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Surfaces in $\mathbb{S}^3$ of L_1-2-type

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Abstract

In this paper we show that an L_1-2-type surface $M^2 \subset \mathbb{S}^3$ is either an open portion of a standard Riemannian product $\mathbb{S}^1(a) \times \mathbb{S}^1(b)$, of any radii, or it has non constant mean curvature H, non constant Gaussian curvature K, and non constant principal curvatures κ_1 and κ_2.

Keywords: spherical surface; Cheng-Yau operator L_1; L_1-finite-type surface; L_1-biharmonic surface; Newton transformation

MSC 2010: 53C40, 53A05, 53B25

1 Introduction

Submanifolds of finite type M (i.e. submanifolds whose isometric immersion in the Euclidean space is constructed by using eigenfunctions of their Laplacian) were introduced by B.Y. Chen during the late 1970s, and the first results on this subject were collected in his book [5]. In subsequent papers, Chen has provided a detailed account of recent development on problems and conjectures about finite type submanifolds, [6, 7]. It is well known that the Laplacian operator Δ can be seen as the first one of a sequence of operators $L_0 = \Delta, L_1, \dots, L_{n-1}$, $n = \dim(M)$, where L_k stands for the linearized operator of the first variation of the $(k+1)$-th mean curvature arising from normal variations (see, for instance, [13]). L_1 is nothing but the differential operator $\Box$ introduced by Cheng and Yau, [8].

The notion of finite type submanifold can be defined for any operator L_k, [10], and then it is natural to try to obtain new results and compare them with the classical ones. For example, it is well known that the only 2-type surfaces in the unit 3-sphere $\mathbb{S}^3$ are open portions of the product of two circles $\mathbb{S}^1(a) \times \mathbb{S}^1(b)$ of different radii, [5, 4, 9].

In the present article we study the same problem for the operator L_1, that is, we study isometric immersions $\psi : M^2 \to \mathbb{S}^3 \subset \mathbb{R}^4$ of L_1-2-type. These surfaces are characterized by the following spectral decomposition of the position vector ψ:

$$\psi = a + \psi_1 + \psi_2, \quad L_1\psi_1 = \lambda_1\psi_1, \quad L_1\psi_2 = \lambda_2\psi_2, \quad \lambda_1 \neq \lambda_2,\ \lambda_i \in \mathbb{R},$$

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where a is a constant vector in $\mathbb{R}^4$, and ψ_1, ψ_2 are $\mathbb{R}^4$-valued non-constant differentiable functions on M^2. It is easy to see that open portions of the product of two circles $\mathbb{S}^1(a) \times \mathbb{S}^1(b)$, of any radii, are surfaces of L_1-2-type (see the example 2). Our main theorem is the following local result:

Theorem. *Let $\psi : M^2 \to \mathbb{S}^3$ be an orientable surface of L_1-2-type. Then either M^2 is an open portion of a standard Riemannian product $\mathbb{S}^1(a) \times \mathbb{S}^1(b)$ of any radii, or M^2 has non constant mean curvature H, non constant Gaussian curvature K, and non constant principal curvatures κ_1 and κ_2.*

2 Preliminaries

Let $\psi : M^2 \to \mathbb{S}^3 \subset \mathbb{R}^4$ be an isometric immersion in the unit 3-sphere $\mathbb{S}^3$ (centered at the origin of $\mathbb{R}^4$) of a connected orientable surface M^2, with Gauss map N. We denote by ∇^0, $\overline{\nabla}$ and ∇ the Levi-Civita connections on $\mathbb{R}^4$, $\mathbb{S}^3$ and M^2, respectively. Then the Gauss and Weingarten formulas are given by

$$\nabla^0_X Y = \nabla_X Y + \langle SX, Y\rangle\, N - \langle X, Y\rangle\, \psi, \tag{1}$$

$$SX = -\overline{\nabla}_X N = -\nabla^0_X N, \tag{2}$$

for all tangent vector fields $X, Y \in \mathfrak{X}(M^2)$, where $S : \mathfrak{X}(M^2) \to \mathfrak{X}(M^2)$ stands for the shape operator (or Weingarten endomorphism) of M^2, with respect to the chosen orientation N. The mean curvature H and the scalar curvature H_2 (also called the extrinsic curvature) of M^2 are defined by $H = \frac{1}{2}(\kappa_1 + \kappa_2)$ and $H_2 = \kappa_1\kappa_2$, respectively, κ_1 and κ_2 being the eigenvalues of S (i.e. the principal curvatures of the surface). From the Gauss equation we know that the Gaussian curvature K is given by $K = 1+\det(S) = 1 + H_2$.

The Newton transformation of M^2 is the operator $P : \mathfrak{X}(M^2) \to \mathfrak{X}(M^2)$ defined by

$$P = 2HI - S. \tag{3}$$

Note that by the Cayley-Hamilton theorem we have $S \circ P = H_2 I$. Observe also that, at any point $p \in M^2$, $S(p)$ and $P(p)$ can be simultaneously diagonalized: if $\{e_1, e_2\}$ are the eigenvectors of $S(p)$ corresponding to the eigenvalues $\kappa_1(p)$ and $\kappa_2(p)$, respectively, then they are also the eigenvectors of $P(p)$ with corresponding eigenvalues $\kappa_2(p)$ and $\kappa_1(p)$, respectively.

According to [12, p. 86], for a tensor T the contraction of the new covariant slot in its covariant differential ∇T with one of its original slots is called a divergence of T. Hence the divergence of a vector field X is the differential function defined by

$$\mathrm{div}(X) = C(\nabla X) = \langle \nabla_{E_1} X, E_1\rangle + \langle \nabla_{E_2} X, E_2\rangle\,,$$

$\{E_1, E_2\}$ being any local orthonormal frame of tangent vectors fields. For an operator $T : \mathfrak{X}(M^2) \to \mathfrak{X}(M^2)$ the divergence associated to the metric contraction C_{12} will be the vector field $\mathrm{div}(T) \in \mathfrak{X}(M^2)$ defined as

$$\mathrm{div}(T) = C_{12}(\nabla T) = (\nabla_{E_1} T)E_1 + (\nabla_{E_2} T)E_2.$$

We have the following properties of P. The first three claims are direct computations; for a proof of claims (d) and (e), see e.g. [1].

Lemma 1 *The Newton transformation P satisfies:*

(a) $\mathrm{tr}(P) = 2H$.

(b) $\mathrm{tr}(S \circ P) = 2H_2$.

(c) $\mathrm{tr}(S^2 \circ P) = 2HH_2$.

(d) $\mathrm{tr}(\nabla_X S \circ P) = \langle \nabla H_2, X\rangle$, *where ∇H_2 stands for the gradient of H_2.*

(e) $\mathrm{div}(P) = 0$.

Associated to the Newton transformation P, we can define a second-order linear differential operator $L_1 : \mathcal{C}^\infty(M^2) \to \mathcal{C}^\infty(M^2)$ by

$$L_1(f) = \mathrm{tr}\big(P \circ \nabla^2 f\big), \tag{4}$$

where $\nabla^2 f : \mathfrak{X}(M^2) \to \mathfrak{X}(M^2)$ denotes the self-adjoint linear operator metrically equivalent to the Hessian of f, given by $\langle \nabla^2 f(X), Y\rangle = \langle \nabla_X(\nabla f), Y\rangle$. An interesting property of L_1 is the following. For every couple of differentiable functions $f, g \in C^\infty(M^2)$ we have

$$L_1(fg) = gL_1(f) + fL_1(g) + 2\langle P(\nabla f), \nabla g\rangle. \tag{5}$$

The operator L_1 can be extended to vector functions as follows. If $F = (f_1, f_2, f_3, f_4) : M^2 \to \mathbb{R}^4$, $f_i \in \mathcal{C}^\infty(M^2)$, then $L_1 F := (L_1 f_1, L_1 f_2, L_1 f_3, L_1 f_4)$.

3 First results

Let $a \in \mathbb{R}^4$ be an arbitrary fixed vector. A direct computation shows that the gradient of the function $\langle \psi, a\rangle$ is given by

$$\nabla \langle \psi, a\rangle = a^\top = a - \langle N, a\rangle N - \langle \psi, a\rangle \psi, \tag{6}$$

where $a^\top \in \mathfrak{X}(M^2)$ denotes the tangential component of a. Taking covariant derivative in (6), and using the Gauss and Weingarten formulae, we obtain

$$\nabla_X \nabla \langle \psi, a\rangle = \nabla_X a^\top = \langle N, a\rangle SX - \langle \psi, a\rangle X, \tag{7}$$

for every vector field $X \in \mathfrak{X}(M^2)$. Finally, by using (4) and Lemma 1, we find that

$$\begin{aligned} L_1 \langle \psi, a\rangle &= \langle N, a\rangle \mathrm{tr}(S \circ P) - \langle \psi, a\rangle \mathrm{tr}(P) \\ &= 2H_2 \langle N, a\rangle - 2H \langle \psi, a\rangle. \end{aligned} \tag{8}$$

Then $L_1\psi$ can be computed as

$$L_1\psi = 2H_2 N - 2H\psi. \tag{9}$$

A straightforward computation yields

$$\nabla \langle N, a \rangle = -Sa^\top.$$

From the Weingarten formula and (7), we find that

$$\begin{aligned}\nabla_X \nabla \langle N, a \rangle &= -(\nabla_X S)a^\top - S(\nabla_X a^\top) \\ &= -(\nabla_{a^\top} S)X - \langle N, a \rangle S^2 X + \langle \psi, a \rangle SX,\end{aligned}$$

for every tangent vector field X. This equation, jointly with (4) and Lemma 1, yields

$$\begin{aligned}L_1 \langle N, a \rangle &= -\mathrm{tr}(\nabla_{a^\top} S \circ P) - \langle N, a \rangle \mathrm{tr}(S^2 \circ P) + \langle \psi, a \rangle \mathrm{tr}(S \circ P) \\ &= -\langle \nabla H_2, a \rangle - 2HH_2 \langle N, a \rangle + 2H_2 \langle \psi, a \rangle. \qquad (10)\end{aligned}$$

In other words,

$$L_1 N = -\nabla H_2 - 2HH_2 N + 2H_2 \psi. \qquad (11)$$

From (9), (11) and (5) we obtain the following result.

Lemma 2 *For any $f \in \mathcal{C}^\infty(M^2)$, we have*

$$\begin{aligned}L_1(f\psi) &= 2P(\nabla f) + 2fH_2 N + (L_1 f - 2Hf)\psi, \\ L_1(fN) &= -(f\nabla H_2 + 2H_2 \nabla f) + (L_1 f - 2HH_2 f)N + 2H_2 f\psi.\end{aligned}$$

On the other hand, equations (5), (8) and (10) lead to

$$\begin{aligned}L_1^2 \langle \psi, a \rangle &= 2H_2 L_1 \langle N, a \rangle + 2L_1(H_2) \langle N, a \rangle + 4\big\langle P(\nabla H_2), \nabla \langle N, a \rangle \big\rangle \\ &\quad - 2HL_1 \langle \psi, a \rangle - 2L_1(H) \langle \psi, a \rangle - 4\big\langle P(\nabla H), \nabla \langle \psi, a \rangle \big\rangle, \\ &= -2H_2 \langle \nabla H_2, a \rangle - 4 \langle (S \circ P)(\nabla H_2), a \rangle - 4 \langle P(\nabla H), a \rangle \\ &\quad + \big[2L_1 H_2 - 4HH_2(H_2 + 1)\big] \langle N, a \rangle \\ &\quad + \big[4H_2^2 + 4H^2 - 2L_1 H\big] \langle \psi, a \rangle.\end{aligned}$$

Finally, we get

$$\begin{aligned}L_1^2 \psi &= -4P(\nabla H) - 3\nabla H_2^2 \\ &\quad + 2\big[L_1 H_2 - 2HH_2\big(H_2 + 1\big)\big]N \\ &\quad + 2\big[2H_2^2 + 2H^2 - L_1 H\big]\psi. \qquad (12)\end{aligned}$$

3.1 L_1-biharmonic surfaces

An isometric immersion $x : M^n \to \mathbb{R}^m$ is said to be *biharmonic* if $\Delta \mathbf{H} = 0$, where Δ and $\mathbf{H}$ are the rough Laplacian on the submanifold M^n and the mean curvature vector field of the immersion, respectively (see e.g. [6]). From the Beltrami formula $\Delta x = n\mathbf{H}$, we know that the submanifold M^n is biharmonic if and only if $\Delta^2 x = 0$. The following definition appears in a natural way (see [3] and [11]).

Definition 3 *An isometric immersion $\psi : M^2 \to \mathbb{R}^4$ is said to be L_1-biharmonic if $L_1^2\psi = 0$. In the case $L_1^2\psi = 0$ and $L_1\psi \neq 0$, we will say that ψ is a proper L_1-biharmonic surface.*

If M^2 is a totally geodesic surface of $\mathbb{S}^3$, then the equation (9) implies $L_1\psi = 0$, and hence M^2 is a (trivial) L_1-biharmonic surface in $\mathbb{R}^4$.

Let $\psi : M^2 \to \mathbb{S}^3 \subset \mathbb{R}^4$ be an L_1-biharmonic surface. Then (12) yields

$$4P(\nabla H) + 3\nabla H_2^2 = 0, \tag{13}$$
$$L_1 H_2 - 2HH_2(H_2 + 1) = 0, \tag{14}$$
$$L_1 H - 2(H^2 + H_2^2) = 0. \tag{15}$$

If H is constant, then (15) yields $H = H_2 = 0$, i.e. M^2 is a totally geodesic surface in $\mathbb{S}^3$; in other words, M^2 is an open portion of a unit 2-sphere $\mathbb{S}^2$. If K is constant (and so H_2 also is), by taking divergence in (13) we get $L_1 H = 0$. Then from (15) we also deduce that M^2 is an open portion of a unit 2-sphere $\mathbb{S}^2$. We have obtained the following result.

Proposition 4 *Let $\psi : M^2 \to \mathbb{S}^3 \subset \mathbb{R}^4$ be an L_1-biharmonic surface. Then either M^2 is an open portion of a unit 2-sphere $\mathbb{S}^2$ or M^2 has non constant curvatures H and K.*

This result can be improved as follows. If H is an L_1-harmonic function (i.e. $L_1 H = 0$), then (15) implies again that M^2 is an open portion of a unit 2-sphere $\mathbb{S}^2$. The same conclusion is also reached when H_2 (or K) is an L_1-harmonic function. In this case, (14) yields

$$HH_2(H_2 + 1) = 0.$$

Let us assume that H is non constant (otherwise, there is nothing to prove) and take the non-empty set $\mathcal{U} = \{p \in M^2 \,|\, \nabla H^2(p) \neq 0\}$. On this set we have $H_2(H_2 + 1) = 0$, and then H_2 is constant on $\mathcal{U}$. Hence Proposition 4 implies that $\mathcal{U}$ is an open portion of a unit 2-sphere $\mathbb{S}^2$, but then the mean curvature H is constant. This is a contradiction. The following result has been proved.

Proposition 5 *Let $\psi : M^2 \to \mathbb{S}^3 \subset \mathbb{R}^4$ be an L_1-biharmonic surface. Then either M^2 is an open portion of a unit 2-sphere $\mathbb{S}^2$ or the curvatures H and K are not L_1-harmonic.*

When M^2 is a closed surface, we can improve that result. By taking divergence in (13) we get

$$L_1 H = -\frac{3}{4}\Delta H_2^2.$$

From here and (15), and by using the divergence theorem, we obtain

$$0 = \int_M L_1 H \, dv = 2\int_M (H^2 + H_2^2)\, dv.$$

This implies $H = H_2 = 0$. We have proved the following result.

Proposition 6 *Let $\psi : M^2 \to \mathbb{S}^3 \subset \mathbb{R}^4$ be a closed surface. Then M^2 is an L_1-biharmonic surface if and only if it is a unit 2-sphere $\mathbb{S}^2$.*

3.2 Equations characterizing the L_1-2-type surfaces

Let us suppose that M^2 is of L_1-2-type in $\mathbb{R}^4$, that is, the position vector ψ of M^2 in $\mathbb{R}^4$ can be written as follows,

$$\psi = a + \psi_1 + \psi_2, \quad L_1\psi_1 = \lambda_1\psi_1, \quad L_1\psi_2 = \lambda_2\psi_2, \quad \lambda_1 \neq \lambda_2,\ \lambda_i \in \mathbb{R},$$

where a is a constant vector in $\mathbb{R}^4$, and ψ_1, ψ_2 are $\mathbb{R}^4$-valued non-constant differentiable functions on M^2.

Since $L_1\psi = \lambda_1\psi_1 + \lambda_2\psi_2$ and $L_1^2\psi = \lambda_1^2\psi_1 + \lambda_2^2\psi_2$, an easy computation shows that

$$L_1^2\psi = (\lambda_1 + \lambda_2)L_1\psi - \lambda_1\lambda_2(\psi - a),$$

and by using (9) we obtain

$$\begin{aligned} L_1^2\psi = \lambda_1\lambda_2 a^\top &+ \big[2(\lambda_1 + \lambda_2)H_2 + \lambda_1\lambda_2 \langle N, a\rangle \big] N \\ &- \big[2(\lambda_1 + \lambda_2)H + \lambda_1\lambda_2 - \lambda_1\lambda_2 \langle \psi, a\rangle \big] \psi. \end{aligned}$$

This equation, jointly with (12), yields the following equations, that characterize the L_1-2-type surfaces in $\mathbb{S}^3$:

$$\lambda_1\lambda_2 a^\top = -3\nabla H_2^2 - 4P(\nabla H), \tag{16}$$

$$\lambda_1\lambda_2 \langle N, a\rangle = 2L_1H_2 - 2H_2\big(2HH_2 + 2H + \lambda_1 + \lambda_2\big), \tag{17}$$

$$\lambda_1\lambda_2 \langle \psi, a\rangle = 4H_2^2 + 4H^2 + 2(\lambda_1 + \lambda_2)H + \lambda_1\lambda_2 - 2L_1H. \tag{18}$$

Example 1 (Surfaces of L_1-1-type)
Totally umbilical surfaces in $\mathbb{S}^3$ are of L_1-1-type. Indeed, let $M^2 \subset \mathbb{S}^3$ be a totally umbilical surface, then its shape operator S is given by $S = HI$. We know that H and H_2 are constants. By taking covariant derivative we get

$$\nabla^0_X(N + H\psi) = 0,$$

for all $X \in \mathfrak{X}(M^2)$, and then $N + H\psi = b$, for a constant vector b. By using this in (9) we deduce

$$L_1\psi = 2H_2 b + \lambda\psi, \quad \lambda = -2H(1 + H_2).$$

If $\lambda \neq 0$, then we write

$$\psi = a + \psi_1, \quad a = -\frac{2H_2}{\lambda}b, \quad \psi_1 = \psi + \frac{2H_2}{\lambda}b,$$

with $L_1\psi_1 = \lambda\psi_1$, i.e. M^2 is of L_1-1-type.

In the case $\lambda = 0$, the surface M^2 is totally geodesic ($H = H_2 = 0$) and then (9) yields $L_1\psi = 0$, showing that M^2 is of L_1-1-type.

By using [2], we easily deduce the following proposition.

Proposition 7 *Let $\psi : M^2 \to \mathbb{S}^3 \subset \mathbb{R}^4$ be an isometric immersion. Then ψ is of L_1-1-type if and only if M^2 is an open portion of a 2-sphere $\mathbb{S}^2(r)$.*

Example 2 (Surfaces of L_1-2-type)
We will see that the standard Riemannian product $M_r^2 = \mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^1(r) \subset \mathbb{S}^3$, $0 < r < 1$, is of L_1-2-type in $\mathbb{R}^4$. Let us consider

$$M^2 = \{x = (x_1, x_2, x_3, x_4) \in \mathbb{S}^3 \,|\, x_3^2 + x_4^2 = r^2\}.$$

In this case, the Gauss map on M^2 is given by

$$N(x) = \left(\frac{-r}{\sqrt{1-r^2}}\, x_1, \frac{-r}{\sqrt{1-r^2}}\, x_2, \frac{\sqrt{1-r^2}}{r}\, x_3, \frac{\sqrt{1-r^2}}{r}\, x_4 \right),$$

and its principal curvatures are

$$\kappa_1 = \frac{r}{\sqrt{1-r^2}} \quad \text{and} \quad \kappa_2 = \frac{-\sqrt{1-r^2}}{r}.$$

If we put $\psi_1 = (x_1, x_2, 0, 0)$ and $\psi_2 = (0, 0, x_3, x_4)$, it is easy to see that $\psi = \psi_1 + \psi_2$, and by using (9) we get that

$$L_1\psi_1 = \lambda_1\psi_1 \quad \text{and} \quad L_1\psi_2 = \lambda_2\psi_2, \quad \text{with} \quad \lambda_1 = \frac{1}{r\sqrt{1-r^2}} \quad \text{and} \quad \lambda_2 = -\lambda_1.$$

Therefore, M^2 is of L_1-2-type in $\mathbb{R}^4$.

4 Main results

Theorem 8 *Let $\psi : M^2 \to \mathbb{S}^3 \subset \mathbb{R}^4$ be an orientable surface of L_1-2-type. Then M^2 has constant mean curvature if and only if M^2 is an open portion of a standard Riemannian product $\mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^1(r)$, $0 < r < 1$.*

Proof. Let M^2 be a surface of L_1-2-type with constant mean curvature. Our goal is to prove that the scalar curvature H_2 of M^2 is constant. Otherwise, let us consider the non-empty open set

$$\mathcal{U}_2 = \{p \in M^2 \mid \nabla H_2^2(p) \neq 0\}.$$

By taking covariant derivative in (18) we have $\lambda_1\lambda_2 a^\top = 4\nabla H_2^2$. Using this in (16) we deduce $H_2 = 0$, which is a contradiction.

Therefore, M^2 is an isoparametric surface in $\mathbb{S}^3$, and then either M^2 is an open portion of a 2-sphere $\mathbb{S}^2(r)$, $0 < r \leq 1$, or M^2 is an open portion of a Riemannian product M_r^2, $0 < r < 1$. Since the totally umbilical surfaces are of L_1-1-type, the result follows. □

Theorem 9 *Let $\psi : M^2 \to \mathbb{S}^3 \subset \mathbb{R}^4$ be an orientable surface of L_1-2-type. Then M^2 has constant Gaussian curvature if and only if M^2 is an open portion of a standard Riemannian product $\mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^1(r)$, $0 < r < 1$.*

Proof. Let M^2 be a surface of L_1-2-type with constant Gaussian curvature K, and consider the open set

$$\mathcal{U} = \{p \in M^2 \mid \nabla H^2(p) \neq 0\}.$$

Our goal is to show that $\mathcal{U}$ is empty. Suppose it is not empty.

By taking covariant derivative in (17), and using that H_2 is constant, we obtain

$$\lambda_1\lambda_2 S a^\top = 4H_2(H_2+1)\nabla H.$$

From (16) and bearing in mind that $S \circ P = H_2 I$, we have $\lambda_1\lambda_2 S a^\top = -4H_2\nabla H$, and therefore

$$H_2(H_2+2)\nabla H = 0.$$

Consequently, on $\mathcal{U}$ we have either $H_2 = -2$ or $H_2 = 0$. We will study each case separately.

Case 1: $H_2 = -2$. By applying the operator L_1 on both sides of (17) and using (18) we get

$$\lambda_1\lambda_2 L_1 \langle N, a\rangle = 4\big[\lambda_1\lambda_2 \langle \psi, a\rangle - 4H^2 - 2(\lambda_1+\lambda_2)H - \lambda_1\lambda_2 - 16\big].$$

On the other hand, (10) leads to

$$\lambda_1\lambda_2 \langle N, a\rangle H - \lambda_1\lambda_2 \langle \psi, a\rangle = \lambda_1\lambda_2 \langle a, \psi\rangle - 4H^2 - 2(\lambda_1+\lambda_2)H - \lambda_1\lambda_2 - 16,$$

and using (17) we find that

$$\lambda_1\lambda_2 \langle \psi, a\rangle = -2H^2 + 3(\lambda_1+\lambda_2)H + \frac{1}{2}(\lambda_1\lambda_2+16). \tag{19}$$

Taking gradients in (19), and using (16) and (3), we obtain

$$\big[-4H + 3(\lambda_1+\lambda_2)\big]\nabla H = -4P_1(\nabla H) = -8H\nabla H + 4S(\nabla H), \tag{20}$$

that is,

$$S(\nabla H) = \frac{4H + 3(\lambda_1+\lambda_2)}{4}\nabla H.$$

Now, by applying the operator S on both sides of the first equality of (20), and bearing in mind that $S \circ P = -2I$, we obtain

$$S(\nabla H) = \frac{8}{-4H + 3(\lambda_1+\lambda_2)}\nabla H.$$

The last two equations for $S(\nabla H)$ imply that H is constant on $\mathcal{U}$, which is a contradiction.

Case 2: $H_2 = 0$. Let us suppose $\kappa_1 = 0$ and $\kappa_2 = 2H \neq 0$ (otherwise, M^2 would be a totally geodesic surface and then of L_1-1-type). Let $\{E_1, E_2\}$ be a local orthonormal frame of principal directions of S such that $SE_i = \kappa_i E_i$. From Codazzi's equation, we easily obtain

$$\nabla_{E_1}E_1 = 0, \qquad \nabla_{E_1}E_2 = 0,$$

$$\nabla_{E_2}E_1 = -\tfrac{\alpha}{H}E_2, \qquad \nabla_{E_2}E_2 = \tfrac{\alpha}{H}E_1 \qquad [E_1, E_2] = \tfrac{\alpha}{H}E_2,$$

where $\alpha = E_1(H)$. Now, from the definition of curvature tensor, we get

$$\begin{aligned} R(E_1, E_2)E_1 &= \nabla_{[E_1,E_2]}E_1 - \nabla_{E_1}\nabla_{E_2}E_1 + \nabla_{E_2}\nabla_{E_1}E_1 \\ &= \Big[E_1\big(\tfrac{\alpha}{H}\big) - \big(\tfrac{\alpha}{H}\big)^2\Big]E_2, \end{aligned}$$

and from the Gauss equation we have $R(E_1, E_2)E_1 = E_2$. By equating the last two equations we deduce

$$HE_1(\alpha) = H^2 + 2\alpha^2. \tag{21}$$

On the other hand, from the definition of L_1, see (4), and after a little calculation, we obtain

$$L_1H = \kappa_2 \langle E_1, \nabla_{E_1}\nabla H\rangle + \kappa_1 \langle E_2, \nabla_{E_2}\nabla H\rangle = 2HE_1(\alpha). \tag{22}$$

By using (21) and (22), (18) can be rewritten as

$$\lambda_1\lambda_2 \langle \psi, a\rangle = 2(\lambda_1 + \lambda_2)H + \lambda_1\lambda_2 - 8\alpha^2. \tag{23}$$

Taking covariant derivative along E_1 here, we have

$$E_1(\lambda_1\lambda_2 \langle \psi, a\rangle) = 2(\lambda_1 + \lambda_2)\alpha - 16\alpha E_1(\alpha). \tag{24}$$

On the other hand, from (18) we get $\lambda_1\lambda_2 a^\top = -8H\alpha E_1$, and therefore

$$E_1(\lambda_1\lambda_2 \langle \psi, a\rangle) = \left\langle \lambda_1\lambda_2 a^\top, E_1\right\rangle = -8H\alpha.$$

This equation, jointly with (24), implies that $(\lambda_1 + \lambda_2)\alpha - 8\alpha E_1(\alpha) = -4H\alpha$. Since $\alpha \neq 0$, see (21), we deduce

$$8E_1(\alpha) = 4H + \lambda_1 + \lambda_2. \tag{25}$$

From here and using (22) we get $4L_1H = 4H^2 + (\lambda_1 + \lambda_2)H$. By using this in (18), we find

$$\lambda_1\lambda_2 \langle \psi, a\rangle = 2H^2 + \frac{3}{2}(\lambda_1 + \lambda_2)H + \lambda_1\lambda_2. \tag{26}$$

Taking gradient here, and using (16) and (3), we obtain

$$\big[4H + \frac{3}{2}(\lambda_1 + \lambda_2)\big]\nabla H = -4P(\nabla H) = -8H\nabla H + 4S(\nabla H), \tag{27}$$

that is,

$$S(\nabla H) = \big(3H + \frac{3}{8}(\lambda_1 + \lambda_2)\big)\nabla H.$$

On the other hand, by applying the operator S on both sides of the first equality of (27), and bearing in mind that $S \circ P = 0$, we obtain

$$\big[4H + \frac{3}{2}(\lambda_1 + \lambda_2)\big]S(\nabla H) = 0.$$

The last two equations imply that H is constant on $\mathcal{U}$, which is a contradiction.

We have proved that if M^2 is a L_1-2-type surface with constant Gaussian curvature, then its mean curvature is constant. Then reasoning as in the proof of Theorem 8 we deduce that M^2 is an open portion of a Riemannian product M^2_r, $0 < r < 1$. This finishes the proof of Theorem 9. □

A surface in $\mathbb{S}^3$ is said to have a *constant principal curvature* if one of its principal curvatures is constant.

Theorem 10 *Let $\psi : M^2 \to \mathbb{S}^3 \subset \mathbb{R}^4$ be an orientable surface of L_1-2-type. Then M^2 has a constant principal curvature if and only if M^2 is an open portion of a standard Riemannian product $\mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^1(r)$, $0 < r < 1$.*

Proof. Let M^2 be a surface of L_1-2-type and assume that κ_1 is a nonzero constant (otherwise, $H_2 = 0$ and Theorem 9 applies). Consider the open set

$$\mathcal{U} = \{p \in M^2 \mid \nabla\kappa_2^2(p) \neq 0\}.$$

Our goal is to show that $\mathcal{U}$ is empty.

Otherwise, the equations (16)–(18) of L_1-2-type can be rewritten in terms of κ_2 as follows

$$\lambda_1\lambda_2 a^\top = [-6\kappa_1^2\kappa_2 - 2(\kappa_1+\kappa_2)]\nabla\kappa_2 + 2S(\nabla\kappa_2), \tag{28}$$

$$\lambda_1\lambda_2 \langle N, a\rangle = 2\kappa_1 L_1\kappa_2 - 2\kappa_1\kappa_2\big[(\kappa_1+\kappa_2)(\kappa_1\kappa_2+1) + \lambda_1 + \lambda_2\big], \tag{29}$$

$$\lambda_1\lambda_2 \langle \psi, a\rangle = 4\kappa_1^2\kappa_2^2 + (\kappa_1+\kappa_2)^2 + (\lambda_1+\lambda_2)(\kappa_1+\kappa_2) + \lambda_1\lambda_2 - L_1\kappa_2. \tag{30}$$

From (29) and (30) we find

$$\lambda_1\lambda_2 \langle N, a\rangle = -2\kappa_1\lambda_1\lambda_2\langle \psi, a\rangle + 2\kappa_1\Big[3\kappa_1^2\kappa_2^2 + \kappa_1^2 + \kappa_1\kappa_2 + (\lambda_1+\lambda_2)\kappa_1 + \lambda_1\lambda_2 - \kappa_1\kappa_2^3\Big].$$

By taking gradient here we obtain

$$-\lambda_1\lambda_2 S a^\top = -2\kappa_1\lambda_1\lambda_2 a^\top + 2\kappa_1^2\Big[1 + 6\kappa_1\kappa_2 - 3\kappa_2^2\Big]\nabla\kappa_2. \tag{31}$$

On the other hand, by using $S \circ P = H_2 I$, we get

$$\lambda_1\lambda_2 S a^\top = -6\kappa_1^2\kappa_2 S(\nabla\kappa_2) - 2\kappa_1\kappa_2\nabla\kappa_2. \tag{32}$$

Now, from (28), (31) and (32) we deduce

$$(3\kappa_1\kappa_2 + 2)S(\nabla\kappa_2) = (-3\kappa_1\kappa_2^2 + (12\kappa_1^2+1)\kappa_2 + 3\kappa_1)\nabla\kappa_2.$$

Since $3\kappa_1\kappa_2 + 2 \neq 0$ (otherwise, κ_2 would be constant), we deduce

$$S(\nabla\kappa_2) = f(\kappa_1,\kappa_2)\nabla\kappa_2, \qquad f(\kappa_1,\kappa_2) = \frac{-3\kappa_1\kappa_2^2 + (12\kappa_1^2+1)\kappa_2 + 3\kappa_1}{(3\kappa_1\kappa_2+2)}.$$

This equation implies that either $f(\kappa_1, \kappa_2) = \kappa_1$ or $f(\kappa_1, \kappa_2) = \kappa_2$. In any case it follows that κ_2 is constant on $\mathcal{U}$, and this is a contradiction. This finishes the proof of Theorem 10. □

As a consequence of theorems 8, 9 and 10, we have the following characterization of L_1-2-type surfaces in $\mathbb{S}^3$.

Theorem 11 *Let $\psi : M^2 \to \mathbb{S}^3 \subset \mathbb{R}^4$ be an orientable surface of L_1-2-type. Then either M^2 is an open portion of a standard Riemannian product $\mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^1(r)$, $0 < r < 1$, or M^2 has non constant mean curvature H, non constant Gaussian curvature K, and non constant principal curvatures κ_1 and κ_2.*

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