y$, and the union of the sections remains likewise unchanged. Thus, an application of Corollary 2.25 yields (2.22). We again trivially have $0 < d \leq |G| < a$, and $d \leq c - b$ follows directly from (2.26).

In case (iii) we have $\mathcal{J}_b = \{0,\dots,m_y - 1\}^{i_y-1} \times \{0,\dots,m_y\}^{n-i_y+1}$ (see Remark 2.6). Therefore, we may choose $d = |G|$, and, by applying Remark 2.6 again, we get that the only non-empty sections of $\mathcal{J}_{a-d}$ are $(\mathcal{J}_{a-d})^{i_x}(m) = (\mathcal{J}_a)^{i_x}(m)$ for all $0 \leq m < m_x$. Moreover, the only non-empty sections of $\mathcal{J}_{b+d}$ are $(\mathcal{J}_{b+d})^{i_y-1}(m) = (\mathcal{J}_b)^{i_y-1}(m)$ for all $0 \leq m < m_y$ and $(\mathcal{J}_{b+d})^{i_y-1}(m_y) = G$. So, since the union of all these sections has not changed, by using Corollary 2.25 we obtain (2.22), once more with equality. It is straightforward that $0 < d = |G| < a$, and since $b = m_y^{i_y-1}(m_y+1)^{n-i_y+1}$ and $b < c$, Remark 2.7 implies $c \geq m_y^{i_y-2}(m_y+1)^{n-i_y+2}$, and so from (2.25) it follows that $d \leq c - b$.

Finally, in case (iv) we have $\mathcal{J}_b = \{0,\dots,m_y\}^n$. Again, we may choose $d = |G|$, which yields the same sections for $\mathcal{J}_{a-d}$ as in the previous case, whereas for the non-empty sections of $\mathcal{J}_{b+d}$ we have $(\mathcal{J}_{b+d})^n(m) = (\mathcal{J}_b)^n(m)$ for all $0 \leq m \leq m_y$ and $(\mathcal{J}_{b+d})^n(m_y+1) = G$. Once more, Corollary 2.25 yields (2.22) with equality. It is again trivial that $0 < d = |G| < a$, and this time, since $b = (m_y+1)^n$ and $b < c$, Remark 2.7 implies $c \geq (m_y+1)^{n-1}(m_y+2)$, and so $d \leq c - b$ follows from (2.25).

This completes the proof of (2.22), and thus, of the result. ■

Now, for any $a,n \in \mathbb{N}$, $a,n > 0$, let $C \subset \mathcal{J}_a \subset \mathbb{N}^n$ be the largest lattice box (with respect to the cardinality) of the form $C = \{0,\dots,\rho - 1\}^j \times \{0,\dots,\rho\}^{n-j}$ for some $\rho, j \in \mathbb{N}$, $1 \leq j \leq n$. Then, we denote by $c(a,n) = |C| = \rho^j(\rho+1)^{n-j}$. Furthermore, for any $i \in \{1,\dots,n\}$ and any non-empty set $X \subset \mathbb{N}^n$, we denote by $c^i(X) = \max_{m\in\mathbb{N}} c\big(|X^i(m)|, n-1\big)$.

We proceed to define the notion of "normalization", which extends the normalization process defined in [104] and also utilized, among others, in [86] (see Figure 2.5 for an example of this construction).

Definition 2.27 Let $n > 1$ and $k \in \{1,\dots,n\}$, and let $X \subset \mathbb{N}^n$ be a non-empty finite set. Let $\rho \in \mathbb{N}$ and $j \in \{1,\dots,n-1\}$ be such that $c^k(X) = \rho^j(\rho+1)^{n-1-j}$. The k-normalization of X, denoted by $\overline{X}_k \subset \mathbb{N}^n$, is the result of the following process:

(i) Replacing each non-empty section $X^k(m)$, $m \in \mathbb{N}$, by the $(n-1)$-dimensional initial segment of the same cardinality.

(ii) Reordering the sections in decreasing order (with respect to set inclusion) such that the largest section corresponds to $m = 0$.

(iii) Starting with $m_1 = 1$ and $m_2 = \max\{m \in \mathbb{N} : X^k(m) \neq \emptyset\}$, and while $m_1 \leq \rho < m_2$, we repeat both of these steps:

1. If $|X^k(m_1)| < c^k(X)$, we replace the sections $X^k(m_1)$ and $X^k(m_2)$ by the initial segments of cardinality $|X^k(m_1)| + h$ and $|X^k(m_2)| - h$, respectively, where
$$h = \min\big\{|X^k(m_2)|,\ c^k(X) - |X^k(m_1)|\big\}.$$
2. If $|X^k(m_2)| = 0$, we decrease m_2 by 1, whereas if $|X^k(m_1)| = c^k(X)$, we increase m_1 by 1.

Furthermore, we say that $X \subset \mathbb{N}^n$ is stable if $X = \overline{X}_k$ for all $k = 1,\dots,n$.

Remark 2.28 We note that the end result $\overline{X}_k$ of a k-normalization, $k \in \{1,\dots,n\}$, is a set such that its non-empty sections, $(\overline{X}_k)^k(m)$, $m \in \mathbb{N}$, are $(n-1)$-dimensional initial segments ordered in decreasing order, i.e., $(\overline{X}_k)^k(0) \supset (\overline{X}_k)^k(1) \supset (\overline{X}_k)^k(2) \supset \dots$, and we have either $(\overline{X}_k)^k(\rho+1) = \emptyset$ or $|(\overline{X}_k)^k(m)| = c^k(X)$ for all $m \in \{1,\dots,\rho\}$ (where ρ is as in Definition 2.27). ♦

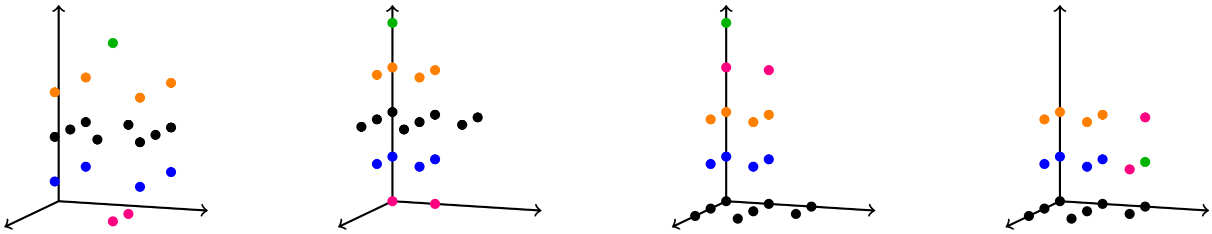

Figure 2.5: From left to right: a finite set, together with the same set after each step of the 3-normalization is applied.

Next we show that neither the rank nor the functional $\mathrm{n}(\cdot)$ increase under the normalization process.

Lemma 2.29 Let $X \subset \mathbb{N}^n$ be a non-empty finite set. If $X \neq \overline{X}_k$ for $k \in \{1,\dots,n\}$, then $r(X) > r(\overline{X}_k)$.

Proof. We proceed by proving that if any of the 3 steps of the normalization changes the set, then the rank of X strictly decreases.

First, it is straightforward from the definition of the order $\prec$ that if some section is not an ($(n-1)$-dimensional) initial segment, then the (n-dimensional) rank of the set will decrease under step (i).

Next, since all sections are initial segments, if $|X^k(m_1)| < |X^k(m_2)|$ for some $m_1 < m_2$ then we have $X^k(m_1) \subset X^k(m_2)$. Therefore, interchanging these sections is equivalent to translating the points in $\{X^k(m_2) \setminus X^k(m_1)\} \times \{m_2\}$ by reducing their k-th coordinate by $m_2 - m_1$, which decreases the rank strictly due to Remark 2.4.

In the third step, if we move a point z with $m_z \geq z_k > \rho$ (with ρ as specified in Definition 2.27) to a point $y \in \mathbb{N}^n$ with $m_y \leq \rho$, then again from the definition of the order $\prec$ the rank strictly decreases. ■

Lemma 2.30 Let $X \subset \mathbb{N}^n$ be a non-empty finite set. Then $\mathrm{n}(X) \geq \mathrm{n}(\overline{X}_k)$ for all $k = 1,\dots,n$.

Proof. Let $k \in \{1,\dots,n\}$. To begin with, we prove that the first two steps of the normalization process do not increase $\mathrm{n}(\cdot)$. Let us denote this resulting intermediate set by Z. By construction, we know there exists a permutation $\sigma : \mathbb{N} \longrightarrow \mathbb{N}$ such that, for every $m \in \mathbb{N}^n$, $Z^k(m)$ is either empty or an initial segment with $|Z^k(m)| = \left|X^k\big(\sigma(m)\big)\right|$. Then, the optimality of the initial segments (cf. Theorem 2.9) implies that $\left|\mathrm{N}^{n-1}_{Z^k(m)}\right| \leq \left|\mathrm{N}^{n-1}_{X^k(\sigma(m))}\right|$ for every $m \in \mathbb{N}$, and, taking into account that the sections of Z form a decreasing sequence, Lemma 2.24 yields $\mathrm{n}(Z) \leq \mathrm{n}(X)$ (cf. (2.18)).

To finish, we prove that the third step of the normalization process does not increase $\mathrm{n}(\cdot)$ either. We observe that the equality case in Lemma 2.24 gives

$$\begin{aligned} \mathrm{n}(Z) &= \left|Z^k(0) + \{0,1\}^{n-1}\right| + \sum_{m\in\mathbb{N}} \left|\mathrm{N}^{n-1}_{Z^k(m)}\right| \text{ and} \\ \mathrm{n}(\overline{X}_k) &= \left|\big(\overline{X}_k\big)^k(0) + \{0,1\}^{n-1}\right| + \sum_{m\in\mathbb{N}} \left|\mathrm{N}^{n-1}_{(\overline{X}_k)^k(m)}\right|. \end{aligned} \tag{2.27}$$

Note that $Z^k(0) = \big(\overline{X}_k\big)^k(0)$. Now, we let m_1, m_2 and h be as in the third step of Definition 2.27, and we set $a = |Z^k(m_1)|$, $b = |Z^k(m_2)|$ and $c = c^k(Z)$. So, clearly $h = \min\{b, c-a\}$. Then, on the one hand, if $h = b \leq c - a$, we have $|Z^k(m_1)| + h = a + b$ and $|Z^k(m_2)| - h = 0$, and thus (2.21) (in dimension $n-1$) ensures that this step strictly decreases the sum of the cardinalities of the above $(n-1)$-dimensional neighborhoods. On the other hand, if $h = c - a < b$, then $c < a + b$, and as per Definition 2.27 we also clearly have $\max\{a,b\} < c$. Therefore (2.20) (in dimension $n-1$) again yields that this step does not increase the sum of the cardinalities of the $(n-1)$-dimensional neighborhoods above. Consequently, from (2.27) we conclude that $\mathrm{n}(Z) \geq \mathrm{n}(\overline{X}_k)$, as desired. ■

The stability property allows us to decompose the set in a precise way:

Lemma 2.31 Let $n \geq 3$, $\rho \geq 1$ and let $X \subset \mathbb{N}^n$ be a non-empty finite set with $|X| = (\rho+1)^n$. If X is stable, then there exist $A, B \subset \mathbb{N}^n$ such that

$$A \subset \{0,\dots,\rho-1\}^{n-1} \times \{\rho+1\}, \qquad \emptyset \neq B \subset \{\rho\} \times \{0,\dots,\rho\}^{n-1}$$

and

$$X = A \cup B \cup \big(\{0,\dots,\rho-1\} \times \{0,\dots,\rho\}^{n-1}\big).$$

Proof. For the sake of brevity we write $C = \{0,\dots,\rho\}^n$. If $X = C$, then the result holds by taking $A = \emptyset$ and $B = \{\rho\} \times \{0,\dots,\rho\}^{n-1}$, and so we assume that $X \neq C$.

For any $i \in \{1,\dots,n\}$, since X is stable, we know that the non-empty sections $X^i(m)$ are initial segments verifying

$$X^i(0) \supset X^i(1) \supset X^i(2) \supset \dots, \tag{2.28}$$

and so $(\rho,\dots,\rho) \notin X$ because $X \neq C$.

First, we show that if $X^i(m) \neq \emptyset$ for some $i \in \{1,\dots,n\}$ and $m \geq 2$, then $\{0,\dots,m-2\}^n \subset X$ must hold. To see this, let $x \in X$ with $x_i = m$. Then, for any $j \neq i$, since $X^j(x_j)$ is an initial segment, we get $\{0,\dots,m-1\}^{n-1} \subset X^j(x_j)$. Fixing such an index $j \neq i$, this implies in particular that we have $(m-1,\dots,m-1,x_j,m-1,\dots,m-1) \in X$, and thus just like before, for any $k \neq j$, we obtain that $\{0,\dots,m-2\}^{n-1} \subset X^k(m-1)$, since $X^k(m-1)$ is an initial segment (observe how it is crucial in this step that $n \geq 3$). This in particular implies that $(m-2,\dots,m-2) \in X$, which together with (2.28) yields $\{0,\dots,m-2\}^n \subset X$, as desired.

It is easy to check that the previous property applied to $m = \rho+1$ and $m = \rho+2$, respectively, together with the fact that $X \neq C$, yields

$$\{0,\dots,\rho-1\}^n \subset X \subset \{0,\dots,\rho+1\}^n.$$

In fact, we further have

$$X \subset \{0,\dots,\rho\}^{n-1} \times \{0,\dots,\rho+1\}. \tag{2.29}$$

Indeed, if $X^i(\rho+1) \neq \emptyset$ for some $i < n$, then there exists $x \in X$ such that $x_i = \rho+1$. Fixing any $j \neq i,n$ and using a very similar argument to the previous one, exploiting that $X^j(x_j)$ is an initial segment, we obtain that $\rho\,\mathrm{e}_i + (\rho+1)\mathrm{e}_n \in X$. But this yields $\{0,\dots,\rho\}^{n-1} \subset X^i(\rho)$ (since $X^i(\rho)$ is an initial segment), contradicting that $(\rho,\dots,\rho) \notin X$. This proves (2.29), as desired.

Now, on the one hand, (2.29) yields $X^1(\rho) \neq \emptyset$, since otherwise we would have

$$|X| \leq \big|\{0,\dots,\rho-1\} \times \{0,\dots,\rho\}^{n-2} \times \{0,\dots,\rho+1\}\big| = \rho(\rho+1)^{n-2}(\rho+2) < (\rho+1)^n,$$

a contradiction. Since $X^1(\rho)$ is an initial segment then $(0,\dots,0) \in X^1(\rho)$, and thus $(\rho,0,\dots,0) \in X$. Hence, $(\rho,0,\dots,0) \in X^n(0)$, and therefore, since $X^n(0)$ is an initial segment, we have

$$\{0,\dots,\rho-1\} \times \{0,\dots,\rho\}^{n-2} \subset X^n(0). \tag{2.30}$$

Both (2.29) and (2.30), together with the fact that $X \neq C$ and thus $X^n(0) \neq \{0,\dots,\rho\}^{n-1}$, yield the bounds

$$\rho(\rho+1)^{n-2} \leq |X^n(0)| < (\rho+1)^{n-1}.$$

This implies that $c^n(X) = \rho(\rho+1)^{n-2}$.

On the other hand, since $X \neq C$, it follows from (2.29) that $X^n(\rho+1) \neq \emptyset$. Therefore, as X is stable (and thus $X = \overline{X}_n$), Remark 2.28 for $k=n$ implies that we have $|X^n(m)| \geq \rho(\rho+1)^{n-2}$ for all $m = 0,\dots,\rho$. Consequently, since $X^n(m)$ is an initial segment, we have

$$\{0,\dots,\rho-1\} \times \{0,\dots,\rho\}^{n-2} \subset X^n(m) \tag{2.31}$$

for all $m = 0,\dots,\rho$.

Finally, we note that, in fact,

$$X^n(\rho+1) \subset \{0,\dots,\rho-1\}^{n-1}. \tag{2.32}$$

Indeed, if $x \in X^n(\rho+1)$ with $x_i = \rho$ for $i < n$, then since $X^i(\rho)$ is an initial segment, we would have $\{0,\dots,\rho\}^{n-1} \subset X^i(\rho)$, contradicting that $(\rho,\dots,\rho) \notin X$.

This concludes the proof by setting $A = X^n(\rho+1) \times \{\rho+1\}$ and $B = \{\rho\} \times X^1(\rho)$, as a consequence of (2.29), (2.31) and (2.32). ■

Finally, we prove that in order to characterize the lattice cubes we only need stability and, either cardinality 2^n, or optimality.

Corollary 2.32 Let $n \geq 3$ and let $X \subset \mathbb{N}^n$ be a non-empty finite set with $|X| = 2^n$. If X is stable, then $X = \{0,1\}^n$.

Proof. Let $A, B \subset \mathbb{N}^n$ be the sets arising from Lemma 2.31 for $\rho = 1$. We notice that $X = \{0,1\}^n$ if and only if $A = \emptyset$. Therefore, if $X \neq \{0,1\}^n$, then we must have $|A| = 1$ and, since $|A|+|B| = 2^{n-1}$, we also have $|B| = 2^{n-1}-1$. Moreover, since X is stable, B is an $(n-1)$-dimensional initial segment, and so

$$X = \big(\{0,1\}^n \setminus (1,\dots,1)\big) \cup (0,\dots,0,2).$$

This contradicts the stability of X since $\overline{X}_n = \{0,1\}^n \neq X$. ■

Lemma 2.33 Let $n \geq 3$, $\rho \geq 2$ and let $X \subset \mathbb{N}^n$ be a non-empty finite set with $|X| = (\rho+1)^n$. If X is optimal and stable, then $X = \{0,\dots,\rho\}^n$.

Proof. Assume that $X \neq \{0,\dots,\rho\}^n$ and let $A, B \subset \mathbb{N}^n$ be the sets arising from Lemma 2.31. Observe that, since X is not a lattice cube, the set $A \neq \emptyset$. Then, $|A| > 0$ and

$$|A|+|B| = (\rho+1)^{n-1}. \tag{2.33}$$

If $\big(A+\{0,1\}^n\big) \cap \big(B+\{0,1\}^n\big) = \emptyset$ then, since $A = X^n(\rho+1) \times \{\rho+1\}$ and $B = \{\rho\} \times X^1(\rho)$, we clearly have, on the one hand, that

$$\mathrm{n}(X) = \left|\mathrm{N}^n_{\{0,\dots,\rho-1\}\times\{0,\dots,\rho\}^{n-1}}\right| + \left|\mathrm{N}^{n-1}_{X^n(\rho+1)}\right| + \left|\mathrm{N}^{n-1}_{X^1(\rho)}\right|.$$

On the other hand, the optimality of X yields

$$\mathrm{n}(X) = \mathrm{n}\big(\{0,\dots,\rho\}^n\big) = \left|\mathrm{N}^n_{\{0,\dots,\rho-1\}\times\{0,\dots,\rho\}^{n-1}}\right| + \left|\mathrm{N}^{n-1}_{\{0,\dots,\rho\}^{n-1}}\right|,$$

a contradiction because (2.21) for $a = \left|X^n(\rho+1)\right|$ and $b = \left|X^1(\rho)\right|$ implies (see also (2.33)) that

$$\left|\mathrm{N}^{n-1}_{X^n(\rho+1)}\right| + \left|\mathrm{N}^{n-1}_{X^1(\rho)}\right| > \left|\mathrm{N}^{n-1}_{\{0,\dots,\rho\}^{n-1}}\right|.$$

Now, if $\big(A+\{0,1\}^n\big) \cap \big(B+\{0,1\}^n\big) \neq \emptyset$, then $(\rho-1,0,\dots,0,\rho+1) \in A$. This, together with the fact that $A = X^n(\rho+1) \times \{\rho+1\}$ is an $(n-1)$-dimensional initial segment, implies, on the one hand, that there are

$(\rho-1)\rho^{n-2}$ points in $X^n(\rho+1)$ strictly smaller, in the order $\prec$, than $(\rho-1,0,\dots,0)\in X^n(\rho+1)$, and therefore, $|A|>(\rho-1)\rho^{n-2}$. On the other hand, they ensure that $(\rho-2,\rho-1,\dots,\rho-1,\rho+1)\in A\subset X$.

From now on we will write $(x_1,\dots,\widehat{x_i},\dots,x_n)$ to indicate that the i-th coordinate x_i does not appear in the point $(x_1,\dots,x_n)$, being henceforth a point in $\mathbb{R}^{n-1}$. Thus, considering the section $X^{n-1}(\rho-1)$, which is also an $(n-1)$-dimensional initial segment, one has that

$$(\rho,\dots,\rho,\widehat{\rho-1},\rho)\prec(\rho-2,\rho-1,\dots,\widehat{\rho-1},\rho+1)\ \text{ in }\ X^{n-1}(\rho-1),$$

and hence $(\rho,\dots,\rho,\widehat{\rho-1},\rho)\in X^{n-1}(\rho-1)$, i.e., $(\rho,\dots,\rho,\rho-1,\rho)\in X$.

Next we observe that, since X is stable and $X^n(\rho+1)\neq\emptyset$, then the set

$$D=\big\{(\rho,\dots,\rho,m)\in\mathbb{N}^n : m=0,\dots,\rho\big\}\subset\{\rho\}\times\{0,\dots,\rho\}^{n-1}$$

satisfies that $D\cap X=\emptyset$, since otherwise we would have $c^n(X)=(\rho+1)^{n-1}$ and thus, by Remark 2.28, that $X=\{0,\dots,\rho\}^n$, a contradiction. Furthermore,

$$B=\{\rho\}\times X^1(\rho)\subset\{\rho\}\times\{0,\dots,\rho\}^{n-1},$$

which yields $|B|\leq\big|\{\rho\}\times\{0,\dots,\rho\}^{n-1}\big|-|D|=(\rho+1)^{n-1}-(\rho+1)$, and we are going to see that, in fact, equality holds. Firstly, it is easy to see that if

$$(\widehat{\rho},\rho,\dots,\rho,\rho-1,\rho)\prec x_0\preceq(\widehat{\rho},\rho,\dots,\rho),$$

for some $x_0\in\mathbb{N}^n$, then $x_0=(\widehat{\rho},\rho,\dots,\rho,m)$ for some $m\in\{0,\dots,\rho\}$. Since $r\big((\widehat{\rho},\rho,\dots,\rho)\big)=(\rho+1)^{n-1}$, this implies that

$$r\big((\widehat{\rho},\rho,\dots,\rho,\rho-1,\rho)\big)=(\rho+1)^{n-1}-(\rho+1).$$

Now, given that $(\widehat{\rho},\rho,\dots,\rho,\rho-1,\rho)\in X^1(\rho)$, and that $X^1(\rho)$ is an $(n-1)$-dimensional initial segment, we know that $|X^1(\rho)|\geq r\big((\widehat{\rho},\rho,\dots,\rho,\rho-1,\rho)\big)$. Consequently, we have that

$$|B|=\big|X^1(\rho)\big|\geq r\big((\widehat{\rho},\rho,\dots,\rho,\rho-1,\rho)\big)=(\rho+1)^{n-1}-(\rho+1).$$

To sum up, $|B|=(\rho+1)^{n-1}-(\rho+1)$, and by (2.33), $|A|=\rho+1$. This contradicts the fact that $|A|>(\rho-1)\rho^{n-1}$, except when $n=3$ and $\rho=2$. In this case, a direct computation proves that X is not optimal (see Figure 2.6): indeed, in that case, $|X|=3^3$ and $\mathrm{n}(X)=41>37=\mathrm{n}(\mathcal{J}_{3^3})$. ∎

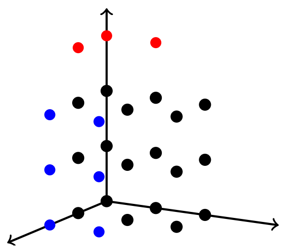

Figure 2.6: The stable set $X\subset\mathbb{N}^3$ from the proof of Lemma 2.33. The sets A and B are shown in red and blue, respectively.

We are now in the position to prove Theorem 2.16.

Proof of Theorem 2.16. We proceed by induction on $s\in\mathbb{N}$. Let $s=1$. Since $\mathcal{J}_{(\rho+1)^n}=\{0,\dots,\rho\}^n$, then $\big|\mathcal{J}_{(\rho+1)^n}+\{0,1\}^n\big|=\big|\{0,\dots,\rho+1\}^n\big|=(\rho+2)^n$, and so we have to prove that

$$\textit{if } \big|X+\{0,1\}^n\big|=(\rho+2)^n, \textit{ then } X \textit{ is a lattice cube.} \tag{2.34}$$

If $\rho=0$ the result is trivial. Thus, we assume $\rho\geq1$ and we proceed by induction on the dimension. If $n=1$ then, in order to have $\big|(X+\{0,1\})\setminus X\big|=1$, necessarily it must be $X=\{0,\dots,\rho\}$ up to translations, i.e., a lattice cube. The case $n=2$ is Lemma 2.23.

So we assume $n \geq 3$. Then, there exists a sequence of sets $\{X_j\}_{j=1}^r$ given recursively by $X_{j+1} = \overline{(X_j)}_{i_j}$ for some $i_j \in \{1,\dots,n\}$, $j = 1,\dots,r$, with $X_1 = X$, such that X_r is stable. Indeed, since the normalization process either leaves the set unchanged or strictly decreases its rank (see Lemma 2.29), which is bounded from below, such a sequence always exists.

By Theorem 2.9 the set X is optimal, and so Lemma 2.30 ensures that all X_j are also optimal for $j = 1,\dots,r$. Therefore, if $\rho = 1$, Corollary 2.32 ensures that X_r is the lattice cube $\{0,1\}^n$, whereas for $\rho \geq 2$, Lemma 2.33 shows that $X_r = \{0,\dots,\rho\}^n$.

Let us now focus on X_{r-1}. Since X_{r-1} is optimal, Corollary 2.25 yields

$$\mathrm{n}(X_{r-1}) = \left| \Big(\bigcup_{m\in\mathbb{N}} (X_{r-1})^{i_{r-1}}(m) \Big) + \{0,1\}^{n-1} \right| + \sum_{m\in\mathbb{N}} \left| \mathrm{N}^{n-1}_{(X_{r-1})^{i_{r-1}}(m)} \right|. \tag{2.35}$$

Moreover, we have

$$\overline{(X_{r-1})}_{i_{r-1}} = X_r = \{0,\dots,\rho\}^n.$$

We show next that this last normalization procedure does not involve the third step of the normalization process. Indeed, since all non-empty sections of the lattice cube $X_r = \{0,\dots,\rho\}^n$ are of the form $\{0,\dots,\rho\}^{n-1}$, applying step (iii) of the normalization process to X_{r-1} would imply the existence of a section $(X_{r-1})^{i_{r-1}}(m_0)$, for some $m_0 \in \mathbb{N}$, that becomes empty during such a step. But then, due to (2.35), an analogous argument to the one of the proof of Lemma 2.30 would show that $\mathrm{n}(X_{r-1}) > \mathrm{n}(X_r)$, contradicting the optimality of X_{r-1}.

Therefore, only the steps (i) and (ii) in Definition 2.27 are used in the last normalization $\overline{(X_{r-1})}_{i_{r-1}}$, which ensures that X_{r-1} has exactly $\rho+1$ non-empty sections $(X_{r-1})^{i_{r-1}}(m)$, each of them with cardinality $(\rho+1)^{n-1}$. We also know that all these (non-empty) sections $(X_{r-1})^{i_{r-1}}(m)$ are optimal sets in $\mathbb{N}^{n-1}$ (see Corollary 2.25), and so

$$\left| (X_{r-1})^{i_{r-1}}(m) + \{0,1\}^{n-1} \right| = (\rho+2)^{n-1}.$$

Thus, the induction hypothesis allows us to conclude that every (non-empty) section $(X_{r-1})^{i_{r-1}}(m)$ is an $(n-1)$-dimensional lattice cube. Furthermore, since X_{r-1} is optimal, Proposition 2.21 ensures it is connected, and hence all these sections are consecutive. Finally, they must all be equal as well: indeed, otherwise, for any non-empty section $(X_{r-1})^{i_{r-1}}(m_0)$, $m_0 \in \mathbb{N}$, we would have

$$(X_{r-1})^{i_{r-1}}(m_0) + \{0,1\}^{n-1} \subsetneq \Big(\bigcup_{m\in\mathbb{N}} (X_{r-1})^{i_{r-1}}(m) \Big) + \{0,1\}^{n-1},$$

and hence we could translate the sections such that for every non-empty $(X_{r-1})^{i_{r-1}}(m)$, $m \in \mathbb{N}$, we had $(X_{r-1})^{i_{r-1}}(m) = (X_{r-1})^{i_{r-1}}(m_0)$, strictly reducing the functional $\mathrm{n}(\cdot)$ (see (2.35)); this would contradict the optimality of X_{r-1}. Therefore, X_{r-1} is itself a lattice cube. The same conclusion is naturally obtained for all X_j, $j = 1,\dots,r$. In particular, $X = X_1$ is a lattice cube, which concludes the proof of the case $s = 1$. Thus we have shown (2.34).

Finally, assume now that $s > 1$ and that the result holds for $s-1$. On the one hand, Corollary 2.11 ensures that

$$\left| X + \{0,\dots,s-1\}^n \right| \geq \left| \mathcal{J}_{(\rho+1)^n} + \{0,\dots,s-1\}^n \right| = (\rho+s)^n.$$

On the other hand, if $\left| X + \{0,\dots,s-1\}^n \right| > (\rho+s)^n$, then Lemma 2.19 would imply that

$$\left| X + \{0,\dots,s\}^n \right| > (\rho+s+1)^n = \left| \mathcal{J}_{(\rho+1)^n} + \{0,\dots,s\}^n \right|,$$

a contradiction. Therefore, $\left| X + \{0,\dots,s-1\}^n \right| = \left| \mathcal{J}_{(\rho+1)^n} + \{0,\dots,s-1\}^n \right|$, and thus, the induction hypothesis yields that X is a lattice cube, as desired. ∎

2.3.3 Characterizations of the equality cases in other inequalities

As a consequence of Theorem 2.16, we derive several equality characterizations in discrete isoperimetric-type and Brunn-Minkowski-type inequalities. We begin by proving Theorems 2.14 and 2.15, thus providing the announced characterizations of Radcliffe and Veomett's results, Theorems H and I.

Proof of Theorem 2.14. Let $r=(\rho+1)^n$. By the translation invariance of the cardinality we may assume, without loss of generality, that $X\subset\mathbb{N}^n$. Then Corollaries 2.11 and 2.13 yield

$$\left|X+\{-1,0,1\}^n\right|=\left|X+\{0,1,2\}^n\right|\geq\left|\mathcal{J}_r+\{0,1,2\}^n\right|=\left|\mathcal{I}_r+\{0,1,2\}^n\right|=\left|\mathcal{I}_r+\{-1,0,1\}^n\right|.$$

Thus, if $\left|X+\{-1,0,1\}^n\right|=\left|\mathcal{I}_r+\{-1,0,1\}^n\right|$, we get that $\left|X+\{0,1,2\}^n\right|=\left|\mathcal{J}_r+\{0,1,2\}^n\right|$, and Theorem 2.16 shows that X is a lattice cube. The converse is obvious. ■

Proof of Theorem 2.15. Let $r=(\rho+1)^n$. By Corollary 2.11 and (2.6) for $\mathcal{J}_r$, we have

$$\left|\left(X+\{-1,0,1\}^n\right)\cap\mathbb{N}^n\right|\geq\left|X+\{0,1\}^n\right|\geq\left|\mathcal{J}_r+\{0,1\}^n\right|=\left|\left(\mathcal{J}_r+\{-1,0,1\}^n\right)\cap\mathbb{N}^n\right|.$$

Thus, if equality holds in (2.5), we get, in particular, that $\left|X+\{0,1\}^n\right|=\left|\mathcal{J}_r+\{0,1\}^n\right|$, and Theorem 2.16 shows that X is a lattice cube. Furthermore, in order to have $\left|\left(X+\{-1,0,1\}^n\right)\cap\mathbb{N}^n\right|=\left|X+\{0,1\}^n\right|$, it must in fact be $X=\{0,\dots,\rho\}^n$, as desired. The converse is obvious. ■

Furthermore, as a consequence of Theorem 2.16, we can also characterize the equality case in (1.14) in some particular cases:

Corollary 2.34 Let $X\subset\mathbb{Z}^n$ be a finite set with $|X|=(\rho+1)^n$ for some $\rho\in\mathbb{N}$ and let Y be a lattice cube. Then

$$\left|X+Y+\{0,1\}^n\right|^{1/n}=|X|^{1/n}+|Y|^{1/n}$$

if and only if X is a lattice cube.

Proof. If X is a lattice cube then Theorem D shows the result. So we assume that

$$\left|X+Y+\{0,1\}^n\right|^{1/n}=|X|^{1/n}+|Y|^{1/n},$$

and let $Y=\{0,\dots,s\}^n$ for some $s\in\mathbb{N}$. Then, by applying Corollary 2.11, we have

$$(\rho+s+2)^n=\left|X+Y+\{0,1\}^n\right|\geq\left|\mathcal{J}_{(\rho+1)^n}+Y+\{0,1\}^n\right|=(\rho+s+2)^n.$$

Thus, $\left|X+\{0,\dots,s+1\}^n\right|=\left|\mathcal{J}_{(\rho+1)^n}+\{0,\dots,s+1\}^n\right|$ and Theorem 2.16 concludes the proof. ■

Remark 2.35 We note that there are examples of sets (even with the cardinality of a lattice cube) reaching equality in Theorem D which are not lattice cubes (see Figure 2.7). ♦

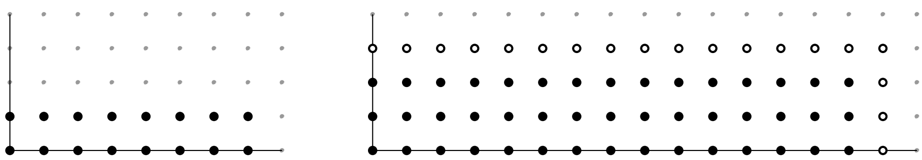

Figure 2.7: A set $X\subset\mathbb{Z}^2$ with $|X|=16$ which is not a lattice cube (left), and $X+X+\{0,1\}^2$ (right), satisfying the equality in (1.14): $\left|X+X+\{0,1\}^2\right|^{1/2}=8=2|X|^{1/2}$.

2.4 New discrete isoperimetric inequalities for the lattice point enumerator

Let K be a non-empty bounded set with $\mathrm{G_n}(K) = \mathrm{G_n}\big(r[-1,1]^n\big) = (2r+1)^n$ for some $r \in \mathbb{N}$. Then, from Theorem 1.2 for $t = 1$ and $s \in \mathbb{N}$,

$$\mathrm{G_n}\Big(K + s[-1,1]^n + \big(-1, \lceil 1+s \rceil\big)^n\Big) \geq \Big(\mathrm{G_n}(K)^{1/n} + s\,\mathrm{G_n}\big([-1,1]^n\big)^{1/n}\Big)^n = (2r+3s+1)^n$$
$$= \mathrm{G_n}\Big(r[-1,1]^n + s[-1,1]^n + \big(-1, \lceil 1+s \rceil\big)^n\Big),$$

which gives a particular discrete analogue of (2.3) for the lattice point enumerator $\mathrm{G_n}(\cdot)$. Here we will show how such a type of inequality can be extended to the setting of arbitrary non-empty bounded sets of $\mathbb{R}^n$, i.e., with an arbitrary amount of integer points, and any $s \geq 0$. This inequality will be generalized to arbitrary lattices (see Remark 2.40). The main tool for this will be the discrete isoperimetric-type inequality for the cardinality obtained in Corollary 2.11.

Before stating the theorem, we need further notation. Using the initial segments $\mathcal{I}_r \subset \mathbb{Z}^n$ we may define the family of *extended cubes* $\mathcal{C}_{\mathcal{I}_r} \subset \mathbb{R}^n$: a uniparametric family of star-shaped sets characterized as the largest sets (with respect to inclusion) such that $\mathcal{C}_{\mathcal{I}_r} + (-1,1)^n \subset \mathcal{I}_r + (-1,1)^n$. Analogously, extended cubes $\mathcal{C}_{\mathcal{J}_r} \subset \mathbb{R}^n_{\geq 0}$ can be defined.

Definition 2.36 For a non-empty bounded set $M \subset \mathbb{R}^n$, we write

$$\mathcal{C}_M = \big\{(\lambda_1 x_1, \ldots, \lambda_n x_n) \in \mathbb{R}^n : (x_1, \ldots, x_n) \in M, \lambda_i \in [0,1] \text{ for } i = 1, \ldots, n\big\}.$$

For the sake of brevity, we just write $\mathcal{C}_x := \mathcal{C}_{\{x\}}$ for any $x \in \mathbb{R}^n$.

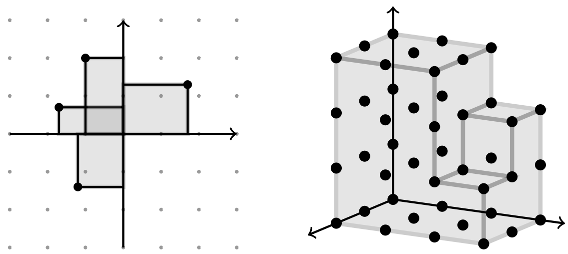

Figure 2.8: The set $\mathcal{C}_X \subset \mathbb{R}^2$ for a finite set $X \subset \mathbb{R}^2$ (left) and the set $\mathcal{C}_{\mathcal{J}_{44}} \subset \mathbb{R}^3$ (right).

Theorem 2.37 [66, Theorem 1.3] Let $K \subset \mathbb{R}^n$ be a non-empty bounded set. If $r = \mathrm{G_n}(K) > 0$, then

$$\mathrm{G_n}\big(K + t[0,1]^n\big) \geq \mathrm{G_n}\big(\mathcal{C}_{\mathcal{J}_r} + t[0,1]^n\big) \tag{2.36}$$

for all $t \geq 0$. When $\mathrm{G_n}(K) = (\rho+1)^n$ for some $\rho \in \mathbb{N}$, equality holds for $t \geq 0$ if and only if $K \cap \mathbb{Z}^n$ is a lattice cube and we have

$$\big(K + t[0,1]^n\big) \cap \mathbb{Z}^n = (K \cap \mathbb{Z}^n) + \big(t[0,1]^n \cap \mathbb{Z}^n\big).$$

Proof. First, we show that for every $\lambda \in [0,1)$ we have

$$\mathcal{C}_{\mathcal{J}_r} + [0,\lambda]^n \subset \mathcal{J}_r + [0,1)^n. \tag{2.37}$$

Indeed, if $y \in \mathcal{C}_{\mathcal{J}_r} + [0,\lambda]^n$, then $y \in \mathcal{C}_x + [0,\lambda]^n$ for some $x \in \mathcal{J}_r$. Hence $y_i \leq x_i + \lambda$ and $x_i \in \mathbb{N}$ for all $i = 1, \ldots, n$, and so $\lfloor y_i \rfloor \leq x_i$, $i = 1, \ldots, n$. Then Remark 2.4 implies that $\big(\lfloor y_1 \rfloor, \ldots, \lfloor y_n \rfloor\big) \preceq x$, and therefore

$$y \in \big(\lfloor y_1 \rfloor, \ldots, \lfloor y_n \rfloor\big) + [0,1)^n \subset \mathcal{J}_r + [0,1)^n.$$

Let $t>0$ (the case $t=0$ is trivial). By applying (2.37) with $\lambda=t-\lfloor t\rfloor$ and adding the cube $[0,\lfloor t\rfloor]^n$, we immediately get

$$\mathcal{C}_{\mathcal{J}_r}+t[0,1]^n\subset\mathcal{J}_r+\big(1+\lfloor t\rfloor\big)[0,1)^n.$$

This completes the proof since, by applying Corollary 2.11 with $s=\lfloor t\rfloor$, we get

$$\begin{aligned}\mathrm{G_n}\big(K+t[0,1]^n\big)&\geq\mathrm{G_n}\big((K\cap\mathbb{Z}^n)+t[0,1]^n\big)=\big|(K\cap\mathbb{Z}^n)+\{0,\dots,\lfloor t\rfloor\}^n\big|\geq\big|\mathcal{J}_r+\{0,\dots,\lfloor t\rfloor\}^n\big|\\&=\mathrm{G_n}\Big(\mathcal{J}_r+\big(1+\lfloor t\rfloor\big)[0,1)^n\Big)\geq\mathrm{G_n}\big(\mathcal{C}_{\mathcal{J}_r}+t[0,1]^n\big).\end{aligned}\tag{2.38}$$

Now, we assume that $\mathrm{G_n}(K)=(\rho+1)^n$ for some $\rho\in\mathbb{N}$. In order to characterize the equality in (2.38), we first note that we have

$$\Big(\mathcal{J}_r+\big(1+\lfloor t\rfloor\big)[0,1)^n\Big)\cap\mathbb{Z}^n=\mathcal{J}_r+\big\{0,\dots,\lfloor t\rfloor\big\}^n=(\mathcal{C}_{\mathcal{J}_r}\cap\mathbb{Z}^n)+\big(t[0,1]^n\cap\mathbb{Z}^n\big)\subset\big(\mathcal{C}_{\mathcal{J}_r}+t[0,1]^n\big)\cap\mathbb{Z}^n,$$

which gives equality in the last inequality of (2.38).

So we have equality in (2.38) if and only if the relations $\mathrm{G_n}\big(K+t[0,1]^n\big)=\mathrm{G_n}\big((K\cap\mathbb{Z}^n)+t[0,1]^n\big)$ and $\big|(K\cap\mathbb{Z}^n)+\big\{0,\dots,\lfloor t\rfloor\big\}^n\big|=\big|\mathcal{J}_r+\big\{0,\dots,\lfloor t\rfloor\big\}^n\big|$ hold. The first one is equivalent to

$$\big(K+t[0,1]^n\big)\cap\mathbb{Z}^n=(K\cap\mathbb{Z}^n)+\big(t[0,1]^n\cap\mathbb{Z}^n\big),$$

whereas the second one holds if and only if we have equality in Corollary 2.11, i.e., when $K\cap\mathbb{Z}^n$ is a lattice cube (see Theorem 2.16), as desired. ∎

Remark 2.38 In order to find a global optimal set for inequality (2.36), i.e., a set attaining equality in Theorem 2.37 for all values of $t\geq 0$, we observe that if we have a non-empty bounded set $K\subset\mathbb{R}^n$ such that $\big(K+[0,1)^n\big)\cap\mathbb{Z}^n=K\cap\mathbb{Z}^n$, then, by repeatedly adding the lattice cube $\{0,1\}^n$, one gets

$$\big(K+t[0,1]^n\big)\cap\mathbb{Z}^n=(K\cap\mathbb{Z}^n)+\big(t[0,1]^n\cap\mathbb{Z}^n\big)$$

for all $t\geq 0$. This shows that we have equality in Theorem 2.37 for all $t\geq 0$ if and only if $K\cap\mathbb{Z}^n$ is a lattice cube and K satisfies $\big(K+[0,1)^n\big)\cap\mathbb{Z}^n=K\cap\mathbb{Z}^n$, provided $\mathrm{G_n}(K)=(\rho+1)^n$ for some $\rho\in\mathbb{N}$. ♦

Remark 2.39 We note that the role of the set $\mathcal{C}_{\mathcal{J}_r}$ in Theorem 2.37 can also be played by any non-empty bounded set $M\subset\mathbb{R}^n$ with $\mathrm{G_n}(M)=r$ such that $M+[0,1)^n\subset\mathcal{J}_r+[0,1)^n$. Nevertheless, $\mathcal{C}_{\mathcal{J}_r}$ are the largest sets (with respect to set inclusion) contained in $\mathbb{R}^n_{\geq 0}$ satisfying this property. Indeed, for any $x\in\mathbb{R}^n_{\geq 0}$ with $x+[0,1)^n\subset\mathcal{J}_r+[0,1)^n$ we have $x\in\mathcal{C}_{\{y\}}$, where $y\in\mathbb{N}^n$ is given by $y_i=\lceil x_i\rceil$ for all $i=1,\dots,n$. Since $y\in\big(x+[0,1)^n\big)\cap\mathbb{N}^n\subset\big(\mathcal{J}_r+[0,1)^n\big)\cap\mathbb{N}^n=\mathcal{J}_r$, we get $x\in\mathcal{C}_{\mathcal{J}_r}$. ♦

Remark 2.40 Theorem 2.37 also holds for an arbitrary lattice $\Lambda\subset\mathbb{R}^n$: if $\mathcal{B}=\{v_1,\dots,v_n\}$ is a basis of Λ, we denote by $\mathrm{G}_\Lambda(M)=|M\cap\Lambda|$ for any $M\subset\mathbb{R}^n$ and by $\varphi:\mathbb{R}^n\longrightarrow\mathbb{R}^n$ the linear (bijective) map given by $\varphi(x)=\sum_{i=1}^n x_iv_i$ for any $x=(x_1,\dots,x_n)\in\mathbb{R}^n$, then Theorem 2.37 yields

$$\mathrm{G}_\Lambda\Big(K+t\varphi\big([0,1]^n\big)\Big)\geq\mathrm{G}_\Lambda\Big(\varphi(\mathcal{C}_{\mathcal{J}_r})+t\varphi\big([0,1]^n\big)\Big)$$

for any bounded set $K\subset\mathbb{R}^n$ with $\mathrm{G}_\Lambda(K)=r>0$ and all $t\geq 0$. ♦

As a consequence of Theorem 2.37, a discrete isoperimetric inequality in the setting of $\mathbb{Z}^n$ can also be obtained, which we independently proved in [67].

Theorem 2.41 [67, Theorem 1.2] Let $K\subset\mathbb{R}^n$ be a non-empty bounded set. If $r=\mathrm{G_n}(K)>0$, then for all $t\geq 0$ we have

$$\mathrm{G_n}\big(K+t[-1,1]^n\big)\geq\mathrm{G_n}\big(\mathcal{C}_{\mathcal{I}_r}+t[-1,1]^n\big).\tag{2.39}$$

Proof. For $t \geq 0$ and $r > 0$, let $r' = \mathrm{G_n}\big(C_{\mathcal{J}_r} + t[0,1]^n\big)$. It is clear from the definition that, if $A, B \subset \mathbb{R}^n_{\geq 0}$, then $\mathcal{C}_{A+B} = \mathcal{C}_A + \mathcal{C}_B$. In particular, since $\mathcal{J}_{r'} = \mathcal{J}_r + \big\{0, \dots, \lfloor t \rfloor\big\}^n$, we have $\mathcal{C}_{\mathcal{J}_{r'}} = \mathcal{C}_{\mathcal{J}_r} + \lfloor t \rfloor [0,1]^n$. Furthermore, Corollary 2.13 implies that

$$\mathrm{G_n}\big(\mathcal{C}_{\mathcal{I}_r} + t[-1,1]^n\big) = \mathrm{G_n}\big(\mathcal{C}_{\mathcal{I}_r} + \lfloor t \rfloor [-1,1]^n\big) = \mathrm{G_n}\big(\mathcal{C}_{\mathcal{J}_r} + \lfloor t \rfloor [-1,1]^n\big).$$

Therefore,

$$\begin{aligned}\mathrm{G_n}\big(\mathcal{C}_{\mathcal{I}_r} + t[-1,1]^n\big) &= \mathrm{G_n}\big(\mathcal{C}_{\mathcal{J}_r} + \lfloor t \rfloor [-1,1]^n\big) = \mathrm{G_n}\big(-\mathcal{C}_{\mathcal{J}_r} + \lfloor t \rfloor [-1,1]^n\big) \\ &= \mathrm{G_n}\big(-\mathcal{C}_{\mathcal{J}_r} + \lfloor t \rfloor [-1,0]^n + \lfloor t \rfloor [0,1]^n\big) = \mathrm{G_n}\big(-\mathcal{C}_{\mathcal{J}_{r'}} + \lfloor t \rfloor [0,1]^n\big) \\ &\leq \mathrm{G_n}\big(\mathcal{C}_{\mathcal{J}_{r'}} + t[0,1]^n\big).\end{aligned}$$

Finally, using Theorem 2.37 with the set $-K$, we have

$$r' = \mathrm{G_n}\big(C_{\mathcal{J}_r} + t[0,1]^n\big) \leq \mathrm{G_n}\big(-K + t[0,1]^n\big) = \mathrm{G_n}\big(K + t[-1,0]^n\big).$$

And, consequently, Theorem 2.37 applied now to the set $K + t[-1,0]^n$ yields

$$\mathrm{G_n}\big(\mathcal{C}_{\mathcal{I}_r} + t[-1,1]^n\big) \leq \mathrm{G_n}\big(\mathcal{C}_{\mathcal{J}_{r'}} + t[0,1]^n\big) \leq \mathrm{G_n}\big(K + t[-1,0]^n + t[0,1]^n\big) = \mathrm{G_n}\big(K + t[-1,1]^n\big). \qquad \blacksquare$$

2.5 From the discrete setting to the continuous one

We finish the chapter by showing that the discrete isoperimetric-type inequalities for the lattice point enumerator obtained in Section 2.4 (which are themselves consequences of the inequalities for the cardinality obtained in Section 2.2) imply the continuous version of the isoperimetric inequality for the volume in its neighborhood form in the setting of compact subsets of $\mathbb{R}^n$, i.e., (2.3) when $E \subset \mathbb{R}^n$ is a cube. It suffices to prove that Theorem 2.41 implies (2.3) for $E = [-1,1]^n$. The homogeneity and translation invariance of the volume then yields the result for any cube, and thus, we also obtain that Theorem 2.37 implies the neighborhood form of the isoperimetric inequality.

We first fix some additional notation. For each $m \in \mathbb{N}$, we denote by $\mathrm{G}_{m,n}(\cdot)$ the lattice point enumerator with respect to the lattice $2^{-m}\mathbb{Z}^n$, that is,

$$\mathrm{G}_{m,n}(L) = \big|L \cap (2^{-m}\mathbb{Z}^n)\big| = \big|(2^m L) \cap \mathbb{Z}^n\big| = \mathrm{G_n}\big(2^m L\big),$$

for any $L \subset \mathbb{R}^n$. Moreover, as in the proof of Theorem 1.18, for each $m \in \mathbb{N}$, we write $\mathrm{R}^m = [0, 2^{-m})^n \subset \mathbb{R}^n$ and $\mathrm{O}^m = \mathrm{R}^m - \mathrm{R}^m = (-2^{-m}, 2^{-m})^n$. Finally, for any compact set $M \subset \mathbb{R}^n$ and each $m \in \mathbb{N}$ we denote by

$$M_m = \big\{z \in 2^{-m}\mathbb{Z}^n : (z + \mathrm{R}^m) \cap M \neq \emptyset\big\},$$

for which we clearly have

$$M \subset M_m + \mathrm{R}^m \subset M + \mathrm{O}^m. \tag{2.40}$$

Before stating the main result we prove the following auxiliary one. The existence of all sequence limits appearing in the succeeding proofs follows from standard arguments (e.g. due to the sequences being monotonic and bounded), and thus, for the sake of clarity, we omit them.

Lemma 2.42 Let $K \subset \mathbb{R}^n$ be a non-empty compact set. If $\{p_m\}_{m\in\mathbb{N}} \subset \mathbb{N}$ is a sequence satisfying $(2p_m + 1)^n \leq |K_m| < (2p_m + 3)^n$ then

$$\lim_{m\to\infty} \frac{p_m}{2^m} = \frac{\mathrm{vol}(K)^{1/n}}{2}. \tag{2.41}$$

Proof. First we show that $\lim_{m\to\infty} 2^{-mn}|K_m| = \mathrm{vol}(K)$. Using (2.40) we have

$$\mathrm{vol}(K) \le \mathrm{vol}(K_m + \mathrm{R}^m) \le \mathrm{vol}(K + \mathrm{O}^m).$$

This, together with the identity $\mathrm{vol}(K_m + \mathrm{R}^m) = 2^{-mn}|K_m|$ and the fact that $\{K + \mathrm{O}^m\}_{m\in\mathbb{N}}$ is a decreasing sequence with

$$\bigcap_{m=0}^{\infty} (K + \mathrm{O}^m) = K,$$

shows that

$$\mathrm{vol}(K) \le \lim_{m\to\infty} \frac{|K_m|}{2^{mn}} \le \lim_{m\to\infty} \mathrm{vol}(K + \mathrm{O}^m) = \mathrm{vol}\left(\bigcap_{m=0}^{\infty} (K + \mathrm{O}^m)\right) = \mathrm{vol}(K).$$

Furthermore, from

$$\lim_{m\to\infty} \frac{(2p_m+1)^n}{2^{mn}} \le \lim_{m\to\infty} \frac{|K_m|}{2^{mn}} = \mathrm{vol}(K),$$

we infer the existence of a constant $c > 0$ such that $p_m < 2p_m + 1 < 2^m c$ for all $m \in \mathbb{N}$. Thus, applying that $(x+2)^n - x^n \le 3^n x^{n-1}$ for any $x \ge 1$, we have

$$0 \le |K_m| - (2p_m+1)^n < (2p_m+3)^n - (2p_m+1)^n \le 3^n\left(2p_m+1\right)^{n-1} \le 3^n\left(2^{m+1}c+1\right)^{n-1},$$

and since we may assume, without loss of generality, that $c \ge 1/4$, then

$$0 \le |K_m| - (2p_m+1)^n < 3^n\left(2^{m+1}c+1\right)^{n-1} \le 3^n\left(2^{m+1}c+2^{m+1}c\right)^{n-1} = 2^{mn-m+2n-2}3^n c^{n-1}.$$

Hence,

$$0 \le \lim_{m\to\infty} \frac{|K_m| - (2p_m+1)^n}{2^{mn}} \le \lim_{m\to\infty} 2^{-m+2n-2}3^n c^{n-1} = 0.$$

Finally, we have

$$\begin{aligned} \frac{\mathrm{vol}(K)^{1/n}}{2} &= \frac{1}{2}\left(\lim_{m\to\infty} \frac{|K_m|}{2^{mn}}\right)^{1/n} = \frac{1}{2}\left(\lim_{m\to\infty} \frac{|K_m| - (2p_m+1)^n}{2^{mn}} + \lim_{m\to\infty} \frac{(2p_m+1)^n}{2^{mn}}\right)^{1/n} \\ &= \frac{1}{2}\lim_{m\to\infty} \frac{2p_m+1}{2^m} = \lim_{m\to\infty} \frac{p_m}{2^m}, \end{aligned}$$

which shows (2.41). This concludes the proof. ■

We observe that, considering the partition

$$\left\{\left[(2k+1)^n, (2k+3)^n\right) \cap \mathbb{N}\right\}_{k\in\mathbb{N}}$$

of $\mathbb{N}\setminus\{0\}$, then the relation $(2p_m+1)^n \le |K_m| < (2p_m+3)^n$ given in Lemma 2.42 uniquely determines such a sequence $\{p_m\}_{m\in\mathbb{N}} \subset \mathbb{N}$.

We are now ready to prove the main result of the section. The underlying idea is simply to successively shrink the lattice and then to approximate the volume by means of the lattice point enumerator.

Theorem 2.43 [67, Theorem 1.4] The discrete isoperimetric inequality (2.39) implies the classical isoperimetric inequality (2.3), with $E = [-1,1]^n$, for non-empty compact sets.

Proof. Let $\{p_m\}_{m\in\mathbb{N}} \subset \mathbb{N}$ be a sequence satisfying the conditions of Lemma 2.42 and, for the sake of brevity, we write $r_m = (2p_m+1)^n$. Since

$$\mathrm{G_n}\left(2^m K_m + 2^m \mathrm{R}^m\right) = \mathrm{G}_{m,n}(K_m + \mathrm{R}^m) = |K_m| \ge (2p_m+1)^n = \left|\mathcal{I}_{r_m}\right| = \mathrm{G_n}\left(\mathcal{C}_{\mathcal{I}_{r_m}}\right),$$

applying (2.39) we get

$$\begin{aligned}\mathrm{G}_{m,n}\big(K_m+\mathrm{R}^m+t[-1,1]^n\big)&=\mathrm{G}_\mathrm{n}\big(2^mK_m+2^m\mathrm{R}^m+2^mt[-1,1]^n\big)\geq\mathrm{G}_\mathrm{n}\big(\mathcal{C}_{\mathcal{I}_{r_m}}+2^mt[-1,1]^n\big)\\&=\mathrm{G}_{m,n}\big(2^{-m}\mathcal{C}_{\mathcal{I}_{r_m}}+t[-1,1]^n\big)\end{aligned}$$

for all $m\in\mathbb{N}$. Therefore

$$\lim_{m\to\infty}\frac{\mathrm{G}_{m,n}\big(K_m+\mathrm{R}^m+t[-1,1]^n\big)}{2^{mn}}\geq\lim_{m\to\infty}\frac{\mathrm{G}_{m,n}\big(2^{-m}\mathcal{C}_{\mathcal{I}_{r_m}}+t[-1,1]^n\big)}{2^{mn}}. \tag{2.42}$$

Applying (2.40) again to the set $M=K_m+\mathrm{R}^m+t[-1,1]^n$, we get

$$K_m+\mathrm{R}^m+t[-1,1]^n\subset\big(K_m+\mathrm{R}^m+t[-1,1]^n\big)_m+\mathrm{R}^m\subset K_m+\mathrm{R}^m+t[-1,1]^n+\mathrm{O}^m\subset K+t[-1,1]^n+2\mathrm{O}^m$$

and then

$$\begin{aligned}\frac{\mathrm{G}_{m,n}\big(K_m+\mathrm{R}^m+t[-1,1]^n\big)}{2^{mn}}&\leq\frac{\Big|\big(K_m+\mathrm{R}^m+t[-1,1]^n\big)_m\Big|}{2^{mn}}=\mathrm{vol}\Big(\big(K_m+\mathrm{R}^m+t[-1,1]^n\big)_m+\mathrm{R}^m\Big)\\&\leq\mathrm{vol}\big(K+t[-1,1]^n+2\mathrm{O}^m\big).\end{aligned}$$

Since $\big\{K+t[-1,1]^n+2\mathrm{O}^m\big\}_{m\in\mathbb{N}}$ is a decreasing sequence with

$$\bigcap_{m=0}^{\infty}\big(K+t[-1,1]^n+2\mathrm{O}^m\big)=K+t[-1,1]^n,$$

we have

$$\lim_{m\to\infty}\mathrm{vol}\big(K+t[-1,1]^n+2\mathrm{O}^m\big)=\mathrm{vol}\big(K+t[-1,1]^n\big).$$

Therefore

$$\lim_{m\to\infty}\frac{\mathrm{G}_{m,n}\big(K_m+\mathrm{R}^m+t[-1,1]^n\big)}{2^{mn}}\leq\mathrm{vol}\big(K+t[-1,1]^n\big). \tag{2.43}$$

Finally, we note that

$$\mathrm{G}_{m,n}\big(2^{-m}\mathcal{C}_{\mathcal{I}_{r_m}}+t[-1,1]^n\big)=\big(2(p_m+t_m)+1\big)^n,$$

where $t_m:=\lfloor 2^mt\rfloor$ for all $m\in\mathbb{N}$ (which clearly satisfies that $t_m/2^m\to t$ as $m\to\infty$). Thus, writing $r=\mathrm{vol}(K)^{1/n}/2$ and applying Lemma 2.42, we get

$$\lim_{m\to\infty}\frac{\mathrm{G}_{m,n}\big(2^{-m}\mathcal{C}_{\mathcal{I}_{r_m}}+t[-1,1]^n\big)}{2^{mn}}=\lim_{m\to\infty}\left(\frac{2(p_m+t_m)+1}{2^m}\right)^n=\big(2(r+t)\big)^n=\mathrm{vol}\big(r[-1,1]^n+t[-1,1]^n\big).$$

This, together with (2.42) and (2.43), shows (2.3), as desired. ■

The following result is a straightforward consequence of Theorem 2.43 and the proof of Theorem 2.41.

Corollary 2.44 The discrete isoperimetric inequality (2.36) implies the classical isoperimetric inequality (2.3), with $E=[0,1]^n$, for non-empty compact sets.

3 Rogers-Shephard type inequalities

One particularly significant application of the Brunn-Minkowski inequality, given any non-empty compact set $K \subset \mathbb{R}^n$, is a lower bound for the volume of the set $K-K$. In particular, (1.1) with $L=-K$ yields

$$\operatorname{vol}(K-K) \geq 2^n \operatorname{vol}(K).$$

When K is a convex body, the set $K-K$ is known as the *difference body*, and it plays an important role in scenarios where central symmetry is critical. In fact, the modified version $(1/2)(K-K)$ is known as the *central symmetral* of K, where the constant is added merely for scaling purposes. This set is trivially origin-symmetric, and it constitutes one of the most essential symmetrizations in Convex Geometry.

A corresponding upper bound for the volume of the difference body $K-K$ is given by the *Rogers-Shephard inequality*, originally proven in [88]. For more details about this inequality, we also refer the reader to [97, Section 10.1].

Theorem J — Rogers-Shephard inequality. Let $K \subset \mathbb{R}^n$ be a convex body. Then

$$\operatorname{vol}(K-K) \leq \binom{2n}{n} \operatorname{vol}(K). \tag{3.1}$$

Equality holds if and only if K is a simplex.

This relation can be generalized to the Minkowski addition of two convex bodies $K, L \subset \mathbb{R}^n$ as follows:

$$\operatorname{vol}(K+L)\operatorname{vol}\big(K \cap (-L)\big) \leq \binom{2n}{n} \operatorname{vol}(K)\operatorname{vol}(L). \tag{3.2}$$

The Rogers-Shephard inequality was recently extended to the functional setting [4, 6, 10, 34], generalized to different types of measures [8, 91], as well as studied in the L_p setting [3, 18]. Moreover, it was recently extended to other geometric functionals [7], and a reverse form of Rogers-Shephard's inequality in the setting of log-concave functions was given in [3]. The role of this inequality in characterization results of the difference body was also studied in [1], and an optimal stability version of it was proved in [24]. It is also interesting to note that a strengthening of this inequality for mixed volumes was conjectured independently by Godbersen and Makai Jr. (see [97, Note 5 for Section 10.1] and the references therein); a conjecture on which engaging progress was recently made in [10].

In [89, Theorem 1], Rogers and Shephard also gave the following lower bound for the volume of a convex body $K \subset \mathbb{R}^n$ in terms of the volumes of a projection and a section of K.

Theorem K Let $k \in \{1,\dots,n-1\}$ and $H \in \mathcal{L}^n_k$. Let $K \subset \mathbb{R}^n$ be a convex body. Then

$$\mathrm{vol}_{n-k}(P_{H^\perp}K)\,\mathrm{vol}_k(K\cap H) \le \binom{n}{k}\mathrm{vol}(K). \tag{3.3}$$

On a related note, a classical result due to Berwald [16] (see also [4, 5] for other extensions and considerations) relates certain weighted power means of a concave function, as follows:

Theorem L — Berwald inequality. Let $K \subset \mathbb{R}^n$ be a convex body with $\dim K = n$ and consider a concave function $f : K \longrightarrow \mathbb{R}_{\ge 0}$. Then, for any $0 < p < q$,

$$\left(\frac{\binom{n+q}{n}}{\mathrm{vol}(K)}\int_K f^q(x)\,\mathrm{d}x\right)^{1/q} \le \left(\frac{\binom{n+p}{n}}{\mathrm{vol}(K)}\int_K f^p(x)\,\mathrm{d}x\right)^{1/p}. \tag{3.4}$$

Remarkably, this result provides an alternative unified proof of both Theorems J and K (see Section 3.2, page 64 for the computations).

In this chapter we will use methods in a similar spirit to the ones employed in the previous ones to obtain several new Rogers-Shephard type inequalities for the lattice point enumerator. We will also get a discretization of Berwald's inequality (3.4) and use it to prove further alternative Rogers-Shephard type inequalities, although as we will see, one cannot expect to retrieve the first ones. Finally, we will show that these new discrete analogues imply their corresponding continuous versions for the volume discussed in this introduction. The results from this chapter are collected in [9].

3.1 New discrete Rogers-Shephard type inequalities

A very elegant discrete analogue of the Rogers-Shephard inequality (3.1) in the planar case (in fact, a stronger version of it) was shown in [46], as a consequence of *Pick's theorem* jointly with (3.1):

Theorem M Let $P \subset \mathbb{R}^2$ be a convex polygon with integer vertices. Then

$$\mathrm{G}_2(P-P) \le 6\,\mathrm{G}_2(P) - \mathrm{b}(P) - 5, \tag{3.5}$$

where $\mathrm{b}(P)$ denotes the number of integer points in the boundary of P.

However, when dealing with an arbitrary convex body $K \subset \mathbb{R}^n$, one cannot expect to get a discrete counterpart of (3.1) for the lattice point enumerator $\mathrm{G}_n(\cdot)$, namely,

$$\mathrm{G}_n(K-K) \le \binom{2n}{n}\mathrm{G}_n(K).$$

Indeed, just considering $K = [-1/2, 1/2]^n$ one would obtain $3^n \le \binom{2n}{n}$, which is false for $n = 1,2,3,4$. Moreover, as pointed out in [42], where the authors consider certain simplices with integer vertices, there is neither a possible extension of (3.5) in dimension $n \ge 3$ nor even a hope to get $\mathrm{G}_n(K-K) \le c_n\,\mathrm{G}_n(K)$ for some constant $c_n > 0$ depending only on the dimension n, for $n \ge 3$.

Altogether, and taking into account the "behavior" of the discrete version of the Brunn-Minkowski inequality collected in (1.15), an alternative to get such an inequality for the lattice point enumerator would be to consider some extension of K (by Minkowski adding a certain cube) on the upper bound side of the different Rogers-Shephard type inequalities.

We will develop the discrete versions of these inequalities with two different approaches. The first one will adapt the argument based on the *covariogram* used in the original proofs by Rogers and Shephard. The second one will exploit the relations between the volume and the lattice point enumerator (1.42), which were already used in Section 1.4 to obtain the discrete log-Brunn-Minkowski analogues (1.41) and (1.48). These methods will produce alternative inequalities which are not necessarily comparable in some cases, as well as provide alternative proofs of the same inequalities in other cases.

For the sake of brevity, we introduce some further notation for this section. We will write

$$\mathrm{hyp}(f) = \big\{(x,t) : x \in M, t \in \mathbb{R}, f(x) \geq t\big\} \subset \mathbb{R}^{n+1}$$

for the *hypograph* of a non-negative function $f : M \longrightarrow \mathbb{R}_{\geq 0}$, where $M \subset \mathbb{R}^n$ is non-empty. Moreover, for a vector subspace of the form $H = \mathrm{lin}\{\mathrm{e}_1, \dots, \mathrm{e}_k\} \in \mathcal{L}^n_k$, $k \in \{1, \dots, n-1\}$, and any $M \subset x + H$ with $x \in H^\perp$, we denote by

$$\mathrm{G}_k(M) = \Big|M \cap \Big(x + \big(\mathbb{Z}^k \times \{0\}^{n-k}\big)\Big)\Big|.$$

Analogously, for each $M \subset y + H^\perp$ with $y \in H$, we use

$$\mathrm{G}_{n-k}(M) = \Big|M \cap \Big(y + \big(\{0\}^k \times \mathbb{Z}^{n-k}\big)\Big)\Big|.$$

Furthermore, we will write $\mathrm{C}_H := (-1,1)^n \cap H$ for the sake of simplicity. Finally, as usual in the literature, we will utilize the following conventional notation:

$$\binom{r}{s} := \frac{\Gamma(r+1)}{\Gamma(s+1)\Gamma(r-s+1)}$$

for any $r, s > 0$, where $\Gamma(\cdot)$ is the *Gamma function*.

3.1.1 Projection-section type inequalities

The main result of this section is the following theorem, which provides a discrete analogue of (3.3) for the lattice point enumerator.

Theorem 3.1 Let $k \in \{1, \dots, n-1\}$ and $H = \mathrm{lin}\{\mathrm{e}_1, \dots, \mathrm{e}_k\} \in \mathcal{L}^n_k$. Let $K \subset \mathbb{R}^n$ be a non-empty convex bounded set. Then

$$\mathrm{G}_{n-k}(P_{H^\perp}K)\,\mathrm{G}_k(K \cap H) \leq \binom{n}{k}\,\mathrm{G}_n\big(K + (-1,1)^n\big). \tag{3.6}$$

As anticipated in the previous section, we will provide two different proofs. The first one is a direct consequence of the following, stronger inequality.

Theorem 3.2 Let $k \in \{1, \dots, n-1\}$ and $H = \mathrm{lin}\{\mathrm{e}_1, \dots, \mathrm{e}_k\} \in \mathcal{L}^n_k$. Let $K \subset \mathbb{R}^n$ be a convex bounded set containing the origin. Then

$$\left[\sum_{i=0}^{n-k} \frac{k}{n-i} \binom{n-k}{i} \binom{n}{i}^{-1} \mathrm{G}_{n-k}(P_{H^\perp}K)^{i/(n-k)}\right] \mathrm{G}_k(K \cap H) \leq \mathrm{G}_n\big(K + (-1,1)^n\big). \tag{3.7}$$

Indeed, taking only the term corresponding to $i = n-k$, Theorem 3.1 follows immediately, under the additional mild assumption that K contains the origin.

Proof. First, for any $r \geq 0$, we define the superlevel set

$$\begin{aligned}\mathcal{D}_r &= \Big\{x \in P_{H^\perp}\big(K+(-1,1)^n\big) : \mathrm{G}_k\Big(\big(K+(-1,1)^n\big)\cap(x+H)\Big) \geq r\Big\}\\ &= \Big\{x \in P_{H^\perp}K + \mathrm{C}_{H^\perp} : \mathrm{G}_k\Big(\big((K+\mathrm{C}_{H^\perp})\cap(x+H)\big)+\mathrm{C}_H\Big) \geq r\Big\}\end{aligned}$$

(see Proposition 1). Now, let $x \in \mathcal{D}_0$, $y \in \mathrm{C}_{H^\perp}$ and $\lambda \in [0,1]$. So, from the convexity of K, we have

$$\begin{aligned}&\mathrm{G}_k\Big(\big((K+\mathrm{C}_{H^\perp})\cap\big((1-\lambda)x+\lambda y+H\big)\big)+\mathrm{C}_H\Big)^{1/k}\\ &\quad\geq \mathrm{G}_k\Big((1-\lambda)\big((K+\mathrm{C}_{H^\perp})\cap(x+H)\big)+\lambda\big((K+\mathrm{C}_{H^\perp})\cap(y+H)\big)+\mathrm{C}_H\Big)^{1/k}.\end{aligned} \tag{3.8}$$

We notice that, since $x \in \mathcal{D}_0 = P_{H^\perp}K + \mathrm{C}_{H^\perp}$ and $y \in \mathrm{C}_{H^\perp} \subset P_{H^\perp}K + \mathrm{C}_{H^\perp}$ (because $0 \in K$), the sets $(K+\mathrm{C}_{H^\perp})\cap(x+H), (K+\mathrm{C}_{H^\perp})\cap(y+H)$ are non-empty and then the above sum

$$(1-\lambda)\big((K+\mathrm{C}_{H^\perp})\cap(x+H)\big)+\lambda\big((K+\mathrm{C}_{H^\perp})\cap(y+H)\big)+\mathrm{C}_H$$

is well-defined. Hence, from (1.15) we get

$$\begin{aligned}&\mathrm{G}_k\Big((1-\lambda)\big((K+\mathrm{C}_{H^\perp})\cap(x+H)\big)+\lambda\big((K+\mathrm{C}_{H^\perp})\cap(y+H)\big)+\mathrm{C}_H\Big)^{1/k}\\ &\quad\geq (1-\lambda)\,\mathrm{G}_k\big((K+\mathrm{C}_{H^\perp})\cap(x+H)\big)^{1/k}+\lambda\,\mathrm{G}_k\big((K+\mathrm{C}_{H^\perp})\cap(y+H)\big)^{1/k}\\ &\quad\geq \lambda\,\mathrm{G}_k(K\cap H)^{1/k},\end{aligned} \tag{3.9}$$

where in the last inequality we have used that

$$\mathrm{G}_k\big((K+\mathrm{C}_{H^\perp})\cap(y+H)\big) \geq \mathrm{G}_k\big((y+K)\cap(y+H)\big) = \mathrm{G}_k(K\cap H)$$

for every $y \in \mathrm{C}_{H^\perp}$. Thus, setting

$$\lambda_s = \left(\frac{s}{\mathrm{G}_k(K\cap H)}\right)^{1/k}$$

for any $0 \leq s \leq \mathrm{G}_k(K\cap H)$ (observe that $\mathrm{G}_k(K\cap H) \neq 0$ since $0 \in K$), from (3.8) and (3.9) for $\lambda = \lambda_s$ we conclude that

$$(1-\lambda_s)\big(P_{H^\perp}K+\mathrm{C}_{H^\perp}\big)+\lambda_s\mathrm{C}_{H^\perp} \subset \mathcal{D}_s.$$

In other words, for any $0 \leq s \leq \mathrm{G}_k(K\cap H)$ we have

$$(1-\lambda_s)P_{H^\perp}K+\mathrm{C}_{H^\perp} \subset \mathcal{D}_s$$

and then, by (1.15), we get

$$(1-\lambda_s)\,\mathrm{G}_{n-k}(P_{H^\perp}K)^{1/(n-k)}+\lambda_s\,\mathrm{G}_{n-k}(\{0\}^{n-k})^{1/(n-k)} \leq \mathrm{G}_{n-k}(\mathcal{D}_s)^{1/(n-k)}.$$

Consequently,

$$\sum_{i=0}^{n-k}\binom{n-k}{i}(1-\lambda_s)^i\lambda_s^{n-k-i}\,\mathrm{G}_{n-k}(P_{H^\perp}K)^{i/(n-k)} \leq \mathrm{G}_{n-k}(\mathcal{D}_s) \tag{3.10}$$

for all $0 \leq s \leq \mathrm{G}_k(K\cap H)$.

Now, on the one hand, doing the change of variables $\theta = \lambda_s$, we get that

$$\begin{aligned}\int_0^{\mathrm{G}_k(K\cap H)}(1-\lambda_s)^i\lambda_s^{n-k-i}\,\mathrm{d}s &= k\,\mathrm{G}_k(K\cap H)\int_0^1\theta^{n-i-1}(1-\theta)^i\,\mathrm{d}\theta\\ &= k\,\mathrm{G}_k(K\cap H)\frac{\Gamma(n-i)\Gamma(i+1)}{\Gamma(n+1)} = \frac{k}{n-i}\binom{n}{i}^{-1}\mathrm{G}_k(K\cap H)\end{aligned}$$

and hence, integration on $s \in [0, \mathrm{G}_k(K\cap H)]$ on the left-hand side of (3.10) yields

$$\left[\sum_{i=0}^{n-k}\frac{k}{n-i}\binom{n-k}{i}\binom{n}{i}^{-1}\mathrm{G}_{n-k}(P_{H^\perp}K)^{i/(n-k)}\right]\mathrm{G}_k(K\cap H).$$

On the other hand, we have

$$\begin{aligned}\int_0^{\mathrm{G}_k(K\cap H)}\mathrm{G}_{n-k}(\mathcal{D}_s)\,\mathrm{d}s &= \int_0^{\mathrm{G}_k(K\cap H)}\sum_{x\in(P_{H^\perp}K+\mathrm{C}_{H^\perp})\cap\mathbb{Z}^n}\chi_{\mathcal{D}_s}(x)\,\mathrm{d}s\\ &= \sum_{x\in\left(P_{H^\perp}K+\mathrm{C}_{H^\perp}\right)\cap\mathbb{Z}^n}\min\left\{\mathrm{G}_k\Big(\big((K+\mathrm{C}_{H^\perp})\cap(x+H)\big)+\mathrm{C}_H\Big),\mathrm{G}_k(K\cap H)\right\}\\ &\le \sum_{x\in\left(P_{H^\perp}K+\mathrm{C}_{H^\perp}\right)\cap\mathbb{Z}^n}\mathrm{G}_k\Big(\big((K+\mathrm{C}_{H^\perp})\cap(x+H)\big)+\mathrm{C}_H\Big)\\ &= \sum_{x\in\left(P_{H^\perp}K+\mathrm{C}_{H^\perp}\right)\cap\mathbb{Z}^n}\mathrm{G}_k\Big(\big(K+(-1,1)^n\big)\cap(x+H)\Big)=\mathrm{G}_n\big(K+(-1,1)^n\big).\end{aligned}$$

This concludes the proof. ■

Remark 3.3 The role of $H=\mathrm{lin}\{\mathrm{e}_1,\dots,\mathrm{e}_k\}$ in the above result can be played by any other k-dimensional coordinate (vector) subspace. ♦

As anticipated, we can also obtain a direct proof of Theorem 3.1 using the relations (1.42) between the volume and the lattice point enumerator, as well as the original projection-section inequality for the volume (3.3) (for which the assumption on the convex set K to be closed is not necessary), taking into account the properties of the Minkowski addition from Proposition 1.

Alternative proof of Theorem 3.1. We have

$$\begin{aligned}\mathrm{G}_{n-k}(P_{H^\perp}K)\,\mathrm{G}_k(K\cap H) &\le \mathrm{vol}_{n-k}\left((P_{H^\perp}K)+\frac{1}{2}\mathrm{C}_{H^\perp}\right)\mathrm{vol}_k\left((K\cap H)+\frac{1}{2}\mathrm{C}_H\right)\\ &\le \mathrm{vol}_{n-k}\left(P_{H^\perp}\left[K+\left(-\frac{1}{2},\frac{1}{2}\right)^n\right]\right)\mathrm{vol}_k\left(\left[K+\left(-\frac{1}{2},\frac{1}{2}\right)^n\right]\cap H\right)\\ &\le \binom{n}{k}\mathrm{vol}\left(K+\left(-\frac{1}{2},\frac{1}{2}\right)^n\right)\le\binom{n}{k}\mathrm{G}_n\big(K+(-1,1)^n\big),\end{aligned}$$

as desired. ■

We point out that, with the argument above, there is no need to assume that $0\in K$. However, such a method does not allow one to show the statement of the stronger inequality collected in Theorem 3.2.

3.1.2 Sum-intersection type inequalities

First we derive a discrete analogue of the Rogers-Shephard inequality (3.2) (and, as a consequence, of (3.1)), by considering a suitable $(2n)$-dimensional convex bounded set and applying the projection-section inequality collected in Theorem 3.1, following the original idea of Rogers and Shephard in [89]:

Theorem 3.4 Let $K,L\subset\mathbb{R}^n$ be convex bounded sets containing the origin. Then

$$\mathrm{G}_n(K+L)\,\mathrm{G}_n\big(K\cap(-L)\big)\le\binom{2n}{n}\mathrm{G}_n\big(K+(-1,1)^n\big)\,\mathrm{G}_n\big(L+(-2,2)^n\big).\tag{3.11}$$

In particular, taking $L=-K$, we have

$$\mathrm{G}_n(K-K)\le\binom{2n}{n}\mathrm{G}_n\big(K+(-1,1)^n\big)\frac{\mathrm{G}_n\big(K+(-2,2)^n\big)}{\mathrm{G}_n(K)}.\tag{3.12}$$

Proof. Consider the $(2n)$-dimensional convex bounded set containing the origin defined by

$$F = \{(x,y) \in \mathbb{R}^{2n} : x \in K, x-y \in -L\}$$

and let $H = \mathrm{lin}\{\mathrm{e}_1,\dots,\mathrm{e}_n\} \in \mathcal{L}^{2n}_n$. Notice that $P_{H^\perp}F$ is the set of points $(0,y)$ such that $(x,y) \in F$ for some $x \in \mathbb{R}^n$, which is equivalent to the fact that $y \in x+L$ for some $x \in K$, and hence we get $P_{H^\perp}F = \{0\}^n \times (K+L)$. Moreover, we clearly have that

$$F \cap H = \big(K \cap (-L)\big) \times \{0\}^n.$$

Now, given $(x,y) \in F+(-1,1)^{2n}$, we have that $x \in x_1+(-1,1)^n$ for some $x_1 \in K$ and that $y \in y_1+(-1,1)^n$ for some $y_1 \in x_1+L \subset x+(-1,1)^n+L$. So, for every $(x,y) \in F+(-1,1)^{2n}$, $x \in K+(-1,1)^n$ and $y \in x+L+(-2,2)^n$. Thus,

$$\mathrm{G}_{2n}\big(F+(-1,1)^{2n}\big) \leq \sum_{x\in(K+(-1,1)^n)\cap\mathbb{Z}^n} \mathrm{G}_n\big(x+L+(-2,2)^n\big) = \mathrm{G}_n\big(K+(-1,1)^n\big)\,\mathrm{G}_n\big(L+(-2,2)^n\big).$$

Therefore, from Theorem 3.1 (applied to the convex bounded set F containing the origin and the vector subspace H) we obtain

$$\mathrm{G}_n(K+L)\,\mathrm{G}_n\big(K\cap(-L)\big) \leq \binom{2n}{n}\mathrm{G}_n\big(K+(-1,1)^n\big)\,\mathrm{G}_n\big(L+(-2,2)^n\big),$$

which clearly further implies (3.12). This concludes the proof. ■

Next we prove another discrete Rogers-Shephard type inequality which is actually stronger than the one collected in Theorem 3.4. Indeed, (3.11) may be obtained as a consequence of it. Before stating the result, we make some additional considerations.

When dealing with the (proof of the) Rogers-Shephard inequality (3.1), one is naturally led to the notion of the *covariogram* of a convex body $K \subset \mathbb{R}^n$, that is, the function $f : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ given by $f(x) = \mathrm{vol}\big(K \cap (x+K)\big)$. Its discrete version for finite sets $A \subset \mathbb{R}^n$, $x \mapsto \big|A\cap(x+A)\big|$, has been studied in [47], where the authors show elegant relations of the latter with the continuous version. Here, we will consider the following slight modification of the corresponding discrete version for $\mathrm{G_n}(\cdot)$ of the covariogram of K:

$$x \mapsto \mathrm{G}_n\Big(\big(K+(-1,1)^n\big)\cap\big(x+K+(-1,1)^n\big)\Big).$$

By using this, and exploiting the classical proof of the Rogers-Shephard inequality (3.1) that is based on the covariogram, we can obtain the following discrete version. Again, we will present it in the more general setting of two convex bounded sets $K,L \subset \mathbb{R}^n$. We recall that given $A,B \subset \mathbb{R}^n$, the set

$$A \sim B := \bigcap_{x\in B}(A-x) = \{x \in \mathbb{R}^n : x+B \subset A\} \tag{3.13}$$

is the *Minkowski difference* (or *Minkowski subtraction*) of A and B. It is clear that $(A+B)\sim B = A$, whereas the inclusion $(A\sim B)+B \subset A$ also holds. For more on this notion and its connection with the Minkowski sum, we refer the reader to [97, Section 3.1].

Theorem 3.5 Let $K,L \subset \mathbb{R}^n$ be convex bounded sets containing the origin such that $(-1,1)^n \subset L$. Then

$$\left[\sum_{i=0}^{n}\frac{n}{2n-i}\binom{n}{i}\binom{2n}{i}^{-1}\mathrm{G}_n\big((K+L)\sim(-1,1)^n\big)^{i/n}\right]\mathrm{G}_n\Big(K\cap\big((-L)\sim(-1,1)^n\big)\Big) \\ \leq \mathrm{G}_n\big(K+(-1,1)^n\big)\,\mathrm{G}_n\big(L+(-1,1)^n\big). \tag{3.14}$$

In particular, taking $L=-K$ (for a convex set $K\subset\mathbb{R}^n$ with $(-1,1)^n\subset K$), we have

$$\sum_{i=0}^{n}\frac{n}{2n-i}\binom{n}{i}\binom{2n}{i}^{-1}\mathrm{G}_n\big((K-K)\sim(-1,1)^n\big)^{i/n}\leq\frac{\mathrm{G}_n\big(K+(-1,1)^n\big)^2}{\mathrm{G}_n\big(K\sim(-1,1)^n\big)}. \tag{3.15}$$

Before showing the result, we observe that taking only the term corresponding to $i=n$ in the above expressions we obtain, respectively,

$$\mathrm{G}_n\big((K+L)\sim(-1,1)^n\big)\,\mathrm{G}_n\Big(K\cap\big((-L)\sim(-1,1)^n\big)\Big)\leq\binom{2n}{n}\mathrm{G}_n\big(K+(-1,1)^n\big)\,\mathrm{G}_n\big(L+(-1,1)^n\big)$$

and

$$\mathrm{G}_n\big((K-K)\sim(-1,1)^n\big)\leq\binom{2n}{n}\frac{\mathrm{G}_n\big(K+(-1,1)^n\big)^2}{\mathrm{G}_n\big(K\sim(-1,1)^n\big)}.$$

Proof. First, for any $r\geq 0$, we consider the superlevel set

$$\mathcal{D}_r=\Big\{x\in(K+L)+(-2,2)^n:\mathrm{G}_n\Big(\big(K+(-1,1)^n\big)\cap\big(x-L+(-1,1)^n\big)\Big)\geq r\Big\}.$$

Now, let $x\in K+L$, $y\in(-1,1)^n$ and $\lambda\in[0,1]$. Then, from the convexity of K and L, we have

$$\begin{aligned}\mathrm{G}_n\Big(&\big(K+(-1,1)^n\big)\cap\big((1-\lambda)x+\lambda y-L+(-1,1)^n\big)\Big)^{1/n}\\&\geq\mathrm{G}_n\Big((1-\lambda)\big(K\cap(x-L)\big)+\lambda\big(K\cap(y-L)\big)+(-1,1)^n\Big)^{1/n}.\end{aligned} \tag{3.16}$$

Notice that, since $x\in K+L$ and $y\in(-1,1)^n\subset L\subset K+L$ (because $0\in K$), the sets $K\cap(x-L),K\cap(y-L)$ are non-empty and then the above sum

$$(1-\lambda)\big(K\cap(x-L)\big)+\lambda\big(K\cap(y-L)\big)+(-1,1)^n$$

is well-defined. Hence, from (1.15) we get

$$\begin{aligned}\mathrm{G}_n\Big((1-\lambda)&\big(K\cap(x-L)\big)+\lambda\big(K\cap(y-L)\big)+(-1,1)^n\Big)^{1/n}\\&\geq(1-\lambda)\,\mathrm{G}_n\big(K\cap(x-L)\big)^{1/n}+\lambda\,\mathrm{G}_n\big(K\cap(y-L)\big)^{1/n}\\&\geq\lambda\,\mathrm{G}_n\Big(K\cap\big((-L)\sim(-1,1)^n\big)\Big)^{1/n},\end{aligned} \tag{3.17}$$

where in the last inequality we have used that, for every $y\in(-1,1)^n$,

$$\mathrm{G}_n\big(K\cap(y-L)\big)\geq\mathrm{G}_n\Big(K\cap\big((-L)\sim(-1,1)^n\big)\Big).$$

Observing also that $\mathrm{G}_n\Big(K\cap\big((-L)\sim(-1,1)^n\big)\Big)\neq 0$ since $0\in K\cap\big((-L)\sim(-1,1)^n\big)$, we may define

$$\lambda_s=\left(\frac{s}{\mathrm{G}_n\Big(K\cap\big((-L)\sim(-1,1)^n\big)\Big)}\right)^{1/n}$$

for any $0\leq s\leq\mathrm{G}_n\Big(K\cap\big((-L)\sim(-1,1)^n\big)\Big)$. Thus, from (3.16) and (3.17) for $\lambda=\lambda_s$ we conclude that

$$(1-\lambda_s)(K+L)+\lambda_s(-1,1)^n\subset\mathcal{D}_s.$$

In particular, for any $0\leq s\leq\mathrm{G}_n\Big(K\cap\big((-L)\sim(-1,1)^n\big)\Big)$ we have

$$\begin{aligned}(1-\lambda_s)\big((K+L)\sim(-1,1)^n\big)+(-1,1)^n&=(1-\lambda_s)\Big(\big((K+L)\sim(-1,1)^n\big)+(-1,1)^n\Big)+\lambda_s(-1,1)^n\\&\subset(1-\lambda_s)(K+L)+\lambda_s(-1,1)^n\subset\mathcal{D}_s.\end{aligned}$$

Then, by (1.15), we get

$$(1-\lambda_s)\,\mathrm{G}_n\big((K+L)\sim(-1,1)^n\big)^{1/n}+\lambda_s\,\mathrm{G}_n(\{0\})^{1/n}\le\mathrm{G}_n(\mathcal{D}_s)^{1/n}$$

and, consequently,

$$\sum_{i=0}^{n}\binom{n}{i}(1-\lambda_s)^i\lambda_s^{n-i}\,\mathrm{G}_n\big((K+L)\sim(-1,1)^n\big)^{i/n}\le\mathrm{G}_n(\mathcal{D}_s) \tag{3.18}$$

for all $0\le s\le\mathrm{G}_n\Big(K\cap\big((-L)\sim(-1,1)^n\big)\Big)$.

Now, writing $s_0=\mathrm{G}_n\Big(K\cap\big((-L)\sim(-1,1)^n\big)\Big)$, we observe that

$$\begin{aligned}\int_0^{s_0}(1-\lambda_s)^i\lambda_s^{n-i}\,\mathrm{d}s&=n\,\mathrm{G}_n\Big(K\cap\big((-L)\sim(-1,1)^n\big)\Big)\int_0^1\theta^{2n-i-1}(1-\theta)^i\,\mathrm{d}\theta\\&=n\,\mathrm{G}_n\Big(K\cap\big((-L)\sim(-1,1)^n\big)\Big)\frac{\Gamma(2n-i)\Gamma(i+1)}{\Gamma(2n+1)}\\&=\frac{n}{2n-i}\binom{2n}{i}^{-1}\mathrm{G}_n\Big(K\cap\big((-L)\sim(-1,1)^n\big)\Big)\end{aligned}$$

and hence, integration on $s\in[0,s_0]$ on the left-hand side of (3.18) yields

$$\left[\sum_{i=0}^{n}\frac{n}{2n-i}\binom{n}{i}\binom{2n}{i}^{-1}\mathrm{G}_n\big((K+L)\sim(-1,1)^n\big)^{i/n}\right]\mathrm{G}_n\Big(K\cap\big((-L)\sim(-1,1)^n\big)\Big).$$

Finally, integrating the right-hand side of (3.18) we can conclude the proof:

$$\begin{aligned}\int_0^{s_0}\mathrm{G}_n(\mathcal{D}_s)\,\mathrm{d}s&=\int_0^{s_0}\sum_{x\in(K+L+(-2,2)^n)\cap\mathbb{Z}^n}\chi_{\mathcal{D}_s}(x)\,\mathrm{d}s\\&=\sum_{x\in(K+L+(-2,2)^n)\cap\mathbb{Z}^n}\min\Big\{\mathrm{G}_n\Big(\big(K+(-1,1)^n\big)\cap\big(x-L+(-1,1)^n\big)\Big),s_0\Big\}\\&\le\sum_{x\in(K+L+(-2,2)^n)\cap\mathbb{Z}^n}\mathrm{G}_n\Big(\big(K+(-1,1)^n\big)\cap\big(x-L+(-1,1)^n\big)\Big)\\&=\sum_{x\in(K+L+(-2,2)^n)\cap\mathbb{Z}^n}\sum_{y\in\mathbb{Z}^n}\chi_{K+(-1,1)^n}(y)\chi_{x-L+(-1,1)^n}(y)\\&=\sum_{x\in(K+L+(-2,2)^n)\cap\mathbb{Z}^n}\sum_{y\in\mathbb{Z}^n}\chi_{K+(-1,1)^n}(y)\chi_{y+L+(-1,1)^n}(x)\\&=\sum_{y\in\mathbb{Z}^n}\chi_{K+(-1,1)^n}(y)\sum_{x\in(K+L+(-2,2)^n)\cap\mathbb{Z}^n}\chi_{y+L+(-1,1)^n}(x)\\&=\mathrm{G}_n\big(K+(-1,1)^n\big)\,\mathrm{G}_n\big(L+(-1,1)^n\big).\end{aligned}$$

■

We observe that, given convex bounded sets $K,L\subset\mathbb{R}^n$ containing the origin, applying Theorem 3.5 to the sets K and $L+(-1,1)^n$, and taking only the term corresponding to $i=n$ (bearing in mind the relations between the Minkowski difference and addition), one gets

$$\mathrm{G}_n(K+L)\,\mathrm{G}_n\big(K\cap(-L)\big)\le\binom{2n}{n}\mathrm{G}_n\big(K+(-1,1)^n\big)\,\mathrm{G}_n\big(L+(-2,2)^n\big), \tag{3.19}$$

i.e., Theorem 3.4.

What we have just seen shows that, in contrast to what happens in the continuous case, in the discrete setting the inequalities obtained directly using the covariogram method (Theorem 3.5) are in fact stronger than the ones derived as a consequence of the projection-section inequality (Theorem 3.4).

As in the previous section, we can also obtain an alternative discrete Rogers-Shephard type inequality by exploiting the relations (1.42) between the volume and the lattice point enumerator and using the original inequality for the volume (3.1). In this regard, here we show the following:

Theorem 3.6 Let $K \subset \mathbb{R}^n$ be a non-empty convex bounded set. Then

$$\mathrm{G}_n(K-K) \leq \binom{2n}{n} \mathrm{G}_n\left(K+\left(-\frac{3}{4},\frac{3}{4}\right)^n\right). \tag{3.20}$$

Proof. Using (1.42) jointly with the classical Rogers-Shephard inequality (3.1) (for which the assumption on the convex bounded set K to be closed is actually not necessary, due to the facts that the boundary of a convex set has null measure and the closure of the Minkowski sum of bounded sets is the Minkowski sum of their closures), we get

$$\begin{aligned}\mathrm{G}_n(K-K) &\leq \mathrm{vol}\left(K-K+\left(-\frac{1}{2},\frac{1}{2}\right)^n\right) = \mathrm{vol}\left(K+\left(-\frac{1}{4},\frac{1}{4}\right)^n-\left[K+\left(-\frac{1}{4},\frac{1}{4}\right)^n\right]\right)\\ &\leq \binom{2n}{n}\mathrm{vol}\left(K+\left(-\frac{1}{4},\frac{1}{4}\right)^n\right) \leq \binom{2n}{n}\mathrm{G}_n\left(K+\left(-\frac{3}{4},\frac{3}{4}\right)^n\right).\end{aligned}$$

■

When considering the Minkowski sum of two non-empty convex bounded sets $K,L \subset \mathbb{R}^n$, instead of $K-K$, we have the following discrete counterpart of (3.2):

Theorem 3.7 Let $K,L \subset \mathbb{R}^n$ be non-empty convex bounded sets and let

$$c_{K,L} = \frac{\mathrm{vol}\left(K+L+\left(-\frac{1}{2},\frac{1}{2}\right)^n\right)}{\mathrm{vol}\left(K+L+(-1,1)^n\right)} \in (0,1).$$

Then

$$\mathrm{G}_n(K+L)\,\mathrm{G}_n\big(K\cap(-L)\big) \leq \binom{2n}{n} c_{K,L}\, \mathrm{G}_n\big(K+(-1,1)^n\big)\,\mathrm{G}_n\big(L+(-1,1)^n\big). \tag{3.21}$$

In particular, taking $L=-K$, with $0 \in K$,

$$\mathrm{G}_n(K-K) \leq \binom{2n}{n} c_{K,-K} \frac{\mathrm{G}_n\big(K+(-1,1)^n\big)^2}{\mathrm{G}_n(K)}. \tag{3.22}$$

Proof. By (1.42) and Proposition 1, we get

$$\begin{aligned}\mathrm{G}_n(K+L)\,\mathrm{G}_n\big(K\cap(-L)\big) &\leq \mathrm{vol}\left(K+L+\left(-\frac{1}{2},\frac{1}{2}\right)^n\right)\mathrm{vol}\left(K\cap(-L)+\left(-\frac{1}{2},\frac{1}{2}\right)^n\right)\\ &\leq c_{K,L}\,\mathrm{vol}\big(K+L+(-1,1)^n\big)\,\mathrm{vol}\left(\left[K+\left(-\frac{1}{2},\frac{1}{2}\right)^n\right]\cap\left[-L+\left(-\frac{1}{2},\frac{1}{2}\right)^n\right]\right).\end{aligned}$$

Now, applying the classical Rogers-Shephard inequality (3.2) (again, the assumption on the convex sets K,L to be closed is not needed) jointly with (1.42) once more, we have

$$\begin{aligned}c_{K,L}\,\mathrm{vol}\big(K+L+(-1,1)^n\big)&\,\mathrm{vol}\left(\left[K+\left(-\frac{1}{2},\frac{1}{2}\right)^n\right]\cap\left[-L+\left(-\frac{1}{2},\frac{1}{2}\right)^n\right]\right)\\ &\leq \binom{2n}{n} c_{K,L}\,\mathrm{vol}\left(K+\left(-\frac{1}{2},\frac{1}{2}\right)^n\right)\mathrm{vol}\left(L+\left(-\frac{1}{2},\frac{1}{2}\right)^n\right)\\ &\leq \binom{2n}{n} c_{K,L}\,\mathrm{G}_n\big(K+(-1,1)^n\big)\,\mathrm{G}_n\big(L+(-1,1)^n\big).\end{aligned}$$

■

We note that, as we will see (Remark 3.18), both (3.20) and (3.21) (and thus also (3.22)) are *asymptotically sharp*, in the sense that, for any of these inequalities, there exist convex bodies containing the origin such that the ratio between the right-hand side and the left-hand side is arbitrarily close to 1.

Since we have at this point obtained several alternative discrete analogues of Rogers-Shephard's inequality that involve extensions of K by Minkowski adding certain cubes, it is natural to wonder about how they compare. We finish the section by performing such a contrast.

On the one hand, given a convex bounded set K containing the origin, if we apply (3.14) to the sets K and $L' := -K + (-1,1)^n$ we get

$$\sum_{i=0}^{n} \frac{n}{2n-i}\binom{n}{i}\binom{2n}{i}^{-1} \mathrm{G}_n(K-K)^{i/n} \leq \frac{\mathrm{G}_n\big(K+(-1,1)^n\big)\,\mathrm{G}_n\big(K+(-2,2)^n\big)}{\mathrm{G}_n(K)}. \tag{3.23}$$

On the other hand, we have the previously obtained inequalities (3.20) and (3.22). So, does there exist a relation between (3.20), (3.22) and (3.23)?

First, to compare (3.23) with (3.20), we need to relate

$$\frac{\mathrm{G}_n\big(K+(-1,1)^n\big)\,\mathrm{G}_n\big(K+(-2,2)^n\big)}{\mathrm{G}_n(K)} - \sum_{i=0}^{n-1} \frac{n}{2n-i}\binom{n}{i}\binom{2n}{i}^{-1} \mathrm{G}_n(K-K)^{i/n}$$

and

$$\mathrm{G}_n\left(K+\left(-\frac{3}{4},\frac{3}{4}\right)^n\right).$$

Although, unfortunately, we do not have a full answer to this question, next we show that in dimension $n=2$ the latter expression provides a smaller upper bound for

$$\mathrm{G}_n(K-K)\binom{2n}{n}^{-1}$$

and hence, in the plane, (3.20) is tighter than (3.23). This is a direct consequence of the following result.

Proposition 3.8 Let $K \subset \mathbb{R}^2$ be a planar convex bounded set containing the origin. Then

$$\mathrm{G}_2\big(K+(-1,1)^2\big) < \mathrm{G}_2\big(K+(-2,2)^2\big) - \frac{1}{2} - \frac{1}{3}\mathrm{G}_2(K-K)^{1/2} \tag{3.24}$$

Proof. Let $H_i = \{x \in \mathbb{R}^2 : \langle x, \mathrm{e}_i\rangle = 0\}$, $i=1,2$, and set

$$m := \max_{i=1,2} \mathrm{G}_1(P_{H_i}K),$$

for which we will assume without loss of generality that $m = \mathrm{G}_1(P_{H_1}K)$. Then, K is contained in a rectangle $[-a_1,b_1]\times[-a_2,b_2]$, with $a_i, b_i \geq 0$ and $\mathrm{G}_1\big([-a_i,b_i]\big) \leq m$, $i=1,2$.

So, we clearly have that $\mathrm{G}_2(K) \leq m^2$ and $\mathrm{G}_2(K-K) \leq (2m+1)^2$. Moreover, since $K+(-1,1)^2$ is open and thus, for any $x \in (P_{H_1}K)\cap\mathbb{Z}^2$, $\big(K+(-2,2)^2\big)\cap(x+\ell_1)$ contains at least two more integer points than $\big(K+(-1,1)^2\big)\cap(x+\ell_1)$, we get

$$\mathrm{G}_2\big(K+(-2,2)^2\big) \geq \mathrm{G}_2\big(K+(-1,1)^2\big) + 2m.$$

Altogether, since $m \geq 1$ (because $0 \in K$), we have

$$\frac{1}{2}+\frac{1}{3}\mathrm{G}_2(K-K)^{1/2} \leq \frac{1}{2}+\frac{1}{3}(2m+1) < 2m \leq \mathrm{G}_2\big(K+(-2,2)^2\big) - \mathrm{G}_2\big(K+(-1,1)^2\big),$$

which shows (3.24). ■

Since clearly

$$\mathrm{G}_2\left(K+\left(-\frac{3}{4},\frac{3}{4}\right)^2\right) < \mathrm{G}_2\big(K+(-1,1)^2\big) \text{ and } \mathrm{G}_2\big(K+(-2,2)^2\big) < \frac{\mathrm{G}_2\big(K+(-1,1)^2\big)\,\mathrm{G}_2\big(K+(-2,2)^2\big)}{\mathrm{G}_2(K)},$$

we obtain the desired comparison. Next we relate (3.20) and (3.22).

Remark 3.9 Inequalities (3.20) and (3.22) are not comparable. Indeed, taking $K=[-r,r]^n$ with $r>0$, one has

$$c_{K,-K}=\left(\frac{4r+1}{4r+2}\right)^n<1.$$

So, on the one hand, if $r\in\mathbb{N}\setminus\{0\}$ we get

$$\mathrm{G}_n(K)=\mathrm{G}_n\left(K+\left(-\frac{3}{4},\frac{3}{4}\right)^n\right)=\mathrm{G}_n\big(K+(-1,1)^n\big)$$

and thus

$$c_{K,-K}\frac{\mathrm{G}_n\big(K+(-1,1)^n\big)^2}{\mathrm{G}_n(K)}<\mathrm{G}_n\left(K+\left(-\frac{3}{4},\frac{3}{4}\right)^n\right).$$

On the other hand, if $r\notin\mathbb{N}\setminus\{0\}$ then

$$c_{K,-K}\frac{\mathrm{G}_n\big(K+(-1,1)^n\big)}{\mathrm{G}_n(K)}=\frac{(4r+1)^n\big(2\lfloor r\rfloor+3\big)^n}{(4r+2)^n\big(2\lfloor r\rfloor+1\big)^n}=\left(\frac{1+\frac{2}{2\lfloor r\rfloor+1}}{1+\frac{1}{4r+1}}\right)^n>1,$$

and hence

$$c_{K,-K}\frac{\mathrm{G}_n\big(K+(-1,1)^n\big)^2}{\mathrm{G}_n(K)}>\mathrm{G}_n\left(K+\left(-\frac{3}{4},\frac{3}{4}\right)^n\right).$$

♦

3.2 A discrete version of Berwald's inequality

The main result of this section is the following discrete analogue of Berwald's inequality (3.4):

Theorem 3.10 Let $K\subset\mathbb{R}^n$ be a convex bounded set containing the origin and let $f:K\longrightarrow\mathbb{R}_{\geq 0}$ be a concave function with $f(0)=|f|_\infty$. Then, for any $0<p<q$,

$$\left(\frac{\binom{n+q}{n}}{\mathrm{G}_n(K)}\sum_{x\in K\cap\mathbb{Z}^n}f^q(x)\right)^{1/q}\leq\left(\frac{\binom{n+p}{n}}{\mathrm{G}_n(K)}\sum_{x\in(K+(-1,1)^n)\cap\mathbb{Z}^n}\big(f^\diamond\big)^p(x)\right)^{1/p}. \tag{3.25}$$

We note that since f in the above result (and in the rest of the section) is defined in $K\subset\mathbb{R}^n$, rather than in $\mathbb{R}^n$, the extension $f^\diamond$ must be understood as being applied to the *zero-extension* of f, i.e., the function that is equal to f in K and 0 otherwise.

Remark 3.11 We note that the assumption on the origin in the theorem above could be substituted by $|f|_\infty=f(x_0)$ for some $x_0\in K\cap\mathbb{Z}^n$. The assumption that the maximum of f is attained in a point of $\mathbb{Z}^n$ is, however, necessary in our proof. ♦

To begin, let $K\subset\mathbb{R}^n$ be a convex bounded set containing the origin, let $f:K\longrightarrow\mathbb{R}_{\geq 0}$ be a non-negative function and set $p>0$. We will write μ to denote the *counting measure* on $\mathbb{Z}^n$, considered as a measure on $\mathbb{R}^n$, namely, $\mu(M)=\mathrm{G}_n(M)$ for any $M\subset\mathbb{R}^n$. First we observe that we have

$$\sum_{x\in K\cap\mathbb{Z}^n}f(x)^p=\int_0^\infty pt^{p-1}\,\mathrm{G}_n\Big(\big\{x\in K:\,f(x)>t\big\}\Big)\,\mathrm{d}t. \tag{3.26}$$

Indeed, by Fubini's theorem, we obtain

$$\begin{aligned}\sum_{x\in K\cap\mathbb{Z}^n}f(x)^p&=\int_{\mathbb{R}^n}f(x)^p\chi_K(x)\,\mathrm{d}\mu(x)=\int_{\mathbb{R}^n}\left(\int_0^{f(x)}pt^{p-1}\,\mathrm{d}t\right)\chi_K(x)\,\mathrm{d}\mu(x)\\&=\int_0^\infty\int_{\mathbb{R}^n}pt^{p-1}\chi_K(x)\chi_{(0,f(x))}(t)\,\mathrm{d}\mu(x)\,\mathrm{d}t=\int_0^\infty pt^{p-1}\int_{\mathbb{R}^n}\chi_{\{x\in K:\,f(x)>t\}}(x)\,\mathrm{d}\mu(x)\,\mathrm{d}t\\&=\int_0^\infty pt^{p-1}\,\mathrm{G}_n\Big(\big\{x\in K:\,f(x)>t\big\}\Big)\,\mathrm{d}t.\end{aligned}$$

To prove Theorem 3.10, we need the following auxiliary results.

Lemma 3.12 Let $K \subset \mathbb{R}^n$ be a convex bounded set containing the origin and let $m > 0$. Consider also the concave function $h_m : K + (-1,1)^n \longrightarrow \mathbb{R}_{\geq 0}$ whose hypograph is the closure of the set $\mathrm{conv}\big((K \times \{0\}) \cup (\{0\}^n, m)\big) + \big((-1,1)^n \times \{0\}\big)$. Then, for every $p > 0$,

$$\left(\frac{\binom{n+p}{n}}{\mathrm{G}_n(K)} \sum_{x \in (K+(-1,1)^n) \cap \mathbb{Z}^n} h_m(x)^p\right)^{1/p} \geq m. \tag{3.27}$$

Proof. Observe that, for any $0 \leq t < m$,

$$\Big(\mathrm{conv}\big((K \times \{0\}) \cup (\{0\}^n, m)\big) + \big((-1,1)^n \times \{0\}\big)\Big) \cap \big(\mathbb{R}^n \times \{t\}\big) = \left(\left(1 - \frac{t}{m}\right) K + (-1,1)^n\right) \times \{t\}$$

and thus

$$\big\{x \in K + (-1,1)^n : h_m(x) > t\big\} = \left(1 - \frac{t}{m}\right) K + (-1,1)^n.$$

Then, using (1.15) we get

$$\begin{aligned}\int_0^\infty p t^{p-1} \mathrm{G}_n\Big(\big\{x \in K + (-1,1)^n : h_m(x) > t\big\}\Big)\,\mathrm{d}t &= \int_0^m p t^{p-1} \mathrm{G}_n\left(\left(1 - \frac{t}{m}\right) K + (-1,1)^n\right)\mathrm{d}t \\ &\geq \int_0^m p t^{p-1} \left(1 - \frac{t}{m}\right)^n \mathrm{G}_n(K)\,\mathrm{d}t \\ &= p m^p\, \mathrm{G}_n(K) \int_0^1 s^{p-1}(1-s)^n\,\mathrm{d}s \\ &= p m^p\, \mathrm{G}_n(K) \frac{\Gamma(p)\Gamma(n+1)}{\Gamma(n+p+1)} \\ &= m^p\, \mathrm{G}_n(K) \binom{n+p}{n}^{-1}.\end{aligned}$$

This, together with (3.26) applied to the function h_m, shows (3.27). ■

Now, given a concave function $f : K \longrightarrow \mathbb{R}_{\geq 0}$ defined on a convex bounded set $K \subset \mathbb{R}^n$, we will relate the number of integer points of the superlevel sets of both the function f and its extension $f^\diamond$ (whose hypograph is the closure of the Minkowski addition of the hypograph of f and $(-1,1)^n \times \{0\}$) in terms of a suitable $(1/n)$-concave function (on its support). To see this, for any fixed $p > 0$, we set

$$m = \left(\frac{\binom{n+p}{n}}{\mathrm{G}_n(K)} \sum_{x \in (K+(-1,1)^n) \cap \mathbb{Z}^n} \big(f^\diamond\big)^p(x)\right)^{1/p},$$

and we consider the function $g : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$ given by

$$g(t) = \begin{cases} \left(1 - \frac{t}{m}\right)^n \mathrm{G}_n(K) & \text{if } t \leq m, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 3.13 Let $K \subset \mathbb{R}^n$ be a convex bounded set containing the origin, and let $f : K \longrightarrow \mathbb{R}_{\geq 0}$ be a concave function with $f(0) = |f|_\infty > 0$. For any $p > 0$, consider m and g defined as above. Then, there exists $t_0 \in \mathbb{R}_{\geq 0}$ such that

$$\mathrm{G}_n\Big(\big\{x \in K + (-1,1)^n : f^\diamond(x) > t\big\}\Big) > g(t) \tag{3.28}$$

for all $0 \leq t < t_0$ and

$$g(t) \geq \mathrm{G}_n\Big(\big\{x \in K : f(x) > t\big\}\Big) \tag{3.29}$$

for all $t \geq t_0$.

Proof. First we will show that $m \geq |f|_\infty$. To this aim, assume by contradiction that $m < |f|_\infty$ and set $h_m : K + (-1,1)^n \longrightarrow \mathbb{R}_{\geq 0}$ the concave function whose hypograph is the closure of

$$\mathrm{conv}\big((K \times \{0\}) \cup (\{0\}^n, m)\big) + \big((-1,1)^n \times \{0\}\big).$$

Then, by the concavity of f, $\mathrm{hyp}(f) \supset \mathrm{conv}\big((K \times \{0\}) \cup (\{0\}^n, m)\big)$, and so $\mathrm{hyp}\big(f^\diamond\big) \supset \mathrm{hyp}(h_m)$. This also implies that $\big\{x \in K + (-1,1)^n : f^\diamond(x) > t\big\} \supset \big\{x \in K + (-1,1)^n : h_m(x) > t\big\}$ for all $0 \leq t < |f|_\infty$. Therefore, assuming that $m < |f|_\infty$, the latter inclusion jointly with (3.26) applied to the functions $f^\diamond$ and h_m, Lemma 3.12 and the fact that $\mathrm{G}_n\Big(\big\{x \in K + (-1,1)^n : \, f^\diamond(x) > t\big\}\Big) \geq 1$ for every $0 \leq t < |f|_\infty$ (since $f(0) = |f|_\infty$), imply the contradiction

$$\begin{aligned}
m &= \left(\frac{\binom{n+p}{n}}{\mathrm{G}_n(K)} \int_0^{|f|_\infty} p t^{p-1} \mathrm{G}_n\Big(\big\{x \in K + (-1,1)^n : f^\diamond(x) > t\big\}\Big)\,\mathrm{d}t\right)^{1/p} \\
&> \left(\frac{\binom{n+p}{n}}{\mathrm{G}_n(K)} \int_0^{m} p t^{p-1} \mathrm{G}_n\Big(\big\{x \in K + (-1,1)^n : f^\diamond(x) > t\big\}\Big)\,\mathrm{d}t\right)^{1/p} \\
&\geq \left(\frac{\binom{n+p}{n}}{\mathrm{G}_n(K)} \int_0^{m} p t^{p-1} \mathrm{G}_n\Big(\big\{x \in K + (-1,1)^n : h_m(x) > t\big\}\Big)\,\mathrm{d}t\right)^{1/p} \\
&= \left(\frac{\binom{n+p}{n}}{\mathrm{G}_n(K)} \sum_{x \in (K + (-1,1)^n) \cap \mathbb{Z}^n} h_m(x)^p\right)^{1/p} \geq m.
\end{aligned}$$

Now, since $f^\diamond \leq |f|_\infty$, we trivially have

$$0 = \mathrm{G}_n\Big(\big\{x \in K + (-1,1)^n : f^\diamond(x) > |f|_\infty\big\}\Big) \leq g\big(|f|_\infty\big),$$

and thus we may consider

$$t_0 := \inf\Big\{t > 0 : \mathrm{G}_n\Big(\big\{x \in K + (-1,1)^n : f^\diamond(x) > t\big\}\Big) \leq g(t)\Big\} < \infty.$$

Then, on the one hand, we obtain by the definition of t_0 that (3.28) holds for all $0 \leq t < t_0 \leq |f|_\infty$. Moreover, since $\big\{x \in K + (-1,1)^n : f^\diamond(x) > t\big\} = \big\{x \in K : f(x) > t\big\} + (-1,1)^n$ (which is an open convex bounded set) for every $0 \leq t < |f|_\infty$, we have that the function $t \mapsto \mathrm{G}_n\Big(\big\{x \in K + (-1,1)^n : f^\diamond(x) > t\big\}\Big)$ is continuous from the right on $\mathbb{R}_{\geq 0}$. Therefore, we obtain that

$$\mathrm{G}_n\Big(\big\{x \in K + (-1,1)^n : f^\diamond(x) > t_0\big\}\Big) \leq g(t_0).$$

On the other hand, given $t \in \big[t_0, |f|_\infty\big]$ and taking $\lambda \in (0,1]$ such that $t_0 = \lambda t$, from (1.15) we get

$$\begin{aligned}
\mathrm{G}_n\Big(\big\{x \in K + (-1,1)^n : f^\diamond(x) > t_0\big\}\Big)^{1/n} &= \mathrm{G}_n\Big(\big\{x \in K : f(x) > t_0\big\} + (-1,1)^n\Big)^{1/n} \\
&\geq \mathrm{G}_n\Big((1-\lambda)\big\{x \in K : f(x) \geq 0\big\} + \lambda\big\{x \in K : f(x) > t\big\} + (-1,1)^n\Big)^{1/n} \\
&\geq (1-\lambda)\,\mathrm{G}_n(K)^{1/n} + \lambda\,\mathrm{G}_n\Big(\big\{x \in K : f(x) > t\big\}\Big)^{1/n}
\end{aligned} \tag{3.30}$$

and also (taking into account that $t_0 \leq |f|_\infty \leq m$)

$$g(t_0)^{1/n} = \left(1 - \frac{t_0}{m}\right)\mathrm{G}_n(K)^{1/n} = (1-\lambda)\,\mathrm{G}_n(K)^{1/n} + \lambda\left(1 - \frac{t}{m}\right)\mathrm{G}_n(K)^{1/n}. \tag{3.31}$$

Thus, using (3.30) and (3.31), we get that (3.29) holds for all $t_0 \leq t \leq |f|_\infty$. This concludes the proof, since (3.29) is further trivially true for any $t \in \big[|f|_\infty, \infty\big)$. ■

Proof of Theorem 3.10. We may assume, without loss of generality, that $|f|_\infty > 0$, and let m and g be defined as in Lemma 3.13. Observe also that, for any $r > 0$,

$$\left(\frac{\binom{n+r}{n}}{\mathrm{G}_n(K)}\int_0^m rt^{r-1}\left(1-\frac{t}{m}\right)^n \mathrm{G}_n(K)\,\mathrm{d}t\right)^{1/r} = m. \tag{3.32}$$

From (3.26) applied to $f^\diamond$ jointly with the definition of g and m, the latter implies, in particular, that

$$\int_0^\infty t^{p-1}\,\mathrm{G}_n\Big(\big\{x\in K+(-1,1)^n : f^\diamond(x)>t\big\}\Big)\,\mathrm{d}t = \int_0^\infty t^{p-1}g(t)\,\mathrm{d}t. \tag{3.33}$$

Hence, with t_0 as provided by Lemma 3.13 we obtain, from (3.28) and (3.29), that

$$\begin{aligned}
&\int_0^{t_0} t^{q-1}\Big[\mathrm{G}_n\Big(\big\{x\in K+(-1,1)^n : f^\diamond(x)>t\big\}\Big)-g(t)\Big]\,\mathrm{d}t\\
&\qquad -\int_{t_0}^\infty t^{q-1}\Big[g(t)-\mathrm{G}_n\Big(\big\{x\in K : f(x)>t\big\}\Big)\Big]\,\mathrm{d}t\\
&=\int_0^{t_0} t^{p-1}t^{q-p}\Big[\mathrm{G}_n\Big(\big\{x\in K+(-1,1)^n : f^\diamond(x)>t\big\}\Big)-g(t)\Big]\,\mathrm{d}t\\
&\qquad -\int_{t_0}^\infty t^{p-1}t^{q-p}\Big[g(t)-\mathrm{G}_n\Big(\big\{x\in K : f(x)>t\big\}\Big)\Big]\,\mathrm{d}t\\
&\le t_0^{q-p}\int_0^{t_0} t^{p-1}\Big[\mathrm{G}_n\Big(\big\{x\in K+(-1,1)^n : f^\diamond(x)>t\big\}\Big)-g(t)\Big]\,\mathrm{d}t\\
&\qquad -t_0^{q-p}\int_{t_0}^\infty t^{p-1}\Big[g(t)-\mathrm{G}_n\Big(\big\{x\in K : f(x)>t\big\}\Big)\Big]\,\mathrm{d}t.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&t_0^{q-p}\int_0^{t_0} t^{p-1}\Big[\mathrm{G}_n\Big(\big\{x\in K+(-1,1)^n : f^\diamond(x)>t\big\}\Big)-g(t)\Big]\,\mathrm{d}t\\
&\qquad +t_0^{q-p}\int_{t_0}^\infty t^{p-1}\Big[\mathrm{G}_n\Big(\big\{x\in K : f(x)>t\big\}\Big)-g(t)\Big]\,\mathrm{d}t\\
&\le t_0^{q-p}\int_0^\infty t^{p-1}\Big[\mathrm{G}_n\Big(\big\{x\in K+(-1,1)^n : f^\diamond(x)>t\big\}\Big)-g(t)\Big]\,\mathrm{d}t = 0,
\end{aligned}$$

where the latter equality follows from (3.33).

Altogether, we have shown that

$$\int_0^{t_0} t^{q-1}\,\mathrm{G}_n\Big(\big\{x\in K+(-1,1)^n : f^\diamond(x)>t\big\}\Big)\,\mathrm{d}t + \int_{t_0}^\infty t^{q-1}\,\mathrm{G}_n\Big(\big\{x\in K : f(x)>t\big\}\Big)\,\mathrm{d}t \le \int_0^\infty t^{q-1}g(t)\,\mathrm{d}t$$

and hence

$$\int_0^\infty t^{q-1}\,\mathrm{G}_n\Big(\big\{x\in K : f(x)>t\big\}\Big)\,\mathrm{d}t \le \int_0^\infty t^{q-1}g(t)\,\mathrm{d}t.$$

Consequently, from (3.32) for $r=q$, we have

$$\left(\frac{\binom{n+q}{n}}{\mathrm{G}_n(K)}\int_0^\infty qt^{q-1}\,\mathrm{G}_n\Big(\big\{x\in K : f(x)>t\big\}\Big)\,\mathrm{d}t\right)^{1/q} \le \left(\frac{\binom{n+q}{n}}{\mathrm{G}_n(K)}\int_0^m qt^{q-1}\left(1-\frac{t}{m}\right)^n \mathrm{G}_n(K)\,\mathrm{d}t\right)^{1/q} = m$$

and thus, from (3.26) applied to f and q, we conclude the proof:

$$\left(\frac{\binom{n+q}{n}}{\mathrm{G}_n(K)}\sum_{x\in K\cap\mathbb{Z}^n} f^q(x)\right)^{1/q} \le \left(\frac{\binom{n+p}{n}}{\mathrm{G}_n(K)}\sum_{x\in (K+(-1,1)^n)\cap\mathbb{Z}^n} \left(f^\diamond\right)^p(x)\right)^{1/p}. \qquad \blacksquare$$

As briefly pointed out within the introduction, the continuous version of Berwald's inequality (Theorem L) allows us to derive the Rogers-Shephard inequalities (3.3) and (3.1). To show this, first notice that Stirling's formula for the gamma function yields the asymptotic formula

$$\lim_{x\to\infty}\frac{\Gamma(x)}{\sqrt{\frac{2\pi}{x}}\left(\frac{x}{e}\right)^x}=1,$$

which implies, in particular, that $\binom{n+q}{n}^{1/q}\to 1$ as $q\to\infty$. Moreover, given a convex body $K\subset\mathbb{R}^n$ with $\dim K=n$ and a concave function $f:K\longrightarrow\mathbb{R}_{\geq 0}$, it is well-known that

$$\lim_{q\to\infty}\left(\int_K f^q(x)\,\mathrm{d}x\right)^{1/q}=|f|_\infty$$

(here we notice that, since f is concave, $|f|_\infty$ agrees with $\operatorname{ess\,sup}_{x\in K} f(x)$). Thus, applying Theorem L with $p=k$ (and $n'=n-k$) and the concave function (cf. (1.1)) $f:P_{H^\perp}K\longrightarrow\mathbb{R}_{\geq 0}$ given by the expression $f(x)=\mathrm{vol}_k\big(K\cap(x+H)\big)^{1/k}$ for $H\in\mathcal{L}^n_k$, we get Theorem K by taking limits as $q\to\infty$. Indeed, we have

$$\begin{aligned}\mathrm{vol}_k(K\cap H)^{1/k}\leq|f|_\infty&=\lim_{q\to\infty}\left(\frac{\binom{n-k+q}{n-k}}{\mathrm{vol}_{n-k}(P_{H^\perp}K)}\int_{P_{H^\perp}K}f^q(x)\,\mathrm{d}x\right)^{1/q}\\&\leq\left(\frac{\binom{n}{k}}{\mathrm{vol}_{n-k}(P_{H^\perp}K)}\int_{P_{H^\perp}K}f^k(x)\,\mathrm{d}x\right)^{1/k}=\left(\frac{\binom{n}{k}}{\mathrm{vol}_{n-k}(P_{H^\perp}K)}\mathrm{vol}(K)\right)^{1/k},\end{aligned}$$

where the last equality follows from Fubini's theorem.

Analogously, from Theorem L for $p=n$ and the concave function (cf. (1.1)) $f:K-K\longrightarrow\mathbb{R}_{\geq 0}$ given by $f(x)=\mathrm{vol}\big(K\cap(x+K)\big)^{1/n}$, for which we have

$$\int_{K-K}f^n(x)\,\mathrm{d}x=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\chi_K(y)\chi_{x+K}(y)\,\mathrm{d}y\,\mathrm{d}x=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\chi_K(y)\chi_{y-K}(x)\,\mathrm{d}x\,\mathrm{d}y=\mathrm{vol}(K)^2,$$

we get

$$\begin{aligned}\mathrm{vol}(K)^{1/n}=|f|_\infty&=\lim_{q\to\infty}\left(\frac{\binom{n+q}{n}}{\mathrm{vol}(K-K)}\int_{K-K}f^q(x)\,\mathrm{d}x\right)^{1/q}\\&\leq\left(\frac{\binom{2n}{n}}{\mathrm{vol}(K-K)}\int_{K-K}f^n(x)\,\mathrm{d}x\right)^{1/n}=\left(\frac{\binom{2n}{n}}{\mathrm{vol}(K-K)}\mathrm{vol}(K)^2\right)^{1/n},\end{aligned}$$

and so Theorem J follows.

We finish the section by pointing out that, arguing in a similar way in the discrete setting, but now applying Theorem 3.10 (for the above-mentioned functions and values of p, and letting $q\to\infty$), we get the following results:

Corollary 3.14 Let $k\in\{1,\dots,n-1\}$ and $H\in\mathcal{L}^n_k$. Let $K\subset\mathbb{R}^n$ be a convex bounded set containing the origin. Then

$$\mathrm{G}_{n-k}(P_{H^\perp}K)\,\mathrm{vol}_k(K\cap H)\leq\binom{n}{k}\sum_{x\in\left(P_{H^\perp}K+\mathrm{C}_{H^\perp}\right)\cap\mathbb{Z}^n}\ \sup_{z\in\mathrm{C}_{H^\perp}}\mathrm{vol}_k\Big(K\cap\big((x+z)+H\big)\Big).$$

Corollary 3.15 Let $K \subset \mathbb{R}^n$ be a convex bounded set containing the origin. Then

$$\mathrm{G}_n(K-K)\,\mathrm{vol}(K) \le \binom{2n}{n} \sum_{x\in(K-K+(-1,1)^n)\cap\mathbb{Z}^n} \sup_{z\in(-1,1)^n} \mathrm{vol}\Big(K\cap\big((x+z)+K\big)\Big).$$

We point out that the more general result, involving convex bounded sets $K, L \subset \mathbb{R}^n$ containing the origin such that $\max_{x\in K+L}\mathrm{vol}\big(K\cap(x-L)\big) = \mathrm{vol}\big(K\cap(-L)\big)$ may be also derived, obtaining that

$$\mathrm{G}_n(K+L)\,\mathrm{vol}\big(K\cap(-L)\big) \le \binom{2n}{n} \sum_{x\in(K+L+(-1,1)^n)\cap\mathbb{Z}^n} \sup_{z\in(-1,1)^n} \mathrm{vol}\Big(K\cap\big((x+z)-L\big)\Big).$$

Remark 3.16 We also observe that one cannot immediately derive, in principle, other discrete versions of the Rogers-Shephard inequalities (3.3) and (3.1) from Theorem 3.10, despite counting with the discrete analogue (1.15) of the classical Brunn-Minkowski inequality, because of the lack of concavity of the functional $\mathrm{G}_n(\cdot)^{1/n}$. This is the reason for which Theorem 3.10 yields the above discrete counterparts of (3.3) and (3.1), where the volume arises jointly with the lattice point enumerator. Some engaging examples of discrete analogues of classical inequalities where these two functionals appear together can be found in [2]. ♦

3.3 From the discrete setting to the continuous one

We begin by proving that the discrete versions of the projection-section and the Rogers-Shephard inequalities we have shown in Section 3.1 imply their corresponding continuous analogues, by exploiting the relations (1.4.2) and (1.43) between the lattice point enumerator and the volume. To this aim, regarding the discrete projection-section type inequalities, we will show that (3.6) already implies (3.3) (and hence, the same is obtained from the stronger inequality (3.7)). In the same way, we will prove that (3.11) is enough to derive (3.2) (and thus, the same happens for the more powerful inequalities (3.21) and (3.14)). Moreover, in particular, (3.12) implies (3.1) (and so, the same is true for the stronger versions (3.20), (3.22) and (3.15)).

Theorem 3.17 Let $K, L \subset \mathbb{R}^n$ be convex bodies containing the origin with $\dim K = \dim L = n$. Then

1. The discrete inequality (3.6) for the lattice point enumerator implies the classical projection-section inequality (3.3) for the volume.
2. The discrete inequality (3.11) for the lattice point enumerator implies the classical Rogers-Shephard inequality (3.2) for the volume.

Proof. Applying (3.6) with rK (for $r>0$), taking limits as $r\to\infty$ and using (1.4.2) and (1.43), we get

$$\begin{aligned}\mathrm{vol}_{n-k}(P_{H^\perp}K)\,\mathrm{vol}_k(K\cap H) &= \lim_{r\to\infty}\frac{\mathrm{G}_{n-k}(rP_{H^\perp}K)}{r^{n-k}}\cdot\frac{\mathrm{G}_k\big(r(K\cap H)\big)}{r^k} = \lim_{r\to\infty}\frac{\mathrm{G}_{n-k}\big(P_{H^\perp}(rK)\big)\,\mathrm{G}_k\big((rK)\cap H\big)}{r^n}\\ &\le \lim_{r\to\infty}\binom{n}{k}\frac{\mathrm{G}_n\big(rK+(-1,1)^n\big)}{r^n} = \binom{n}{k}\mathrm{vol}(K).\end{aligned}$$

Analogously, but now applying (3.11) with rK and rL (for $r>0$), we obtain

$$\begin{aligned}\mathrm{vol}(K+L)\mathrm{vol}\big(K\cap(-L)\big) &= \lim_{r\to\infty}\frac{\mathrm{G}_n\big(r(K+L)\big)\mathrm{G}_n\Big(r\big(K\cap(-L)\big)\Big)}{r^{2n}} = \lim_{r\to\infty}\frac{\mathrm{G}_n(rK+rL)\mathrm{G}_n\big((rK)\cap(-rL)\big)}{r^{2n}}\\ &\le \lim_{r\to\infty}\binom{2n}{n}\frac{\mathrm{G}_n\big(rK+(-1,1)^n\big)\,\mathrm{G}_n\big(rL+(-2,2)^n\big)}{r^{2n}} = \binom{2n}{n}\mathrm{vol}(K)\,\mathrm{vol}(L).\end{aligned}$$

This concludes the proof. ■

Remark 3.18 Since (3.3) and (3.2) are sharp, from the proof above, we have that their discrete analogues (3.6) and (3.11) (and hence their corresponding stronger related versions) are asymptotically sharp. ♦

To conclude the chapter, we show that the discrete version of Berwald's inequality obtained in Section 3.2, Theorem 3.10, implies its continuous analogue, Theorem L.

Before stating and proving this result we observe the following. Given a convex body $K \subset \mathbb{R}^n$ and a concave function $f: K \longrightarrow \mathbb{R}_{\geq 0}$, we have

$$\lim_{r\to\infty}\left[\frac{1}{r^n}\sum_{x\in(rK)\cap\mathbb{Z}^n} f\left(\frac{x}{r}\right)\right] = \lim_{r\to\infty}\left[\frac{1}{r^n}\sum_{y\in K\cap((1/r)\mathbb{Z}^n)} f(y)\right] = \int_K f(x)\,\mathrm{d}x, \tag{3.34}$$

since f is Riemann integrable (because it is concave on the convex set K, whose boundary has null Lebesgue measure).

Moreover, we may assume without loss of generality that f is upper semicontinuous. Indeed, otherwise we would work with its upper closure, which is determined via the closure of the superlevel sets of f (see [87, page 14 and Theorem 1.6]) and thus has the same integral on $\mathbb{R}^n$ because of Fubini's theorem together with the facts that all the superlevel sets of f are convex (since f is concave) and the boundary of a convex set has null (Lebesgue) measure. Notice then that, for any decreasing sequence $\{r_k\}_{k\in\mathbb{N}} \subset \mathbb{R}_{\geq 0}$ with $r_k \to 0$ as $k \to \infty$, we have

$$\bigcap_{k=0}^{\infty}\Big(\big\{x\in K:\ f(x)\geq t\big\}+r_k(-1,1)^n\Big) = \big\{x\in K:\ f(x)\geq t\big\} \tag{3.35}$$

due to the fact that $\{x \in K : f(x) \geq t\}$ is closed for all $t \geq 0$.

Theorem 3.19 Let $K \subset \mathbb{R}^n$ be a convex body with $\dim K = n$ and let $f: K \longrightarrow \mathbb{R}_{\geq 0}$ be a concave function. Then the discrete inequality (3.25) implies the classical Berwald inequality (3.4).

Proof. On the one hand, from (3.25) applied to the function $h: rK \longrightarrow \mathbb{R}_{\geq 0}$ given by $h(x) := f(x/r)$, $r > 0$, we get

$$\left(\frac{\binom{n+q}{n}}{\mathrm{G}_n(rK)}\sum_{y\in(rK)\cap\mathbb{Z}^n} h^q(y)\right)^{1/q} \leq \left(\frac{\binom{n+p}{n}}{\mathrm{G}_n(rK)}\sum_{y\in(rK+(-1,1)^n)\cap\mathbb{Z}^n} \left(h^{\diamond}\right)^p(y)\right)^{1/p}.$$

On the other hand, given $\varepsilon > 0$, for sufficiently large $r > 0$ we have that

$$\begin{aligned}
\sum_{y\in(rK+(-1,1)^n)\cap\mathbb{Z}^n}\left(h^{\diamond}\right)^p(y) &= \sum_{y/r\in[K+(1/r)(-1,1)^n]\cap[(1/r)\mathbb{Z}^n]}\left(\sup_{u\in(-1,1)^n} f\left(\frac{y+u}{r}\right)\right)^p \\
&= \sum_{x\in[K+(1/r)(-1,1)^n]\cap[(1/r)\mathbb{Z}^n]}\left(\sup_{v\in(1/r)(-1,1)^n} f(x+v)\right)^p \\
&\leq \sum_{x\in[K+\varepsilon(-1,1)^n]\cap[(1/r)\mathbb{Z}^n]}\left(\sup_{v\in\varepsilon(-1,1)^n} f(x+v)\right)^p \\
&\leq \sum_{x\in[K+\varepsilon(-1,1)^n]\cap[(1/r)\mathbb{Z}^n]}\left(f^{\diamond_\varepsilon}\right)^p(x),
\end{aligned}$$

where $f^{\diamond_\varepsilon}: K+\varepsilon(-1,1)^n \longrightarrow \mathbb{R}_{\geq 0}$ is the function given by

$$f^{\diamond_\varepsilon}(z) = \sup_{u\in\varepsilon(-1,1)^n} f(z+u)$$

for all $z \in \mathbb{R}^n$.

Thus, for r large enough, we get

$$\left(\frac{\binom{n+q}{n}}{\mathrm{G}_n(rK)}\sum_{y\in(rK)\cap\mathbb{Z}^n}h^q(y)\right)^{1/q}\leq\left(\frac{\binom{n+p}{n}}{\mathrm{G}_n(rK)}\sum_{x\in[K+\varepsilon(-1,1)^n]\cap[(1/r)\mathbb{Z}^n]}\left(f^{\diamond_\varepsilon}\right)^p(x)\right)^{1/p}$$

which implies, using (3.34) and (1.4.2), that

$$\left(\frac{\binom{n+q}{n}}{\mathrm{vol}(K)}\int_K f^q(x)\,\mathrm{d}x\right)^{1/q}\leq\left(\frac{\binom{n+p}{n}}{\mathrm{vol}(K)}\int_{K+\varepsilon(-1,1)^n}\left(f^{\diamond_\varepsilon}\right)^p(x)\,\mathrm{d}x\right)^{1/p}.$$

Since $\varepsilon>0$ was arbitrary, to conclude the proof it is enough to show that

$$\inf_{\varepsilon>0}\int_{K+\varepsilon(-1,1)^n}\left(f^{\diamond_\varepsilon}\right)^p(x)\,\mathrm{d}x\leq\int_K f^p(x)\,\mathrm{d}x. \tag{3.36}$$

To this aim first observe that, by Fubini's theorem, we have (cf. (3.26))

$$\int_{K+\varepsilon(-1,1)^n}\left(f^{\diamond_\varepsilon}\right)^p(x)\,\mathrm{d}x=\int_0^\infty pt^{p-1}\,\mathrm{vol}\Big(\big\{x\in K+\varepsilon(-1,1)^n:\,f^{\diamond_\varepsilon}(x)>t\big\}\Big)\,\mathrm{d}t. \tag{3.37}$$

Now, since

$$\big\{x\in K+\varepsilon(-1,1)^n:\,f^{\diamond_\varepsilon}(x)>t\big\}=\big\{x\in K:\,f(x)>t\big\}+\varepsilon(-1,1)^n,$$

we have

$$\mathrm{vol}\Big(\big\{x\in K+\varepsilon(-1,1)^n:\,f^{\diamond_\varepsilon}(x)>t\big\}\Big)\leq\mathrm{vol}\Big(\big\{x\in K:\,f(x)\geq t\big\}+\varepsilon(-1,1)^n\Big)$$

and hence, from (3.35),

$$\lim_{\varepsilon\to0^+}\mathrm{vol}\Big(\big\{x\in K+\varepsilon(-1,1)^n:\,f^{\diamond_\varepsilon}(x)>t\big\}\Big)\leq\mathrm{vol}\Big(\big\{x\in K:\,f(x)\geq t\big\}\Big). \tag{3.38}$$

Therefore, taking limits as $\varepsilon\to0^+$ in both sides of (3.37), applying the monotone convergence theorem and using (3.38), we get

$$\begin{aligned}\inf_{\varepsilon>0}\int_{K+\varepsilon(-1,1)^n}\left(f^{\diamond_\varepsilon}\right)^p(x)\,\mathrm{d}x&=\lim_{\varepsilon\to0^+}\int_{K+\varepsilon(-1,1)^n}\left(f^{\diamond_\varepsilon}\right)^p(x)\,\mathrm{d}x\\&\leq\int_0^\infty pt^{p-1}\,\mathrm{vol}\Big(\big\{x\in K:\,f(x)\geq t\big\}\Big)\,\mathrm{d}t=\int_K f^p(x)\,\mathrm{d}x.\end{aligned}$$

So (3.36) follows, which concludes the proof. ■

4

Inequalities for the successive minima

As the work in the previous chapters suggests, inequalities relating the volume and the lattice point enumerator functionals can be a very powerful asset. This type of inequalities has been explored since the inception of Convex Geometry. Minkowski himself showed in 1891 ([81]) that if $K \subset \mathbb{R}^n$ is an origin-symmetric bounded and convex set with $\mathrm{vol}(K) > 2^n$, then $\mathrm{G_n}(\mathrm{int}\, K) > 1$, in what is now known as *Minkowski's 1st Theorem*. This inequality is tight, as shown e.g. by the set $(-1,1)^n$.

Analogously, one of Blichfeldt's most well-known results, obtained in 1921 ([21]), states that if $K \subset \mathbb{R}^n$ is a bounded convex set with $\dim(K \cap \mathbb{Z}^n) = n$, then $\mathrm{G_n}(K) \leq (n+1)!\,\mathrm{vol}(K)$. It can be checked that the inequality is again tight by considering, for instance, the standard simplex $S_n = \mathrm{conv}\{0, \mathrm{e}_1, \dots, \mathrm{e}_n\}$. Conversely, van der Corput obtained a lower bound in 1935 ([54, Chapter 2, Theorem 7.1]). In particular, he showed that if K is, in addition, origin-symmetric, then $\mathrm{G_n}(\mathrm{int}\, K) \geq 2^{-n}\,\mathrm{vol}(K)$, which is once again tight, for example, for $[-1,1]^n$. Clearly, this result strengthens that of Minkowski.

Since all these inequalities are already sharp (yet sometimes vastly inaccurate), one way to improve them is to introduce parameters depending on the actual sets involved. This technique was already exploited by Minkowski to improve the bound in his original fundamental theorem. To present this, let us first observe that if we denote by $\lambda_1 = \min\{\lambda > 0 : \mathrm{G_n}(\lambda K) > 1\}$, then said theorem translates into $\mathrm{vol}(\lambda_1 K) \leq 2^n$, and thus $\mathrm{vol}(K) \leq 2^n/\lambda_1^n$. Minkowski generalized this and defined the *succesive minima* of an origin-symmetric convex body:

Definition 4.1 Let $K \subset \mathbb{R}^n$ be an origin-symmetric convex body. Then

$$\lambda_i(K) = \min\{\lambda > 0 : \dim(\lambda K \cap \mathbb{Z}^n) \geq i\}, \tag{4.1}$$

for each $i = 1, \dots, n$.

If a linearly independent set $\{u_i\}_{i=1}^n \subset \mathbb{Z}^n$ satisfies $u_i \in \lambda_i(K)K$ for all $i = 1, \dots, n$, we say that the u_i's *realize* the successive minima $\lambda_i(K)$. Though we will not need it, the above definition (and the subsequent result) can be extended to arbitrary lattices in a natural way.

More generally, if K is not origin-symmetric, one may define $\lambda_i(K) := \lambda_i\big(\mathrm{cs}(K)\big)$, where $\mathrm{cs}(K)$ is the central symmetral $(1/2)(K-K)$ of K. It is clear that $0 < \lambda_1(K) \leq \lambda_2(K) \leq \cdots \leq \lambda_n(K)$ and that

$\lambda_i(\mu K) = \mu^{-1}\lambda_i(K)$ for all $\mu > 0$. With this notion, Minkowski refined the bound of his first theorem and proved the following powerful result in 1896:

Theorem N — Minkowski Second Theorem. Let $K \subset \mathbb{R}^n$ be a convex body. Then

$$\frac{1}{n!}\prod_{i=1}^{n}\frac{2}{\lambda_i(K)} \le \mathrm{vol}(K) \le \prod_{i=1}^{n}\frac{2}{\lambda_i(K)}. \tag{4.2}$$

The lower and upper bounds in (4.2) are attained, e.g., by the standard simplex S_n and the cube $[-1,1]^n$, respectively.

For origin-symmetric K, this was proved by Minkowski [54, Chapter 2, Theorems 9.1 and 9.2]. In the general case, the upper bound follows directly from the inequality $\mathrm{vol}(K) \le \mathrm{vol}\big(\mathrm{cs}(K)\big)$, which in turn is a special case of the Brunn-Minkowski inequality (1.1). The lower bound can also be proved by an inclusion argument, similar to the symmetric case: one considers the convex hull of the $2n$ vectors $\pm u_i/\lambda_i(K)$ of $\mathrm{cs}(K)$ that realize the $\lambda_i(K)$ [60, Remark 1.1].

Many alternatives to Minkowski's complicated original proof have been obtained. One of the first short proofs was given by Davenport [38]. More analytic proofs were obtained by Weyl [105] and Estermann [40]; whereas Bambah, Woods and Zassenhaus provided three new proofs in [13]. A more recent example was obtained by Henk [59].

The result has been extended, for instance, to more general successive minima functionals by Hlawka [54, Section 9.5]; to more general discrete sets, not necessarily lattices, by Woods [106]; to intrinsic volumes by Henk [58]; or to surface area measures by Henk, Henze and Hernández Cifre [60].

In this chapter we will study and obtain inequalities that simultaneously relate the volume, the lattice point enumerator, and the successive minima (see Theorems 4.11 and 4.15), following conjectures by Betke, Henk and Wills (see Conjecture 4.3). These inequalities recover, in particular, all the classical inequalities by Minkowski, Blichfeldt and van der Corput, as well as (1) (see Remark 4.4). Furthermore, one powerful application of these inequalities will be a discrete analogue of Minkowski's 2nd Theorem for the lattice point enumerator (see Corollary 4.12). The key method to obtain these results will be a reduction to a special class of sets called "anti-blocking" (see Definition 4.8). The results of this chapter are collected in [43].

4.1 Discrete analogues of Minkowski's 2nd Theorem

Betke, Henk and Wills studied in [17] the relation between the lattice point enumerator and the successive minima of K and obtained, for origin-symmetric convex bodies, that

$$\frac{1}{n!}\prod_{i=1}^{n}\left(\frac{1}{\lambda_i(K)}-1\right) \le \mathrm{G_n}(K) \le \prod_{i=1}^{n}\left(\frac{2i}{\lambda_i(K)}+1\right),$$

where, for the lower bound, $\lambda_n(K) \le 2$ is needed. While the lower bound is best-possible, it is conjectured in [17, Conjecture 2.1] that the upper bound can be strengthened as follows:

Conjecture 4.2 — Betke, Henk, Wills. Let $K \subset \mathbb{R}^n$ be a convex body. Then

$$\mathrm{G_n}(K) \le \prod_{i=1}^{n}\left\lfloor\frac{2}{\lambda_i(K)}+1\right\rfloor. \tag{4.3}$$

Equality would be attained, e.g., for boxes of the form $[-m_1,m_1]\times\cdots\times[-m_n,m_n]$, where $m_i \in \mathbb{Z}_{>0}$. In dimension 2 the conjecture was confirmed by Betke, Henk and Wills themselves [17, Theorem 2.2], whereas in dimension 3 it was shown by Malikiosis [77, Section 3.2]. Moreover Malikiosis also proved in [77, Theorem 3.2.1] that

$$\mathrm{G_n}(K) \leq \frac{4}{e} 3^{(n-1)/2} \prod_{i=1}^{n} \left(\frac{2}{\lambda_i(K)} + 1 \right). \tag{4.4}$$

To this day, (4.4) is the best known upper bound for $\mathrm{G_n}(K)$ in terms of the successive minima in general dimension. Betke, Henk and Wills additionally pointed out in [17, Proposition 2.2] that any inequality of the form

$$\mathrm{G_n}(K) \leq \prod_{i=1}^{n} \left(\frac{2}{\lambda_i(K)} + c_i \right), \tag{4.5}$$

for some numbers $c_i \in \mathbb{R}$, $i = 1, \ldots, n$, independent of K (but not necessarily of n), would imply the upper bound in Minkowski's Second Theorem (4.2). Indeed, one can asymptotically approximate the volume of K by the lattice point enumerator using (1), to which (4.5) could then be applied, and the resulting limit is precisely Minkowski's bound.

In this paper, we use Minkowski's Second Theorem (Theorem N) to show (4.5) with $c_i = n$ (see Corollary 4.12). In order to do so, we aim to express the deviation between $\mathrm{G_n}(K)$ and $\mathrm{vol}(K)$ in terms of the successive minima $\lambda_i(K)$, $i = 1, \ldots, n$. Our approach stems from another conjecture by Betke, Henk and Wills that relates the volume, the lattice point enumerator and the successive minima simultaneously.

Conjecture 4.3 Let $K \subset \mathbb{R}^n$ be a convex body. Then,

$$\mathrm{G_n}(K) \leq \mathrm{vol}(K) \prod_{i=1}^{n} \left(1 + \frac{i\lambda_i(K)}{2} \right) \tag{4.6}$$

and, if $\lambda_n(K) \leq 2/n$,

$$\mathrm{G_n}(\mathrm{int}\, K) \geq \mathrm{vol}(K) \prod_{i=1}^{n} \left(1 - \frac{i\lambda_i(K)}{2} \right). \tag{4.7}$$

Moreover, if K is origin-symmetric and $\lambda_n \leq 2$, we have

$$\mathrm{G_n}(\mathrm{int}\, K) \geq \mathrm{vol}(K) \prod_{i=1}^{n} \left(1 - \frac{\lambda_i(K)}{2} \right). \tag{4.8}$$

On the one hand, the bound (4.8) is stated as Conjecture 2.2 in [17], where it is formulated for arbitrary n-dimensional lattices. However, there is no loss of generality in restricting to the integer lattice $\mathbb{Z}^n$. On the other hand, (4.6) and (4.7) have been posed to the authors by Martin Henk personally. In this chapter, we will confirm a slightly weakened version of these inequalities in the general n-dimensional setting (see Theorem 4.11), as well as obtain tight confirmations (either asymptotically, see Theorem 4.15, or in some special cases, completely, see Proposition 4.16) in the 2-dimensional setting.

All inequalities in Conjecture 4.3 have equality cases that are invariant with respect to integer scaling. Indeed, (4.6) is tight, e.g., for integer multiples of the standard simplex S_n, since $\lambda_i(S_n) = 2$ and therefore,

$$\mathrm{vol}(mS_n) \prod_{i=1}^{n} \left(1 + \frac{i\lambda_i(mS_n)}{2} \right) = \frac{1}{n!} \prod_{i=1}^{n} (m+i),$$

where the right-hand side is exactly the *Ehrhart polynomial* of S_n (see [15, Theorem 2.2 (a)]). In view of [15, Theorem 2.2 (b)], we also have

$$\mathrm{G_n}\big(\mathrm{int}(mS_n)\big) = \frac{1}{n!} \prod_{i=1}^{n} (m-i) = \frac{m^n}{n!} \prod_{i=1}^{n} \left(1 - \frac{i}{m} \right) = \mathrm{vol}(mS_n) \prod_{i=1}^{n} \left(1 - \frac{i\lambda_i(mS_n)}{2} \right)$$

and so (4.7) is tight for integer multiples of S_n as well. As it was mentioned already in [17], equality cases for (4.8) are given, for example, by boxes parallel to the coordinate axes with integer side lengths.

One immediate application of Conjecture 4.3 is that it would yield a discrete analogue of Minkowski's 2nd Theorem. Indeed, (4.6) together with the upper bound in Minkowski's 2nd Theorem (4.2) yields

$$\mathrm{G_n}(K) \leq \prod_{i=1}^{n} \left(\frac{2}{\lambda_i(K)} + i \right),$$

which, though weaker than (4.3), is nonetheless of the form (4.5), and thus, it would imply the upper bound in (4.2). Consequently, it would result in an equivalent discrete analogue of the upper bound in Minkowski's 2nd Theorem.

Remark 4.4 Apart from this discrete analogue, Conjecture 4.3 is interesting in its own right. On the one hand, one can deduce (1) from it, since $\lambda_i(rK)$ tends to 0 as $r \to \infty$. On the other hand, if K contains an n-dimensional set of lattice points it follows that $\lambda_i(K) \leq 2$, and if K is also origin-symmetric, one has $\lambda_i(K) \leq 1$ for all $i = 1, \ldots, n$. Therefore, from Conjecture 4.3 we retrieve the universal bounds

$$\mathrm{G_n}(K) \leq (n+1)!\ \mathrm{vol}(K),$$

in the general case, and

$$\mathrm{G_n}(\mathrm{int}\, K) \geq 2^{-n}\ \mathrm{vol}(K),$$

in the symmetric case, i.e., the classical results by Blichfeldt and van der Corput. ♦

4.2 Reduction of the problem to a simpler class of sets

The main goal of this section is to obtain results that will allow us to reduce the analysis of Conjecture 4.3 to a simpler class of convex bodies, the so-called *anti-blocking* convex bodies. For this, we will make use of a powerful tool, similar to Steiner's symmetrization, known as the *Blaschke shaking*, which is part of a wider class of transformations known as "shakings".

Before doing this, it is necessary to introduce the *gauge function*. If $K \subset \mathbb{R}^n$ is an origin-symmetric convex body, its gauge function $|\cdot|_K : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ is given by

$$|x|_K = \min\{r \geq 0 : x \in rK\}.$$

The gauge function of K is a norm in $\mathbb{R}^n$ whose unit ball is precisely K.

Remark 4.5 As an alternative to (4.1), one can use the gauge function of $\mathrm{cs}(K)$ to define the successive minima. Namely, one has that $\lambda_1(K) = \min |z|_{\mathrm{cs}(K)}$, where z ranges over $\mathbb{Z}^n \setminus \{0\}$, and $\lambda_i(K) = \min |z|_{\mathrm{cs}(K)}$, where z ranges over $\mathbb{Z}^n \setminus \mathrm{lin}\big(\lambda_{i-1}(K)\mathrm{cs}(K) \cap \mathbb{Z}^n\big)$ ♦

We finally note that if $\Lambda \subset \mathbb{R}^n$ is a lattice, then so is $P_{(\mathrm{lin} X)^\perp}\Lambda$ for every set $X \subset \Lambda$.

4.2.1 Properties of the Blaschke shaking

An important tool for the simplification of Conjecture 4.3 is the following one (see Figure 4.1):

Definition 4.6 — Blaschke shaking. Let $K \subset \mathbb{R}^n$ be a convex body and let $u \in \mathbb{R}^n \setminus \{0\}$. The Blaschke shaking of K with respect to $u^\perp$ is given by

$$\mathrm{sh}_u(K) = \bigcup_{x \in P_{u^\perp} K} \left[x, x + \mathrm{vol}_1\Big(\big(x + \mathrm{lin}\{u\}\big) \cap K \Big) \frac{u}{\|u\|} \right].$$

The Blaschke shaking was introduced in [20]. This process, which bares resemblance to Steiner's symmetrization, belongs to a wider class of transformations known as "shakings". These processes were explored, for instance, to obtain discrete isoperimetric inequalities by Kleitman [72], and more recently by Bollobás and Leader [22]. Stability results, akin to that of Gross for Steiner's symmetrization, were obtained by Biehl [19], Schöpf [98], and more recently, by Campi, Colesanti and Gronchi [31], for example. Other applications can be found in [30] and [102].

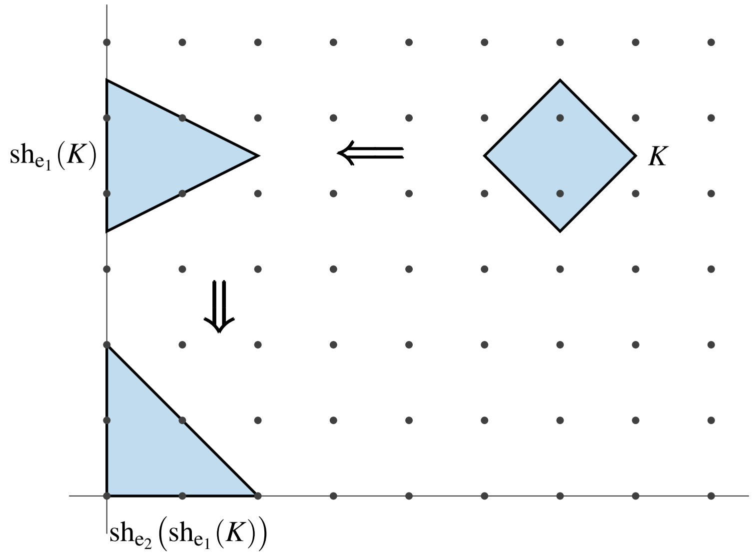

Figure 4.1: Illustration of two consecutive Blaschke shakings.

The operator sh_u is known to preserve convexity [31, Lemma 1.1], and we have the following result:

Proposition 4.7 Let $K \subset \mathbb{R}^n$ be a convex body and let $u \in \mathbb{R}^n \setminus \{0\}$. For the Blaschke shaking $\mathrm{sh}_u(K)$, the following relations hold:

i) $P_{u^\perp} K \subset \mathrm{sh}_u(K)$,
ii) $\mathrm{vol}(K) = \mathrm{vol}\big(\mathrm{sh}_u(K)\big)$,
iii) $|u|_{\mathrm{cs}(K)} = |u|_{\mathrm{cs}(\mathrm{sh}_u(K))}$ and
iv) $|x|_{\mathrm{cs}(K)} \geq |P_{u^\perp} x|_{\mathrm{cs}(\mathrm{sh}_u(K))}$ for all $x \in \mathbb{R}^n$.

If $u = \mathrm{e}_i$, for some $i \in \{1, \dots, n\}$, we also have

v) $\mathrm{G_n}(K) \leq \mathrm{G_n}\big(\mathrm{sh}_{\mathrm{e}_i}(K)\big)$,
vi) $\mathrm{G_n}(\mathrm{int}\, K) \geq \mathrm{G_n}\big(\mathrm{int}\, \mathrm{sh}_{\mathrm{e}_i}(K)\big)$.

Proof. i) and ii) follow directly from the definition of $\mathrm{sh}_u(K)$. For iii), if $u = (r/2)(x-y)$ for any $x, y \in \mathbb{R}^n$ and any $r \in \mathbb{R} \setminus \{0\}$, projecting onto $u^\perp$ yields $P_{u^\perp} x = P_{u^\perp} y$. For each $z \in P_{u^\perp} K$, and denoting by $\ell_z = \big(z + \mathrm{lin}\{u\}\big) \cap K$, let $d_z \in \mathbb{R}$ be such that $\ell_z = d_z u + \mathrm{sh}_u(\ell_z)$. Then $d_x = d_y$ and, by definition, $z \in K$ if and only if $z - d_z u \in \mathrm{sh}_u(K)$. Therefore, given $r > 0$ and considering

$$r\,\frac{x-y}{2} = r\,\frac{(x - d_x u) - (y - d_x u)}{2}$$

we obtain that $u \in r\,\mathrm{cs}(K)$ if and only if $u \in r\,\mathrm{cs}\big(\mathrm{sh}_u(K)\big)$, i.e. iii).

For iv) let $r = |x|_{\mathrm{cs}(K)}^{-1}$. Then there are $a, b \in K$ such that $rx = (a-b)/2$, and from i) it follows that

$$r P_{u^\perp} x = \frac{1}{2}\big(P_{u^\perp} a - P_{u^\perp} b\big) \in \mathrm{cs}\big(\mathrm{sh}_u(K)\big).$$

Thus $|r P_{u^\perp} x|_{\mathrm{cs}(\mathrm{sh}_u(K))} \leq 1$, which implies iv) by the choice of r.

In order to prove v), we show that the number of lattice points in a segment $[a,b]\subset\mathbb{R}$ of length $b-a$ is maximized, e.g., when $a\in\mathbb{Z}$. Otherwise, we could let $\delta=a-\lfloor a\rfloor$ and observe that

$$\mathrm{G}_1\big([a-\delta,b-\delta]\big)=\lfloor b-\delta\rfloor-\lceil a-\delta\rceil+1=\lfloor b-\delta\rfloor-\lfloor a\rfloor+1\geq\lfloor b\rfloor-\lfloor a\rfloor=\lfloor b\rfloor-\lceil a\rceil+1=\mathrm{G}_1\big([a,b]\big).$$

It is then enough to note that the lattice points in K and $\mathrm{sh}_{\mathrm{e}_i}(K)$ are contained in segments parallel to e_i of equal length, and that the ones in $\mathrm{sh}_{\mathrm{e}_i}(K)$ start at a lattice point. Therefore, they contain at least as many lattice points as the corresponding segments in K (cf. Figure 4.1).

vi) is proved with the same argument, but since the segments involved are now open, translating them such that one endpoint is a lattice point will never increase their lattice point count. ■

4.2.2 Properties of anti-blocking convex bodies

We now prove a series of useful properties for the following class of sets.

Definition 4.8 — Anti-blocking convex body. A convex body $K\subset\mathbb{R}^n_{\geq 0}$ is anti-blocking if for every $(x_1,\dots,x_n)\in K$,

$$\big\{(x'_1,\dots,x'_n)\in\mathbb{R}^n_{\geq 0}: x'_i\leq x_i,\ i=1,\dots,n\big\}\subset K.$$

Given the convexity of K, the condition above translates into $K\cap\mathrm{e}_i^\perp=P_{\mathrm{e}_i^\perp}K$ for all $i=1,\dots,n$. Equivalently, the anti-blocking convex bodies are exactly the first orthants of the unconditional convex bodies.

Anti-blocking convex bodies were introduced in [44]. Their volumes were extensively studied in [11]. We observe that, in the discrete setting, the set of lattice points $K\cap\mathbb{Z}^n$ inside an anti-blocking convex body K is a (downward) compressed set (see Definition 2.3). Compressed sets were considered in [50] (in the context of sum-set estimates) and in [86] (in the context of discrete isoperimetric inequalities), for instance.

The goal of this section is to prove the following statement, from which it follows that it is enough to show (4.6) and (4.7) for the special class of anti-blocking convex bodies.

Proposition 4.9 For any convex body $K\subset\mathbb{R}^n$, there exists an anti-blocking convex body $A\subset\mathbb{R}^n_{\geq 0}$ such that the following holds:

i) $\mathrm{vol}(K)=\mathrm{vol}(A)$,
ii) $\mathrm{G_n}(K)\leq\mathrm{G_n}(A)$,
iii) $\mathrm{G_n}(\mathrm{int}\,K)\geq\mathrm{G_n}(\mathrm{int}\,A)$ and
iv) $\lambda_i(K)\geq\lambda_i(A)$ for all $i=1,\dots,n$.

Proof. Let $v_1,\dots,v_n\in\mathbb{Z}^n$ be linearly independent such that $|v_i|_{\mathrm{cs}(K)}=\lambda_i(K)$. Since all the functionals involved are invariant with respect to integer unimodular transformations, we may assume that the matrix $(v_1\cdots v_n)$ is an upper triangular matrix (e.g., a Hermite normal form, see e.g. [99, Section 4.1]). Let $K_0=K$, and for $j\in\{1,\dots,n\}$, let $K_j=\mathrm{sh}_{\mathrm{e}_j}(K_{j-1})$. We will show that $A:=K_n$ is the desired body.

First, we observe that items i)–iii) follow immediately from a repeated application of items ii), v) and vi) of Proposition 4.7. In order to prove item iv), it suffices to show that for any $j=0,\dots,n$ there exist linearly independent vectors $u_1,\dots,u_n\in\mathbb{Z}^n$ such that $|u_i|_{\mathrm{cs}(K_j)}\leq\lambda_i(K)$, and such that the matrix $(u_1\cdots u_n)$ is of the form

$$\begin{pmatrix}D_j & 0\\ 0 & T_{n-j}\end{pmatrix},\tag{4.9}$$

where D_j is a $(j\times j)$-diagonal matrix and T_{n-j} is an $\big((n-j)\times(n-j)\big)$-upper triangular matrix; note that in this way we would get that $u_i\notin\mathrm{lin}\big(\lambda_{i-1}(K)\mathrm{cs}(K)\cap\mathbb{Z}^n\big)$, $i=2,\dots,n$ (see Remark 4.5).

For $j=0$, the statement is clearly true for $u_i = v_i$, $i=1,\dots,n$. So we assume that the claim holds for some $j-1<n$ and we apply induction. We choose $u_i' = P_{\mathrm{e}_j^\perp} u_i$, for $i \neq j$, and $u_j' = u_j$. On the one hand, since $K_j = \mathrm{sh}_{\mathrm{e}_j}(K_{j-1})$, then in view of Proposition 4.7 iv) and our induction hypothesis we have

$$|u_i'|_{\mathrm{cs}(K_j)} = \left|P_{\mathrm{e}_j^\perp} u_i\right|_{\mathrm{cs}(K_j)} \leq |u_i|_{\mathrm{cs}(K_{j-1})} \leq \lambda_i(K)$$

for all $i \neq j$. From Proposition 4.7 iii) it also follows that

$$|u_j'|_{\mathrm{cs}(K_j)} = |u_j|_{\mathrm{cs}(K_{j-1})} \leq \lambda_j(K).$$

On the other hand, the matrix $(u_1' \cdots u_n')$ differs from $(u_1 \cdots u_n)$ only by the zeros in the j-th row after the diagonal entry. Therefore, the system $u_1',\dots,u_n' \in \mathbb{Z}^n$ is also linearly independent and is of the form (4.9). Altogether, the claim is proved, and taking $j=n$ shows item iv).

It only remains to prove that A is indeed anti-blocking. To this end, it suffices to show that for any $j=0,\dots,n$ and any $x \in K_j$ we have $P_{\mathrm{e}_i^\perp} x \in K_j$ for all $i=1,\dots,j$.

For $j=0$ the statement is trivial. So we assume the claim holds for some $j-1 \in \{1,\dots,n-1\}$ and apply induction. Let $x \in K_j$. By item i) of Proposition 4.7 it follows that $P_{\mathrm{e}_j^\perp} x \in K_j$, and so, we consider $i \in \{1,\dots,j-1\}$. Now, let x_j be the j-th entry of x, and consider the point y with minimum j-th coordinate in $(x+\ell_j) \cap K_{j-1}$ (cf. Figure 4.2).

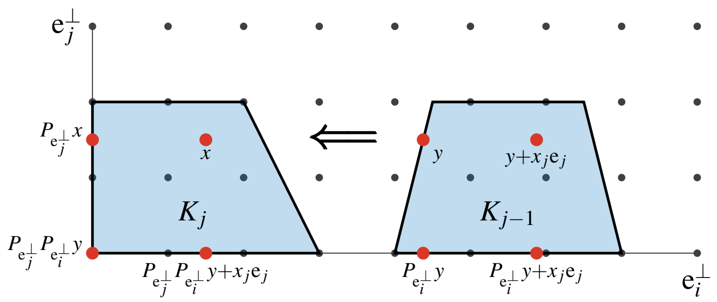

Figure 4.2: Construction for the proof that A is anti-blocking in Proposition 4.9.

Then, $[y, y+x_j\mathrm{e}_j] \subset K_{j-1}$. By induction, it follows that $\left[P_{\mathrm{e}_i^\perp} y, P_{\mathrm{e}_i^\perp} y + x_j \mathrm{e}_j\right] \subset K_{j-1}$. Due to the fact that $P_{\mathrm{e}_j^\perp} P_{\mathrm{e}_i^\perp} y = P_{\mathrm{e}_j^\perp} P_{\mathrm{e}_i^\perp} x$, we have that $\left[P_{\mathrm{e}_j^\perp} P_{\mathrm{e}_i^\perp} x, P_{\mathrm{e}_j^\perp} P_{\mathrm{e}_i^\perp} x + x_j \mathrm{e}_j\right] \subset K_j$, and since $P_{\mathrm{e}_j^\perp} P_{\mathrm{e}_i^\perp} x + x_j \mathrm{e}_j = P_{\mathrm{e}_i^\perp} x$, the statement is proved for j. Altogether, we get that the desired claim holds, and taking $j=n$ yields that A is anti-blocking. ■

The inductive proof of the second claim in the previous proof essentially corresponds to the argument given in the proof of [31, Lemma 1.2].

One of the reasons why anti-blocking convex bodies are beneficial when dealing with successive minima problems is that the successive minima are always realized by the canonical basis of $\mathbb{Z}^n$.

Lemma 4.10 Let $K \subset \mathbb{R}^n$ be an anti-blocking convex body. Then the coordinates can be permuted such that $|\mathrm{e}_i|_{\mathrm{cs}(K)} = \lambda_i(K)$. In this case, one also has $\big(2/\lambda_i(K)\big)\mathrm{e}_i \in K$, $i=1,\dots,n$.

Proof. Let $v_1,\dots,v_n \in \mathbb{Z}^n$ be linearly independent with $|v_i|_{\mathrm{cs}(K)} = \lambda_i(K)$, $i=1,\dots,n$. Then there exists a permutation σ of $\{1,\dots,n\}$ such that the σ_i-th entry of v_i is non-zero, since otherwise we would get $\det(v_1 \cdots v_n) = 0$, a contradiction. For the sake of simplicity we assume that σ is the identity. Since K is anti-blocking,

$$w_i := P_{\ell_i} v_i = P_{\bigcap_{j\neq i} \mathrm{e}_j^\perp} v_i \subset K,$$

and thus a repeated application of item iv) of Proposition 4.7 shows that $|w_i|_{\mathrm{cs}(K)} \leq |v_i|_{\mathrm{cs}(K)} = \lambda_i(K)$. By the minimality of the λ_i's and the fact that $w_i \in \ell_i \cap \mathbb{Z}^n$, we obtain that $w_i = \mathrm{e}_i$ and $|\mathrm{e}_i|_{\mathrm{cs}(K)} = \lambda_i(K)$.

For the second part, we deduce from $|\mathrm{e}_i|_{\mathrm{cs}(K)} = \lambda_i(K)$ that

$$\frac{1}{\lambda_i(K)}\mathrm{e}_i = \frac{1}{2}(a-b),$$

for some $a, b \in K$. Since $1/\lambda_i(K)$ is the maximum number r such that $r\mathrm{e}_i \in \mathrm{cs}(K)$, and K is anti-blocking, b_i must be zero. Therefore $b \in \mathrm{e}_i^{\perp}$, and the anti-blocking property again implies that

$$\frac{2}{\lambda_i(K)}\mathrm{e}_i = P_{\ell_i}\left(\frac{2}{\lambda_i(K)}\mathrm{e}_i + b\right) = P_{\ell_i}a \in K,$$

as desired. ∎

4.3 Slightly weakened n-dimensional results

The main result of this section is a slightly weakened version of Conjecture 4.3 in the general n-dimensional setting. In particular, we prove the following theorem, which provides a weakening of (4.6) and (4.7):

Theorem 4.11 [43, Theorem 1.1] Let $K \subset \mathbb{R}^n$ be a convex body. Then

$$\mathrm{G_n}(K) \leq \mathrm{vol}(K)\prod_{i=1}^{n}\left(1 + \frac{n\lambda_i(K)}{2}\right). \tag{4.10}$$

Moreover, if $\lambda_n(K) \leq 2/n$, we have

$$\mathrm{G_n}(\mathrm{int}\,K) \geq \mathrm{vol}(K)\prod_{i=1}^{n}\left(1 - \frac{n\lambda_i(K)}{2}\right). \tag{4.11}$$

As anticipated in Section 4.1, from this we can deduce immediately, by applying the upper bound in (4.2) to the volume in (4.10), the following inequality:

Corollary 4.12 Let $K \subset \mathbb{R}^n$ be a convex body. Then

$$\mathrm{G_n}(K) \leq \prod_{i=1}^{n}\left(\frac{2}{\lambda_i(K)} + n\right). \tag{4.12}$$

While our bound is tight for convex bodies rK, $r \to \infty$, it is weaker than Malikiosis's bound (4.4) if, e.g., $\lambda_i(K) = 1/c$ for some fixed number $c > 0$. Then the bound in (4.12) is of order n^n, while the bound in (4.4) is of order $\sqrt{3}^n$.

We will use the following result due to van der Corput (see [54, Ch.2, Theorem 6.1]):

Theorem O Let $M \subset \mathbb{R}^n$ be a Jordan-measurable set. Then there exists a vector $z \in \mathbb{R}^n$ such that

$$\mathrm{vol}(M) \leq \mathrm{G_n}(M+z). \tag{4.13}$$

We first provide the following bounds in terms of the covering radius $\mu(K)$ of a convex body $K \subset \mathbb{R}^n$, i.e., the smallest number $\mu > 0$ such that $\mu K + \mathbb{Z}^n = \mathbb{R}^n$.

Proposition 4.13 Let $K \subset \mathbb{R}^n$ be a convex body. Then

$$\mathrm{G_n}(K) \le \mathrm{vol}(K)\big(1+\mu(K)\big)^n. \tag{4.14}$$

If $\mu(K) \le 1$, i.e., $K + \mathbb{Z}^n = \mathbb{R}^n$, we also have

$$\mathrm{G_n}(\mathrm{int}\, K) \ge \mathrm{vol}(K)\big(1-\mu(K)\big)^n. \tag{4.15}$$

Both inequalities are tight.

Proof. We write $\mu = \mu(K)$ for the sake of brevity. For the upper bound, it is enough to show that μK contains a measurable set S that $\mathbb{Z}^n$-tiles $\mathbb{R}^n$, i.e., with $S + \mathbb{Z}^n = \mathbb{R}^n$ and $(\mathrm{int}\, S) \cap (z + \mathrm{int}\, S) = \emptyset$ for all $z \in \mathbb{Z}^n \setminus \{0\}$. Indeed, in that case, since $\mathrm{vol}(S) = 1$ (see, e.g., [54, Theorem 1, page 82]), we have

$$\mathrm{G_n}(K) = \sum_{z \in K \cap \mathbb{Z}^n} \mathrm{vol}(S) = \mathrm{vol}\big((K \cap \mathbb{Z}^n) + S\big) \le \mathrm{vol}(K + \mu K) = (1+\mu)^n\, \mathrm{vol}(K).$$

In order to find S, let $P = [0,1]^n$. There are finitely many translates $\mu K + x_i$, $x_i \in \mathbb{Z}^n$, $i = 1, \dots, m$, that cover P. We define inductively $P_1 = P \cap (\mu K + x_1)$ and

$$P_i = \left(P \setminus \bigcup_{j<i} P_j\right) \cap (\mu K + x_i).$$

Now, let $S_i = P_i - x_i \subset \mu K$ and $S = \bigcup_{i=1}^m S_i$. We claim that S is the desired set. To prove this, we show that S has volume 1 and that its $\mathbb{Z}^n$-translates do not overlap (see again [54, Theorem 1, page 82]).

Clearly the P_i's are non-overlapping, i.e., $(\mathrm{int}\, P_i) \cap (\mathrm{int}\, P_j) = \emptyset$, and satisfy $\bigcup_{i=1}^m P_i = P$. The S_i's are non-overlapping too. Indeed, if there were $i \ne j$ such that $\mathrm{int}\, S_i$ intersected $\mathrm{int}\, S_j$, then $\mathrm{int}\, P_i$ would intersect $x_i - x_j + \mathrm{int}\, P_j$. Since the $\mathbb{Z}^n$-translates of P are non-overlapping, we would have $x_i = x_j$, a contradiction. Therefore the S_i's are non-overlapping and it follows that

$$\mathrm{vol}(S) = \sum_{i=1}^m \mathrm{vol}(S_i) = \sum_{i=1}^m \mathrm{vol}(P_i) = \mathrm{vol}(P) = 1.$$

Now assume that $(\mathrm{int}\, S) \cap (x + \mathrm{int}\, S) \ne \emptyset$ for some $x \in \mathbb{Z}^n$. Then there exist $i, j \in \{1, \dots, m\}$ such that $-x_i + \mathrm{int}\, P_i$ intersects $-x_j + x + \mathrm{int}\, P_j$. Again, since the $\mathbb{Z}^n$-translates of P are non-overlapping, as well as the P_i's, we must have $i = j$ and $x = 0$. Hence, the $\mathbb{Z}^n$-translates of S are non-overlapping, and so S is as desired. This finishes the proof of the upper bound.

For the lower bound, we apply (4.13) to $K' = (1-\mu)\,\mathrm{int}\, K$ and obtain a vector $z \in \mathbb{R}^n$ such that $\mathrm{vol}(K') \le \mathrm{G_n}(K' + z)$. Since $\mu K + \mathbb{Z}^n = \mathbb{R}^n$, we may assume that $z \in \mu K$. Thus,

$$\mathrm{vol}(K)(1-\mu)^n = \mathrm{vol}(K') \le \mathrm{G_n}(K' + z) \le \mathrm{G_n}\big((1-\mu)\,\mathrm{int}\, K + \mu K\big) = \mathrm{G_n}(\mathrm{int}\, K).$$

In order to see that both inequalities are tight, consider $K = [0,m]^n$, where $m \in \mathbb{Z}_{>0}$. For such cubes one has $\mathrm{vol}(K) = m^n$, $\mathrm{G_n}(K) = (m+1)^n$, $\mathrm{G_n}(\mathrm{int}\, K) = (m-1)^n$ and $\mu(K) = 1/m$. So equality is achieved in both bounds. ∎

The upper bound (4.14) was also shown independently by Dadush in [36, Lemma 7.4.1]. The strategy of finding an appropriate tiling that we used in the proof of (4.14) was also applied earlier, for instance, in the proof of a classical result by Blichfeldt (see [54, Chapter 2, Theorem 5.2]). Moreover, in [107], the authors showed that convex tilings in these conditions (i.e., of the form $S + \mathbb{Z}^n$ with $S \subset K$, where $K + \mathbb{Z}^n$ is a prescribed covering) need not exist.

On the one hand, the disadvantage of (4.14) in comparison to the upper bound (4.10) is that it cannot benefit from K being large in a lattice subspace. To see this, it suffices to consider the convex body

$K = [-r,r]^{n-1} \times [-1/2,1/2]$, where r is "large". Then it holds that $\mu(K) = 1$, and so the constant in (4.14) is 2^n. But the constant in (4.10) is of order $n+1$, since $\lambda_i(K) \to 0$ as $r \to \infty$ for $i < n$.

On the other hand, (4.15) is actually stronger than the lower bound (4.11) in Theorem 4.11. In fact, we will use (4.15) to prove (4.11).

Remark 4.14 Applied to the special class of convex lattice tiles, i.e., convex bodies K with $K + \mathbb{Z}^n = \mathbb{R}^n$ and $(\operatorname{int} K) \cap (z + \operatorname{int} K) = \emptyset$ for all $z \in \mathbb{Z}^n \setminus \{0\}$, Proposition 4.13 yields, for $r \geq 1$, that

$$(r-1)^n \leq \mathrm{G_n}\big(\operatorname{int}(rK)\big) \quad \text{and} \quad \mathrm{G_n}(rK) \leq (r+1)^n,$$

since $\operatorname{vol}(K) = \mu(K) = 1$ and $\mu(rK) = \mu(K)/r$, which is sharp for $K = [0,1]^n$ and $r \in \mathbb{Z}_{>0}$. ♦

We will also need to make use of an inequality of Davenport (see [39]), which states that for any convex body $K \subset \mathbb{R}^n$ one has the bound

$$\mathrm{G_n}(K) \leq \sum_{I \subset \{1,\ldots,n\}} \operatorname{vol}_{n-|I|}\big(P_{\ell_I^\perp} K\big), \tag{4.16}$$

where $\ell_I = \operatorname{lin}\{\mathrm{e}_i : i \in I\}$. We note that the extremal terms in the above expression are $\operatorname{vol}(K)$ (corresponding to $I = \emptyset$, by setting $\ell_\emptyset = \{0\}$) and $1 = \operatorname{vol}_0\big(\{0\}\big)$ (corresponding to $I = \{1,\ldots,n\}$). The same convention will be used from now on. For the sake of consistency, we will further convene that an empty sum be equal to 0 and an empty product be equal to 1.

Since K will be anti-blocking in our case, (4.16) can also be derived directly as follows:

$$\mathrm{G_n}(K) = \operatorname{vol}\big((K \cap \mathbb{Z}^n) + [-1,0]^n\big) \leq \operatorname{vol}\big(K + [-1,0]^n\big) = \sum_{I \subset \{1,\ldots,n\}} \operatorname{vol}_{n-|I|}\big(P_{\ell_I^\perp} K\big),$$

where the last identity follows from the fact that, since K is anti-blocking, the Minkowski sum can be decomposed into a union of disjoint prisms:

$$K + [-1,0]^n = \bigcup_{I \subset \{1,\ldots,n\}} \big\{x = (x_1,\ldots,x_n) \in \mathbb{R}^n : P_{\ell_I^\perp} x \in P_{\ell_I^\perp} K \text{ and } x_i \in [-1,0] \text{ for all } i \in I\big\}.$$

We are now ready to prove Theorem 4.11.

Proof of Theorem 4.11. Let us denote by $\lambda_i = \lambda_i(K)$, $i = 1,\ldots,n$, for the sake of brevity. In order to prove (4.10), we may suppose that K is anti-blocking by Proposition 4.9. After renumbering the coordinates, we can also assume that $|\mathrm{e}_i|_{\mathrm{cs}(K)} = \lambda_i$ holds for all $i = 1,\ldots,n$ (see Lemma 4.10). For any set $I \subset \{1,\ldots,n\}$, let $\ell_I = \operatorname{lin}\{\mathrm{e}_i : i \in I\}$. The Rogers-Shephard inequality (3.3) then yields

$$\operatorname{vol}_k(K \cap \ell_I) \operatorname{vol}_{n-k}\big(P_{\ell_I^\perp} K\big) \leq \binom{n}{k} \operatorname{vol}(K), \tag{4.17}$$

for any $I \subset \{1,\ldots,n\}$ with $|I| = k$, $k \in \{0,\ldots,n\}$. By Lemma 4.10 we have $(2/\lambda_i)\mathrm{e}_i \in K$ and so, from (4.17), we deduce that

$$\begin{aligned} \operatorname{vol}_{n-k}\big(P_{\ell_I^\perp} K\big) &\leq \binom{n}{k} \frac{\operatorname{vol}(K)}{\operatorname{vol}_k(K \cap \ell_I)} \leq \binom{n}{k} \frac{\operatorname{vol}(K)}{\operatorname{vol}_k\Big(\operatorname{conv}\big\{(2/\lambda_i)\mathrm{e}_i : i \in I\big\}\Big)} \\ &= k!\binom{n}{k} \operatorname{vol}(K) \prod_{i \in I} \frac{\lambda_i}{2} \leq \operatorname{vol}(K) \prod_{i \in I} \frac{n\lambda_i}{2}. \end{aligned}$$

Combining this with Davenport's inequality (4.16) we get

$$\mathrm{G_n}(K) \leq \operatorname{vol}(K) \sum_{I \subset \{1,\ldots,n\}} \prod_{i \in I} \frac{n\lambda_i}{2} = \operatorname{vol}(K) \prod_{i=1}^{n} \left(1 + \frac{n\lambda_i}{2}\right)$$

as desired. In order to prove (4.11), we use the lower bound (4.15) in terms of $\mu(K)$, as well as the relation

$$\mu(K) \le \sum_{i=1}^{n} \frac{\lambda_i}{2} \tag{4.18}$$

(see, e.g., [70, Lemma 2.4]). Since $\lambda_n \le 2/n$, (4.18) yields $\mu(K) \le 1$. Thus, we may apply Proposition 4.13 to obtain

$$\begin{aligned} \mathrm{G_n}(\operatorname{int} K) &\ge \operatorname{vol}(K)\big(1-\mu(K)\big)^n \ge \operatorname{vol}(K)\left(1-\sum_{i=1}^{n}\frac{\lambda_i}{2}\right)^n \\ &= \operatorname{vol}(K)\left(\frac{1}{n}\sum_{i=1}^{n}\left(1-\frac{n\lambda_i}{2}\right)\right)^n \ge \operatorname{vol}(K)\prod_{i=1}^{n}\left(1-\frac{n\lambda_i}{2}\right), \end{aligned}$$

where the last step follows from the inequality (6) between the arithmetic and geometric means. ■

4.4 The stronger 2-dimensional results

The main result of this section is the following theorem, which confirms Conjecture 4.3 in the planar case, either exactly in the case of the upper bound (4.6), or asymptotically in the case of the lower bound (4.7).

Theorem 4.15 [43, Theorem 1.4] Let $K \subset \mathbb{R}^2$ be a planar convex body. Then

$$\mathrm{G}_2(K) \le \operatorname{vol}(K)\left(1+\frac{\lambda_1(K)}{2}\right)\big(1+\lambda_2(K)\big) \tag{4.19}$$

and

$$\mathrm{G}_2(\operatorname{int} K) \ge \operatorname{vol}(K)\left(1-\frac{\lambda_1(K)}{2}-\lambda_2(K)\right). \tag{4.20}$$

In particular, for any $\varepsilon > 0$, if $\lambda_1(K) \le 2\varepsilon/(1+\varepsilon)$ it follows from (4.20) that

$$\mathrm{G}_2(\operatorname{int} K) \ge \operatorname{vol}(K)\left(1-\frac{\lambda_1(K)}{2}\right)\big(1-(1+\varepsilon)\lambda_2(K)\big),$$

which is the announced asymptotic confirmation.

Before proving this result though, we show that (4.8) can in fact be confirmed exactly for the special class of origin-symmetric lattice polygons (i.e. origin-symmetric sets which are the convex hull of finitely many integer lattice points).

Proposition 4.16 Let $P \subset \mathbb{R}^2$ be a planar origin-symmetric lattice polygon. Then we have

$$\mathrm{G}_2(\operatorname{int} P) \ge \operatorname{vol}(P)\left(1-\frac{\lambda_1(P)}{2}\right)\left(1-\frac{\lambda_2(P)}{2}\right).$$

Proof. On the one hand, Pick's Theorem (see, e.g., [15, Theorem 2.8]) states that for a lattice polygon P one has

$$\mathrm{G}_2(\operatorname{int} P) = \operatorname{vol}(P) - \frac{\mathrm{G}_2(\operatorname{bd} P)}{2} + 1. \tag{4.21}$$

On the other hand, an inequality of Henk, Schürmann and Wills ([61, (1.6)]) for origin-symmetric lattice polygons yields

$$\frac{\mathrm{G}_2(\operatorname{bd} P)}{2} \le \operatorname{vol}(P)\left(\frac{\lambda_1(P)}{2}+\frac{\lambda_2(P)}{2}\right). \tag{4.22}$$

Combining (4.21) with (4.22) we obtain

$$\mathrm{G}_2(\operatorname{int} P) \geq \operatorname{vol}(P) - \operatorname{vol}(P)\left(\frac{\lambda_1(P)}{2} + \frac{\lambda_2(P)}{2}\right) + 1.$$

By the upper bound in Minkowski's second theorem (4.2), we have $1 \geq \operatorname{vol}(P)\lambda_1(P)\lambda_2(P)/4$. Hence,

$$\mathrm{G}_2(\operatorname{int} P) \geq \operatorname{vol}(P)\left(1 - \frac{\lambda_1(P)}{2} - \frac{\lambda_2(P)}{2} + \frac{\lambda_1(P)\lambda_2(P)}{4}\right) \geq \operatorname{vol}(P)\left(1 - \frac{\lambda_1(P)}{2}\right)\left(1 - \frac{\lambda_2(P)}{2}\right),$$

and the proof is finished. ■

Now, in order to prove the general planar results established in Theorem 4.15, we need to take the reduction developed in Section 4.2 a step further.

4.4.1 Non-orthogonal shakings

We extend the notion of the Blaschke shaking to the setting of non-orthogonal projections. This will enable us to transform a convex body K in such a way that it is not only anti-blocking, but in addition, also located below the diagonal line passing through $(2/\lambda_1(K))\mathrm{e}_1$ and $(2/\lambda_1(K))\mathrm{e}_2$ (see Figure 4.3).

For any affine line $\ell \subset \mathbb{R}^2$ and any vector $u \in \mathbb{R}^2 \setminus \{0\}$ which is not parallel to ℓ, let $\pi_{u,\ell}$ denote the projection onto ℓ in the direction of u.

Definition 4.17 Let $K \subset \mathbb{R}^2$ be a planar convex body and fix $u \in \mathbb{R}^2 \setminus \{0\}$. Let $\ell \subset \mathbb{R}^2$ be an affine line that is not parallel to u. Then, the shaking of K with respect to ℓ along u is given by

$$\mathrm{sh}_{u,\ell}(K) = \bigcup_{x \in \pi_{u,\ell}(K)} \left[x, x + \operatorname{vol}_1\left(\left(x + \operatorname{lin}\{u\}\right) \cap K\right)\frac{u}{\|u\|}\right].$$

Clearly, $\mathrm{sh}_u = \mathrm{sh}_{u,u^\perp}$. We will use the fact that non-orthogonal Blaschke shakings are monotonous, which is a widely known fact for classical orthogonal Blaschke shakings (see, e.g., [31, Lemma 1.1 (iii)]).

Lemma 4.18 Let $K, L \subset \mathbb{R}^2$ be planar convex bodies with $K \subset L$. Also, consider an affine line $\ell \subset \mathbb{R}^2$ and a vector $u \in \mathbb{R}^n \setminus \{0\}$ not parallel to ℓ. Then we have

$$\mathrm{sh}_{u,\ell}(K) \subset \mathrm{sh}_{u,\ell}(L).$$

Proof. If $x \in \mathrm{sh}_{u,\ell}(K)$, then clearly $x \in \pi_{u,\ell}(K) \subset \pi_{u,\ell}(L)$. By inclusion, we also have

$$l_1 := \operatorname{vol}_1\left(\left(x + \operatorname{lin}\{u\}\right) \cap K\right) \leq \operatorname{vol}_1\left(\left(x + \operatorname{lin}\{u\}\right) \cap L\right) =: l_2.$$

Hence, since $x \in \mathrm{sh}_{u,\ell}(K)$, we have

$$x \in \left[\pi_{u,\ell}(x), \pi_{u,\ell}(x) + l_1 \frac{u}{\|u\|}\right] \subset \left[\pi_{u,\ell}(x), \pi_{u,\ell}(x) + l_2 \frac{u}{\|u\|}\right] \subset \mathrm{sh}_{u,\ell}(L),$$

as desired. ■

As we saw in Section 4.2, it is enough to prove Theorem 4.15 when K is anti-blocking. Starting with an anti-blocking convex body K that satisfies $|\mathrm{e}_i|_{\mathrm{cs}(K)} = \lambda_i(K)$, $i = 1,2$ (see Lemma 4.10), here we construct a new convex body A by first shaking K vertically and then horizontally with respect to a lattice diagonal of the form

$$D = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 = m\right\}$$

for some $m \in \mathbb{Z}$, and finally shaking back down to $e_2^\perp$. More formally, we set $A = T(K)$, where

$$T = \mathrm{sh}_{e_2} \circ \mathrm{sh}_{-e_1,D} \circ \mathrm{sh}_{-e_2,D}.$$

Note that the value $m \in \mathbb{Z}$ may be chosen arbitrarily since the parameters involved are invariant by lattice translations.

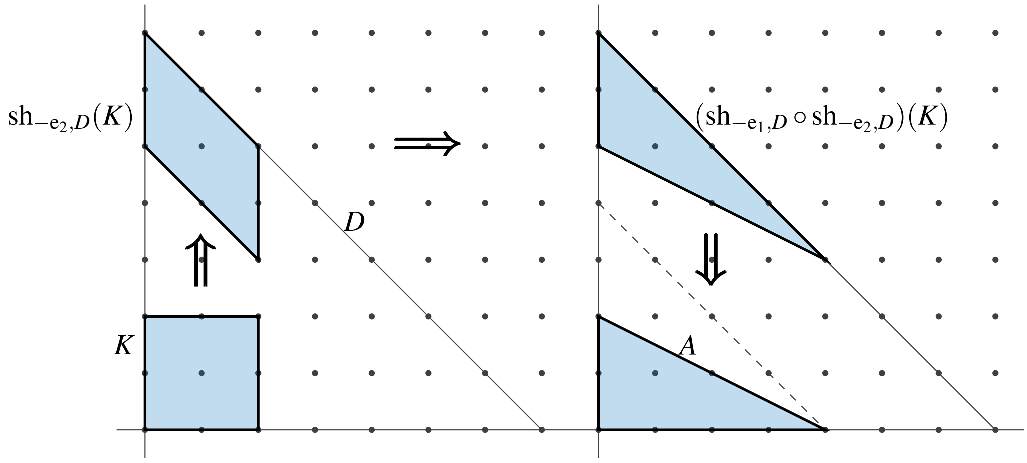

Figure 4.3: Illustration of the shaking process T.

Lemma 4.19 Let $K \subset \mathbb{R}^2$ be a planar convex body and set $A = T(K)$. Then the following statements hold:

i) A is convex,
ii) A is anti-blocking,
iii) $\mathrm{vol}(A) = \mathrm{vol}(K)$,
iv) $\mathrm{G}_2(A) \geq \mathrm{G}_2(K)$,
v) $\mathrm{G}_2(\mathrm{int}A) \leq \mathrm{G}_2(\mathrm{int}K)$,
vi) $\lambda_1(A) \leq \lambda_1(K)$,
vii) $\lambda_2(A) = \lambda_2(K)$ and
viii) $A \subset \{(x_1,x_2) \in \mathbb{R}^2 : x_1 + x_2 \leq 2/\lambda_1(A)\}$.

Proof. For i) we show that $\mathrm{sh}_{u,\ell}(K)$ is convex for all $u \in \mathbb{R}^2 \setminus \{0\}$ and every affine line $\ell \subset \mathbb{R}^2$ not parallel to u. To see this, we consider $x, y \in \mathrm{sh}_{u,\ell}(K)$. Let $\bar{x}$ and $\bar{y}$ denote the extreme points of the segments $K \cap (x + \mathrm{lin}\{u\})$ and $K \cap (y + \mathrm{lin}\{u\})$, respectively, that minimize $\langle \cdot, u \rangle$. Then, the points

$$\tilde{z} = \bar{z} + \|z - \pi_{u,\ell}(z)\| \frac{u}{\|u\|}, \quad z \in \{x, y\},$$

are contained in K, and thus, Lemma 4.18 implies that

$$\mathrm{conv}\{\pi_{u,\ell}(x), \pi_{u,\ell}(y), x, y\} = \mathrm{sh}_{u,\ell}(\mathrm{conv}\{\bar{x}, \tilde{x}, \bar{y}, \tilde{y}\}) \subset \mathrm{sh}_{u,\ell}(K).$$

In particular, $[x,y] \subset \mathrm{sh}_{u,\ell}(K)$, which shows that $\mathrm{sh}_{u,\ell}(K)$ is convex, and therefore, that A is convex too.

Next we consider, on the one hand, the box

$$B = \left[0, \frac{2}{\lambda_1(K)}\right] \times \left[0, \frac{2}{\lambda_2(K)}\right].$$

Clearly, we have $K \subset B$, and by Lemma 4.18, it follows that $A \subset T(B)$. Both $\mathrm{sh}_{-e_2,D}$ and $\mathrm{sh}_{e_2,D}$ do not change the length of the vertical segments, whereas it is easy to see that $\mathrm{sh}_{-e_1,D}$ cannot increase them either (cf. Figure 4.3 for $K=B$). Since the vertical segments of B are all of length $2/\lambda_2(K)$, we conclude that all vertical segments in $T(B)$ (and thus also in A) are of length at most $2/\lambda_2(K)$.

On the other hand, by considering the triangle

$$\Delta = \operatorname{conv}\left\{0, \frac{2}{\lambda_1(K)}\mathrm{e}_1, \frac{2}{\lambda_2(K)}\mathrm{e}_2\right\} \subset K,$$

which verifies $T(\Delta)=\Delta$ due to the fact that $\lambda_1(K)\leq\lambda_2(K)$, we see that $A\cap\ell_2$ has length precisely $2/\lambda_2(K)$. Since by construction we have $A\cap\mathrm{e}_2^\perp = P_{\mathrm{e}_2^\perp}A$, we obtain from this that $A\cap\mathrm{e}_1^\perp = P_{\mathrm{e}_1^\perp}A$ as well. Therefore, A is anti-blocking and satisfies $|\mathrm{e}_2|_{\mathrm{cs}(A)} = \lambda_2(K)$, together with $|\mathrm{e}_1|_{\mathrm{cs}(A)} \leq |\mathrm{e}_1|_{\mathrm{cs}(\Delta)} = \lambda_1(K)$. Altogether, we have proved items ii), vi) and vii).

iii) follows from Fubini's theorem applied to $\mathrm{lin}\{u\}$ and $u^\perp$, since also for arbitrary non-orthogonal shakings one has that $P_{u^\perp}\mathrm{sh}_{u,\ell}(K) = P_{u^\perp}K$ and

$$\mathrm{vol}_1\Big(\mathrm{sh}_{u,\ell}(K)\cap\big(x+\mathrm{lin}\{u\}\big)\Big) = \mathrm{vol}_1\Big(K\cap\big(x+\mathrm{lin}\{u\}\big)\Big)$$

for any $x\in u^\perp$.

iv) and v) are proved in the same way as items v) and vi) of Proposition 4.7: since $\pi_{-\mathrm{e}_i,D}(\mathbb{Z}^2)=\mathbb{Z}^2\cap D$ for $i=1,2$, all the segments in the direction of e_i of the set $\mathrm{sh}_{-\mathrm{e}_i,D}(K)$ containing lattice points have an endpoint in $D\cap\mathbb{Z}^2$.

Finally, for viii), let $z\in D$ be the point with minimum 2nd coordinate in $\mathrm{sh}_{-\mathrm{e}_1,D}\big(\mathrm{sh}_{-\mathrm{e}_2,D}(K)\big)$. Then

$$\mathrm{sh}_{-\mathrm{e}_1,D}\big(\mathrm{sh}_{-\mathrm{e}_2,D}(K)\big) \subset \operatorname{conv}\{m\mathrm{e}_2, P_{\mathrm{e}_1^\perp}z, z\},$$

where $m\in\mathbb{Z}$ is such that $m\mathrm{e}_2\in D$ (cf. Figure 4.3), and applying $\mathrm{sh}_{\mathrm{e}_2}$ to both sides of the inclusion yields

$$A\subset\operatorname{conv}\left\{0, \frac{2}{\lambda_1(A)}\mathrm{e}_1, \frac{2}{\lambda_1(A)}\mathrm{e}_2\right\} \subset \left\{(x_1,x_2)\in\mathbb{R}^2 : x_1+x_2\leq\frac{2}{\lambda_1(A)}\right\},$$

as desired. ∎

4.4.2 Proof of Theorem 4.15

For the proof of Theorem 4.15 we will need the following estimates, which follow from elementary properties of concave functions:

Lemma 4.20 Let $f:[a,b]\longrightarrow\mathbb{R}$ be a concave function. Then, we have

$$\int_a^b f(t)\,\mathrm{d}t \geq \frac{1}{2}\big(f(a)+f(b)\big)(b-a).$$

Moreover, if $f'(a)$ exists, then also

$$\int_a^b f(t)\,\mathrm{d}t \leq (b-a)\left(f(a)+\frac{1}{2}(b-a)f'(a)\right).$$

Proof. For the lower bound, let g be the affine function given by $g(a)=f(a)$ and $g(b)=f(b)$, i.e.,

$$g(t) = \frac{f(b)-f(a)}{b-a}(t-a)+f(a).$$

By concavity we have $f\geq g$, and therefore

$$\int_a^b f(t)\,\mathrm{d}t \geq \int_a^b g(t)\,\mathrm{d}t = \frac{1}{2}\big(f(a)+f(b)\big)(b-a).$$

For the upper bound let h be the tangent of f at a, i.e.,

$$h(t) = f'(a)(t-a) + f(a).$$

Again, by concavity we get $h \geq f$, and thus

$$\int_a^b f(t)\,\mathrm{d}t \leq \int_a^b h(t)\,\mathrm{d}t \leq (b-a)\left(f(a) + \frac{1}{2}(b-a)f'(a)\right). \qquad \blacksquare$$

Before proving Theorem 4.15 we need one more ingredient, which is in fact valid in general dimension. Schymura generalized Davenport's inequality (4.16) and obtained, for an arbitrary linearly independent set $\{b_1, \ldots, b_n\} \subset \mathbb{Z}^n$, that

$$\mathrm{G_n}(K) \leq \sum_{I \subset \{1,\ldots,n\}} \mathrm{vol}_{n-|I|}\left(P_{\ell_I^\perp} K\right) \mathrm{vol}_k(P_I),$$

where $\ell_I = \mathrm{lin}\{b_i : i \in I\}$ and $P_I = \sum_{i\in I}[0, b_i]$ (see [62, Lemma 1.1]). We reverse it in the following way.

Theorem 4.21 [43, Theorem 1.3] Let $K \subset \mathbb{R}^n$ be a convex body and let $\mathcal{B} = \{b_1, \ldots, b_n\} \subset \mathbb{Z}^n$ be a linearly independent set. Then

$$\mathrm{vol}(K) \leq \sum_{I \subset \{1,\ldots,n\}} \mathrm{G}_{P_{\ell_I^\perp}\mathbb{Z}^n}\left(P_{\ell_I^\perp}(\mathrm{int}\, K)\right), \tag{4.23}$$

where $\ell_I = \mathrm{lin}\{b_i : i \in I\}$. The inequality is tight.

Proof. It suffices to prove that, given a general n-dimensional lattice $\Lambda \subset \mathbb{R}^n$ and a linearly independent set $\mathcal{B} = \{b_1, \ldots, b_n\} \subset \Lambda$, then, for every convex and bounded (not necessarily closed) set $K \subset \mathbb{R}^n$ and any $z = (z_1, \ldots, z_n) \in \mathbb{R}^n$,

$$\mathrm{G}_\Lambda(K+z) \leq \sum_{I \subset \mathrm{supp}_{\mathcal{B}}(z)} \mathrm{G}_{P_{\ell_I^\perp}\Lambda}\left(P_{\ell_I^\perp} K\right), \tag{4.24}$$

where $\mathrm{supp}_{\mathcal{B}}(z) = \left\{i \in \{1, \ldots, n\} : \alpha_i \neq 0\right\}$ for $z = \sum_{i=1}^n \alpha_i b_i$. Indeed, (4.24) for $\Lambda = \mathbb{Z}^n$ yields

$$\mathrm{G_n}(z + \mathrm{int}\, K) \leq \sum_{I \subset \{1,\ldots,n\}} \mathrm{G}_{P_{\ell_I^\perp}\mathbb{Z}^n}\left(P_{\ell_I^\perp}(\mathrm{int}\, K)\right)$$

for any $z \in \mathbb{R}^n$, and hence (4.23) follows from (4.13).

If $z = 0$ there is only one term in (4.24) corresponding to $I = \emptyset$, and so (4.24) reads as $\mathrm{G}_\Lambda(K) \leq \mathrm{G}_\Lambda(K)$, a tautology. Thus, from now on we assume that $z \neq 0$.

First we note that if $n = 1$, then (4.24) states for non-zero z that

$$\mathrm{G}_\Lambda(K+z) \leq \mathrm{G}_\Lambda(K) + 1. \tag{4.25}$$

Since any convex body $K \subset \mathbb{R}$ is an interval, the statement is trivial. Now, for any $n > 1$, we will prove (4.24) by induction on $\left|\mathrm{supp}_{\mathcal{B}}(z)\right|$. If $\left|\mathrm{supp}_{\mathcal{B}}(z)\right| = 1$ then $z = \alpha_1 b_1$ for some $\alpha_1 \neq 0$, and thus

$$\mathrm{G}_\Lambda(K+z) = \sum_{x \in (P_{b_1^\perp} K) \cap (P_{b_1^\perp}\Lambda)} \mathrm{G}_\Lambda\Big((K+z) \cap \big(x + \mathrm{lin}\{b_1\}\big)\Big).$$

Since the bodies on the right-hand side are segments parallel to b_1, we can apply (4.25) to obtain

$$\mathrm{G}_\Lambda(K+z) \leq \sum_{x \in (P_{b_1^\perp} K) \cap (P_{b_1^\perp}\Lambda)} \left(\mathrm{G}_\Lambda\Big(K \cap \big(x + \mathrm{lin}\{b_1\}\big)\Big) + 1\right) = \mathrm{G}_\Lambda(K) + \mathrm{G}_{P_{b_1^\perp}\Lambda}\left(P_{b_1^\perp} K\right), \tag{4.26}$$

which corresponds to (4.24) in this case.

Finally, let $z=\sum_{i=1}^{n}\alpha_i b_i$ be an arbitrary non-zero vector, and consider any $j\in\operatorname{supp}_{\mathcal{B}}(z)$. We define $z'=z-\alpha_j b_j$ and $z''=P_{b_j^\perp}z'$, as well as $\mathcal{B}'=\mathcal{B}\setminus\{b_j\}$ and $\mathcal{B}''=P_{b_j^\perp}\mathcal{B}'$. Then, we observe that

$$\operatorname{supp}_{\mathcal{B}}(z')=\operatorname{supp}_{\mathcal{B}''}(z'')=\operatorname{supp}_{\mathcal{B}}(z)\setminus\{j\}.$$

Therefore, setting $\widetilde{\ell}_I=\operatorname{lin}\{P_{b_j^\perp}b_i : i\in I\}$, using (4.26) and applying the induction hypothesis (4.24) to both K, Λ, $\mathcal{B}$, z', and $P_{b_j^\perp}K$, $P_{b_j^\perp}\Lambda$, $\mathcal{B}''$, z'', we obtain that

$$\begin{aligned}
\mathrm{G}_\Lambda(K+z)&=\mathrm{G}_\Lambda\big(K+z'+\alpha_j b_j\big)\le\mathrm{G}_\Lambda\big(K+z'\big)+\mathrm{G}_{P_{b_j^\perp}\Lambda}\Big(P_{b_j^\perp}\big(K+z'\big)\Big)=\mathrm{G}_\Lambda\big(K+z'\big)+\mathrm{G}_{P_{b_j^\perp}\Lambda}\big(P_{b_j^\perp}K+z''\big)\\
&\le\sum_{I\subset\operatorname{supp}_{\mathcal{B}}(z)\setminus\{j\}}\mathrm{G}_{P_{\ell_I^\perp}\Lambda}\big(P_{\ell_I^\perp}K\big)+\sum_{I\subset\operatorname{supp}_{\mathcal{B}}(z)\setminus\{j\}}\mathrm{G}_{P_{\widetilde{\ell}_I^\perp}P_{b_j^\perp}\Lambda}\big(P_{\widetilde{\ell}_I^\perp}P_{b_j^\perp}K\big)\\
&=\sum_{I\subset\operatorname{supp}_{\mathcal{B}}(z)\setminus\{j\}}\Big(\mathrm{G}_{P_{\ell_I^\perp}\Lambda}\big(P_{\ell_I^\perp}K\big)+\mathrm{G}_{P_{\ell_{I\cup\{j\}}^\perp}\Lambda}\big(P_{\ell_{I\cup\{j\}}^\perp}K\big)\Big)=\sum_{I\subset\operatorname{supp}_{\mathcal{B}}(z)}\mathrm{G}_{P_{\ell_I^\perp}\Lambda}\big(P_{\ell_I^\perp}K\big).
\end{aligned}$$

This finishes the proof of (4.24), and hence, of (4.23).

Finally, to see that (4.23) is tight, take $b_i=\mathrm{e}_i$ and let $K=[0,k_1]\times\cdots\times[0,k_n]$, where $k_1,\dots,k_n\in\mathbb{Z}_{>0}$. Then we have

$$\operatorname{vol}(K)=\prod_{i=1}^{n}k_i=\prod_{i=1}^{n}\big((k_i-1)+1\big)=\sum_{I\subset\{1,\dots,n\}}\prod_{i\in I}(k_i-1)=\sum_{I\subset\{1,\dots,n\}}\mathrm{G}_{P_{\ell_I^\perp}\mathbb{Z}^n}\big(P_{\ell_I^\perp}(\operatorname{int}K)\big),$$

as desired. ■

We are now in the conditions to prove Theorem 4.15.

Proof of Theorem 4.15. We write $\lambda_i=\lambda_i(K)$, $i=1,\dots,n$, for the sake of brevity. In view of Proposition 4.9 and Lemma 4.19, we can assume that K is an anti-blocking convex body with $|\mathrm{e}_i|_{\mathrm{cs}(K)}=\lambda_i$ and such that $K\subset\{(x_1,x_2)\in\mathbb{R}^2 : x_1+x_2\le 2/\lambda_1\}$. Consequently, $h_K(\mathrm{e}_2)=2/\lambda_2$ holds. Consider the function $f:[0,2/\lambda_2]\longrightarrow\mathbb{R}$ given by $f(t)=\operatorname{vol}_1\big(K\cap(t\mathrm{e}_2+\ell_1)\big)$. Since K is convex, f is concave. Moreover, since K is anti-blocking, f is decreasing. Furthermore, from the inclusion $K\subset\{(x_1,x_2)\in\mathbb{R}^2 : x_1+x_2\le 2/\lambda_1\}$ it follows that

$$f(t)\le f(0)-t=\frac{2}{\lambda_1}-t \tag{4.27}$$

for all $t\in[0,2/\lambda_2]$. First, we observe that we have

$$\begin{aligned}
\mathrm{G}_2(K)&=\sum_{i=0}^{\lfloor 2/\lambda_2\rfloor}\mathrm{G}_1\big(K\cap(t\mathrm{e}_2+\ell_1)\big)\le\sum_{i=0}^{\lfloor 2/\lambda_2\rfloor}\big(f(i)+1\big)\\
&=\frac{1}{2}\left(f(0)+f\left(\left\lfloor\frac{2}{\lambda_2}\right\rfloor\right)\right)+\sum_{i=1}^{\lfloor 2/\lambda_2\rfloor}\left(\frac{1}{2}\big(f(i-1)+f(i)\big)\right)+\left\lfloor\frac{2}{\lambda_2}\right\rfloor+1.
\end{aligned}$$

Applying the lower bound in Lemma 4.20 to the terms $\big(f(i-1)+f(i)\big)/2$ and using (4.27) for $t=\lfloor 2/\lambda_2\rfloor$ we deduce that

$$\begin{aligned}
\mathrm{G}_2(K)&\le\frac{2}{\lambda_1}-\frac{1}{2}\left\lfloor\frac{2}{\lambda_2}\right\rfloor+\int_0^{\lfloor 2/\lambda_2\rfloor}f(t)\,\mathrm{d}t+\left\lfloor\frac{2}{\lambda_2}\right\rfloor+1\le\operatorname{vol}(K)+\frac{2}{\lambda_1}+\frac{1}{\lambda_2}+1\\
&\le\operatorname{vol}(K)\left(1+\lambda_2+\frac{\lambda_1}{2}+\frac{\lambda_1\lambda_2}{2}\right)=\operatorname{vol}(K)\left(1+\frac{\lambda_1}{2}\right)\big(1+\lambda_2\big),
\end{aligned}$$

where the last inequality follows from the lower bound in Minkowski's second theorem (4.2). This shows the upper bound (4.19).

For the lower bound, we can assume that f is differentiable. Otherwise, we could approximate f with a linear spline φ from below, which in turn could be approximated by a smooth concave function g from below by rounding its corners. This function would satisfy that $g(0) = f(0)$, and thus, the anti-blocking convex body

$$K' = \left\{ (x_1,x_2) \in \mathbb{R}^2 : 0 \le x_2 \le \frac{2}{\lambda_2},\ 0 \le x_1 \le g(x_2) \right\} \subset K$$

would be located underneath the diagonal $\{(x_1,x_2) \in \mathbb{R}^2 : x_1 + x_2 = 2/\lambda_2(K')\}$ as well, as desired.

We observe that $\lceil 2/\lambda_2 - 1 \rceil$ is the height of the highest horizontal integer line that intersects $\operatorname{int} K$. Therefore, we can estimate the number of interior lattice points of K as follows:

$$\mathrm{G}_2(\operatorname{int} K) = \sum_{i=1}^{\lceil 2/\lambda_2 - 1 \rceil} \mathrm{G}_1\big((\operatorname{int} K) \cap (i e_2 + \ell_1)\big) \ge \sum_{i=1}^{\lceil 2/\lambda_2 - 1 \rceil} \big(f(i) - 1\big) = \sum_{i=1}^{\lceil 2/\lambda_2 - 1 \rceil} f(i) - \left\lceil \frac{2}{\lambda_2} - 1 \right\rceil.$$

Now, on the one hand, we can use the upper bound in Lemma 4.20 in order to get

$$f(i) \ge \int_i^{i+1} f(t)\,\mathrm{d}t - \frac{1}{2} f'(i)$$

for all $i = 1, \ldots, \lceil 2/\lambda_2 - 1 \rceil$. On the other hand, since $2/\lambda_2 - \lceil 2/\lambda_2 - 1 \rceil \le 1$, we also have from Lemma 4.20 that

$$f\left(\left\lceil \frac{2}{\lambda_2} - 1 \right\rceil \right) \ge \int_{\lceil 2/\lambda_2 - 1 \rceil}^{2/\lambda_2} f(t)\,\mathrm{d}t - \frac{1}{2}\left(\frac{2}{\lambda_2} - \left\lceil \frac{2}{\lambda_2} - 1 \right\rceil \right) f'\left(\left\lceil \frac{2}{\lambda_2} - 1 \right\rceil \right).$$

Altogether, we obtain the estimate

$$\begin{aligned} \mathrm{G}_2(\operatorname{int} K) \ge & \sum_{i=1}^{\lceil 2/\lambda_2 - 1 \rceil - 1} \int_i^{i+1} f(t)\,\mathrm{d}t + \int_{\lceil 2/\lambda_2 - 1 \rceil}^{2/\lambda_2} f(t)\,\mathrm{d}t \\ & - \frac{1}{2}\left(\sum_{i=1}^{\lceil 2/\lambda_2 - 1 \rceil - 1} f'(i) + \left(\frac{2}{\lambda_2} - \left\lceil \frac{2}{\lambda_2} - 1 \right\rceil \right) f'\left(\left\lceil \frac{2}{\lambda_2} - 1 \right\rceil \right) \right) - \left\lceil \frac{2}{\lambda_2} - 1 \right\rceil \\ = & \operatorname{vol}(K) - \int_0^1 f(t)\,\mathrm{d}t \\ & - \frac{1}{2}\left(\sum_{i=1}^{\lceil 2/\lambda_2 - 1 \rceil - 1} f'(i) + \left(\frac{2}{\lambda_2} - \left\lceil \frac{2}{\lambda_2} - 1 \right\rceil \right) f'\left(\left\lceil \frac{2}{\lambda_2} - 1 \right\rceil \right) \right) - \left\lceil \frac{2}{\lambda_2} - 1 \right\rceil. \end{aligned} \tag{4.28}$$

Next, due to (4.27), we have

$$\int_0^1 f(t)\,\mathrm{d}t \le \frac{2}{\lambda_1} - \frac{1}{2}. \tag{4.29}$$

Furthermore, in order to bound the last sum in (4.28), we observe that f' is decreasing and non-positive, due to the facts that f is concave and K is anti-blocking, respectively. Therefore, we obtain that

$$\begin{aligned} & \sum_{i=1}^{\lceil 2/\lambda_2 - 1 \rceil - 1} f'(i) + \left(\frac{2}{\lambda_2} - \left\lceil \frac{2}{\lambda_2} - 1 \right\rceil \right) f'\left(\left\lceil \frac{2}{\lambda_2} - 1 \right\rceil \right) \\ & \qquad \le \sum_{i=1}^{\lceil 2/\lambda_2 - 1 \rceil - 1} \int_{i-1}^{i} f'(t)\,\mathrm{d}t + \left(\left(\frac{2}{\lambda_2} - 1 \right) - \left(\left\lceil \frac{2}{\lambda_2} - 1 \right\rceil - 1 \right) \right) f'\left(\frac{2}{\lambda_2} - 1 \right) \\ & \qquad \le \int_0^{2/\lambda_2 - 1} f'(t)\,\mathrm{d}t = f\left(\frac{2}{\lambda_2} - 1 \right) - f(0) \le 1 - \frac{2}{\lambda_2}, \end{aligned}$$

where we have used (4.27) in the last step. Substituting this and (4.29) into (4.28) yields

$$\begin{aligned} \mathrm{G}_2(\operatorname{int} K) &\ge \operatorname{vol}(K) - \frac{2}{\lambda_1} + \frac{1}{2} + \frac{1}{2}\left(\frac{2}{\lambda_2} - 1 \right) - \left\lceil \frac{2}{\lambda_2} - 1 \right\rceil \ge \operatorname{vol}(K) - \frac{2}{\lambda_1} - \frac{1}{\lambda_2} \\ &= \operatorname{vol}(K)\left(1 - \frac{1}{\operatorname{vol}(K)}\left(\frac{2}{\lambda_1} + \frac{1}{\lambda_2} \right) \right) \ge \operatorname{vol}(K)\left(1 - \frac{\lambda_1}{2} - \lambda_2 \right), \end{aligned}$$

where the last inequality follows from the lower bound in Minkowski's second theorem (4.2). Therefore, the proof of the lower bound (4.20) is finished. ∎

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