Geometry of extended Bianchi-Cartan-Vranceanu spaces

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Abstract

The differential geometry of 3-dimensional Bianchi, Cartan and Vranceanu (BCV) spaces is well known. We introduce the extended Bianchi, Cartan and Vranceanu ($EBCV$) spaces as a natural seven dimensional generalization of BCV spaces and study some of their main geometric properties, such as the Levi-Civita connection, Ricci curvatures, Killing fields and geodesics.

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1 The Bianchi-Cartan-Vranceanu (BCV) spaces (see [2, 5])

It was Cartan ([6]) who obtained the families of today known as BCV-spaces by classifying three-dimensional Riemannian manifolds with four-dimensional isometry group. They also appeared in the work of L. Bianchi ([3, 4]), and G. Vranceanu ([18]). These kind of spaces have been extensively studied and classified (see for instance [14, 17]). In theoretical cosmology they are known as Bianchi-Kantowski-Saks spaces, which are used to construct some homogeneous spacetimes ([10]).

For real numbers m and l, consider the set

$$BCV(m,l) = \{(x,y,z) \in \mathbb{R}^3 \,:\, 1 + m(x^2+y^2) > 0\}$$

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equipped with the metric

$$\mathrm{ds}^2_{m,l} = \frac{dx^2+dy^2}{\lambda^2} + \left(dz - \frac{l}{2}\,\frac{xdy-ydx}{\lambda}\right)^2,$$

where m, l are real numbers and $\lambda = 1 + m(x^2+y^2) > 0$.

Observe that this metric is obtained as a conformal deformation of the planar Euclidean metric by adding the imaginary part of $z\,d\bar{z}$, for a complex number z.

The complete classification of BCV spaces is as follows:

(i) If $m = l = 0$, then $BCV(m,l) \cong \mathbb{R}^3$;

(ii) If $m = \frac{l}{4}$, then $BCV(m,l) \cong (\mathbb{S}^3(m) - \{\infty\})$;

(iii) If $m > 0$ and $l = 0$, then $BCV(m,l) \cong (\mathbb{S}^2(4m) - \{\infty\}) \times \mathbb{R}$;

(iv) If $m < 0$ and $l = 0$, then $BCV(m,l) \cong (\mathbb{H}^2(4m) - \{\infty\}) \times \mathbb{R}$;

(v) If $m > 0$ and $l \neq 0$, then $BCV(m,l) \cong \mathrm{SU}(2) - \{\infty\}$;

(vi) If $m < 0$ and $l \neq 0$, then $BCV(m,l) \cong \widetilde{\mathrm{SL}}(2,\mathbb{R})$;

(vii) If $m = 0$ and $l \neq 0$, then $BCV(m,l) \cong \mathrm{Nil}_3$.

The following vector fields form an orthonormal frame of $BCV(m,l)$:

$$E_1 = \lambda\,\partial_x - \frac{l}{2} y\partial_z, \qquad E_2 = \lambda\,\partial_y + \frac{l}{2} x\partial_z, \qquad E_3 = \partial_z.$$

Let $\mathcal{D}$ be the distribution generated by $\{E_1, E_2\}$, then the manifold $\left(BCV(m,l), \mathcal{D}, \mathrm{ds}^2_{m,l}\right)$ is an example of sub-Riemannian geometry (see [5, 15]) and the horizontal distribution is a 2-step breaking-generating distribution everywhere.

2 Extended Bianchi-Cartan-Vranceanu spaces

2.1 Set up

Observe that letting $z = x + iy$, we see that $\mathrm{Im}(z\,d\bar{z}) = ydx - xdy$, which reminds us the map $\mathbb{C} \times \mathbb{C} \to \mathbb{R} \times \mathbb{C}$ given by $(z_1, z_2) \mapsto (|z_1|^2 - |z_2|^2, 2(z_1\bar{z}_2))$, that easily leads to the classical Hopf fibration $\mathbb{S}^1 \hookrightarrow \mathbb{S}^3 \to \mathbb{S}^2$, where coordinates in $\mathbb{S}^2$ are given by $(|z_1|^2 - |z_2|^2, 2\mathrm{Re}(z_1\bar{z}_2), 2\mathrm{Im}(z_1\bar{z}_2))$.

In the same line, using quaternions $\mathbb{H}$ instead of complex numbers, we get the fibration $\mathbb{S}^3 \hookrightarrow \mathbb{S}^7 \to \mathbb{S}^4$. Quaternions are usually presented with the imaginary units i, j, k in the form $q = x_0 + x_1 i + x_2 j + x_3 k$, $x_0, x_1, x_2, x_3 \in \mathbb{R}$ with $i^2 = j^2 = k^2 = ijk = -1$. They can also be defined equivalently, using the complex numbers $c_1 = x_0 + x_1 i$ and $c_2 = x_2 + x_3 i$, in the form $q = c_1 + c_2 j$. Then for a point $(q_1 = \alpha + \beta j, q_2 = \gamma + \delta j) \in \mathbb{S}^7$, we get the following coordinate expressions $(|q_1|^2 - |q_2|^2, 2\mathrm{Re}(\bar{\alpha}\gamma + \bar{\beta}\delta), 2\mathrm{Im}(\bar{\alpha}\gamma + \bar{\beta}\delta), 2\mathrm{Re}(\alpha\delta - \beta\gamma), 2\mathrm{Im}(\alpha\delta - \beta\gamma))$.

For any $q = w + xi + yj + zk \in \mathbb{H}$ we find that $qd\bar{q} = wdw + xdx + ydy + zdz + (xdw - wdx + zdy - ydz)i + (ydw - wdy + xdz - zdx)j + (zdw - wdz + ydx - xdy)k$. The quaternionic contact group $\mathbb{H} \times \mathrm{Im}\mathbb{H}$, with coordinates (w, x, y, z, r, s, t), can be equipped with the metric

$$\mathrm{ds}^2 = (dw^2 + dx^2 + dy^2 + dz^2) + \left(dr + \frac{1}{2}(xdw - wdx + zdy - ydz)\right)^2 + \left(ds + \frac{1}{2}(ydw - wdy + xdz - zdx)\right)^2 + \left(dt + \frac{1}{2}(zdw - wdz + ydx - xdy)\right)^2.$$

Then, by extending this metric, and following [9], it seems natural to find a 7-dimensional generalization of the 3-dimensional BCV spaces endowed with the two-parameter family of metrics

$$\mathrm{ds}^2_{m,l} = \frac{dw^2 + dx^2 + dy^2 + dz^2}{K^2} + \left(dr + \frac{l}{2}\frac{wdx - xdw + ydz - zdy}{K}\right)^2 + \left(ds + \frac{l}{2}\frac{wdy - ydw + zdx - xdz}{K}\right)^2 + \left(dt + \frac{l}{2}\frac{wdz - zdw + xdy - ydx}{K}\right)^2,$$

where m, l are real numbers and $K = 1 + m(w^2 + x^2 + y^2 + z^2) > 0$.

Then $(EBCV, \mathrm{ds}^2_{m,l})$ will be called extended BCV spaces ($EBCV$ for short).

Note that the first summand in the metric $\mathrm{ds}^2_{m,l}$ is as a conformal change of the Euclidean metric on $\mathbb{R}^4$, whereas the three other summands (depending on m and l) are coming from the imaginary part of $q\, d\bar{q}$, for a quaternion q. When $m = 0$ we get a one-parameter family of Riemannian metrics depending on l. Furthermore, if $l = 1$, we find the 7-dimensional quaternionic Heisenberg group (see [9] and [19]). The manifold $EBCV$ provides another example of sub-Riemannian geometry and the horizontal distribution is a 2-step breaking-generating distribution everywhere.

Observe that when $m = l = 0$, $EBCV$ is nothing but $\mathbb{R}^7$; when $m > 0, l = 0$, $EBCV \cong \mathbb{S}^4(4m) \times \mathbb{R}^3$ and when $m < 0, l = 0$, $EBCV \cong \mathbb{H}^4(4m) \times \mathbb{R}^3$.

The metric $\mathrm{ds}^2_{m,l}$ can also be written as

$$\mathrm{ds}^2_{m,l} = \sum_{\alpha=1}^{7} \omega^\alpha \otimes \omega^\alpha,$$

where

$$\omega^1 = dr + \tfrac{l}{2K}(wdx - xdw + ydz - zdy), \qquad \omega^4 = \tfrac{1}{K}dw,$$
$$\omega^2 = ds + \tfrac{l}{2K}(wdy - ydw + zdx - xdz), \qquad \omega^5 = \tfrac{1}{K}dx,$$
$$\omega^3 = dt + \tfrac{l}{2K}(wdz - zdw + xdy - ydx), \qquad \omega^6 = \tfrac{1}{K}dy,$$
$$\omega^7 = \tfrac{1}{K}dz,$$

with the corresponding dual orthonormal frame

$$X_1 = \partial_r, \qquad X_2 = \partial_s, \qquad X_3 = \partial_t,$$

$$X_4 = K\partial_w + \frac{lx}{2}\partial_r + \frac{ly}{2}\partial_s + \frac{lz}{2}\partial_t, \qquad X_5 = K\partial_x - \frac{lw}{2}\partial_r - \frac{lz}{2}\partial_s + \frac{ly}{2}\partial_t,$$

$$X_6 = K\partial_y + \frac{lz}{2}\partial_r - \frac{lw}{2}\partial_s - \frac{lx}{2}\partial_t, \qquad X_7 = K\partial_z - \frac{ly}{2}\partial_r + \frac{lx}{2}\partial_s - \frac{lw}{2}\partial_t.$$

Writing $1 \leq i, j \leq 3$, $4 \leq a \leq 7$, we find that

$$[X_i, X_j] = 0; \qquad [X_i, X_a] = 0,$$

as well as

$$[X_4, X_5] = -l\{1 + m(y^2 + z^2)\}X_1 + ml(wz + xy)X_2 - ml(wy - xz)X_3 - 2mxX_4 + 2mwX_5,$$

and so on (see Appendix).

For later use, when $m = 0$ brackets reduce to

$$\begin{array}{lll} [X_4, X_5] = -lX_1, & [X_4, X_6] = -lX_2, & [X_4, X_7] = -lX_3, \\ [X_5, X_6] = -lX_3, & [X_5, X_7] = lX_2, & [X_6, X_7] = -lX_1. \end{array}$$

Remark 1 *When $l = 1$, we have the brackets of the quaternionic contact manifold.*

As for the Levi-Civita connection we find out

$$\nabla_{X_i} X_j = 0, \qquad \nabla_{X_i} X_a = \nabla_{X_a} X_i,$$

and

$$\begin{aligned} \nabla_{X_1} X_4 &= \frac{l}{2}\{1 + m(y^2 + z^2\}X_5 + \frac{ml}{2}(wz - xy)X_6 - \frac{ml}{2}(wy + xz)X_7, \\ \nabla_{X_1} X_5 &= -\frac{l}{2}\{1 + m(y^2 + z^2)\}X_4 + \frac{ml}{2}(wy + xz)X_6 + \frac{ml}{2}(wz - xy)X_7, \\ \nabla_{X_1} X_6 &= -\frac{ml}{2}(wz - xy)X_4 - \frac{ml}{2}(wy + xz)X_5 + \frac{l}{2}\{1 + m(w^2 + x^2)\}X_7, \\ \nabla_{X_1} X_7 &= \frac{ml}{2}(wy + xz)X_4 - \frac{ml}{2}(wz - xy)X_5 - \frac{l}{2}\{1 + m(w^2 + x^2)\}X_6, \end{aligned}$$

and son on (see Appendix).

When $m = 0$, the Levi-Civita connection reduces to

$$\begin{array}{llll} \nabla_{X_1} X_4 = \frac{l}{2} X_5, & \nabla_{X_3} X_4 = \frac{l}{2} X_7, & \nabla_{X_5} X_4 = \frac{l}{2} X_1, & \nabla_{X_7} X_4 = \frac{l}{2} X_3, \\ \nabla_{X_1} X_5 = -\frac{l}{2} X_4, & \nabla_{X_3} X_5 = \frac{l}{2} X_6, & \nabla_{X_5} X_5 = 0, & \nabla_{X_7} X_5 = -\frac{l}{2} X_2, \\ \nabla_{X_1} X_6 = \frac{l}{2} X_7, & \nabla_{X_3} X_6 - \frac{l}{2} X_5, & \nabla_{X_5} X_6 = -\frac{l}{2} X_3, & \nabla_{X_7} X_6 = \frac{l}{2} X_1, \\ \nabla_{X_1} X_7 = -\frac{l}{2} X_6, & \nabla_{X_3} X_7 = -\frac{l}{2} X_4, & \nabla_{X_5} X_7 = \frac{l}{2} X_2, & \nabla_{X_7} X_7 = 0. \\ \nabla_{X_2} X_4 = \frac{l}{2} X_6, & \nabla_{X_4} X_4 = 0, & \nabla_{X_6} X_4 = \frac{l}{2} X_2, & \\ \nabla_{X_2} X_5 = -\frac{l}{2} X_7, & \nabla_{X_4} X_5 = -\frac{l}{2} X_1, & \nabla_{X_6} X_5 = \frac{l}{2} X_3, & \\ \nabla_{X_2} X_6 = -\frac{l}{2} X_4, & \nabla_{X_4} X_6 = -\frac{l}{2} X_2, & \nabla_{X_6} X_6 = 0, & \\ \nabla_{X_2} X_7 = \frac{l}{2} X_5, & \nabla_{X_4} X_7 = -\frac{l}{2} X_3, & \nabla_{X_6} X_7 = -\frac{l}{2} X_1, & \end{array}$$

Remark 2 *When $l = 1$, we find the Levi-Civita connection of the quaternionic contact manifold.*

As for the curvature tensor R we have

$$
\begin{aligned}
R_{X_1X_4X_1X_4} = R_{X_1X_5X_1X_5} &= \frac{l^2}{4}\{1 + m(K+1)(y^2+z^2)\}, \\
R_{X_1X_6X_1X_6} = R_{X_1X_7X_1X_7} &= \frac{l^2}{4}\{1 + m(K+1)(w^2+x^2)\},
\end{aligned}
$$

and so on (see Appendix).

Remark 3 *When $m = 0$, the curvature of the quaternionic contact manifold reduces to*

$$
R_{X_iX_aX_iX_a} = \frac{l^2}{4}; \quad R_{X_aX_bX_aX_b} = -\frac{3l^2}{4}.
$$

2.2 The Ricci tensor

Proposition 4 *The matrix representing the Ricci tensor is given by*

$$
\begin{pmatrix}
\frac{l^2}{2}(K^2+1) & 0 & 0 & -mlx(K+2) & mlw(K+2) & -mlz(K+2) & mly(K+2) \\
0 & \frac{l^2}{2}(K^2+1) & 0 & -mly(K+2) & mlz(K+2) & mlw(K+2) & -mlx(K+2) \\
0 & 0 & \frac{l^2}{2}(K^2+1) & -mlz(K+2) & -mly(K+2) & mlx(K+2) & mlw(K+2) \\
-mlx(K+2) & -mly(K+2) & -mlz(K+2) & A(K-1-mw^2)+B & ml^2(K+1)wx & ml^2(K+1)wy & ml^2(K+1)wz \\
mlw(K+2) & mlz(K+2) & -mly(K+2) & ml^2(K+1)wx & A(K-1-mx^2)+B & ml^2(K+1)xy & ml^2(K+1)xz \\
-mlz(K+2) & mlw(K+2) & mlx(K+2) & ml^2(K+1)wy & ml^2(K+1)xy & A(K-1-my^2)+B & ml^2(K+1)yz \\
mly(K+2) & -mlx(K+2) & mlw(K+2) & ml^2(K+1)wz & ml^2(K+1)xz & ml^2(K+1)yz & A(K-1-mz^2)+B
\end{pmatrix}.
$$

where $A = -l^2(K+1)$ and $B = 12m - 3/2l^2$.

Some particular cases could be interesting, for instance we get the following Ricci matrix when $K = 1$ (or $m = 0$)

$$\mathrm{Ric}_1 = \begin{pmatrix} l^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & l^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & l^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3/2l^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3/2l^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3/2l^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -3/2l^2 \end{pmatrix}$$

Remark 5 *When $l = 1$, we find the Ricci curvature of the quaternionic contact manifold.*

An easy computation leads to

Corollary 6 *The EBCV manifold has constant scalar curvature $S = 48m$.*

3 The characteristic connection on the $EBCV$ manifold

We consider on $EBCV$ the characteristic connection D defined by (see [7])

$$D_L M = \nabla_L M + \frac{P}{2}(\nabla_L P)M,$$

where P is the natural almost product structure given by $P = \mathcal{V} - \mathcal{H}$, $Id = \mathcal{V} + \mathcal{H}$ and L, M are arbitrary vector fields. Let us remember that the vertical distribution in $EBCV$ is spanned by X_1, X_2, X_3 and the horizontal distribution by X_4, X_5, X_6, X_7. Then we have

$$\begin{aligned} D_{X_i} X_j &= \mathcal{V}(\nabla_{X_i} X_j), \ i, j = 1, 2, 3, \\ D_{X_a} X_j &= \mathcal{V}(\nabla_{X_a} X_j), \ a = 4 \ldots, 7; \ j = 1, 2, 3, \\ D_{X_i} X_b &= \mathcal{H}(\nabla_{X_i} X_b), \ i = 1, 2, 3; \ b = 4, \ldots, 7, \\ D_{X_a} X_b &= \mathcal{H}(\nabla_{X_a} X_b), \ a, b = 4, \ldots, 7. \end{aligned}$$

This is a metric connection which can be completely obtained by using the table giving the Levi-Civita connection.

Following the classification given by A. M. Naveira for almost product structures, [12], we have

Proposition 7 *$(EBCV, P)$ is in (TGF, AF) class.*

To prove this proposition it is enough to see that $\nabla_A(P)B = 0$, when A, B are vertical, and $\nabla_X(P)X = 0$, if X is horizontal. The result follows using the tables given in the Appendix for the Levi-Civita connection.

When we consider $m = 0$, it is known that $EBCV$ is a homogeneous manifold. Indeed, it is the quaternionic contact group (see [9, 19]). In [1] W. Ambrose and I. M. Singer proved that a connected, complete and simply-connected Riemannian manifold (M, g) is homogeneous if and only if there exists a (1,2) tensor field T such that

(i) $g(T_X Y, Z) + g(Y, T_X Z) = 0$,

(ii) $(\nabla_X R)_{YZ} = [T_x, R_{YZ}] - R_{T_X YZ} - R_{Y T_X Z}$,

(iii) $(\nabla_X T)_Y = [T_X, T_Y] - T_{T_X Y}$,

for $X, Y, Z \in \mathfrak{X}(M)$, where ∇ stands for the Levi-Civita connection and R is the Riemann curvature tensor of M (see [16]). As a consequence, Tricerri and Vanhecke define a homogeneous Riemannian structure on (M, g) as a (1,2) tensor field T which is a solution of the above three equations. Instead of taking (1,2) tensors it is prefered to work with (0,3) tensors via the isomorphism $T_{uvw} = g(T_u v, w)$, for $u, v, w \in T_p M$ and $p \in M$. So far we have not been able to find the tensor field T satisfying the above conditions, which are equivalent to those given in page 14 of [16]. Then allow us to state the following question: is $EBCV$ a homogeneous manifold?

Let $T = T^D$ be the torsion tensor of the connection D, that is,

$$T^D_L M \equiv T^D(L, M) = D_L M - D_M L - [L, M],$$

or equivalently

$$T^D(L, M) = \frac{P}{2}\left((\nabla_L P)M - (\nabla_M P)L\right).$$

Then we find out

$$T^D(X_k, X_k) = 0,\ k = 1, \dots 7.$$

There are non-vanishing components such as $T^D(X_i, X_a)$ or $T^D(X_a, X_b)$, for instance,

$$T^D(X_1, X_4) = \frac{l}{2}\{1 + m(y^2 + z^2)\}X_5 + \frac{lm}{2}(wz - xy)X_6 - \frac{lm}{2}(wy + xz)X_7$$

or

$$T^D(X_4, X_5) = l\{(1 + m(y^2 + z^2))X_1 - m(wz + xy)X_2 + m(wy - xz)X_3\}.$$

On the other hand, it is easy to see that

(a) $T^D_{X_1 X_4 X_5} + T^D_{X_5 X_1 X_4} + T^D_{X_4 X_5 X_1} = \langle T^D_{X_1} X_4, X_5\rangle + \langle T^D_{X_5} X_1, X_4\rangle + \langle T^D_{X_4} X_5, X_1\rangle = 2l \neq 0$;

(b) $T^D_{XYZ} + T^D_{YXZ} = 0$.

4 Killing vector fields in $EBCV$

Remember that a Killing vector field is a vector field on a Riemannian manifold that preserves the metric. Killing vector fields are the infinitesimal generators of isometries, that is, flows generated by Killing fields are continuous isometries of the manifold. Specifically, a vector field X is a Killing vector field if the Lie derivative with respect to X of the metric g vanishes: $\mathcal{L}_X g = 0$ or equivalently

$$\mathcal{L}_X \mathrm{ds}^2_{l,m} = (\mathcal{L}_X \omega^\alpha) \otimes \omega^\alpha = 0, \tag{1}$$

where

$$\mathcal{L}_X \omega^\alpha = \iota_X d\omega^\alpha + d(\iota_X \omega^\alpha).$$

In terms of the Levi-Civita connection, Killing's condition is equivalent to

$$g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0. \tag{2}$$

It is easy to prove that

Proposition 8 *$\mathcal{L}_X g(Y,Z) = 0$ if and only if $\mathcal{L}_X g(X_i, X_j) = 0$ for basic vector fields X_i, X_j.*

We know that the dimension of the Lie algebra of the Killing vector fields is $m \leq n(n+1)/2$ and the maximum is reached on constant curvature manifolds ([8], p. 238, Vol. II) , then for our manifold $m < 28$. Then obviously

Proposition 9 *The basic vertical vector fields X_1, X_2, X_3 are Killing fields.*

From (2) it is easy to prove that the horizontal basic vector fields $X_4, \ldots, X_7$ are not Killing vector fields.

In [13] the Levi-Civita connection, curvature tensor and Killing vector fields on Bianchi-Cartan-Vranceanu spaces are introduced. In [14] Piu and Profir proved that the Lie algebra of Killing vector fields of BCV spaces is 4-dimensional for generic parameters m and l.

Now we are going to determine the space of Killing vector fields in $EBCV$.

4.1 The Killing equations

In the usual coordinate system (r, s, t, w, x, y, z) on $EBCV$, a vector field $X = \sum_{\alpha=1}^{7} f_\alpha X_\alpha$ will be a Killing field if and only if the real functions f_i satisfy the following system of 28-partial differential equations:

$$\partial_r(f_1)=0,$$
$$\partial_s(f_2)=0,$$
$$\partial_t(f_3)=0,$$
$$\partial_r(f_2)+\partial_s(f_1)=0,$$
$$\partial_r(f_3)+\partial_t(f_1)=0,$$
$$\partial_s(f_3)+\partial_t(f_2)=0,$$
$$\partial_r(f_4)+K\partial_w(f_1)+\tfrac{ly}{2}\partial_s(f_1)+\tfrac{lz}{2}\partial_t(f_1)-l\{1+m(y^2+z^2)\}f_5-ml(wz-xy)f_6+ml(wy+xz)f_7=0,$$
$$\partial_r(f_5)+K\partial_x(f_1)-\tfrac{lz}{2}\partial_s(f_1)+\tfrac{ly}{2}\partial_t(f_1)+l\{1+m(y^2+z^2)\}f_4-ml(wy+xz)f_6-ml(wz-xy)f_7=0,$$
$$\partial_r(f_6)+K\partial_y(f_1)-\tfrac{lw}{2}\partial_s(f_1)-\tfrac{lx}{2}\partial_t(f_1)+ml(wz-xy)f_4+ml(wy+xz)f_5-l\{1+m(w^2+x^2\}f_7=0,$$
$$\partial_r(f_7)+K\partial_z(f_1)+\tfrac{lx}{2}\partial_s(f_1)-\tfrac{lw}{2}\partial_t(f_1)-ml(wy+xz)f_4+ml(wz-xy)f_5+l\{1+m(w^2+x^2\}f_6=0,$$
$$\partial_s(f_4)+K\partial_w(f_2)+\tfrac{lx}{2}\partial_r(f_2)+\tfrac{lz}{2}\partial_t(f_2)+ml(wz+xy)f_5-l\{1+m(x^2+z^2)\}f_6-ml(wx-yz)f_7=0,$$
$$\partial_s(f_5)+K\partial_x(f_2)-\tfrac{lw}{2}\partial_r(f_2)+\tfrac{ly}{2}\partial_t(f_2)-ml(wz+xy)f_4+ml(wx-yz)f_6+l\{1+m(w^2+y^2)\}f_7=0,$$
$$\partial_s(f_6)+K\partial_y(f_2)-\tfrac{lz}{2}\partial_r(f_2)-\tfrac{lx}{2}\partial_t(f_2)+l\{1+m(x^2+z^2)\}f_4-ml(wx-yz)f_5-ml(wz+xy)f_7=0,$$
$$\partial_s(f_7)+K\partial_z(f_2)-\tfrac{ly}{2}\partial_r(f_2)-\tfrac{lw}{2}\partial_t(f_2)+ml(wx-yz)f_4-l\{1+m(w^2+y^2)\}f_5+ml(wz+xy)f_6=0,$$
$$\partial_t(f_4)+K\partial_w(f_3)+\tfrac{lx}{2}\partial_r(f_3)+\tfrac{ly}{2}\partial_s(f_3)+ml(wy-xz)f_5+ml(wx+yz)f_6-l\{1+m(x^2+y^2)\}f_7=0,$$
$$\partial_t(f_5)+K\partial_x(f_3)-\tfrac{lw}{2}\partial_r(f_3)-\tfrac{lz}{2}\partial_s(f_3)+ml(wy-xz)\}f_4-l\{1+m(w^2+z^2)\}f_6+ml(wx+yz)f_7=0,$$
$$\partial_t(f_6)+K\partial_y(f_3)+\tfrac{lz}{2}\partial_r(f_3)-\tfrac{lw}{2}\partial_s(f_3)-ml(wx+yz)f_4+l\{1+m(w^2+z^2\}f_5+ml(wy-xz)f_7=0,$$
$$\partial_t(f_7)+K\partial_z(f_3)-\tfrac{ly}{2}\partial_r(f_3)+\tfrac{lx}{2}\partial_s(f_3)+l\{1+m(x^2+y^2)\}f_4-ml(wx+yz)f_5-ml(wy-xz)f_6=0,$$
$$K\partial_w(f_4)+\tfrac{lx}{2}\partial_r(f_4)+\tfrac{ly}{2}\partial_s(f_4)+\tfrac{lz}{2}\partial_t(f_4)-2mxf_5-2myf_6-2mzf_7=0,$$
$$K\partial_w(f_5)+\tfrac{lx}{2}\partial_r(f_5)+\tfrac{ly}{2}\partial_s(f_5)+\tfrac{lz}{2}\partial_t(f_5)+K\partial_x(f_4)-\tfrac{lw}{2}\partial_r(f_4)-\tfrac{lz}{2}\partial_s(f_4)+\tfrac{ly}{2}\partial_t(f_4)+2mxf_4+2mwf_5=0,$$
$$K\partial_w(f_6)+\tfrac{lx}{2}\partial_r(f_6)+\tfrac{ly}{2}\partial_s(f_6)+\tfrac{lz}{2}\partial_t(f_6)+K\partial_y(f_4)+\tfrac{lz}{2}\partial_r(f_4)-\tfrac{lw}{2}\partial_s(f_4)-\tfrac{lx}{2}\partial_t(f_4)+2myf_4+2mwf_6=0,$$
$$K\partial_w(f_7)+\tfrac{lx}{2}\partial_r(f_7)+\tfrac{ly}{2}\partial_s(f_7)+\tfrac{lz}{2}\partial_t(f_7)+K\partial_z(f_4)-\tfrac{ly}{2}\partial_r(f_4)+\tfrac{lx}{2}\partial_s(f_4)-\tfrac{lw}{2}\partial_t(f_4)+2mzf_4+2mwf_7=0,$$
$$K\partial_x(f_5)-\tfrac{lw}{2}\partial_r(f_5)-\tfrac{lz}{2}\partial_s(f_5)+\tfrac{ly}{2}\partial_t(f_5)-2mwf_4-2myf_6-2mzf_7=0,$$
$$K\partial_x(f_6)-\tfrac{lw}{2}\partial_r(f_6)-\tfrac{lz}{2}\partial_s(f_6)+\tfrac{ly}{2}\partial_t(f_6)+K\partial_y(f_5)+\tfrac{lz}{2}\partial_r(f_5)-\tfrac{lw}{2}\partial_s(f_5)-\tfrac{lx}{2}\partial_t(f_5)+2myf_5+2mxf_6=0,$$
$$K\partial_x(f_7)-\tfrac{lw}{2}\partial_r(f_7)-\tfrac{lz}{2}\partial_s(f_7)+\tfrac{ly}{2}\partial_t(f_7)+K\partial_z(f_5)-\tfrac{ly}{2}\partial_r(f_5)+\tfrac{lx}{2}\partial_s(f_5)-\tfrac{lw}{2}\partial_t(f_5)+2mzf_5+2mxf_7=0,$$
$$K\partial_y(f_6)+\tfrac{lz}{2}\partial_r(f_6)-\tfrac{lw}{2}\partial_s(f_6)-\tfrac{lx}{2}\partial_t(f_6)-2mwf_4-2mxf_5-2mzf_7=0,$$
$$K\partial_y(f_7)+\tfrac{lz}{2}\partial_r(f_7)-\tfrac{lw}{2}\partial_s(f_7)-\tfrac{lx}{2}\partial_t(f_7)+K\partial_z(f_6)-\tfrac{ly}{2}\partial_r(f_6)+\tfrac{lx}{2}\partial_s(f_6)-\tfrac{lw}{2}\partial_t(f_6)+2mzf_6+2myf_7=0,$$
$$K\partial_z(f_7)-\tfrac{ly}{2}\partial_r(f_7)+\tfrac{lx}{2}\partial_s(f_7)-\tfrac{lw}{2}\partial_t(f_7)-2mwf_4-2mxf_5-2myf_6=0.$$

It seems that the solution of the system is very difficult, so that we focus on solving the system for $m = 0$, that is:

$$
\begin{aligned}
&\partial_r(f_1) = 0,\\
&\partial_s(f_2) = 0,\\
&\partial_t(f_3) = 0,\\
&\partial_r(f_2) + \partial_s(f_1) = 0,\\
&\partial_r(f_3) + \partial_t(f_1) = 0,\\
&\partial_s(f_3) + \partial_t(f_2) = 0,\\
&\partial_r(f_4) + \partial_w(f_1) + \tfrac{ly}{2}\partial_s(f_1) + \tfrac{lz}{2}\partial_t(f_1) - lf_5 = 0,\\
&\partial_r(f_5) + \partial_x(f_1) - \tfrac{lz}{2}\partial_s(f_1) + \tfrac{ly}{2}\partial_t(f_1) + lf_4 = 0,\\
&\partial_r(f_6) + \partial_y(f_1) - \tfrac{lw}{2}\partial_s(f_1) - \tfrac{lx}{2}\partial_t(f_1) - lf_7 = 0,\\
&\partial_r(f_7) + \partial_z(f_1) + \tfrac{lx}{2}\partial_s(f_1) - \tfrac{lw}{2}\partial_t(f_1) + lf_6 = 0,\\
&\partial_s(f_4) + \partial_w(f_2) + \tfrac{lx}{2}\partial_r(f_2) + \tfrac{lz}{2}\partial_t(f_2) - lf_6 = 0,\\
&\partial_s(f_5) + \partial_x(f_2) - \tfrac{lw}{2}\partial_r(f_2) + \tfrac{ly}{2}\partial_t(f_2) + lf_7 = 0,\\
&\partial_s(f_6) + \partial_y(f_2) + \tfrac{lz}{2}\partial_r(f_2) - \tfrac{lx}{2}\partial_t(f_2) + lf_4 = 0,\\
&\partial_s(f_7) + \partial_z(f_2) - \tfrac{ly}{2}\partial_r(f_2) - \tfrac{lw}{2}\partial_t(f_2) - lf_5 = 0,\\
&\partial_t(f_4) + \partial_w(f_3) + \tfrac{lx}{2}\partial_r(f_3) + \tfrac{ly}{2}\partial_s(f_3) - lf_7 = 0,\\
&\partial_t(f_5) + \partial_x(f_3) - \tfrac{lw}{2}\partial_r(f_3) - \tfrac{lz}{2}\partial_s(f_3) - lf_6 = 0,\\
&\partial_t(f_6) + \partial_y(f_3) + \tfrac{lz}{2}\partial_r(f_3) - \tfrac{lw}{2}\partial_s(f_3) + lf_5 = 0,\\
&\partial_t(f_7) + \partial_z(f_3) - \tfrac{ly}{2}\partial_r(f_3) + \tfrac{lx}{2}\partial_s(f_3) + lf_4 = 0,\\
&\partial_w(f_4) + \tfrac{lx}{2}\partial_r(f_4) + \tfrac{ly}{2}\partial_s(f_4) + \tfrac{lz}{2}\partial_t(f_4) = 0,\\
&\partial_w(f_5) + \tfrac{lx}{2}\partial_r(f_5) + \tfrac{ly}{2}\partial_s(f_5) + \tfrac{lz}{2}\partial_t(f_5) + \partial_x(f_4) - \tfrac{lw}{2}\partial_r(f_4) - \tfrac{lz}{2}\partial_s(f_4) + \tfrac{ly}{2}\partial_t(f_4) = 0,\\
&\partial_w(f_6) + \tfrac{lx}{2}\partial_r(f_6) + \tfrac{ly}{2}\partial_s(f_6) + \tfrac{lz}{2}\partial_t(f_6) + \partial_y(f_4) + \tfrac{lz}{2}\partial_r(f_4) - \tfrac{lw}{2}\partial_s(f_4) - \tfrac{lx}{2}\partial_t(f_4) = 0,\\
&\partial_w(f_7) + \tfrac{lx}{2}\partial_r(f_7) + \tfrac{ly}{2}\partial_s(f_7) + \tfrac{lz}{2}\partial_t(f_7) + \partial_z(f_4) - \tfrac{ly}{2}\partial_r(f_4) + \tfrac{lx}{2}\partial_s(f_4) - \tfrac{lw}{2}\partial_t(f_4) = 0,\\
&\partial_x(f_5) - \tfrac{lw}{2}\partial_r(f_5) - \tfrac{lz}{2}\partial_s(f_5) + \tfrac{ly}{2}\partial_t(f_5) = 0,\\
&\partial_x(f_6) - \tfrac{lw}{2}\partial_r(f_6) - \tfrac{lz}{2}\partial_s(f_6) + \tfrac{ly}{2}\partial_t(f_6) + \partial_y(f_5) + \tfrac{lz}{2}\partial_r(f_5) - \tfrac{lw}{2}\partial_s(f_5) - \tfrac{lx}{2}\partial_t(f_5) = 0,\\
&\partial_x(f_7) - \tfrac{lw}{2}\partial_r(f_7) - \tfrac{lz}{2}\partial_s(f_7) + \tfrac{ly}{2}\partial_t(f_7) + \partial_z(f_5) - \tfrac{ly}{2}\partial_r(f_5) + \tfrac{lx}{2}\partial_s(f_5) - \tfrac{lw}{2}\partial_t(f_5) = 0,\\
&\partial_y(f_6) + \tfrac{lz}{2}\partial_r(f_6) - \tfrac{lw}{2}\partial_s(f_6) - \tfrac{lx}{2}\partial_t(f_6) = 0,\\
&\partial_y(f_7) + \tfrac{lz}{2}\partial_r(f_7) - \tfrac{lw}{2}\partial_s(f_7) - \tfrac{lx}{2}\partial_t(f_7) + \partial_z(f_6) - \tfrac{ly}{2}\partial_r(f_6) + \tfrac{lx}{2}\partial_s(f_6) - \tfrac{lw}{2}\partial_t(f_6) = 0,\\
&\partial_z(f_7) - \tfrac{ly}{2}\partial_r(f_7) + \tfrac{lx}{2}\partial_s(f_7) - \tfrac{lw}{2}\partial_t(f_7) = 0.
\end{aligned}
$$

whose solution is given by

$$
\begin{aligned}
f_1(r,s,t,w,x,y,z) &= (P+R)s + (S-N)t + \frac{l}{2}\{-M(w^2+x^2) - U(y^2+z^2) + (R-P)(wy+xz) \\
&\quad + (N+S)(wz-xy) + 2Tw - 2Qx + 2Wy - 2Vz\} + C_1, \\
f_2(r,s,t,w,x,y,z) &= -(P+R)r + (M+U)t - \frac{l}{2}\{N(w^2+y^2) - S(x^2+z^2) + (R-P)(wx-yz) \\
&\quad + (M-U)(wz+xy) - 2Vw + 2Wx + 2Qy - 2Tz\} + C_2, \\
f_3(r,s,t,w,x,y,z) &= -(S-N)r - (M+U)s - \frac{l}{2}\{P(w^2+z^2) + R(x^2+y^2) + (N+S)(wx+yz) \\
&\quad + (U-M)(wy-xz) - 2Ww - 2Vx + 2Ty + 2Qz\} + C_3, \\
f_4(r,s,t,w,x,y,z) &= Mx + Ny + Pz + Q, \\
f_5(r,s,t,w,x,y,z) &= -Mw + Ry + Sz + T, \\
f_6(r,s,t,w,x,y,z) &= -Nw - Rx + Uz + V, \\
f_7(r,s,t,w,x,y,z) &= -Pw - Sx - Uy + W,
\end{aligned}
$$

where $M, N, P, Q, R, S, T, U, V, W, C_1, C_2, C_3 \in \mathbb{R}$.

As a consequence, when $m = 0$, we obtain

Proposition 10 *The Lie algebra of Killing vector fields is 13-dimensional.*

5 Computing horizontal geodesics of the quaternionic Heisenberg group

Following the computations in [11], the vector fields

$$
\begin{aligned}
W &= \partial_w + \frac{1}{2}(x\partial_r + y\partial_s + z\partial_t), \\
X &= \partial_x - \frac{1}{2}(w\partial_r + z\partial_s - y\partial_t), \\
Y &= \partial_y + \frac{1}{2}(z\partial_r - w\partial_s - x\partial_t), \\
Z &= \partial_z - \frac{1}{2}(y\partial_r - x\partial_s - w\partial_t),
\end{aligned}
$$

which are the old $X_4, \ldots, X_7$ ones, provided $m = 0, l = 1$, along with $\{\partial_r, \partial_s, \partial_t\}$, form an orthonormal frame for the quaternionic contact manifold $\mathbb{H} \times \mathrm{Im}\mathbb{H}$. This means that $\{W, X, YZ\}$ frame the fourth plane $\mathcal{H}$ and they are orthonormal with respect to the inner product $\mathrm{ds}^2 = (dw^2 + dx^2 + dy^2 + dz^2)|_{\mathcal{H}}$ on the distribution. The sub-Riemannian Hamiltonian writes down as

$$
H = \frac{1}{2}(P_W^2 + P_X^2 + P_Y^2 + P_Z^2), \tag{3}
$$

where P_W, P_X, P_Y, P_Z are the momentum functions of the vector fields W, X, Y, Z, respectively. Thus

$$\begin{aligned}
P_W &= p_w + \frac{1}{2}(xp_r + yp_s + zp_t), \\
P_X &= p_x - \frac{1}{2}(wp_r + zp_s - yp_t), \\
P_Y &= p_y + \frac{1}{2}(zp_r - wp_s - xp_t), \\
P_Z &= p_z - \frac{1}{2}(yp_r - xp_s + wp_t),
\end{aligned}$$

where $p_w, p_x, p_y, p_z, p_r, p_s, p_t$ are the fiber coordinates on the cotangent bundle of $\mathbb{R}^7$ corresponding to the cartesian coordinates w, x, y, z, r, s, t on $\mathbb{R}^7$. Again, these fiber coordinates are defined by writing a covector as $p = p_w dw + p_x dx + p_y dy + p_z dz + p_r dr + p_s ds + p_t dt$. Together, $(w, x, y, z, r, s, t, p_w, p_x, p_y, p_z, p_r, p_s, p_t)$ are global coordinates on the cotangent bundle $T^*\mathbb{R}^7 = \mathbb{R}^7 \oplus \mathbb{R}^7$. Hamilton's equations can be written

$$\frac{df}{du} = \{f, H\}, \quad f \in C^\infty(T^*\mathbb{R}^7), \tag{4}$$

which holds for any smooth function f. The function H defines a vector field X_H, called the Hamiltonian vector field, which has a flow $\Phi_u : T^*\mathbb{R}^7 \to T^*\mathbb{R}^7$. Let $f : T^*\mathbb{R}^7 \to \mathbb{R}$ be any smooth function on the cotangent bundle. Form the u-dependent function $f_u = \Phi_u^* f$ by pulling f back via the flow. Thus $f_u(w, x, y, z, r, s, t, p_w, p_x, p_y, p_z, p_r, p_s, p_t) = f(\Phi_u(w, x, y, z, r, s, t, p_w, p_x, p_y, p_z, p_r, p_s, p_t))$. In other words $\frac{df}{du} = X_H[f_u]$, which gives meaning to the left-hand side of Hamilton's equations.

To define the right hand side, which is to say the vector field X_H, we will need the Poisson bracket. The Poisson bracket on the cotangent bundle $T^*\mathbb{R}^7$ of a manifold $\mathbb{R}^7$ is a canonical Lie algebra structure defined on the vector space $C^\infty(T^*\mathbb{R}^7)$ of smooth functions on $T^*\mathbb{R}^7$. The Poisson bracket is denoted $\{\cdot, \cdot\} : C^\infty \times C^\infty \to C^\infty$, where $C^\infty = C^\infty(T^*\mathbb{R}^7)$, and can be defined by the coordinate formula

$$\{f, g\} = \frac{\partial f}{\partial x^i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial x^i}\frac{\partial f}{\partial p_i}.$$

This formula is valid in any canonical coordinate system, and can be shown to be coordinate independent. The Poisson bracket satisfies the Leibniz identity

$$\{f, gh\} = g\{f, h\} + h\{f, g\},$$

which means that the operation $\{., H\}$ defines a vector field X_H, called the Hamiltonian vector field. By letting the functions f vary over the collection of coordinate functions x^i and we get the more common form of Hamilton's equations

$$\dot{x}^\alpha = \frac{\partial H}{\partial p_\alpha}, \quad \dot{p}_\alpha = -\frac{\partial H}{\partial x^\alpha}.$$

Indeed, for the first one we take $f = w$ and $g = H$. Then $\{w, H\} = \frac{\partial w}{\partial x^i}\frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial x^i}\frac{\partial w}{\partial p_i}$ if and only if $\dot{w} = \frac{\partial H}{\partial p_w}$. Also we have

$$\dot{x} = \frac{\partial H}{\partial p_x}, \quad \dot{y} = \frac{\partial H}{\partial p_y}, \quad \dot{z} = \frac{\partial H}{\partial p_z}.$$

These equations are in turn equivalent to the above formulation (4), which is more convenient to use, because the momentum function $W \mapsto P_W$ is a Lie algebra anti-homomorphism from the Lie algebra of all smooth vector fields on $\mathbb{R}^7$ to $C(T^*\mathbb{R}^7)$ with the Poisson brackets:

$$\begin{aligned} \{P_W, P_X\} = -\, P_{[W,X]}, \quad \{P_W, P_Y\} = -P_{[W,Y]}, \quad \{P_W, P_Z\} = -P_{[W,Z]}, \\ \{P_X, P_Y\} = -\, P_{[X,Y]}, \quad \{P_X, P_Z\} = -P_{[X,Z]}, \quad \{P_Y, P_Z\} = -P_{[Y,Z]}. \end{aligned} \tag{5}$$

Since all calculations are similar, we only prove the first one:

$$\{P_W, P_X\} = \{p_w + \frac{x}{2}p_r + \frac{y}{2}p_s + \frac{z}{2}p_t, p_x - \frac{w}{2}p_r - \frac{z}{2}p_s + \frac{y}{2}p_t\} = p_r = -P_{[W,X]}.$$

For the quaternionic contact group, with our choose of W, X, Y, Z as a frame for $\mathcal{H}$, we compute

$$[W, X] = -\partial_r, \quad [W, Y] = -\partial_s, \quad [W, Z] = -\partial_t,$$

$$[X, Y] = -\partial_t, \quad [X, Z] = \partial_s, \quad [Y, Z] = -\partial_t,$$

$$[W, \partial_r] = [W, \partial_r s] = [W, \partial_t] = [X, \partial_r] = [X, \partial_s] = [X, \partial_t] = 0,$$

$$[Y, \partial_r] = [Y, \partial_s] = [Y, \partial_t] = [Z, \partial_r] = [Z, \partial_s] = [Z, \partial_t] = 0.$$

Thus

$$\{P_W, P_X\} = \partial_r := P_r, \quad \{P_W, P_Y\} = \partial_s := P_s, \quad \{P_W, P_Z\} = \partial_t := P_t,$$

$$\{P_X, P_Y\} = P_t, \quad \{P_X, P_Z\} = -p_s = -P_s, \quad \{P_Y, P_Z\} = p_r = P_r$$

We can prove that

$$\begin{aligned} \{P_W, P_r\} = \{P_W, P_s\} = \{P_W, P_t\} = \{P_X, P_r\} = \{P_X, P_s\} = \{P_X, P_t\} = 0, \\ \{P_Y, P_r\} = \{P_Y, P_s\} = \{P_Y, P_t\} = \{P_Z, P_r\} = \{P_Z, P_s\} = \{P_Z, P_t\} = 0. \end{aligned}$$

These relations can also easily be computed by hand, from our formulae for P_W, P_X, P_Y, P_Z and the bracket in terms of $w, x, y, z, r, s, r, p_w, p_x, p_y, p_z, p_r, p_s, p_t$.

Lemma 11 *By letting f vary over the functions $w, x, y, z, r, s, r, P_W, P_X, P_Y, P_Z, P_r, P_s, P_t$, using the bracket relations and equation* (5)*, we find that Hamilton's equations are equivalent to the system*

$$\begin{aligned} &\dot{w} = P_W, && \dot{P}_W = p_r P_X + p_s P_Y + p_t P_Z, \\ &\dot{x} = P_X, && \dot{P}_X = -p_r P_W - p_s P_Z + p_t P_Y, \\ &\dot{y} = P_Y, && \dot{P}_Y = p_r P_Z - p_s P_W - p_t P_X, \\ &\dot{z} = P_Z, && \dot{P}_Z = -p_r P_Y + p_s P_X - p_t P_W, \\ &\dot{r} = \tfrac{1}{2}(xP_W - wP_X + zP_Y - yP_Z), && \dot{P}_r = 0, \\ &\dot{s} = \tfrac{1}{2}(yP_W - zP_X + xP_Y - wP_Z), && \dot{P}_s = 0, \\ &\dot{t} = \tfrac{1}{2}(zP_W + yP_X - xP_Y - wP_Z), && \dot{P}_t = 0. \end{aligned}$$

To see it, remember that $H = \frac{1}{2}(P_W^2 + P_X^2 + P_Y^2 + P_z^2)$. Then

$$\begin{aligned}
\dot{w} &= \{w, H\} = P_w \frac{\partial P_W}{\partial p_w} = P_W, \\
\dot{x} &= \{x, H\} = P_X \frac{\partial P_X}{\partial p_x} = P_X, \\
\dot{y} &= P_Y, \\
\dot{z} &= P_Z.
\end{aligned}$$

Also, considering that:

$$\frac{\partial P_W}{\partial p_r} = \frac{x}{2}, \qquad \frac{\partial P_W}{\partial p_s} = \frac{y}{2}, \qquad \frac{\partial P_W}{\partial p_t} = \frac{z}{2},$$

$$\frac{\partial P_X}{\partial p_r} = -\frac{w}{2}, \qquad \frac{\partial P_X}{\partial p_s} = -\frac{z}{2}, \qquad \frac{\partial P_X}{\partial p_t} = \frac{y}{2},$$

$$\frac{\partial P_Y}{\partial p_r} = \frac{z}{2}, \qquad \frac{\partial P_Y}{\partial p_s} = -\frac{w}{2}, \qquad \frac{\partial P_Y}{\partial p_t} = -\frac{x}{2},$$

$$\frac{\partial P_Z}{\partial p_r} = -\frac{y}{2}, \qquad \frac{\partial P_Z}{\partial p_s} = \frac{x}{2}, \qquad \frac{\partial P_Z}{\partial p_t} = -\frac{w}{2},$$

we have

$$\dot{r} = \frac{1}{2}(xP_W - wP_X + zP_Y - yP_Z).$$

Indeed,

$$\begin{aligned}
\dot{r} = \{r, H\} &= P_W \frac{\partial P_W}{\partial p_r} + P_X \frac{\partial P_X}{\partial p_r} + P_Y \frac{\partial P_Y}{\partial p_r} + P_Z \frac{\partial P_Z}{\partial p_r} \\
&= \frac{1}{2}(xP_W - wP_X + zP_Y - yP_Z) \\
\dot{s} = \{s, H\} &= P_W \frac{\partial P_W}{\partial p_s} + P_X \frac{\partial P_X}{\partial p_s} + P_Y \frac{\partial P_Y}{\partial p_s} + P_Z \frac{\partial P_Z}{\partial p_s} \\
&= \frac{1}{2}(yP_W - zP_X + xP_Y - wP_Z) \\
\dot{t} = \{t, H\} &= P_W \frac{\partial P_W}{\partial p_t} + P_X \frac{\partial P_X}{\partial p_t} + P_Y \frac{\partial P_Y}{\partial p_t} + P_Z \frac{\partial P_Z}{\partial p_t} \\
&= \frac{1}{2}(zP_W + yP_X - xP_Y - wP_Z).
\end{aligned}$$

Working as above we obtain

$$\begin{aligned}
\dot{P}_W &= \{P_W, H\} = p_r P_X + p_s P_Y + p_t P_Z, \\
\dot{P}_X &= \{P_X, H\} = -p_r P_W - p_s P_Z + p_t P_Y, \\
\dot{P}_Y &= \{P_Y, H\} = p_r P_Z - p_s P_W - p_t P_X, \\
\dot{P}_Z &= \{P_Z, H\} = -p_r P_Y + p_s P_X - p_t P_W.
\end{aligned}$$

Then we are ready to show the following

Theorem 12 *The horizontal geodesics of the quaternionic Heisenberg group are exactly the horizontal lifts of arcs of circles, including line segments as a degenerate case.*

Proof. It is not difficult to see that $\dot{P}_r = \dot{P}_s = \dot{P}_t = 0$. These equations assert that $P_r = p_r$, $P_s = p_s$ and $P_t = p_t$ are constant. The variables r, s, t appears nowhere in the right-hand sides of these equations. It follows that the variables $w, x, y, z, P_W, P_X, P_Y, P_Z$ evolve independently of r, s, t, and so we can view the system as defining a one-parameter family of dynamical systems on $\mathbb{R}^8$ parameterized by the constant value of P_r, P_s, P_t.

Combine w, x, y, z into a single quaternionic variable $\omega = w + ix + jy + kz$ and taking into account the fourteen equations one has

$$\frac{d\omega}{du} = P_W + iP_X + jP_Y + kP_Z$$

The u-derivative of $P_W + iP_X + jP_Y + kP_Z$ is $-(ip_r + jp_s + kp_t)(P_W + iP_X + jP_Y + kP_Z)$. Then we have $\frac{d^2\omega}{du^2} = -(ip_r + jp_s + kp_t)\frac{d\omega}{du}$, where p_r, p_s and p_t are constant.

By integrating the above expression we get

$$P_W + iP_X + jP_Y + kP_Z = P(0)\exp(-(ip_r + jp_s + kp_t)t),$$

where $P(0) = P_W(0) + iP_X(0) + jP_Y(0) + kP_Z(0)$.

A second integration yields the general form of the geodesics on the quaternionic contact group:

$$\begin{aligned}
&\omega(u) = w(u) + ix(u) + jy(u) + kz(u) = \\
&\qquad \frac{P(0)}{ip_r + jp_s + kp_t}\left(\exp(-(ip_r + jp_s + kp_t)t - 1) + w(0) + ix(0) + jy(0) + kz(0)\right), \\
&r(u) = r(0) + \frac{1}{2}\int_0^t \mathrm{Im}_I(\bar{\omega}\, d\omega), \\
&s(u) = s(0) + \frac{1}{2}\int_0^t \mathrm{Im}_J(\bar{\omega}\, d\omega), \\
&r(u) = t(0) + \frac{1}{2}\int_0^t \mathrm{Im}_K(\bar{\omega}\, d\omega).
\end{aligned}$$

6 Appendix

The brackets:

$$[X_4, X_5] = -l\{1 + m(y^2 + z^2)\}X_1 + ml(wz + xy)X_2 - ml(wy - xz)X_3 - 2mxX_4 + 2mwX_5,$$
$$[X_4, X_6] = -ml(wz - xy)X_1 - l\{1 + m(x^2 + z^2)\}X_2 + ml(wx + yz)X_3 - 2myX_4 + 2mwX_6,$$
$$[X_4, X_7] = ml(wy + xz)X_1 - ml(wx - yz)X_2 - l\{1 + (x^2 + y^2)\}X_3 - 2mzX_4 + 2mwX_7,$$
$$[X_5, X_6] = -ml(wy + xz)X_1 + ml(wx - yz)X_2 - l\{1 + m(w^2 + z^2)\}X_3 - 2myX_5 + 2mxX_6,$$
$$[X_5, X_7] = ml(xy - wz)X_1 + l\{1 + m(w^2 + y^2)\}X_2 + ml(wx + yz)X_3 - 2mzX_5 + 2mxX_7,$$
$$[X_6, X_7] = -l\{1 + m(w^2 + x^2)\}X_1 - ml(wz + xy)X_2 + ml(wy - xz)X_3 - 2mzX_6 + 2myX_7.$$

The Levi-Civita connection:

$$\nabla_{X_1}X_4 = \frac{l}{2}\{1 + m(y^2 + z^2)\}X_5 + \frac{ml}{2}(wz - xy)X_6 - \frac{ml}{2}(wy + xz)X_7,$$
$$\nabla_{X_1}X_5 = -\frac{l}{2}\{1 + m(y^2 + z^2)\}X_4 + \frac{ml}{2}(wy + xz)X_6 + \frac{ml}{2}(wz - xy)X_7,$$
$$\nabla_{X_1}X_6 = -\frac{ml}{2}(wz - xy)X_4 - \frac{ml}{2}(wy + xz)X_5 + \frac{l}{2}\{1 + m(w^2 + x^2)\}X_7,$$
$$\nabla_{X_1}X_7 = \frac{ml}{2}(wy + xz)X_4 - \frac{ml}{2}(wz - xy)X_5 - \frac{l}{2}\{1 + m(w^2 + x^2)\}X_6,$$
$$\nabla_{X_2}X_4 = -\frac{ml}{2}(wz + xy)X_5 + \frac{l}{2}\{1 + m(x^2 + z^2)\}X_6 + \frac{ml}{2}(wx - yz)X_7,$$
$$\nabla_{X_2}X_5 = \frac{ml}{2}(wz + xy)X_4 - \frac{ml}{2}(wx - yz)X_6 - \frac{l}{2}\{1 + m(w^2 + y^2)\}X_7,$$
$$\nabla_{X_2}X_6 = -\frac{l}{2}\{1 + m(x^2 + z^2)\}X_4 + \frac{ml}{2}(wx - yz)X_5 + \frac{ml}{2}(wz + xy)X_7,$$
$$\nabla_{X_2}X_7 = -\frac{ml}{2}(wx - yz)X_4 + \frac{l}{2}\{1 + m(w^2 + y^2)\}X_5 - \frac{ml}{2}(wz + xy)X_6,$$
$$\nabla_{X_3}X_4 = \frac{ml}{2}(wy - xz)X_5 - \frac{ml}{2}(wx + yz)X_6 + \frac{l}{2}\{1 + m(x^2 + y^2)\}X_7,$$
$$\nabla_{X_3}X_5 = -\frac{ml}{2}(wy - xz)X_4 + \frac{l}{2}\{1 + m(w^2 + z^2)\}X_6 - \frac{ml}{2}(wx + yz)X_7,$$
$$\nabla_{X_3}X_6 = \frac{ml}{2}(wx + yz)X_4 - \frac{l}{2}\{1 + m(w^2 + z^2)\}X_5 - \frac{ml}{2}(wy - xz)X_7,$$
$$\nabla_{X_3}X_7 = -\frac{l}{2}\{1 + m(x^2 + y^2)\}X_4 + \frac{ml}{2}(wx + yz)X_5 + \frac{ml}{2}(wy - xz)X_6,$$
$$\nabla_{X_4}X_4 = 2m(xX_5 + yX_6 + zX_7),$$
$$\nabla_{X_4}X_5 = -\frac{l}{2}\{1 + m(y^2 + z^2)\}X_1 + \frac{ml}{2}(wz + xy)X_2 - \frac{ml}{2}(wy - xz)X_3 - 2mxX_4,$$
$$\nabla_{X_4}X_6 = -\frac{ml}{2}(wz - xy)X_1 - \frac{l}{2}\{1 + m(x^2 + z^2)\}X_2 + \frac{ml}{2}(wx + yz)X_3 - 2myX_4,$$
$$\nabla_{X_4}X_7 = \frac{ml}{2}(wy + xz)X_1 - \frac{ml}{2}(wx - yz)X_2 - \frac{l}{2}\{1 + m(x^2 + y^2)\}X_3 - 2mzX_4,$$
$$\nabla_{X_5}X_4 = \frac{l}{2}\{1 + m(y^2 + z^2)\}X_1 - \frac{ml}{2}(wz + xy)X_2 + \frac{ml}{2}(wy - xz)X_3 - 2mwX_5,$$
$$\nabla_{X_5}X_5 = 2m(wX_4 + yX_6 + zX_7),$$
$$\nabla_{X_5}X_6 = -\frac{ml}{2}(wy + xz)X_1 + \frac{ml}{2}(wx - yz)X_2 - \frac{l}{2}\{1 + m(w^2 + z^2)\}X_3 - 2myX_5,$$
$$\nabla_{X_5}X_7 = -\frac{ml}{2}(wz - xy)X_1 + \frac{l}{2}\{1 + m(w^2 + y^2)\}X_2 + \frac{ml}{2}(wx + yz)X_3 - 2mzX_5,$$

$$
\begin{aligned}
&\nabla_{X_6} X_4 = \frac{ml}{2}(wz - xy)X_1 + \frac{l}{2}\{1 + m(x^2 + z^2)\}X_2 - \frac{ml}{2}(wx + yz)X_3 - 2mwX_6,\\
&\nabla_{X_6} X_5 = \frac{ml}{2}(wy + xz)X_1 - \frac{ml}{2}(wx - yz)X_2 + \frac{l}{2}\{1 + m(w^2 + z^2)\}X_3 - 2mxX_6,\\
&\nabla_{X_6} X_6 = 2m(wX_4 + xX_5 + zX_7),\\
&\nabla_{X_6} X_7 = -\frac{l}{2}\{1 + m(w^2 + x^2)\}X_1 - \frac{ml}{2}(wz + xy)X_2 + \frac{ml}{2}(wy - xz)X_3 - 2mzX_6,\\
&\nabla_{X_7} X_4 = -\frac{ml}{2}(wy + xz)X_1 + \frac{ml}{2}(wx - yz)X_2 + \frac{l}{2}\{1 + m(x^2 + y^2)\}X_3 - 2mwX_7,\\
&\nabla_{X_7} X_5 = \frac{ml}{2}(wz - xy)X_1 - \frac{l}{2}\{1 + m(w^2 + y^2)\}X_2 - \frac{ml}{2}(wx + yz)X_3 - 2mxX_7,\\
&\nabla_{X_7} X_6 = \frac{l}{2}\{1 + m(w^2 + x^2)\}X_1 + \frac{ml}{2}(wz + xy)X_2 - \frac{ml}{2}(wy - xz)X_3 - 2myX_7,\\
&\nabla_{X_7} X_7 = 2m(wX_4 + xX_5 + yX_6).
\end{aligned}
$$

The curvature tensor:

$$
\begin{aligned}
R_{X_1X_4X_1X_4} = R_{X_1X_5X_1X_5} &= \frac{l^2}{4}\{1 + m(K+1)(y^2 + z^2)\},\\
R_{X_1X_6X_1X_6} = R_{X_1X_7X_1X_7} &= \frac{l^2}{4}\{1 + m(K+1)(w^2 + x^2)\},\\
R_{X_2X_4X_2X_4} = R_{X_2X_6X_2X_6} &= \frac{l^2}{4}\{1 + m(K+1)(x^2 + z^2)\},\\
R_{X_2X_5X_2X_5} = R_{X_2X_7X_2X_7} &= \frac{l^2}{4}\{1 + m(K+1)(w^2 + y^2)\},\\
R_{X_3X_4X_3X_4} = R_{X_3X_7X_3X_7} &= \frac{l^2}{4}\{1 + m(K+1)(x^2 + y^2)\},\\
R_{X_3X_5X_3X_5} = R_{X_3X_6X_3X_6} &= \frac{l^2}{4}\{1 + m(K+1)(w^2 + z^2)\},\\
R_{X_4X_5X_4X_5} &= 4m - 3R_{X_1X_4X_1X_4},\\
R_{X_4X_6X_4X_6} &= 4m - 3R_{X_2X_4X_2X_4},\\
R_{X_4X_7X_4X_7} &= 4m - 3R_{X_3X_4X_3X_4},\\
R_{X_5X_6X_5X_6} &= 4m - 3R_{X_3X_5X_3X_5},\\
R_{X_5X_7X_5X_7} &= 4m - 3R_{X_2X_5X_2X_5},\\
R_{X_6X_7X_6X_7} &= 4m - 3R_{X_1X_6X_1X_6}.
\end{aligned}
$$

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