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Free Banach lattices over Banach spaces and ordered sets

Retículos de Banach libres sobre espacios de Banach y conjuntos ordenados

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No puedo empezar esta tesis doctoral de otra manera que agradeciendo infinitamente a mi tutor y director, Antonio Avilés López, todo lo que ha hecho por mí y la paciencia que ha tenido conmigo durante estos últimos cuatro años. Me faltarían palabras para poder expresar todo lo que me ha ayudado y todo lo que he aprendido gracias a él, no solo a nivel matemático, sino también a nivel general. Me ha hecho crecer como persona y madurar en todos los sentidos. Son miles los recuerdos y anécdotas que me llevo gracias a él, que tiene gran parte de la culpa de que esta etapa haya sido de las mejores de mi vida.

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Contents

Resumen

Esta memoria ha sido elaborada durante el período de disfrute de una beca FPI de la Fundación Séneca - Agencia de Ciencia y Tecnología de la Región de Murcia. Una ayuda complementaria de dicho programa nos ha permitido realizar una estancia en la *Facultad de Matemáticas y Física de la Universidad Carolina* de Praga (enero-abril de 2020).

Esta investigación también ha sido financiada parcialmente por el proyecto de investigación 20797/PI/18 de la Fundación Séneca - Agencia de Ciencia y Tecnología de la Región de Murcia, y por el proyecto MTM2017-86182-P del Ministerio de Economía y Competitividad y FEDER.

La tesis doctoral tiene como marco general la teoría de los retículos de Banach. Más concretamente, se estudian los retículos de Banach libres generados por determinadas estructuras, tales como los espacios de Banach y los retículos, si bien en este último caso nos centramos más en el caso particular de los conjuntos linealmente ordenados.

El concepto de objeto libre es bien conocido, puede expresarse en el lenguaje general de la teoría de categorías, y se ha probado de gran utilidad en diversas áreas tanto en análisis como en álgebra. Sin embargo, en el contexto de los retículos de Banach su introducción ha sido reciente. En [15] es donde se menciona por primera vez, definiéndose tal concepto para conjuntos, y posteriormente, en [8], se generaliza a espacios de Banach, el cual contiene al caso de los conjuntos como caso particular.

Dentro del estudio de los retículos de Banach, nos centramos en estudiar dos propiedades: condiciones de cadena y proyectividad, las cuales marcan en gran parte la estructura de esta memoria. Dicho esto, podemos ambientar este texto dentro del Análisis Funcional, y en cierto sentido, la Topología General.

Los resultados originales incluidos en esta tesis pueden encontrarse en nuestros trabajos:

[5] AVILÉS, A., MARTÍNEZ-CERVANTES, G., AND RODRÍGUEZ ABELLÁN, J. D. On projective Banach lattices of the form $C(K)$ and $FBL[E]$. *J. Math. Anal. Appl. 489*, 124129 (2020)

[6] Avilés, A., Martínez-Cervantes, G., and Rodríguez Abellán, J. D. On the Banach lattice c_0. To appear in Rev. Mat. Complut. https://doi.org/10.1007/s13163-019-00342-x, 2020

[7] Avilés, A., Plebanek, G., and Rodríguez Abellán, J. D. Chain conditions in free Banach lattices. *J. Math. Anal. Appl. 465* (2018), 1223–1229

[9] Avilés, A., and Rodríguez Abellán, J. D. The free Banach lattice generated by a lattice. *Positivity 23* (2019), 581–597

[10] Avilés, A., and Rodríguez Abellán, J. D. Projectivity of the free Banach lattice generated by a lattice. *Archiv der Mathematik 113* (2019), 515–524

En la introducción de cada capítulo indicaremos explícitamente en cuáles de ellos se basa cada uno.

El **Capítulo 1** sirve como introducción a la teoría general de los retículos de Banach. En él se dan las definiciones básicas y las propiedades que uno necesita conocer para poder comprender el resto de contenidos que aquí se muestran. Con esto, además, se intenta que la memoria sea tan autocontenida como sea posible, así como establecer en gran medida la notación que se usa a lo largo de todo el texto.

Está dividido en tres secciones: la Sección 1.1, como pequeña introducción histórica al estudio de los retículos vectoriales y retículos de Banach, la Sección 1.2, correspondiente a los retículos vectoriales, y la Sección 1.3, en relación a los retículos de Banach. En esta última se introduce, además, el concepto de retículo de Banach libre generado por un conjunto, y por un espacio de Banach, dos de los principales protagonistas de este texto, y se enuncian sus principales propiedades, estudiadas originalmente en [15] y [8], respectivamente.

En el **Capítulo 2** se define un nuevo objeto, el retículo de Banach libre generado por un retículo, siguiendo las mismas ideas de [15] y [8]. Grosso modo, el retículo de Banach libre generado por un retículo es un retículo de Banach que está generado, como retículo de Banach, por una copia de los elementos del retículo.

El resultado principal de este capítulo es una descripción explícita, como un cierto espacio de funciones, del retículo de Banach libre generado por un retículo. En [8] se hace lo mismo para el retículo de Banach libre generado por un conjunto, y para el retículo de Banach libre generado por un espacio de Banach. La descripción explícita en nuestro caso es análoga a la comentada en ese artículo. Sin embargo, la prueba de ello requiere herramientas completamente diferentes.

En la Sección 2.1 damos la definición de retículo de Banach libre generado por un retículo, esto es, como aquel que cumple una cierta propiedad universal de objeto libre, y enunciamos el teorema principal del capítulo, que nos da la descripción explícita de tal retículo de Banach mencionada líneas arriba.

Como apoyo a la demostración, lo que hacemos en primer lugar, en la Sección 2.2, es demostrar que, efectivamente, tal objeto existe y es único salvo isometrías. Demostramos que cualesquiera dos retículos de Banach candidatos a ser libres son, de hecho, isométricos entre sí, y, posteriormente, describimos el retículo de Banach libre generado por un retículo como un cierto cociente del retículo de Banach libre generado por el retículo de partida visto como conjunto, olvidándonos de su estructura reticular.

En la Sección 2.3 lo que hacemos es observar que el caso más natural de considerar el retículo de Banach libre generado por un retículo es cuando este es distributivo (es decir, las operaciones de supremo e ínfimo son distributivas entre sí), lo cual nos facilita la prueba del teorema principal, y que el caso general se reduce a este de manera sencilla.

La Sección 2.4 está dedicada a la prueba del teorema principal, que nos da la descripción explícita del retículo de Banach libre generado por un retículo como un cierto espacio de funciones. La idea de la prueba es demostrar que tal espacio de funciones, que describiremos en su momento, es isométrico al cociente comentado anteriormente. Ahora bien, nos vemos obligados a distinguir dos casos. En primer lugar, consideramos el caso en el que el retículo es finito, pues gracias a B. de Pagter y a A. W. Wickstead [15] sabemos que, con esta hipótesis, el retículo de Banach libre generado por el retículo visto como conjunto es isomorfo al reticulo de Banach $C_{ph}([-1,1]^n)$ de las funciones continuas y positivamente homogéneas sobre $[-1,1]^n$, siendo n la cardinalidad del retículo, y el cual no es difícil de manejar. El caso infinito lo reducimos, en cierto sentido, al caso finito.

Finalmente, en la Sección 2.5 probamos que, en el caso en el que el retículo es un conjunto linealmente ordenado, la copia del mismo dentro del retículo de Banach libre generado por él se comporta como la base sumante de c_0 desde el punto de vista de los espacios de Banach.

En el **Capítulo 3** estudiamos determinadas condiciones de cadena para los retículos de Banach libres generados por conjuntos linealmente ordenados y por espacios de Banach.

En la Sección 3.1 recordamos las definiciones de las condiciones de cadena para retículos de Banach que se estudian en el capítulo, esto es, las definiciones de condición de cadena σ-acotada y condición de cadena contable. Además, en la misma enunciamos los principales teoremas que demostramos después.

Por un lado, en la Sección 3.2, demostramos que el retículo de Banach libre generado por un conjunto linealmente ordenado satisface la condición de cadena contable si, y solo si, el conjunto linealmente ordenado se embebe dentro del conjunto de los números reales con su orden usual. Para ello, nos será de gran utilidad la conocida caracterización de los subconjuntos linealmente ordenados de la recta real como aquellos separables en la topología del orden para los cuales el conjunto de saltos es contable. Además, en la prueba del mismo resultado usaremos un hecho que ya tiene bastante interés por sí solo, esto es, que si $\mathbb{L}$ es un subconjunto del conjunto linealmente ordenado $\mathbb{M}$, entonces el retículo de Banach libre generado por $\mathbb{L}$ es isométrico al retículo de Banach generado por la copia de $\mathbb{L}$ dentro del retículo de Banach libre generado por $\mathbb{M}$, al igual que ocurre en el caso de los retículos de Banach libres generados por conjuntos.

Más aún, demostraremos que si el conjunto linealmente ordenado se embebe en la recta real, el retículo de Banach libre generado por él no solo satisface la condición de cadena contable, sino que es además σ-centrado.

La Sección 3.3 está dedicada al caso del retículo de Banach libre generado por un espacio de Banach. B. de Pagter y A. W. Wickstead ya probaron en [15] que el retículo de Banach libre generado por un conjunto siempre satisface la condición de cadena contable. Nosotros vamos más allá, y demostramos que, en efecto, tal retículo de Banach satisface la condición de cadena σ-acotada, la cual implica trivialmente la condición de cadena contable. Es más, esto lo probamos para el caso en el que el objeto de partida es un espacio de Banach, del cual se deduce el caso anterior.

Más concretamente, usando el teorema de Ramsey, demostramos que el retículo de Banach de las funciones continuas y positivamente homogéneas sobre la bola cerrada unidad del dual de cualquier espacio de Banach E, $C_{ph}(B_{E^*})$, satisface tal condición. Pero ahora, como el retículo de Banach libre generado por cualquier espacio de Banach E es un subretículo de Banach de $C_{ph}(B_{E^*})$, y la condición de cadena σ-acotada es hereditaria, se tiene como caso particular que este último también satisface la misma condición.

Finalmente, en la Sección 3.4 estudiamos la posibilidad de la existencia de condiciones de cadena más fuertes sobre el retículo de Banach $C_{ph}(B_{E^*})$, y en consecuencia, sobre el retículo de Banach libre generado por el espacio de Banach E.

El **Capítulo 4** está dedicado al estudio de la proyectividad de ciertos retículos de Banach.

De manera similar a lo que hacemos en los capítulos anteriores, la introducción, que corresponde con la Sección 4.1, la dedicamos a dar la definición del concepto clave del capítulo, en este caso, la de retículo de Banach λ-proyectivo para $\lambda > 1$, y enunciamos los diferentes resultados que demostramos a lo largo de él.

En la Sección 4.2 enunciamos y demostramos una serie de resultados simples que son de utilidad para probar los teoremas principales del capítulo.

En las Secciones 4.3 y 4.4 nos centramos en el caso del retículo de Banach libre generado por un retículo. Por un lado, demostramos que el retículo de Banach libre generado por un retículo finito es siempre λ-proyectivo para cualquier $\lambda > 1$ (o 1^+-proyectivo), mientras que si el retículo de partida es un conjunto linealmente ordenado infinito, entonces el retículo de Banach libre generado por él no es λ-proyectivo para ningún $\lambda > 1$ (es decir, no es ∞-proyectivo, según nuestra terminología). En la demostración de ambos resultados usamos fuertemente el hecho de que el retículo de Banach libre generado por un retículo es un cociente del retículo de Banach libre generado por el retículo visto simplemente como conjunto. Otra de las claves del primer resultado es que el retículo de Banach libre generado por cualquier conjunto es siempre 1^+-proyectivo, como prueban B. de Pagter y A. W. Wickstead en [15]. Para demostrar el segundo de los resultados, lo que hacemos es ver que los retículos de Banach libres generados por el conjunto de los números naturales, y el conjunto de los números naturales junto con el $+\infty$, vistos como conjuntos linealmente ordenados, no son ∞-proyectivos, y después, probamos que, o bien el retículo de Banach libre generado por el conjunto de los números naturales, o bien el retículo de Banach libre generado por el conjunto de los números naturales junto con el $+\infty$, vistos como conjuntos linealmente ordenados, están complementados en el retículo de Banach libre generado por el conjunto linealmente ordenado de partida, de donde se sigue fácilmente el resultado.

La Sección 4.5 contiene dos resultados principales. Por un lado, demostramos que si un retículo de Banach es ∞-proyectivo, entonces toda sucesión acotada que pueda ser llevada vía un homomorfismo de retículos de Banach a la base canónica de c_0 de manera sobreyectiva debe contener una ℓ_1-subsucesión, de donde obtenemos, en particular, que ni c_0 ni ℓ_p (para $2 \leq p < \infty$) son ∞-proyectivos.

El otro resultado que probamos en esta sección nos dice que si E es un espacio de Banach con la propiedad de que el retículo de Banach libre generado por él es ∞-proyectivo, entonces E tiene la propiedad de Schur (es decir, toda sucesión débilmente convergente converge en norma). Dicho en otras palabras, para que el retículo de Banach libre generado por un espacio de Banach E sea ∞-proyectivo, la estructura de E debe ser bastante parecida a la de $\ell_1(A)$ para algún conjunto A.

Más aún, al final de esta sección proporcionamos un contraejemplo que nos dice que, en la categoría de los espacios de Banach no separables, el recíproco de este resultado no es cierto.

En la Sección 4.6 probamos que si K es un espacio topológico compacto de Hausdorff, entonces el retículo de Banach de las funciones continuas sobre K, $C(K)$, con la norma del supremo es 1^+-proyectivo si, y solo si, K es un retracto de entornos absoluto en la categoría de los espacios topológicos compactos de Hausdorff. B. de Pagter y A. W. Wickstead ya probaron en [15] que si $C(K)$ es 1^+-proyectivo, entonces K es un retracto de entornos absoluto en categoría de los espacios topológicos compactos de Hausdorff. Sin embargo, solamente consiguieron probar el recíproco bajo la hipótesis adicional de que K está dentro de $\mathbb{R}^n$.

Finalmente, en la Sección 4.7 estudiamos la complementabilidad del retículo de Banach c_0 en el retículo de Banach libre generado por él visto como espacio de Banach, mientras que en la Sección 4.8 proponemos varios problemas abiertos que tenemos en relación a los retículos de Banach proyectivos.

Abstract

This memoir has been prepared during the period of an FPI grant from the Fundación Séneca - Agencia de Ciencia y Tecnología de la Región de Murcia. A complementary grant from this program has allowed us to carry out a stay at the *Faculty of Mathematics and Physics of the Charles University* in Prague (January-April 2020).

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The general framework of this doctoral thesis is the theory of Banach lattices. More specifically, we study the free Banach lattices generated by certain structures, such as Banach spaces and lattices, although in the latter case we mainly focus on the particular case of linearly ordered sets.

The concept of free object is well known, can be expressed in the general language of the theory of categories, and has been proved very useful in various areas in both analysis and algebra. However, in the context of Banach lattices, it has been recently introduced. It is mentioned for the first time in [15], defining such a concept for sets, and later, in [8], it is generalized for Banach spaces, which contains the set case as a particular case.

Within the study of Banach lattices, we focus on studying two properties: chain conditions and projectivity, which mark the structure of this memoir. That said, we can set this text within Functional Analysis, and in a certain sense, General Topology.

The original results included in this thesis can be found in our works:

[5] Avilés, A., Martínez-Cervantes, G., and Rodríguez Abellán, J. D. On projective Banach lattices of the form $C(K)$ and $FBL[E]$. *J. Math. Anal. Appl. 489*, 124129 (2020)

[6] Avilés, A., Martínez-Cervantes, G., and Rodríguez Abellán, J. D. On the Banach lattice c_0. To appear in Rev. Mat. Complut. https://doi.org/10.1007/s13163-

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[7] Avilés, A., Plebanek, G., and Rodríguez Abellán, J. D. Chain conditions in free Banach lattices. *J. Math. Anal. Appl. 465* (2018), 1223–1229

[9] Avilés, A., and Rodríguez Abellán, J. D. The free Banach lattice generated by a lattice. *Positivity 23* (2019), 581–597

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In the introduction of each chapter we will explicitly indicate which of them is based on each one.

Chapter 1 is an introduction to the general theory of Banach lattices. In it we give the basic definitions and properties one needs to know in order to understand the rest of the content shown here. With this, in addition, we try to make the memoir as self-contained as possible, as well as to establish the notation that is used throughout the text.

It is divided into three sections: Section 1.1, as a short historical introduction to the study of vector lattices and Banach lattices, Section 1.2, devoted to vector lattices, and Section 1.3, in relation to Banach lattices. In this section we also introduce the concept of free Banach lattice generated by a set, and by a Banach space, two of the main protagonists of this text, and its main properties, originally studied in [15] and [8], respectively.

In **Chapter 2** a new object is defined, the free Banach lattice generated by a lattice, following the same ideas as [15] and [8]. Roughly speaking, the free Banach lattice generated by a lattice is a Banach lattice that is generated, as a Banach lattice, by a copy of the elements of the lattice.

The main result of this chapter is an explicit description, as a certain function space, of the free Banach lattice generated by a lattice. In [8] the same is done for the free Banach lattice generated by a set, and for the free Banach lattice generated by a Banach space. The explicit description in our case is analogous to that discussed in that paper. However, the proof of this requires completely different tools.

In Section 2.1 we give the definition of a free Banach lattice generated by a lattice, that is, as one that satisfies a certain universal property of free object, and we state the main theorem of the chapter, which gives us the explicit description of such a Banach lattice mentioned above.

In support of the proof, what we do first, in Section 2.2, is to prove that indeed such an object exists and is unique up to isometries. We prove that any two candidate Banach lattices to be free are, in fact, isometric to each other, and subsequently describe the free Banach lattice generated by a lattice as a certain quotient of the free Banach lattice generated by the starting lattice seen as a set, forgetting about its lattice structure.

In Section 2.3 we observe that the most natural case of considering the free Banach lattice generated by a lattice is when it is distributive (that is, the operations of supremum and infimum are distributive to each other), which makes the proof of the main theorem easier, and that the general case is reduced to this in a simple way.

Section 2.4 is devoted to the proof of the main theorem, which gives us the explicit description of the free Banach lattice generated by a lattice as a certain function space. The idea of the proof is to show that such a function space, which we will describe in due course, is isometric to the quotient mentioned above. Now, we are forced to distinguish two cases. First, we consider the case in which the lattice is finite, since thanks to B. de Pagter and A. W. Wickstead [15] we know that, with this hypothesis, the free Banach lattice generated by the lattice seen as a set is isomorphic to the Banach lattice $C_{ph}([-1,1]^n)$ of the continuous and positively homogeneous functions on $[-1,1]^n$, where n is the cardinality of the lattice, and which is not difficult to handle. We reduce the infinite case, in a certain sense, to the finite case.

Finally, in Section 2.5 we prove that, in the case in which the lattice is a linearly ordered set, the copy of it within the free Banach lattice generated by it behaves like the summing basis of c_0 from a Banach space point of view.

In **Chapter 3** we study certain chain conditions in free Banach lattices generated by linearly ordered sets and Banach spaces.

In Section 3.1 we recall the definitions of the chain conditions in Banach lattices discussed in the chapter, that is, the definitions of the σ-bounded chain condition and the countable chain condition. In addition, in it we state the main theorems that will be later proved.

On the one hand, in Section 3.2, we prove that the free Banach lattice generated by a linearly ordered set satisfies the countable chain condition if, and only if, the linearly ordered set can be embedded into the set of the real numbers with its usual order. For this, the well-known characterization of the linearly ordered subsets of the real line as those that are separable in the order topology and for which the set of leaps is countable will be very useful. Furthermore, in the proof of the same result we will use a fact that is already quite interesting on its own, that is, if $\mathbb{L}$ is a subset of the linearly ordered set $\mathbb{M}$, then the free Banach lattice generated by $\mathbb{L}$ is isometric to the Banach lattice generated by the copy of $\mathbb{L}$ inside the free Banach lattice generated by $\mathbb{M}$, just as it happens in the case of the free Banach lattices generated by sets.

Furthermore, we will prove that if the linearly ordered set can be embedded into the real line, the free Banach lattice generated by it not only satisfies the countable chain condition, but it is also σ-centered.

Section 3.3 is devoted to the case of the free Banach lattice generated by a Banach space. B. de Pagter and A. W. Wickstead already proved in [15] that the free Banach lattice generated by a set always satisfies the countable chain condition. We go further, and show that, in fact, such a Banach lattice satisfies the σ-bounded chain condition, which trivially implies the countable chain condition. Furthermore, we prove this for the case in which the starting object is a Banach space, from which the previous case is deduced.

More specifically, using the Ramsey's theorem, we show that the Banach lattice of continuous and positively homogeneous functions on the closed unit ball of the dual of any

Banach space E, $C_{ph}(B_{E^*})$, satisfies such a condition. But now, as the free Banach lattice generated by any Banach space E is a Banach sublattice of $C_{ph}(B_{E^*})$, and the σ-bounded chain condition is hereditary, we have as a particular case that the latter also satisfies the same condition.

Finally, in Section 3.4 we study the possibility of the existence of stronger chain conditions in the Banach lattice $C_{ph}(B_{E^*})$, and consequently, in the free Banach lattice generated by the Banach space E.

Chapter 4 is devoted to the study of the projectivity of certain Banach lattices.

Similar to what we did in the previous chapters, in the introduction, which corresponds to Section 4.1, we give the definition of the key concept of the chapter, in this case, that of λ-projective Banach lattice for $\lambda > 1$, and we state the different results we prove throughout it.

In Section 4.2 we state and prove some simple results that are useful to prove the main theorems of the chapter.

In Sections 4.3 and 4.4 we focus on the case of the free Banach lattice generated by a lattice. On the one hand, we prove that the free Banach lattice generated by a finite lattice is always λ-projective for any $\lambda > 1$ (or 1^+-projective), whereas if the starting lattice is an infinite linearly ordered set, then the free Banach lattice generated by it is not λ-projective for any $\lambda > 1$ (i.e. it is not ∞-projective, according to our terminology). In the proof of both results, we strongly use the fact that the free Banach lattice generated by a lattice is a quotient of the free Banach lattice generated by the lattice viewed as a set. Another key to the first result is that the free Banach lattice generated by any set is always 1^+-projective, as B. de Pagter and A. W. Wickstead proved in [15]. To prove the second of the results, we show that the free Banach lattices generated by the set of the natural numbers, and the set of the natural numbers together with $+\infty$, seen as linearly ordered sets, are not ∞-projective, and then we prove that either the free Banach lattice generated by the set of the natural numbers or the free Banach lattice generated by the set of the natural numbers together with $+\infty$, seen as linearly ordered sets, are complemented in the free Banach lattice generated by the starting linearly ordered set, from which the result easily follows.

Section 4.5 contains two main results. On the one hand, we prove that if a Banach lattice is ∞-projective, then every bounded sequence that can be mapped by a Banach lattice homomorphism onto the basis of c_0 must contain an ℓ_1-subsequence, from which we obtain, in particular, that neither c_0 nor ℓ_p (for $2 \leq p < \infty$) are ∞-projective.

The other result we prove in this section tells us that if E is a Banach space with the property that the free Banach lattice generated by it is ∞-projective, then E has the Schur property (i.e. every weakly convergent sequence converges in norm). In other words, for the free Banach lattice generated by a Banach space E to be ∞-projective, the structure of E must be very close to that of $\ell_1(A)$ for some set A.

Moreover, at the end of this section we provide a counterexample which shows that, in the category of nonseparable Banach spaces, the converse of this result does not hold.

In Section 4.6 we prove that if K is a compact Hausdorff topological space, then the Banach lattice of continuous functions on K, $C(K)$, with the supremum norm is 1^+-

projective if, and only if, K is an absolute neighbourhood retract in the category of compact Hausdorff topological spaces. B. de Pagter and A. W. Wickstead have already proved in [15] that if $C(K)$ is 1^+-projective, then K is an absolute neighbourhood retract in the category of compact Hausdorff topological spaces. However, they only managed to show the converse with the additional hypothesis that K is inside $\mathbb{R}^n$.

Finally, in Section 4.7 we study the complementability of the Banach lattice c_0 in the free Banach lattice generated by itself seen as a Banach space, while in Section 4.8 we propose some open problems we have about projective Banach lattices.

Chapter 1

Vector and Banach lattices

1.1 Introduction

We all know that the starting point of functional analysis was the investigation of the classical function spaces, which provide its most important applications. However, the natural order in these spaces was neglected almost completely. A first attempt to include a compatible order structure in the study of linear and normed spaces was due to F. Riesz, H. Freudenthal and L. V. Kantorovič in the mid-thirties. In the following years, schools of research on vector lattices were subsequently founded and these investigations were continued by various mathematicians in the Soviet Union (B. Z. Vulikh, A. G. Pinsker, A. I. Judin), in Japan (H. Nakano, T. Ogasawara, K. Yosida), and in United States (G. Birkhoff, S. Kakutani, H. F. Bohnenblust, M. H. Stone) (see [28] and [29]).

L. V. Kantorovič and his school first recognized the importance of studying vector lattices in connection with Banach's theory of normed spaces; they investigated normed vector lattices as well as order-related linear operators between such vector lattices (see [28]).

This chapter is about the basic theory of vector and Banach lattices. We will try to collect the basic definitions and properties one has to know for understanding the content of this memoir. The interested reader can find more about this subject in books [12], [25], [28] and [29], for example.

It is divided into two sections, one concerning vector lattices (basic concepts and the free vector lattice generated by a set) and the other one concerning Banach lattices (basic concepts and the free Banach lattices generated by a set and a Banach space).

1.2 Vector lattices

In this section we show the basic definitions and properties concerning vector lattices and the free vector lattice generated by a set.

1.2.1 Basic concepts

Most of the definitions and examples we show here are extracted from [38].

Definition 1.1. An *order* on a non-empty set M is a relation $\leq$ such that

1. $x \leq x$ for all $x \in M$,
2. $x \leq y$ and $y \leq x$ implies that $x = y$,
3. $x \leq y$ and $y \leq z$ implies that $x \leq z$.

We use $y \geq x$ as a synonym for $x \leq y$, and $x < y$ for $x \leq y$ but $x \neq y$. Similarly, we write $y > x$ for $x < y$.

Definition 1.2. If A is a non-empty subset of M, then

1. $x \in M$ is an *upper bound* (resp. *lower bound*) of A if $y \leq x$ (resp. $x \leq y$) for every $y \in A$.
2. A is *bounded from above* (resp. *bounded from below*) if there is an upper bound (resp. *lower bound*) of A.
3. An upper bound (resp. lower bound) x of A is the *supremum* (resp. *infimum*) of A if for any other upper bound (resp. lower bound) y of A we have $x \leq y$ (resp. $y \leq x$).
4. If A is bounded from above and bounded from below, we will say that A is *order bounded*.
5. A is an *order interval* if it is of the form $[x, y] := \{m \in M : x \leq m \leq y\}$ for some $x, y \in M$.

Definition 1.3. A *lattice* is a non-empty set $\mathbb{L}$ with an order $\leq$ such that for every pair of elements $x, y \in \mathbb{L}$, the set $\{x, y\}$ has both a supremum (which is denoted by $x \vee y$, and also called the supremum of x and y) and an infimum (which is denoted by $x \wedge y$, and also called the infimum of x and y).

The supremum of a general subset A of M, when it exists, is denoted by any of $\sup(A)$, $\sup\{a : a \in A\}$, $\bigvee\{a : a \in A\}$ or $\bigvee_{a \in A} a$. The notation for the infimum is analogous, replacing $\bigvee$ by $\bigwedge$ and $\sup$ by $\inf$.

Often, throughout this memoir we will focus on studying properties related to the following particular case of lattice:

Definition 1.4. A *linearly ordered set* (or *linear order*) is a non-empty set $\mathbb{L}$ with an order $\leq$ with the property that for any $x, y \in \mathbb{L}$, either $x \leq y$ or $y \leq x$.

Definition 1.5. A map $T\colon \mathbb{L}_1 \longrightarrow \mathbb{L}_2$ between two lattices, $\mathbb{L}_1$ and $\mathbb{L}_2$, is said to be a *lattice homomorphism* if it preserves the lattice operations, i.e.

$$T(x \vee y) = T(x) \vee T(y) \text{ and } T(x \wedge y) = T(x) \wedge T(y) \text{ for every } x, y \in \mathbb{L}_1.$$

If T is also bijective, we will say that T is a *lattice isomorphism*.

Definition 1.6. A real vector space E which is ordered by some order relation $\leq$ is called a *vector lattice* (or *Riesz space*) if any two elements $x, y \in E$ have a supremum and an infimum, and the following properties are satisfied:

1. $x \leq y$ implies $x + z \leq y + z$ for all $x, y, z \in E$,

2. $0 \leq x$ implies $0 \leq tx$ for all $x \in E$ and $t \in \mathbb{R}^+$.

The set $E^+ := \{x \in E : x \geq 0\}$ is called the *positive cone* of E and its elements are termed *positive* (rather than non-negative), while for $x \in E$,

$$x^+ := x \vee 0,\ x^- := (-x) \vee 0, \text{and } |x| := x \vee (-x)$$

are called the *positive part*, the *negative part*, and the *absolute value* of x, respectively. If $x \in E^+$ is not zero, we will often say that x is *strictly positive*.

We will also say that $x, y \in E$ are *disjoint* if $|x| \wedge |y| = 0$.

Example 1.7. The most obvious example of a vector lattice is the reals with all the usual operations. The *usual* or *standard* order on $\mathbb{R}^n$ is that in which $(x_1, \ldots, x_n) \leq (y_1, \ldots, y_n)$ means that $x_k \leq y_k$ for $k = 1, \ldots, n$. This order makes $\mathbb{R}^n$ into a vector lattice in which the supremum of two vectors is $(x_k) \vee (y_k) = (x_k \vee y_k)$ and the infimum is $(x_k) \wedge (y_k) = (x_k \wedge y_k)$. Hence, the positive part, the negative part and the absolute value are given by $(x_k)^+ = (x_k^+)$, $(x_k)^- = (x_k^-)$ and $|(x_k)| = (|x_k|)$, respectively.

We now show some basic properties of the absolute value:

Proposition 1.8 ([28, Proposition 1.4])**.** *Let E be a vector lattice. For all x, y, x_1, $y_1 \in E$ and all $\lambda \in \mathbb{R}$, the following relations are valid:*

1. $x = x^+ - x^-$,

2. $|x| = x^+ + x^-$,

3. $|x| = 0 \Leftrightarrow x = 0$; $|\lambda x| = |\lambda||x|$; $|x + y| \leq |x| + |y|$,

4. $x + y = x \vee y + x \wedge y$,

5. $|x - y| = x \vee y - x \wedge y$,

6. $|x \vee y - x_1 \vee y_1| \leq |x - x_1| + |y - y_1|$,

7. $|x \wedge y - x_1 \wedge y_1| \leq |x - x_1| + |y - y_1|$.

Moreover, the equality in 1 is the unique representation of x as a difference of disjoint positive elements of E.

Definition 1.9. A map $T\colon E_1 \longrightarrow E_2$ between two vector lattices, E_1 and E_2, is said to be a *vector lattice homomorphism* if it is linear and preserves the lattice operations, i.e.

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y) \text{ for every } x, y \in E_1, \alpha, \beta \in \mathbb{R},$$

$$T(x \vee y) = T(x) \vee T(y) \text{ and } T(x \wedge y) = T(x) \wedge T(y) \text{ for every } x, y \in E_1.$$

If T is also bijective, we will say that T is a *vector lattice isomorphism*.

A very important concept in the theory of vector lattices, as we will see later, is the following:

Definition 1.10. A vector lattice E has the *Archimedean property* (or is *Archimedean*) if

$$\big[(\forall n \geq 1) ny \leq x \in E^+\big] \Rightarrow \big[y \leq 0\big].$$

Example 1.11. Clearly, $\mathbb{R}^n$ with the usual order has the Archimedean property. However, $\mathbb{R}^n$ with the *lexicographic* order fails to have this property. For example, on $\mathbb{R}^2$, the lexicographic order is given by $(x_1, x_2) \leq (y_1, y_2)$ if, and only if, either $x_1 < y_1$ or $x_1 = y_1$ and $x_2 \leq y_2$. Here, $n(0,1) \leq (1,0)$ for all $n \in \mathbb{N}$, but $(0,1) \not\leq (0,0)$.

Example 1.12. Function spaces are important examples of Archimedean vector lattices. Let X be a non-empty set and take $E = \mathbb{R}^X$, that is to say, the space of all real-valued functions on X. Order this with the pointwise order under which $f \leq g$ if, and only if, $f(x) \leq g(x)$ for every $x \in X$ and give it the pointwise vector operations, we have a vector lattice, where the supremum of two functions f and g is the function given by $(f \vee g)(x) = f(x) \vee g(x)$ for every $x \in X$, and the infimum of two functions f and g is the function given by $(f \wedge g)(x) = f(x) \wedge g(x)$ for every $x \in X$.

E is Archimedean as if $nf \leq g$ for all $n \in \mathbb{N}$ then $nf(x) \leq g(x)$ for all $n \in \mathbb{N}$ and for all $x \in X$. As $\mathbb{R}$ is Archimedean it follows that $f(x) \leq 0$ for all $x \in X$ and hence that $f \leq 0$ (where this 0 is the zero function on X).

E will have many vector subspaces which are also vector lattices under the same order, for example the bounded functions; if X has a topology then we could take the continuous functions or continuous bounded functions.

Definition 1.13. A subset $\mathbb{A}$ of a lattice $\mathbb{L}$ is a *sublattice* if $x, y \in \mathbb{A}$ implies that $x \vee y$, $x \wedge y \in \mathbb{A}$, where these lattice operations are computed in $\mathbb{L}$. A *vector sublattice* of a vector lattice is simply a vector subspace which is also a sublattice.

Example 1.14. Both c_0 and c are vector sublattices of ℓ_∞.

Definition 1.15. Let A be a non-empty subset of a vector lattice E. The *sublattice generated* (or *lattice generated*) *by* A is the smallest subset of E containing A and closed under the operations $\vee$ and $\wedge$. The *vector sublattice generated* (or *vector lattice generated*) *by* A is the smallest vector space of E containing A and closed under the operations $\vee$ and $\wedge$.

If we denote by $\bigvee A$ (resp. $\bigwedge A$) the set of suprema (resp. infima) of all non-empty finite subsets of A, then it is easy to check that $\bigvee\left(\bigwedge A\right) = \bigwedge\left(\bigvee A\right)$ is the sublattice generated by A. Moreover, if A is a vector subspace of E, then $\bigvee\left(\bigwedge A\right)$ is a vector sublattice of E (see [21, page 47]).

Throughout this memoir, we will denote the vector lattice generated by a non-empy set A by $lat\ A$ or $lat\,\{a : a \in A\}$.

Definition 1.16. An *ideal* $\mathcal{I}$ in a vector lattice E is a vector subspace such that $y \in \mathcal{I}$, $x \in E$ and $|x| \leq |y|$ together imply that $x \in \mathcal{I}$.

Example 1.17. In ℓ_∞, c_0 is an ideal, but c is not as $\left|\left((-1)^n\right)\right| \leq (1) \in c$ but $\left((-1)^n\right) \notin c$.

Definition 1.18. Let A be a non-empty subset of a vector lattice E. The *ideal generated by* A is the smallest ideal in E containing A.

Now we are going to show a very important result about Archimedean vector lattices which is very useful.

Definition 1.19. An *elementary inequality* in a vector lattice is an inequality or an equality which involves only linear and lattice operations and a finite number of elements of the vector lattice.

For example, $x + (y \vee z) = (x + y) \vee (x + z)$ is an elementary inequality, which can be true or false. Then, if we want to know if such an elementary inequality is true or false in every Archimedean vector lattice one just has to see if it is true or false in $\mathbb{R}$. More concretely:

Theorem 1.20 ([25])**.** *An elementary inequality is true in every Archimedean vector lattice if, and only if, it is true in the reals.*

On pages 66 and 67 of [25] there is a proof of this theorem for a special class of vector lattices (the class of uniformly complete vector lattices). However, all Archimedean vector lattices may be embedded as vector sublattices inside a vector lattice in this class, namely, their Dedekind completion (see [28, Proposition 1.10], or [24, Theorem 32.5] for further details), from which the general result follows.

1.2.2 The free vector lattice generated by a set

In [13] the existing theory of free vector lattices is recapitulated. The basic facts we show here are extracted from [15].

Definition 1.21. Let A be a non-empty set. A *free vector lattice over* or *generated by* A is a vector lattice F together with a function $\phi\colon A \longrightarrow F$ with the property that for every vector lattice E and every map $T\colon A \longrightarrow E$ there is a unique vector lattice homomorphism $\hat{T}\colon F \longrightarrow E$ such that $T = \hat{T} \circ \phi$.

$$\begin{tikzcd} A \arrow[r, "T"] \arrow[d, "\phi"'] & E \\ F \arrow[ur, "\hat{T}"'] & \end{tikzcd}$$

Clearly, ϕ must be injective, as we can certainly choose E and T to make ϕ injective.

Now we are going to list some basic properties one has to know about free vector lattices:

Proposition 1.22 ([15, Proposition 3.2]). *If F together with $\phi\colon A \longrightarrow F$ is a free vector lattice over a non-empty set A, then F is generated, as a vector lattice, by $\phi(A)$.*

Proposition 1.23 ([15, Proposition 3.3]). *If F_1 together with the function $\phi_1\colon A \longrightarrow F_1$ and F_2 together with the function $\phi_2\colon A \longrightarrow F_2$ are free vector lattices over a non-empty set A, then there is a unique vector lattice isomorphism $T\colon X_1 \longrightarrow X_2$ such that $T(\phi_1(a)) = \phi_2(a)$ for every $a \in A$.*

Then, we can say *the* free vector lattice over A instead of *a* free vector lattice over A because all free vector lattices over A are isomorphic as vector lattices between them. We will denote it by $FVL(A)$. Moreover, if A and B are sets of equal cardinality, then $FVL(A)$ and $FVL(B)$ are isomorphic vector lattices, so that $FVL(A)$ depends only on the cardinality of the set A.

Now, the question is whether such an object exists. Let us denote by $\mathbb{R}^{\mathbb{R}^A}$ the vector lattice consisting of all functions $f\colon \mathbb{R}^{\mathbb{A}} \longrightarrow \mathbb{R}$ together with the pointwise order and the pointwise operations. Then, we have that:

Theorem 1.24 ([15, Theorem 3.6]). *For any non-empty set A, $FVL(A)$ exists and is the vector sublattice of $\mathbb{R}^{\mathbb{R}^A}$ generated by δ_a $(a \in A)$, where $\delta_a(x^*) = x^*(a)$ for every $x^* \in \mathbb{R}^A$.*

Thus, identifying a with δ_a, one may view A as a subset of $FVL(A)$. Since $FVL(A)$ is a sublattice of $\mathbb{R}^{\mathbb{R}^A}$, and the latter is Archimedean, $FVL(A)$ is also Archimedean.

It is easy to see that if $a_1, \dots, a_n \in A$, then $FVL\big(\{a_1, \dots, a_n\}\big)$ may be identified with the vector sublattice of $FVL(A)$ generated by $\{\delta_{a_1}, \dots, \delta_{a_n}\}$ (see [15, Proposition 3.5]). Moreover, if $f \in FVL(A)$, then there exists a finite subset $\{a_1, \dots, a_n\}$ of A such that $f \in FVL\big(\{a_1, \dots, a_n\}\big)$ (see [15, Proposition 3.7]).

Note that $FVL(A)$ may be interpreted as the set of all lattice-linear expressions of elements of A, where we identify two expressions if they are equal when we substitute the elements of A by real numbers.

1.3 Banach lattices

In this section we show the basic definitions and properties concerning Banach lattices and the free Banach lattices generated by a set and a Banach space.

1.3.1 Basic concepts

Most of the definitions, and the example we show here, are extracted from [38].

Definition 1.25. A *Banach lattice* is a vector lattice X together with a norm that is also a Banach space in which $|x| \leq |y| \Rightarrow \|x\| \leq \|y\|$ (i.e. $\|\cdot\|$ is a *lattice norm*). A *Banach sublattice* Y of a Banach lattice is simply a vector subspace which is also a sublattice and closed under the norm of the Banach lattice. This makes Y a Banach lattice.

Example 1.26. Classical examples of Banach lattices are $\ell_p, c_0, c, C(K)$ and $L^p(\mu)$ with their usual norm and the pointwise (almost everywhere in the last case) order.

Definition 1.27. Let A be a non-empty subset of a Banach lattice X. The *Banach sublattice generated* (or *Banach lattice generated*) *by* A is the smallest Banach sublattice of X containing A.

Definition 1.28. A map $T\colon X \longrightarrow Y$ between two Banach lattices, X and Y, is said to be a *Banach lattice homomorphism* if it is a bounded linear operator and preserves the lattice operations.

If T is also bijective and T^{-1} is a Banach lattice homomorphism, we will say that T is a *Banach lattice isomorphism*. If moreover, T preserves the norm (that is, $\|T(x)\| = \|x\|$ for every $x \in X$), we will say that T is a *Banach lattice isometry*.

If X is a Banach lattice and $\mathcal{I}$ is a closed ideal in X, we can define an equivalence relation on X given by $x \sim_{\mathcal{I}} y$ if, and only if, $x - y \in \mathcal{I}$.

For $x \in X$, if we denote by $x + \mathcal{I} := \{y \in X : x \sim_{\mathcal{I}} y\}$ the equivalence class of x, the set $X/\mathcal{I} := \{x + \mathcal{I} : x \in X\}$ together with the operations

$$(x + \mathcal{I}) + (y + \mathcal{I}) := (x + y) + \mathcal{I} \text{ and } \lambda(x + \mathcal{I}) := \lambda x + \mathcal{I} \text{ for every } \lambda \in \mathbb{R},$$

is a real vector space.

If we equip this vector space with the quotient norm, which is defined as

$$\|x\|_{\mathcal{I}} = \|x + \mathcal{I}\|_{\mathcal{I}} := \inf\{\|y\| : y \in X, x \sim_{\mathcal{I}} y\} = \inf\{\|x + y\| : y \in \mathcal{I}\},$$

we obtain a Banach space (see [16, Proposition 1.21]).

Moreover, this Banach space together with the operations

$$(x + \mathcal{I}) \vee (y + \mathcal{I}) := (x \vee y) + \mathcal{I} \text{ and } (x + \mathcal{I}) \wedge (y + \mathcal{I}) := (x \wedge y) + \mathcal{I}$$

is a Banach lattice (see [28, page 85, Proposition 5.4]).

It is easy to check that the map $Q\colon X \longrightarrow X/\mathcal{I}$ given by $Q(x) = x + \mathcal{I}$ for every $x \in X$ is a surjective Banach lattice homomorphism of norm 1. We will call this map the *quotient map*.

1.3.2 The free Banach lattice generated by a set

As the title of the thesis indicates, the main objects of study in this thesis are the free Banach lattices over Banach spaces and ordered sets.

The first authors who introduced the concept of free object within the category of Banach lattices were B. de Pagter and A. W. Wickstead in 2015, who defined and studied properties about the free Banach lattice generated by a set [15].

Definition 1.29. Let A be a non-empty set. A *free Banach lattice over* or *generated by* A is a Banach lattice F together with a bounded map $\phi\colon A \longrightarrow F$ with the property that for every

Banach lattice X and every bounded map $T\colon A \longrightarrow X$ there is a unique Banach lattice homomorphism $\hat{T}\colon F \longrightarrow X$ such that $T = \hat{T} \circ \phi$ and $\|\hat{T}\| = \|T\|$.

$$\begin{tikzcd} A \arrow[r, "T"] \arrow[d, "\phi"'] & X \\ F \arrow[ur, "\hat{T}"'] & \end{tikzcd}$$

Here, the norm of T is $\|T\| := \sup\{\|T(a)\| : a \in A\}$, while the norm of $\hat{T}$ is the usual for Banach spaces.

It is easy to check (see [15, Remark 4.2]) that this definition forces that $\|\phi(a)\| = 1$ for every $a \in A$, so the norm $\|\phi\| = \sup\{\|\phi(a)\| : a \in A\} = 1$.

Similar to Proposition 1.22 for the free vector lattice generated by A, we have that:

Proposition 1.30 ([15]). *If F together with $\phi\colon A \longrightarrow F$ is a free Banach lattice over a non-empty set A, then F is generated, as a Banach lattice, by $\phi(A)$.*

Moreover, we have that:

Proposition 1.31 ([15, Proposition 4.3]). *If F_1 together with the function $\phi_1\colon A \longrightarrow F_1$ and F_2 together with the function $\phi_2\colon A \longrightarrow F_2$ are free Banach lattices over a non-empty set A, then there is a unique Banach lattice isometry $T\colon F_1 \longrightarrow F_2$ such that $T(\phi_1(a)) = \phi_2(a)$ for every $a \in A$.*

Then, we can speak of *the* free Banach lattice over A instead of *a* free Banach lattice over A because all free Banach lattices over A are isometric as Banach lattices between them. We will denote it by $FBL(A)$.

Now, the question is whether such an object exists. The answer is affirmative. B. de Pagter and A. W. Wickstead prove it in [15], but A. Avilés, J. Rodríguez and P. Tradacete give an alternative and more tangible way of constructing it in [8]. They describe it as a space of functions:

For $a \in A$, let $\delta_a\colon [-1,1]^A \longrightarrow \mathbb{R}$ be the evaluation function given by $\delta_a(x^*) = x^*(a)$ for every $x^* \in [-1,1]^A$, and for $f\colon [-1,1]^A \longrightarrow \mathbb{R}$ define

$$\|f\| = \sup\left\{\sum_{i=1}^{n} |f(x_i^*)| : n \in \mathbb{N},\ x_1^*, \ldots, x_n^* \in [-1,1]^A,\ \sup_{a\in A} \sum_{i=1}^{n} |x_i^*(a)| \le 1\right\},$$

which we will denote by $\|f\|$ or $\|f\|_{FBL(A)}$.

Theorem 1.32 ([8, Corollary 2.9]). *The free Banach lattice generated by a set A is the closure of the vector lattice generated by $\{\delta_a : a \in A\}$ under the above norm inside the Banach lattice of all functions $f \in \mathbb{R}^{[-1,1]^A}$ with $\|f\| < \infty$, endowed with the norm $\|\cdot\|$, the pointwise order and the pointwise operations.*

The natural identification of A inside $FBL(A)$ is given by the map $\phi\colon A \longrightarrow FBL(A)$ where $\phi(a) = \delta_a$ for every $a \in A$. Since every function in $FBL(A)$ is a uniform limit of such functions, they are all continuous (in the product topology) and positively homogeneous (that is, $f(\lambda x^*) = \lambda f(x^*)$ for every $x^* \in [-1,1]^A$ and for every $\lambda \geq 0$ such that $\lambda x^* \in [-1,1]^A$, or equivalently, $f(\lambda x^*) = \lambda f(x^*)$ for every $x^* \in [-1,1]^A$ and for every $0 \leq \lambda \leq 1$).

On the other hand, we can view $FBL(A)$ as $FVL(A)$ equipped with the greatest lattice norm one can put on it. Indeed, V. G. Troitsky proved the following:

Theorem 1.33 ([36, Theorem 2.1]). *There exists a maximal lattice seminorm ν on $FVL(A)$ with $\nu(a) \leq 1$ for every $a \in A$. It is a lattice norm, and the completion of $FVL(A)$ with respect to it is $FBL(A)$.*

1.3.3 The free Banach lattice generated by a Banach space

The concept of a Banach lattice freely generated by a given Banach space has been recently introduced and investigated by A. Avilés, J. Rodríguez and P. Tradacete in [8], and provides a new tool for better understanding the relation between Banach spaces and Banach lattices.

Consider any Banach space E. Roughly speaking, the free Banach lattice generated by E is a Banach lattice F which contains a subspace linearly isometric with E in such a way that its elements work as lattice-free generators. More formally:

Definition 1.34. Let E be a Banach space. A *free Banach lattice over* or *generated by E* is a Banach lattice F together with a bounded operator $\phi\colon E \longrightarrow F$ with the property that for every Banach lattice X and every bounded operator $T\colon E \longrightarrow X$ there is a unique Banach lattice homomorphism $\hat{T}\colon F \longrightarrow X$ such that $T = \hat{T} \circ \phi$ and $\|\hat{T}\| = \|T\|$.

$$\begin{tikzcd} E \arrow[r, "T"] \arrow[d, "\phi"'] & X \\ F \arrow[ur, "\hat{T}"'] & \end{tikzcd}$$

This property uniquely determines F up to Banach lattices isometries, and so we can speak of *the* free Banach lattice generated by E, denoted by $FBL[E]$. This definition generalizes the notion of the free Banach lattice generated by a set A. Namely, the free Banach lattice generated by a set A is the free Banach lattice generated by the Banach space $\ell_1(A)$ (see [8, Corollary 2.9]).

Again, it is possible to give an explicit description of it as a space of functions:

Let us denote by $H[E]$ the vector subspace of $\mathbb{R}^{E^*}$ consisting of all positively homogeneous functions $f\colon E^* \longrightarrow \mathbb{R}$ (that is, all functions that satisfy $f(\lambda x^*) = \lambda f(x^*)$ for every $x^* \in E^*$ and for every $\lambda \geq 0$). For any $f \in H[E]$ let us define

$$\|f\|_{FBL[E]} = \sup\left\{ \sum_{i=1}^{n} |f(x_i^*)| : n \in \mathbb{N},\, x_1^*, \ldots, x_n^* \in E^*,\ \sup_{x \in B_E} \sum_{i=1}^{n} |x_i^*(x)| \leq 1 \right\}.$$

Let us take $H_0[E] = \left\{ f \in H[E] : \|f\|_{FBL[E]} < \infty \right\}$. It is easy to check that $H_0[E]$ is a Banach lattice when equipped with the norm $\|\cdot\|_{FBL[E]}$ and the pointwise order.

Now, given $x \in E$, let $\delta_x\colon E^* \longrightarrow \mathbb{R}$ be the evaluation function given by $\delta_x(x^*) = x^*(x)$ for every $x^* \in E^*$.

Theorem 1.35 ([8, Theorem 2.5]). *The free Banach lattice generated by a Banach space E is the closure of the vector lattice generated by $\{\delta_x : x \in E\}$ under the above norm inside $H_0[E]$.*

The natural identification of E inside $FBL[E]$ is given by the map $\phi\colon E \longrightarrow FBL[E]$ where $\phi(x) = \delta_x$ for every $x \in E$ (it is a linear isometry between E and its image in $FBL[E]$). Moreover, all the functions in $FBL[E]$ are weak*-continuous when restricted to the closed unit ball B_{E^*} (see [8, Lemma 4.10]).

Similar to the previous case, V. G. Troitsky also proved the following:

Theorem 1.36 ([36, Theorem 3.1]). *Let E be a Banach space, and let L be the vector sublattice of $\mathbb{R}^{E^*}$ generated by $\{\delta_x : x \in E\}$. There is a maximal lattice seminorm ν on L satisfying $\nu(\delta_x) \leq \|x\|$ for every $x \in E$. It is a lattice norm and the completion of L with respect to it is $FBL[E]$.*

It is natural to wonder about the motivation for the explicit expression for the norm of $FBL[E]$ and $FBL(A) = FBL[\ell_1(A)]$. It is explained in [8] and [11], but we summarize it here for the convenience of the reader:

The free vector lattice generated by E (seen as a set), $FVL(E)$, can be identified with the vector sublattice of $H[E]$ generated by the evaluation functions δ_x ($x \in E$). The norm of $FBL[E]$ must be the largest possible lattice norm that we can define on this space. In particular, given arbitrary $(x_k^*)_{k=1}^n \in E^*$, we can define a bounded operator $T\colon E \longrightarrow \ell_1^n$ by the expression $T(x) = (x_k^*(x))_{k=1}^n$ for every $x \in E$. It is easy to check that the Banach lattice homomorphism $\hat{T}\colon FBL[E] \longrightarrow \ell_1^n$ extending T is necessarily given by $\hat{T}(f) = (f(x_k^*))_{k=1}^n$ for every $f \in FVL(E)$. Hence, the norm of $FBL[E]$ must satisfy the inequality

$$\|\hat{T}(f)\|_{\ell_1^n} \leq \|T\| \|f\|_{FBL[E]}.$$

Therefore, we have that

$$\|f\|_{FBL[E]} \geq \frac{\|\hat{T}(f)\|_{\ell_1^n}}{\|T\|} = \frac{\sum_{k=1}^n |f(x_k^*)|}{\sup_{x \in B_E} \sum_{k=1}^n |x_k^*(x)|},$$

which motivates the explicit expression for the norm of $FBL[E]$ and $FBL(A)$.

Chapter 2

The free Banach lattice generated by a lattice

2.1 Introduction

The purpose of this chapter is to introduce the free Banach lattice generated by a lattice, prove its existence and give an explicit description of it as a space of functions. Its content is basically extracted from our publication:

[9] AVILÉS, A., AND RODRÍGUEZ ABELLÁN, J. D. The free Banach lattice generated by a lattice. *Positivity 23* (2019), 581–597

The idea is similar to the free Banach lattices explained in [Chapter 1, Sections 1.3.2 and 1.3.3], using lattices instead of sets without any structure or Banach spaces.

Remember that a lattice is a set $\mathbb{L}$ together with two operations $\wedge$ and $\vee$ that are the infimum and supremum of some partial order relation on $\mathbb{L}$, and a lattice homomorphism is a function between lattices that commutes with the two operations.

Definition 2.1. Given a lattice $\mathbb{L}$, a *free Banach lattice over* or *generated by* $\mathbb{L}$ is a Banach lattice F together with a bounded lattice homomorphism $\phi\colon \mathbb{L} \longrightarrow F$ with the property that for every Banach lattice X and every bounded lattice homomorphism $T\colon \mathbb{L} \longrightarrow X$ there is a unique Banach lattice homomorphism $\hat{T}\colon F \longrightarrow X$ such that $T = \hat{T} \circ \phi$ and $\|\hat{T}\| = \|T\|$.

$$\begin{tikzcd} \mathbb{L} \arrow[r, "T"] \arrow[d, "\phi"'] & X \\ F \arrow[ur, "\hat{T}"'] & \end{tikzcd}$$

Here, the norm of T is $\|T\| := \sup\{\|T(x)\| : x \in \mathbb{L}\}$, while the norm of $\hat{T}$ is the usual for Banach spaces.

This definition determines a Banach lattice that we denote by $FBL\langle\mathbb{L}\rangle$ in an essentially unique way. When $\mathbb{L}$ is a distributive lattice (which is a natural assumption in this context, see Section 2.3) the function ϕ is injective and, loosely speaking, we can view $FBL\langle\mathbb{L}\rangle$ as a Banach lattice which contains a subset lattice-isomorphic to $\mathbb{L}$ in a way that its elements work as free generators modulo the lattice relations on $\mathbb{L}$.

Using the existence of the free Banach lattice generated by the set $\mathbb{L}$ (viewing $\mathbb{L}$ as a set with no extra structure), $FBL(\mathbb{L})$, we will prove that $FBL\langle\mathbb{L}\rangle$ also exists and that can be viewed as a certain quotient of $FBL(\mathbb{L})$.

In order to give an explicit description of it similar to the mentioned in [Chapter 1, Theorems 1.32 and 1.35], define

$$\mathbb{L}^* = \{x^* \colon \mathbb{L} \longrightarrow [-1,1] : x^* \text{ is a lattice homomorphism}\}.$$

For every $x \in \mathbb{L}$ consider the evaluation function $\dot{\delta}_x \colon \mathbb{L}^* \longrightarrow \mathbb{R}$ given by $\dot{\delta}_x(x^*) = x^*(x)$, and for $f \in \mathbb{R}^{\mathbb{L}^*}$, define

$$\|f\|_* = \sup\left\{\sum_{i=1}^n |f(x_i^*)| : n \in \mathbb{N},\ x_1^*, \ldots, x_n^* \in \mathbb{L}^*,\ \sup_{x\in\mathbb{L}} \sum_{i=1}^n |x_i^*(x)| \leq 1\right\}.$$

Theorem 2.2. *Consider F to be the closure of the vector lattice generated by $\{\dot{\delta}_x : x \in \mathbb{L}\}$ under the norm $\|\cdot\|_*$ inside the Banach lattice of all functions $f \in \mathbb{R}^{\mathbb{L}^*}$ with $\|f\|_* < \infty$, endowed with the norm $\|\cdot\|_*$, the pointwise order and the pointwise operations. Then F, together with the assignment $\phi(x) = \dot{\delta}_x$, is the free Banach lattice generated by $\mathbb{L}$.*

In spite of the similarity to the Banach space case from [8], our proof requires completely different techniques. Section 2.4 is entirely devoted to this. In Section 2.5 we check, when $\mathbb{L}$ is a linearly ordered set, that the elements of $\mathbb{L}$ inside $FBL\langle\mathbb{L}\rangle$ behave like the summing basis of c_0 from a Banach space point of view.

Later, in [Chapter 3, Section 3.2], we will study when the free Banach lattice also generated by a linearly ordered set satisfies the countable chain condition. Moreover, in [Chapter 4, Sections 4.3 and 4.4], we will focus on the projectivity of such an object.

2.2 The Banach lattice $FBL\langle\mathbb{L}\rangle$ as a quotient of a space of functions

Throughout this section $\mathbb{L}$ is a fixed lattice. Let us start by checking that Definition 2.1 forces each $\phi(x)$ to have norm precisely 1. Indeed, if we take $T \colon \mathbb{L} \longrightarrow \mathbb{R}$ the bounded lattice homomorphism given by $T(x) = 1$ for every $x \in \mathbb{L}$, then the Banach lattice homomorphism $\hat{T}$ that is guaranteed to exist has norm 1, so that $1 = \|\hat{T}(\phi(x))\| \leq \|\phi(x)\|$. On the other hand, if we take $T = \phi$, then $\hat{T}$ is the identity Banach lattice homomorphism, with norm 1, so that $\|\phi\| = \sup\{\|\phi(x)\| : x \in \mathbb{L}\} = 1$.

On the other hand, Definition 2.1 provides a uniquely determined object. If $\phi \colon \mathbb{L} \longrightarrow F$ and $\phi' \colon \mathbb{L} \longrightarrow F'$ satisfy this definition, then we can get a Banach lattice homomorphism

$\hat{\phi}' : F \longrightarrow F'$ with $\phi' = \hat{\phi}' \circ \phi$. Reversing the roles, we also get $\hat{\phi} : F' \longrightarrow F$ with $\phi = \hat{\phi} \circ \phi'$. The function $\hat{\phi} \circ \hat{\phi}'$ and the identity function id_F on F both satisfy Definition 2.1 as $\hat{T}$ when $T = \phi$. So $\hat{\phi} \circ \hat{\phi}' = id_F$. Similarly, reversing roles, $\hat{\phi}' \circ \hat{\phi} = id_{F'}$. Thus, we obtained inverse Banach lattice homomorphism of norm 1 between F and F' that commute with ϕ and ϕ'.

Now, we are going to construct a Banach lattice F that satisfies Definition 2.1. We will show later that the Banach lattice described in Theorem 2.2 also satisfies Definition 2.1. We take as a starting point that, when we view $\mathbb{L}$ as a set with no extra structure, we have the free Banach lattice $FBL(\mathbb{L})$, together with a bounded map $u : \mathbb{L} \longrightarrow FBL(\mathbb{L})$ $(u(x) = \delta_x)$, constructed by de B. Pagter and A. W. Wickstead, whose universal property was described in [Chapter 1, Section 1.3.2]. Take $\mathcal{I}$ the closed ideal in $FBL(\mathbb{L})$ generated by

$$\{u(x) \vee u(y) - u(x \vee y), \ \ u(x) \wedge u(y) - u(x \wedge y) \ : \ x, y \in \mathbb{L}\}.$$

We take $F = FBL(\mathbb{L})/\mathcal{I}$, and $\phi : \mathbb{L} \longrightarrow FBL(\mathbb{L})/\mathcal{I}$ given by $\phi(x) = u(x) + \mathcal{I}$. The very definition of $\mathcal{I}$ provides that ϕ is a bounded lattice homomorphism. Now, let X be a Banach lattice and $T : \mathbb{L} \longrightarrow X$ a bounded lattice homomorphism. We know that $FBL(\mathbb{L})$ satisfies the universal property of the free Banach lattice. Therefore, there exists a Banach lattice homomorphism $\hat{T}^1 : FBL(\mathbb{L}) \longrightarrow X$ such that $\hat{T}^1 \circ u = T$ and $\|\hat{T}^1\| = \|T\|$. The fact that T was a lattice homomorphism implies that $\hat{T}^1$ vanishes on $\mathcal{I}$. Thus, we can have a Banach lattice homomorphism $\hat{T} : FBL(\mathbb{L})/\mathcal{I} \longrightarrow X$ given by $\hat{T}(f + \mathcal{I}) = \hat{T}^1(f)$. It is clear that $\hat{T} \circ \phi = T$. Let us see that $\|T\| = \|\hat{T}\|$. We only need to check that $\|T\| \geq \|\hat{T}\|$. Let $f + \mathcal{I} \in FBL(\mathbb{L})/\mathcal{I}$ with $\|f\|_{\mathcal{I}} < 1$. We have that

$$\|f\|_{\mathcal{I}} = \inf \{\|f + g\| : g \in \mathcal{I}\},$$

and, therefore, there exists $g \in \mathcal{I}$ such that $\|f + g\| < 1$. Thus, $\|\hat{T}(f + \mathcal{I})\| = \|\hat{T}^1(f + g)\| \leq \|T\|$. Only the uniqueness of the extension T remains to be checked. But this follows from the uniqueness of the extension to $FBL(\mathbb{L})$, because if $\hat{T} \circ \phi = T$, then $\hat{T} \circ \pi \circ u = T$, where $\pi : FBL(\mathbb{L}) \longrightarrow FBL(\mathbb{L})/\mathcal{I}$ is the quotient map.

We have proved that $F = FBL(\mathbb{L})/\mathcal{I}$ together with ϕ above satisfy Definition 2.1. Now, our aim is to make this represantion more concrete. We are going to give an explicit description of it as a space of functions, but before that, we need to show some additional observations.

2.3 Distributivity

A lattice $\mathbb{L}$ is said to be *distributive* if the two operations $\wedge$ and $\vee$ distribute each other. That is, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for all $x, y, z \in \mathbb{L}$. For a lattice $\mathbb{L}$, let $\widetilde{\mathbb{L}} = \phi(\mathbb{L})$ be the image of $\mathbb{L}$ inside $FBL\langle\mathbb{L}\rangle$. The following proposition collects some well known facts and observations:

Proposition 2.3. *For a lattice $\mathbb{L}$ the following are equivalent:*

1. *$\mathbb{L}$ is distributive,*

2. *$\mathbb{L}$ is lattice-isomorphic to a subset of a Boolean algebra,*

3. *$\mathbb{L}$ is lattice-isomorphic to a bounded subset of a Banach lattice,*

4. *The canonical map $\phi\colon \mathbb{L} \longrightarrow FBL\langle\mathbb{L}\rangle$ is injective.*

Proof. The equivalence of 1, 2 and 3 is well known, see [17, Theorem II.19] for $1 \Rightarrow 2$, [23, Theorem 1.b.3] for $2 \Rightarrow 3$ and [28, Proposition II.1.5] for $3 \Rightarrow 1$. It is obvious that 4 implies 3. If 3 holds, then we have a bounded injective lattice homomorphism $T\colon \mathbb{L} \longrightarrow X$ for some Banach lattice X. Using Definition 2.1, there is $\hat{T}\colon FBL\langle\mathbb{L}\rangle \longrightarrow X$ such that $\hat{T} \circ \phi = T$. Since T is injective, ϕ is injective and therefore 4 holds. □

Proposition 2.4. *$FBL\langle\mathbb{L}\rangle = FBL\langle\widetilde{\mathbb{L}}\rangle$. More precisely, if F with ϕ is the free Banach lattice over the lattice $\mathbb{L}$, then F with the inclusion map is the free Banach lattice over the lattice $\widetilde{\mathbb{L}}$.*

The proof is immediate from Definition 2.1. The conclusion of these observations is that the most natural case in which to consider $FBL\langle\mathbb{L}\rangle$ is when $\mathbb{L}$ is distributive, and that the case of general $\mathbb{L}$ reduces to the distributive case in a natural easy way. Still, we find that it may be useful to state the results for any lattice $\mathbb{L}$. Two more facts:

Proposition 2.5. *Every lattice homomorphism $x^*\colon \mathbb{L} \longrightarrow [-1,1]$ factors through $\widetilde{\mathbb{L}}$. That is, there exists $y^*\colon \widetilde{\mathbb{L}} \longrightarrow [-1,1]$ such that $x^* = y^* \circ \phi$.*

Proof. Find a Banach lattice homomorphism of norm at most 1 $\hat{x}^*\colon FBL\langle\mathbb{L}\rangle \longrightarrow \mathbb{R}$ with $x^* = \hat{x}^* \circ \phi$, as in Definition 2.1. Take $y^* = \hat{x}^*|_{\widetilde{\mathbb{L}}}$. □

Proposition 2.6 ([12, Lemma III.3]). *Every finitely generated sublattice of a distributive lattice is finite.*

2.4 The Banach lattice $FBL\langle\mathbb{L}\rangle$ as a space of functions

This section is devoted to the proof of Theorem 2.2. First of all, note that the vector space $\left\{f \in \mathbb{R}^{\mathbb{L}^*} : \|f\|_* < \infty\right\}$ together with the norm $\|\cdot\|_*$ and the pointwise order is a Banach lattice. Indeed, the only possibly delicate point is the fact that if $f \in \mathbb{R}^{\mathbb{L}^*}$ has norm 0, then $f = 0$, but this is straightforward because if $\|f\|_* = 0$, then for every $n \in \mathbb{N}$ and for every $x_1^*, \ldots, x_n^* \in \mathbb{L}^*$ with $\sup_{x\in\mathbb{L}} \sum_{i=1}^n |x_i^*(x)| \leq 1$ we have that $\sum_{i=1}^n |f(x_i^*)| = 0$. Taking $n = 1$, since $\sup_{x\in\mathbb{L}} |x^*(x)| \leq 1$ for every $x^* \in \mathbb{L}^*$, we have in particular that $|f(x^*)| = 0$ for every $x^* \in \mathbb{L}^*$, so $f = 0$.

Now, let $FBL_*\langle\mathbb{L}\rangle$ be the Banach lattice described in that theorem. By Propositions 2.4 and 2.5, both $FBL\langle\mathbb{L}\rangle$ and $FBL_*\langle\mathbb{L}\rangle$ remain unchanged if we change $\mathbb{L}$ by $\widetilde{\mathbb{L}}$. So we can assume throughout this section that $\mathbb{L}$ is distributive. Since we already know that $FBL(\mathbb{L})/\mathcal{I}$, where $\mathcal{I}$ is the ideal defined in Section 2.2, is the free Banach lattice over the lattice $\mathbb{L}$, what we have to do is to find a Banach lattice isometry $T\colon FBL(\mathbb{L})/\mathcal{I} \longrightarrow FBL_*\langle\mathbb{L}\rangle$ such that $T(\delta_x + \mathcal{I}) = \dot{\delta}_x$.

We know that $FBL(\mathbb{L}) = \overline{lat}^{\|\cdot\|}\{\delta_x : x \in \mathbb{L}\} \subset \mathbb{R}^{[-1,1]^{\mathbb{L}}}$, where

$$\|f\| = \sup\left\{\sum_{i=1}^n |f(x_i^*)| : n \in \mathbb{N},\ x_1^*, \ldots, x_n^* \in [-1,1]^{\mathbb{L}},\ \sup_{x\in\mathbb{L}} \sum_{i=1}^n |x_i^*(x)| \leq 1\right\},$$

and recall that $FBL_*\langle\mathbb{L}\rangle = \overline{lat}^{\|\cdot\|_*}\left\{\dot{\delta}_x : x \in \mathbb{L}\right\} \subset \mathbb{R}^{\mathbb{L}^*}$, where

$$\|f\|_* = \sup\left\{\sum_{i=1}^n |f(x_i^*)| : n \in \mathbb{N},\ x_1^*, \dots, x_n^* \in \mathbb{L}^*,\ \sup_{x\in\mathbb{L}} \sum_{i=1}^n |x_i^*(x)| \leq 1\right\}.$$

For every function $f\colon [-1,1]^{\mathbb{L}} \longrightarrow \mathbb{R}$, consider its restriction $R(f) = f|_{\mathbb{L}^*}$. It is clear that the function R commutes with linear combinations and the lattice operations and that $\|R(f)\|_* \leq \|f\|$. Moreover, $R(\delta_x) = \dot{\delta}_x$ for every $x \in \mathbb{L}$. From this, we conclude that if $f \in FBL(\mathbb{L})$, then $R(f) \in FBL_*\langle\mathbb{L}\rangle$, and we can view $R\colon FBL(\mathbb{L}) \longrightarrow FBL_*\langle\mathbb{L}\rangle$ as a Banach lattice homomorphism of norm 1. Moreover, since $\mathbb{L}^*$ consists of lattice homomorphisms, R vanishes on the ideal $\mathcal{I}$. Thus, we have a Banach lattice homomorphism of norm at most 1

$$R_{\mathcal{I}}\colon FBL(\mathbb{L})/\mathcal{I} \longrightarrow FBL_*\langle\mathbb{L}\rangle$$

given by $R_{\mathcal{I}}(f+\mathcal{I}) = R(f)$ for every $f + \mathcal{I} \in FBL(\mathbb{L})/\mathcal{I}$. What we want to prove is that $R_{\mathcal{I}}$ is an isometry. That is, we have to show that

$$\|f\|_{\mathcal{I}} \leq \|f|_{\mathbb{L}^*}\|_*$$

for every $f \in FBL(\mathbb{L})$.

First, suppose that $\mathbb{L}$ is finite (say $\mathbb{L} = \{0, \dots, n-1\} = n$ with some lattice operations). B. de Pagter and A. W. Wickstead showed that in this case, $FBL(\mathbb{L})$ consists exactly of all the positively homogeneous continuous functions on $[-1,1]^{\mathbb{L}} = [-1,1]^n$ (see [15, Proposition 5.3]). Moreover, if we consider the boundary $\partial[-1,1]^n$, and the Banach lattice of continuous functions $C(\partial[-1,1]^n)$, the restriction map $P\colon FBL(\mathbb{L}) \longrightarrow C(\partial[-1,1]^n)$ is a Banach lattice isomorphism (it is not however, an isometry: the norm of $FBL(\mathbb{L})$ is transferred to a lattice norm that is equivalent to the supremum norm).

A closed ideal in a Banach lattice of continuous functions on a compact space always consists of the functions that vanish on a certain closed set. Thus, there exists a closed set $S \subset \partial[-1,1]^n$ such that

$$\mathcal{I} = \{f \in FBL(\mathbb{L}) : f|_S = 0\}.$$

In fact, the points of S must be those where f vanish for all $f \in \mathcal{I}$, or equivalently, for all generators f of $\mathcal{I}$:

$$S = \{(\xi_x)_{x\in\mathbb{L}} \in \partial[-1,1]^n : \xi_x \vee \xi_y = \xi_{x\vee y},\ \xi_x \wedge \xi_y = \xi_{x\wedge y},\ x, y \in \mathbb{L}\} = \mathbb{L}^* \cap \partial[-1,1]^n.$$

Now fix $f \in FBL(\mathbb{L})$, and let us prove that $\|f\|_{\mathcal{I}} \leq \|f|_{\mathbb{L}^*}\|_*$. Remember that

$$\|f|_{\mathbb{L}^*}\|_* = \sup\left\{\sum_{i=1}^m |f(x_i^*)| : m \in \mathbb{N},\ x_1^*, \dots, x_m^* \in \mathbb{L}^*,\ \sup_{x\in\mathbb{L}} \sum_{i=1}^m |x_i^*(x)| \leq 1\right\},$$

and

$$\|f\|_{\mathcal{I}} = \inf\{\|f+g\| : g \in \mathcal{I}\} = \inf\{\|g\| : g \in FBL(\mathbb{L}), f \sim_{\mathcal{I}} g\},$$

where $f \sim_{\mathcal{I}} g$ if, and only if, $f - g \in \mathcal{I}$.

Given $k \in \mathbb{N}$, let

$$S_k^+ = \left\{ x^* \in \partial[-1,1]^n : d(x^*, S) < \frac{1}{k} \right\}$$

and

$$S_k^- = \left\{ x^* \in \partial[-1,1]^n : d(x^*, S) \geq \frac{1}{k} \right\},$$

where d is the square metric in $\mathbb{R}^n$.

Since S and S_k^- are disjoint closed subsets of $\partial[-1,1]^n$, by Urysohn's lemma we can find a continuous function $\widetilde{1}_k \colon \partial[-1,1]^n \longrightarrow [0,1]$ such that $\widetilde{1}_k(S) = 1$ and $\widetilde{1}_k(S_k^-) = 0$.

Define $f_k = P^{-1}(\widetilde{1}_k f|_S) \in FBL(\mathbb{L})$ be the positively homogeneous extension of $\widetilde{1}_k f|_S$ to the cube $[-1,1]^n$. Since $f_k|_S = f|_S$, we have that $f_k \sim_{\mathcal{I}} f$ for every k. Therefore, it is enough to prove that for a given $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $\|f_k\| \leq \|f|_{\mathbb{L}^*}\|_* + \varepsilon$.

We have that

$$\|f|_{\mathbb{L}^*}\|_* = \sup \left\{ \sum_{i=1}^m |r_i f(x_i^*)| : x_1^*, \ldots, x_m^* \in S,\ r_1, \ldots, r_m \in \mathbb{R},\ \sup_{x \in \mathbb{L}} \sum_{i=1}^m |r_i x_i^*(x)| \leq 1 \right\},$$

$$\|f_k\| = \sup \left\{ \sum_{i=1}^m |r_i f_k(x_i^*)| : x_1^*, \ldots, x_m^* \in \partial[-1,1]^n,\ r_1, \ldots, r_m \in \mathbb{R},\ \sup_{x \in \mathbb{L}} \sum_{i=1}^m |r_i x_i^*(x)| \leq 1 \right\}.$$

Notice that the scalars $r_1, \ldots, r_m \in \mathbb{R}$ that appear in these formulas always satisfy $\sum_{i=1}^m |r_i| \leq n$. This is because for every i we can find $\xi_i \in \mathbb{L}$ with $x_i^*(\xi) = \pm 1$, and then,

$$\sum_{i=1}^m |r_i| = \sum_{\xi \in \mathbb{L}} \sum_{\xi_i = \xi} |r_i x_i^*(\xi)| \leq \sum_{\xi \in \mathbb{L}} 1 = n.$$

The function f is bounded and uniformly continuous on $[-1,1]^n$, so we can pick $k \in \mathbb{N}$ satisfying the following two conditions:

1. For all $x^*, y^* \in [-1,1]^n$, if $d(x^*, y^*) \leq \frac{1}{k}$, then $|f(x^*) - f(y^*)| < \varepsilon/2n$.
2. $\frac{Mn^2}{n+k} < \frac{\varepsilon}{2}$, where $M = \max\{|f(y^*)| : y^* \in [-1,1]^n\}$.

By the definition of S_k^+, given $x_i^* \in S_k^+$, there exists $y_i^* \in S$ such that $d(x_i^*, y_i^*) \leq \frac{1}{k}$. When $x_i^* \in S$, we can take $y_i^* = x_i^*$. In this way, we can estimate any sum in the supremum that gives $\|f_k\|$ as follows:

$$\begin{aligned}
\sum_{i=1}^m |r_i f_k(x_i^*)| &= \sum_{x_i^* \in S_k^+} |r_i f_k(x_i^*)| + \sum_{x_i^* \in S_k^-} |r_i f_k(x_i^*)| \\
&= \sum_{x_i^* \in S_k^+} |r_i f_k(x_i^*)| \leq \sum_{x_i^* \in S_k^+} |r_i f(x_i^*)| \\
&\leq \sum_{x_i^* \in S_k^+} |r_i f(y_i^*)| + \sum_{x_i^* \in S_k^+} |r_i|\, |f(x_i^*) - f(y_i^*)| \\
&\leq \sum_{x_i^* \in S_k^+} |r_i f(y_i^*)| + \frac{\varepsilon}{2n} \sum_{x_i^* \in S_k^+} |r_i| \\
&\leq \sum_{x_i^* \in S_k^+} |r_i f(y_i^*)| + \frac{\varepsilon}{2}.
\end{aligned}$$

We have estimated a sum in the supremum that gives $\|f_k\|$ by something that looks very much like a sum in the supremum that gives $\|f|_{\mathbb{L}^*}\|_*$. Still, in order to have a sum in that supremum we would need that $\sup_{x\in\mathbb{L}}\sum|r_i y_i^*(x)|\leq 1$. This is not the case, but we will get it after a small perturbation. For $x\in\mathbb{L}$,

$$\begin{aligned}\sum_{x_i^*\in S_k^+}|r_i y_i^*(x)| &\leq \sum_{x_i^*\in S_k^+}|r_i x_i^*(x)|+\sum_{x_i^*\in S_k^+}|r_i|\,|y_i^*(x)-x_i^*(x)|\\ &\leq \sum_{x_i^*\in S_k^+}|r_i x_i^*(x)|+\frac{1}{k}\sum_{x_i^*\in S_k^+}|r_i|\\ &\leq 1+\frac{n}{k}.\end{aligned}$$

Thus, the scalars $\widetilde{r_i}=\frac{r_i}{1+n/k}$ and the elements y_i^*, for every i with $x_i^*\in S_k^+$, are as required in the supremum that gives $\|f|_{\mathbb{L}^*}\|_*$. Coming back to our estimate of the sum in the sup of $\|f_k\|$:

$$\begin{aligned}\sum_{i=1}^m|r_i f_k(x_i^*)| &\leq \sum_{x_i^*\in S_k^+}|r_i f(y_i^*)|+\frac{\varepsilon}{2}\\ &\leq \sum_{x_i^*\in S_k^+}|\widetilde{r_i}f(y_i^*)|+\sum_{x_i^*\in S_k^+}|(r_i-\widetilde{r_i})f(y_i^*)|+\frac{\varepsilon}{2}\\ &\leq \|f|_{\mathbb{L}^*}\|_*+\left(1-\frac{1}{1+n/k}\right)\sum_{x_i^*\in S_k^+}|r_i f(y_i^*)|+\frac{\varepsilon}{2}\\ &= \|f|_{\mathbb{L}^*}\|_*+\frac{n}{n+k}\sum_{x_i^*\in S_k^+}|r_i f(y_i^*)|+\frac{\varepsilon}{2}\\ &\leq \|f|_{\mathbb{L}^*}\|_*+\frac{Mn}{n+k}\sum_{x_i^*\in S_k^+}|r_i|+\frac{\varepsilon}{2}\\ &\leq \|f|_{\mathbb{L}^*}\|_*+\frac{Mn^2}{n+k}+\frac{\varepsilon}{2}\leq \|f|_{\mathbb{L}^*}\|_*+\varepsilon,\end{aligned}$$

as we needed to prove. This finishes the proof of Theorem 2.2 in the case when $\mathbb{L}$ is finite. Before getting to the infinite case, we state a lemma.

Lemma 2.7. *Let $\mathbb{L}$ be a distributive lattice and $\mathbb{F}_0\subset\mathbb{L}$ be a finite subset. Then, there exists a finite sublattice $\mathbb{F}_1\subset\mathbb{L}$ that contains $\mathbb{F}_0$ and such that for every lattice $\mathbb{M}$ and every lattice homomorphism $y^*\colon\mathbb{F}_1\longrightarrow\mathbb{M}$ there exists a lattice homomorphism $z^*\colon\mathbb{L}\longrightarrow\mathbb{M}$ such that $z^*|_{\mathbb{F}_0}=y^*|_{\mathbb{F}_0}$.*

Proof. We start with a claim: If $\mathbb{M}$ is a finite lattice and $x^*\colon\mathbb{F}_0\longrightarrow\mathbb{M}$ is a function which is not the restricion of any lattice homomorphism $z^*\colon\mathbb{L}\longrightarrow\mathbb{M}$, then there exists a finite sublattice $\mathbb{F}_1[x^*]\subset\mathbb{L}$ that contains $\mathbb{F}_0$ and such that x^* is not the restriction of any lattice homomorphism $y^*\colon\mathbb{F}_1[x^*]\longrightarrow\mathbb{M}$.

Proof of the claim: For every finite subset $\mathbb{F} \subset \mathbb{L}$ that contains $\mathbb{F}_0$, consider the set

$$\begin{aligned} K_{\mathbb{F}} = \{z^* \colon \mathbb{L} \longrightarrow \mathbb{M} \quad : \quad & z^*|_{\mathbb{F}_0} = x^*, \\ & z^*(a \vee b) = z^*(a) \vee z^*(b), \text{ for all } a, b \in \mathbb{F}, \\ & z^*(a \wedge b) = z^*(a) \wedge z^*(b), \text{ for all } a, b \in \mathbb{F}\}. \end{aligned}$$

Since every finitely generated sublattice of a distributive lattice is finite, the negation of the claim above implies that $K_{\mathbb{F}} \neq \emptyset$ whenever $\mathbb{F}$ is finite. It is easy to check that $K_{\mathbb{F}}$ is a closed subset of $\mathbb{M}^{\mathbb{L}}$ (with the product topology of the discrete topology on $\mathbb{M}$). We also have that $\bigcap K_{\mathbb{F}^i} \supset K_{\bigcup \mathbb{F}^i}$ for any $\mathbb{F}^1, \ldots, \mathbb{F}^k$. Thus, the sets of the form $K_{\mathbb{F}}$ form a family of closed subsets of $\mathbb{M}^{\mathbb{L}}$ with the finite intersection property. By compactness, there exists $z^* \colon \mathbb{L} \longrightarrow \mathbb{M}$ that belongs to all sets $K_{\mathbb{F}}$. But then, z^* is a lattice homomorphism with $z^*|_{\mathbb{F}_0} = x^*$ in contradiction with the hypothesis of the claim.

Once the claim is proved, we return to the proof of the Lemma. First, let us notice that we can suppose that $\mathbb{F}_0$ is a finite sublattice of $\mathbb{L}$ and that $\mathbb{M}$ is finite. The first assumption is because we can pass to the sublattice generated by $\mathbb{F}_0$, and remember that every finitely generated sublattice of a distributive lattice is finite. The second assumption is because we can consider the restriction of y^* onto its range. Let us say that two surjective lattice homomorphisms $x_1^* \colon \mathbb{F}_0 \longrightarrow \mathbb{M}_1$ and $x_2^* \colon \mathbb{F}_0 \longrightarrow \mathbb{M}_2$ are equivalent if there exists a lattice isomorphism $\phi \colon \mathbb{M}_1 \longrightarrow \mathbb{M}_2$ such that $\phi \circ x_1^* = x_2^*$. Clearly, there are only finitely many equivalence classes of such surjective lattice homomorphisms, so let $\mathcal{C} = \{x_1^*, x_2^*, \ldots, x_p^*\}$ be a finite list that contains a representative of each equivalence class. Let $\mathcal{C}'$ be the smallest list made of all the $x_i^* \in \mathcal{C}$ that are not the restriction of any lattice homomorphism $z^* \colon \mathbb{L} \longrightarrow \mathbb{M}_i$. We can construct then $\mathbb{F}_1$ to be the sublattice of $\mathbb{L}$ generated by $\mathbb{F}_0$ and by all the $\mathbb{F}_1[x_i^*]$ for $x_i^* \in \mathcal{C}'$. □

Now, we consider the case when $\mathbb{L}$ is infinite. Again, we fix $g \in FBL(\mathbb{L})$, and have to show that $\|g\|_{\mathcal{I}} \leq \|g|_{\mathbb{L}^*}\|_*$.

For this proof it will be convenient to explicitly indicate the domain of the evaluation functions, so we write $\delta_x^{\mathbb{L}} \colon [-1,1]^{\mathbb{L}} \longrightarrow \mathbb{R}$ for the function $\delta_x^{\mathbb{L}}(x^*) = x^*(x)$. We can suppose that g can be written as $g = P(\delta_{x_1}^{\mathbb{L}}, \ldots, \delta_{x_n}^{\mathbb{L}})$ for some $x_1, \ldots, x_n \in \mathbb{L}$, where P is a formula that involves linear combinations and the lattice operations $\wedge$ and $\vee$. This is because this kind of functions are dense in $FBL(\mathbb{L})$, that was generated by the functions $\delta_x^{\mathbb{L}}$ as a Banach lattice. Let $\mathbb{F}_0 = \{x_1, \ldots, x_n\}$ and let $\mathbb{F}_1$ be the finite sublattice of $\mathbb{L}$ provided by Lemma 2.7. For any set $\mathbb{A}$ such that $\mathbb{F}_0 \subset \mathbb{A} \subset \mathbb{L}$, we consider

$$g^{\mathbb{A}} = P(\delta_{x_1}^{\mathbb{A}}, \ldots, \delta_{x_n}^{\mathbb{A}}) \colon [-1,1]^{\mathbb{A}} \longrightarrow \mathbb{R}$$

Claim X: If $\mathbb{A} \subset \mathbb{B}$ and $x^* \in [-1,1]^{\mathbb{B}}$, then $g^{\mathbb{B}}(x^*) = g^{\mathbb{A}}(x^*|_{\mathbb{A}})$.

Proof of the claim X: This is easily checked by induction on the complexity of the expression P. If P is just a variable $P(u_1, \ldots, u_n) = u_i$, then we have the fact that $\delta_{x_i}^{\mathbb{B}}(x^*) = x^*(x_i) = \delta_{x_i}^{\mathbb{A}}(x^*|_{\mathbb{A}})$. And it is trivial that if the claim X is satisfied by P and Q, it is also satisfied for $P \wedge Q$, $P \vee Q$ and any linear combination of P and Q. This finishes the proof of the claim X.

Let $\mathcal{I}_1$ be the closed ideal in $FBL(\mathbb{F}_1)$ generated by the elements of the form $\delta_{x\vee y}^{\mathbb{F}_1} - \delta_x^{\mathbb{F}_1} \vee \delta_y^{\mathbb{F}_1}$ and $\delta_{x\wedge y}^{\mathbb{F}_1} - \delta_x^{\mathbb{F}_1} \wedge \delta_y^{\mathbb{F}_1}$. By the finite case that we already proved, we have that

$$\left\| g^{\mathbb{F}_1} \right\|_{\mathcal{I}_1} \leq \left\| g^{\mathbb{F}_1}|_{\mathbb{F}_1^*} \right\|_* .$$

Thus, it is enough to prove that $\|g\|_{\mathcal{I}} \leq \left\|g^{\mathbb{F}_1}\right\|_{\mathcal{I}_1}$ and that $\left\|g^{\mathbb{F}_1}|_{\mathbb{F}_1^*}\right\|_* \leq \|g|_{\mathbb{L}^*}\|_*$.

Let us see first that $\left\|g^{\mathbb{F}_1}|_{\mathbb{F}_1^*}\right\|_* \leq \|g|_{\mathbb{L}^*}\|_*$. We have that

$$\left\| g^{\mathbb{F}_1}|_{\mathbb{F}_1^*} \right\|_* = \sup \left\{ \sum_{i=1}^m \left| g^{\mathbb{F}_1}(y_i^*) \right| : m \in \mathbb{N},\ y_i^* \in \mathbb{F}_1^*,\ \sup_{x\in\mathbb{F}_1} \sum_{i=1}^m |y_i^*(x)| \leq 1 \right\},$$

$$\|g|_{\mathbb{L}^*}\|_* = \sup \left\{ \sum_{i=1}^m |g(z_i^*)| : m \in \mathbb{N},\ z_i^* \in \mathbb{L}^*,\ \sup_{x\in\mathbb{L}} \sum_{i=1}^m |z_i^*(x)| \leq 1 \right\}.$$

We take a sum $\sum_{i=1}^m \left|g^{\mathbb{F}_1}(y_i^*)\right|$ and we will find a sum $\sum_{i=1}^m |g(z_i^*)|$ like in the second supremum with the same value. Consider

$$\mathbb{M} = \{(y_1^*(x), \ldots, y_m^*(x)) : x \in \mathbb{F}_1\} \subset [-1,1]^m.$$

Notice that, since each y_i^* is a lattice homomorphism, the set $\mathbb{M}$ is a sublattice of $\mathbb{R}^m$ and we have a lattice homomorphism $y^*\colon \mathbb{F}_1 \longrightarrow \mathbb{M}$ given by $y^*(x) = (y_1^*(x), \ldots, y_m^*(x))$. Also, since we are assuming that the y_i^* are as in the supremum above, we have that $\sum_{i=1}^m |\xi_i| \leq 1$ whenever $(\xi_1, \ldots, \xi_m) \in \mathbb{M}$. We are in a position to apply Lemma 2.7, and we find a lattice homomorphism $z^*\colon \mathbb{L} \longrightarrow \mathbb{M} \subset [-1,1]^m$ such that $z^*|_{\mathbb{F}_0} = y^*|_{\mathbb{F}_0}$. Write $z^*(x) = (z_1^*(x), \ldots, z_m^*(x))$, so that we have $z_1^*, \ldots, z_m^* \in \mathbb{L}^*$. Since the range of z^* is inside $\mathbb{M}$, we have that $\sum_{i=1}^m |z_i^*(x)| \leq 1$ for all $x \in \mathbb{L}$. Finally, using Claim X above

$$\sum_{i=1}^m |g(z_i^*)| = \sum_{i=1}^m \left|g^{\mathbb{L}}(z_i^*)\right| = \sum_{i=1}^m \left|g^{\mathbb{F}_0}(z_i^*|_{\mathbb{F}_0})\right| = \sum_{i=1}^m \left|g^{\mathbb{F}_0}(y_i^*|_{\mathbb{F}_0})\right| = \sum_{i=1}^m \left|g^{\mathbb{F}_1}(y_i^*)\right|,$$

as required.

Now, we prove the remaining inequality $\|g\|_{\mathcal{I}} \leq \|g^{\mathbb{F}_1}\|_{\mathcal{I}_1}$. In this proof, it will be useful to use a subindex on norms to indicate in which free Banach lattice these norms are calculated. Remember that

$$\begin{aligned} \|g\|_{\mathcal{I}} &= \inf \left\{ \|f\|_{FBL(\mathbb{L})} : f \in FBL(\mathbb{L}),\ f - g \in \mathcal{I} \right\}, \\ \|g^{\mathbb{F}_1}\|_{\mathcal{I}_1} &= \inf \left\{ \|h\|_{FBL(\mathbb{F}_1)} : h \in FBL(\mathbb{F}_1),\ h - g^{\mathbb{F}_1} \in \mathcal{I}_1 \right\}, \end{aligned}$$

where

$$\|f\|_{FBL(\mathbb{L})} = \sup \left\{ \sum_{i=1}^m |f(z_i^*)| : m \in \mathbb{N},\ z_i^* \in [-1,1]^{\mathbb{L}},\ \sup_{x\in\mathbb{L}} \sum_{i=1}^m |z_i^*(x)| \leq 1 \right\},$$

$$\|h\|_{FBL(\mathbb{F}_1)} = \sup\left\{\sum_{i=1}^m |h(y_i^*)| : m \in \mathbb{N},\ y_i^* \in [-1,1]^{\mathbb{F}_1},\ \sup_{x\in\mathbb{F}_1}\sum_{i=1}^m |y_i^*(x)| \le 1\right\}.$$

Thus, the question is if given $h \in FBL(\mathbb{F}_1)$ such that $h - g^{\mathbb{F}_1} \in \mathcal{I}_1$, there exists $f \in FBL(\mathbb{L})$ such that $f - g \in \mathcal{I}$ and $\|f\|_{FBL(\mathbb{L})} \le \|h\|_{FBL(\mathbb{F}_1)}$.

For every $h\colon [-1,1]^{\mathbb{F}_1} \longrightarrow \mathbb{R}$, we consider $e(h)\colon [-1,1]^{\mathbb{L}} \longrightarrow \mathbb{R}$ given by $e(h)(z^*) = h(z^*|_{\mathbb{F}_1})$. It is clear that $e(\delta_x^{\mathbb{F}_1}) = \delta_x^{\mathbb{L}}$, and e preserves linear combinations, the lattice operations and $\|e(h)\|_{FBL(\mathbb{L})} = \|h\|_{FBL(\mathbb{F}_1)}$. Thus, we can view e as a Banach lattice homomorphism $e\colon FBL(\mathbb{F}_1) \longrightarrow FBL(\mathbb{L})$ that preserves the norm.

Now, we see that $f = e(h)$ is what we are looking for. It only remains to check that $f - g \in \mathcal{I}$. We know that $h - g^{\mathbb{F}_1} \in \mathcal{I}_1$, which is the closed ideal generated by

$$\left\{\delta^{\mathbb{F}_1}_{x\vee y} - \delta^{\mathbb{F}_1}_x \vee \delta^{\mathbb{F}_1}_y,\ \delta^{\mathbb{F}_1}_{x\wedge y} - \delta^{\mathbb{F}_1}_x \wedge \delta^{\mathbb{F}_1}_y : x, y \in \mathbb{F}_1\right\}.$$

Therefore, $e(h) - e(g^{\mathbb{F}_1})$ is in the closed ideal generated by

$$\left\{e\left(\delta^{\mathbb{F}_1}_{x\vee y} - \delta^{\mathbb{F}_1}_x \vee \delta^{\mathbb{F}_1}_y\right),\ e\left(\delta^{\mathbb{F}_1}_{x\wedge y} - \delta^{\mathbb{F}_1}_x \wedge \delta^{\mathbb{F}_1}_y\right) : x, y \in \mathbb{F}_1\right\}$$

$$= \left\{\delta^{\mathbb{L}}_{x\vee y} - \delta^{\mathbb{L}}_x \vee \delta^{\mathbb{L}}_y,\ \delta^{\mathbb{L}}_{x\wedge y} - \delta^{\mathbb{L}}_x \wedge \delta^{\mathbb{L}}_y : x, y \in \mathbb{F}_1\right\}.$$

Notice that $e(g^{\mathbb{F}_1}) = g$ by Claim X above. So we conclude that $e(h) - e(g^{\mathbb{F}_1}) = f - g \in \mathcal{I}$ as required.

2.5 Linear structure of a line in its free Banach lattice

In this section, $\mathbb{L}$ is a linearly ordered set, and $FBL\langle\mathbb{L}\rangle = FBL_*\langle\mathbb{L}\rangle$ its free Banach lattice, in the form of Theorem 2.2. For $x \in \mathbb{L}$, we will denote the evaluation functions as $\delta_x\colon \mathbb{L}^* \longrightarrow \mathbb{R}$ instead of $\dot{\delta}_x$, as we do not distinguish it from other evaluations. We will show that in this case, the linear combinations of the copy of $\mathbb{L}$ inside $FBL\langle\mathbb{L}\rangle$ behave similarly to the summing basis of c_0. More precisely:

Proposition 2.8. *Let $\mathbb{L}$ be a linearly ordered set. Then, for every $u_1 < \ldots < u_m \in \mathbb{L}$ and $a_1, \ldots, a_m \in \mathbb{R}$ we have that*

$$\left\|\sum_{i=1}^m a_i s_i\right\|_\infty \le \left\|\sum_{i=1}^m a_i \delta_{u_i}\right\|_* \le 6\left\|\sum_{i=1}^m a_i s_i\right\|_\infty,$$

where $s_i = (\underbrace{1, 1, \ldots, 1}_{i}, 0, 0, 0, \ldots) \in c_0$.

Proof. Let $T\colon \mathbb{L} \longrightarrow c_0$ be the map given by

$$T(x) = \begin{cases} s_1 & \text{if } x < u_2; \\ s_k & \text{if } u_k \le x < u_{k+1} \text{ for some } 2 \le k \le m-1; \\ s_m & \text{if } x \ge u_m. \end{cases}$$

Clearly, T is a bounded and increasing map. Let $\hat{T}\colon FBL\langle\mathbb{L}\rangle \longrightarrow c_0$ be its extension as in Definition 2.1. Since $\|\hat{T}\| \leq 1$, we have that $\|\hat{T}(\sum_{i=1}^m a_i\delta_{u_i})\|_\infty \leq \|\sum_{i=1}^m a_i\delta_{u_i}\|_*$, where $\hat{T}(\sum_{i=1}^m a_i\delta_{u_i}) = \sum_{i=1}^m a_i s_i$. This proves the first inequality in the proposition.

For $f \in FBL_*\langle\mathbb{L}\rangle$ we have that

$$\begin{aligned}
\|f\|_* &= \sup\left\{\sum_{j=1}^n \left|f(x_j^*)\right| : n\in\mathbb{N},\ x_1^*,\dots,x_n^*\in\mathbb{L}^*,\ \sup_{x\in\mathbb{L}}\sum_{j=1}^n\left|x_j^*(x)\right|\leq 1\right\}\\
&\leq 2\sup\left\{\left|\sum_{j=1}^n f(x_j^*)\right| : n\in\mathbb{N},\ x_1^*,\dots,x_n^*\in\mathbb{L}^*,\ \sup_{x\in\mathbb{L}}\sum_{j=1}^n\left|x_j^*(x)\right|\leq 1\right\}.
\end{aligned}$$

This is because

$$\sum_{j=1}^n |f(x_j^*)| = \left|\sum_{f(x_j^*)>0} f(x_j^*)\right| + \left|\sum_{f(x_j^*)<0} f(x_j^*)\right|.$$

Therefore

$$\begin{aligned}
\left\|\sum_{i=1}^m a_i\delta_{u_i}\right\|_* &\leq 2\sup\left\{\left|\sum_{j=1}^n\sum_{i=1}^m a_i x_j^*(u_i)\right| : n\in\mathbb{N},\ x_1^*,\dots,x_n^*\in\mathbb{L}^*,\ \sup_{x\in\mathbb{L}}\sum_{j=1}^n\left|x_j^*(x)\right|\leq 1\right\}\\
&= 2\sup\left\{\left|\sum_{i=1}^m a_i(\sum_{j=1}^n x_j^*)(u_i)\right| : n\in\mathbb{N},\ x_1^*,\dots,x_n^*\in\mathbb{L}^*,\ \sup_{x\in\mathbb{L}}\sum_{j=1}^n\left|x_j^*(x)\right|\leq 1\right\}\\
&\leq 2\sup\left\{\left|\sum_{i=1}^m a_i x^*(u_i)\right| : x^*\in\mathbb{L}^*\right\}.
\end{aligned}$$

On the other hand,

$$3\left\|\sum_{i=1}^m a_i s_i\right\|_\infty = \sup\left\{\left|z^*\left(\sum_{i=1}^m a_i s_i\right)\right| : z^*\in 3B_{c_0{}^*}\right\} = \sup\left\{\left|\sum_{i=1}^m a_i z^*(s_i)\right| : z^*\in 3B_{\ell_1}\right\}.$$

Given $x^*\in\mathbb{L}^*$, if we define $z_1 = x^*(u_1)$ and $z_k = x^*(u_k) - x^*(u_{k-1})$ for every $k\geq 2$, then $z^* = (z_1, z_2, z_3, \dots) \in 3B_{\ell_1}$, and $z^*(s_i) = x^*(u_i)$ for all $i = 1,\dots,m$. Combining all these facts, we get the second inequality in the proposition. $\square$

Chapter 3

Chain conditions in free Banach lattices

3.1 Introduction

In this chapter we investigate what chain conditions hold in free Banach lattices generated by Banach spaces and linearly ordered sets. Its content is basically extracted from our publications:

[7] AVILÉS, A., PLEBANEK, G., AND RODRÍGUEZ ABELLÁN, J. D. Chain conditions in free Banach lattices. *J. Math. Anal. Appl. 465* (2018), 1223–1229

[9] AVILÉS, A., AND RODRÍGUEZ ABELLÁN, J. D. The free Banach lattice generated by a lattice. *Positivity 23* (2019), 581–597

Let us recall that the countable chain condition and its various strengthenings, typically considered in the context of Boolean algebras or topological spaces, in a more general setting define combinatorial properties of partially ordered sets, see e.g. Todorcevic's survey article [33]. Given a Banach lattice X, it is natural to discuss chain conditions of the partially ordered set X^+ of positive elements of the lattice. We shall consider the following chain conditions formed in this way:

Definition 3.1. We say that a Banach lattice X

(i) satisfies the *countable chain condition* (*ccc*) if for every uncountable family $\mathcal{F} \subset X^+$ there are distinct $f, g \in \mathcal{F}$ such that $f \wedge g \neq 0$;

(ii) satisfies *Knaster's condition* K_2 if every uncountable family $\mathcal{F} \subset X^+$ contains an uncountable family $\mathcal{G}$ with the property that $f \wedge g \neq 0$ for every $f, g \in \mathcal{G}$;

(iii) satisfies the *σ-bounded chain condition* (*σ-bcc*) if X^+ admits a countable decomposition $X^+ = \bigcup_{n\geq 2} \mathcal{F}_n$ such that, for every n, in every subset $\mathcal{G} \subset \mathcal{F}_n$ of size n there are two distinct elements $f, g \in \mathcal{G}$ such that $f \wedge g \neq 0$.

We have listed those chain conditions according to their increasing strength; in fact, the implications

$$\sigma\text{-}bcc \Rightarrow K_2 \Rightarrow ccc,$$

are valid for arbitrary partially ordered sets. While it is clear that K_2 implies *ccc*, the first implication is less obvious. Nonetheless, the σ-bounded chain condition implies K_2, as a consequence of the Dushnik-Miller partition theorem, cf. [33, page 52]. We are grateful to Stevo Todorcevic for bringing this fact to our attention. The first version of [7] contained a separate argument that the lattice $FBL[E]$ satisfies Knaster's condition under an additional assumption that E is weakly compactly generated. The role of the σ-bounded chain condition is briefly discussed in Section 3.4.

B. de Pagter and A. W. Wickstead showed that the free Banach lattice $FBL[\ell_1(A)] = FBL(A)$ generated by any set A always satisfies the *ccc* ([15, Corollary 6.6]). This is in analogy with the well-known property of free Boolean algebras, which satisfy the countable chain condition regardless of their size ([26, Chapter 4, Corollary 9.18]). Assuming some linear and metric restrictions does not seem to help in constructing large sets of disjoint elements, and for this reason it is natural to guess that the free Banach lattice generated by any Banach space E should also satisfy the *ccc*. Although the original proof from [15] does not admit a straightforward generalization, we shall prove in this chapter that this is the case; in fact, our main result in Section 3.3 reads as follows:

Theorem 3.2. *For every Banach space E, the free Banach lattice $FBL[E]$ satisfies the σ-bounded chain condition.*

Recall that all the functions in $FBL[E]$ are positively homogeneous and weak*-continuous when restricted to the closed unit ball B_{E^*}. So, there is a natural inclusion

$$FBL[E] \subset C_{ph}(B_{E^*}),$$

where the right-hand side is the set of all weak*-continuous and positively homogeneous functions on B_{E^*}. This inclusion preserves the order relation $\leq$ and the infimum and supremum operations $(\wedge, \vee)$, that are always defined pointwise. Theorem 3.2 follows from Theorem 3.3 below, because the σ-bounded chain condition is transferred by the inclusion mentioned above.

Theorem 3.3. *The lattice $C_{ph}(B_{E^*})$ satisfies the σ-bounded chain condition for every Banach space E.*

Before proving it, in Section 3.2, we will focus on the countable chain condition for the free Banach lattice generated by a linearly ordered set. The main result in this section is the following:

Theorem 3.4. *For $\mathbb{L}$ linearly ordered set, $FBL\langle\mathbb{L}\rangle$ satisfies the countable chain condition if, and only if, $\mathbb{L}$ is order-isomorphic to a subset of the real line.*

3.2 Chain conditions in $FBL\langle\mathbb{L}\rangle$

Throughout this section $\mathbb{L}$ is a linearly ordered set, and $FBL\langle\mathbb{L}\rangle = FBL_*\langle\mathbb{L}\rangle$ is the free Banach lattice generated by $\mathbb{L}$, in the concrete form described in [Chapter 2, Theorem 2.2]. Again, for $x \in \mathbb{L}$, we will denote the evaluation functions as $\delta_x \colon \mathbb{L}^* \longrightarrow \mathbb{R}$ instead of $\dot{\delta}_x$.

Before proving Theorem 3.4, we state a couple of lemmas:

Lemma 3.5. *For a linearly ordered set $\mathbb{L}$ the following are equivalent:*

1. *$\mathbb{L}$ is order-isomorphic to a subset of the real line.*
2. *$\mathbb{L}$ is separable in the order topology, and the set of leaps $\{(a,b) \in \mathbb{L}^2 : a < b, [a,b] = \{a,b\}\}$ is countable.*
3. *For every uncountable family of triples*
$$\mathcal{F} = \left\{\{x_1^i, x_2^i, x_3^i\} : x_1^i, x_2^i, x_3^i \in \mathbb{L},\ x_1^i < x_2^i < x_3^i,\ i \in J\right\}$$
there exist $i \neq j$ such that $x_1^i \leq x_2^j \leq x_3^i$ and $x_1^j \leq x_2^i \leq x_3^j$.

Proof. The equivalence of 1 and 2 is easy and is well known folklore, cf. [32, Corollary 3.1]. Assume now 2 and let us prove 3. Take a countable dense subset $D \subset \mathbb{L}$ that contains all the elements $a, b \in \mathbb{L}$ for which $a < b$ and $[a,b] = \{a,b\}$. Let $f \colon \mathcal{F} \longrightarrow D^2$ be the map given by $f(x_1^i, x_2^i, x_3^i) = (d_1, d_2)$, where d_k is an element of D such that $x_k^i < d_k < x_{k+1}^i$ if such an element exists, and $d_k = x_k^i$ otherwise, when $x_k^i, x_{k+1}^i \in D$ form a leap. Since $\mathcal{F}$ is uncountable and D^2 is countable, there exists an uncountable $\mathcal{F}_0 \subseteq \mathcal{F}$ such that $f|_{\mathcal{F}_0}$ is constant. Any pair of distinct elements $\{x_1^i, x_2^i, x_3^i\}$, $\{x_1^j, x_2^j, x_3^j\} \in \mathcal{F}_0$ is as required because we can interpolate $x_k^u \leq d_k < x_{k+1}^v$. Let us prove now that 3 implies 2. First, let us see that the set of leaps is countable. Let us say two leaps (a,b) and (a',b') are equivalent if there exist $c_0 < c_1 < \cdots < c_p$ finitely many elements of $\mathbb{L}$ such that each (c_k, c_{k+1}) is a leap and either $c_0 = a$ and $c_p = b'$, or $c_0 = a'$ and $c_p = b$. It is clear that each equivalence class of leaps is countable. So if there were uncountably many leaps, we could find an uncountable family $\mathcal{G} = \{\{x_1^i, x_2^i\} : i \in J\}$ of nonequivalent leaps $x_1^i < x_2^i$. We can assume that x_2^i is never the maximum of $\mathbb{L}$ and we choose an arbitrary $x_3^i > x_2^i$. Applying 3 to the family $\mathcal{F} = \{\{x_1^i, x_2^i, x_3^i\} : i \in J\}$, we could find $i \leq j$ such that $x_1^i \leq x_2^j$ and $x_1^j \leq x_2^i$. But when we have two nonequivalent leaps, one has to be strictly to the right of the other, so either $x_2^j < x_1^i$ or $x_2^i < x_1^j$, a contradiction. Now we prove that $\mathbb{L}$ is separable. Using Zorn's lemma, we can find a maximal family $\mathcal{F}$ that fails the property stated in 3. This family must be then countable. Let D be the set of all elements of $\mathbb{L}$ that either appear in some triple of the family $\mathcal{F}$ or are one of the two sides of a leap. We know now that D is countable. Let us check that it is dense. Take a non-empty open interval $(a,b) \subset \mathbb{L}$. If the interval (a,b) is finite, then all its elements are parts of leaps, so it intersects D. Suppose that (a,b) is infinite but does not intersect D. Then if we pick $a < x_1 < x_2 < x_3 < b$, then the triple $\{x_1, x_2, x_3\}$ could be added to $\mathcal{F}$, in contradiction with its maximality. $\square$

We notice that the use of triples in Lemma 3.5 is essential. The analogous property of condition 3 for couples instead of triples would be that for every uncountable family

$\mathcal{F} = \{\{x_1^i, x_2^i\} : x_1^i < x_2^i\}$ there are $i \neq j$ such that $x_1^i \leq x_2^j$ and $x_1^j \leq x_2^i$. A connected Suslin line has this weaker property but it does not embed into the real line.

Lemma 3.6. *Let $\mathbb{L} \subset \mathbb{M}$ be two linearly ordered sets. Then $FBL\langle\mathbb{L}\rangle$ is isometric to a Banach sublattice of $FBL\langle\mathbb{M}\rangle$.*

Proof. We will prove that, in fact, $FBL\langle\mathbb{L}\rangle$ is isometric to the closure of the vector lattice generated by the image of $\mathbb{L}$ via the inclusion mapping of $\mathbb{M}$ inside $FBL\langle\mathbb{M}\rangle$. That is, if we denote by $\phi_{\mathbb{M}}\colon \mathbb{M} \longrightarrow FBL\langle\mathbb{M}\rangle$ the inclusion of $\mathbb{M}$ inside $FBL\langle\mathbb{M}\rangle$ (given by $\phi_{\mathbb{M}}(y)(y^*) = \delta_y^{\mathbb{M}}(y^*) = y^*(y)$, for every $y \in \mathbb{M}$, $y^* \in \mathbb{M}^*$), we have that

$$FBL\langle\mathbb{L}\rangle \cong \overline{lat}^{\|\cdot\|_*}\left\{\delta_x^{\mathbb{M}} : x \in \mathbb{L}\right\} \subset FBL\langle\mathbb{M}\rangle.$$

To prove that, let us denote by $\iota\colon \mathbb{L} \longrightarrow \mathbb{M}$ the inclusion mapping of $\mathbb{L}$ inside $\mathbb{M}$, and let $\varphi\colon FBL\langle\mathbb{L}\rangle \longrightarrow FBL\langle\mathbb{M}\rangle$ be the extension of the bounded lattice homomorphism $\phi_{\mathbb{M}} \circ \iota\colon \mathbb{L} \longrightarrow FBL\langle\mathbb{M}\rangle$ to $FBL\langle\mathbb{L}\rangle$ given by the universal property of the free Banach lattice.

It is clear that the action of φ is given by $\varphi(f)(y^*) = f(y^*|_{\mathbb{L}})$ for every $f \in FBL\langle\mathbb{L}\rangle$, $y^* \in \mathbb{M}^*$, and that $\|\varphi(f)\|_* \leq \|f\|_*$ for every $f \in FBL\langle\mathbb{L}\rangle$, where

$$\|f\|_* = \sup\left\{\sum_{i=1}^n |f(x_i^*)| : n \in \mathbb{N},\ x_1^*, \ldots, x_n^* \in \mathbb{L}^*,\ \sup_{x\in\mathbb{L}} \sum_{i=1}^n |x_i^*(x)| \leq 1\right\},$$

$$\begin{aligned}\|\varphi(f)\|_* &= \sup\left\{\sum_{i=1}^n |\varphi(f)(y_i^*)| : n \in \mathbb{N},\ y_1^*, \ldots, y_n^* \in \mathbb{M}^*,\ \sup_{y\in\mathbb{M}} \sum_{i=1}^n |y_i^*(y)| \leq 1\right\} \\ &= \sup\left\{\sum_{i=1}^n |f(y_i^*|_{\mathbb{L}})| : n \in \mathbb{N},\ y_1^*, \ldots, y_n^* \in \mathbb{M}^*,\ \sup_{y\in\mathbb{M}} \sum_{i=1}^n |y_i^*(y)| \leq 1\right\}.\end{aligned}$$

Thus, to see that φ gives an isometry from $FBL\langle\mathbb{L}\rangle$ onto $\overline{lat}^{\|\cdot\|_*}\left\{\delta_x^{\mathbb{M}} : x \in \mathbb{L}\right\} \subset FBL\langle\mathbb{M}\rangle$, it only remains to prove that we also have $\|f\|_* \leq \|\varphi(f)\|_*$ for every $f \in FBL\langle\mathbb{L}\rangle$.

First, observe that

$$\mathbb{L}^* = \{x^*\colon \mathbb{L} \longrightarrow [-1,1] : u \leq v \Rightarrow x^*(u) \leq x^*(v)\}$$

and

$$\mathbb{M}^* = \{y^*\colon \mathbb{M} \longrightarrow [-1,1] : u \leq v \Rightarrow y^*(u) \leq y^*(v)\}.$$

Fix $f \in FBL\langle\mathbb{L}\rangle$ and let $x_1^*, \ldots, x_n^* \in \mathbb{L}^*$ like in the expression of the norm $\|f\|_*$. Let $\gamma\colon \mathbb{L}^* \longrightarrow \mathbb{M}^*$ be the map given by

$$\gamma(x^*)(y) = \begin{cases} \sup\{x^*(x) : x \in \mathbb{L}, x \leq y\} & \text{if there exists } x \in \mathbb{L} \text{ with } x \leq y, \\ \inf\{x^*(x) : x \in \mathbb{L}, x \geq y\} & \text{otherwise,}\end{cases}$$

for every $x^* \in \mathbb{L}^*$, $y \in \mathbb{M}$.

Put $y_i^* := \gamma(x_i^*) \in \mathbb{M}^*$ for every $i = 1, \ldots, n$, and let us see that $y_1^*, \ldots, y_n^*$ are like in the expression of the norm $\|\varphi(f)\|_*$ satisfying that $\sum_{i=1}^n |f(x_i^*)| \leq \sum_{i=1}^n |f(y_i^*|_{\mathbb{L}})|$.

Since $\gamma(x^*)|_{\mathbb{L}} = x^*$ for every $x^* \in \mathbb{L}^*$, we have that

$$f(y_i^*|_{\mathbb{L}}) = f(\gamma(x_i^*)|_{\mathbb{L}}) = f(x_i^*)$$

for every $i = 1, \ldots, n$.

Finally, we have to check that $\sup_{y\in\mathbb{M}} \sum_{i=1}^n |y_i^*(y)| \leq 1$.

Suppose not, and let $y \in \mathbb{M}$ and $\varepsilon > 0$ such that $\sum_{i=1}^n |y_i^*(y)| > 1 + \varepsilon$. Suppose also that there exists $x \in \mathbb{L}$ with $x \leq y$ (the other case is analogous).

Since, in this case, $y_i^*(y) = \sup\{x_i^*(x) : x \in \mathbb{L}, x \leq y\}$ for every $i = 1, \ldots, n$, we have that there exists $\tilde{x_i} \in \mathbb{L}$, with $\tilde{x_i} \leq y$, such that $y_i^*(y) - x_i^*(\tilde{x_i}) < \frac{\varepsilon}{n}$. Now, if $x \in \mathbb{L}$ is such that $\tilde{x_i} \leq x \leq y$, since x_i^* is increasing, we have that $x_i^*(\tilde{x_i}) \leq x_i^*(x)$. But then, $|x_i^*(x) - y_i^*(y)| = y_i^*(y) - x_i^*(x) \leq y_i^*(y) - x_i^*(\tilde{x_i}) < \frac{\varepsilon}{n}$.

Let $\tilde{x} := \max\{\tilde{x_1}, \ldots, \tilde{x_n}\} \in \mathbb{L}$. Due to the above, we have that $|x_i^*(\tilde{x}) - y_i^*(y)| < \frac{\varepsilon}{n}$ for every $i = 1, \ldots, n$. Then, using that $\sum_{i=1}^n |y_i^*(y)| > 1 + \varepsilon$ and $\sum_{i=1}^n |x_i^*(\tilde{x}) - y_i^*(y)| < \varepsilon$, we have that

$$\begin{aligned}
\sum_{i=1}^n |x_i^*(\tilde{x})| &= \sum_{i=1}^n |x_i^*(\tilde{x}) - y_i^*(y) + y_i^*(y)| \\
&\geq \sum_{i=1}^n |y_i^*(y)| - \sum_{i=1}^n |x_i^*(\tilde{x}) - y_i^*(y)| \\
&> 1 + \varepsilon - \varepsilon = 1,
\end{aligned}$$

which is a contradiction. □

We prove now Theorem 3.4. Endow $\mathbb{L}^*$ with the pointwise topology. If a function $f : \mathbb{L}^* \longrightarrow \mathbb{R}$ belongs to $FBL\langle\mathbb{L}\rangle$, then it is continuous. This is because the functions δ_x are continuous, and the property of being continuous is preserved under all Banach lattice operations (including limits, because every limit in $FBL\langle\mathbb{L}\rangle$ is a uniform limit).

A basis for the topology of $\mathbb{L}^*$ is given by the sets of the form

$$U(x_1, I_1, \ldots, x_n, I_n) := \{x^* \in \mathbb{L}^* : x^*(x_i) \in I_i \text{ for all } i = 1, \ldots, n\},$$

for $x_1, \ldots, x_n \in \mathbb{L}$ and $I_1, \ldots, I_n$ open intervals with rational endpoints. Write $I_i < I_j$ if $\sup(I_i) < \inf(I_j)$, and consider the family

$$\mathcal{W} = \{U(x_1, I_1, \ldots, x_n, I_n) : x_1 < x_2 < \cdots < x_n, I_1 < I_2 < \cdots < I_n\}.$$

This is not a basis anymore. But since $\mathbb{L}^*$ consists of nondecreasing functions, it is clear that $\mathcal{W}$ is a π-basis. That means that every non-empty open subset of $\mathbb{L}^*$ contains a non-empty open subset from $\mathcal{W}$.

Let us suppose that $\mathbb{L}$ is a subset of the real line, and we prove that $FBL\langle\mathbb{L}\rangle$ satisfies the countable chain condition. Let $D \subset \mathbb{L}$ be a countable dense subset of $\mathbb{L}$ that contains all elements that are part of a leap, $D \supset \{a, b : a < b, [a, b] = \{a, b\}\}$. Observe that in this case

$$\mathcal{W}_0 = \{U(d_1, I_1, \ldots, d_n, I_n) \in \mathcal{W} : d_1, d_2, \ldots, d_n \in D\}$$

is also a π-basis of $\mathbb{L}^*$. This is because for every $U(x_1, I_1, \ldots, x_n, I_n) \in \mathcal{W}$, we can interpolate $d_1^- \leq x_1 \leq d_1^+ \leq d_2^- \leq x_2 \leq d_2^+ \leq \cdots \leq d_n^- \leq x_n \leq d_n^+$ with $d_k^\pm \in D$, and then

$$U(d_1^-, I_1, d_1^+, I_1, \ldots, d_n^-, I_n, d_n^+, I_n) \subset U(x_1, I_1, \ldots, x_n, I_n).$$

Take an uncountable family of strictly positive elements $\mathcal{G} \subset FBL\langle\mathbb{L}\rangle^+$. For each $f \in \mathcal{G}$ there exists $V_f \in \mathcal{W}_0$ such that $V_f \subset \{x^* \in \mathbb{L}^* : f(x^*) > 0\}$. Notice that $f \wedge g \neq 0$ whenever $V_f \cap V_g \neq \emptyset$. Since $\mathcal{G}$ is uncountable and $\mathcal{W}_0$ is countable, there are plenty of pairs f, g such that in fact $V_f = V_g$. This finishes the proof that $FBL\langle\mathbb{L}\rangle$ satisfies the countable chain condition whenever $\mathbb{L}$ embeds into the real line.

We may notice that we proved a property stronger that the *ccc*: If a linear order $\mathbb{L}$ embeds into the real line, then $FBL\langle\mathbb{L}\rangle$ is σ-centered. That means, we can decompose the strictly positive elements into countably many pieces in such a way that every finite infimum inside each piece is nonzero. Indeed, if we take as $\mathcal{G}$ the uncountable family of all strictly positive elements of $FBL\langle\mathbb{L}\rangle^+$, the decomposition is given by

$$\mathcal{G} = \bigcup_{w \in \mathcal{W}_0} \left\{ f \in FBL\langle\mathbb{L}\rangle^+ : f > 0, V_f = w \right\}.$$

Now we turn to the proof that if $\mathbb{L}$ does not embed into the real line, then $FBL\langle\mathbb{L}\rangle$ does not satisfy the countable chain condition. We are going to prove it first under the extra assumption that $\mathbb{L}$ has a maximum M or a minimum m. First, suppose it has a maximum M. We fix an uncountable family of triples $\mathcal{F}$ that fails property 3 in Lemma 3.5. For every $i \in J$ consider

$$h_i = 0 \vee \left(\delta_{x_1^i} \wedge \left(\delta_{x_2^i} - \delta_{x_1^i} - 0.4\, \delta_M \right) \wedge \left(\delta_{x_3^i} - \delta_{x_2^i} - 0.4\, \delta_M \right) \right).$$

Let us see that these elements of $FBL\langle\mathbb{L}\rangle$ witness the failure of the *ccc*. Obviously $h_i \geq 0$. First, we fix i and we check that $h_i > 0$. For this, define $x^* \colon \mathbb{L} \longrightarrow [-1, 1]$ by

$$x^*(x) = \begin{cases} 0.1 & \text{if } x < x_2^i, \\ 0.55 & \text{if } x_2^i \leq x < x_3^i, \\ 1 & \text{if } x_3^i \leq x. \end{cases}$$

We have that $h_i(x^*) = 0 \vee (0.1 \wedge (0.55 - 0.1 - 0.4) \wedge (1 - 0.55 - 0.4)) = 0.05$, so $h_i \neq 0$.

Now, we prove that $h_i \wedge h_j = 0$ for $i \neq j$. Suppose on the contrary that $h_i \wedge h_j > 0$. Then, there exists $x^* \in \mathbb{L}^*$ such that $h_i(x^*) \wedge h_j(x^*) > 0$. Then

$$x^*(x_1^i) > 0,\ x^*(x_1^j) > 0,$$

$$x^*(x_2^i) - x^*(x_1^i) > 0.4\, x^*(M),$$

$$x^*(x_3^i) - x^*(x_2^i) > 0.4\, x^*(M),$$

$$x^*(x_2^j) - x^*(x_1^j) > 0.4\, x^*(M),$$

$$x^*(x_3^j) - x^*(x_2^j) > 0.4\, x^*(M).$$

Remember that property 3 of Lemma 3.5 fails, and therefore either $x_2^j \notin [x_1^i, x_3^i]$ or $x_2^i \notin [x_1^j, x_3^j]$. For example, say that $x_2^i < x_1^j$ (all other cases are analogous). Then, combining the fact that x^* is nondecreasing with the above inequalities, we get that

$$\begin{aligned} x^*(M) &> x^*(M) - x^*(x_1^i) \\ &\geq x^*(x_3^j) - x^*(x_1^i) \\ &= x^*(x_3^j) - x^*(x_2^j) + x^*(x_2^j) - x^*(x_1^j) + \\ &\quad x^*(x_1^j) - x^*(x_2^i) + x^*(x_2^i) - x^*(x_1^i) \\ &> 1.2\, x^*(M), \end{aligned}$$

a contradicition because $x^*(M) \geq x^*(x_1^i) > 0$.

The proof of the case when $\mathbb{L}$ has a maximum is over. Let $\overleftarrow{\mathbb{L}}$ be the linear order whose underlying set is the same as $\mathbb{L}$, but with the reverse order. It is easy to check that the map $\Phi\colon FBL\langle\mathbb{L}\rangle \longrightarrow FBL\langle\overleftarrow{\mathbb{L}}\rangle$ given by $\Phi(f)(x^*) = -f(-x^*)$ is an isomorphism of Banach lattices with $\Phi(\delta_x) = \delta_x$ for all $x \in \mathbb{L}$. Thus, $FBL\langle\mathbb{L}\rangle$ and $FBL\langle\overleftarrow{\mathbb{L}}\rangle$ are isomorphic, so we will have that $\mathbb{L}$ embeds into the real line whenever $FBL\langle\mathbb{L}\rangle$ satisfies the countable chain condition and $\mathbb{L}$ has a minimum. The case when $\mathbb{L}$ has neither a maximum nor a minimum remains. In that case, we just pick an arbitrary element $a \in \mathbb{L}$ and consider $\mathbb{L}_1 = \{x \in \mathbb{L} : x \leq a\}$ and $\mathbb{L}_2 = \{x \in \mathbb{L} : x \geq a\}$. By Lemma 3.6, if $FBL\langle\mathbb{L}\rangle$ satisfies the countable chain condition then both $FBL\langle\mathbb{L}_1\rangle$ and $FBL\langle\mathbb{L}_2\rangle$ satisfy the countable chain condition. But $\mathbb{L}_1$ and $\mathbb{L}_2$ have a maximum and a minimum respectively, so by the cases that we already proved, we conclude that both $\mathbb{L}_1$ and $\mathbb{L}_2$ embed into the real line. This implies that $\mathbb{L}$ embeds into the real line, as required.

3.3 Chain conditions in $FBL[E]$

This section is devoted to the proof of Theorem 3.3, in which case we will also have proved Theorem 3.2. In the sequel, we often identify a natural number n with the set $\{0, 1, 2, \ldots, n-1\}$. For any set A and $s \in \mathbb{N}$, we use the following standard notation: $[A]^s = \{B \subset A : |B| = s\}$.

We start by recalling the classical Ramsey's theorem which we use in the proof of Lemma 3.8 below.

Theorem 3.7 ([34, Corollary 1.4]). *Given $p, q, r \in \mathbb{N}$, with $p \leq r$, there exists $N = N(p, q, r) \in \mathbb{N}$ such that for every map*

$$\varphi\colon\ [N]^p \longrightarrow q$$

there exists $B \in [N]^r$ such that $\varphi|_{[B]^p}$ is constant.

Given any set A, we write Δ_A for the diagonal in $A \times A$.

Lemma 3.8. *For every $a \in \mathbb{N}$, there exists $N = N(a) \in \mathbb{N}$ such that for every map*

$$c\colon N \times N \setminus \Delta_N \longrightarrow a,$$

there exist $i < j < k \in N$ such that

$$c(i,j) = c(i,k) \text{ and } c(k,i) = c(k,j).$$

Proof. We shall check that the Ramsey number $N = N(3, a^2, 5)$ given by Theorem 3.7 has the required property. Fix any function $c\colon N \times N \setminus \Delta_N \longrightarrow a$.

Let $\varphi\colon [N]^3 \longrightarrow a^2$ be the map given by

$$\varphi(\{i,j,k\}) = (c(i,j), c(k,j)),$$

whenever $\{i,j,k\} \in [N]^3$ and $i < j < k$. By Theorem 3.7, there exists $B \in [N]^5$ such that φ is constant on $[B]^3$. Write $B = \{b_1, \ldots, b_5\}$ so that $b_1 < \ldots < b_5$.

We now check that b_2, b_3, b_4 is the triple satisfying the assertion of the lemma. Since

$$\varphi(\{b_2, b_3, b_4\}) = \varphi(\{b_2, b_4, b_5\}),$$

we get $c(b_2, b_3) = c(b_2, b_4)$ by the definition of φ.

Analogously, since

$$\varphi(\{b_1, b_3, b_4\}) = \varphi(\{b_1, b_2, b_4\}),$$

we conclude that $c(b_4, b_3) = c(b_4, b_2)$, and the proof is complete. □

Let us now fix a Banach space E and consider the compact space $K = (B_{E^*}, w^*)$.

Theorem 3.9. *There is a countable decomposition $X = \bigcup_{\nu \in \mathbb{N}} X_\nu$ of the family*

$$X = \left\{ f \in C(K) : f|_{\frac{1}{3}K} \neq 0 \right\}$$

such that for every $\mathcal{G} \subset X_\nu$ of cardinality ν there exist two distinct $f, g \in \mathcal{G}$ such that $f \cdot g \neq 0$.

Proof. What we are going to find is a countable decomposition $X = \bigcup_{w \in W} Y_w$, indicated on a suitable countable set W, together with a function $M\colon W \longrightarrow \mathbb{N}$ such that for every $\mathcal{G} \subset Y_w$ of cardinality $M(w)$ there exist two distinct $f, g \in \mathcal{G}$ such that $f \cdot g \neq 0$. From such a decomposition we can define one like stated in the theorem, by picking either $X_\nu = \emptyset$ or $X_\nu = Y_w$ where w is the least element (in some enumeration of W) that has not been previously chosen and $M(w) < \nu$.

Recall that for $x^* \in K$, sets of the form

$$V_{x^*}(x_1 \ldots, x_n, \delta) = \{y^* \in K : |y^*(x_i) - x^*(x_i)| < \delta \text{ for every } i = 1, \ldots, n\},$$

where $x_1, \ldots, x_n \in E$ and $\delta > 0$, form a base for the weak* topology at $x^* \in K$.

For every $f \in X$ we have $f|_{\frac{1}{3}K} \neq 0$, so there is $x^*_f \in E^*$ such that $\|x^*_f\| \leq 1/3$ and $\left|f(x^*_f)\right| > 0$. Without loss of generality we can assume that there is $\varepsilon > 0$ such that $\left|f(x^*_f)\right| > \varepsilon$ for every f.

Every function $f \in X$ is weak*-continuous at x^*_f so there is a weak*-neighbourhood U_f of x^*_f such that $\left|f(y^*) - f(x^*_f)\right| < \varepsilon/2$ for every $y^* \in U_f$. We may assume that every U_f is a basic neighbourhood determined by n_f vectors from E and some $\delta_f > 0$ that can be supposed to be rational. Our index set will be $W = \mathbb{N} \times \mathbb{Q}$ and $Y_{(n,\delta)} = \{f \in X : n_f = n, \delta_f = \delta\}$.

So we fix $w = (n, \delta)$, and what we have is that for every $f \in Y_w$ there exist $x_1^f, \dots, x_n^f \in B_E$ satisfying

(i) $\left|f(x_f^*)\right| > \varepsilon$ for every $f \in Y_w$;

(ii) writing $U_f = V_{x_f^*}(x_1^f, \dots, x_n^f, \delta)$, we have $\left|f(y^*) - f(x_f^*)\right| < \varepsilon/2$ for every $y^* \in U_f$.

In order to complete the proof it is enough to show that there is a large enough number N (that will be our $M(w)$) that satisfies the following claim:

Claim A. Suppose that $\{f_0, \dots, f_{N-1}\} \subseteq Y_w$. Putting $x_a^* := x_{f_a}^*$ for every $0 \le a \le N-1$, there exist $0 \le i < j < k \le N-1$ such that for $y^* = x_i^* - x_j^* + x_k^* \in K$ we have $f_i(y^*) \neq 0$ and $f_k(y^*) \neq 0$.

Indeed, the general case follows then by reindexing the functions in question. In turn, Claim A follows from the following:

Claim B. In the setting of Claim A, putting $U_a := U_{f_a}$ for every $0 \le a \le N-1$, there are $0 \le i < j < k \le N-1$ such that

$$y^* = x_i^* - x_j^* + x_k^* \in U_i \cap U_k.$$

Indeed, if $y^* \in U_i \cap U_k$ then

$$\left.\begin{array}{rcl} |f_i(x_i^*) - f_i(y^*)| & < & \varepsilon/2 \\ |f_i(x_i^*)| & > & \varepsilon \end{array}\right\} \Rightarrow |f_i(y^*)| > \varepsilon/2 \Rightarrow f_i(y^*) \neq 0,$$

and $f_k(y^*) \neq 0$ for the same reason. To complete the proof we shall now verify Claim B.

Write $[-1, 1] = \bigcup_{s=0}^{m-1} I_s$, where I_s are pairwise disjoint intervals (with or without endpoints) of diameter less than δ.

The number $N = M(w)$ that we need to take is the number $N = N(m^n)$ given by Lemma 3.8. Put $x_p^a := x_p^{f_a}$ for every $1 \le p \le n$ and $0 \le a \le N-1$, and consider the mapping

$$c\colon N \times N \setminus \Delta_N \longrightarrow m^n, \quad c(a,b) = (c_1(a,b), \dots, c_n(a,b))\,,$$

where for every $1 \le p \le n$, the value of $0 \le c_p(a,b) \le m-1$ is defined by the condition $x_b^*(x_p^a) \in I_{c_p(a,b)}$.

By Lemma 3.8, there exist $i < j < k \le N-1$ such that

$$c(i,j) = c(i,k) \text{ and } c(k,i) = c(k,j).$$

As $c(i,j) = c(i,k)$, for every $p \le n$ we have $\left|x_j^*(x_p^i) - x_k^*(x_p^i)\right| < \delta$, and

$$\left|x_i^*(x_p^i) - y^*(x_p^i)\right| = \left|x_j^*(x_p^i) - x_k^*(x_p^i)\right| < \delta,$$

which means that $y^* \in V_{x_i^*}(x_1^i, \dots, x_n^i, \delta) = U_i$.

In the same manner, from $c(k,i) = c(k,j)$ we get $\left|x_i^*(x_p^k) - x_j^*(x_p^k)\right| < \delta$, and

$$\left|x_k^*(x_p^k) - y^*(x_p^k)\right| = \left|x_i^*(x_p^k) - x_j^*(x_p^k)\right| < \delta.$$

Again, this means that $y^* \in V_{x_k^*}(x_1^k, \ldots, x_n^k, \delta) = U_k$, and this verifies Claim B. $\square$

Theorem 3.3 follows immediately from Theorem 3.9, because all strictly positive elements of $C_{ph}(K)$ satisfy $f|_{\frac{1}{3}K} \neq 0$. As it was observed in the introduction, Theorem 3.2 follows from Theorem 3.3.

3.4 Concluding remarks

Let us note that the proof of Theorem 3.9 works even if we replace B_{E^*} by any weak*-closed and absolutely convex subset K of B_{E^*}. The only delicate point in the proof that one has to be careful about is that the vector $y^* = x_i^* - x_j^* + x_k^*$ chosen in Claim B is still an element of K and this is guaranteed by $x_i^*, x_j^*, x_k^* \in \frac{1}{3}K$. Thus, Theorem 3.3 may be stated as follows:

Theorem 3.10. *Given a Banach space E and a weak*-closed absolutely convex set $K \subset B_{E^*}$, the lattice $C_{ph}(K)$ satisfies the σ-bounded chain condition.*

The question arises if there are natural stronger chain conditions that would hold in $C_{ph}(B_{E^*})$, and so in $FBL[E]$, for every Banach space E.

The σ-bounded chain condition was introduced by Horn and Tarski in connection with Boolean algebras carrying strictly positive measures. It is worth recalling that the related Horn-Tarski problem, whether the condition *σ-bcc* is equivalent to its certain formally weaker version was solved in the negative only a few years ago by Thümmel [31] and Todorcevic [35].

Suppose that $\mathfrak{A}$ is a Boolean algebra and $\mu\colon \mathfrak{A} \to [0,1]$ is a finitely additive probability measure such that $\mu(a) > 0$ for every $a \in \mathfrak{A}^+$. Then we can write

$$\mathfrak{A}^+ = \bigcup_{n \geq 2} \mathcal{F}_n, \text{ where } \mathcal{F}_n = \{a \in \mathfrak{A} : \mu(a) > 1/n\}.$$

Clearly, $\mathcal{F}_n$ contains no n many pairwise disjoint elements, so $\mathfrak{A}$ satisfies the σ-bounded chain condition. This cannot be reversed, there are algebras with *σ-bcc* not carrying strictly positive measures; cf. Chapter 6 of [14].

If X is a sublattice of the space $C(K)$ for some compact space K then one can think of an analogous chain-like condition, stating that there is a finitely additive probability measure μ on K which is *strictly positive on* $X^+ \setminus \{0\}$, that is $\int_K f \,\mathrm{d}\mu > 0$ for every $f \in X^+ \setminus \{0\}$. Note that to have $\int_K f \,\mathrm{d}\mu$ well-defined for every continuous function f we need only to assume that the domain of μ contains the algebra $\mathcal{A}(K)$ generated by closed subsets of K. Once we have such μ, it is not difficult to verify the condition *σ-bcc*. Let us first observe that whether the measure in question is actually countably additive or merely finitely additive is not essential here.

Remark 3.11. Suppose that μ is finitely additive probability measure which is strictly positive on $X^+ \setminus \{0\}$ for some sublattice X of the lattice $C(K)$ of continuous functions on a compact space K. Then there is a countably additive Borel measure μ' on K which is again strictly positive on $X^+ \setminus \{0\}$.

For $f \in X^+ \setminus \{0\}$ write $\varepsilon = \int_K f \, \mathrm{d}\mu$ and $A = \{x \in K : f(x) \geq \varepsilon/2\}$; then

$$\varepsilon = \int_A f \, \mathrm{d}\mu + \int_{K \setminus A} f \, \mathrm{d}\mu \leq \|f\|_\infty \cdot \mu(A) + \varepsilon/2,$$

which gives $\mu(A) > 0$. This implies that whenever a finitely additive measure μ' satisfies $\mu'(A) \geq \mu(A)$ for every closed $A \subset K$ then again $\int_K f \, \mathrm{d}\mu' > 0$ for every $f \in X^+ \setminus \{0\}$. Now the point is that there is such μ' that is closed-inner-regular on the algebra $\mathcal{A}(K)$, see [27]; μ' is then countably additive (by compactness) and, consequently, extends to a countable additive Borel measure on K which is positive on $X^+ \setminus \{0\}$.

Using Remark 3.11 it is not difficult to give an example showing that the σ-bounded chain condition that holds in every $FBL[E]$ does not admit the obvious measure-theoretic strengthening mentioned above.

Example 3.12. Consider the Banach space $E = c_0(\Gamma)$, where Γ is a uncountable set; then $E^* = \ell_1(\Gamma)$. There is no measure on $K = B_{E^*}$ which would be positive on all elements from $C_{ph}(K)^+$.

Indeed, every $\gamma \in \Gamma$ defines $f_\gamma \in C_{ph}(K)^+$, where $f_\gamma(x) = |x_\gamma|$. Suppose that μ is a measure on K such that $\int_K f_\gamma \, \mathrm{d}\mu > 0$ for every γ. By Remark 3.11 we can assume that μ is countable additive. Then for every γ there is $\delta(\gamma) > 0$ such that

$$\mu\left(\{x \in K : f_\gamma(x) \geq \delta(\gamma)\}\right) > 0.$$

Using the fact that Γ is uncountable, we conclude easily that there is $\delta > 0$ and a sequence of distinct $\gamma_n \in \Gamma$ such that, writing $A_n = \{x \in K : f_{\gamma_n}(x) \geq \delta\}$, we have $\mu(A_n) \geq \delta$. But then $\mu(\bigcap_n \bigcup_{k \geq n} A_k) \geq \delta$; in particular, there is $x \in K$ belonging to infinitely many sets A_n. This clearly contradicts the fact that $x \in K \subseteq \ell_1(\gamma)$.

Chapter 4

Projective Banach lattices

4.1 Introduction

There is a very extensive theory about projective objects in the context of the general theory of categories, see e.g. Semadeni's book [30]. In this chapter we continue the program proposed by B. de Pagter and A. W. Wickstead [15] of studying the projective Banach lattices. Its content is basically extracted from our publications:

[5] Avilés, A., Martínez-Cervantes, G., and Rodríguez Abellán, J. D. On projective Banach lattices of the form $C(K)$ and $FBL[E]$. *J. Math. Anal. Appl. 489*, 124129 (2020)

[6] Avilés, A., Martínez-Cervantes, G., and Rodríguez Abellán, J. D. On the Banach lattice c_0. To appear in Rev. Mat. Complut. https://doi.org/10.1007/s13163-019-00342-x, 2020

[10] Avilés, A., and Rodríguez Abellán, J. D. Projectivity of the free Banach lattice generated by a lattice. *Archiv der Mathematik 113* (2019), 515–524

Definition 4.1. Let $\lambda > 1$ be a real number. A Banach lattice P is *λ-projective* if whenever X is a Banach lattice, $\mathcal{J}$ a closed ideal in X and $Q\colon X \longrightarrow X/\mathcal{J}$ the quotient map, then for every Banach lattice homomorphism $T\colon P \longrightarrow X/\mathcal{J}$, there is a Banach lattice homomorphism $\hat{T}\colon P \longrightarrow X$ such that $T = Q \circ \hat{T}$ and $\|\hat{T}\| \leq \lambda \|T\|$.

A Banach lattice is called *projective* in [15] if it is $(1+\varepsilon)$-projective for every $\varepsilon > 0$. For a more intuitive terminology, and by analogy to similar notions in Banach spaces, we will call this 1^+-*projective* instead of just projective. Note that if P is λ-projective, then P is μ-projective for every $\mu \geq \lambda$. We will call a Banach lattice ∞-*projective* if it is λ-projective for some $\lambda > 1$.

Since the canonical 1^+-projective Banach lattice is the free Banach lattice $FBL(A)$ ([15, Proposition 10.2]), it is natural to think that its variants $FBL\langle\mathbb{L}\rangle$ and $FBL[E]$ may also be 1^+-projective at least in some cases. Sections 4.3, 4.4 and 4.5 are devoted to this.

On the one hand, we prove that:

Theorem 4.2. *If $\mathbb{L}$ is a finite lattice, then $FBL\langle\mathbb{L}\rangle$ is a 1^+-projective Banach lattice.*

In this case, $FBL\langle\mathbb{L}\rangle$ is a renorming of a Banach lattice of continuous functions $C(K)$ on a compact neighbourhood retract K of $\mathbb{R}^n$, which is 1^+-projective ([15, Theorem 11.4]). However, 1^+-projectivity is not preserved under renorming, because of the $(1+\varepsilon)$ bound required in the definition. Getting this bound will be the key point in the proof.

In the infinite case, we consider only linearly ordered sets, as they are easier to handle than general lattices. We prove the following:

Theorem 4.3. *Let $\mathbb{L}$ be an infinite linearly ordered set. Then, $FBL\langle\mathbb{L}\rangle$ is not ∞-projective.*

On the other hand, as we said before, the free Banach lattice $FBL(A) = FBL[\ell_1(A)]$ is 1^+-projective. Here, we show that, for $FBL[E]$ to be ∞-projective, the structure of E must be very close to that of $\ell_1(A)$:

Theorem 4.4. *Let E be a Banach space. If $FBL[E]$ is ∞-projective, then E has the Schur property (i.e. every weakly convergent sequence in E converges in norm).*

Moreover, at the end of Section 4.5 we provide a counterexample which shows that, in the category of nonseparable Banach spaces, the converse of this result does not hold. We still do not know if there exists a separable Banach space E which has the Schur property and such that $FBL[E]$ is not ∞-projective.

Other examples of 1^+-projective Banach lattices given in [15] include every finite dimensional Banach lattice ([15, Theorem 11.1]), ℓ_1 and any countable ℓ_1-sum of separable 1^+-projective Banach lattices ([15, Theorem 11.11]). However, questions as whether c_0, ℓ_2 or $C([0,1]^{\mathbb{N}})$ are 1^+-projective were left open in the same publication. In Sections 4.5 and 4.6 we answer these questions. In fact, we prove the following more general result:

Theorem 4.5. *Let $(u_i)_{i\in\mathbb{N}}$ be a bounded sequence of vectors in an ∞-projective Banach lattice X. Suppose that there exists a Banach lattice homomorphism $T\colon X \longrightarrow c_0$ such that $T(u_i) = e_i$ for every $i \in \mathbb{N}$, where $(e_i)_{i\in\mathbb{N}}$ is the canonical basis of c_0. Then there is a subsequence $(u_{i_k})_{k\in\mathbb{N}}$ equivalent to the canonical basis of ℓ_1,*

where we get, in particular, that neither c_0 nor ℓ_p (for $2 \leq p < \infty$) are ∞-projective; and the result below, answering [15, Question 12.12], where it was asked whether the fact that a compact Hausdorff topological space K is an absolute neighbourhood retract in the category of compact Hausdorff topological spaces implies that the space of the continuous functions on K with the supremum norm is 1^+-projective:

Theorem 4.6. *If K is a compact Hausdorff topological space, then $C(K)$ is 1^+-projective if, and only if, K is an absolute neighbourhood retract in the category of compact Hausdorff topological spaces.*

In [15, Proposition 11.7] B. de Pagter and A. W. Wickstead prove that if $C(K)$ is 1^+-projective, then K is an absolute neighbourhood retract in the category of compact Hausdorff topological spaces, while they only manage to show the converse with the additional hypothesis that K is inside $\mathbb{R}^n$ ([15, Theorem 11.4]).

Finally, in Section 4.7 we study the complementability of the Banach lattice c_0 in $FBL[c_0]$, while in Section 4.8 we propose some open problems we have about projective Banach lattices.

It should be noted that the complementability of c_0 in $FBL[c_0]$ is not a fact directly related to the topic of this chapter, but nevertheless, in [6] we used this to show that $FBL[c_0]$ is not 1^+-projective, which is already deduced here from Theorem 4.4 in a more general way.

4.2 Preliminaries

We start this chapter with some preliminaries about projective Banach lattices, quotients of projective Banach lattices and absolute neighbourhood retracts.

In Section 4.2.1 we show a simple but very useful result to study when the quotient of a 1^+-projective Banach lattice is also 1^+-projective, and which we will use several times throughout the chapter.

In Section 4.2.2 we show the definition of absolute neighbourhood retract and some basic facts which we will need in the proofs of some of the theorems mentioned in the introduction.

But before that, we prove that, in the case of ∞-projective, Q can be taken any surjective Banach lattice homomorphism. Moreover, we prove that if a Banach lattice has the property that we can lift Banach lattice homomorphisms from it to another Banach lattice via surjective Banach lattice homomorphisms (or just quotient maps), then it is λ-projective for some λ. More concretely:

Proposition 4.7. *A Banach lattice P is ∞-projective if, and only if, whenever X and Y are Banach lattices, and $S\colon X \longrightarrow Y$ is a surjective Banach lattice homomorphism, then for every Banach lattice homomorphism $T\colon P \longrightarrow Y$ there is a Banach lattice homomorphism $\hat{T}\colon P \longrightarrow X$ such that $S \circ \hat{T} = T$.*

Proof. Suppose that P is ∞-projective, and let $S\colon X \longrightarrow Y$ be a surjective Banach lattice homomorphism between two Banach lattices X and Y, and $T\colon P \longrightarrow Y$ a Banach lattice homomorphism. Since S is surjective, we have that Y is isomorphic to the quotient $X/\text{Ker } S$. Let $V\colon Y \longrightarrow X/\text{Ker } S$ be the isomorphism, and let $Q := V \circ S\colon X \longrightarrow X/\text{Ker } S$ be the quotient map and $T' := V \circ T\colon P \longrightarrow X/\text{Ker } S$.

Using that P is ∞-projective, we have that there exists a Banach lattice homomorphism $\hat{T}'\colon P \longrightarrow X$ such that $Q \circ \hat{T}' = T'$. But then $\hat{T} := \hat{T}'$ is as required. Indeed, we have that $V \circ S \circ \hat{T}' = V \circ T$, which implies that $S \circ \hat{T}' = T$, so $S \circ \hat{T} = T$.

For the converse, suppose that P satisfies such a property but it is not ∞-projective. Then, for every $\lambda \in \mathbb{N}$ there exist a Banach lattice X_λ, a closed ideal $\mathcal{J}_\lambda$ in X_λ, and a Banach lattice homomorphism $T_\lambda\colon P \longrightarrow X_\lambda/\mathcal{J}_\lambda$ of norm 1 for which every Banach lattice homomorphism $\hat{T_\lambda}\colon P \longrightarrow X$ satisfying $Q_\lambda \circ \hat{T_\lambda} = T_\lambda$, where $Q_\lambda\colon X_\lambda \longrightarrow X_\lambda/\mathcal{J}_\lambda$ is the quotient map, has norm $\|\hat{T_\lambda}\| > \lambda$.

Let

$$X := \oplus_{\lambda\in\mathbb{N}} X_\lambda = \left\{ x = (x_1, x_2, \ldots) : x_\lambda \in X_\lambda \text{ for every } \lambda \in \mathbb{N}, \sup_{\lambda\in\mathbb{N}} \|x_\lambda\|_{X_\lambda} < \infty \right\},$$

together with the norm $\|x\|_X := \sup_{\lambda\in\mathbb{N}} \|x_\lambda\|_{X_\lambda}$, and the order given by

$$(x_1, x_2, \ldots) \leq_X (x_1', x_2', \ldots) \text{ if, and only if, } x_\lambda \leq_{X_\lambda} x_\lambda' \text{ for every } \lambda \in \mathbb{N},$$

be the ℓ_∞-sum of X_λ, which is a Banach lattice, and let $Y := \oplus_{\lambda\in\mathbb{N}} X_\lambda/\mathcal{J}_\lambda$ be the ℓ_∞-sum of $X_\lambda/\mathcal{J}_\lambda$, defined in a similar way to X.

Let $T\colon P \longrightarrow Y$ be the Banach lattice homomorphism given by $T(p) = (T_1(p), T_2(p), \ldots)$ for every $p \in P$, and let $S\colon X \longrightarrow Y$ be the surjective Banach lattice homomorphism given by $S(x) = (Q_1(x_1), Q_2(x_2), \ldots)$ for every $x = (x_1, x_2, \ldots) \in X$ (S is, in fact, a quotient map).

We are going to see that there is not any Banach lattice homomorphism $\hat{T}\colon P \longrightarrow X$ such that $S \circ \hat{T} = T$, which will be a contradiction.

Suppose that it exists, and let us denote $\hat{T}\colon P \longrightarrow X$ by $\hat{T}(p) = (\hat{T_1}(p), \hat{T_2}(p), \ldots)$ for every $p \in P$. Then, we have that

$$\begin{aligned}
\|\hat{T}\| &= \sup \left\{ \|\hat{T}(p)\|_X : p \in P, \|p\| \leq 1 \right\} \\
&= \sup \left\{ \|(\hat{T_1}(p), \hat{T_2}(p), \ldots)\|_X : p \in P, \|p\| \leq 1 \right\} \\
&= \sup \left\{ \sup_{\lambda\in\mathbb{N}} \|\hat{T_\lambda}(p)\|_{X_\lambda} : p \in P, \|p\| \leq 1 \right\} \\
&= \sup \left\{ \|\hat{T_\lambda}(p)\|_{X_\lambda} : \lambda \in \mathbb{N}, p \in P, \|p\| \leq 1 \right\},
\end{aligned}$$

which implies that $\|\hat{T_\lambda}(p)\|_{X_\lambda} \leq \|\hat{T}\|$ for every $\lambda \in \mathbb{N}$ and $p \in P$ with $\|p\| \leq 1$.

But now, from the fact that $S \circ \hat{T} = T$, we have that $Q_\lambda(\hat{T_\lambda}(p)) = T_\lambda(p)$ for every $\lambda \in \mathbb{N}$ and $p \in P$, and then, if we view the λ-th coordinate of $\hat{T}$ as a Banach lattice homomorphism $\hat{T_\lambda}\colon P \longrightarrow X_\lambda$, we have that $Q_\lambda \circ \hat{T_\lambda} = T_\lambda$ and $\|\hat{T_\lambda}\| \leq \|\hat{T}\|$ for every $\lambda \in \mathbb{N}$, which is a contradiction. □

4.2.1 Quotients of projective Banach lattices

The following result is a variation of [15, Theorem 10.3]:

Proposition 4.8. *Let P be a 1^+-projective Banach lattice, $\mathcal{I}$ a closed ideal in P and $\pi\colon P \longrightarrow P/\mathcal{I}$ the quotient map. The quotient $P/\mathcal{I}$ is 1^+-projective if, and only if, for every $\varepsilon > 0$ there exists a Banach lattice homomorphism $u_\varepsilon\colon P/\mathcal{I} \longrightarrow P$ such that $\pi \circ u_\varepsilon = id_{P/\mathcal{I}}$ and $\|u_\varepsilon\| \leq 1 + \varepsilon$.*

Proof. If $P/\mathcal{I}$ is 1^+-projective, then we can just apply Definition 4.1. On the other hand, if we have the above property and we want to check Definition 4.1, take $\varepsilon_0 > 0$, a quotient map $Q\colon X \longrightarrow X/\mathcal{J}$ and a Banach lattice homomorphism $T\colon P/\mathcal{I} \longrightarrow X/\mathcal{J}$. Take ε with $(1+\varepsilon)^2 \leq 1+\varepsilon_0$. Since P is 1^+-projective we can find $S\colon P \longrightarrow X$ with $Q \circ S = T \circ \pi$ and $\|S\| \leq (1+\varepsilon)\|T \circ \pi\| = (1+\varepsilon)\|T\|$. If we take $\hat{T} = S \circ u_\varepsilon$, then $Q \circ \hat{T} = Q \circ S \circ u_\varepsilon = T \circ \pi \circ u_\varepsilon = T$ and $\|\hat{T}\| \leq (1+\varepsilon)^2\|T\| \leq (1+\varepsilon_0)\|T\|$ as desired. $\square$

In the context of the free Banach lattice generated by a lattice $FBL_*\langle\mathbb{L}\rangle$, since $FBL(\mathbb{L})$ is 1^+-projective, and the restriction map $R\colon FBL(\mathbb{L}) \longrightarrow FBL_*\langle\mathbb{L}\rangle$ described in [Chapter 2, Section 2.4] ($R(f) = f|_{\mathbb{L}^*}$) is a quotient map, we get, as a particular instance of Proposition 4.8,

Proposition 4.9. *Let $\mathbb{L}$ be a lattice and let $R\colon FBL(\mathbb{L}) \longrightarrow FBL_*\langle\mathbb{L}\rangle$ be the restriction map $R(f) = f|_{\mathbb{L}^*}$. The Banach lattice $FBL_*\langle\mathbb{L}\rangle$ is 1^+-projective if, and only if, for every $\varepsilon > 0$ there exists a Banach lattice homomorphism $u_\varepsilon\colon FBL_*\langle\mathbb{L}\rangle \longrightarrow FBL(\mathbb{L})$ such that $R \circ u_\varepsilon = id_{FBL_*\langle\mathbb{L}\rangle}$ and $\|u_\varepsilon\| \leq 1+\varepsilon$.*

4.2.2 Absolute neighbourhood retracts

Definition 4.10. We say that a topological space K is an *absolute neighbourhood retract* (ANR) if whenever $i\colon K \longrightarrow X$ is a homeomorphism between K and a subspace of the topological space X, there exist an open set V in X and a continuous function $\phi\colon V \longrightarrow K$ such that $i(K) \subset V \subset X$ and $\phi(i(k)) = k$ for all $k \in K$.

Note that when in Theorem 4.6 we say that "K is an absolute neighbourhood retract in the category of compact Hausdorff topological spaces", we mean that the condition of this definition holds whenever X is a compact Hausdorff topological space.

The following result will be very useful in the proof of this theorem:

Lemma 4.11. *In the situation of Definition 4.10, whenever X is a normal topological space, there exist a continuous function $u\colon X \longrightarrow [0,1]$ and a continuous function $\varphi\colon X \setminus u^{-1}(0) \longrightarrow K$ such that $u(i(k)) = 1$ and $\varphi(i(k)) = k$ for every $k \in K$.*

Proof. By Urysohn's lemma, we can find a continuous function $u\colon X \longrightarrow [0,1]$ such that $u(i(k)) = 1$ for every $k \in K$, $u(x) = 0$ for every $x \in X \setminus V$, and $u(x) \in (0,1)$ for every $x \in V \setminus i(K)$. Notice that $X \setminus u^{-1}(0) \subset V$, so we can take $\varphi = \phi|_{X \setminus u^{-1}(0)}$ and the statement of the lemma is satisfied. $\square$

On the other hand, the following are two basic facts of the theory that can be found in [37] as Theorems 1.5.1 and 1.5.9:

- Every closed convex subset of $\mathbb{R}^n$ is ANR.
- If K_1, K_2 are closed subsets of K, and K_1, K_2 and $K_1 \cap K_2$ are ANR, then $K_1 \cup K_2$ is also ANR.

From this, one can easily prove that every finite union of closed convex subsets of $\mathbb{R}^n$ is ANR, by induction on the number of convex sets in that union. We will use this fact in the proof of Theorem 4.2.

4.3 Projectivity of the free Banach lattice generated by a finite lattice

In this section we prove Theorem 4.2, that is to say, we prove that if $\mathbb{L}$ is a finite lattice, then $FBL\langle\mathbb{L}\rangle$ is a 1^+-projective Banach lattice.

Proposition 4.12. *If $\mathbb{L} = \{0, \dots, n-1\}$ with some lattice operations is a finite lattice, then $\mathbb{L}^* \cap \partial[-1,1]^n$ is ANR.*

Proof. Clearly, $\partial[-1,1]^n$ is a finite union of closed convex subsets of $\mathbb{R}^n$. On the other hand, for every $i, j, k \in \mathbb{L}$ let

$$A_{ijk} = \{(x_0, \dots, x_{n-1}) \in [-1,1]^n : x_i \vee x_j = x_k\}$$

and

$$B_{ijk} = \{(x_0, \dots, x_{n-1}) \in [-1,1]^n : x_i \wedge x_j = x_k\}\,.$$

It is clear that

$$\begin{aligned} A_{ijk} &= \{(x_0, \dots, x_{n-1}) \in [-1,1]^n : x_i = x_k,\, x_j \le x_i\} \\ &\quad \bigcup \{(x_0, \dots, x_{n-1}) \in [-1,1]^n : x_j = x_k,\, x_i \le x_j\} \end{aligned}$$

and

$$\begin{aligned} B_{ijk} &= \{(x_0, \dots, x_{n-1}) \in [-1,1]^n : x_i = x_k,\, x_j \ge x_i\} \\ &\quad \bigcup \{(x_0, \dots, x_{n-1}) \in [-1,1]^n : x_j = x_k,\, x_i \ge x_j\} \end{aligned}$$

are union of two closed convex sets. Since

$$\mathbb{L}^* = \left(\bigcap_{i \vee j = k} A_{ijk} \right) \bigcap \left(\bigcap_{i \wedge j = k} B_{ijk} \right),$$

we have that $\mathbb{L}^*$ is the intersection of finite unions of closed convex sets. One can easily rearrange this formula to obtain that it is also a finite union of intersections of closed convex subsets of $\mathbb{R}^n$. Now, by using that the intersection of closed convex subsets of $\mathbb{R}^n$ is again a closed convex subset of $\mathbb{R}^n$, we conclude that $\mathbb{L}^*$ is a finite union of closed convex subsets of $\mathbb{R}^n$. Thus, we have that both $\mathbb{L}^*$ and $\partial[-1,1]^n$ are finite union of closed convex subsets of $\mathbb{R}^n$, so $\mathbb{L}^* \cap \partial[-1,1]^n$ is also a finite union of closed convex subsets of $\mathbb{R}^n$, and thus, ANR. □

In the context of compact metric spaces, the retractions in the definition of ANR can be taken arbitrarily close to the identity. We state this fact as a lemma in the particular case that we need:

Lemma 4.13. *Let $\mathbb{L} = \{0, \dots, n-1\}$ with some lattice operations be a finite lattice. Then, given $\varepsilon > 0$, there exist an open set $V_\varepsilon = V_\varepsilon(\mathbb{L}^*)$ with $\mathbb{L}^* \cap \partial[-1,1]^n \subset V_\varepsilon \subset \mathbb{R}^n$ and a continuous map $\varphi\colon V_\varepsilon \longrightarrow \mathbb{L}^* \cap \partial[-1,1]^n$ such that $\varphi|_{\mathbb{L}^* \cap \partial[-1,1]^n} = id_{\mathbb{L}^* \cap \partial[-1,1]^n}$ and $d(x^*, \varphi(x^*)) < \varepsilon$ for every $x^* \in V_\varepsilon$, where d is the square metric in $\mathbb{R}^n$.*

Proof. As $\mathbb{L}^* \cap \partial[-1,1]^n$ is an ANR by Proposition 4.12, we cand find a bounded neighbourhood V of $\mathbb{L}^* \cap \partial[-1,1]^n$ in $\mathbb{R}^n$ and a retraction $\phi\colon V \longrightarrow \mathbb{L}^* \cap \partial[-1,1]^n$. Let us take an open set W such that $\mathbb{L}^* \cap \partial[-1,1]^n \subset W \subset \overline{W} \subset V \subset \mathbb{R}^n$. Now, $\phi|_{\overline{W}}\colon \overline{W} \longrightarrow \mathbb{L}^* \cap \partial[-1,1]^n$ is a continuous map between compact metric spaces, so it is uniformly continuous. Given $\varepsilon > 0$, there exists $\delta > 0$ such that $d(\phi(x^*), \phi(y^*)) < \varepsilon/2$ if $x^*, y^* \in \overline{W}$ and $d(x^*, y^*) < \delta$. Put $\eta = \min(\varepsilon/2, \delta)$ and take

$$V_\varepsilon = \{x^* \in W : \text{there exists } y^* \in \mathbb{L}^* \cap \partial[-1,1]^n \text{ with } d(x^*, y^*) < \eta\},$$

and $\varphi = \phi|_{V_\varepsilon}\colon V_\varepsilon \longrightarrow \mathbb{L}^* \cap \partial[-1,1]^n$. Clearly, φ is continuous and $\varphi|_{\mathbb{L}^* \cap \partial[-1,1]^n} = id_{\mathbb{L}^* \cap \partial[-1,1]^n}$. Let $x^* \in V_\varepsilon$, and let $y^* \in \mathbb{L}^* \cap \partial[-1,1]^n$ such that $d(x^*, y^*) < \eta$. Then,

$$d(x^*, \varphi(x^*)) \leq d(x^*, y^*) + d(y^*, \varphi(x^*)) = d(x^*, y^*) + d(\varphi(y^*), \varphi(x^*)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

□

We are ready to prove Theorem 4.2. Let n be the cardinality of $\mathbb{L}$. We may suppose that $\mathbb{L} = \{0, \dots, n-1\}$ with some lattice operations, and in this way we identify $[-1,1]^{\mathbb{L}}$ with $[-1,1]^n$. We fix $\varepsilon > 0$, and we will construct the map $u_\varepsilon\colon FBL_*\langle\mathbb{L}\rangle \longrightarrow FBL(\mathbb{L})$ of Proposition 4.9. Let V_ε and φ be given by Lemma 4.13. By Urysohn's lemma, we can find a continuous function $1_\varepsilon\colon \partial[-1,1]^n \longrightarrow [0,1]$ such that $1_\varepsilon(x^*) = 1$ if $x^* \in \mathbb{L}^* \cap \partial[-1,1]^n$, and $1_\varepsilon(x^*) = 0$ if $x^* \notin V_\varepsilon$. We define $u_\varepsilon(f)(x^*) = 1_\varepsilon(x^*) \cdot f(\varphi(x^*))$ if $x^* \in V_\varepsilon$, and $u_\varepsilon(f)(x^*) = 0$ if $x^* \notin V_\varepsilon$, for every $f \in FBL_*\langle\mathbb{L}\rangle$ and $x^* \in \partial[-1,1]^n$. We extend the definition for elements $x^* \in [-1,1]^n \setminus \partial[-1,1]^n$ in such a way that $u_\varepsilon(f)$ is positively homogeneous. Since $\mathbb{L}$ is finite, the fact that $u_\varepsilon(f)$ is continuous on $\partial[-1,1]^n$ and positively homogeneous guarantees that $u_\varepsilon(f) \in FBL(\mathbb{L})$. It is easy to check that u_ε is a Banach lattice homomorphism and that $R \circ u_\varepsilon = id_{FBL_*\langle\mathbb{L}\rangle}$. It would remain to check that $\|u_\varepsilon\| \leq 1 + \varepsilon$. We will prove instead that for this u_ε we have $\|u_\varepsilon\| \leq 1 + n\varepsilon$, which is still good enough. We know that

$$\|u_\varepsilon\| = \sup\{\|u_\varepsilon(f)\| : f \in FBL_*\langle\mathbb{L}\rangle, \|f\|_* \leq 1\},$$

where

$$\|u_\varepsilon(f)\| = \sup\left\{\sum_{i=1}^m |r_i u_\varepsilon(f)(x_i^*)| : x_i^* \in \partial[-1,1]^n,\ r_i \in \mathbb{R},\ \sup_{x \in \mathbb{L}} \sum_{i=1}^m |r_i x_i^*(x)| \leq 1\right\}.$$

So we fix $f \in FBL_*\langle\mathbb{L}\rangle$ with $\|f\|_* \leq 1$, where

$$\|f\|_* = \sup\left\{\sum_{i=1}^m |s_i f(y_i^*)| : y_i^* \in \mathbb{L}^*,\ s_i \in \mathbb{R},\ \sup_{x \in \mathbb{L}} \sum_{i=1}^m |s_i y_i^*(x)| \leq 1\right\},$$

and we want to prove that $\|u_\varepsilon(f)\| \leq 1 + n\varepsilon$. Using the expression of $\|u_\varepsilon(f)\|$ as a supremum, we pick $x_1^*, \dots, x_m^* \in \partial[-1,1]^n$, $r_1, \dots, r_m \in \mathbb{R}$ such that $\sup_{x \in \mathbb{L}} \sum_{i=1}^m |r_i x_i^*(x)| \leq 1$, and we want to prove that

$$\sum_{i=1}^m |r_i u_\varepsilon(f)(x_i^*)| \leq 1 + n\varepsilon.$$

The first estimation is that

$$\sum_{i=1}^{m} |r_i u_\varepsilon(f)(x_i^*)| = \sum_{x_i^* \in V_\varepsilon} |r_i 1_\varepsilon(x_i^*) f(\varphi(x_i^*))| \leq \sum_{x_i^* \in V_\varepsilon} |r_i f(\varphi(x_i^*))| \, .$$

If we write $y_i{}^* := \varphi(x_i^*)$ for $x_i^* \in V_\varepsilon$, the inequality above becomes

$$(\star) \quad \sum_{i=1}^{m} |r_i u_\varepsilon(f)(x_i^*)| \leq \sum_{x_i^* \in V_\varepsilon} |r_i f(y_i{}^*)| \, .$$

On the other hand, if $x \in \mathbb{L}$ then

$$\begin{aligned} \sum_{x_i^* \in V_\varepsilon} |r_i y_i^*(x)| &= \sum_{x_i^* \in V_\varepsilon} |r_i \varphi(x_i^*)(x)| \\ &\leq \sum_{x_i^* \in V_\varepsilon} |r_i x_i^*(x)| + \sum_{x_i^* \in V_\varepsilon} |r_i| \, |\varphi(x_i^*)(x) - x_i^*(x)| \\ &\leq 1 + \varepsilon \sum_{x_i^* \in V_\varepsilon} |r_i| \leq 1 + \varepsilon n. \end{aligned}$$

The last inequality is because $x_i^* \in \partial[-1,1]^n$, and therefore

$$\sum_{i=1}^{m} |r_i| = \sum_{i=1}^{m} |r_i| \sup_{x \in \mathbb{L}} |x_i^*(x)| \leq \sum_{x \in \mathbb{L}} \sum_{i=1}^{m} |r_i||x_i^*(x)| \leq |\mathbb{L}| \cdot 1 = n.$$

Taking $s_i = \frac{r_i}{1+n\varepsilon}$, we have that, for all $x \in \mathbb{L}$,

$$\sum_{x_i^* \in V_\varepsilon} |s_i y_i^*(x)| = \sum_{x_i^* \in V_\varepsilon} \left| \frac{r_i}{1+n\varepsilon} y_i^*(x) \right| \leq 1.$$

Thus, the s_i and the y_i are as in the supremum that defines $\|f\|_* \leq 1$. Therefore

$$\sum_{x_i^* \in V_\varepsilon} |s_i f(y_i^*)| \leq 1,$$

and getting back to our initial estimation ($\star$), we get

$$\sum_{i=1}^{m} |r_i u_\varepsilon(f)(x_i^*)| \leq \sum_{x_i^* \in V_\varepsilon} |r_i f(y_i^*)| \leq 1 + n\varepsilon.$$

4.4 Projectivity of the free Banach lattice generated by an infinite linear order

Now, we are going to prove Theorem 4.3, that is, if $\mathbb{L}$ is an infinite linear order, then $FBL\langle\mathbb{L}\rangle$ is not ∞-projective. This will be a direct consequence of the fact that the free Banach lattices generated by the set of the natural numbers and the set of the natural numbers together with $+\infty$ as linearly ordered sets are not ∞-projective. In the proof, we will use the following:

Lemma 4.14. *Suppose that $\varphi_i \colon [-1,1]^{\mathbb{N}} \longrightarrow \mathbb{R}$, $i = 1, 2, \ldots$, are continuous functions such that, for every i,*

1. *$\varphi_i((x_n)_{n\in\mathbb{N}}) = x_i$ whenever $x_1 \leq x_2 \leq \ldots$,*
2. *$\varphi_i(x) \leq \varphi_{i+1}(x)$ for all $x \in [-1,1]^{\mathbb{N}}$.*

Then, when we view the φ_i's as elements of the free Banach lattice $FBL(\mathbb{N})$, the sequence of norms $\|\varphi_i\|_{FBL(\mathbb{N})}$ is unbounded.

Proof. Let $\pi_i \colon [-1,1]^{\mathbb{N}} \longrightarrow [-1,1]$ be the projection on the i-th coordinate. Consider the set $M := \left\{(x_n)_{n\in\mathbb{N}} \in [-1,1]^{\mathbb{N}} : x_1 \leq x_2 \leq \ldots\right\} \subset [-1,1]^{\mathbb{N}}$. Since M is closed and $[-1,1]^{\mathbb{N}}$ with the product topology is compact, we have that M is compact. Condition 1 in the Lemma means that $\varphi_i|_M = \pi_i|_M$ for all i. Using the compactness of M and the continuity of φ_i and π_i, we have that that there exists a neighbourhood U_i of M such that

$$d(\varphi_i|_{U_i}, \pi_i|_{U_i}) = \sup_{x\in U_i} |\varphi_i(x) - \pi_i(x)| < \frac{1}{2}.$$

For an integer $k \geq 3$, let

$$W_k := \left\{(x_n)_{n\in\mathbb{N}} \in [-1,1]^{\mathbb{N}} : x_i < x_j + k^{-1} \text{ whenever } i < j < k\right\}.$$

The family $\{W_k : k \geq 3\}$ is a neighbourhood basis of M. We define inductively an increasing sequence of natural numbers $k_0 < k_1 < k_2 < k_3 < \cdots$, and a sequence of points $y^1, y^2, \ldots \in [-1,1]^{\mathbb{N}}$ as follows. We take $k_0 = 1$ as a starting point of the induction. Suppose that we have defined $k_0 < k_1 < \cdots < k_j$ and $y^1, \ldots, y^j$. We choose $k_{j+1} > k_j$ such that $W_{k_{j+1}} \subset U_{k_j}$, and we define $y^{j+1} \colon \mathbb{N} \longrightarrow [-1,1]$ to be the map given by

$$y^{j+1}(n) = \begin{cases} 0 & if \quad n < k_j, \\ 1 & if \quad k_j \leq n < k_{j+1}, \\ 0 & if \quad n \geq k_{j+1}. \end{cases}$$

We have $y^{j+1} \in W_{k_{j+1}}$, so $|\varphi_{k_j}(y^{j+1}) - \pi_{k_j}(y^{j+1})| = |\varphi_{k_j}(y^{j+1}) - 1| < \frac{1}{2}$, and then, $\varphi_{k_j}(y^{j+1}) > \frac{1}{2}$.

Fix $m \in \mathbb{N}$. When $j+1 \leq m$, using condition 2 of the Lemma, we get that

$$\varphi_{k_m}(y^{j+1}) \geq \varphi_{k_j}(y^{j+1}) > \frac{1}{2}.$$

Remember how the norm is defined:

$$\|\varphi\|_{FBL(\mathbb{N})} = \sup\left\{\sum_{j=1}^{m} |\varphi(x_j)| : m \in \mathbb{N},\ x_j \in [-1,1]^{\mathbb{N}},\ \sup_{n\in\mathbb{N}} \sum_{j=1}^{m} |x_j(n)| \leq 1\right\}.$$

We have that $\sup_{n\in\mathbb{N}} \sum_{j=1}^{m} |y^j(n)| = 1$, and therefore

$$\|\varphi_{k_m}\|_{FBL(\mathbb{N})} \geq |\varphi_{k_m}(y^1)| + \cdots + |\varphi_{k_m}(y^m)| > \frac{m}{2}.$$

□

Now, let $\mathbb{N}^+ = \mathbb{N} \cup \{+\infty\}$.

Lemma 4.15. *$FBL\langle\mathbb{N}\rangle$ and $FBL\langle\mathbb{N}^+\rangle$ are not ∞-projective.*

Proof. First, if $FBL_*\langle\mathbb{N}\rangle$ was λ-projective for some $\lambda > 1$, by the very definition of λ-projectivity we would have a Banach lattice homomorphism $u\colon FBL_*\langle\mathbb{N}\rangle \longrightarrow FBL(\mathbb{N})$ such that $R \circ u = id_{FBL_*\langle\mathbb{N}\rangle}$ and $\|u\| \leq \lambda$. Remember that if $i \in \mathbb{N}$, $\dot{\delta}_i\colon \mathbb{N}^* \longrightarrow \mathbb{R}$ is the map given by $\dot{\delta}_i(x^*) = x^*(i)$ for every $x^* \in \mathbb{N}^*$, that is an element of $FBL_*\langle\mathbb{N}\rangle$. We consider $\varphi_i = u(\dot{\delta}_i) \in FBL(\mathbb{N})$, that we view as continuous functions $\varphi_i\colon [-1,1]^{\mathbb{N}} \longrightarrow \mathbb{R}$. The fact that u is a lattice homomorphism gives condition 2 of Lemma 4.14, while the fact that $R \circ u = id_{FBL_*\langle\mathbb{N}\rangle}$ gives condition 1 of Lemma 4.14. The fact that $\|u\| \leq \lambda$ contradicts the conclusion of Lemma 4.14.

On the other hand, if $FBL_*\langle\mathbb{N}^+\rangle$ was λ-projective for some $\lambda > 1$, there would exists a Banach lattice homomorphism $w\colon FBL_*\langle\mathbb{N}^+\rangle \longrightarrow FBL(\mathbb{N}^+)$ such that $R \circ w = id_{FBL_*\langle\mathbb{N}^+\rangle}$ and $\|w\| \leq \lambda$. For every $i \in \mathbb{N}^+$, let $\psi_i = w(\dot{\delta}_i)$. Then again $\psi_i((x_n)_{n\in\mathbb{N}^+}) = x_i$ if $x_1 \leq x_2 \leq \ldots \leq x_{+\infty}$, and $\psi_1 \leq \psi_2 \leq \ldots \leq \psi_{+\infty}$.

Fix $\mathcal{U}$ a nonprincipal ultrafilter on $\mathbb{N}$ and define $\varphi_i((x_n)_{n\in\mathbb{N}}) = \psi_i((x_1, x_2, \ldots, \lim_{\mathcal{U}} x_n))$ for every $i \in \mathbb{N}$. Then the functions φ_i are as in Lemma 4.14, so $\|\varphi_i\|_{FBL(\mathbb{N})}$ is unbounded. We check now that $\|\varphi_i\|_{FBL(\mathbb{N})} \leq \|\psi_i\|_{FBL(\mathbb{N}^+)} \leq \lambda$, a contradiction. Take $\sum_{j=1}^m |\varphi_i(x_j)|$ one of the sums that appear in the definition of $\|\varphi_i\|_{FBL(\mathbb{N})}$ as a supremum. Consider $y_j = (x_j(1), x_j(2), \ldots, \lim_{\mathcal{U}} x_j(n))$. Then, $\sum_{j=1}^m |\varphi_i(x_j)| = \sum_{j=1}^m |\psi_i(y_j)|$ and this is one of the sums that appears in the supremum defining $\|\psi_i\|_{FBL(\mathbb{N}^+)}$ because

$$\sum_{j=1}^{m} |y_j(n)| = \sum_{j=1}^{m} |x_j(n)| \leq 1$$

if $n \in \mathbb{N}$ and

$$\sum_{j=1}^{m} |y_j(+\infty)| = \sum_{j=1}^{m} |\lim_{\mathcal{U}} x_j(n)| = \lim_{\mathcal{U}} \sum_{j=1}^{m} |x_j(n)| \leq 1.$$

□

The following result is well known in the general theory of categories [30, Proposition 24.6.2], and tells us that λ-projectivity is transferred to complemented Banach sublattices:

Lemma 4.16. *Let P and P' be Banach lattices, and let $\tilde{\pi}\colon P \longrightarrow P'$ and $\tilde{\imath}\colon P' \longrightarrow P$ be Banach lattice homomorphisms such that $\|\tilde{\imath}\| = \|\tilde{\pi}\| = 1$ and $\tilde{\pi} \circ \tilde{\imath} = id_{P'}$. If P is λ-projective for some $\lambda > 1$, then P' is λ-projective.*

Proof. In order to check the λ-projectivity of P', let $Q\colon X \longrightarrow X/\mathcal{J}$ and $T'\colon P' \longrightarrow X/\mathcal{J}$ be as in Definition 4.1. Then, we can apply the λ-projectivity of P considering $T = T' \circ \tilde{\pi}$, so we get $\hat{T}\colon P \longrightarrow X$ such that $Q \circ \hat{T} = T' \circ \tilde{\pi}$ and $\|\hat{T}\| \leq \lambda\|T\| \leq \lambda\|T'\|$. The desired lift is $\hat{T}' = \hat{T} \circ \tilde{\imath}$. On the one hand $\|\hat{T}'\| \leq \|\hat{T}\| \leq \lambda\|T'\|$, and on the other hand $Q \circ \hat{T}' = Q \circ \hat{T} \circ \tilde{\imath} = T' \circ \tilde{\pi} \circ \tilde{\imath} = T'$. □

We prove now Theorem 4.3. Let us denote by $\preceq$ the linear order on $\mathbb{L}$. $\mathbb{L}$ contains either an increasing or a decreasing sequence. Let us suppose without loss of generality that it contains an increasing sequence $x_1 \prec x_2 \prec x_3 \prec \cdots$.

First, suppose that it has no upper bound. The map $\imath\colon (\mathbb{N}, \leq) \longrightarrow (\mathbb{L}, \preceq)$ given by $\imath(n) = x_n$ for every $n \in \mathbb{N}$ is a lattice homomorphism. Let $\pi\colon \mathbb{L} \longrightarrow \mathbb{N}$ be the map given by

$$\pi(x) = \begin{cases} 1 & \text{if} \quad x \prec x_2, \\ n & \text{if} \quad x \in [x_n, x_{n+1}) \text{ for any } n \geq 2. \end{cases}$$

Notice that π is also a lattice homomorphism and $\pi \circ \imath = id_{\mathbb{N}}$. We are going to use the universal property of the free Banach lattice over a lattice as stated in Definition 2.1. Let $\phi_{\mathbb{L}}$ and $\phi_{\mathbb{N}}$ be the canonical inclusion of $\mathbb{L}$ and $\mathbb{N}$ into $FBL\langle \mathbb{L} \rangle$ and $FBL\langle \mathbb{N} \rangle$, respectively, and let $\tilde{\imath}\colon FBL\langle \mathbb{N} \rangle \longrightarrow FBL\langle \mathbb{L} \rangle$ and $\tilde{\pi}\colon FBL\langle \mathbb{L} \rangle \longrightarrow FBL\langle \mathbb{N} \rangle$ be the corresponding extensions of $\phi_{\mathbb{L}} \circ \imath$ and $\phi_{\mathbb{N}} \circ \pi$ according to Definition 2.1. The composition $\tilde{\pi} \circ \tilde{\imath}$ and the identity mapping $FBL\langle \mathbb{N} \rangle \longrightarrow FBL\langle \mathbb{N} \rangle$ are both extensions of $\phi_{\mathbb{N}}$ so by the uniqueness in Definition 2.1, $\tilde{\pi} \circ \tilde{\imath} = id_{FBL\langle \mathbb{N} \rangle}$. We can apply Lemma 4.16, so if $FBL\langle \mathbb{L} \rangle$ was ∞-projective, then $FBL\langle \mathbb{N} \rangle$ would also be ∞-projective, in contradiction with Lemma 4.15.

On the other hand, if the sequence $x_1 \prec x_2 \prec x_3 \prec \cdots$ has an upper bound, $x_{+\infty}$, we can take $\imath\colon (\mathbb{N}^+, \leq) \longrightarrow (\mathbb{L}, \preceq)$ given by $\imath(n) = x_n$ for every $n \in \mathbb{N}^+$ and $\pi\colon \mathbb{L} \longrightarrow \mathbb{N}^+$ given by

$$\pi(x) = \begin{cases} 1 & \text{if} \quad x \prec x_2, \\ n & \text{if} \quad x \in [x_n, x_{n+1}) \text{ for any } n \geq 2, \\ +\infty & \text{if} \quad x > x_n \text{ for all } n, \end{cases}$$

and apply the same reasoning substituting $\mathbb{N}$ by $\mathbb{N}^+$.

4.5 Schur property in Banach spaces with projective free Banach lattice

In this section we prove Theorems 4.5 and 4.4, in this order. As a preparation towards Theorem 4.5 we provide a criterion to obtain ℓ_1-subsequences in the free Banach lattice $FBL(L)$. We denote the index set L instead of A for convenience in latter application.

Lemma 4.17. *Let L be an infinite set, $(x_n^*)_{n\in\mathbb{N}}$ a sequence in $[-1,1]^L$ and $(f_n)_{n\in\mathbb{N}}$ a sequence in $FBL(L)$ with the following properties:*

1. *$(f_n)_{n\in\mathbb{N}}$ converges pointwise to 0, i.e. $\lim_{n\to\infty} f_n(x^*) = 0$ for every $x^* \in [-1,1]^L$;*
2. *$f_n(x_n^*) = 1$ for every $n \in \mathbb{N}$;*
3. *For every finite set $F \subset L$ there is a natural number n such that $x_n^*|_F = 0$, i.e. the restriction of x_n^* to F is null.*

Then, for every $\varepsilon > 0$ there is a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ such that for every $l \in \mathbb{N}$ and for every $\lambda_1, \ldots, \lambda_l \in \mathbb{R}$,

$$\left\| \sum_{k=1}^{l} \lambda_k f_{n_k} \right\| \geq (1-\varepsilon) \sum_{k=1}^{l} |\lambda_k|.$$

Proof. Fix $\varepsilon > 0$ and $(\varepsilon_{ij})_{i,j=1}^{\infty}$ a family of positive real numbers such that $\varepsilon = \sum_{i,j=1}^{\infty} \varepsilon_{ij}$ and $\varepsilon_{ij} = \varepsilon_{ji}$ for every i, j.

We are going to define a subsequence $(f_{m_k})_{k\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$ as follows:

Let $m_1 := 1$. Since the elements of $FBL(L)$ are continuous with respect to the product topology, there is a neighbourhood U_{m_1} of $x^*_{m_1}$ such that $f_{m_1}(x^*) \in [1-\varepsilon_{11}, 1+\varepsilon_{11}]$ whenever $x^* \in U_{m_1}$. In particular, there is a finite set $F_{m_1} \subset L$ such that $f_{m_1}(x^*) \in [1-\varepsilon_{11}, 1+\varepsilon_{11}]$ whenever $x^*|_{F_{m_1}} = x^*_{m_1}|_{F_{m_1}}$.

By property 3, there exists $m_2 \in \mathbb{N}$ such that $x^*_{m_2}|_{F_{m_1}} = 0$. Since f_{m_2} is continuous, there exists a finite set $F_{m_2} \supset F_{m_1}$ such that $f_{m_2}(x^*) \in [1-\varepsilon_{22}, 1+\varepsilon_{22}]$ whenever $x^*|_{F_{m_2}} = x^*_{m_2}|_{F_{m_2}}$.

Suppose that we have $f_{m_1}, \dots, f_{m_{k-1}}$ for some $k \geq 2$, and $F_{m_1}, \dots, F_{m_{k-1}}$ finite subsets of L such that $F_{m_1} \subset \cdots \subset F_{m_{k-1}}$, $x^*_{m_i}|_{F_{m_{i-1}}} = 0$ for every $i = 2, \dots, k-1$ and $f_{m_i}(x^*) \in [1-\varepsilon_{ii}, 1+\varepsilon_{ii}]$ whenever $x^*|_{F_{m_i}} = x^*_{m_i}|_{F_{m_i}}$.

Property 3 guarantees the existence of a number $m_k \in \mathbb{N}$ such that $x^*_{m_k}|_{F_{m_{k-1}}} = 0$. It follows from property 2 that there is a finite set $F_{m_k} \subset L$, with $F_{m_{k-1}} \subset F_{m_k}$, such that $f_{m_k}(x^*) \in [1-\varepsilon_{kk}, 1+\varepsilon_{kk}]$ whenever $x^*|_{F_{m_k}} = x^*_{m_k}|_{F_{m_k}}$.

For each $k \in \mathbb{N}$ define $y^*_{m_k} : L \longrightarrow [-1, 1]$ such that $y^*_{m_k}|_{F_{m_k}} = x^*_{m_k}|_{F_{m_k}}$ and $y^*_{m_k}(x) = 0$ whenever $x \in L \setminus F_{m_k}$. Notice that $f_{m_k}(y^*_{m_k}) \in [1-\varepsilon_{kk}, 1+\varepsilon_{kk}]$ for every $k \in \mathbb{N}$. On the other hand, if $m_k < m_{k'}$ and $y^*_{m_k}(x) \neq 0$ then $x \in F_{m_k}$ (by the definition of $y^*_{m_k}$) and therefore $x^*_{m_{k'}}(x) = 0$, so $y^*_{m_{k'}}(x) = 0$. It follows that $y^*_{m_k}$ and $y^*_{m_{k'}}$ have disjoint supports. In particular,

$$\sup_{x\in L} \sum_{k=1}^{l} \left|y^*_{m_k}(x)\right| \leq 1.$$

Let $\nu_1 := m_1 = 1$. Combining property 1 with the fact that the functions f_n are continuous in $[-1, 1]^L$ and the functions $y^*_{m_n}$ converge to 0 in the product topology, we have that there exists $\nu_2 \in \mathbb{N}$ such that

$$|f_{m_n}(y^*_{m_{\nu_1}})| \leq \varepsilon_{12} \text{ and } |f_{m_{\nu_1}}(y^*_{m_n})| \leq \varepsilon_{21} = \varepsilon_{12} \text{ for every } n \geq \nu_2.$$

Again, using the above, there exists a natural number $\nu_3 \geq \nu_2$ such that

$$|f_{m_n}(y^*_{m_{\nu_1}})| \leq \varepsilon_{13}, \; |f_{m_{\nu_1}}(y^*_{m_n})| \leq \varepsilon_{31} = \varepsilon_{13}$$

and

$$|f_{m_n}(y^*_{m_{\nu_2}})| \leq \varepsilon_{23}, \; |f_{m_{\nu_2}}(y^*_{m_n})| \leq \varepsilon_{32} = \varepsilon_{23}$$

for every $n \geq \nu_3$.

Suppose that we have $\nu_1 \leq \nu_2 \leq \cdots \leq \nu_p \in \mathbb{N}$ such that

$$|f_{m_n}(y^*_{m_{\nu_j}})| \leq \varepsilon_{jp} \text{ and } |f_{m_{\nu_j}}(y^*_{m_n})| \leq \varepsilon_{pj} = \varepsilon_{jp} \text{ for every } j < p \text{ and every } n \geq \nu_p.$$

Then, there exists a natural number $\nu_{p+1} \geq \nu_p$ such that

$$|f_{m_n}(y^*_{m_{\nu_j}})| \leq \varepsilon_{j(p+1)} \text{ and } |f_{m_{\nu_j}}(y^*_{m_n})| \leq \varepsilon_{(p+1)j} = \varepsilon_{j(p+1)} \text{ for every } j < p+1$$

and every $n \geq \nu_{p+1}$.

Since $f_{m_{\nu_i}}(y^*_{m_{\nu_i}}) \in [1-\varepsilon_{\nu_i\nu_i}, 1+\varepsilon_{\nu_i\nu_i}]$ for every i, we can write $f_{m_{\nu_i}}(y^*_{m_{\nu_i}}) = 1+\eta_{\nu_i\nu_i}$ with $|\eta_{\nu_i\nu_i}| \leq \varepsilon_{\nu_i\nu_i}$.

On the other hand, if $k \neq i$, we have that $f_{m_{\nu_k}}(y^*_{m_{\nu_i}}) \in [-\varepsilon_{ik}, \varepsilon_{ik}]$, and we will write $f_{m_{\nu_k}}(y^*_{m_{\nu_i}}) = \eta_{\nu_i\nu_k}$ with $|\eta_{\nu_i\nu_k}| \leq \varepsilon_{ik}$.

We take the subsequence $f_{n_k} := f_{m_{\nu_k}}$ for every $k \in \mathbb{N}$.

Now, let $\lambda_1, \dots, \lambda_l \in \mathbb{R}$. We have that

$$\begin{aligned}
\left\|\sum_{k=1}^{l} \lambda_k f_{n_k}\right\| &= \sup\left\{\sum_{i=1}^{q}\left|\sum_{k=1}^{l}\lambda_k f_{n_k}(z_i^*)\right| : q\in\mathbb{N},\ z_i^*\in[-1,1]^L,\ \sup_{x\in L}\sum_{i=1}^{q}|z_i^*(x)|\leq 1\right\}\\
&\geq \sum_{i=1}^{l}\left|\sum_{k=1}^{l}\lambda_k f_{m_{\nu_k}}(y^*_{m_{\nu_i}})\right| = \sum_{i=1}^{l}\left|\lambda_i f_{m_{\nu_i}}(y^*_{m_{\nu_i}}) + \sum_{k\neq i}\lambda_k f_{m_{\nu_k}}(y^*_{m_{\nu_i}})\right|\\
&= \sum_{i=1}^{l}\left|\lambda_i(1+\eta_{\nu_i\nu_i}) + \sum_{k\neq i}\lambda_k\eta_{\nu_i\nu_k}\right| = \sum_{i=1}^{l}\left|\lambda_i + \sum_{k=1}^{l}\lambda_k\eta_{\nu_i\nu_k}\right|\\
&\geq \sum_{i=1}^{l}|\lambda_i| - \sum_{i=1}^{l}\sum_{k=1}^{l}|\lambda_k|\,|\eta_{\nu_i\nu_k}| = \sum_{i=1}^{l}|\lambda_i| - \sum_{k=1}^{l}|\lambda_k|\left(\sum_{i=1}^{l}|\eta_{\nu_i\nu_k}|\right)\\
&\geq \sum_{i=1}^{l}|\lambda_i| - \sum_{k=1}^{l}|\lambda_k|\left(\varepsilon_{\nu_k\nu_k} + \sum_{i\neq k}\varepsilon_{ik}\right) \geq (1-\varepsilon)\sum_{k=1}^{l}|\lambda_k|\,.
\end{aligned}$$

□

Now, let $L = \mathcal{P}^+_{fin}(\omega) = \mathcal{P}_{fin}(\omega)\setminus\{\emptyset\}$ be the set of the finite parts of ω without the empty set.

For $A \in L$ let us define the map $\chi_A\colon L \longrightarrow [-1,1]$ given by $\chi_A(B) = 1$ if $B \subset A$ and $\chi_A(B) = 0$ if $B \not\subset A$.

Let $\Phi\colon FBL(L) \longrightarrow c_0$ be the map given by

$$\Phi(f) = \left(f\left((\chi_A(\{1\}))_{A\in L}\right), f\left((\chi_A(\{2\}))_{A\in L}\right), \dots\right) = \left(f\left((\chi_A(\{n\}))_{A\in L}\right)\right)_{n\in\mathbb{N}}$$

for every $f\colon [-1,1]^L \longrightarrow \mathbb{R}$ belonging to $FBL(L)$.

Lemma 4.18. *The map $\Phi\colon FBL(L) \longrightarrow c_0$ is a surjective Banach lattice homomorphism.*

Proof. For every $n \in \mathbb{N}$ let $g_n\colon L \longrightarrow [-1,1]$ be the function $g_n = (\chi_A(\{n\}))_{A\in L}$. Then the sequence $(g_n)_{n\in\mathbb{N}}$ is pointwise convergent to zero, so $f(g_n)$ converges to zero for every $f \in FBL(L)$, and thus $\Phi(f) \in c_0$. Since Φ preserves linear combinations, suprema, infima and $\|\Phi(f)\|_\infty \leq \|f\|$ for every $f \in FBL(L)$, we have that Φ is a Banach lattice homomorphism.

Finally, let us see that Φ is surjective. By the very definition of Φ, if for $A \in L$ we put $e_A := \sum_{i\in A} e_i$, we have that $\Phi(\delta_A) = e_A$. Let $x = (x_1, x_2, \dots) \in c_0$ and suppose, without loss of generality, that $x \geq 0$. Fix a sequence of natural numbers $(n_i)_{i\in\mathbb{N}}$ with $n_i \neq n_j$ for every $i \neq j$ and such that $x_{n_1} \geq x_n$ for every $n \in \mathbb{N}$ and $x_{n_{i+1}} \geq x_n$ for every

$n \in \mathbb{N} \setminus \{n_1 \ldots, n_i\}$. Now, let $A_i = \{n_1, \ldots, n_i\}$ and $\lambda_i = x_{n_i} - x_{n_{i+1}}$ for every $i = 1, 2, \ldots$ We have that $x = \sum_{j=1}^{\infty} \lambda_j e_{A_j}$, and then,

$$x = \sum_{j=1}^{\infty} \lambda_j \Phi(\delta_{A_j}) = \sum_{j=1}^{\infty} \Phi(\lambda_j \delta_{A_j}) = \Phi\big(\sum_{j=1}^{\infty} \lambda_j \delta_{A_j}\big),$$

where the last element $\sum_{j=1}^{\infty} \lambda_j \delta_{A_j}$ is well-defined since $\sum_{j=1}^{\infty} \lambda_j < \infty$ and each δ_{A_j} has norm 1. $\square$

We are ready to prove Theorem 4.5 from the introduction. Suppose that there is no subsequence equivalent to the canonical basis of ℓ_1. Then, by Rosenthal's ℓ_1-theorem [2, Theorem 10.2.1], the sequence $(u_i)_{i\in\mathbb{N}}$ has a weakly Cauchy subsequence $(u_{i_k})_{k\in\mathbb{N}}$. Thus, the sequence $(y_n)_{n\in\mathbb{N}}$, with $y_n = u_{i_{2n+1}} - u_{i_{2n}}$ for every $n \in \mathbb{N}$, is weakly null and bounded.

Let us denote $T(x) = (T(x)_j)_{j\in\mathbb{N}} \in c_0$ for $x \in X$, and let $\tilde{T}\colon X \longrightarrow c_0$ be the map given by $\tilde{T}(x) = (T(x)_{i_{2k+1}})_{k\in\mathbb{N}}$.

Let L and $\Phi\colon FBL(L) \longrightarrow c_0$ like in Lemma 4.18. Since Φ is a surjective Banach lattice homomorphism and X is ∞-projective, by Proposition 4.7 there exists a bounded Banach lattice homomorphism $\ddot{T}\colon X \longrightarrow FBL(L)$ such that $\Phi \circ \ddot{T} = \tilde{T}$. We are going to find now f_n and x_n^* for the application of Lemma 4.17.

Let $f_n := \ddot{T}(y_n)$ for every $n \in \mathbb{N}$. The sequence $(f_n)_{n\in\mathbb{N}}$ converges pointwise to 0, since $(y_n)_{n\in\mathbb{N}}$ is weakly null. It follows from the equality $\Phi(f_n) = (\Phi \circ \ddot{T})(y_n) = \tilde{T}(y_n) = e_n$ and the definition of Φ that

$$f_n\left((\chi_A(\{n\}))_{A\in L}\right) = \Phi(f_n)_n = e_n(n) = 1$$

for every $n \in \mathbb{N}$. Set $x_n^* = (\chi_A(\{n\}))_{A\in L} \in [-1,1]^L$ for every $n \in \mathbb{N}$. Notice that if $F \subset L$ is finite (i.e. F is a finite collection of finite subsets of ω), and we take $n \in \omega \setminus \bigcup_{S\in F} S$, then for every $S \in F$, $n \notin S$, so $x_n^*(S) = \chi_S(\{n\}) = 0$, and therefore condition 3 of Lemma 4.17 is also satisfied.

We can now apply Lemma 4.17, so for every $\varepsilon > 0$ there is a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ such that for every $l \in \mathbb{N}$ and for every $\lambda_1, \ldots, \lambda_l \in \mathbb{R}$,

$$\left\|\sum_{k=1}^{l} \lambda_k f_{n_k}\right\| \geq (1-\varepsilon)\sum_{k=1}^{l} |\lambda_k|.$$

On the other hand, since $\ddot{T}$ and $(y_n)_{n\in\mathbb{N}}$ are bounded, there are two constants $C, M > 0$ such that

$$\left\|\sum_{k=1}^{l} \lambda_k f_{n_k}\right\| = \left\|\ddot{T}\left(\sum_{k=1}^{l} \lambda_k y_{n_k}\right)\right\| \leq C\left\|\sum_{k=1}^{l} \lambda_k y_{n_k}\right\| \leq CM\sum_{k=1}^{l} |\lambda_k|.$$

Thus,

$$(1-\varepsilon)\sum_{k=1}^{l} |\lambda_k| \leq \left\|\sum_{k=1}^{l} \lambda_k f_{n_k}\right\| \leq CM\sum_{k=1}^{l} |\lambda_k|,$$

so that $(f_{n_k})_{k\in\mathbb{N}}$ is equivalent to the canonical basis of ℓ_1, and in consequence, $(y_{n_k})_{k\in\mathbb{N}}$ is also equivalent to the canonical basis of ℓ_1, which is a contradiction.

Corollary 4.19. *The Banach lattices c_0 and l_p (for $2 \le p < \infty$) are not ∞-projective.*

Proof. On the one hand, the canonical basis $(u_i)_{i\in\mathbb{N}}$ of c_0 does not have subsequences equivalent to the canonical basis of ℓ_1, and the identity map $T = id_{c_0}$ is a Banach lattice homomorphism such that $T(u_i) = e_i$ for every $i \in \mathbb{N}$, where $(e_i)_{i\in\mathbb{N}}$ is the canonical basis of c_0. On the other hand, the canonical basis $(u_i)_{i\in\mathbb{N}}$ of l_p does not have subsequences equivalent to the canonical basis of ℓ_1, and the formal inclusion T of l_p into c_0 is a Banach lattice homomorphism such that $T(u_i) = e_i$ for every $i \in \mathbb{N}$, where $(e_i)_{i\in\mathbb{N}}$ is the canonical basis of c_0. □

Remark 4.20. N. J. Laustsen pointed out to us that Theorem 4.5 gives us something stronger than Corollary 4.19. Indeed, let E be a Banach space with a normalized 1-unconditional basis $(u_i)_{i\in\mathbb{N}}$, and let X be the Banach lattice consisting of the Banach space E endowed with the coordinatewise order, that is, the order given by

$$\sum_{i=1}^{\infty} \alpha_i u_i \le \sum_{i=1}^{\infty} \beta_i u_i \text{ if, and only if, } \alpha_i \le \beta_i \text{ for every } i \in \mathbb{N}.$$

Then we have the following result:

Corollary (N. J. Laustsen). *If X is a Banach lattice as above and the basis $(u_i)_{i\in\mathbb{N}}$ does not have subsequences equivalent to the canonical basis of ℓ_1, then X is not ∞-projective.*

Proof. Since $(u_i)_{i\in\mathbb{N}}$ is normalized, we have a formal inclusion map $T\colon X \longrightarrow c_0$ given by $T(\sum_{i=1}^{\infty} \alpha_i u_i) = (\alpha_i)_{i\in\mathbb{N}}$. By the very definition of the order on E, we have that T is a lattice homomorphism. The fact that $(u_i)_{i\in\mathbb{N}}$ is 1-unconditional asserts that T has norm 1. Finally, $T(u_i) = e_i$ for every $i \in \mathbb{N}$, where $(e_i)_{i\in\mathbb{N}}$ is the canonical basis of c_0. □

The following is a corollary of Theorem 4.5 in the context of free Banach lattices $FBL[E]$:

Lemma 4.21. *Let E be a Banach space such that $FBL[E]$ is ∞-projective, and let $(u_i)_{i\in\mathbb{N}}$ be a bounded sequence of vectors in E. Suppose that there exists an operator $S\colon E \longrightarrow c_0$ such that $S(u_i) = e_i$ for every $i \in \mathbb{N}$, where $(e_i)_{i\in\mathbb{N}}$ is the canonical basis of c_0. Then there is a subsequence $(u_{i_k})_{k\in\mathbb{N}}$ equivalent to the canonical basis of ℓ_1.*

Proof. Let $\phi\colon E \longrightarrow FBL[E]$ be the inclusion of E into $FBL[E]$, and let $T\colon FBL[E] \longrightarrow c_0$ be the Banach lattice homomorphism given by the universal property of the free Banach lattice over a Banach space which extends the operator S.

The sequence $(\phi(u_i))_{i\in\mathbb{N}}$ is bounded in $FBL[E]$ and $T(\phi(u_i)) = S(u_i) = e_i$ for every $i \in \mathbb{N}$, so that applying Theorem 4.5 we have that $(\phi(u_i))_{i\in\mathbb{N}}$ has a subsequence $(\phi(u_{i_k}))_{k\in\mathbb{N}}$ equivalent to the canonical basis of ℓ_1, which implies that $(u_{i_k})_{k\in\mathbb{N}}$ is a subsequence of $(u_i)_{i\in\mathbb{N}}$ equivalent to the canonical basis of ℓ_1. □

We pass now to the proof of Theorem 4.4, which states that E has the Schur property when $FBL[E]$ is ∞-projective. Lemmas 4.22, 4.23 and 4.24 are necessary only to deal with the case when E is nonseparable. The reader interested in the separable case may skip those lemmas and just apply Sobczyk's extension theorem [2, Theorem 2.5.8] where appropriate.

Lemma 4.22. *Let E be a Banach space. If $FBL[E]$ is ∞-projective, then E is isomorphic to a subspace of $C([-1,1]^\Gamma)$ for some set Γ.*

Proof. Let Γ be a dense subset of the unit ball B_E of E. Let B_{E^*} be the closed unit ball of the dual space E^*, endowed with the weak* topology. We have a surjective Banach lattice homomorphism $P\colon C([-1,1]^\Gamma) \longrightarrow C(B_{E^*})$ given by $P(f)(x^*) = f((x^*(x))_{x\in\Gamma})$. This is just the composition operator with the continuous injection $x^* \mapsto (x^*(x))_{x\in\Gamma}$ from B_{E^*} into $[-1,1]^\Gamma$. Let $\iota\colon E \longrightarrow C(B_{E^*})$ be the canonical inclusion $\iota(x)(x^*) = x^*(x)$, and let $\hat{\iota}\colon FBL[E] \longrightarrow C(B_{E^*})$ be the Banach lattice homomorphism given by the universal property of the free Banach lattice. Since $FBL[E]$ is supposed to be ∞-projective, by Proposition 4.7 there exists $\hat{T}\colon FBL[E] \longrightarrow C([-1,1]^\Gamma)$ such that $P \circ \hat{T} = \hat{\iota}$. We take the restriction $T := \hat{T}|_E\colon E \longrightarrow C([-1,1]^\Gamma)$. Notice that $PTx = \iota x$, and therefore

$$\|Tx\| \geq \|PTx\| = \|\iota x\| = \|x\|$$

for every $x \in E$. This implies that T gives an isomorphism of E onto a subspace of $C([-1,1]^\Gamma)$. □

The following fact is well known in the context of a more general theory about Valdivia compacta, Plichko spaces and projectional skeletons (cf. for instance [22]), but we provide a short proof for the reader's convenience:

Lemma 4.23. *For every set Γ, the Banach space $C([-1,1]^\Gamma)$ has the separable complementation property. That is, for every separable subspace $G \subset C([-1,1]^\Gamma)$ there exists a separable complemented subspace G_0 of $C([-1,1]^\Gamma)$ such that $G \subset G_0$.*

Proof. Let S be a countable dense subset of G. By Mibu's theorem [4, page 80, Theorem 4], for every $f \in S$ there exists a countable subset $\Gamma_f \subset \Gamma$ such that $f(x) = f(y)$ whenever $x|_{\Gamma_f} = y|_{\Gamma_f}$. The set $A = \bigcup_{f\in S}\Gamma_f$ is a countable set such that $f(x) = f(y)$ whenever $x|_A = y|_A$ and $f \in G$. The desired separable complemented subspace is

$$G_0 = \left\{f \in C([-1,1]^\Gamma) : x|_A = y|_A \Rightarrow f(x) = f(y)\right\} \cong C([-1,1]^A).$$

The projection $P\colon C([-1,1]^\Gamma) \longrightarrow G_0$ is given by $P(f)(x) = f(\tilde{x})$ where $\tilde{x}_i = x_i$ if $i \in A$ and $\tilde{x}_i = 0$ if $i \notin A$. □

Lemma 4.24. *Let E be a Banach space such that $FBL[E]$ is ∞-projective, and let $F \subset E$ be a separable subspace. Every operator $S_0\colon F \longrightarrow c_0$ can be extended to an operator $S\colon E \longrightarrow c_0$.*

Proof. By Lemma 4.22, there is an operator $T\colon E \longrightarrow C([-1,1]^\Gamma)$ that is an isomorphism onto its range, so that $G = T(F)$ is a separable subspace of $C([-1,1]^\Gamma)$. By Lemma 4.23, we can find a complemented separable subspace G_0 of $C([-1,1]^\Gamma)$ with $G \subset G_0$. Let $P\colon C([-1,1]^\Gamma) \longrightarrow G_0$ be the projection. If $S_0'\colon G_0 \longrightarrow c_0$ is the extension of S_0 given by Sobczyk's theorem, then $S := S_0' \circ P \circ T\colon E \longrightarrow c_0$ is the desired operator. □

Theorem 4.4 follows from the previous results. Indeed, if E does not have the Schur property, then there is a weakly null sequence $(u_i)_{i\in\mathbb{N}}$ that does not converge to 0 in norm. By passing to a subsequence we may assume that 0 is not in the norm closure of $\{u_i\}_{i\in\mathbb{N}}$. By the theorem of Kadets and Pełczyński [2, Theorem 1.5.6], by passing to a further subsequence, we can suppose that $(u_i)_{i\in\mathbb{N}}$ is a basic sequence. We are going to see that there exists an operator $S\colon E \longrightarrow c_0$ such that $S(u_i) = e_i$ for every $i \in \mathbb{N}$, where $(e_i)_{i\in\mathbb{N}}$ is the canonical basis of c_0, and then by Lemma 4.21, this will mean that $(u_i)_{i\in\mathbb{N}}$ has a subsequence equivalent to the canonical basis of ℓ_1, a contradiction with the fact that it is weakly null.

Let $F = \overline{\operatorname{span}}\,\{u_i : i \in \mathbb{N}\} \subset E$. For every $n \in \mathbb{N}$ let $u_n^*\colon F \longrightarrow \mathbb{R}$ be the n-th coordinate functional, given by $u_n^*(\sum_{i=1}^{\infty} \alpha_i u_i) = \alpha_n$, and let $S_0\colon F \longrightarrow \ell_\infty$ be the map given by $S_0(x) = (u_n^*(x))_{n\in\mathbb{N}}$ for every $x \in F$. Since the sequence $(u_n^*)_{n\in\mathbb{N}}$ is weak*-null, we have that $S_0(F) \subset c_0$. On the other hand, $S_0(u_i) = e_i$ for every $i \in \mathbb{N}$. Now, since F is separable and $FBL[E]$ is ∞-projective, applying Lemma 4.24 we can extend S_0 to an operator $S\colon E \longrightarrow c_0$ such that $S(u_i) = e_i$ for every $i \in \mathbb{N}$, which ends the proof.

As a remark, throughout the first lines of the proof we justify that the Schur property is characterized by the property that every basic sequence contains a subsequence equivalent to the canonical basis of ℓ_1. We may refer to [20] for a study of this kind of facts in a more general context.

Finally, let us see that, in the category of nonseparable Banach spaces, the converse does not hold. By [19, Theorem 1, e) and f)], there exist a separable Banach space F and a bounded set Λ in F^* such that $E := \overline{\operatorname{span}}(\Lambda)$ is nonseparable, has the Schur property and does not contain any copy of $\ell_1(\omega_1)$. Now, since for every set Γ the space $[-1,1]^\Gamma$ is a continuous image of $\{0,1\}^m$ for some infinite cardinal number m, by [18, Corollary 3] we have that E is not isomorphic to any subspace of $C([-1,1]^\Gamma)$ for any set Γ, and then, by Lemma 4.22, we have that $FBL[E]$ cannot be ∞-projective.

4.6 Projectivity of $C(K)$

This section is devoted to the proof of Theorem 4.6. We first start with two basic facts:

Lemma 4.25. *Let A be a set and $f\colon [-1,1]^A \longrightarrow \mathbb{R}$ a continuous and positively homogeneous function that depends on finitely many coordinates, i.e. there exist a finite subset $A_0 \subset A$ and $\tilde{f}\colon [-1,1]^{A_0} \longrightarrow \mathbb{R}$ such that $f(x^*) = \tilde{f}(x^*|_{A_0})$ for every $x^* \in [-1,1]^A$. Then, f is in $FBL(A)$.*

Proof. The function $\tilde{f}\colon [-1,1]^{A_0} \longrightarrow \mathbb{R}$ is continuous and positively homogeneous. By [15, Proposition 5.3], $\tilde{f}$ is in $FBL(A_0)$.

Let $T\colon \mathbb{R}^{[-1,1]^{A_0}} \longrightarrow \mathbb{R}^{[-1,1]^A}$ be the function given by $T(g)(x^*) = g(x^*|_{A_0})$ for every $g \in \mathbb{R}^{[-1,1]^{A_0}}$, $x^* \in [-1,1]^A$.

It is clear that T commutes with linear combinations and the lattice operations and that $\|T(g)\|_{FBL(A)} \leq \|g\|_{FBL(A_0)}$. Moreover, $T(\delta_x^{A_0}) = \delta_x^A$ for every $x \in A_0$. From this, we conclude that if $g \in FBL(A_0)$, then $T(g) \in FBL(A)$. Now, since $f = T(\tilde{f})$, the affirmation of the lemma holds. $\square$

Lemma 4.26. *Let A be a set, $f\colon [-1,1]^A \longrightarrow \mathbb{R}$ a continuous function, and $a \in A$. Then, the $FBL(A)$-norm of the pointwise product $f \cdot |\delta_a|$ is less than or equal to the supremum norm $\|f\|_\infty$.*

Proof.

$$\begin{aligned}
\|f\cdot|\delta_a|\| &= \sup\left\{\sum_{k=1}^m |f\cdot|\delta_a|(x_k^*)| : m\in\mathbb{N},\ x_k^*\in[-1,1]^A,\ \sup_{x\in A}\sum_{k=1}^m |x_k^*(x)|\le 1\right\}\\
&= \sup\left\{\sum_{k=1}^m |f(x_k^*)|\cdot|\delta_a(x_k^*)| : m\in\mathbb{N},\ x_k^*\in[-1,1]^A,\ \sup_{x\in A}\sum_{k=1}^m |x_k^*(x)|\le 1\right\}\\
&\le \sup\left\{\sum_{k=1}^m |f(x_k^*)|\cdot|x_k^*(a)| : m\in\mathbb{N},\ x_k^*\in[-1,1]^A,\ \sum_{k=1}^m |x_k^*(a)|\le 1\right\}\\
&\le \sup\left\{\max\{|f(x_k^*)| : k=1,\dots,m\} : m\in\mathbb{N},\ x_k^*\in[-1,1]^A,\ \sum_{k=1}^m |x_k^*(a)|\le 1\right\}\\
&\le \|f\|_\infty.
\end{aligned}$$

□

We are ready to prove Theorem 4.6. In [15, Proposition 11.7] it is proved that if $C(K)$ is 1^+-projective (or even, ∞-projective), then K is an ANR in the category of compact Hausdorff topological spaces, but we show the proof here for the sake of completeness.

Suppose that $i\colon K \longrightarrow X$ is an homeomorphism between K and a subspace of the compact Hausdorff topological space X. We want to see that there exist an open set V in X and a continuous function $\phi\colon V \longrightarrow K$ such that $i(K)\subset V\subset X$ and $\phi(i(k)) = k$ for all $k\in K$.

The restriction map $R\colon C(X)\longrightarrow C(i(K))$ is a surjective Banach lattice homomorphism. If $C(K)$ is 1^+-projective (or ∞-projective), $C(i(K))$ so is, so that, by Proposition 4.7, the identity map on $C(i(K))$ lifts to a Banach lattice homomorphism $T\colon C(i(K))\longrightarrow C(X)$ with $R\circ T = id_{C(i(K))}$. Now, by [1, Theorem 4.25], there is a continuous function $\omega\colon X\longrightarrow\mathbb{R}^+$ and a continuous function $\phi'\colon V=\{x\in X : \omega(x)>0\}\longrightarrow i(K)$ such that

$$Tf(x) = \begin{cases} \omega(x)\cdot f(\phi'(x)) & \text{if } \omega(x)>0,\\ 0 & \text{otherwise}, \end{cases}$$

for every $f\in C(i(K))$, $x\in X$.

If $k\in K$, then $Tf(i(k)) = R(Tf(i(k))) = f(i(k))$, so that $\omega(i(k))=1$ (which implies that $i(K)\subset V$), and $\phi'(i(k)) = i(k)$. Finally, if $i^{-1}\colon i(K)\longrightarrow K$ is the inverse of i, taking $\phi = i^{-1}\circ\phi'\colon V\longrightarrow K$ we have that $\phi(i(k)) = k$ as desired.

For the converse, let $X := [-1,1]^{B_{C(K)}}$, where $B_{C(K)} = \{f\in C(K) : \|f\|_\infty\le 1\}$ is the closed unit ball of the space of continuous functions. The map $i\colon K\longrightarrow X$ given by $i(k) = (\gamma(k))_{\gamma\in B_{C(K)}}$ is an homeomorphism between K and $i(K)$. By Lemma 4.11 there exist a continuous function $u\colon X\longrightarrow[0,1]$ and a continuous function $\varphi\colon X\setminus u^{-1}(0)\longrightarrow K$ such that $u(i(k)) = 1$ and $\varphi(i(k)) = k$ for every $k\in K$.

By the universal property of the free Banach lattice, there is a Banach lattice homomorphism $T\colon FBL(B_{C(K)}) \longrightarrow C(K)$ that extends the inclusion $B_{C(K)} \hookrightarrow C(K)$. This is clearly a quotient map and its action is given by $Tf(k) = f(i(k))$ for every $f \in FBL(B_{C(K)})$, $k \in K$.

Since $FBL(B_{C(K)})$ is 1^+-projective, by Proposition 4.8 it is enough to prove that there exists a Banach lattice homomorphism $S\colon C(K) \longrightarrow FBL(B_{C(K)})$ such that $T \circ S = id_{C(K)}$ and $\|S\| \leq 1$.

Let $\bar{1} \in B_{C(K)}$ be the constant function equal to 1, and let $v\colon \{x \in X : x_{\bar{1}} \neq 0\} \longrightarrow X$ be the map given by $v(x) = (v(x)_\gamma)_{\gamma \in B_{C(K)}}$, where

$$v(x)_\gamma = \begin{cases} -1 & \text{if} \quad \frac{x_\gamma}{x_{\bar{1}}} < -1, \\ \frac{x_\gamma}{x_{\bar{1}}} & \text{if} \quad \frac{x_\gamma}{x_{\bar{1}}} \in [-1,1], \\ 1 & \text{if} \quad \frac{x_\gamma}{x_{\bar{1}}} > 1, \end{cases}$$

for every $x = (x_\gamma)_{\gamma \in B_{C(K)}} \in X$ with $x_{\bar{1}} \neq 0$.

For a given $h \in C(K)$, define $f\colon X \longrightarrow \mathbb{R}$ by

$$f(x) := \begin{cases} (h \circ \varphi \circ v) \cdot (u \circ v)(x) & \text{if } x_{\bar{1}} \neq 0 \text{ and } u(v(x)) \neq 0, \\ 0 & \text{otherwise .} \end{cases}$$

Formally, we should call the function f_h as it depends on h. But we omit the subindex for a more friendly notation (in fact the subindex would always be "h" throughout the proof). Notice also that $x_{\bar{1}} \neq 0$ is required for x to be in the domain of v and $u(v(x)) \neq 0$ is required for $v(x)$ to be in the domain of φ.

The desired $S\colon C(K) \longrightarrow FBL(B_{C(K)})$ will be the map given by $Sh(x) = (f \cdot |\delta_{\bar{1}}|)(x)$ for every $h \in C(K)$, $x \in X$. The function Sh is a real-valued function on $X = [-1,1]^{B_{C(K)}}$, and we will need to prove that, in fact, $Sh \in FBL(B_{C(K)})$. Once that is proved, the rest of properties required for S are relatively easy to check: It is clear that S is a linear map, and it preserves the lattice operations $\wedge$ and $\vee$. The fact that $\|S\| \leq 1$ comes from Lemma 4.26:

$$\|Sh\| = \|f \cdot |\delta_{\bar{1}}|\| \leq \|f\|_\infty = \|(h \circ \varphi \circ v)(u \circ v))\|_\infty \leq \|h\|_\infty .$$

To see that $T \circ S = id_{C(K)}$, take $h \in C(K)$ and $k \in K$. Remember that $u(i(k)) = 1$ and $\varphi(i(k)) = k$ and notice that $i(k)_{\bar{1}} = 1$ and $v(i(k)) = i(k)$ for every $k \in K$, so

$$TSh(k) = Sh(i(k)) = (f \cdot |\delta_{\bar{1}}|)(i(k)) = h(\varphi(i(k))) \cdot u(i(k)) = h(k).$$

So we turn now to the remaining more delicate question whether $Sh \in FBL(B_{C(K)})$ for every $h \in C(K)$. Functions in the free Banach lattice must be continuous and positively homogeneous. We check first that Sh has these two properties. Clearly, Sh is continuous on the open set $\{x \in X : x_{\bar{1}} \neq 0, u(v(x)) \neq 0\}$ because Sh is expressed there by the formula $(h \circ \varphi \circ v) \cdot (u \circ v) \cdot |\delta_{\bar{1}}|$. If $x_{\bar{1}} = 0$, then for every $\varepsilon > 0$ there is a neighbourhood W such that $|f(y)| \cdot |y_{\bar{1}}| \leq \|h\|_\infty \cdot \varepsilon$ for all $y \in W$, so Sh is also continuous at those x. If $x_{\bar{1}} \neq 0$ but $u(v(x)) = 0$, again, given $\varepsilon > 0$, we can find a neighbourhood W of x where $y_{\bar{1}} \neq 0$ and $|f(y)| \cdot |y_{\bar{1}}| \leq \|h\|_\infty \cdot \varepsilon$ for all $y \in W$. For positive homogeneity, on the one hand, if $x_{\bar{1}} \neq 0$, then $v(\lambda x) = v(x)$ for every $0 < \lambda \leq 1$ and $x \in X$, while $|\delta_{\bar{1}}|$ is positively homogeneous. If $x_{\bar{1}} = 0$, then for every $0 < \lambda \leq 1$ we have that $Sh(\lambda x) = 0 = \lambda Sh(x)$.

Finally, what we are going to prove is that Sh can be obtained as the limit, in the $FBL(B_{C(K)})$-norm, of a sequence of continuous and positively homogeneous functions that only depend on finitely many coordinates of the cube $[-1,1]^{B_{C(K)}}$. Combining Lemma 4.25 with the fact that $FBL(B_{C(K)})$ is a closed space, we will have that $Sh \in FBL(B_{C(K)})$.

Consider $L = \{x \in X : x_{\bar{1}} = 1\} \subset X$. Since the restriction $f|_L$ is a continuous function on the compact space L, by Stone-Weierstrass' theorem, for every $n \in \mathbb{N}$ we can find a continuous function $f_n^+ \in C(L)$ that depends only on finitely many coordinates of the cube $[-1,1]^{B_{C(K)}}$ such that

$$\left\| f|_L - f_n^+ \right\|_\infty < \frac{1}{n}.$$

Define $f_n \colon X \longrightarrow \mathbb{R}$ by

$$f_n(x) := \begin{cases} f_n^+(v(x)) & \text{if } x_{\bar{1}} \neq 0, \\ 0 & \text{otherwise} . \end{cases}$$

It is clear that $f_n(\lambda x) = f_n(x)$ for all $0 < \lambda \leq 1$ and $x \in X$, since $v(\lambda x) = v(x)$. Moreover, f_n depends on finitely many coordinates because f_n^+ does so, and each coordinate of v depends on two coordinates ($v(x)_\gamma$ only depends on x_γ and $x_{\bar{1}}$). On the other hand, $f_n \cdot |\delta_{\bar{1}}|$ is continuous in X. This is because $f_n \cdot |\delta_{\bar{1}}|$ is continuous in $\{x \in X : x_{\bar{1}} \neq 0\}$ clearly, and, if $x_{\bar{1}} = 0$, then for every $\varepsilon > 0$ there is a neighbourhood W such that $|f_n(y)| \cdot |y_{\bar{1}}| \leq \|f_n^+\|_\infty \cdot \varepsilon$ for all $y \in W$. Thus, the functions $f_n \cdot |\delta_{\bar{1}}|$ are all continuous, positively homogeneous and depend on finitely many coordinates. It follows from Lemma 4.25 that $f_n \cdot |\delta_{\bar{1}}| \in FBL(B_{C(K)})$ for every $n \in \mathbb{N}$. It only remains to prove that $\|Sh - f_n \cdot |\delta_{\bar{1}}|\| \to 0$ when $n \to \infty$. For this, first notice that $v(v(x)) = v(x)$ for all $x \in X$ with $x_{\bar{1}} \neq 0$. This is just because $v(x)_{\bar{1}} = 1$. From this, it follows that $f(x) = f(v(x))$ for all x with $x_{\bar{1}} \neq 0$. This together with Lemma 4.26 gives:

$$\begin{aligned}
\|Sh - f_n \cdot |\delta_{\bar{1}}|\| &= \|f \cdot |\delta_{\bar{1}}| - f_n \cdot |\delta_{\bar{1}}|\| = \|(f - f_n) \cdot |\delta_{\bar{1}}|\| \\
&\leq \|f - f_n\|_\infty \\
&= \sup\{|f(x) - f_n(x)| : x \in X\} \\
&= \sup\{|f(x) - f_n(x)| : x \in X, x_{\bar{1}} \neq 0\} \\
&= \sup\left\{|f(x) - f_n^+(v(x))| : x \in X, x_{\bar{1}} \neq 0\right\} \\
&= \sup\left\{|f(v(x)) - f_n^+(v(x))| : x \in X, x_{\bar{1}} \neq 0\right\} \\
&\leq \sup\left\{|f(y) - f_n^+(y)| : y \in L\right\} = \|f|_L - f_n^+\|_\infty < \frac{1}{n},
\end{aligned}$$

which ends the proof.

4.7 Complementability of c_0 in $FBL[c_0]$

Following the terminology of [3], a Banach lattice Y is said to be *lattice-embeddable* in a Banach lattice X if it is isomorphic to a Banach sublattice of X, or equivalently, there exist a Banach lattice homomorphism $u\colon Y \longrightarrow X$ and two constants $K, M > 0$ such that

$$K \|y\|_Y \leq \|u(y)\|_X \leq M \|y\|_Y$$

for every $y \in Y$. Such a map u is called a *Banach lattice embedding*.

In this section we will prove that the Banach lattice c_0 is isometrically lattice-embeddable in $FBL[c_0]$, and that the copy of c_0 inside $FBL[c_0]$ is 1-complemented (as a Banach lattice) in $FBL[c_0]$. Indeed, this is the main theorem of this section and can be stated as follows:

Theorem 4.27. *There exist an isometric Banach lattice embedding $u\colon c_0 \longrightarrow FBL[c_0]$ and a Banach lattice homomorphism $P\colon FBL[c_0] \longrightarrow c_0$ of norm 1 such that $P \circ u = id_{c_0}$.*

Remark 4.28. V. G. Troitsky pointed out to us that, by a direct consequence of [3, Theorem 4.60], we have a Banach lattice embedding $u_1\colon c_0 \longrightarrow FBL[c_0]$ together with an operator projection $P_1\colon FBL[c_0] \longrightarrow c_0$ such that $P_1 \circ u_1 = id_{c_0}$. On the other hand, if we extend the identity operator $id_{c_0}\colon c_0 \longrightarrow c_0$, by using the universal property of the free Banach lattice, to a Banach lattice homomorphism $P_2\colon FBL[c_0] \longrightarrow c_0$, we have that $P_2 \circ \phi = id_{c_0}$, where $\phi\colon c_0 \longrightarrow FBL[c_0]$ is the canonical inclusion of c_0 inside $FBL[c_0]$, which is not a lattice homomorphism. So, the difficulty is in getting Banach lattice homomorphism inclusion and projection at the same time, and this is what Theorem 4.27 tells us that it is possible to get.

In order to prove Theorem 4.27, the following weaker version of [3, Theorem 4.50] will be very useful:

Theorem 4.29 ([3, Theorem 4.50])**.** *Let X be a Banach lattice. If there exists a disjoint sequence $(f_n)_{n\in\mathbb{N}} \in X^+$ such that*

a) $(f_n)_{n\in\mathbb{N}}$ does not converge in norm to zero, and

b) the sequence of partial sums of $(f_n)_{n\in\mathbb{N}}$ is norm bounded, i.e. there exists some $M > 0$ satisfying $\|\sum_{i=1}^n f_i\|_X \leq M$ for every $n \in \mathbb{N}$,

then the map $u\colon c_0 \longrightarrow X$ given by the formula $u(x) = \sum_{i=1}^\infty x_i f_i$ for every $x = (x_1, x_2, \ldots) \in c_0$ is a Banach lattice embedding.

First, what we are going to do is to construct a sequence $(f_n)_{n\in\mathbb{N}} \in FBL[c_0]$ with the above properties.

For it, let $(N_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence of natural numbers.

For $r \in \mathbb{R}$ let $r^+ = \max\{r, 0\}$ be the positive part of r, and for every $n \in \mathbb{N}$ let $f_n\colon c_0^* \longrightarrow \mathbb{R}$ be the map given by

$$f_n(x^*) = (|x_n^*| - N_n \max\{|x_m^*| : m < n\})^+ \cdot \Pi_{m>n} g_{nm}(x^*)$$

for every $x^* = (x_1^*, x_2^*, \ldots) \in c_0^* = \ell_1$, where $g_{nm}\colon c_0^* \longrightarrow [0,1]$ is any continuous function such that $g_{nm}(x^*) = 0$ if $N_m|x_n^*| \leq |x_m^*|$, $g_{nm}(x^*) = 1$ if $|x_m^*| \leq (N_m - 1)|x_n^*|$ and $g_{nm}(x^*) = g_{nm}(\frac{x^*}{\|x^*\|})$ whenever $x^* \neq 0$.

Let us see that $(f_n)_{n\in\mathbb{N}}$ is a disjoint sequence of positive elements of $FBL[c_0]$ which satisfies both properties a) and b) in Theorem 4.29:

Lemma 4.30. *The sequence $(f_n)_{n\in\mathbb{N}}$ satisfies the following properties:*

1. *f_n is in $FBL[c_0]$ for every $n \in \mathbb{N}$.*

2. *$f_n \geq 0$ for every $n \in \mathbb{N}$.*

3. *$f_n \wedge f_l = 0$ for every $n \neq l$.*

4. *$\| \sum_{i=1}^{n} f_i \|_{FBL[c_0]} \leq 1$ for every $n \in \mathbb{N}$.*

5. *$\|f_n\|_{FBL[c_0]} = 1$ for every $n \in \mathbb{N}$. In particular, the sequence $(f_n)_{n\in\mathbb{N}}$ does not converge in norm to zero.*

Proof. For item 1, fix $n \in \mathbb{N}$. We are going to find a sequence of functions $(h_k)_{k\in\mathbb{N}} \in FBL[c_0]$ such that $\lim_{k\to\infty} \|h_k - f_n\|_{FBL[c_0]} = 0$. Then, we will have that $f_n \in FBL[c_0]$.

Let $h_k \colon c_0^* \longrightarrow \mathbb{R}$ be the map given by

$$h_k(x^*) = (|x_n^*| - N_n \max\{|x_m^*| : m < n\})^+ \cdot \Pi_{n<m\leq n+k}\, g_{nm}(x^*)$$

for every $x^* = (x_1^*, x_2^*, \ldots) \in c_0^* = \ell_1$. Let us see that $h_k \in FBL[c_0]$ for every $k \in \mathbb{N}$.

Notice that h_k is continuous, positively homogeneous and satisfies that $h_k(x^*) = h_k(y^*)$ whenever $x_1^* = y_1^*, \ldots, x_{n+k}^* = y_{n+k}^*$. Let $T\colon FBL(B_{c_0}) \longrightarrow \mathbb{R}^{c_0^*}$ be the map given by $T(f)(x^*) = f((\frac{x^*(y)}{\|x^*\|})_{y\in B_{c_0}}) \cdot \|x^*\|$ if $x^* \neq 0$ and $T(f)(0) = f(0)$, where 0 denotes the identically zero function in the corresponding space. We have that $T(FBL(B_{c_0})) \subset FBL[c_0]$ because T maps the evaluation functions in $FBL(B_{c_0})$ to the evaluation functions in $FBL[c_0]$, preserves linear combinations, the lattice structure and $\|T(f)\|_{FBL[c_0]} \leq \|f\|_{FBL(B_{c_0})}$.

Now, let $\tilde{h_k}\colon [-1,1]^{B_{c_0}} \longrightarrow \mathbb{R}$ be the map given by

$$\tilde{h_k}(z^*) = h_k((z^*(e_1), \ldots, z^*(e_{n+k}), 0, 0, \ldots))$$

for every $z^* \in [-1,1]^{B_{c_0}}$. Since $\tilde{h_k}$ is continuous, positively homogeneous and depends only on finitely many coordinates, by Lemma 4.25 we have that $\tilde{h_k} \in FBL(B_{c_0})$. It follows that $T(\tilde{h_k}) = h_k \in FBL[c_0]$.

Now, by definition, we have that

$$\|h_k - f_n\|_{FBL[c_0]} = \sup\left\{ \sum_{i=1}^{l} |(h_k - f_n)|(x_i^*) : l \in \mathbb{N}, x_1^*, \ldots, x_l^* \in B_{\ell_1}, \sup_{x\in B_{c_0}} \sum_{i=1}^{l} |x_i^*(x)| \leq 1 \right\}.$$

Take $x_1^*, \ldots, x_l^* \in B_{\ell_1}$, with $x_i^* = (x_{i1}^*, x_{i2}^*, \ldots)$ for every $i = 1, \ldots, l$ and such that $(h_k - f_n)(x_i^*) \neq 0$ for every $i = 1, \ldots, l$ and $\sup_{x\in B_{c_0}} \sum_{i=1}^{l} |x_i^*(x)| \leq 1$.

Note that $h_k \geq f_n$, so we can remove absolute values in the previous expression. Then, we have that

$$
\begin{aligned}
\sum_{i=1}^{l}(h_k - f_n)(x_i^*) &= \sum_{i=1}^{l}(|x_{in}^*| - N_n \max\{|x_{im}^*| : m < n\})^+ \cdot \\
&\cdot \quad \big[\Pi_{n<m\le n+k}\, g_{nm}(x_i^*) - \Pi_{n<m}\, g_{nm}(x_i^*)\big] \\
&= \sum_{i=1}^{l}(|x_{in}^*| - N_n \max\{|x_{im}^*| : m < n\})^+ \cdot \\
&\cdot \quad \big[\Pi_{n<m\le n+k}\, g_{nm}(x_i^*) - \Pi_{n<m\le n+k}\, g_{nm}(x_i^*)\Pi_{n+k<m}\, g_{nm}(x_i^*)\big] \\
&= \sum_{i=1}^{l}(|x_{in}^*| - N_n \max\{|x_{im}^*| : m < n\})^+ \cdot \\
&\cdot \quad \Pi_{n<m\le n+k}\, g_{nm}(x_i^*)\big[1 - \Pi_{n+k<m}\, g_{nm}(x_i^*)\big] \\
&\le \sum_{i=1}^{l}|x_{in}^*|\big[1 - \Pi_{n+k<m}\, g_{nm}(x_i^*)\big].
\end{aligned}
$$

Since $(h_k - f_n)(x_i^*) \neq 0$ for every $i = 1, \ldots, l$, we have that $|x_{in}^*| \neq 0$ and also that $1 - \Pi_{n+k<m}\, g_{nm}(x_i^*) \neq 0$. Thus, for every i there exists $m_i > n + k$ such that $g_{nm_i}(x_i^*) \neq 1$, that is to say, $\frac{|x_{im_i}^*|}{|x_{in}^*|} > N_{m_i} - 1$. Since $N_{m_i} > N_{n+k}$, this implies that $|x_{in}^*| < \frac{1}{N_{n+k}-1}|x_{im_i}^*|$.

Thus,

$$\sum_{i=1}^{l}(h_k - f_n)(x_i^*) \le \sum_{i=1}^{l}|x_{in}^*| < \frac{1}{N_{n+k}-1}\sum_{i=1}^{l}|x_{im_i}^*|.$$

Therefore, since $\lim_{k\to\infty}\frac{1}{N_{n+k}-1} = 0$, the proof will follow from the following Claim.

Claim. For every $x_1^*, x_2^*, \ldots, x_l^* \in B_{\ell_1}$ and every natural numbers $m_1, m_2, \ldots, m_l \in \mathbb{N}$ we have

$$\sum_{i=1}^{l}|x_{im_i}^*| \le 1$$

whenever $\sup_{x\in B_{c_0}}\sum_{i=1}^{l}|x_i^*(x)| \le 1$.

Proof of the Claim. Fix $m = \max\{m_i : 1 \le i \le l\}$. We show first that

$$\sum_{i=1}^{l}|x_{im_i}^*| \le \max\left\{\sum_{i=1}^{l}\Big|\sum_{j=1}^{m}\varepsilon(j)x_{ij}^*\Big| : \varepsilon \in \{-1,+1\}^m\right\}.$$

In fact,

$$\begin{aligned}
\max\left\{\sum_{i=1}^{l}\Big|\sum_{j=1}^{m}\varepsilon(j)x_{ij}^*\Big| : \varepsilon\in\{-1,+1\}^m\right\} &\geq \frac{1}{2^m}\sum_{\varepsilon\in\{-1,+1\}^m}\sum_{i=1}^{l}\Big|\sum_{j=1}^{m}\varepsilon(j)x_{ij}^*\Big| \\
&= \frac{1}{2^m}\sum_{\tilde\varepsilon\in\{-1,+1\}}\sum_{i=1}^{l}\sum_{\substack{\varepsilon\in\{-1,+1\}^m\\ \varepsilon(m_i)=\tilde\varepsilon}}\Big|\sum_{j=1}^{m}\varepsilon(j)x_{ij}^*\Big| \\
&\geq \frac{1}{2}\sum_{\tilde\varepsilon\in\{-1,+1\}}\sum_{i=1}^{l}\frac{1}{2^{m-1}}\Big|\sum_{\substack{\varepsilon\in\{-1,+1\}^m\\ \varepsilon(m_i)=\tilde\varepsilon}}\sum_{j=1}^{m}\varepsilon(j)x_{ij}^*\Big| \\
&= \sum_{i=1}^{l}|x_{im_i}^*|.
\end{aligned}$$

Let $\varepsilon\in\{-1,+1\}^m$ be the function which gives the maximum above. We have that

$$\sum_{i=1}^{l}|x_{im_i}^*| \leq \sum_{i=1}^{l}\Big|\sum_{j=1}^{m}\varepsilon(j)x_{ij}^*\Big| = \sum_{i=1}^{l}|x_i^*(x)| \leq 1,$$

taking, in the equality, $x=(\varepsilon(1),\varepsilon(2),\ldots,\varepsilon(m),0,\ldots)\in B_{c_0}$.

Item 2 is clear.

For item 3, suppose, for example, that $n<l$, and let $x^*\in c_0^*$ such that $f_l(x^*)\neq 0$. We have that $|x_l^*|>N_l\max\{|x_m^*| : m<l\}$. In particular, $|x_l^*|>N_l|x_n^*|$. Now, if $f_n(x^*)\neq 0$, we have that $g_{nm}(x^*)\neq 0$ for every $m>n$, and then, that $|x_m^*|<N_m|x_n^*|$ for every $m>n$. Taking $m=l$ we have a contradiction.

For item 4, by definition, we have that

$$\left\|\sum_{i=1}^{n}f_i\right\|_{FBL[c_0]} = \sup\left\{\sum_{j=1}^{k}(\sum_{i=1}^{n}f_i)(x_j^*) : k\in\mathbb{N}, x_1^*,\ldots,x_k^*\in B_{\ell_1}, \sup_{x\in B_{c_0}}\sum_{j=1}^{k}|x_j^*(x)|\leq 1\right\}.$$

Fix $x_1^*,x_2^*,\ldots,x_k^*\in B_{\ell_1}$ with $\sup_{x\in B_{c_0}}\sum_{j=1}^{k}|x_j^*(x)|\leq 1$. Since the functions f_i are disjoint, for each $j=1,2,\ldots,k$ there is at most one $i_j\in\{1,2,\ldots,n\}$ such that $f_{i_j}(x_j^*)\neq 0$. Thus,

$$\sum_{j=1}^{k}(\sum_{i=1}^{n}f_i)(x_j^*) = \sum_{j=1}^{k}f_{i_j}(x_j^*).$$

Without loss of generality, we suppose that $f_{i_j}(x_j^*)\neq 0$ for every $j=1,2,\ldots,k$.
Notice that each $f_{i_j}(x_j^*)\leq |x_{ji_j}^*|$ for every $j=1,2,\ldots,k$, so

$$\sum_{j=1}^{k}(\sum_{i=1}^{n}f_i)(x_j^*) = \sum_{j=1}^{k}f_{i_j}(x_j^*) \leq \sum_{j=1}^{k}|x_{ji_j}^*|.$$

Now the Claim in the proof of item 1 applied to $x_1^*, \ldots, x_k^*$ asserts that $\sum_{j=1}^k |x_{ji_j}^*| \leq 1$ and therefore

$$\sum_{j=1}^k (\sum_{i=1}^n f_i)(x_j^*) \leq \sum_{j=1}^k |x_{ji_j}^*| \leq 1.$$

Thus, $\|\sum_{i=1}^n f_i\|_{FBL[c_0]} \leq 1$.

For item 5, we know that

$$\|f_n\|_{FBL[c_0]} = \sup \left\{ \sum_{j=1}^k f_n(x_j^*) : k \in \mathbb{N}, x_1^*, \ldots, x_k^* \in B_{\ell_1}, \sup_{x \in B_{c_0}} \sum_{j=1}^k |x_j^*(x)| \leq 1 \right\}.$$

Taking $e_n^* \in B_{\ell_1}$ we have that $f_n(e_n^*) = 1$, so that $\|f_n\|_{FBL[c_0]} \geq 1$. In general, since $f_n(x^*) \leq |x^*(e_n)|$ for every $x^* \in B_{\ell_1}$, we have that

$$\sum_{j=1}^k f_n(x_j^*) \leq \sum_{j=1}^k |x_j^*(e_n)| \leq \sup_{x \in B_{c_0}} \sum_{j=1}^k |x_j^*(x)| \leq 1,$$

so $\|f_n\|_{FBL[c_0]} = 1$. □

Theorem 4.27 easily follows from the above. Indeed, by Lemma 4.30 and Theorem 4.29, the map $u\colon c_0 \longrightarrow FBL[c_0]$ given by the formula $u(x) = \sum_{i=1}^\infty x_i f_i$ for every $x = (x_1, x_2, \ldots) \in c_0$ is a Banach lattice embedding. It only remains to prove that it is an isometry between c_0 and its image in $FBL[c_0]$ and the existence of the projection $P\colon FBL[c_0] \longrightarrow c_0$ of norm 1.

To prove the first fact, note that it is enough to check that

$$\left\| \sum_{i=1}^n x_i f_i \right\|_{FBL[c_0]} = \max_{1 \leq i \leq n} |x_i|$$

for every $n \in \mathbb{N}$ and every $x_1, x_2, \ldots, x_n \in \mathbb{R}$. Fix any $x_1, x_2, \ldots, x_n \in \mathbb{R}$. Without loss of generality, suppose that $|x_1| = \max_{1 \leq i \leq n} |x_i|$. Since each function f_i is positive and has norm 1, we have that

$$\begin{aligned} \max_{1 \leq i \leq n} |x_i| = |x_1| = \| |x_1| f_1 \|_{FBL[c_0]} \leq \left\| \sum_{i=1}^n |x_i| f_i \right\|_{FBL[c_0]} &= \left\| \left| \sum_{i=1}^n x_i f_i \right| \right\|_{FBL[c_0]} \\ &= \left\| \sum_{i=1}^n x_i f_i \right\|_{FBL[c_0]}. \end{aligned}$$

On the other hand,

$$\begin{aligned}\left\|\sum_{i=1}^{n} x_i f_i\right\|_{FBL[c_0]} = \left\|\sum_{i=1}^{n} |x_i| f_i\right\|_{FBL[c_0]} &\leq \left\|\sum_{i=1}^{n} |x_1| f_i\right\|_{FBL[c_0]} \\ &= |x_1| \left\|\sum_{i=1}^{n} f_i\right\|_{FBL[c_0]} \\ &\leq |x_1| = \max_{1\leq i\leq n} |x_i|,\end{aligned}$$

so $u\colon c_0 \longrightarrow FBL[c_0]$ is an isometry between c_0 and its image in $FBL[c_0]$.

Finally, the Banach lattice homomorphism $P\colon FBL[c_0] \longrightarrow c_0$ given by the formula $P(f) = (f(e_1^*), f(e_2^*), \ldots)$ for every $f\colon c_0^* \longrightarrow \mathbb{R}$ belonging to $FBL[c_0]$ has norm 1 and satisfies that $P \circ u = id_{c_0}$, so the proof of Theorem 4.27 is finished.

4.8 Problems

Concerning the different variations of projectivity, it was already observed in Proposition 4.7 that if a Banach lattice P has the property that every Banach lattice homomorphism into a quotient $T\colon P \longrightarrow X/\mathcal{J}$ can be lifted to a Banach lattice homomorphism $\hat{T}\colon P \longrightarrow X$, then P is λ-projective for some λ. It is obvious that the class of ∞-projective Banach lattices is closed under renorming but the 1^+-projective class is not. It was asked in [15] whether every ∞-projective Banach lattice is the renorming of a 1^+-projective Banach lattice. But, in fact, we do not know a single example that separates these two classes, even by renorming.

Problem 4.31. Find an equivalent norm on a 1^+-projective Banach lattice that makes it ∞-projective but not 1^+-projective.

A natural candidate would be $FBL[E]$ with E a suitable Banach space renorming of ℓ_1.

Theorems 4.4 and 4.5 suggest a large presence of the Banach space ℓ_1 inside projective Banach lattices. This does not exclude other subspaces ($C([0,1])$ is 1^+-projective and contains isometric copies of any separable Banach space) but we may at least ask:

Problem 4.32. If X is ∞-projective and infinite-dimensional, must X contain a Banach subspace isomorphic to ℓ_1?

We proved that E has the Schur property if $FBL[E]$ is ∞-projective. But the only positive case that we know is that $FBL[\ell_1(A)] = FBL(A)$ is 1^+-projective.

Problem 4.33. Is there a Banach space E with the Schur property, not isometric to $\ell_1(A)$, for which $FBL[E]$ is 1^+-projective? Is there a Banach space E with the Schur property, not isomorphic to $\ell_1(A)$, for which $FBL[E]$ is ∞-projective?

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